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REAL ANALYTIC MACHINES AND DEGREES: A TOPOLOGICAL VIEW ON ALGEBRAIC LIMITING COMPUTATION *

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Abstract. We study and compare in degree-theoretic ways (iterated Halting oracles analogous to Kleene's arithmetical hierarchy, and the Borel hierarchy of descriptive set theory) the capabilities and limitations of three models of real computation: BSS machines (aka real-RAM) and strongly/weakly analytic machines as introduced by Hotz et al. (1995).

1. Introduction

The Turing machine as standard model of (finite) computation and computational complexity over discrete universes like $\mathbb{N}$ or $\{0,1\}^*$ suggests two both natural but distinct ways of extension to real numbers: Already Alan Turing [Turi37] considered *in*finitely long calculations producing as output a sequence of integer fractions $r_n/s_n \in \mathbb{Q}$ (i.e. discrete objects) approximating some real $x \in \mathbb{R}$ up to error 2^{-n}.

The first model (dominant in Recursive Analysis) reflects that actual digital computers can operate in each step only on finite information, and in particular with limited precision on real numbers [Weih00, Theorem 4.3.1]. It is, however, often criticized [Koep01] for the consequence that any computable function must necessarily be continuous. In the second model, simple discontinuous functions like Heaviside or Gauß Staircase are computable, whereas intricate and intuitively non-computable functions (such as the characteristic of Mandelbrot's fractal set) can indeed be proven uncomputable. Criticism of this model

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arises from its ability to compute certain pathological functions [Weih00, Example 9.7.2] but not as simple functions as square root or exponential. A formal introduction to this model is deferred to Section 1.4. We refer to [Zhon98, BoVi99, Brat00] for comparisons between both models.

1.1. **Oracle Computation and Turing Degrees.** Classical (i.e. discrete) computability and complexity theory had, almost from the very beginning [Turi39], started considering machines with access to oracles: not so much to model actual computational practice, but because it permits to formally compare problems according to their degree of uncomputability or complexity [Soar87, Papa94]. For a universe $\mathcal{U}$ like $\mathbb{N}$ or, equivalently (cmp. Example 2.1 below), $\{0,1\}^*$, the Halting problem $H \subseteq \mathcal{U}$ (sometimes denoted as *jump* of $\emptyset$) for instance becomes trivially decidable when granted oracle access to H; whereas the iterated Halting problem (or *jump* of H)

$$H^H \;:=\; \big\{\langle M,\vec{x}\rangle : \text{oracle Turing machine } M^H \text{ terminates on input } \vec{x}\in\mathcal{U}\big\} \;, \tag{1.1}$$

that is the question of termination of a given such Turing machine with H-oracle, remains undecidable by an H-oracle Turing machine.

The class of semidecidable problems is often denoted as Σ_1; Π_1 are their complements, and $\Delta_1 = \Sigma_1 \cap \Pi_1$ the decidable problems. Σ_2 are problems of the form

$$\big\{\vec{x}\in\mathcal{U} \;\big|\; \exists\vec{y}\in\mathcal{U}\;\forall\vec{z}\in\mathcal{U}: \;\langle\vec{x},\vec{y},\vec{z}\rangle \in T\big\} \tag{1.2}$$

with decidable $T \subseteq \mathcal{U}$; Π_2 consists of complements of Σ_2-problems, that is problems defined by "$\forall\exists$"-formulas. Similarly, "$\exists\forall\exists$" defines Σ_3; and so on: Kleene's *Arithmetical Hierarchy.* Post's *Theorem* [Soar87, Section IV.2] asserts for a set $S \subseteq \mathcal{U}$ the following to be equivalent:

a) S is semidecidable relative to the Turing Halting Problem H.
b) $S \in \Sigma_2$, i.e. has the form (1.2) with decidable T.
c) S is many-one reducible to the iterated Halting problem (written "$S \preceq H^H$").

Analogously, S is semidecidable relative to H^H iff $S \in \Sigma_3$ iff S is many-one reducible to H^{H^H}. Some natural Σ_2-complete and Σ_3-complete problems are identified in [Soar87, Section IV.3].

Similar investigations have been pursued in both the BSS model [Cuck92] and in Recursive Analysis [Ho99, ZhWe01]. However, and as opposed to the discrete setting, oracles turn out to be of limited help in the case of real functions: both square root and exponential still remain uncomputable to a BSS machine; and computability in Recursive Analysis remains restricted to continuous functions [Zie07a, Zie07b].

1.2. **Limiting Computation.** This subsumes calculations whose result occurs after $\geq \omega$ steps. It may include transfinite cases [HaLe00] although we shall restrict to computations producing a countable sequence of outputs that 'converge' to the result. For finite binary strings (i.e. w.r.t. the discrete topology), this appears in the literature as *limiting recursion* [Gold65, Schm02], *inductive algorithms* [Burg04], or *trial-and-error predicates* [Putn65].

On continuous universes like Cantor or Baire space or reals, a classical Turing machine processing finite information in each step needs an infinite amount of time. This led to the field of Recursive Analysis ($\mathbb{R}^d$) and Weihrauch's Type-2 Theory ($\{0,1\}^\omega$, $\mathbb{N}^\omega$). Here, convergence is required to be *effective* in the sense that the n-approximate output differs from the ultimate result by no more than an absolute error of 2^{-n}; equivalently, the approximations are to be accompanied by absolute error bounds tending to 0.

On the other hand, relaxing the output (but not input [BrHe02, SECTION 6]) to merely converging approximations does render some discontinuous functions computable; and in fact corresponds to climbing up the effective Borel Hierarchy [Brat05]:

1.3. **Topological Complexity and Borel Degrees.** Recall the Borel Hierarchy on an arbitrary topological space X: $\mathbf{\Sigma}_1(X)$ denotes the family of open subsets of X, $\mathbf{\Pi}_1(X)$ that of closed sets (i.e. complements of open ones), $\mathbf{\Sigma}_2(X)$ the class of countable unions of closed sets (aka F_σ), $\mathbf{\Pi}_2(X)$ that of countable intersections of open sets (aka G_δ, i.e. complements of F_σ), and iteratively $\mathbf{\Sigma}_k(X)$ the class of countable unions of $\mathbf{\Pi}_{k-1}$-sets, and $\mathbf{\Pi}_k(X)$ their complements; $\mathbf{\Delta}_k(X) = \mathbf{\Sigma}_k(X) \cap \mathbf{\Pi}_k(X)$.

We shall frequently make use of the folklore

Fact 1.1.
a) It holds $\mathbb{Q} \in \mathbf{\Sigma}_2(\mathbb{R}) \setminus \mathbf{\Pi}_2(\mathbb{R})$.
b) It holds $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) \notin \mathbf{\Pi}_2(\mathbb{R}^2) \cup \mathbf{\Sigma}_2(\mathbb{R}^2)$.
c) If $X \in \mathbf{\Sigma}_k(Y)$ and $Y \in \mathbf{\Sigma}_k(Z)$, then $X \in \mathbf{\Sigma}_k(Z)$; similarly for $\mathbf{\Pi}_k$.

Proof.
a) $\mathbb{Q}$ is a countable union of (closed) singletons, hence in $\mathbf{\Sigma}_2$. Suppose $\mathbb{Q} \in \mathbf{\Pi}_2$, i.e. $\mathbb{Q}^\complement = \bigcup_n A_n$ with closed A_n. But $\mathbb{R}\setminus\mathbb{Q}$ being of second category according to *Baire's Theorem*, there has to exist at least one with nonempty interior: $\emptyset \neq \overline{A_n}^\circ = A_n^\circ \subseteq (\mathbb{R} \setminus \mathbb{Q})^\circ$, a contradiction.
b) $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ contains as sections both $\mathbb{Q} \notin \mathbf{\Pi}_2$ and $\mathbb{R} \setminus \mathbb{Q} \notin \mathbf{\Sigma}_2$, hence cannot be in $\mathbf{\Pi}_2$ nor in $\mathbf{\Sigma}_2$.
c) Since Y is equipped with the relative topology of Z, induction on k shows that $X \in \mathbf{\Sigma}_k(Y)$ implies $X = Y \cap X'$ for some $X' \in \mathbf{\Sigma}_k(Z)$; and the latter class is closed under finite intersection. □

If X is a polish space, the Borel hierarchy is strict and contains complete members [Kech95]. According to Alexandrov's Theorem, every G_δ–subset of a Polish space (such as $\mathbb{R}^d$) is again Polish. In the sequel, all spaces X under consideration will be (not necessarily closed) subspaces of some $\mathbb{R}^d$ ($d \in \mathbb{N}$) equipped with the Euclidean topology.

Also recall that continuity of a (total) function $f : X \to Y$ means that preimages $f^{-1}[V]$ of open sets $V \subseteq Y$ are open in X; and, more generally, f is called $\mathbf{\Sigma}_k$-measurable if preimages of open subsets of Y are in $\mathbf{\Sigma}_k(X)$.

As common in classical computability theory, partial functions $f :\subseteq \mathbb{R}^d \to \mathbb{R}$ arise naturally also in the real case. We distinguish between the Borel complexity of their domain $\mathrm{dom}(f) \subseteq \mathbb{R}^d$ and that of the *total* function $f : \mathrm{dom}(f) \to \mathbb{R}$; cmp. Remark 1.4 below.

1.4. **Blum-Shub-Smale Machines.** The BSS model (over $\mathbb{R}$) considers real numbers as entities that can be read, stored, output, added, subtracted, multiplied, divided, and compared exactly. It captures the semantics of, e.g., the `FORTRAN` programming language and essentially coincides with the real-RAM model underlying, e.g., Algorithmic Geometry [BKOS97]. Specifically, a program consists of a sequence of arithmetic instructions ($+, -, \times, \div$) and branchings based on tests ($=, <$). A countably infinite sequence of working registers can be accessed directly or via indirect addressing through dedicated integer-valued index registers. Assignments ($a := b$) may copy data between registers or initialize them to one of finitely many real constants listed in the program. Upon start, the input vector $(x_1, \ldots, x_d) \in \mathbb{R}^d$ is provided in the real registers and, for uniformity purposes, its dimension d in the index registers. If the machine terminates within finitely many steps, the contents of the real registers (up to index given by the index register) is considered as output. For further details we refer to [BCSS98] or [Cuck92, Definition 1.1].

Definition 1.2. For $\mathbb{X} \subseteq \mathbb{R}^*$, a set $\mathbb{L} \subseteq \mathbb{X}$ is called BSS-*decidable in* $\mathbb{X}$ if some BSS machine can, given any $\vec{x} \in \mathbb{X}$, report within finite time which one of $\vec{x} \in \mathbb{L}$ or $\vec{x} \notin \mathbb{L}$ holds. $\mathbb{L}$ is BSS-*semidecidable in* $\mathbb{X}$ if some BSS machine terminates for every $\vec{x} \in \mathbb{L}$ and does not terminate ('*diverges*') for every $\vec{x} \in \mathbb{X} \setminus \mathbb{L}$. A (possibly partial) function $f :\subseteq \mathbb{R}^* \to \mathbb{R}^*$ is BSS-*computable in* $\mathbb{X}$ if some BSS machine, on inputs $\vec{x} \in \mathrm{dom}(f) \cap \mathbb{X}$, terminates with output $f(\vec{x})$ and diverges on inputs $\vec{x} \in \mathbb{X} \setminus \mathrm{dom}(f)$.

Note that this common notion [BCSS98, Definition 4.2] ignores the behavior on inputs outside of $\mathbb{X}$. In the case $\mathbb{X} = \mathbb{R}^*$, we simply speak of BSS (semi-)decidability/computability of $\mathbb{L}/f$.

1.5. **The Arithmetical Hierarchy for** BSS **Machines.** [Cuck92, Theorems 2.11+2.13] has succeeded in generalizing Post's Theorem to oracle BSS machines, however based on entirely different arguments:

Fact 1.3. For a set $\mathbb{S} \subseteq \mathbb{R}^*$ the following are equivalent:

a) $\mathbb{S}$ is semidecidable by a BSS machine with oracle access to $\mathbb{H}$.
b) There exists a BSS-decidable set $\mathbb{W} \subseteq \mathbb{R}^*$ such that
$$\mathbb{S} \;=\; \big\{\vec{x} \in \mathbb{R}^* \;\big|\; \exists y \in \mathbb{N}\, \forall z \in \mathbb{N} :\; (\vec{x}, y, z) \in \mathbb{W}\big\}\;. \tag{1.3}$$
c) $\mathbb{S}$ is BSS many-one reducible to the *iterated* BSS Halting problem:
$$\mathbb{S} \;\preccurlyeq\; \mathbb{H}^{\mathbb{H}} \;:=\; \big\{(\langle\mathcal{M}\rangle, \vec{x}) : \text{oracle BSS machine } \mathcal{M}^{\mathbb{H}} \text{ terminates on input } \vec{x}\big\}\;.$$

Observe that, in spite of referring to a real complexity class, quantifiers in Equation (1.3) range over integers.

1.6. **Analytic Machines.** In [HVS95, ChHo99], a third model and kind of synthesis of the above two had been proposed: An *analytic machine* is essentially a BSS machine (i.e. with *exact* arithmetic operations and tests) permitted to *approximate* the output; either with (*strongly analytic*) or without (*weakly analytic*) error bounds. More precisely, let $\|\vec{y}\| := \sum_i |y_i|$ denote the 1-norm and $\dim(\vec{y}) = d$ for $\vec{y} \in \mathbb{R}^d$. Then computing $f : \mathbb{R}^* \to \mathbb{R}^*$ means producing, on input $\vec{x} \in \mathrm{dom}(f)$, some infinite sequence $\vec{y}_n \in \mathbb{R}^*$ with, for $\vec{y} := f(\vec{x})$,
$$\begin{array}{lcll} \dim(\vec{y}_n) = \dim(\vec{y}) & \text{and} & \|\vec{y}_n - \vec{y}\| \le 2^{-n} \quad \text{for all } n \in \mathbb{N} & \text{(strong)} \\ \dim(\vec{y}_n) = \dim(\vec{y}) & \multicolumn{2}{l}{\text{for all but finitely many } n \text{ and } \lim_n \vec{y}_n = \vec{y}} & \text{(weak)} \end{array} \tag{1.4}$$
Another variation depends on whether the 'program' may employ finitely many pre-stored real constants or not. (Several further variants considered in [ChHo99] are outside the scope of the present work...) Now Heaviside and square root and exponential function are easily seen to be computable by a strongly analytic machine. In fact, every computation in Recursive Analysis or by a BSS machine (without constants) can be simulated on a strongly analytic machine (without constants).

Remark 1.4. Recall that a BSS machine computing a partial function $f :\subseteq \mathbb{R}^* \to \mathbb{R}^*$ is required to diverge on inputs $\vec{x} \notin \mathrm{dom}(f)$. This convention is common in the BSS community and corresponds to what in Recursive Analysis is called *strong computability* [Weih00, Exercise 4.3.18]. Similarly, a strongly/weakly analytic machine computing f must violate Equation (1.4) for $\vec{x} \notin \mathrm{dom}(f)$.

Note that in the strong case, this amounts to the output of either only a finite sequence (i.e. a terminating computation or an indefinite one which, however, eventually ceases to print further approximations) or of one for which the following fails:

$$\|\vec{y}_n - \vec{y}_m\| \;\le\; 2^{-n} + 2^{-m} \quad \forall n, m \;. \tag{1.5}$$

As a consequence, the domain (not to mention its complement) of a strongly analytically computable partial function in general need not be BSS-semidecidable (i.e. the domain of a BSS-computable function):

Examples 1.5.

a) There exists a BSS-computable function $f :\subseteq \mathbb{R}^* \to \{1\}$ with $\mathrm{dom}(f) = \mathbb{H}$, the Halting problem for BSS machines:

$$\mathbb{H} \quad := \quad \big\{(\langle\mathcal{M}\rangle, \vec{x}) : \text{constant-free BSS machine } \mathcal{M} \text{ terminates on input } \vec{x} \in \mathbb{R}^*\big\}$$

b) There is a function $g :\subseteq \mathbb{R}^* \to \{1\}$ computable by a strongly analytic machine with $\mathrm{dom}(g) = \mathbb{H}^\complement$.

Proof.

a) Define $f(\langle\mathcal{M}\rangle, \vec{x}) := 1$ for $(\langle\mathcal{M}\rangle, \vec{x}) \in \mathbb{H}$, $f(\langle\mathcal{M}\rangle, \vec{x}) := \perp$ otherwise. A universal BSS machine can obviously compute this.

b) Define $g(\langle\mathcal{M}\rangle, \vec{x}) := \perp$ for $(\langle\mathcal{M}\rangle, \vec{x}) \in \mathbb{H}$, $g(\langle\mathcal{M}\rangle, \vec{x}) := 1$ otherwise. Consider a variant of the universal BSS machine which, upon input $(\langle\mathcal{M}\rangle, \vec{x})$, simulates the computation of $\mathcal{M}$ on $\vec{x}$ and at each step outputs 1s but switches to alternating its outputs between 0s and 1s at each step when $\mathcal{M}$ terminates. $\square$

We will explore this discrepancy more closely in Corollary 2.6a+d). Technically this means that certain of our results (have to) refer to *some extension* of a given partial function; cmp. Definition 1.2.

1.7. **Overview.** Section 2 explores the power of strongly analytic machines by comparison to classical BSS machines. Roughly speaking, it turns out that BSS-computable total functions are $\mathbf{\Delta}_2$-measurable and partial functions have $\mathbf{\Sigma}_2$-measurable domains; whereas total functions computable by strongly analytic machines cover all $\mathbf{\Sigma}_1$-measurable (i.e. continuous) ones while including also some $\mathbf{\Sigma}_2$-measurable ones—but in order to cover all of them, the composition of two such functions is sufficient and in general necessary. Partial functions here have $\mathbf{\Pi}_3$-measurable domains.

We then (Section 3) proceed to weakly analytic machines. These are characterized as strongly analytic ones relative to the BSS halting problem by establishing a real variant of the Shoenfield Limit Lemma. The proof is assisted by a particular notion of weak semidecidability (Section 3.1) many-one equivalent to the iterated (i.e. jump of the) BSS Halting problem. Similar to the strongly analytic case from Example 1.5, Section 3.3 reveals the convergence problem (i.e. domain of a function computable by a weakly analytic machine) to not be weakly semidecidable in general. In fact we show the complement of this problem equivalent to the jump of the iterated BSS Halting problem.

2. Exploring the Power of Strongly Analytic Machines

Both Recursive Analysis and the BSS model exhibit many properties similar to the classical theory of computation—although not all of them. Analytic Machines behave even more nicely in this sense:

Examples 2.1. A major part of discrete recursion theory relies on the existence of a bicomputable pairing function, that is a bijective encoding

$$\mathbb{N}\times\mathbb{N} \ \ni\ (x,y) \ \mapsto\ \langle x,y\rangle \ \in\ \mathbb{N}$$

and decoding of pairs of integers into a single one, both computable by a Turing machine. ($\langle x,y\rangle := 2^{x-1}\cdot(2y-1)$ for instance will do...)
This significantly differs from the real case where there is no BSS-computable (even local) real pairing function:

a) Let $\emptyset\neq U\subseteq\mathbb{R}^2$ be open and $f:U\to\mathbb{R}$ injective. Then f is not BSS-computable.
b) Let $\emptyset\neq U\subseteq\mathbb{R}^2$ be open and $g:\subseteq\mathbb{R}\to U$ surjective. Then g is not BSS-computable relative to any (set) oracle.
c) There is, however, a strongly analytic machine without constants computing a total bijection $h:\mathbb{R}^2\to\mathbb{R}$ and its inverse.

Proof.

a) Suppose f is computable by some BSS machine; then its path decomposition yields a non-empty open ball $B\subseteq U$ such that $f|_B$ is a rational function and in particular continuous. But a continuous f from a convex open $B\subseteq\mathbb{R}^2$ to $\mathbb{R}$ cannot be injective: Consider $\vec{u},\vec{v}\in B$ and two disjoint paths $g,h:[0,1]\to B$ connecting them, that is with $g(0)=\vec{u}=h(0)$ and $g(1)=\vec{v}=h(1)$ and $g([0,1])\cap h([0,1])=\{\vec{u},\vec{v}\}$. By continuity, it follows from the intermediate value theorem that $f\circ g$ and $f\circ h$ must have a common value on the open interval $(0,1)$. This contradicts the assumption of f being injective.
b) Recall that for fields $\mathbb{F}\subseteq\mathbb{E}$, the transcendence degree of $S\subseteq\mathbb{E}$ (over $\mathbb{F}$) $\operatorname{trdeg}_{\mathbb{F}}(S)$ denotes the cardinality of a largest subset of S algebraically independent over $\mathbb{F}$; equivalently: of a least $T\subseteq S$ such that $\mathbb{F}(S)$ is algebraic over $\mathbb{F}(T)$. In particular, a finite S is algebraically independent over a finite field extension $\mathbb{F}(T)$ iff $\operatorname{trdeg}_{\mathbb{F}}(S\cup T)=\operatorname{Card}(S)+\operatorname{trdeg}_{\mathbb{F}}(T)$ [Cohn91, Proposition 5.1.2].
Now observe that the output of a BSS-machine $\mathcal{M}$ with constants $c_1,\dots,c_d$ on input $x_1,\dots,x_n$ is limited to the rational field extension $\mathbb{Q}(c_1,\dots,c_d,x_1,\dots,x_n)$. On the other hand, the transcendence degree $\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{R})$ of $\mathbb{R}$ over $\mathbb{Q}$ is infinite. In particular, there exist $y,z\in\mathbb{R}$ with $\{y,z\}$ algebraically independent over $\mathbb{Q}(c_1,\dots,c_d)$. Now $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and $U\subseteq\mathbb{R}^2$ is open; hence there are $p,q\in\mathbb{Q}$ with $(y+p,z+q)\in U$; and $\{y+p,z+q\}$ remains algebraically independent over $\mathbb{Q}(c_1,\dots,c_d)$. So suppose there is a machine $\mathcal{M}$ with constants $c_1,\dots,c_d$ computing g and in particular $(y+p,z+q)=g(x)$ for some x. Then $y+p,z+q\in\mathbb{Q}(c_1,\dots,c_d,x)$ by the above observation, hence $\operatorname{trdeg}_{\mathbb{Q}}(y+p,z+q,c_1,\dots,c_d)\leq\operatorname{trdeg}_{\mathbb{Q}}(c_1,\dots,c_d,x)\leq\operatorname{trdeg}_{\mathbb{Q}}(c_1,\dots,c_d)+1$; whereas algebraic independence of $\{y+p,z+q\}$ over $\mathbb{Q}(c_1,\dots,c_d)$ requires $\operatorname{trdeg}_{\mathbb{Q}}(y+p,z+q,c_1,\dots,c_d)=2+\operatorname{trdeg}_{\mathbb{Q}}(c_1,\dots,c_d)$: contradiction.

c) Observe that the mapping

$$[0,1) \ni x = \sum_{n=1}^{\infty} b_n 2^{-n} \mapsto (b_1, b_2, \ldots, b_n, \ldots) \in \{0,1\}^{\mathbb{N}} ,$$

extracting from a given real number its* binary expansion is computable digit by digit by (exact) comparison: For $n = 1$, let $b_n := 1$ in case $x \geq 1/2$ and $b_n := 0$ otherwise; then iterate with $x \mapsto 2x - 1$ and $n \mapsto n + 1$.

Conversely, a strongly analytic machine (but no BSS machine) can encode any binary sequence b_n (say, of intermediate results) into a real number $\sum_{n=1}^{\infty} b_n 2^{-n} \in [0,1]$ by approximations $\sum_{n=1}^{N} b_n 2^{-n}$ up to error 2^{-N}.

We will thus decode both $x, y \in [0,1)$ into their binary expansions a_n and b_n, merge the latter into $c_{2n} := a_n$ and $c_{2n-1} := b_n$, and then re-code $(c_m)_m$ into $z \in [0,1)$.

This procedure obviously extends from $[0,1)$ to $[0,\infty)$ with binary expansion $\sum_{n=-k}^{\infty} a_n 2^{-n}$ and, incorporating also $\operatorname{sign}(x)$ and $\operatorname{sign}(y)$, to a pairing function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Its inverse can be computed similarly by a strongly analytic machine: From a given $z \in \mathbb{R}$, extract from its binary expansion the (sub)sequence of those digits with even/odd index and compose them into (approximations up to error 2^{-n} of) $x, y \in \mathbb{R}$ with $h(x,y) = z$. □

It is clear that a BSS machine without constants cannot compute the constant function $f(x) \equiv c$ unless $c \in \mathbb{Q}$; and oracles do not help. This is different for analytic machines:

Proposition 2.2. *Let $\mathbb{H}$ denote the Halting problem for* BSS *machines from Example 1.5a). To every strongly/weakly analytic machine $\mathcal{M}$ with* recursive *constants, there exists a strongly/weakly analytic oracle machine $\mathcal{N}$ such that $\mathcal{N}^{\mathbb{H}}$ is equivalent to $\mathcal{M}$.*

Proof. Let $c_1, \ldots, c_k \in \mathbb{R}$ denote the constants of an analytic machine $\mathcal{M}$. Observe that each computation of $\mathcal{M}$ on some input $\vec{x} \in \mathbb{R}^d$ can be described as an infinite sequence of elementary operations (arithmetic on two intermediate results, branch based on testing some intermediate result, output of an intermediate result); where each intermediate result is a rational function $R(x_1, \ldots, x_d, c_1, \ldots, c_k)$ with rational coefficients. Now given such an input $\vec{x}$, let $\mathcal{N}^{\mathbb{H}}$ symbolically record the intermediate calculations R performed by $\mathcal{M}$ up to the first test "$R(\vec{x}, \vec{c}) : 0$?" Since $\vec{c}$ is presumed recursive, so is $R(\vec{x}, \vec{c})$ and can be output up to any desired precision whenever $\mathcal{M}$ would output $R(\vec{x}, \vec{c})$ exactly. Another consequence, both "$R(\vec{x}, \vec{c}) < 0$" and "$R(\vec{x}, \vec{c}) > 0$" are semidecidable; thus by querying $\mathbb{H}$, $\mathcal{N}^{\mathbb{H}}$ can decide "$R(\vec{x}, \vec{c}) = 0$" and proceed with the simulation of $\mathcal{M}$'s control flow accordingly. □

The following technical tool will be used in the sequel:

Fact 2.3. For a set $S \subseteq \mathbb{R}^k$, consider its *distance function*

$$\operatorname{dist}(\cdot; S) : \mathbb{R}^k \to [0, \infty], \quad \vec{x} \mapsto \inf \{ \|\vec{x} - \vec{y}\| : \vec{y} \in S \} . \tag{2.1}$$

a) The function $\operatorname{dist}(\cdot; S)$ is always continuous.
b) If S is closed, the infimum in Equation (2.1) is a minimum, i.e. attained.
c) For closed $A, B \subseteq \mathbb{R}^k$, it holds $\operatorname{dist}(\vec{x}; A \cup B) = \min\{\operatorname{dist}(\vec{x}; A), \operatorname{dist}(\vec{x}; B)\}$.

Example 2.4. CANTOR's *Excluded Middle* set $\mathcal{C} \subseteq [0,1]$ is closed; its distance function $x \mapsto \operatorname{dist}(x; \mathcal{C})$ is computable by a strongly analytic machine without constants.

*Dyadic rationals $x = (2\ell + 1)/2^k$ have two distinct such expansions. For the purpose of well-definition, we here refer to the one with only finitely many 1s.

Proof. Recall that $\mathcal{C}$ consists of all real numbers $x \in [0,1]$ having a ternary expansion $x = \sum_{n=1}^{\infty} c_n 3^{-n}$ with $c_n \in \{0,2\}$ and is obviously closed.
Concerning computability of $x \mapsto \operatorname{dist}(x;\mathcal{C})$, observe that the sequence of open rational intervals $I_{\langle m,k\rangle} := \big((3k+1)/3^m, (3k+2)/3^m\big)$, $m \in \mathbb{N}$, $k = 0,\dots,3^{m-1}-1$ is recursive and exhausts $[0,1]\setminus\mathcal{C}$. Therefore the sequence y_m, defined by $y_m := \min\{|x-(3k+1)/3^m|, |x-(3k+2)/3^m|\}$ if $x \in I_{\langle m,k\rangle}$ for some k and $y_m := 0$ otherwise, is computable from x and has $\max\{y_1, y_2, \dots, y_m\}$ converging to $\operatorname{dist}(x;\mathcal{C})$ from below.
In order to approximate $\operatorname{dist}(x;\mathcal{C})$ from above, observe that the sequence $x_{\bar{c}} := \big(\sum_{n=1}^{|\bar{c}|} c_n 3^{-n}\big)_{\bar{c}\in\{0,2\}^*}$ (w.r.t. lexicographically ordered index set) is recursive and dense in $\mathcal{C}$. Therefore, $z_m := \min\{|x - x_{\bar{c}}| : \bar{c} \in \{0,2\}^{\leq m}\}$ is computable from x and converges to $\operatorname{dist}(x;\mathcal{C})$ from above.
So $y_m \leq \sup_m y_m = \operatorname{dist}(x;\mathcal{C}) = \inf_m z_m \leq z_m$ shows the existence of a subsequence y_{m_n} with $z_{m_n} - z_{m_n} \leq 2^{-n}$: this can be sought for computationally and satisfies $|y_{m_n} - \operatorname{dist}(y;\mathcal{C})| \leq 2^{-n}$ as required. □

2.1. **Topological Complexity of** BSS **and Strongly Analytic Computation.** While strongly analytic machines can compute strictly more (e.g., the exponential function) than BSS machines, the present section reveals that the topological complexity (in the sense of descriptive set theory) does not increase for decision problems, and increases only slightly for function problems.

Recall that, classically, a (possibly partial) function $f :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is computable iff its graph is semidecidable. The appropriate counterpart for a real function $f :\subseteq \mathbb{R}^* \to \mathbb{R}$ are the *strict epigraph* and *strict hypograph* [Brat00]:

$$\begin{aligned} \text{s-epigraph}(f) &:= \big\{(\vec{x}, y) : \vec{x} \in \operatorname{dom}(f), y > f(\vec{x})\big\}, \\ \text{s-hypograph}(f) &:= \big\{(\vec{x}, y) : \vec{x} \in \operatorname{dom}(f), y < f(\vec{x})\big\} \ . \end{aligned}$$

In the BSS model, the square root has both strict epigraph and strict hypograph decidable yet is not computable.

Theorem 2.5.

a) *Conversely, if both* s-epigraph(f) *and* s-hypograph(f) *are semidecidable by a* BSS *machine with/out constants, then f is computable by a strongly analytic machine with/out constants.*
b) *And if both* s-epigraph(f) *and* s-hypograph(f) *are semidecidable* in dom(f) *by a* BSS *machine with/out constants, then* some extension *of f is computable by a strongly analytic machine with/out constants.*
c) *A set $\mathbb{S} \subseteq \mathbb{R}^*$ is decidable by a* BSS *machine with/out constants iff its characteristic function $\mathbf{1}_{\mathbb{S}} : \mathbb{R}^* \to \{0,1\}$ is computable by a strongly analytic machine with/out constants.*
d) *Every open subset of $\mathbb{R}^k$ is* BSS-*semidecidable.*
e) *Every continuous (i.e. $\mathbf{\Sigma}_1$-measurable) total function $f : \mathbb{R}^d \to \mathbb{R}^k$ is computable by a strongly analytic machine;*
f) *and so is every countable family $f_\ell : \mathbb{R}^d \to \mathbb{R}^k$ ($\ell \in \mathbb{N}$) of continuous total functions.*

Note the subtle mismatch between a) and b+c) with respect to relative/absolute semidecidability, imposed by Example 1.5b). Similarly, Item f) does not extend to arbitrary partial functions.

We remark also that Item f), together with Fact 2.3, yields the distance function of the *Mandelbrot Set* to be computable by a strongly analytic machine—whereas its computability in Recursive Analysis is still an open question (and consequence of the *Hyperbolicity Conjecture*) [Hert05]; yet the *Mandelbrot Set* cannot be decided by a BSS[BCSS98, Theorem 2.4.2] nor (Item d) by a strongly analytic machine.

Proof. (Theorem 2.5)

a) Given $(\vec{x}, z)$ with $\vec{x} \in \mathrm{dom}(f)$, let the BSS machine simulate the strongly analytic computation of $y := f(\vec{x})$ and denote by (y_n) the output sequence it generates, i.e satisfying $|y_n - y| \leq 2^{-n}$. Now search for some n with $y_n + 2^{-n} < z$; if found, $y < z$ follows and the machine accepts $(\vec{x}, z) \in$ s-epigraph(f); and conversely, in case $(\vec{x}, z) \in$ s-epigraph(f), $y < z$ holds and there exists n with $y_n + 2^{-n} < z$.
Note that for $\vec{x} \notin \mathrm{dom}(f)$, such n may or may not exist; hence the described procedure semidecides s-epigraph(f) only *in* $\mathrm{dom}(f)$.

b) Let $\mathcal{M}$ and $\mathcal{N}$ denote BSS machines semideciding s-epigraph(f) and s-hypograph(f), respectively. In order to approximate $f(\vec{x})$ for given $\vec{x}$ up to error 2^{-n+1} simulate, for all $q \in \mathbb{Q}$ in parallel, both $\mathcal{M}$ on $(\vec{x}, q + 2^{-n})$ and $\mathcal{N}$ on $(\vec{x}, q - 2^{-n})$. If both terminate, $q - 2^{-n} < f(\vec{x}) < q + 2^{-n}$ justifies to output q; and, conversely, for $\vec{x} \in \mathrm{dom}(f)$, there is a rational approximation q to $f(\vec{x})$ up to error 2^{-n} for which both simulations terminate; whereas for $\vec{x} \notin \mathrm{dom}(f)$, both simulations stall by hypothesis.

c) Similarly to b), but now the behavior of $\mathcal{M}$ and $\mathcal{N}$ on $\vec{x} \notin \mathrm{dom}(f)$ is undefined. Hence the search for approximations may or may not yield a strongly converging output sequence, i.e. produce $\perp$ or some value y for $f(\vec{x})$.

d) If $\mathbb{S}$ is decidable by a BSS machine, its characteristic function is BSS-computable; hence strongly analytic, recall Section 1.6.
Conversely let $\mathbf{1}_\mathbb{S}$ be computable by a strongly analytic machine $\mathcal{M}$. Upon input of $\vec{x}$, $\mathcal{M}$ will thus output a sequence (y_n) of reals with $|y_n - \mathbf{1}_\mathbb{S}(\vec{x})| \leq 2^{-n}$. Since $\mathbf{1}_\mathbb{S}(\vec{x}) \in \{0, 1\}$, this means y_2 uniquely exhibits whether $\vec{x} \in \mathbb{S}$ or $\vec{x} \notin \mathbb{S}$ holds. A BSS machine thus suffices to simulate $\mathcal{M}$ for the finitely many steps it takes to generate y_2 and then output either 0 or 1 accordingly.

e) The open rectangles $(\vec{a}, \vec{b}) := \{\vec{x} : \vec{a} < \vec{x} < \vec{b}\}$ with rational corners $\vec{a}, \vec{b}$ are well-known to form a base of the Euclidean topology on $\mathbb{R}^k$; that is, every open $V \subseteq \mathbb{R}^k$ can be written as a countable union of certain such open rational rectangles $(\vec{a}_i, \vec{b}_i)$. Their corners' coordinates, being integer fractions, can be encoded in binary into a single real number [Cuck92, Lemma 2.3]. Perusing this as a pre-stored constant, a BSS machine can, given $\vec{x} \in \mathbb{R}^k$, iteratively extract the coordinates of the open rational rectangles and search for one to contain $\vec{x}$.

f) W.l.o.g. $k = 1$. By the Weierstraß Approximation Theorem, there exists a double sequence $p_{n,m}$ of d-variate rational polynomials such that $p_{n,m}$ converges to f as $n \to \infty$ uniformly on $[-m, +m]^d$, w.l.o.g. with uniform error $\leq 2^{-n}$. Now encoding the list of rational coefficients into a real constant as in d), a BSS machine can, given $\vec{x}$ first find m with $\vec{x} \in [-m, +m]^d$ and then evaluate and output $p_{n,m}(\vec{x})$ for $n = 1, 2, \ldots$.

g) Similarly to f), encode the (still countable) list of rational coefficients of $p_{n,m,\ell}$ into a real constant. □

Note that Item e) is an extension of [GaHo10, Proposition 1], where in the current case for the one-dimensional case all inputs in the open interval between two integers have the same computation path.

Corollary 2.6.

a) *Fix* $X \subseteq \mathbb{R}^d$. *Every set* $S \subseteq X$ BSS-*semidecidable in* X *belongs to the Borel class* $\mathbf{\Sigma}_2(X)$.
b) *Every* $S \subseteq X$ *decidable in* X *by a strongly analytic machine belongs to* $\mathbf{\Delta}_2(X)$.
c) *Each function* $f :\subseteq \mathbb{R}^d \to \mathbb{R}$ *computable by a strongly analytic machine is* $\mathbf{\Sigma}_2$*-measurable in* $\mathrm{dom}(f)$.
d) *For* $f :\subseteq \mathbb{R}^d \to \mathbb{R}$ *computable by a strongly analytic machine, it holds* $\mathrm{dom}(f) \in \mathbf{\Pi}_3(\mathbb{R}^d)$.

Compare also [Cuck92, SECTION 4].

Proof.

a) Note that every semialgebraic set is (a finite union of basic semialgebraic sets and thus) the intersection of a closed and an open set [BCSS98, p.51 l.6] and in particular in $\mathbf{\Sigma}_2$, which is closed under countable unions. Now the *Path Decomposition Theorem* for BSS machines [BCSS98, THEOREM 2.3.1] shows that every BSS-semidecidable set (in X) is the countable union of semialgebraic sets (intersected with X).
b) Decidability of S means that both S and its complement are semidecidable, hence in $\mathbf{\Sigma}_2$ by a); that means $S \in \mathbf{\Sigma}_2 \cap \mathbf{\Pi}_2 = \mathbf{\Delta}_2$.
c) Let $V \subseteq \mathbb{R}$ be open, $V = \bigcup_n (a_n, b_n)$ with $a_n, b_n \in \mathbb{Q}$ enumerated by a BSS machine according to the proof of Theorem 2.5e). Now $f^{-1}[(a,b)] = f^{-1}\big[(a,\infty)\big] \cap f^{-1}\big[(-\infty,b)\big]$ is semidecidable in $\mathrm{dom}(f)$ according to Theorem 2.5a); hence so is $f^{-1}[V] = \bigcup_n f^{-1}[(a_n,b_n)]$; and thus $\mathbf{\Sigma}_2$ in view of a).
d) Let $\mathcal{M}$ denote a strongly analytic machine computing $f :\subseteq \mathbb{R}^* \to \mathbb{R}$. According to Equation (1.5), $\vec{x} \in \mathrm{dom}(f)$ iff for all $n, m \in \mathbb{N}$, $(\vec{x}, n, m)$ belongs to the set
$$\big\{(\vec{x}, n, m) : \mathcal{M} \text{ on input } \vec{x} \text{ prints } y_1, \ldots, y_n, \ldots, y_m \text{ with } |y_n - y_m| \le 2^{-n} + 2^{-m}\big\}$$
which is clearly BSS semidecidable and thus $\mathbf{\Sigma}_2$ according to a). Adding the universal quantification over n, m, it follows that $\mathrm{dom}(f)$ is $\mathbf{\Pi}_3$. □

The following result due to ARNO PAULY (personal communication) exhibits the topological difference between functions computable by BSS machines and by strongly analytical ones, recall Corollary 2.6c).

Theorem 2.7.

a) *Every (possibly partial) function* $f :\subseteq \mathbb{R}^d \to \mathbb{R}$ *computable by a* BSS *machine is* $\mathbf{\Delta}_2$*-measurable in* $\mathrm{dom}(f)$.
b) *More generally, let* $\mathcal{F}$ *denote a family of continuous, partial real functions* $f :\subseteq \mathbb{R}^{d_f} \to \mathbb{R}$ *of various arities* $d_f \in \mathbb{N}$ *with domains in* $\mathbf{\Delta}_2$. *Let* $\mathcal{R}$ *denote a family of* $\mathbf{\Delta}_2$*-measurable real relations* $R \subseteq \mathbb{R}^{k_R}$ *of various arities* $k_R \in \mathbb{N}$. *Consider a uniform machine model over the structure* [†] $(\mathbb{R}, \mathcal{F}, \mathcal{R})$, *i.e. capable of performing a finite sequence of operations from* $\mathcal{F}$ *and branchings based on tests from* $\mathcal{R}$. *Then any (possibly partial) function* $g :\subseteq \mathbb{R}^d \to \mathbb{R}^k$ *computable by such a machine is necessarily* $\mathbf{\Delta}_2$*-measurable in* $\mathrm{dom}(f)$.

Proof.

a) follows from b) with $\mathcal{F} := \{+, -, \times, \div\}$ and $\mathcal{R} := \{=, <\}$.

[†] We refrain from formally defining the intuitive but tedious concept of a (nonuniform) machine model over a structure but refer, e.g., to [Poiz95, §4.A] (which technically restricts to structures having only total functions); cmp. also [TuZu00, §3]

b) Similarly to the proof of [BCSS98, Theorem 3.3.1], the computation of such a machine can be unrolled into a (possibly infinite) binary tree $\mathcal{T}$: Each internal node u describes the branching based on the outcome of a test "$\vec{y} \in R_u$?" ($R_u \in \mathcal{F}$) of intermediate results $\vec{y}$; intermediate results which arise from the input $\vec{x}$ evaluated on functions g_u which are compositions of functions from $\mathcal{F}$. In particular, for a leaf v of $\mathcal{T}$, the set G_v of inputs $\vec{x} \in \mathbb{R}^d$ ending up in v is the intersection $G_v = \bigcap_u \{\vec{x} : g_u(\vec{x}) \in R_u\}$ with u running over the (finitely many) internal nodes from $\mathcal{T}$'s root to v; and the output printed in v is of the form $g_v(\vec{x})$. This yields a disjoint decomposition $g = \biguplus_v g_v|_{G_v}$, now with v running over all (countably many) leaves of $\mathcal{T}$. In particular, $g^{-1}[Y] = \biguplus_v (g_v^{-1}[Y] \cap G_v)$ holds for any $Y \subseteq \mathbb{R}^k$.
Now by hypothesis, g_v is continuous as the composition of continuous functions; and for continuous f_1, f_2 with $\mathrm{dom}(f_1)$ and $\mathrm{dom}(f_2)$ both $\mathbf{\Delta}_2$-measurable, $\mathrm{dom}(f_2 \circ f_1) = \{\vec{x} : f_1(\vec{x}) \in \mathrm{dom}(f_2)\}$ is easily verified to be again $\mathbf{\Delta}_2$-measurable: recall that $\mathbf{\Delta}_2$ is closed under both finite unions and finite intersections. Similarly, it follows that each $G_v \subseteq \mathbb{R}^d$ is $\mathbf{\Delta}_2$-measurable as well. Thus, both for open and for closed $Y \subseteq \mathbb{R}^k$, $g^{-1}[Y]$ is a countable union of $\mathbf{\Sigma}_2$ sets. $\square$

Corollary 2.6, Theorem 2.5e), and Theorem 2.7a) are (almost) best possible:

Examples 2.8.

a) The set $\mathbb{Q}$ of rational numbers is BSS-semidecidable (but not in $\mathbf{\Pi}_2$).
b) The characteristic function $\mathbf{1}_{[0,1)} : \mathbb{R} \to \{0,1\}$ is BSS-computable but is not $\mathbf{\Sigma}_1 \cup \mathbf{\Pi}_1$-measurable.
c) Cantor's *Excluded Middle* set $\mathcal{C} \subseteq [0,1]$ belongs to $\mathbf{\Pi}_1 \subseteq \mathbf{\Sigma}_2$, but is not BSS semidecidable.
d) Recall Thomae's or *Popcorn Function* $h : \mathbb{R} \to \mathbb{R}$, defined as $h(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and $h(\pm p/q) = 1/q$ for coprime $p, q \in \mathbb{N}$, $h(0) = 1$.
This function is computable by a strongly analytic machine but is not $\mathbf{\Pi}_2$-measurable.
e) There is a function $f :\subseteq \mathbb{R}^2 \to \{0\}$ computable by a strongly analytic machine with $\mathrm{dom}(f) = \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ not $\mathbf{\Sigma}_2 \cup \mathbf{\Pi}_2$-measurable.

Proof.

a) A BSS machine can, given $x \in \mathbb{R}$, enumerate all pairs $r, s \in \mathbb{Z}$ and compare $x = r/s$ to semidecide "$x \in \mathbb{Q}$".
b) The set $[0,1)$ is neither closed nor open, hence its characteristic function is not $\mathbf{\Sigma}_1 \cup \mathbf{\Pi}_1$-measurable.
c) Note that each singleton $\{x\} \subseteq \mathcal{C}$ is a connected component of $\mathcal{C}$ of its own. Hence $\mathcal{C}$ has uncountably many connected components; whereas any BSS-semidecidable set, being a countable union of semialgebraic sets (recall the proof of Corollary 2.6a) of only finitely many connected components each [BPR03, Section 5.2], can have at most countably many connected components.
d) Recall from a) that $\mathbb{Q}$ is not in $\mathbf{\Pi}_2$ but the preimage of an open set: $\mathbb{Q} = h^{-1}[(0,2)]$. We now describe a machine computing $h(x)$ on input $x > 0$:
Iteratively for $q = 1, 2, 3, \ldots$ test whether $q \cdot x$ is an integer; if not, output q as approximation to $h(x)$ up to error $1/q$ and continue with the iteration; otherwise switch to outputting $1/q, 1/q, 1/q, \ldots$ as approximations to $h(x)$ up to error $1/m$ for all $m \geq q$.
It is easy to convert this sequence $(y_q)_q$ of approximations up to error $1/q$ into a sequence $(z_n)_n$ of approximations up to error 2^{-n} by printing only the subsequence $(y_{2^n})_n$.

e) Consider a machine which, given (x, y), first searches (without output) for $p, q \in \mathbb{Z}$ with $x = p/q$. When found, it starts similarly enumerating each $r_n \in \mathbb{Q}$ and printing 2^{-n} until (and if) arriving at one with $r_n = y$. □

The rough conclusion of this subsection is that both BSS model and analytic machines lie slightly skew to the Borel Hierarchy, having topological power strictly between $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$; and partial functions are even more skewed relative to the hierarchy.

2.2. **Composition of Strongly Analytic Machines.** The analytic machine models presume the input to be given exactly but produce only approximations to the output. It is thus reasonable to expect that the composition of two functions computable by analytic machines in general need not itself be computable by an analytic machine. This has been proven for weakly analytic machines in [ChHo99, LEMMA 6]; cmp. [GaHo09, COROLLARY 2.4]. It is not surprising that we can establish the same for strongly analytic machines in Proposition 2.9b) below. However the use of descriptive set theory is of interest because of the new perspective it provides: It is well-known that the composition of two $\mathbf{\Sigma}_2$-measurable functions is in general no more than $\mathbf{\Sigma}_3$-measurable [Brat05, COROLLARY 3.9]; whereas the composition of a $\mathbf{\Sigma}_2$-measurable function with a continuous one is again $\mathbf{\Sigma}_2$-measurable. In view of Theorem 2.5f) and Corollary 2.6c), VASSILIOS GREGORIADES (personal communication) thus raised the natural question of whether the composition of a strongly analytic and a continuous function is again strongly analytic. A complete answer is given by the already mentioned

Proposition 2.9.

a) *Let $g : \mathbb{R}^k \to \mathbb{R}^\ell$ be computable by a strongly analytic machine and let $h : \mathbb{R}^d \to \mathbb{R}^k$ denote a continuous function. Then $h \circ g$ is computable by a strongly analytic machine.*
b) *There exists a function $g : \mathbb{R} \to \mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$, both computable by strongly analytic machines without constants, such that $g \circ f$ is not computable by a strongly analytic machine.*

In particular, we obtain:

Corollary 2.10. *The class of total real functions computable by strongly analytic machines is not closed under composition.*

Proof. (Prop. 2.9)

a) We refine the proof of Theorem 2.5f) by storing, in addition to the coefficients of rational polynomials $p_{n,m}$ approximating $h|_{[-m,+m]^d}$ up to error 2^{-n} also some moduli of uniform continuity, that is, integers $\mu_{n,m}$ subject to:
$$\vec{y}, \vec{y}' \in [-m, +m]^d, \quad |\vec{y} - \vec{y}'| \leq 2^{-\mu_{n,m}} \qquad \Longrightarrow \qquad |h(\vec{y}) - h(\vec{y}')| \leq 2^{-n} \ . \tag{2.2}$$
Now, given $\vec{x}$ and a desired precision 2^{-n}, determine m with $\vec{x} \in [-m, +m]^d$ and evaluate $\vec{y} := g(\vec{x})$: By hypothesis, the strongly analytic machine computing g can produce an approximation $\vec{y}'$ up to precision $2^{-\mu_{n+1,m+1}}$. Finally output $z := p_{n+1,m+1}(\vec{y}')$ and verify
$$|h \circ g(\vec{x}) - z| \quad \leq \quad |h(\vec{y}) - h(\vec{y}')| \ + \ |h(\vec{y}') - z| \quad \overset{(2.2)}{\leq} \quad 2^{-n-1} + 2^{-n-1} \ .$$
b) Let $g(0) := 1$ and $g(y) := 0$ for $y \neq 0$ denote the characteristic function of $\{0\}$. Let $f := \mathrm{dist}(\cdot, \mathcal{C})$ denote the distance function of the Cantor set, recall Example 2.4. Since $\mathcal{C}$ is closed, it follows $g \circ f = \mathbf{1}_\mathcal{C}$: a function not computable by a strongly analytic machine according to Example 2.8c). □

We now extend Theorem 2.5f):

Theorem 2.11. *Every $\mathbf{\Sigma}_2$-measurable $f : \mathbb{R}^d \to \mathbb{R}$ can be expressed as the composition $f = h \circ g$ of a strongly analytic $g : \mathbb{R}^d \to \mathbb{R}^\omega$ and a strongly analytic partial function $h :\subseteq \mathbb{R}^\omega \to \mathbb{R}$.*

Proof. Being $\mathbf{\Sigma}_2$-measurable means

$$f^{-1}\big[(q,p)\big] \;=\; \bigcup\nolimits_k A_{k,q,p}, \qquad \text{for } q,p \in \mathbb{Q} \text{ and closed } A_{k,q,p} \subseteq \mathbb{R}^d \;. \tag{2.3}$$

- Since $Q := \{(q,p,k) : k \in \mathbb{N}, q,p \in \mathbb{Q}, q < p\}$ is countable, Theorem 2.5g) yields a strongly analytic machine computing the function
$$\hat{g} : Q \times \mathbb{R}^d \;\ni\; \big((q,p,k),\vec{x}\big) \;\mapsto\; \operatorname{dist}(\vec{x}, A_{k,q,p}) \;\in\; [0,\infty)$$
which we shall identify with the function $g : \mathbb{R}^d \to \mathbb{R}^Q$, $\vec{x} \mapsto \big((q,p,k) \mapsto \hat{g}(q,p,k,\vec{x})\big)$. Note that $\exists k : \hat{g}(q,p,k,\vec{x}) = 0 \Leftrightarrow q < f(\vec{x}) < p$.
- Now consider the function
$$h :\subseteq \mathbb{R}^Q \to \mathbb{R}, \quad \delta \mapsto \sup\{q : \exists k,p : \delta(q,p,k) = 0\} \quad \text{with} \quad \operatorname{dom}(h) \;:=$$
$$\big\{\delta : Q \to [0,\infty) \;\big|\; \sup\{q : \exists k,p : \delta(q,p,k) = 0\} = \inf\{p : \exists k,q : \delta(q,p,k) = 0\}\big\}$$
and observe that $\delta := g(\vec{x})$ has $\sup\{q : \exists k,p : \delta(q,p,k) = 0\} = f(\vec{x}) = \inf\{p : \exists k,q : \delta(q,p,k) = 0\}$; hence $\operatorname{dom}(h) \subseteq \operatorname{range}(g)$ holds and, moreover, $(h \circ g)(\vec{x}) = f(\vec{x})$.
- Finally, h is computable by a strongly analytic machine: Given $\delta \in \operatorname{dom}(h)$ and for each $n \in \mathbb{N}$, search for q,p,k with $\delta(q,p,k) = 0$ and $p - q \leq 2^{-n}$ and, when found, print q, then continue with $n+1$. On the one hand such (q,p,k) exist because, according to the hypothesis $\delta \in \operatorname{dom}(h)$, it holds $\sup\{q : \exists k,p : \delta(q,p,k) = 0\} = \inf\{p : \exists k,q : \delta(q,p,k) = 0\}\big\} = h(\delta) =: y$. On the other hand such a tuple satisfies $q < y < p \leq q + 2^{-n}$, hence the output sequence converges effectively to this y. □

Note that we have silently extended the classical analytic machine model to infinite dimensional arguments and values—which raises

Question 2.12. *In Theorem 2.11, can the infinite-dimensional intermediate results be avoided? Can h be chosen total? How far up on the Borel hierarchy of measurability do compositions of k strongly analytic functions reach/cover?*

Indeed, strongly analytic machines can encode infinite sequences into single reals and back; but a priori, each such operation incurs an additional machine, thus resulting in the composition in Theorem 2.11 to become three-fold.

3. Comparing Weakly and Strongly Analytic Machines

It will turn out (Theorem 3.5) that weakly analytic machines are essentially strongly ones equipped with oracle access to the BSS Halting problem.

We first record the following relativizations of Theorem 2.5 and Corollary 2.6:

Corollary 3.1.

a) *Each set $\mathbb{S} \subseteq \mathbb{R}^d$* BSS*-semidecidable with oracle $\mathbb{H}$ necessarily belongs to Borel class $\mathbf{\Sigma}_3$.*
b) *Every $\mathbb{S} \in \mathbf{\Sigma}_2$ is* BSS*-semidecidable with oracle $\mathbb{H}$.*
c) *Each total function $f : \mathbb{R}^d \to \mathbb{R}$ computable by a strongly analytic machine with $\mathbb{H}$-oracle is $\mathbf{\Sigma}_3$-measurable.*

d) *Every* $\mathbf{\Sigma}_2$*-measurable total function* $f : \mathbb{R}^d \to \mathbb{R}$ *is computable by a strongly analytic machine with* $\mathbb{H}$*-oracle.*

Proof.

a) follows from Corollary 2.6a) and Fact 1.3b), observing that $\{(\vec{x}, y) : \forall z \in \mathbb{N} : (\vec{x}, y, z) \in \mathbb{W}\}$ is in $\mathbf{\Pi}_2$ because its complement is BSS-semidecidable.

b) Let $\mathbb{S} = \bigcup_n \mathbb{A}_n$ with $\mathbb{A}_n$ closed, i.e. the complement of $\mathbb{A}_n$ is of the form $\bigcup_m (\vec{a}_{n,m}, \vec{b}_{n,m})$ with rational corners $\vec{a}_{n,m}, \vec{b}_{n,m}$. Just like in the proof of Theorem 2.5e), this rational double sequence can be encoded into one single real constant in order to enable a BSS machine deciding $\mathbb{W} := \{(\vec{x}, n, m) : \vec{x} \notin (\vec{a}_{n,m}, \vec{b}_{n,m})\}$. Now apply Fact 1.3c) to $\bigcup_n \mathbb{A}_n = \{\vec{x} \big| \exists n \forall m : (\vec{x}, n, m) \in \mathbb{W}\}$.

c) Like in the proof of Corollary 2.6c) and relativizing Theorem 2.5a), we observe that $f^{-1}[\bigcup_n (a_n, b_n)]$ is semidecidable by a BSS machine with $\mathbb{H}$-oracle. Now apply a).

d) For $q \in \mathbb{Q}$ and $n \in \mathbb{N}$, consider the $\mathbf{\Sigma}_2$-set $f^{-1}[(q - 2^{-n}, q + 2^{-n})]$. Extending Item b), we see that these sets are BSS-semidecidable with $\mathbb{H}$-oracle *uniformly* in q and n. Hence given $\vec{x}$, an $\mathbb{H}$-oracle machine can search and output, for each $n \in \mathbb{N}$, some $q \in \mathbb{Q}$ with $\vec{x} \in f^{-1}[(q - 2^{-n-1}, q + 2^{-n-1})]$. □

[Cuck92, Theorems 2.15+2.16] establishes two natural problems over the reals as BSS-equivalent for (i.e. many-one reducible from and to) $\mathbb{H}^{\mathbb{H}}$. The next section will add *Boundedness*; and Theorem 3.11 shows (the complement of) *Convergence* BSS-equivalent to $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$.

3.1. **The Boundedness Problem and Weak Semidecidability.**

Consider the boundedness problem for analytic machines:

$$\mathbb{B} \;:=\; \big\{(\langle \mathcal{M} \rangle, \vec{x}) : \text{machine } \mathcal{M} \text{ produces on input } \vec{x} \text{ some bounded sequence } (\vec{y}_n)_n\big\} \;.$$

By convention, we regard also a finite sequence as bounded.

Proposition 3.2.

a) *A* BSS *machine with oracle access to* $\mathbb{H}$ *can semidecide* $\mathbb{B}$

b) *but cannot decide* $\mathbb{B}$*. More precisely, it holds* $\mathbb{H}^{\mathbb{H}} \lessdot \mathbb{B}$*.*

Proof.

a) Given $\mathcal{M}$ and $\vec{x}$, iteratively try the bounds $n = 1, 2, \ldots$ and use oracle access to $\mathbb{H}$ in order to detect whether some output of $\mathcal{M}$ on $\vec{x}$ has norm exceeding n: If so, retry with $n + 1$; otherwise accept.

b) Since $\mathbb{H}^{\mathbb{H}}$ is semidecidable relative to $\mathbb{H}$, it has the form of Equation (1.3). Now consider the BSS machine $\mathcal{M}$ executing the following algorithm: Given $\vec{x}$ and iteratively for each $y = 1, 2, \ldots$, $\mathcal{M}$ looks for some $z = 1, 2, \ldots$ such that $(\vec{x}, y, z) \notin \mathbb{W}$. If such a z is found, $\mathcal{M}$ outputs y and restarts with $y + 1$; otherwise $\mathcal{M}$ keeps looking for z indefinitely.
For $\vec{x} \in \mathbb{H}^{\mathbb{H}}$, the above machine will thus eventually find an y that leads to an infinite loop on z; and hence a bounded (even finite) output sequence. Whereas for $\vec{x} \notin \mathbb{H}^{\mathbb{H}}$, every $y \in \mathbb{N}$ will eventually be output by $\mathcal{M}$, that is, an unbounded sequence. □

Theorem 2.5d) suggests the following

Definition 3.3. A set $\mathbb{S} \subseteq \mathbb{R}^*$ is *weakly decidable* iff its characteristic function $\mathbf{1}_\mathbb{S} : \mathbb{R}^* \to \{0,1\}$ is computable by a weakly analytic machine.
$\mathbb{S}$ is *weakly semidecidable* iff $\mathbb{S}$ is BSS-semidecidable with $\mathbb{H}$-oracle.

In view of Fact 1.3 and Proposition 3.2, $\mathbb{S}$ is weakly semidecidable iff it is BSS many-one reducible to $\mathbb{H}^\mathbb{H}$ or, equivalently, to $\mathbb{B}$.

Examples 3.4.
a) For a function $f : \mathbb{R}^* \to \mathbb{R}^k$ computable by a weakly analytic machine and for open $V \subseteq \mathbb{R}^k$, the pre-image $f^{-1}[V] \subseteq \mathbb{R}^*$ is weakly semidecidable.
b) Every $\mathbf{\Sigma}_2$-set is weakly semidecidable.

Proof. Let $(\vec{y}_n)$ be a sequence output by the weakly analytic machine on input $\vec{x}$, i.e. with $\lim_n \vec{y}_n = \vec{y} := f(\vec{x})$. In view of Theorem 2.5e), we can assume to have an enumeration (V_m) of rational open rectangles exhausting $V = \bigcup_m V_m$ at our disposition.
a) For each $m, k = 1, 2, \ldots$ test whether it holds that the rectangle $[\vec{y}_n - 2^{-m}, \vec{y}_n + 2^{-m}]$ is contained in V_k for all $n \geq m$. This can be achieved by setting up a machine $\mathcal{N}$ searching for a counter-example n and querying oracle $\mathbb{H}$ for non-termination of $\mathcal{N}$. If so, since $\vec{y} \in [\vec{y}_n - 2^{-m}, \vec{y}_n + 2^{-m}]$ for all sufficiently large n, it follows $\vec{y} \in V$ and we can safely accept. Conversely in case $\vec{y} \in V_k$, it holds $[\vec{y}_n - 2^{-m}, \vec{y}_n + 2^{-m}] \subseteq V_k$ for all sufficiently large n, m; hence the above search succeeds.
b) Let $V = \bigcup_j A_j \in \mathbf{\Sigma}_2$. Analogously to the proof of Theorem 2.5e), the closed sets A_j can be represented as complements of a countable union of open rectangles with rational corners $A_j = \Big(\bigcup_i (\vec{a}_{j,i}, \vec{b}_{j,i})\Big)^\complement$. The rational coordinates of $\vec{a}_{j,i}$ and $\vec{b}_{j,i}$ can all be encoded into one real constant. A machine that semidecides $\vec{x} \in V$ tries, for increasing $n = 1, 2, \ldots$, whether $\vec{x} \in A_n$. To this end, the coordinates of the rectangles exhausting A_n are extracted and for increasing m the condition $\vec{x} \in (\vec{a}_{n,m}, \vec{b}_{n,m})$ is checked. After each check, the machine outputs n, and as soon as a rectangle containing x is found, the machine proceeds to the next n. If $\vec{x} \in V$, then it is in some A_n, and therefore in no rectangle $(\vec{a}_{n,m}, \vec{b}_{n,m})$, $m \in \mathbb{N}$. In this case, the machine never exceeds stage n. If, on the other hand, $\vec{x} \notin V$, then for each n there is such a rectangle, and the machine reaches (and outputs) each $n \in \mathbb{N}$. □

3.2. **Weakly Analytic Machines are the Jump of Strongly Analytic Ones.**

Definition 3.3 is justified by Item a) of the following

Theorem 3.5.
a) $\mathbb{S} \subseteq \mathbb{R}^*$ *is weakly decidable iff both* $\mathbb{S}$ *and its complement are weakly semidecidable.*
b) *If a (possibly partial) function* $f : \mathbb{R}^* \to \mathbb{R}^*$ *is computable by a weakly analytic machine,* f *is also computable by a strongly analytic machine with oracle access to* $\mathbb{H}$.
c) *Conversely, if* f *is computable by a strongly analytic machine with oracle access to* $\mathbb{H}$, *then some extension of* f *is computable by a weakly analytic machine.*

The equivalence in b+c) constitutes an analytic analogue of the *Shoenfield Limit Lemma*. The slight mismatch with respect to partial functions resembles Theorem 2.5abc) and raises

Question 3.6. *Is every partial function computable by a strongly analytic machine with oracle access to $\mathbb{H}$ also computable by a weakly analytic machine?*

Proof. (Theorem 3.5)

a) Suppose $\mathcal{M}$ is a weakly analytic machine computing $\mathbf{1}_{\mathbb{S}}$, and let (y_n) denote the sequence output by $\mathcal{M}$ on input $\vec{x}$. We show that both $\mathbb{S}$ and its complement $\mathbb{S}^{\complement}$ are reducible to $\mathbb{B}$. To this end modify $\mathcal{M}$ to output $u_n := 1/\max\{y_n, 1/n\}$: Since $\{0,1\} \ni \lim_n y_n$ exists, u_n is bounded iff $y_n \to 1$ iff $\vec{x} \in \mathbb{S}$. Similarly, $v_n := 1/\max\{1-y_n, 1/n\}$ is bounded iff $y_n \to 0$ iff $\vec{x} \notin \mathbb{S}$.
Conversely consider BSS machines $\mathcal{M}$ and $\mathcal{N}$ computing reductions from $\mathbb{S}$ and $\mathbb{S}^{\complement}$ to $\mathbb{B}$, respectively. Then the following machine weakly computes $\mathbf{1}_{\mathbb{S}}$: Given input $\vec{x}$, test (by parallel simulation) for increasing bounds $n = 1, 2, \ldots$ whether some output of $\mathcal{M}$ or of $\mathcal{N}$ exceed the bound n. If so, append "0" to the output if it was $\mathcal{M}$, and "1" if it was $\mathcal{N}$; then increment the bound n. Since $\vec{x}$ belongs to exactly one of $\mathbb{S}$ and $\mathbb{S}^{\complement}$, precisely one of $\mathcal{M}, \mathcal{N}$ produces an unbounded sequence; and our output thus becomes a stationary sequence of 1s (in case $\vec{x} \in \mathbb{S}$) or of 0s ($\vec{x} \in \mathbb{S}^{\complement}$), respectively.

b) Let $\mathcal{M}$ denote a weakly analytic machine computing f and $(\vec{y}_n)$ the (possibly finite) sequence output on input $\vec{x}$. We describe another machine $\mathcal{N}$ that uses oracle queries to $\mathbb{H}$ in order to output a subsequence of $(\vec{y}_n)$ satisfying Equation (1.5). Note that the violation of this condition can be detected by searching for n, m and hence is semidecidable. Using $\mathbb{H}$, $\mathcal{N}$ can thus decide, for each required precision index $k = 1, 2, \ldots$ and each $K \in \mathbb{N}$, whether $\|\vec{y}_K - \vec{y}_m\| \leq 2^{-k} + 2^{-m}$ holds for all m. On the other hand such $K = K(k)$ exists to every k iff $(\vec{y}_n)$ converges. $\mathcal{N}$ will thus, iteratively for $k = 1, 2, \ldots$, search for such a K and, when found, output the corresponding $\vec{y}_K$. Note that, if $\mathcal{M}$ outputs only a finite sequence, so will $\mathcal{N}$. In effect, the subsequence printed by $\mathcal{N}$ satisfies the bottom of Equation (1.4) iff the original sequence printed by $\mathcal{M}$ satisfies the top of Equation (1.4).

c) Assume f is computed by the strongly analytic machine $\mathcal{M}$ with oracle access to $\mathbb{H}$. We describe a weakly analytic machine $\mathcal{N}$ that computes f. Fix an input $\vec{x} \in \mathbb{R}^*$. For $\sigma \in \{0,1\}^{\mathbb{N}}$, let $\vec{y}_n^{\sigma}$ be the output of the machine $\mathcal{M}$, simulated under the assumption that the j-th oracle query is negative or positive, depending on $\sigma(j)$. For increasing $n = 1, 2, \ldots$ (*simulation level*), the machine $\mathcal{N}$ simulates (without output) $\mathcal{M}$ up to the n-th output. It simulates all machines queried by the oracle, up to n output steps (or until they halt), and stores the knowledge about the oracle answer in the sequence $\sigma_n \in \{0,1\}^{\mathbb{N}}$ (0: does not halt, 1: halts), initially assuming all oracle queries to be answered negatively. In addition, the conditions
$$\|\vec{y}_i^{\sigma_n} - \vec{y}_j^{\sigma_n}\| \;\leq\; 2^{-i} + 2^{-j} \quad \forall 1 \leq i < j \leq n \ . \tag{3.1}$$
are checked. If one of these conditions is violated, the number of steps of all simulated oracle queries is increased until all these conditions are fulfilled. As soon as this is the case, $\mathcal{N}$ outputs $\vec{y}_n^{\sigma_m}$, m being the number of simulated steps of the oracle queries. Then, $\mathcal{N}$ proceeds to level $n+1$.
Given $\vec{x} \in \mathrm{dom}(f)$ and $N \in \mathbb{N}$, there is a number of simulation steps $n_0(N)$ after which all oracle queries made until output N of $\mathcal{M}$ have been answered correctly. At level $n \geq n_0(N)$, $\mathcal{M}$ produces an output $\vec{y}_n^{\sigma_n}$ which, because of Equation (3.1), satisfies $\|\vec{y}_N^{\sigma_n} - \vec{y}_n^{\sigma_n}\| \leq 2^{-N} + 2^{-n}$. Furthermore, because at level n, all oracle assumptions up to output N are correct, we know that $\vec{y}_N^{\sigma_n} = \vec{y}_N$. Therefore, the outputs of $\mathcal{N}$ correctly converge to $\vec{y}$. $\square$

In connection with Corollary 3.1, we conclude

Corollary 3.7.

a) *Every weakly semidecidable* $\mathbb{S} \subseteq \mathbb{R}^d$ *belongs to Borel class* $\mathbf{\Sigma}_3$.
b) *Every function* $f : \mathbb{R}^d \to \mathbb{R}^k$ *computable by a weakly analytic machine is* $\mathbf{\Sigma}_3$*-measurable.*
c) *Conversely, every* $\mathbf{\Sigma}_2$*-measurable* $f : \mathbb{R}^d \to \mathbb{R}^k$ *is computable by a weakly analytic machine.*

Again, Corollary 3.7 is (almost) best possible:

Example 3.8. The set $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ is decidable by a weakly analytic machine (but not in $\mathbf{\Pi}_2 \cup \mathbf{\Sigma}_2$).

Proof. Since $\mathbb{Q}$ is semidecidable by a BSS machine, it is decidable relative to $\mathbb{H}$; and so is $\mathbb{R} \setminus \mathbb{Q}$. $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ can be decided relative to $\mathbb{H}$ by testing both components separately; hence this set is weakly decidable according to Theorem 3.5a). □

Question 3.9. *Is there a set* $S \subseteq \mathbb{R}$ *weakly semidecidable yet such that* $S \notin \mathbf{\Pi}_3$*?*

3.3. **The Convergence Problem and Naïve Semidecidability.** Our proof of Example 3.4a) erroneously accepts in case the output sequence $\vec{y}_n$ fails to converge by having several accumulation points all contained in some V_m. This cannot happen for $\vec{y}_n$ produced by the weak evaluation of a *total* function f.

Definition 3.10. In view of the second part of Equation (1.4), consider

$\mathbb{K} := \big\{(\langle\mathcal{M}\rangle, \vec{x}) : \text{machine } \mathcal{M} \text{ produces on input } \vec{x} \text{ some convergent infinite sequence } (\vec{y}_n)_n\big\}$

Call a set $\mathbb{S} \subseteq \mathbb{R}^*$ *naïvely semidecidable* if there is a weakly analytic machine calculating (i.e. printing a sequence of approximations which converge to)

i) the real number 0 for inputs $\vec{x} \in \mathbb{S}$
ii) $\perp$ (i.e. fails to converge) for inputs $\vec{x} \notin \mathbb{S}$.

A machine that produces only finitely many outputs is considered divergent.

Diagonalization shows [HVS95] that $\mathbb{K}$ is undecidable to a weakly analytic machine; yet it can be written as the composition of two functions computable by weakly analytic machines [GaHo09, Theorem 2.3].

Theorem 3.11.

a) *If* $\mathbb{S} \subseteq \mathbb{R}^*$ *is naïvely semidecidable, it is* BSS *many-one reducible to the convergence problem* $\mathbb{K}$.
b) *Conversely, every* $\mathbb{S} \subseteq \mathbb{R}^*$ BSS *many-one reducible to* $\mathbb{K}$ *is naïvely semidecidable.*
c) *The complement of* $\mathbb{K}$ *is* BSS *many-one reducible to* $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$.
d) *Conversely,* $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$ *is* BSS *many-one reducible to* $\mathbb{K}^{\complement}$.

Since $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$ is strictly harder than $\mathbb{H}^{\mathbb{H}}$, convergence is strictly harder than boundedness; and weak semidecidability is strictly stronger a notion than naïve semidecidability.

Proof.

a) Let $\mathcal{M}$ naïvely semidecide $\mathbb{S}$. Then $\vec{x} \mapsto (\langle\mathcal{M}\rangle, \vec{x})$ constitutes a BSS-computable many-one reduction of $\mathbb{S}$ to $\mathbb{K}$: For $\vec{x} \in \mathbb{S}$, $\mathcal{M}$ on input $\vec{x}$ outputs a sequence converging to 0; whereas for $\vec{x} \notin \mathbb{S}$, $\mathcal{M}$ on input $\vec{x}$ outputs a divergent sequence.

b) Consider a many-one reduction from $\mathbb{S}$ to $\mathbb{K}$, i.e. mapping an instance $\vec{u}$ for $\mathbb{S}$ to an instance $(\langle\mathcal{M}\rangle, \vec{x})$ of $\mathbb{K}$. Consider a BSS machine which simulates $\mathcal{M}$ and replaces its output sequence (y_n) by the sequence $(|y_n - y_m|)_{\langle n,m\rangle}$, where $\langle\,\cdot\,,\,\cdot\,\rangle : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ denotes a recursive pairing function. To see that this machine naïvely semidecides $\mathbb{S}$, observe that (y_n) converges (i.e. is Cauchy) iff $(|y_n - y_m|)_{\langle n,m\rangle}$ converges to 0: To every $N\in\mathbb{N}$ there is some $M\in\mathbb{N}$ such that $n,m\geq N$ implies $\langle n,m\rangle\geq M$; and, conversely, to every $M\in\mathbb{N}$ there is some $N\in\mathbb{N}$ such that $\langle n,m\rangle\geq M$ implies $n,m\geq N$.

c) We employ from [Cuck92, Theorem 2.11] the following extension of Fact 1.3b+c):

$\mathbb{S}\subseteq\mathbb{R}^*$ is BSS many-one reducible to $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$ iff there exists some BSS-decidable $\mathbb{W}\subseteq\mathbb{R}^*$ such that

$$\mathbb{S} \;=\; \big\{\vec{x}\in\mathbb{R}^* \;\big|\; \exists u\in\mathbb{N}\,\forall v\in\mathbb{N}\,\exists w\in\mathbb{N}:\ (\vec{x},u,v,w)\in\mathbb{W}\big\}\ . \tag{3.2}$$

Now observe that an infinite real sequence (y_n) fails to converge iff

$$\exists k\in\mathbb{N}\,\forall K\in\mathbb{N}\,\exists\langle n,m,\ell\rangle\in\mathbb{N}:\quad n,m\geq K\ \wedge\ |y_n-y_m|\geq 1/k\ .$$

Finally, to take into account the computation of (y_n), consider the BSS-decidable

$$\mathbb{W} \;:=\; \big\{(\langle\mathcal{M}\rangle,x,k,K,n,m,\ell) : \vec{x}\in\mathbb{R}^*, k,K,n,m,\ell\in\mathbb{N};\ n\geq m\geq K \text{ and } \mathcal{M} \text{ on input } \vec{x} \text{ within } \ell \text{ steps outputs } y_1,\ldots,y_m,\ldots,y_n \text{ with } |y_n-y_m|\geq 1/k\big\}$$

d) Again we invoke the characterization from [Cuck92, Theorem 2.11] and show that every $\mathbb{S}$ of the form (3.2) is BSS many-one reducible to $\mathbb{K}^{\complement}$. To this end execute the following procedure for each $u\in\mathbb{N}$ in parallel:

Let $v:=1$ and, for each $w=1,2,\ldots$ output "0". Moreover, if $(\vec{x},u,v,w)\in\mathbb{W}$, output "$2^{-u}$", increment v, and restart with $w=1,2,\ldots$

Observe that, if $\forall v\exists w:(\vec{x},u,v,w)\in\mathbb{W}$ holds, this will for each such u produce a sequence with accumulation points precisely 0 and 2^{-u}; and otherwise a sequence containing finitely many 2^{-u}'s and 0's otherwise. Hence, if $\exists u\forall v\exists w:(\vec{x},u,v,w)\in\mathbb{W}$ holds, the parallel search for such u will result in a non-converging output; and otherwise in an output converging to 0. □

4. Conclusion

In Recursive Analysis, adding oracle access[‡] (to the, say, Halting Problem) does not increase the topological power of computation: Computable real functions are still necessarily continuous (i.e. $\mathbf{\Sigma}_1$-measurable). Relaxing the output representation from approximations with error bounds to converging approximations without error bounds, however, does increase the topological capabilities by proceeding one step up the (effective) Borel Hierarchy.

For Analytic Machines, on the other hand, we have revealed both to be equivalent: relaxing output with to without error bounds and permitting oracle access to the BSS Halting Problem. Both amount to climbing up one step in the (non-effective) Borel Hierarchy—although the algebraic model lies slightly skewly to its levels.

Question 4.1. *How about degrees of quasi-strongly analytic machines?*

[‡]in the sense of querying digits of a single infinite sequence. As a referee kindly pointed out, this corresponds more to a pre-stored constant of a BSS machine than to real number oracle queries. Other notions of oracles in Recursive Analysis are discussed in [BdBP10].

These are a blend of weak and strong ones, required to provide error bounds which, however, they are permitted to violate a finite (yet unbounded) number of times.

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