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UNSOLVABILITY CORES IN CLASSIFICATION PROBLEMS

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ABSTRACT. Classification problems have been introduced by M. Ziegler as a generalization of promise problems. In this paper we are concerned with solvability and unsolvability questions with respect to a given set or language family, especially with cores of unsolvability. We generalize the results about unsolvability cores in promise problems to classification problems. Our main results are a characterization of unsolvability cores via cohesiveness and existence theorems for such cores in unsolvable classification problems. In contrast to promise problems we have to strengthen the conditions to assert the existence of such cores. In general unsolvable classification problems with more than two components exist, which possess no cores, even if the set family under consideration satisfies the assumptions which are necessary to prove the existence of cores in unsolvable promise problems. But, if one of the components is fixed we can use the results on unsolvability cores in promise problems, to assert the existence of such cores in general. In this case we speak of conditional classification problems and conditional cores. The existence of conditional cores can be related to complexity cores. Using this connection we can prove for language families, that conditional cores with recursive components exist, provided that this family admits an uniform solution for the word problem.

Introduction

The concept of classification problems was introduced by M. Ziegler ([1]) as a generalization of promise problems due to S. Even ([5]). Promise problems are a generalization of decision problems. A classification problem is a vector $\mathbf{A} = (A_1, \dots, A_k)$ where the A_i are pairwise disjoint infinite subsets of a given basic set S. For a set family $\mathcal{F} \subseteq \mathbf{2}^S$ such a classification problem is \mathcal{F} -solvable, if a vector $\mathbf{Q} = (Q_1, \dots, Q_k)$ exists with $A_i \subseteq Q_i$, $Q_i \in \mathcal{F}, Q_i \cap Q_j = \emptyset$ for $1 \le i \ne j \le k$ and $Q_1 \cup \dots \cup Q_k = S$. If k = 2 we are faced with promise problems. In applications $S = X^*$ where X is a finite nonempty alphabet and $\mathcal{F} = \mathcal{L}$ a language family and/or a complexity class. From an algorithmic point of view solutions of classification problems can be used to obtain constant size advices. In this case advices indicate the inputs to belong to certain subsets (c.f. [1] for further details). We extend the results about unsolvability cores in promise problems ([4]) to unsolvability cores in classification problems. Again cohesiveness is the characterizing indicator. For unsolvable promise problems we can

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find in general unsolvability cores, if the set family is closed under union, intersection and finite variation. But for unsolvable classification problems with k > 2 the existence of unsolvability cores needs further conditions. We show, that we can assert the existence of unsolvability cores for k > 2 under the same assumption as needed for promise problems, if we fix one of the components. In this approach the fixed component is called the *condition* for the classification problem. The results are proven under assumptions which involve closure properties of \mathcal{F} against some or all boolean operations union, intersection and complementation. Moreover, we can relate unsolvability cores for conditional classification problems to so called proper hard cores introduced by R. Book and D.-Z. Du in a general form ([3]) and first defined by N. Lynch ([6]) for complexity classes. Using results and proof techniques from [3] we can apply our results to language families and complexity classes. Especially, we are able to construct unsolvability cores where the components are recursive. To do this, the language family or complexity class under consideration must allow an enumeration where the word problem has a uniform solution. We assume the reader to be familiar with the theory of recursive functions, languages and complexity (cf.[2],[7]).

1. Set and Language Families, Basic Notations

In the following an infinite basic set S is given. We assume that the elements of set families \mathcal{F} are subsets of S. Moreover, sets $A, A', B, B', C, \dots, Q, \dots$ are always subsets of S and singletons $\{s\}$ are identified with s. We mainly deal with denumerable set families \mathcal{F} ; i.e. a function $\mathbf{e}_{\mathcal{F}}: \mathbb{N}_0 \to \mathbf{2}^S$ with $\mathbf{e}_{\mathcal{F}}(\mathbb{N}_0) = \mathcal{F}$ exists (enumeration of \mathcal{F}). Consider the boolean operations $A \cup B$ union, $A \cap B$ intersection and $A^c = S \setminus A$ complementation in connection with set families \mathcal{F} . These operations can be lifted to binary operations between set families \mathcal{F}_1 and \mathcal{F}_2 and unary operations for \mathcal{F} . Define

$$\mathcal{F}_1 \oplus \mathcal{F}_2 = \{ A \cup B | A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2 \},$$

 $\mathcal{F}_1 \odot \mathcal{F}_2 = \{ A \cap B | A \in \mathcal{F}_1 \text{ and } B \in \mathcal{F}_2 \}$

and the closure operations

$$\mathcal{F}^{\mathbf{u}} = \{A_1 \cup \ldots \cup A_n | n \ge 1, A_i \in \mathcal{F} \text{ for } 1 \le i \le n\} (union),$$

$$\mathcal{F}^{\mathbf{s}} = \{A_1 \cap \ldots \cap A_n | n \ge 1, A_i \in \mathcal{F} \text{ for } 1 \le i \le n\} (intersection),$$

$$\mathcal{F}^{\mathbf{co}} = \{A^{\mathbf{c}} | A \in \mathcal{F}\}, \ \mathcal{F}^{\mathbf{cc}} = \mathcal{F} \cup \mathcal{F}^{\mathbf{co}} (complementation) \text{ and }$$

$$\mathcal{F}^{\mathbf{b}} = ((\mathcal{F}^{\mathbf{cc}})^{\mathbf{s}})^{\mathbf{u}} (boolean \ closure).$$

We will frequently use $\mathcal{F}^{dc} = \mathcal{F} \cap \mathcal{F}^{co}$. Note, that $(\mathcal{F}^{\mathbf{u}})^{\mathbf{s}} = (\mathcal{F}^{\mathbf{s}})^{\mathbf{u}}(distributivity), (\mathcal{F}^{co})^{\mathbf{u}} = (\mathcal{F}^{\mathbf{s}})^{\mathbf{co}}(deMorgan), (\mathcal{F}^{\mathbf{cc}})^{\mathbf{dc}} = \mathcal{F}^{\mathbf{cc}}$ and $(\mathcal{F}^{\mathbf{co}})^{\mathbf{co}} = \mathcal{F}$. Furthermore, $\mathcal{F} = \mathcal{F}^{\mathbf{cc}}$ ($\mathcal{F} = \mathcal{F}^{\mathbf{u}}$, $\mathcal{F} = \mathcal{F}^{\mathbf{s}}$) if and only if $\mathcal{F} = \mathcal{F}^{\mathbf{co}}$ ($\mathcal{F} \oplus \mathcal{F} \subseteq \mathcal{F}$, $\mathcal{F} \odot \mathcal{F} \subseteq \mathcal{F}$, respectively).

Let $\mathbf{fin}(S) = \{A \subseteq S | A \text{ finite}\}\$. Then \mathcal{F} is closed under \mathbf{finite} variation if $\mathcal{F} \oplus \mathbf{fin}(S) \subseteq \mathcal{F}$ and $\mathcal{F} \odot \mathbf{fin}(S)^{\mathbf{co}} \subseteq \mathcal{F}$. We call \mathcal{F} nontrivial if $\emptyset, S \in \mathcal{F}$ and \mathcal{F} is closed under finite variation. In this case $\mathbf{fin}(S) \subseteq \mathcal{F}$. Note, that $\mathbf{fin}(S) = \mathbf{fin}(S)^{\mathbf{b}}$. Moreover, $\mathcal{F}^{\mathbf{cc}}$, $\mathcal{F}^{\mathbf{u}}$, $\mathcal{F}^{\mathbf{s}}$ and $\mathcal{F}^{\mathbf{b}}$ are nontrivial, if \mathcal{F} is nontrivial.

Consider the case $S = X^*$, where X^* is the free monoid over X (a nonempty, finite alphabet) with concatenation of words as monoid operation and $\mathbf{1}$ as identity. As usual $L \subseteq X^*$ is called a language and $\mathcal{L} \subseteq \mathbf{2}^{X^*}$ a language family. For a word $w = x_1 \dots x_n$ ($x_i \in X$ for $1 \leq i \leq n$) |w| = n is the length of w and $|\mathbf{1}| = 0$. For languages L_1 and L_2 the complex product is defined by $L_1L_2 = \{w_1w_2|w_1 \in L_1, w_2 \in L_2\}$. There are various kinds

of quotients available, for example the *left quotient* defined by $L_1^{-1}L_2 = \{w \mid \exists w_1 \in L_1: w_1w \in L_1\}$. In this context we are mainly interested in handling *leftmarkers*, i.e. we consider the products wL and the quotients $w^{-1}L$ where $w \in X^*$ and L is a language. With respect to language families \mathcal{L} we get the closure operations $\mathcal{L}^{\mathbf{ltr}} = \{wL | w \in X^*, L \in \mathcal{L}\}$ and $\mathcal{L}^{\mathbf{-ltr}} = \{w^{-1}L | w \in X^*, L \in \mathcal{L}\}$. In handling the leftmarkers (for example complementation of a leftmarked language) we use variation by $\mathcal{L}_{\mathbf{reg}}(X)$, the family of regular languages (for details see [4]). A language family \mathcal{L} is closed under regular variation if $\mathcal{L} \oplus \mathcal{L}_{\mathbf{reg}}(X) \subseteq \mathcal{L}$ and $\mathcal{L} \odot \mathcal{L}_{\mathbf{reg}}(X) \subseteq \mathcal{L}$.

Looking at (partial) orderings on X^* the lexicographic ordering is important for our purposes. For $n \geq 0$ let $[n]_0 = \{0, \ldots, n-1\}$ and $[n] = \{1, \ldots, n\}$. Given a bijection $\omega: X \to [b]_0$ (b = #(X)) define $w \leq v$ if and only if (|w| < |v|) or (|w| = |v|) and $(\forall u \in X^*, x, y \in X: w \in uxX^*)$ and $v \in uyX^* \Rightarrow \omega(x) \leq \omega(y))$. This is a well-ordering, hence we can define a successor function succ for $w \in X^*$ by $succ(w) = min\{v \in X^* | w \neq v\}$ and $w \leq v\}$ where the minimum is taken with respect to the lexicographic ordering. Then $\lambda i.lex(i) = succ^i(1)$ defines a bijection $lex: \mathbb{N}_0 \to X^*$ with inverse $ord = lex^{-1}$.

Consider the language families $\mathcal{L}_{\mathbf{r.e.}}(X)$ (recursively enumerable languages) and $\mathcal{L}_{\mathbf{rec}}(X) = \mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{dc}}$ (recursive languages). Let $\mathbf{rec}_n(n \geq 0)$ be the set of n-ary recursive functions. Using $0, 1 \in \mathbb{N}_0$ as truth values define for a language L the function $\lambda i.\delta_L(i) = "\mathbf{lex}(i) \in L"$. Then a language L is recursive if and only if $\delta_L \in \mathbf{rec}_1$. Alternatively, a nonempty language L is recursive if and only if a function $f: \mathbb{N}_0 \to X^*$ exists such that $\lambda i.\mathbf{ord}(f(i))$ is nondecreasing and recursive. Classical language families and complexity classes are always denumerable. Of special interest are families with enumerations which are in a certain sense "effective". For our purpose it is important to assert that these enumerations allow a uniform solution for the word problem. More formular, we define for an enumeration \mathbf{e} of a language family \mathcal{L} the function $\lambda i, j.\mathbf{word}_{\mathbf{e}}(i,j) = "\mathbf{lex}(j) \in \mathbf{e}(i)$ ". If $\mathbf{word}_{\mathbf{e}} \in \mathbf{rec}_2$ then \mathbf{e} is called \mathbf{WP} -recursive. \mathcal{L} is called \mathbf{WP} -recursive, if a WP-recursive enumeration \mathbf{e} of \mathcal{L} exists. Note, that any WP-recursive \mathcal{L} is a (proper) subfamily of $\mathcal{L}_{\mathbf{rec}}(X)$ and every complexity class with reasonable ressource bounds (\mathbf{time} - and \mathbf{space} -constructability [2]) is WP-recursive.

2. Solvability of Classification Problems

Let k > 0. We consider vectors $\mathbf{A} = (A_1, \dots, A_k)$ with $A_i \subseteq S$ for $1 \le i \le k$. To such an \mathbf{A} we associate two functions $\mathbf{set}(\mathbf{A}) = A_1 \cup \dots \cup A_k$ and $|\mathbf{A}| = k$. Moreover, if $\mathbf{B} = (B_1, \dots, B_m)$ with $1 \le m \le k$ is another vector, then $\mathbf{B} \le \mathbf{A}$ if and only if an injective $\sigma : [m] \to [k]$ exists with $B_i \subseteq A_{\sigma(i)}$ for $1 \le i \le m$. \mathbf{A} is a classification problem if A_i is infinite and $A_i \cap A_j = \emptyset$ for all $1 \le i \ne j \le k$. For a given \mathcal{F} a vector $\mathbf{Q} = (Q_1, \dots, Q_k)$ is an \mathcal{F} -partition if $\mathbf{set}(\mathbf{Q}) = S$, $Q_i \in \mathcal{F}$ and $Q_i \cap Q_j = \emptyset$ for $1 \le i \ne j \le k$.

Definition 2.1. A classification problem **A** is \mathcal{F} -solvable $(A \in class_k(\mathcal{F}))$ if and only if an \mathcal{F} -partition **Q** exists with $|\mathbf{Q}| = k$ and $\mathbf{A} \leq \mathbf{Q}$, where $k = |\mathbf{A}|$.

If $S = \mathbb{N}_0$ then \mathcal{F} -solvability of promise problems corresponds to the separation principle defined in [7] (exercise 5-33). Our definition of \mathcal{F} -solvability for classification problems is stronger than the definition of \mathcal{F} -separability given in [1], where a classification problem \mathbf{A} is \mathcal{F} -separable, if there exists a \mathbf{Q} , which satisfies the conditions of Definition 2.1. except the condition " $\mathbf{set}(\mathbf{Q}) = S$ ", which may not necessarily be valid. Note that for such a \mathbf{Q} , we always obtain $Q_k \subseteq (Q_1 \cup \cdots \cup Q_{k-1})^c$. Hence, the class of \mathcal{F} -solvable classification problems

with more than one components is identical with the class of \mathcal{F} -separable classification problems, if \mathcal{F} is a boolean algebra. That \mathcal{F} -solvability is stronger than \mathcal{F} -separability, follows from results in [7]. Consider $\mathcal{L}_{\mathbf{r.e.}}(X)$ where X is a one-letter alphabet. Then a promise problem (A, B) consisting of recursively enumerable sets exists, which is not $\mathcal{L}_{\mathbf{r.e.}}(X)$ -solvable ([7] exercise 5-34). But (A, B) is clearly $\mathcal{L}_{\mathbf{r.e.}}(X)$ -separable. We also find the interesting result that any promise problem (A, B) with $A, B \in \mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ is $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ -solvable ([7] exercise 5-33). Hence all promise problems, which are $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ -separable are $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ -solvable. But $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{co}}$ is not closed under complementation.

For k=1 we identify A_1 with (A_1) . If \mathcal{F} is nontrivial then every A_1 is \mathcal{F} -solvable. If k>2 and \mathcal{F} satisfies appropriate closure properties, then we can reduce the question of solvability of classification problems to solvability of promise problems. Directly from the definition we get

Proposition 2.2. If $\mathcal{F} = \mathcal{F}^u$ then for all classification problems A and B with $B \leq A$ $A \in class_{|A|}(\mathcal{F})$ implies $B \in class_{|B|}(\mathcal{F})$.

Proof. Suppose $\mathbf{B} \leq \mathbf{A} \leq \mathbf{Q}$ where \mathbf{Q} is an \mathcal{F} -partition. Let $B = (B_1, \dots, B_m), A = (A_1, \dots, A_k)$ and $Q = (Q_1, \dots, Q_k)$. Then we can assume without loss of generality $B_i \subseteq A_i \subseteq Q_i$ for all i. Consider $P = Q_1 \cup \dots \cup Q_k$. Then $P^{\mathbf{c}} = Q_{m+1} \cup \dots \cup Q_k \in \mathcal{F}$. Hence, $\mathbf{Q}' = (Q_1, \dots, Q_{k-1}, Q_k \cup P^{\mathbf{c}})$ is an \mathcal{F} -partition with $\mathbf{B} \leq \mathbf{Q}'$.

Lemma 2.3. If $\mathcal{F} = \mathcal{F}^u = \mathcal{F}^s$ and $\mathbf{A} = (A_1, \dots A_k)$ is a classification problem then $\mathbf{A} \in \mathbf{class}_k(\mathcal{F})$ if and only if $(A_i, A_j) \in \mathbf{class}_2(\mathcal{F})$ for all $1 \leq i \neq j \leq k$.

Proof. The "if part" follows by Proposition 2.2. Suppose that $(A_i, A_j) \in \operatorname{class}_2(\mathcal{F})$ for $1 \leq i \neq j \leq k$. Now we proceed by induction over $|\mathbf{A}| = k$. If k = 2 nothing is to prove. Let $\mathbf{A} = (A_1, \dots, A_{k+1})$ and suppose $(A_1, \dots, A_k) \in \operatorname{class}_k(\mathcal{F})$. Then an \mathcal{F} -partition $\mathbf{Q}' = (Q'_1, \dots, Q'_k)$ with $(A_1, \dots, A_k) \leq \mathbf{Q}'$ exists. Assume without loss of generality $A_i \subseteq Q'_i$ for $1 \leq i \leq k$. On the other side $Q''_i \in \mathcal{F}^{\operatorname{dc}}$ exist with $A_i \subseteq Q''_i$ and $A_{k+1} \subseteq (Q''_i)^{\operatorname{c}}$ for $1 \leq i \leq k$. Consider $P = Q''_1 \cup \dots \cup Q''_k$. Then $A_i \subseteq P \in \mathcal{F}$ for $1 \leq i \leq k$ and $P^{\operatorname{c}} = (Q''_1)^{\operatorname{c}} \cap \dots \cap (Q''_k)^{\operatorname{c}} \in \mathcal{F}$ with $A_{k+1} \subseteq P^{\operatorname{c}}$. This shows $\mathbf{Q} = (Q'_1 \cap P, \dots, Q'_k \cap P, P^{\operatorname{c}})$ is an \mathcal{F} -partition with $\mathbf{A} \leq \mathbf{Q}$.

As indicated in the introduction we generalize the notion of a classification problem to conditional classification problems by fixing one component as condition. Consider $C \subseteq S$ and a classification problem A. Then (C, \mathbf{A}) is a conditional classification problem if $C \cap \mathbf{set}(\mathbf{A}) = \emptyset$, referring to C as the problem condition. C could be finite, even empty. If $C^{\mathbf{c}}$ is finite, then no conditional classification problems (C, \mathbf{A}) exist.

Definition 2.4. A conditional classification problem (C, \mathbf{A}) is called \mathcal{F} -solvable $(A \in \mathbf{cclass}_k(C, \mathcal{F}))$ if and only if an \mathcal{F} -partition $\mathbf{Q} = (Q_0, Q_1, \dots, Q_k)$ exists with $C \subseteq Q_0$ and $\mathbf{A} \leq (Q_1, \dots, Q_k)$ where $k = |\mathbf{A}|$.

The following facts follow directly from the definition

Proposition 2.5. Let \mathcal{F} and k > 0 be given.

- (1) $C_1 \subseteq C_2 \subseteq S \Rightarrow cclass_k(C_2, \mathcal{F}) \subseteq cclass_k(C_1, \mathcal{F}).$
- (2) $C^c \in fin(S) \Rightarrow cclass_k(C, \mathcal{F}) = \emptyset$.
- (3) $\emptyset \in \mathcal{F} \Rightarrow class_k(\mathcal{F}) \subseteq cclass_k(\emptyset, \mathcal{F}).$
- (4) $\mathcal{F} = \mathcal{F}^u \Rightarrow class_k(\mathcal{F}) = cclass_k(\emptyset, \mathcal{F}).$
- (5) \mathcal{F} nontrivial and $C \in \mathbf{fin}(S) \Rightarrow \mathbf{cclass}_k(C, \mathcal{F}) = \mathbf{cclass}_k(\emptyset, \mathcal{F})$.

Example 2.6. Consider $X = \{a, b\}$. Let $\mathcal{L} = \mathcal{L}^{\mathbf{ltr}} = \mathcal{L}^{\mathbf{-ltr}}$ a nontrivial language family, which is closed under regular variation. If A is a set with $A^{\mathbf{c}}, A \notin \mathcal{L}$, then $(A^{\mathbf{c}}, A) \notin \mathbf{class}_2(\mathcal{L})$ and by our assumption on $\mathcal{L}(xA^{\mathbf{c}}, xA) \notin \mathbf{class}_2(\mathcal{L})$ for x = a, b (Lemma 5.4. in [4]). Clearly, $(aA^{\mathbf{c}}, bA) \in \mathbf{class}_2(\mathcal{L})$, but $(aA \cup bA^{\mathbf{c}}, aA^{\mathbf{c}}, bA) \notin \mathbf{class}_3(\mathcal{L})$. Hence $(aA^{\mathbf{c}}, bA) \notin \mathbf{cclass}_2(aA \cup bA^{\mathbf{c}}, \mathcal{L})$.

3. Unsolvability Cores in Classification Problems

As in the case of promise problems unsolvability of classification problems is closely related to cohesiveness.

Definition 3.1. $A \subseteq S$ is \mathcal{F} -cohesive $(A \in \mathbf{cohesive}(\mathcal{F}))$ if and only if A is infinite and for all $Q \in \mathcal{F}^{\mathbf{dc}}$ either $A \cap Q$ or $A \cap Q^{\mathbf{c}}$ is finite (cf.[4] and [7]).

Remark 3.2. It is interesting to compare our definition of cohesiveness with related classical definitions, as they are presented in [7]. Consider the families $\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{cc}}$, $\mathcal{L}_{\mathbf{r.e.}}(X)$ and $\mathcal{L}_{\mathbf{rec}}(X)$. Then $L \in \mathbf{cohesive}(\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{cc}})$ if and only if L is cohesive in the classical sense. Moreover, $\mathbf{cohesive}(\mathcal{L}_{\mathbf{rec}}(X)) = \mathbf{cohesive}(\mathcal{L}_{\mathbf{r.e.}}(X))$, since a language Q is recursive if and only if Q and $Q^{\mathbf{c}}$ are recursively enumerable. Furthermore the definition of recursively indecomposability coincides with the definition of $\mathcal{L}_{\mathbf{rec}}(X)$ -cohesiveness. In [7] we also find the notion of indecomposability. L is indecomposable if there exist no infinite sets $L_1, L_2 \in \mathcal{L}_{\mathbf{r.e.}}(X)$ such that $L_1 \cap L_2 = \emptyset, L \subseteq L_1 \cup L_2, L \cap L_1$ is infinite and $L \cap L_2$ is infinite. Then we find the following results in [7]. If $L \in \mathbf{cohesive}(\mathcal{L}_{\mathbf{r.e.}}(X)^{\mathbf{cc}})$ then it is indecomposable and any indecomposable L is $\mathcal{L}_{\mathbf{rec}}(X)$ -cohesive. None of the converse implications hold.

In [4] (Theorem 5.1.) it is proven, that for a promise problem (A, B) and a nontrivial set family $\mathcal{F}(A \cup B) \in \mathbf{cohesive}(\mathcal{F})$ if and only if $A, B \in \mathbf{cohesive}(\mathcal{F})$ and $(A, B) \notin \mathbf{class}_2(\mathcal{F})$. This result leads to a much stronger one. In the theory of complexity we find the notion of hard cores inside those sets which can be computed with bounded ressources (time, space, e.t.c. [3]). Similarly, we can consider unsolvability cores of classification problems which are not solvable.

Definition 3.3. For k > 1 a classification problem \mathbf{A} with $|\mathbf{A}| = k$ is a k-core of \mathcal{F} $(\mathbf{A} \in \mathbf{core}_k(\mathcal{F}))$ if and only if for all classification problems \mathbf{A}' with $\mathbf{A}' \leq \mathbf{A}$ and $|\mathbf{A}'| > 1$: $\mathbf{A}' \notin \mathbf{class}_{|\mathbf{A}'|}(\mathcal{F})$.

Clearly, any subproblem of a core is itself a core. This is especially true for subproblems, which are promise problems. This enables us to use the results about unsolvability cores for promise problems from [4].

Lemma 3.4. If $\mathcal{F} = \mathcal{F}^u$ and $\mathbf{A} = (A_1, \dots, A_k)(k > 1)$ is a classification problem then $\mathbf{A} \in \mathbf{core}_k(\mathcal{F})$ if and only if $(A_i, A_i) \in \mathbf{core}_2(\mathcal{F})$ for all $1 \le i \ne j \le k$.

Proof. Suppose $\mathbf{A} \in \mathbf{core}_k(\mathcal{F})$, then by definition $(A_i, A_j) \leq \mathbf{A}$ and therefore $(A_i, A_j) \in \mathbf{core}_2(\mathcal{F})$. Conversely, suppose that $\mathbf{A} \notin \mathbf{core}_k(\mathcal{F})$, i. e. $\mathbf{A}' = (A'_1, \ldots, A'_m)$ exists with $\mathbf{A}' \leq \mathbf{A}$, m > 1 and $\mathbf{A}' \in \mathbf{class}_{|\mathbf{A}'|}(\mathcal{F})$. Since $\mathcal{F} = \mathcal{F}^{\mathbf{u}}$ we know $(A'_1, A'_2) \in \mathbf{class}_2(\mathcal{F})$. Moreover, $A'_1 \subseteq A_i$ and $A'_2 \subseteq A_j$ for some $1 \leq i \neq j \leq k$. But then $(A_i, A_j) \notin \mathbf{core}_2(\mathcal{F})$. \square

Now we can characterize cores by cohesiveness. Using Theorem 5.1. and Theorem 6.7. of [4] we can prove

Theorem 3.5. If $\mathcal{F} = \mathcal{F}^u$ is nontrivial and \mathbf{A} a classification problem with $|\mathbf{A}| = k > 1$ then $\mathbf{A} \in \mathbf{core}_k(\mathcal{F})$ if and only if $\mathbf{set}(\mathbf{A}) \in \mathbf{cohesive}(\mathcal{F})$.

Proof. If $\mathbf{A} = (A_1, \dots, A_k) \in \mathbf{core}_k(\mathcal{F})$, then $(A_i, A_j) \in \mathbf{core}_2(\mathcal{F})$ for all $1 \leq i \neq j \leq k$. By Theorem 6.7. in [4] we know $A_1 \cup A_i \in \mathbf{cohesive}(\mathcal{F})$ for all $2 \leq i \leq k$. But then $A_1 \cup \dots \cup A_k = (A_1 \cup A_2) \cup \dots \cup (A_1 \cup A_k)$. Since $A_1 \subseteq (A_1 \cup A_i) \cap (A_1 \cup A_j)$ for all $2 \leq i \neq j \leq k$ and A_1 is infinite, a simple induction proof shows $\mathbf{set}(\mathbf{A}) \in \mathbf{cohesive}(\mathcal{F})$.

Conversely, if $A_1 \cup \cdots \cup A_k \in \boldsymbol{cohesive}(\mathcal{F})$ then for all $1 \leq i \neq j \leq k$, $A_i \cup A_j \in \boldsymbol{cohesive}(\mathcal{F})$. Again by Theorem 6.7. of [4] $(A_i, A_j) \in \boldsymbol{core}_2(\mathcal{F})$ and therefore by Lemma 3.4. $\mathbf{A} \in \boldsymbol{core}_k(\mathcal{F})$.

We can find to any classification problem \mathbf{A} with $|\mathbf{A}| = 2$ and $\mathbf{A} \notin class_2(\mathcal{F})$ a $\mathbf{B} \leq \mathbf{A}$ such that $\mathbf{B} \in core_2(\mathcal{F})$ if $\mathcal{F} = \mathcal{F}^{\mathbf{u}} = \mathcal{F}^{\mathbf{s}}$ is denumerable ([4]). But this is not true for classification problems \mathbf{A} with $|\mathbf{A}| > 2$. To see this we prove the following theorem, where we use $S = X^*$ with $X = \{a, b, c\}$. Define for $A \subseteq X^*$ the classification problem $\mathbf{C}(A) = (A_{ab}, A_{bc}, A_{ca})$, where $A_{xy} = xA \cup yA^{\mathbf{c}}$ for $x, y \in X$.

Theorem 3.6. Let \mathcal{L} be a nontrivial language family with $\mathcal{L} = \mathcal{L}^{u} = \mathcal{L}^{ltr} = \mathcal{L}^{-ltr}$, which is closed under regular variation. If $A \subseteq S$ with $A \notin \mathcal{L}$ or $A^{c} \notin \mathcal{L}$, then $C(A) \notin class_{3}(\mathcal{L})$ and for all $B \leq C(A)$ with $|B| = 3 : B \notin core_{3}(\mathcal{L})$.

Proof. (1) We know $(A^{\mathbf{c}}, A) \notin \mathbf{class}_2(\mathcal{L})$ ([4]). But then by Lemma 5.4. of [4] $(xA^{\mathbf{c}}, xA) \notin \mathbf{class}_2(\mathcal{L})$ for all $x \in X$. Now $(bA^{\mathbf{c}}, bA) \leq (A_{ab}, A_{bc})$, $(cA^{\mathbf{c}}, cA) \leq (A_{bc}, A_{ca})$ and $(aA^{\mathbf{c}}, aA) \leq (A_{ca}, A_{ab})$. This shows $(A_{xy}, A_{xz}) \notin \mathbf{class}_2(\mathcal{L})$ for all $x \neq y$, $z \neq y$ and $x \neq z$.

(2) Suppose $\mathbf{B} \leq \mathbf{C}(A)$ exists with $\mathbf{B} \in \mathbf{core}_3(\mathcal{L})$. Then by Theorem 3.5. $\mathbf{set}(\mathbf{B}) \in \mathbf{cohesive}(\mathcal{L})$. Assume without loss of generality that $\mathbf{B} = (B(a,b),B(b,c),B(c,a))$ and $B(x,y) \subseteq A_{xy}$ for $x,y \in X$ with $x \neq y$. In the following let $B'(x,y) = B(x,y) \cap xX^*$ and $B''(x,y) = B(x,y) \cap (xX^*)^{\mathbf{c}}$.

Assertion: $B'(x,y) \in fin(X^*)$ for all $x,y \in X$ with $x \neq y$. Suppose to the contrary (without loss of generality) $B'(a,b) \notin fin(X^*)$. But then $B'(b,c) \in fin(X^*)$. Otherwise we obtain $(B'(a,b),B'(b,c)) \leq (aX^*,bX^*) \leq (aX^*,(aX^*)^c)$. Since $\mathcal{L}_{reg} \subseteq \mathcal{L}$, $\mathbf{B} \notin core_3(\mathcal{L})$ - a contradiction. But now B''(b,c) is infinite and $B''(b,c) \subseteq fin(X^*)$

 $cX^* \subseteq (aX^*)^{\mathbf{c}}$, hence both $set(\mathbf{B}) \cap aX^*$ and $set(\mathbf{B}) \cap (aX^*)^{\mathbf{c}}$ are infinite - a contradiction to $set(\mathbf{B}) \in cohesive(\mathcal{L})$. Now consider B''(a,b) and B''(c,a). Then both sets are infinite and $(B''(a,b), B''(c,a)) \le cohesive(\mathcal{L})$.

Now consider B''(a,b) and B''(c,a). Then both sets are infinite and $(B''(a,b), B''(c,a)) \le (bX^*, aX^*) \le (bX^*, (bX^*)^c)$ - a contradiction to $\mathbf{B} \in \mathbf{core}_3(\mathcal{L})$. This completes the proof.

Remark 3.7. The basic idea behind the proof of Theorem 3.6. is due to M. Ziegler ([1]). Note, that complexity classes and most of the known language families satisfy the conditions of Theorem 3.6.

Using conditional unsolvability, we can derive an existence theorem for cores.

Theorem 3.8. Let $\mathcal{F} = \mathcal{F}^u = \mathcal{F}^s$ be denumerable and nontrivial. If $\mathbf{A} = (A_1, \dots, A_k)$ is a classification problem and $C \subseteq \mathbf{set}(\mathbf{A})^c$ is \mathcal{F} -cohesive with $(C, A_i) \notin \mathbf{class}_2(\mathcal{F})$ for $1 \le i \le k$, then there exists $\mathbf{B} \le \mathbf{A}$ with $|\mathbf{B}| = k$ and $\mathbf{B} \in \mathbf{core}_k(\mathcal{F})$.

Proof. Since $(C, A_i) \notin class_2(\mathcal{F})$, we can find $C_i \subseteq C$ and $B_i \subseteq A_i$ with $(C_i, B_i) \in core_2(\mathcal{F})$ (Theorem 6.14. in [4]). By Theorem 3.5. $C_i \cup B_i \in cohesive(\mathcal{F})$ and therefore $B_i \in cohesive(\mathcal{F})$. Now $(C, B_i) \notin class_2(\mathcal{F})$ and $C \in cohesive(\mathcal{F})$. By Theorem 5.1. in [4] we know $C \cup B_i \in cohesive(\mathcal{F})$. But then $C \cup B_1 \cup \cdots \cup B_k = (C \cup B_1) \cup \cdots \cup (C \cup B_k) \in cohesive(\mathcal{F})$, since for all $1 \leq i \neq j \leq k$ C is infinite and $C \subseteq (C \cup B_i) \cap (C \cup B_j)$. It follows $B_1 \cup \cdots \cup B_k \in cohesive(\mathcal{F})$ and we obtain $\mathbf{B} = (B_1, \ldots, B_k) \leq \mathbf{A}$ and by Theorem 3.5. $\mathbf{B} \in core_k(\mathcal{F})$.

Remark 3.9. Consider the situation of Theorem 3.6. Then $set(\mathbf{C}(A)) = XX^*$ and there is no room for an infinite condition C to make the conditional classification problem $(C, \mathbf{C}(A))$ \mathcal{L} -solvable.

4. Cores in Conditional Classification Problems

Unsolvability of conditional classification problems can be related to cohesiveness, too.

Definition 4.1. Let $C, A \subseteq S$. Then A is \mathcal{F} -cohesive under condition C (in short: $A \in \mathbf{ccohesive}(C, \mathcal{F})$), if and only if A is infinite and for all $Q \in \mathcal{F}^{\mathbf{dc}}$ with $Q \subseteq C$ either $A \cap Q$ or $A \cap Q^c$ is finite.

Clearly, if $C_1 \subseteq C_2 \subseteq S$, then $\operatorname{ccohesive}(C_2, \mathcal{F}) \subseteq \operatorname{ccohesive}(C_1, \mathcal{F})$. Especially, we get $\operatorname{ccohesive}(S, \mathcal{F}) = \operatorname{cohesive}(\mathcal{F})$ and therefore $\operatorname{cohesive}(\mathcal{F}) \subseteq \operatorname{ccohesive}(C, \mathcal{F})$ for all $C \subseteq S$. Rewriting the definition, we also find $\operatorname{ccohesive}(C, \mathcal{F})) = \operatorname{cohesive}(\mathcal{F}(C)^{\operatorname{cc}})$ where $\mathcal{F}(C) = \{Q \mid Q \subseteq C \text{ and } Q \in \mathcal{F}\}$. Analogously, we define conditional cores by

Definition 4.2. Let $C \subseteq S$ and \mathbf{A} a classification problem. Then \mathbf{A} is a C-conditional core of \mathcal{F} ($\mathbf{A} \in \mathbf{ccore}_{|\mathbf{A}|}(C, \mathcal{F})$) if and only if for all $\mathbf{A}' \leq \mathbf{A}$ with $|\mathbf{A}'| > 0$: $\mathbf{A}' \notin \mathbf{cclass}_{|\mathbf{A}'|}(C, \mathcal{F})$.

In contrast to the definition of $core(\mathcal{F})$ subproblems \mathbf{A}' with $|\mathbf{A}'| = 1$ are considered, too. Note, that (C, \mathbf{A}') is a conditional-classification problem, if $\mathbf{A}' \leq \mathbf{A}$. Moreover, if $\mathbf{A} \in ccore_{|\mathbf{A}|}(C, \mathcal{F})$, then $\mathbf{A}' \in ccore_{|\mathbf{A}'|}(C, \mathcal{F})$. The following lemma characterizes $A \in ccore_1(C, \mathcal{F})$ by conditional cohesiveness.

Lemma 4.3. Let \mathcal{F} be nontrivial and $C, A \subseteq S$ with A infinite and $A \cap C = \emptyset$. Then the following statements are equivalent

- (i) $A \in ccore_1(C, \mathcal{F})$
- (ii) $A \notin cclass_1(C, \mathcal{F})$ and $A \in ccohesive(C^c, \mathcal{F})$.
- Proof. (i) \Rightarrow (ii): Suppose $A \in \mathbf{ccore}_1(C, \mathcal{F})$. Then $A \notin \mathbf{cclass}_1(C, \mathcal{F})$. Assume to the contrary that $A \notin \mathbf{ccohesive}(C^{\mathbf{c}}, \mathcal{F})$. Then $Q \in \mathcal{F}^{\mathbf{dc}}$ exists with $Q \subseteq C^{\mathbf{c}}, A \cap Q \notin \mathbf{fin}(S)$ and $A \cap Q^{\mathbf{c}} \notin \mathbf{fin}(S)$. Let $B = A \cap Q$. Then $B \subseteq Q$, but $Q \subseteq C^{\mathbf{c}}$, hence $C \subseteq Q^{\mathbf{c}}$. Moreover, $Q, Q^{\mathbf{c}} \in \mathcal{F}$, i.e. $B \in \mathbf{cclass}_1(C, \mathcal{F})$.
- (ii) \Rightarrow (i): Suppose that $A \notin cclass_1(C, \mathcal{F})$ and $A \in ccohesive(C^c, \mathcal{F})$. Assume to the contrary that an infinite set $B \subseteq A$ exists, such that $B \subseteq Q^c$ and $C \subseteq Q$ for some $Q \in \mathcal{F}^{dc}$. Then $Q^c \subseteq C^c$. Since $B \cap Q^c \notin fin(S)$, $A \cap Q^c \notin fin(S)$, too. Hence $A \cap Q \in fin(S)$, because $A \in ccohesive(C^c, \mathcal{F})$. Consider $Q' = Q^c \cup (A \cap Q)$. Since \mathcal{F} is nontrivial, $Q' \in \mathcal{F}$. Note that $A = (A \cap Q) \cup (A \cap Q^c) \subseteq Q^c \cup (A \cap Q) = Q'$. On the other side, $Q^c \subseteq C^c$ and $A \cap Q \subseteq A \subseteq C^c$, i.e. $Q' \subseteq C^c$. Hence $C \subseteq Q'^c$. This shows that $A \notin cclass_1(C, \mathcal{F})$ a contradiction.

Theorem 4.4. Let \mathcal{F} be nontrivial with $\mathcal{F} = \mathcal{F}^u$ and (C, \mathbf{A}) a conditional k-classification problem. If $\mathbf{A} = (A_1, \dots, A_k)$ then the following statements are equivalent

- (i) $\mathbf{A} \in \mathbf{ccore}_k(C, \mathcal{F})$
- (ii) $A_i \notin cclass_1(C, \mathcal{F})$ and $A_i \in ccohesive(C^c, \mathcal{F})$ for all $1 \leq i \leq k$.

Proof.

- (i) \Rightarrow (ii): Suppose that $\mathbf{A} \in \mathbf{ccore}_k(C, \mathcal{F})$. Then for all $1 \leq i \leq k : (C, A_i) \in \mathbf{ccore}_1(C, \mathcal{F})$, since $A_i \leq \mathbf{A}$. Applying Lemma 4.3. we get the result.
- (ii) \Rightarrow (i): Let the A_i be given according to the assumption. Assume to the contrary that $\mathbf{B} \leq \mathbf{A}$ exists with $\mathbf{B} = (B_1, \dots, B_m) \in \mathbf{cclass}_m(C, \mathcal{F})$. Then an injective $\sigma : [m] \rightarrow [k]$ exists with $B_i \subseteq A_{\sigma(i)}$ for $1 \leq i \leq k$. Since $\mathcal{F} = \mathcal{F}^{\mathbf{u}}, B_i \in \mathbf{cclass}_1(C, \mathcal{F})$. But $A_{\sigma(i)} \in \mathbf{core}_1(C, \mathcal{F})$ and $B_i \subseteq A_{\sigma(i)}$. This is a contradiction.

Now, we are able to assert the existence of conditional cores in the case that both C and $C^{\mathbf{c}}$ are infinite. Observe that under this assumption $A \in \mathbf{cclass}_1(C, \mathcal{F})$ if and only if (C, A) considered as a promise problem is solvable for \mathcal{F} , i.e. $(C, A) \in \mathbf{class}_1(\mathcal{F})$.

Lemma 4.5. Let \mathcal{F} be denumerable and nontrivial with $\mathcal{F} = \mathcal{F}^u = \mathcal{F}^s$. If $A \notin fin(S)$, $C \notin fin(S)^{cc}$, $A \cap C = \emptyset$ and $A \notin cclass_1(C, \mathcal{F})$, then $B \subseteq A$ exists with $B \in ccore_1(C, \mathcal{F})$.

Proof. If $A \notin cclass_1(C, \mathcal{F})$, i.e. $(C, A) \notin class_1(\mathcal{F})$. By cor.5.16. in[4] we can find $B \subseteq A$ such that for all infinite $B' \subseteq B$ $(C, B') \notin class_2(\mathcal{F})$, i.e. $B \in ccore_1(C, \mathcal{F})$.

Using this lemma in connection with Theorem 4.4. we get

Lemma 4.6. Let \mathcal{F} be denumerable and nontrivial with $\mathcal{F} = \mathcal{F}^u = \mathcal{F}^s$ and (C, \mathbf{A}) a conditional classification problem where C and C^c are infinite. If $\mathbf{A} = (A_1, \ldots, A_k)$ with $A_i \notin \mathbf{cclass}_1(C, \mathcal{F})$ for $1 \leq i \leq k$ then a $\mathbf{B} \leq \mathbf{A}$ exists with $|\mathbf{B}| = k$ and $\mathbf{B} \in \mathbf{ccore}_k(C, \mathcal{F})$.

Proof. By Lemma 4.5. we find for each $1 \le i \le k$ $B_i \in ccore_1(C, \mathcal{F})$ and $B_i \subseteq A_i$. Let $\mathbf{B} = (B_1, \dots, B_k)$. Then $\mathbf{B} \le \mathbf{A}$ and $|\mathbf{B}| = k$. By Theorem 4.4. $\mathbf{B} \in ccore_k(C, \mathcal{F})$.

5. CONDITIONAL CORES AND HARD CORES

For WP-recursive language families we can prove a much stronger result. This depends on the relation between $A \in \mathbf{ccore}_1(C, \mathcal{F})$ and proper hard cores introduced by N. Lynch [6] for complexity classes and in a very general form by R. Book- D.-Z. Du [3].

Definition 5.1. B is a \mathcal{F} -hardcore of A if and only if B is infinite and for all $C \in \mathcal{F}(A)$: $B \cap C \in fin(S)$. If additionally $B \subseteq A$ then B is a proper \mathcal{F} -hardcore of A. (Remind $\mathcal{F}(A) = \{Q \subseteq A \mid Q \in \mathcal{F}\}$ for \mathcal{F} and A.)

Note, that for $A' \subseteq A$ with A' infinite every \mathcal{F} -hardcore of A is a \mathcal{F} -hardcore of A'. Rephrasing Lemma 7.2. of [4] we get the following

Lemma 5.2. If \mathcal{F} is nontrivial with $\mathcal{F} = \mathcal{F}^{co}$ and (C, A) a conditional classification problem then A is a proper \mathcal{F} -hardcore of C^c if and only if $A \in \mathbf{ccore}_1(C, \mathcal{F})$.

Now we can use a construction for proper hard cores from [3] in a modified form.

Theorem 5.3. If \mathcal{L} is a nontrivial and WP-recursive language family with $\mathcal{L} = \mathcal{L}^b$ and (C, A) a conditional classification problem with $A \notin \mathbf{cclass}_1(C, \mathcal{L})$ and C, A are recursive then a recursive $B \subseteq A$ exists with $B \in \mathbf{ccore}_1(C, \mathcal{L})$.

Proof. Consider an enumeration \mathbf{e} of \mathcal{L} such that $\mathbf{word_e} \in \mathbf{rec_2}$. Furthermore, let $\delta_C, \delta_A \in \mathbf{rec_1}$. Now define for all $n \geq 0$ B(n), cancel(n) and card(n) by the following algorithm:

```
if lex(0) \in C then
   cancel(0) := 0
end if
if lex(0) \in A and lex \notin e(0) then
   B(0) := 0; card(0) := 1
end if
n := 1;
while n \neq 0 do
   B(n) := B(n-1); cancel(n) := \text{cancel}(n-1); card(n) := \text{card}(n-1);
  if lex(n) \in C then
      \operatorname{cancel}(n) := \operatorname{cancel}(n) \cup \{i | 0 \le i \le \operatorname{card}(n) \text{ and } \operatorname{lex}(n) \in \mathbf{e}(i)\}
   end if
  if lex(n) \in A and \forall 0 \le i \le card(n) : (i \notin cancel(n) \Rightarrow lex(n) \notin e(i)) then
      B(n) := B(n) \cup lex(n); card(n) := card(n) + 1
  end if;
  n := n + 1
end while
  (For A = C^{\mathbf{c}} we get the construction of [3]).
```

Now, let $B = \bigcup_{i=0}^{\infty} B(n)$ and cancel $= \bigcup_{i=0}^{\infty} \operatorname{cancel}(i)$. Assume for the moment that B is infinite. B is recursive and $B \subseteq A$, since all basic functions are recursive, $\operatorname{cancel}(n)$ is finite for all n and the elements of B are added in increasing order with respect to lex . Moreover, $\lim_{n\to\infty} \operatorname{card}(n) = \infty$. Hence $\{k|\mathbf{e}(k)\cap C\neq\emptyset\}$ = cancel and we get $\mathbf{e}(i)\subseteq C^{\mathbf{c}}$ and by construction $\mathbf{e}(i)\cap B\in\operatorname{fin}(X^*)$ for $i\notin\operatorname{cancel}(cf.[3])$. In conclusion, B is a proper \mathcal{L} -hardcore of $C^{\mathbf{c}}$ and by Lemma 4.9. $B\in\operatorname{ccore}_1(C,\mathcal{L})$. It remains to show the

Assertion: $B \notin fin(X^*)$.

Suppose to the contrary, that B is finite. Then M exists with card(n) = M for almost all n. Moreover, for every $i \in [M+1]_0$ with $\mathbf{e}(i) \cap C \neq \emptyset$ there must exist K(i) with $i \in \operatorname{cancel}(K(i))$. Let $K = \max\{K(i)|i \in [M+1]_0 \text{ with } \mathbf{e}(i) \cap C \neq \emptyset\}$. Then we know that for all $i \in [M+1]_0$ with $i \notin \operatorname{cancel}(K(i)) : \mathbf{e}(i) \subseteq C^{\mathbf{c}}$. Choose $N \geq K$ sufficiently large such that additionally $\operatorname{card}(n) = M$ for every $n \geq N$. Consider $\operatorname{lex}(n) \in A$ with $n \geq N$. Since $\operatorname{lex}(n) \notin B$, $i \in [M+1]_0$ exists with $\operatorname{lex}(n) \in \mathbf{e}(i)$. This shows $A \subseteq \{\operatorname{lex}(k)|k < N \text{ and } \operatorname{lex}(k) \in A\} \cup \bigcup_{i=0, i \notin \operatorname{cancel}}^M \mathbf{e}(i) = Q \subseteq C^{\mathbf{c}}$ and therefore $C \subseteq Q^{\mathbf{c}}$. Since \mathcal{L} is nontrivial and $\mathcal{L} = \mathcal{L}^{\mathbf{u}}$, we know $Q \in \mathcal{L}$. Moreover, $\mathcal{L} = \mathcal{L}^{\mathbf{co}}$ implies $Q^{\mathbf{c}} \in \mathcal{L}$, hence $A \notin \operatorname{cclass}_1(C, \mathcal{L})$ - a contradiction.

Now we can derive a stronger result than Lemma 4.6.:

Theorem 5.4. Let \mathcal{L} be a nontrivial and WP-recursive language family with $\mathcal{L} = \mathcal{L}^b$ and (C, \mathbf{A}) a conditional k-classification problem. If C is recursive and $\mathbf{A} = (A_1, \ldots, A_k)$ such that $A_i \in \mathbf{cclass}_1(C, \mathcal{L})$ and A_i is recursive for $1 \le i \le k$ then $\mathbf{B} = (B_1, \ldots, B_k)$ exists with $\mathbf{B} \le \mathbf{A}, \mathbf{B} \in \mathbf{ccore}_k(C, \mathcal{L})$ and B_i is recursive for $1 \le i \le k$.

Proof. By Theorem 5.3. we find for each $1 \le i \le k$ $B_i \in cclass_1(C, \mathcal{L})$ with $B_i \subseteq A_i$ and B_i is recursive. Let $\mathbf{B} = (B_1, \dots, B_k)$. Then $\mathbf{B} \le \mathbf{A}$ and by Theorem 4.4. $\mathbf{B} \in ccore_k(C, \mathcal{L})$. \square

Remark 5.5. The B_i 's constructed in Theorem 5.4. are all infinite. By the Dekker-Myhill theorem (§12.3 Theorem VI in [7]), we can find in every B_i a \mathcal{L} -cohesive B_i' , but we cannot show, that B_i' is recursive under the conditions of Theorem 5.4. The best result to our knowledge is the result of Friedberg (§12.4 Theorem XI in [7]). The construction (due to Yates) in the proof given in [7] can be easily modified in such a way, that to any infinite, recursive A a $\mathcal{L}_{\mathbf{r.e.}}(X)$ -cohesive subset B with $B^{\mathbf{c}} \in \mathcal{L}_{\mathbf{r.e.}}(X)$ can be found. Since any WP-recursive language family \mathcal{L} is a subfamily of $\mathcal{L}_{\mathbf{r.e.}}(X)$ this B is \mathcal{L} -cohesive, too.

Concluding Remarks

This paper continues our research about unsolvability cores in promise problems ([4]) generalizing the results to classification problems. Our approach is very general, though the applications in this paper deal mainly with language families and complexity classes. The main open problem in our approach is to construct cohesive sets with "nice" properties.

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