
AN EFFECT SYSTEM FOR ALGEBRAIC EFFECTS AND HANDLERS *

ANDREJ BAUER AND MATIJA PRETNAR

Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
e-mail address: Andrej.Bauer@andrej.com, matija.pretnar@fmf.uni-lj.si

ABSTRACT. We present an effect system for *core Eff*, a simplified variant of *Eff*, which is an ML-style programming language with first-class algebraic effects and handlers. We define an expressive effect system and prove safety of operational semantics with respect to it. Then we give a domain-theoretic denotational semantics of *core Eff*, using Pitts’s theory of minimal invariant relations, and prove it adequate. We use this fact to develop tools for finding useful contextual equivalences, including an induction principle. To demonstrate their usefulness, we use these tools to derive the usual equations for mutable state, including a general commutativity law for computations using non-interfering references. We have formalized the effect system, the operational semantics, and the safety theorem in Twelf.

1. INTRODUCTION

An *effect system* supplements a traditional type system for a programming language with information about which computational effects may, will, or will not happen when a piece of code is executed. A well designed and solidly implemented effect system helps programmers understand source code, find mistakes, as well as safely rearrange, optimize, and parallelize code [11, 8]. As many before us [11, 24, 25, 7] we take on the task of striking just the right balance between simplicity and expressiveness by devising an effect system for *Eff* [2], an ML-style programming language with first-class algebraic effects [17, 15] and handlers [19].

Our effect system is *descriptive* in the sense that it provides information about possible computational effects but it does not prescribe them. In contrast, Haskell’s monads *prescribe* the possible effects by wrapping types into computational monads. In the implementation we envision effect inference which never fails, although in some cases it may be uninformative. Of course, typing errors are still errors.

An important feature of our effect system is *non-monotonicity*: it detects the fact that a handler removes some effects. For instance, a piece of code which uses mutable state is determined to actually be pure when wrapped by a handler that handles away lookups and updates.

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Our contributions are as follows:

- (1) We define *core Eff*, a fragment of the language which retains the essential features of *Eff*, including first-class handlers and instances (Section 2), although we leave out dynamic creation of new instances.
- (2) We give small-step and big-step operational semantics for core *Eff* and show them to be equivalent (Section 3).
- (3) We devise an expressive effect system for core *Eff* and prove safety of the operational semantics with respect to it (Section 4).
- (4) Using the standard domain-theoretic apparatus and Pitts’s theory of minimal invariant relations [14], we provide denotational semantics for core *Eff* and prove an adequacy theorem (Section 5).
- (5) We identify a set of observational equivalences and an induction principle that allow us to reason about effectful computations (Section 6).
- (6) We demonstrate how the equivalences are used by deriving the standard equations for state from general principles. The induction principle is used in a proof of a general commutativity law which allows us to interchange two computations that use non-interfering references (Section 7).
- (7) We formalized core *Eff*, the operational semantics, the effect system, and the safety theorem in Twelf [13] (Section 8).

2. CORE EFF

The current implementation of *Eff* includes a number of features, such as syntactic sugar, products, records, inductive types, type definitions, effect definitions, etc., which are inessential for a conceptual analysis. We therefore restrict attention to *core Eff*, a fragment of the language described here. We refer the readers to [2] for a more thorough introduction of how one actually programs in *Eff*.

In *Eff* all computational effects are accessed uniformly and exclusively through *operations*. These are a primitive concept, of which typical examples are reading and writing on a communication channel, updating and looking up the contents of a reference, and raising an exception. Thus, in *Eff* each terminating computation results either in an effect-free value, or it calls an operation. Each operation has an associated delimited *continuation*, which is a suspended computation awaiting the result of the operation.

Operations do not actually perform effects, but are just suspended computations whose behavior is controlled by a second primitive notion, the *effect handlers*. These are like exception handlers, except that an effect handler has access to the continuation of the handled operation, and so may restart the computation after the operation is handled. With handlers we may implement all the usual computational effects, as well as great variety of others, such as transactional memory, non-deterministic execution strategies, stream redirection, cooperative multi-threading, and delimited continuations. At the top level there may be built-in handlers that provide interaction with the external environment, although we do not consider these in core *Eff*.

Since *Eff* is geared towards practical programming, it and core *Eff* depart in several respects from previous work on handlers and algebraic effects [19, 7]. First, rather than imposing equations on handlers by a typing discipline, the programmer may write arbitrary handlers, and then prove that a particular handler satisfies the desired equations. We demonstrate the technique in Section 7.1, where we implement a state handler and show that

it satisfies the standard equations. Second, *Eff* uses fine-grained call-by-value evaluation strategy [10] rather than the theoretically more desirable call-by-push-value [9] because we found the former to be closer to programming practice as well as easier to implement. Third, every effect has multiple *instances*. For example, a program may write and read from multiple communication channels, raise different kinds of exceptions, and manipulate multipartite state. Thus in core *Eff* an *operation symbol* op is always paired with an *instance* ι to give an *operation* $\iota\#\text{op}$. From a theoretical point of view instances are straightforward (as long as we do not generate them dynamically) and inessential, but are absolutely necessary for practical programming.

2.1. Effects and types. To get things going we presume given a collection of *effects*

$$\text{Effect } E ::= \text{exception} \mid \text{ref} \mid \dots$$

which in full *Eff* are declared by the programmer. For each E there is a given set \mathcal{I}_E of *instances* $\iota_1, \iota_2, \iota_3, \dots$ of E . The instances may be thought of as atomic names. In full *Eff* they may be dynamically created, but to keep the semantics reasonably simple we assume a fixed set. Additionally, with each E we associate a set of \mathcal{O}_E of *operation symbols* $\text{op}_1, \text{op}_2, \dots$. An operation symbol is associated with at most one effect.

The terms of core *Eff* are split into effect-free *expressions* and possibly effectful *computations*, as described in the next subsection. Consequently, the type system of core *Eff* consists of *pure types* for expressions and *dirty types* for computations:

$$\begin{aligned} \text{Pure type } A, B &::= \text{bool} \mid \text{nat} \mid \text{unit} \mid \text{empty} \mid A \rightarrow \underline{C} \mid E^R \mid \underline{C} \Rightarrow \underline{D} \\ \text{Dirty type } \underline{C}, \underline{D} &::= A! \Delta \\ \text{Region } R &::= \{\iota_1, \dots, \iota_n\} \\ \text{Dirt } \Delta &::= \{\iota_1\#\text{op}_1, \dots, \iota_n\#\text{op}_n\} \end{aligned}$$

A dirty type $A! \Delta$ is just a pure type A tagged with a finite set Δ of operations that might be called during evaluation. We require that any operation $\iota\#\text{op}$ appearing in Δ is well-formed in the sense that $\iota \in \mathcal{I}_E$ and $\text{op} \in \mathcal{O}_E$ for some effect E .

The pure types comprise the usual ground types, the function types $A \rightarrow \underline{C}$, the effect types E^R , and the *handler types* $\underline{C} \Rightarrow \underline{D}$. Note that the function type takes pure types to dirty ones because a function accepts a pure expression as an argument and may call operations when evaluated. We let $!$ bind more strongly than \rightarrow , so that $A \rightarrow B! \Delta$ means $A \rightarrow (B! \Delta)$. Each effect type E^R is tagged with a finite set of instances $R = \{\iota_1, \dots, \iota_n\} \subseteq \mathcal{I}_E$ which tells us that the expression equals one of the instances in R . Finally, $\underline{C} \Rightarrow \underline{D}$ is the type of handlers which take computations of *ingoing* type \underline{C} to computations of *outgoing* type \underline{D} .

We assume that each effect E has an associated *effect signature*

$$\Sigma_E = \{\text{op}_1 : A^{\text{op}_1} \rightarrow B^{\text{op}_1}, \dots, \text{op}_n : A^{\text{op}_n} \rightarrow B^{\text{op}_n}\}$$

which assigns to each operation $\text{op}_i \in \mathcal{O}_E$ its *parameter type* A^{op_i} and *result type* B^{op_i} . In full *Eff* the signature is part of the definition of an effect, so for instance we might have

$$\begin{aligned} \Sigma_{\text{ref}} &= \{\text{lookup} : \text{unit} \rightarrow \text{nat}, \text{update} : \text{nat} \rightarrow \text{unit}\}, \\ \Sigma_{\text{exception}} &= \{\text{raise} : \text{unit} \rightarrow \text{empty}\}. \end{aligned}$$

Note that the signature may create circularities such as

$$\Sigma_E = \{\text{op} : \text{unit} \rightarrow (\text{unit} \rightarrow \text{unit}! \{\iota\#\text{op}\})\}.$$

Consequently, the denotational semantics of types in Section 5 will involve recursive domain equations.

2.2. Terms. The abstract syntax of terms of core *Eff* is as follows:

$$\begin{aligned} \text{Expression } e &::= x \mid \text{true} \mid \text{false} \mid 0 \mid \text{succ } e \mid () \mid \text{fun } x : A \mapsto c \mid \iota \mid h \\ \text{Handler } h &::= (\text{handler val } x : A \mapsto c_v \mid \text{ocs}) \\ \text{Operation cases } \text{ocs} &::= \text{nil}_{\underline{C}} \mid (e\#\text{op } x \ k \mapsto c \mid \text{ocs}) \\ \text{Computation } c &::= \text{val } e \mid e_1\#\text{op } e_2(y.c) \mid \text{with } e \text{ handle } c \mid \\ &\quad \text{if } e \text{ then } c_1 \text{ else } c_2 \mid \text{absurd}_{\underline{C}} e \mid e_1 e_2 \mid \\ &\quad (\text{match } e \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) \mid \\ &\quad \text{let } x = c_1 \text{ in } c_2 \mid \text{let rec } f x : A \rightarrow \underline{C} = c_1 \text{ in } c_2 \end{aligned}$$

In order to ensure that each term has at most one skeletal typing derivation, cf. Subsection 4.3, certain terms include typing annotations. We shall omit these when they do not play a role. The *Eff* implementation does not have typing annotations because its effect system automatically infers types and effects [22].

An expression is either a variable, a constant of ground type, a function abstraction (note that we abstract over computations), an *effect instance*, or a *handler*. It is worth noting that both instances and handlers are first-class values. We sometimes abbreviate $\text{succ}^n 0$ as n . A handler consists of a single *value case* and multiple *operation cases*, which describe how values and operations are handled, respectively. We defined operation cases inductively as lists, which is how they are formalized in Twelf, but we also write them as $(e_i\#\text{op}_i x_i \ k_i \mapsto c_i)_i$.

A computation is either a pure expression, an *operation call*, a **handle** construct, an eliminator for a ground type, an application, a **let** binding, or a recursive function definition.

3. OPERATIONAL SEMANTICS

We first describe the operational semantics informally. A computation $\text{val } e$ is pure and indicates a “final” result e , while an operation call $e_1\#\text{op } e_2(y.c)$ is the principal way of triggering an effect. The instance e_1 and the operation symbol op together form an operation $e_1\#\text{op}$, its parameter is e_2 , and $(y.c)$ the delimited continuation. We do not expect programmers to write explicit continuations, so the concrete syntax of *Eff* only gives access to calls through functions of the form $\text{fun } x \mapsto e\#\text{op } x(y.\text{val } y)$, known also as *generic effects* [17]. In examples we shall use generic effects rather than explicit continuations, and there we write them as $e\#\text{op}$. A general operation call $e_1\#\text{op } e_2(y.c)$ may then be expressed in terms of a generic effect and a **let** binding as $\text{let } y = e_1\#\text{op } e_2 \text{ in } c$.

A binding $\text{let } x = c_1 \text{ in } c_2$ is evaluated as follows:

- (1) If c_1 evaluates to $\text{val } e$ then the binding evaluates as c_2 with x bound to e .

(2) If c_1 evaluates to an operation call $\iota\#\text{op } e(y. c'_1)$, then the binding evaluates to

$$\iota\#\text{op } e(y. \text{let } x = c'_1 \text{ in } c_2),$$

where we assume that y does not occur free in c_2 .

It may be helpful to think of `val` and `let` as being similar to Haskell `return` and `do`, respectively. In ML `val` is invisible, while `let` is essentially the same as ours.

The `handle` construct applies a handler to a computation. If h is the *handler*

$$\text{handler val } x \mapsto c_v \mid (\iota_i\#\text{op}_i x_i k_i \mapsto c_i)_i$$

and c is a computation, then `with h handle c` first evaluates c which is then handled according to h :

- (1) If c evaluates to `val e` , then the `handle` construct evaluates as c_v with x bound to e .
- (2) If c evaluates to $\iota\#\text{op } e'(y. c')$, and $\iota_i\#\text{op}_i x_i k_i \mapsto c_i$ is the first operation case in h for which $\iota\#\text{op} = \iota_i\#\text{op}_i$ then the `handle` construct evaluates to c_i with x_i and k_i bound to e' and `fun $y \mapsto$ with h handle c'` , respectively. We assume that y does not occur free in h .
- (3) If c evaluates to an operation call $\iota\#\text{op } e'(y. c')$ which is not listed by h , then the `handle` construct propagates the call and acts as if h contained the clause

$$\iota\#\text{op } x k \mapsto \iota\#\text{op } x(y. k y).$$

Thus it evaluates to $\iota\#\text{op } e'(y. \text{with } h \text{ handle } c')$, where again we assume that y does not occur free in h .

Note that the handler always wraps itself around the continuation so that subsequent operations are handled as well. A binding `let $x = c_1$ in c_2` is equivalent to

$$\text{with (handler val } x \mapsto c_2) \text{ handle } c_1$$

so we could theoretically omit `let`.

3.1. Small-step semantics. The small-step operational semantics of core *Eff* is defined in terms of a relation $c \rightsquigarrow c'$, which intuitively means that the computation c takes a single step to c' . There is no operational semantics for expressions, which are just inert pieces of

data. The relation \rightsquigarrow is defined inductively by the following rules:

$$\begin{array}{c}
\frac{}{\text{if true then } c_1 \text{ else } c_2 \rightsquigarrow c_1} \qquad \frac{}{\text{if false then } c_1 \text{ else } c_2 \rightsquigarrow c_2} \\
\\
\frac{}{(\text{match } 0 \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) \rightsquigarrow c_1} \quad \frac{}{(\text{match } (\text{succ } e) \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) \rightsquigarrow c_2[e/x]} \\
\\
\frac{}{(\text{fun } x \mapsto c) e \rightsquigarrow c[e/x]} \qquad \frac{c_1 \rightsquigarrow c'_1}{\text{let } x = c_1 \text{ in } c_2 \rightsquigarrow \text{let } x = c'_1 \text{ in } c_2} \\
\\
\frac{}{\text{let } x = (\text{val } e) \text{ in } c \rightsquigarrow c[e/x]} \qquad \frac{}{\text{let } x = (\iota\#\text{op } e(y.c_1)) \text{ in } c_2 \rightsquigarrow \iota\#\text{op } e(y.\text{let } x = c_1 \text{ in } c_2)} \\
\\
\frac{}{\text{let rec } f x = c_1 \text{ in } c_2 \rightsquigarrow c_2[(\text{fun } x \mapsto \text{let rec } f x = c_1 \text{ in } c_1)/f]} \\
\\
\frac{c \rightsquigarrow c'}{\text{with } e \text{ handle } c \rightsquigarrow \text{with } e \text{ handle } c'} \qquad \frac{}{\text{with } (\text{handler val } x \mapsto c_v \mid \text{ocs}) \text{ handle } (\text{val } e) \rightsquigarrow c_v[e/x]} \\
\\
\frac{h \stackrel{\text{def}}{=} (\text{handler val } x \mapsto c_v \mid \text{ocs}) \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E}{\text{with } h \text{ handle } (\iota\#\text{op } e(y.c)) \rightsquigarrow \text{ocs}_{\iota\#\text{op}}(e, (\text{fun } y : B^{\text{op}} \mapsto \text{with } h \text{ handle } c))}
\end{array}$$

In the last rule for **let** binding and the last rule for the **handle** construct variable y must not occur free in c_2 and h respectively, and it goes without saying that the substitutions are capture avoiding. In the last rule we have an auxiliary definition of $\text{ocs}_{\iota\#\text{op}}$:

$$\begin{aligned}
(\text{nil})_{\iota\#\text{op}}(e, \kappa) &= \iota\#\text{op } e(y.\kappa y) \\
(\iota\#\text{op}' x k \mapsto c \mid \text{ocs})_{\iota\#\text{op}}(e, \kappa) &= \begin{cases} c[e/x, \kappa/k] & \text{if } \iota\#\text{op} = \iota\#\text{op}' \\ \text{ocs}_{\iota\#\text{op}}(e, \kappa) & \text{otherwise} \end{cases}
\end{aligned}$$

In words, $\text{ocs}_{\iota\#\text{op}}(e, \kappa)$ finds the first handler case in ocs that matches the operation $\iota\#\text{op}$ and executes it, or calls the operation again if no match is found.

Example 3.1. The non-standard state handler (recall that in examples we use generic effects)

$$\begin{aligned}
h \stackrel{\text{def}}{=} & \text{handler} \\
& \mid \text{val } x \mapsto \iota\#\text{update } x \\
& \mid \iota\#\text{lookup } x k \mapsto k \ 1 \\
& \mid \iota\#\text{update } x k \mapsto k \ ()
\end{aligned}$$

treats the reference ι as if its content were always 1, and updates ι with the final result of the handled computation. This update is not handled by h because it escapes its scope. If we use h to handle the computation

$$\begin{aligned}
c \stackrel{\text{def}}{=} & \text{let } x_1 = \iota\#\text{lookup } () \text{ in} \\
& \text{let } x_2 = \iota\#\text{update } x_1 \text{ in} \\
& \text{val } (\text{succ } x_1)
\end{aligned}$$

the outcome of the first lookup is 1, which is bound to x_1 , the update is ignored and finally $\iota\#\text{update } 2$ is called. The exact reduction sequence is as follows, where we underline the

active parts at each step and indicate desugaring of generic effects with \equiv :

```

with  $h$  handle
  let  $x_1 = \underline{\iota\#\text{lookup}}(\text{C})$  in let  $x_2 = \iota\#\text{update } x_1$  in val (succ  $x_1$ )  $\equiv$ 
with  $h$  handle
  let  $x_1 = \underline{\iota\#\text{lookup}}(\text{C})(y_1, \text{val } y_1)$  in let  $x_2 = \iota\#\text{update } x_1$  in val (succ  $x_1$ )  $\rightsquigarrow$ 
with  $\underline{h}$  handle
   $\underline{\iota\#\text{lookup}}(\text{C})(y_1, \text{let } x_1 = \text{val } y_1 \text{ in let } x_2 = \iota\#\text{update } x_1 \text{ in val (succ } x_1)) \rightsquigarrow$ 
 $\underline{(\text{fun } y_1 \mapsto \text{with } h \text{ handle (let } x_1 = \text{val } y_1 \text{ in let } x_2 = \iota\#\text{update } x_1 \text{ in val (succ } x_1))} \text{)) } \underline{1} \rightsquigarrow$ 
with  $h$  handle (let  $x_1 = \underline{\text{val } 1}$  in let  $x_2 = \iota\#\text{update } x_1$  in val (succ  $x_1$ ))  $\rightsquigarrow$ 
with  $h$  handle (let  $x_2 = \underline{\iota\#\text{update } 1}$  in val 2)  $\equiv$ 
with  $h$  handle (let  $x_2 = \underline{\iota\#\text{update } 1}(y_2, \text{val } y_2)$  in val 2)  $\rightsquigarrow$ 
with  $\underline{h}$  handle ( $\underline{\iota\#\text{update } 1}(y_2, \text{let } x_2 = \text{val } y_2 \text{ in val } 2)$ )  $\rightsquigarrow$ 
 $\underline{(\text{fun } y_2 \mapsto \text{with } h \text{ handle (let } x_2 = \text{val } y_2 \text{ in val } 2))} \text{C} \rightsquigarrow$ 
with  $h$  handle (let  $x_2 = \underline{\text{val } \text{C}}$  in val 2)  $\rightsquigarrow$ 
with  $\underline{h}$  handle ( $\underline{\text{val } 2}$ )  $\rightsquigarrow \underline{\iota\#\text{update } 2} \equiv \iota\#\text{update } 2(y_3, \text{val } y_3)$ 

```

3.2. Big-step semantics. In addition to small-step operational semantics, we also provide a big-step variant, which is closer to the actual implementation of *Eff*. Define a *result* to be a pure expression or an operation call:

Result $r ::= \text{val } e \mid \iota\#\text{op } e(x.c)$

Big-step semantics $c \Downarrow r$ evaluates a computation c to a result r , according to the following inductive rules:

$$\begin{array}{c}
\frac{c_1 \Downarrow r}{\text{if true then } c_1 \text{ else } c_2 \Downarrow r} \qquad \frac{c_2 \Downarrow r}{\text{if false then } c_1 \text{ else } c_2 \Downarrow r} \\
\\
\frac{c_1 \Downarrow r}{(\text{match } 0 \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) \Downarrow r} \qquad \frac{c_2[e/x] \Downarrow r}{(\text{match succ } e \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) \Downarrow r} \\
\\
\frac{c[e/x] \Downarrow r}{(\text{fun } x \mapsto c) e \Downarrow r} \qquad \frac{}{\text{val } e \Downarrow \text{val } e} \qquad \frac{}{\iota\#\text{op } e(x.c) \Downarrow \iota\#\text{op } e(x.c)} \qquad \frac{c_1 \Downarrow \text{val } e \quad c_2[e/x] \Downarrow r}{\text{let } x = c_1 \text{ in } c_2 \Downarrow r} \\
\\
\frac{c_1 \Downarrow \iota\#\text{op } e(y.c)}{\text{let } x = c_1 \text{ in } c_2 \Downarrow \iota\#\text{op } e(y, \text{let } x = c \text{ in } c_2)} \qquad \frac{c_2[(\text{fun } x \mapsto \text{let rec } f x = c_1 \text{ in } c_1)/f] \Downarrow r}{\text{let rec } f x = c_1 \text{ in } c_2 \Downarrow r} \\
\\
\frac{c \Downarrow \text{val } e \quad c_v[e/x] \Downarrow r}{\text{with (handler val } x \mapsto c_v \mid \text{ocs) handle } c \Downarrow r} \\
\\
\frac{h \stackrel{\text{def}}{=} (\text{handler val } x \mapsto c_v \mid \text{ocs}) \quad c \Downarrow \iota\#\text{op } e(y.c) \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E \quad \text{ocs}_{\iota\#\text{op}}(e, (\text{fun } y : B^{\text{op}} \mapsto \text{with } h \text{ handle } c)) \Downarrow r}{\text{with } h \text{ handle } c \Downarrow r}
\end{array}$$

To relate the two semantics we define an auxiliary relation \rightsquigarrow^* by the rules

$$\frac{}{\text{val } e \rightsquigarrow^* \text{val } e} \quad \frac{}{\iota\#\text{op } e(x.c) \rightsquigarrow^* \iota\#\text{op } e(x.c)} \quad \frac{c \rightsquigarrow c' \quad c' \rightsquigarrow^* r}{c \rightsquigarrow^* r}$$

This is roughly the reflexive transitive closure of \rightsquigarrow , except that it relates computations to results rather than to computations. The small-step and big-step semantics agree in the following sense.

Proposition 3.2. *For all computations c and results r , $c \Downarrow r$ if and only if $c \rightsquigarrow^* r$.*

Proof. Both directions of the equivalence proceed by a routine induction. The formalized proofs of the two implications can be found in the file `small-big.elf`. \square

4. AN EFFECT SYSTEM

4.1. **Subtyping.** As in most effect systems, we need to take care of the *poisoning problem* [26]. For example, what should be the type of *ignore* in

```
let ignore = val (fun msg ↦ val () in
let f = if b then (val ignore) else (val std#write) in
val ignore
```

assuming we have the ground type `string` and a boolean expression b ? If we give it the desired type `string` \rightarrow `unit!` \emptyset then there is a type mismatch between the branches in the conditional statement, whereas the dirty type `string` \rightarrow `unit!` $\{\text{std}\#\text{write}\}$ loses the valuable knowledge that *ignore* is a pure function. The simplest antidote to the poisoning problem is subtyping so that *ignore* may be given the function type with empty dirt which is coerced in the conditional statement to a supertype that matches the other branch.

For our purposes, a straightforward variant of *structural* subtyping [6] suffices. We have subtyping of pure types $A \leq A'$ and of dirty types $\underline{C} \leq \underline{C}'$, given by the rules

$$\frac{}{\text{bool} \leq \text{bool}} \quad \frac{}{\text{nat} \leq \text{nat}} \quad \frac{}{\text{unit} \leq \text{unit}} \quad \frac{}{\text{empty} \leq \text{empty}} \quad \frac{A' \leq A \quad \underline{C} \leq \underline{C}'}{A \rightarrow \underline{C} \leq A' \rightarrow \underline{C}'}$$

$$\frac{R \subseteq R'}{E^R \leq E^{R'}} \quad \frac{\underline{C}' \leq \underline{C} \quad \underline{D} \leq \underline{D}'}{\underline{C} \Rightarrow \underline{D} \leq \underline{C}' \Rightarrow \underline{D}'} \quad \frac{A \leq A' \quad \Delta \subseteq \Delta'}{A! \Delta \leq A'! \Delta'}$$

It is easily checked that reflexivity and transitivity of subtyping are admissible. Apart from resolving the poisoning problem, subtyping allows us to better deduce the behavior of handlers. Consider the computation

```
let u = val  $\iota$  in
let v = (if b then val u else val  $\iota'$ ) in
let h = val (handler val x ↦  $\dots$  | u#op x k ↦ c) in
 $\dots$ 
```

Without subtyping we are forced to give both u and v the type $E^{\{\iota, \iota'\}}$. Therefore, by looking at the type of u we cannot tell whether h handles $\iota\#\text{op}$ or $\iota'\#\text{op}$, and so we must assume

that both may be unhandled by h . With subtyping we may give u the type $E^{\{\iota\}}$ which makes it clear that h handles $\iota\#\text{op}$.

4.2. Typing rules. There are two typing judgments,

$$\Gamma \vdash e : A \quad \text{and} \quad \Gamma \vdash c : \underline{C}$$

stating that an expression e has a pure type A and a computation c has a dirty type \underline{C} , respectively. Here Γ is a typing context of the form $x_1 : A_1, \dots, x_n : A_n$. There is also a third, auxiliary typing judgment

$$\Gamma \vdash \text{ocs} : \underline{C}/\Delta$$

which states that operation cases ocs all have the same outgoing type \underline{C} and are guaranteed to handle operations in Δ . Most of the typing rules in Figure 1 are standard, except for:

INST: we check that ι is one of instances in R , which in turn must be contained in \mathcal{I}_E .

HAND: to check that a handler has type $A! \Delta \Rightarrow B! \Delta'$, we verify that it converts a computation of type $A! \Delta$ to one of type $B! \Delta'$, which involves checking three premises. First, the value case must take a value of type A to a computation of type $B! \Delta'$. Second, all operation cases must have outgoing type $B! \Delta'$. Third, every operation in Δ is either guaranteed to be handled by ocs , or is contained in Δ' .

OPCASES-NIL, OPCASES-CONS: the auxiliary typing judgment verifies that the operation cases have the given outgoing type, and that they cover the given dirt. The empty list nil does not cover anything and has any outgoing type. The rule **OPCASES-CONS** checks the first operation case, checks the others inductively, and verifies that $\Delta \subseteq \Delta' \cup R\#\text{op}$, where

$$\Delta' \cup R\#\text{op} \stackrel{\text{def}}{=} \begin{cases} \Delta' \cup \{\iota\#\text{op}\} & \text{if } R = \{\iota\}, \\ \Delta' & \text{otherwise.} \end{cases}$$

The idea is that we can be sure that an operation case handles $\iota\#\text{op}$ only when the type of its instance is of the form $E^{\{\iota\#\text{op}\}}$.

OP: we first check that e and op belong to the same effect. Then we check that Δ covers not just all possible operations that the operation call may cause (recall that R may contain more than one instance), but also any operations in the continuation c . We may assume that c has the same dirt, as we can use **SUBCOMP** otherwise. We use the same reasoning in rules **IFTHENELSE**, **MATCH** and **LET**.

WITH: handlers behave like functions from computations to computations.

SUBEXPR, SUBCOMP: these *subsumption rules* allow us to always assign a bigger type.

The effect system is safe with respect to the operational semantics:

Theorem 4.1 (Progress & Preservation).

Progress: *If $\vdash c : A! \Delta$ then either*

- *there exists a computation c' such that $c \rightsquigarrow c'$, or*
- *c is of the form $\text{val } e$ for some expression e , or*
- *c is of the form $\iota\#\text{op } e(x.c')$ for some $\iota\#\text{op} \in \Delta$.*

Preservation: *If $\vdash c : \underline{C}$ and $c \rightsquigarrow c'$ then $\vdash c' : \underline{C}$.*

Proof. Both statements are proved by induction. The formalized proofs can be found in the files `progress.elf` and `preservation.elf`. \square

VAR $\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}$	TRUE $\frac{}{\Gamma \vdash \text{true} : \text{bool}}$	FALSE $\frac{}{\Gamma \vdash \text{false} : \text{bool}}$	ZERO $\frac{}{\Gamma \vdash 0 : \text{nat}}$	SUCC $\frac{\Gamma \vdash e : \text{nat}}{\Gamma \vdash \text{succ } e : \text{nat}}$
UNIT $\frac{}{\Gamma \vdash () : \text{unit}}$	FUN $\frac{\Gamma, x : A \vdash c : \underline{C}}{\Gamma \vdash \text{fun } x : A \mapsto c : A \rightarrow \underline{C}}$	INST $\frac{\iota \in R \subseteq \mathcal{I}_E}{\Gamma \vdash \iota : E^R}$		
HAND $\frac{\Gamma, x : A \vdash c_v : B! \Delta' \quad \Gamma \vdash \text{ocs} : B! \Delta' / \Delta'' \quad \Delta \subseteq \Delta'' \cup \Delta'}{\Gamma \vdash (\text{handler val } x : A \mapsto c_v \mid \text{ocs}) : A! \Delta \Rightarrow B! \Delta'}$			SUBEXPR $\frac{\Gamma \vdash e : A \quad A \leq A'}{\Gamma \vdash e : A'}$	
OPCASES-NIL $\frac{}{\Gamma \vdash \text{nil}_{\underline{C}} : \underline{C} / \emptyset}$	OPCASES-CONS $\frac{\Gamma \vdash e : E^R \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E \quad \Gamma, x : A^{\text{op}}, k : B^{\text{op}} \rightarrow \underline{C} \vdash c : \underline{C} \quad \Gamma \vdash \text{ocs} : \underline{C} / \Delta' \quad \Delta \subseteq \Delta' \cup R\#\text{op}}{\Gamma \vdash (e\#\text{op } x k \mapsto c \mid \text{ocs}) : \underline{C} / \Delta}$			
IFTHENELSE $\frac{\Gamma \vdash e : \text{bool} \quad \Gamma \vdash c_1 : \underline{C} \quad \Gamma \vdash c_2 : \underline{C}}{\Gamma \vdash \text{if } e \text{ then } c_1 \text{ else } c_2 : \underline{C}}$		MATCH $\frac{\Gamma \vdash e : \text{nat} \quad \Gamma \vdash c_1 : \underline{C} \quad \Gamma, x : \text{nat} \vdash c_2 : \underline{C}}{\Gamma \vdash \text{match } e \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2 : \underline{C}}$		
ABSURD $\frac{\Gamma \vdash e : \text{empty}}{\Gamma \vdash \text{absurd}_{\underline{C}} e : \underline{C}}$	APP $\frac{\Gamma \vdash e_1 : A \rightarrow \underline{C} \quad \Gamma \vdash e_2 : A}{\Gamma \vdash e_1 e_2 : \underline{C}}$	VAL $\frac{\Gamma \vdash e : A}{\Gamma \vdash \text{val } e : A! \Delta}$		
OP $\frac{\Gamma \vdash e_1 : E^R \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E \quad \Gamma \vdash e_2 : A^{\text{op}} \quad \Gamma, y : B^{\text{op}} \vdash c : A! \Delta \quad \forall \iota \in R. \iota\#\text{op} \in \Delta}{\Gamma \vdash e_1\#\text{op } e_2(y.c) : A! \Delta}$				
LET $\frac{\Gamma \vdash c_1 : A! \Delta \quad \Gamma, x : A \vdash c_2 : B! \Delta}{\Gamma \vdash \text{let } x = c_1 \text{ in } c_2 : B! \Delta}$		LETREC $\frac{\Gamma, f : A \rightarrow \underline{C}, x : A \vdash c_1 : \underline{C} \quad \Gamma, f : A \rightarrow \underline{C} \vdash c_2 : \underline{D}}{\Gamma \vdash \text{let rec } f x : A \rightarrow \underline{C} = c_1 \text{ in } c_2 : \underline{D}}$		
WITH $\frac{\Gamma \vdash e : \underline{C} \Rightarrow \underline{D} \quad \Gamma \vdash c : \underline{C}}{\Gamma \vdash \text{with } e \text{ handle } c : \underline{D}}$		SUBCOMP $\frac{\Gamma \vdash c : \underline{C} \quad \underline{C} \leq \underline{C}'}{\Gamma \vdash c : \underline{C}'}$		

Figure 1: The typing rules of core Eff .

Corollary 4.2 (Safety). *A terminating computation of type $A! \Delta$ returns a value of type A , or calls an operation in Δ .*

In particular, a terminating computation of type $A! \emptyset$ does not call any operations and returns a pure value of type A .

Example 4.3. Let us see what the effect system tells us about Example 3.1. We may give the reference $\iota \in \mathcal{I}_{\text{ref}}$ type $\text{ref}^{\{\iota\}}$. Then the computation c has the dirty type

$\text{nat}!\{\iota\#\text{lookup}, \iota\#\text{update}\}$, and the handler h the type

$$(\text{nat}!\{\iota\#\text{lookup}, \iota\#\text{update}\}) \Rightarrow (\text{unit}!\{\iota\#\text{update}\})$$

because it handles both `lookup` and `update`, but then calls `update` in the value case. This `update` also changes the type of computation from `nat` to `unit`.

If we give ι the less precise type $\text{ref}^{\{\iota, \iota'\}}$, the dirt of c is

$$\Delta \stackrel{\text{def}}{=} \{\iota\#\text{lookup}, \iota'\#\text{lookup}, \iota\#\text{update}, \iota'\#\text{update}\}$$

while the best type we can give to h is $(\text{nat}!\Delta) \Rightarrow (\text{unit}!\Delta)$. Since $\{\iota, \iota'\}$ is not a singleton, we cannot give any guarantees on what operations are handled.

4.3. Skeletal types. We relate the pure and dirty types to ML-style types by an operation which erases all effect information to produce a *skeletal* type. We will use these later to obtain a coherent semantics of types. The skeletal types are defined as follows:

$$\text{Skeletal type } S, T ::= \text{bool} \mid \text{nat} \mid \text{unit} \mid \text{empty} \mid S \rightarrow T \mid E \mid S \Rightarrow T$$

There is no distinction between pure and dirty types anymore. The typing rules for skeletal types are like those for pure and dirty types with the effect information omitted, for instance

$$\begin{array}{c} \text{INST}' \\ \frac{\iota \in \mathcal{I}_E}{\Gamma \vdash \iota : E} \end{array} \quad \begin{array}{c} \text{HAND}' \\ \frac{\Gamma, x : A^s \vdash c_v : S \quad \Gamma \vdash \text{ocs} : S}{\Gamma \vdash (\text{handler val } x : A \mapsto c_v \mid \text{ocs}) : A^s \Rightarrow S} \end{array} \quad \begin{array}{c} \text{OPCASES-NIL}' \\ \frac{}{\Gamma \vdash \text{nil}_{\underline{C}} : \underline{C}^s} \end{array}$$

$$\begin{array}{c} \text{OPCASES-CONS}' \\ \frac{\Gamma \vdash e : E \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E \quad \Gamma, x : (A^{\text{op}})^s, k : (B^{\text{op}})^s \rightarrow S \vdash c : S \quad \Gamma \vdash \text{ocs} : S}{\Gamma \vdash (e\#\text{op } x \ k \mapsto c \mid \text{ocs}) : S} \end{array}$$

$$\begin{array}{c} \text{OP}' \\ \frac{\Gamma \vdash e_1 : E \quad (\text{op} : A^{\text{op}} \rightarrow B^{\text{op}}) \in \Sigma_E \quad \Gamma \vdash e_2 : (A^{\text{op}})^s \quad \Gamma, y : (B^{\text{op}})^s \vdash c : S}{\Gamma \vdash e_1\#\text{op } e_2(y.c) : S} \end{array}$$

The remaining rules remain unchanged as they do not mention effects, while subsumption rules are *removed*. To every pure type A and a dirty type \underline{C} we assign their skeletal versions A^s and \underline{C}^s , which are like A and \underline{C} with region and dirt removed. The skeletal version Γ^s of a typing context Γ is obtained by taking the skeletons of the types in Γ . We summarize the properties of skeletal types:

Theorem 4.4.

- (1) If $A \leq B$ and $\underline{C} \leq \underline{D}$ then $A^s = B^s$ and $\underline{C}^s = \underline{D}^s$.
- (2) If

$$\Gamma \vdash e : A \quad \text{and} \quad \Gamma \vdash c : \underline{C}$$

then

$$\Gamma^s \vdash e : A^s \quad \text{and} \quad \Gamma^s \vdash c : \underline{C}^s.$$

- (3) In a given context, an expression and a computation has at most one skeletal type, with a unique typing derivation.

Proof. The first statement holds by an induction on the derivation of $A \leq B$ and $\underline{C} \leq \underline{D}$.

To prove the second statement, observe that a typing derivation may be mapped to the corresponding skeletal version rule by rule, except for subsumption rules. But these can be simply omitted from the typing derivations because $A \leq B$ and $\underline{C} \leq \underline{D}$ imply $A^s = B^s$ and $\underline{C}^s = \underline{D}^s$ by the first statement.

The last statement holds by inversion: in any situation at most one skeletal typing rule applies in at most one way. \square

A consequence of the theorem is that if a computation c has dirty types \underline{C} and \underline{D} then $\underline{C}^s = \underline{D}^s$, hence \underline{C} and \underline{D} differ only in the effect information. An analogous property holds for expressions and pure types.

5. DENOTATIONAL SEMANTICS

We use standard domain theory to provide an adequate denotational semantics of core *Eff*. We shall use ω -cpos as domains, but presumably a different kind of domains could be used, as long as they support the standard constructions, in particular solutions of domain equations, and are amenable to Pitts's theory of minimal invariant properties [14]. We refer to [1] for background on domain theory and denotational semantics.

We define a *predomain* to be a poset in which chains (ascending sequences) have suprema, while a *domain* is a predomain with a least element \perp . A *continuous* map is a monotone map which commutes with suprema of chains. If D is a predomain and E is a domain the set $D \rightarrow E$ of all continuous maps forms a domain. The ordering on continuous maps is pointwise. A continuous map is *strict* if it maps \perp to \perp . The set $D \multimap E$ of strict maps between domains D and E forms a subdomain of $D \rightarrow E$.

5.1. Computation domains. We first build domains that will serve as the meanings of computation types. Let A be a predomain, I an index set, and for each $i \in I$ let A_i and B_i be predomains. We seek a domain T satisfying the domain equation $T \cong F(T)$ where F is the functor

$$F(D) = (A + \coprod_{i \in I} A_i \times (B_i \rightarrow D))_{\perp}.$$

Following [14] we work in the category of domains and strict maps, and take T to be a minimal solution in the sense that it possesses the *minimal invariant property*. The usual limit-colimit construction [1, Chapter 7] yields such a domain. As domain equations go, this one is quite simple because T occurs only covariantly. The elements of T can be thought of as trees whose leaves are tagged with elements of A or \perp , and whose nodes have branching types B_i and are tagged with elements of A_i . The trees need not be well founded.

The minimality of T yields a recursion and an induction principle. The recursion principle says that for any domain D , a continuous map $f_{\text{val}} : A \rightarrow D$, and continuous maps $f_i : A_i \times (B_i \rightarrow D) \rightarrow D$ for $i \in I$, there is a unique strict continuous map $f : T \multimap D$ such that

$$\begin{aligned} f(\text{in}_{\text{val}}(x)) &= f_{\text{val}}(x) && \text{for } x \in A, \\ f(\text{in}_i(y, \kappa)) &= f_i(y, f \circ \kappa) && \text{for } y \in A_i \text{ and } \kappa : B_i \rightarrow T. \end{aligned}$$

The induction principle for T applies to *admissible* predicates on T , i.e., those that hold for \perp and are closed under suprema of chains. Precisely, if ϕ is an admissible predicate on T such that

- (1) $\phi(\text{in}_{\text{val}}(x))$ for all $x \in A$, and
(2) for all $i \in I$, $x \in A_i$, and $\kappa : B_i \rightarrow T$, if $\forall y \in B_i. \phi(\kappa(y))$ then $\phi(\text{in}_i(x, \kappa))$,
then $\phi(t)$ holds for all $t \in T$.

For any I , the construction of a minimal solution $T_I(A, (A_i)_i, (B_i)_i)$ from the input data $A, (A_i)_{i \in I}, (B_i)_{i \in I}$ forms a locally continuous functor

$$T_I : \mathbf{pCpo} \times \mathbf{pCpo}^I \times (\mathbf{pCpo}^{\text{op}})^I \rightarrow \mathbf{Cppo}.$$

Here \mathbf{Cppo} is the category of domains and strict continuous maps, while \mathbf{pCpo} is the category of predomains and partial continuous maps which are defined on open subsets (an upper set which is inaccessible by suprema of chains). The functor will appear later on in a larger system of recursive domain equations.

5.2. Semantics of skeletal types. We interpret pure and dirty types as predomains and domains, respectively. A typing context is interpreted as a cartesian product of predomains, and a typing judgment as a continuous map. However, typing judgments do not have unique derivations because of the subsumption rules, and so we have to worry about coherence. That is, when we define the meaning of a typing judgment by induction on its derivation, we need to make sure that the result does not depend on the choice of derivation. We accomplish this by providing a semantics which factors through the skeletal types from Section 4.3.

Let

$$\Omega = \{\iota \# \text{op} \mid \exists E. \iota \in \mathcal{I}_E \wedge \text{op} \in \Sigma_E\}$$

be the set of *all* operations. If dirts could be infinite, $A! \Omega$ would be a dirty type expressing the fact that any effect could happen.

To each skeletal type S we assign a predomain $\llbracket S \rrbracket_e$ and a domain $\llbracket S \rrbracket_c$ as follows:

$$\begin{aligned} \llbracket \text{bool} \rrbracket_e &= \{\text{ff}, \text{tt}\}, & \llbracket \text{nat} \rrbracket_e &= \mathbb{N}, \\ \llbracket \text{unit} \rrbracket_e &= \{\star\}, & \llbracket \text{empty} \rrbracket_e &= \emptyset, \\ \llbracket S \rightarrow T \rrbracket_e &= \llbracket S \rrbracket_e \rightarrow \llbracket T \rrbracket_c, & \llbracket E \rrbracket_e &= \mathcal{I}_E, \\ \llbracket S \Rightarrow T \rrbracket_e &= \llbracket S \rrbracket_c \multimap \llbracket T \rrbracket_c, \end{aligned}$$

and

$$\llbracket S \rrbracket_c = T_\Omega(\llbracket S \rrbracket_e, (\llbracket (A^{\text{op}})^s \rrbracket_e)_{\iota \# \text{op}}, (\llbracket (B^{\text{op}})^s \rrbracket_e)_{\iota \# \text{op}}).$$

These should be read as a system of domain and predomain equations indexed by the skeletal types. There are possible circularities in the system because the equation for $\llbracket S \rrbracket_c$ refers to possibly larger skeletal types $(A^{\text{op}})^s$ and $(B^{\text{op}})^s$. As in the case of computation domains, we take the minimal solutions which enjoy the minimal invariant property.

To each pure type A and dirty type \underline{C} we assign a *skeletal predomain* $\llbracket A \rrbracket$ and *skeletal domain* $\llbracket \underline{C} \rrbracket$ by setting

$$\llbracket A \rrbracket = \llbracket A^s \rrbracket_e \quad \text{and} \quad \llbracket \underline{C} \rrbracket = \llbracket \underline{C}^s \rrbracket_c.$$

The first part of Theorem 4.4 guarantees that $A \leq B$ and $\underline{C} \leq \underline{D}$ imply $\llbracket A \rrbracket = \llbracket B \rrbracket$ and $\llbracket \underline{C} \rrbracket = \llbracket \underline{D} \rrbracket$.

5.3. Semantics of expressions and computations. The meaning of a typing context Γ

$$x_1 : A_1, \dots, x_n : A_n$$

is

$$\llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \dots \times \llbracket A_n \rrbracket.$$

We interpret typing judgments

$$\Gamma \vdash e : A \quad \text{and} \quad \Gamma \vdash c : \underline{C}$$

as continuous maps

$$\llbracket \Gamma \vdash e : A \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket \quad \text{and} \quad \llbracket \Gamma \vdash c : \underline{C} \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \underline{C} \rrbracket.$$

When no confusion can arise we abbreviate these as $\llbracket e \rrbracket$ and $\llbracket c \rrbracket$. The definition proceeds by induction on the derivation of the typing judgment. Given an environment $\eta \in \llbracket \Gamma \rrbracket$, the base rules for expressions and the successor rule are taken care of by

$$\begin{aligned} \llbracket \Gamma \vdash x_i : A_i \rrbracket \eta &= \eta_i & \llbracket \Gamma \vdash \mathbf{false} : \mathbf{bool} \rrbracket \eta &= \mathbf{ff} \\ \llbracket \Gamma \vdash () : \mathbf{unit} \rrbracket \eta &= \star & \llbracket \Gamma \vdash \mathbf{true} : \mathbf{bool} \rrbracket \eta &= \mathbf{tt} \\ \llbracket \Gamma \vdash 0 : \mathbf{nat} \rrbracket \eta &= 0 & \llbracket \Gamma \vdash \mathbf{succ } e : \mathbf{nat} \rrbracket \eta &= (\llbracket \Gamma \vdash e : \mathbf{nat} \rrbracket \eta) + 1 \\ \llbracket \Gamma \vdash \iota : E^R \rrbracket \eta &= \iota, \end{aligned}$$

and the abstraction rule by

$$\llbracket \Gamma \vdash (\mathbf{fun } x : A \mapsto c) : A \rightarrow \underline{C} \rrbracket \eta = \lambda a \in \llbracket A \rrbracket. \llbracket \Gamma, x : A \vdash c : \underline{C} \rrbracket (\eta, a)$$

For the handler rule we set

$$\llbracket \Gamma \vdash (\mathbf{handler } \mathbf{val } x : A \mapsto c_v \mid \mathit{ocs}) : A! \Delta \Rightarrow B! \Delta' \rrbracket = h$$

where $h : \llbracket \Gamma \rrbracket \rightarrow \llbracket A! \Delta \rrbracket \multimap \llbracket B! \Delta' \rrbracket$ is defined by recursion on $\llbracket A! \Delta \rrbracket$:

$$\begin{aligned} h(\eta)(\perp) &= \perp \\ h(\eta)(\mathit{in}_{\mathbf{val}}(a)) &= \llbracket \Gamma, x : A \vdash c_v : B! \Delta' \rrbracket (\eta, a) \\ h(\eta)(\mathit{in}_{\iota \# \mathbf{op}}(a, \kappa)) &= \llbracket \mathit{ocs} \rrbracket_{\iota \# \mathbf{op}}(\eta, a, h(\eta) \circ \kappa) \end{aligned}$$

The auxiliary map

$$\llbracket \mathit{ocs} \rrbracket_{\iota \# \mathbf{op}} : \llbracket \Gamma \rrbracket \times \llbracket A^{\mathbf{op}} \rrbracket \times \llbracket B^{\mathbf{op}} \rightarrow B! \Delta' \rrbracket \rightarrow \llbracket B! \Delta' \rrbracket$$

is defined by

$$\begin{aligned} \llbracket \mathit{nil}_{\underline{C}} \rrbracket_{\iota \# \mathbf{op}}(\eta, a, \kappa) &= \mathit{in}_{\iota \# \mathbf{op}}(a, \kappa) \\ \llbracket e' \# \mathbf{op}' x k \mapsto c \mid \mathit{ocs} \rrbracket_{\iota \# \mathbf{op}}(\eta, a, \kappa) &= \\ &\begin{cases} \llbracket \Gamma, x : A^{\mathbf{op}}, k : B^{\mathbf{op}} \rightarrow B! \Delta' \vdash c : B! \Delta' \rrbracket (\eta, a, \kappa) & \text{if } (\llbracket e' \rrbracket \eta) \# \mathbf{op}' = \iota \# \mathbf{op}, \\ \llbracket \mathit{ocs} \rrbracket_{\iota \# \mathbf{op}}(\eta, a, \kappa) & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, if $\Gamma \vdash e : A'$ is derived from the premises $\Gamma \vdash e : A$ and $A \leq A'$ by the subsumption rule, we set

$$\llbracket \Gamma \vdash e : A' \rrbracket \eta = \llbracket \Gamma \vdash e : A \rrbracket \eta.$$

The definition is meaningful because $A \leq A'$ implies $\llbracket A \rrbracket = \llbracket A' \rrbracket$. The meaning of pure computations and operations is

$$\llbracket \Gamma \vdash \text{val } e : A! \Delta \rrbracket \eta = \text{in}_{\text{val}}(\llbracket \Gamma \vdash e : A \rrbracket)$$

$$\llbracket \Gamma \vdash e_1 \#_{\text{op}} e_2 (y. c) : A! \Delta \rrbracket \eta = \text{in}_{\llbracket e_1 \rrbracket \eta \#_{\text{op}} (\llbracket e_2 \rrbracket \eta, \lambda b \in \llbracket B^{\text{op}} \rrbracket . \llbracket \Gamma, y : B^{\text{op}} \vdash c : A! \Delta \rrbracket (\eta, b))}.$$

The meaning of elimination forms is

$$\llbracket \Gamma \vdash \text{with } e \text{ handle } c : \underline{D} \rrbracket \eta = (\llbracket \Gamma \vdash e : \underline{C} \Rightarrow \underline{D} \rrbracket \eta)(\llbracket \Gamma \vdash c : \underline{C} \rrbracket \eta)$$

$$\llbracket \Gamma \vdash \text{if } e \text{ then } c_1 \text{ else } c_2 : \underline{C} \rrbracket \eta = \begin{cases} \llbracket \Gamma \vdash c_1 : \underline{C} \rrbracket \eta & \text{if } \llbracket \Gamma \vdash e : \text{bool} \rrbracket \eta = \text{tt}, \\ \llbracket \Gamma \vdash c_2 : \underline{C} \rrbracket \eta & \text{if } \llbracket \Gamma \vdash e : \text{bool} \rrbracket \eta = \text{ff} \end{cases}$$

$$\llbracket \Gamma \vdash \text{absurd } e : \underline{C} \rrbracket \eta = \perp$$

$$\llbracket \Gamma \vdash e_1 e_2 : \underline{C} \rrbracket \eta = (\llbracket \Gamma \vdash e_1 : A \rightarrow \underline{C} \rrbracket \eta)(\llbracket \Gamma \vdash e_2 : A \rrbracket)$$

and

$$\llbracket \Gamma \vdash (\text{match } e \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2) : \underline{C} \rrbracket \eta =$$

$$\begin{cases} \llbracket \Gamma \vdash c_1 : \underline{C} \rrbracket \eta & \text{if } \llbracket \Gamma \vdash e : \text{nat} \rrbracket \eta = 0, \\ \llbracket \Gamma, x : \text{nat} \vdash c_2 : \underline{C} \rrbracket (\eta, n) & \text{if } \llbracket \Gamma \vdash e : \text{nat} \rrbracket \eta = n + 1. \end{cases}$$

To give semantics of **let** binding, we first define the *lifting* of a map $f : \llbracket A \rrbracket \rightarrow \llbracket B! \Delta \rrbracket$ to be the map $f^\dagger : \llbracket A! \Delta \rrbracket \rightarrow \llbracket B! \Delta \rrbracket$ defined recursively by

$$f^\dagger(\perp) = \perp,$$

$$f^\dagger(\text{in}_{\text{val}}(x)) = f(x),$$

$$f^\dagger(\text{in}_{\iota \#_{\text{op}}}(x, \kappa)) = \text{in}_{\iota \#_{\text{op}}}(x, f^\dagger \circ \kappa).$$

Then we set

$$\llbracket \Gamma \vdash \text{let } x = c_1 \text{ in } c_2 : B! \Delta \rrbracket \eta =$$

$$(\lambda a \in \llbracket A \rrbracket . \llbracket \Gamma, x : A \vdash c_2 : B! \Delta \rrbracket (\eta, a))^\dagger(\llbracket \Gamma \vdash c_1 : A! \Delta \rrbracket \eta).$$

The meaning of a recursive function definition is

$$\llbracket \Gamma \vdash (\text{let rec } f x : A \rightarrow \underline{C} = c_1 \text{ in } c_2) : \underline{C} \rrbracket \eta = \llbracket \Gamma, f : A \rightarrow \underline{C} \vdash c_2 : \underline{C} \rrbracket (\eta, g)$$

where $g : \llbracket A \rrbracket \rightarrow \llbracket \underline{C} \rrbracket$ is the least fixed-point of the map

$$g \mapsto (\lambda a \in \llbracket A \rrbracket . \llbracket c_1 \rrbracket (\eta, g, a)).$$

Finally, just like for expressions, if $\Gamma \vdash c : \underline{C}'$ is derived from the premises $\Gamma \vdash c : \underline{C}$ and $\underline{C} \leq \underline{C}'$ by the subsumption rule, we set

$$\llbracket \Gamma \vdash c : \underline{C}' \rrbracket \eta = \llbracket \Gamma \vdash c : \underline{C} \rrbracket \eta.$$

This concludes the definition of denotational semantics of expressions and computations.

Theorem 5.1 (Coherence). *All derivations of a typing judgment give it the same meaning.*

Proof. The semantics of $\Gamma \vdash e : A$ and $\Gamma \vdash c : \underline{C}$ factor through the associated skeletal derivations $\Gamma^s \vdash e : A^s$ and $\Gamma^s \vdash c : \underline{C}^s$, respectively. This is so because the semantic rules given above clearly factor through the associated skeletal rules. In other words, they ignore the effect information and the subsumption rules. Uniqueness of meaning is thus a consequence of the uniqueness of skeletal derivations, cf. Theorem 4.4. \square

5.4. Semantics of effects. In the terminology of John Reynolds [23] the semantics given so far is *intrinsic*, while the semantics of effects given below is *extrinsic*. This is in accordance with our understanding that effect information is descriptive rather than prescriptive: a function f of type $(A \rightarrow B! \Delta) \rightarrow B'! \Delta'$ should accept *any* function g of type $A \rightarrow B! \Delta''$, even if $\Delta'' \not\subseteq \Delta$, although it has the *property* that if $\Delta'' \subseteq \Delta$ then $f(g)$ calls only operations in Δ' .

For each pure type A and dirty type \underline{C} we define subpredomains $\llbracket A \rrbracket \subseteq \llbracket A \rrbracket$ and subdomains $\llbracket \underline{C} \rrbracket \subseteq \llbracket \underline{C} \rrbracket$ of those elements that behave according to the effect information. The ground types are easy:

$$\begin{aligned} \llbracket \text{bool} \rrbracket &= \llbracket \text{bool} \rrbracket, & \llbracket \text{nat} \rrbracket &= \llbracket \text{nat} \rrbracket, & \llbracket \text{unit} \rrbracket &= \llbracket \text{unit} \rrbracket, \\ \llbracket \text{empty} \rrbracket &= \llbracket \text{empty} \rrbracket, & \llbracket E^R \rrbracket &= R. \end{aligned}$$

For function types, handler types, and dirty types we would like to solve the following system of equations with unknowns $\llbracket A \rrbracket$ and $\llbracket \underline{C} \rrbracket$ where A and \underline{C} range over pure and dirty types, respectively:

$$\begin{aligned} \llbracket A \rightarrow \underline{C} \rrbracket &= \{f \in \llbracket A \rrbracket \rightarrow \llbracket \underline{C} \rrbracket \mid \forall x \in \llbracket A \rrbracket. f(x) \in \llbracket \underline{C} \rrbracket\}, \\ \llbracket \underline{C} \Rightarrow \underline{D} \rrbracket &= \{h \in \llbracket \underline{C} \rrbracket \multimap \llbracket \underline{D} \rrbracket \mid \forall t \in \llbracket \underline{C} \rrbracket. h(t) \in \llbracket \underline{D} \rrbracket\}, \\ \llbracket A! \Delta \rrbracket &= \{t \in \llbracket A! \Delta \rrbracket \mid t = \perp \vee (\exists x \in \llbracket A \rrbracket. t = \text{in}_{\text{val}}(x)) \vee \\ &\quad \exists \iota \# \text{op} \in \Delta, y \in \llbracket A^{\text{op}} \rrbracket, \kappa \in \llbracket B^{\text{op}} \rightarrow A! \Delta \rrbracket. \\ &\quad (\forall z \in \llbracket B^{\text{op}} \rrbracket. \kappa(b) \in \llbracket A! \Delta \rrbracket) \wedge t = \text{in}_{\iota \# \text{op}}(y, \kappa)\}. \end{aligned} \tag{5.1}$$

The last rule says that $t \in \llbracket A! \Delta \rrbracket$ when it is \perp , or of the form $\text{in}_{\text{val}}(x)$ for some $x \in \llbracket A \rrbracket$, or of the form $\text{in}_{\iota \# \text{op}}(y, \kappa)$ for some $y \in \llbracket A^{\text{op}} \rrbracket$ and $\kappa \in \llbracket B^{\text{op}} \rightarrow A! \Delta \rrbracket$.

The system is potentially problematic because the types A^{op} and B^{op} introduce circularities in the last equation. We apply Pitts's theorem [14, Theorem 4.16] about existence of invariant relations to obtain a solution that satisfies an induction principle, see Theorem 5.2 below. For Pitts's theorem to apply we must verify that our conditions form an admissible action on admissible relations, as defined in [14, Definition 4.6]. The relational structure in question is that of [14, Example 4.2(ii)] for which the accompanying notion of admissibility is the one we are using, cf. [14, Example 4.5(ii)]. The locally continuous functor is the evident one, while its admissible action on relations is read off (5.1). The admissibility of the action follows from [14, Lemma 6.6]: the first two actions in (5.1) are considered explicitly, while the third one is a composition of actions from the cited lemma. The solution so obtained possesses the following induction principle.

Theorem 5.2. *Suppose ϕ is an admissible predicate on $\llbracket A! \Delta \rrbracket$ such that*

- (1) $\phi(\text{in}_{\text{val}}(a))$ for every $a \in \llbracket A \rrbracket$, and
- (2) for all $\iota \# \text{op} \in \Delta$, $a \in \llbracket A^{\text{op}} \rrbracket$, $\kappa \in \llbracket B^{\text{op}} \rightarrow A! \Delta \rrbracket$, if $\forall b \in \llbracket B^{\text{op}} \rrbracket. \phi(\kappa(b))$ then $\phi(\text{in}_{\iota \# \text{op}}(a, \kappa))$.

Then $\phi(t)$ for all $t \in \llbracket A! \Delta \rrbracket$.

Proof. Because we defined $\llbracket - \rrbracket$ mutually for all types, we first extend ϕ to be constantly true on types other than $A! \Delta$. Then our theorem becomes an instance of the induction property [14, Theorem 6.5]. The admissible action required for the application of the theorem is obtained using [14, Lemma 6.6]. \square

The semantics of effects and the semantics of expressions and computations fit together:

Theorem 5.3. *If $\Gamma \vdash e : A$ and $\Gamma \vdash c : \underline{C}$ then for all $\eta \in \llbracket \Gamma \rrbracket = \llbracket A_1 \rrbracket \times \cdots \times \llbracket A_n \rrbracket$, we have*

$$\llbracket \Gamma \vdash e : A \rrbracket \eta \in \llbracket A \rrbracket \quad \text{and} \quad \llbracket \Gamma \vdash c : \underline{C} \rrbracket \eta \in \llbracket C \rrbracket.$$

Proof. The proof proceeds by induction on the derivation of the typing judgment. All cases are easy, except for the typing rule for a handler

$$\Gamma \vdash (\text{handler val } x \mapsto c_v \mid \text{ocs}) : A! \Delta \Rightarrow B! \Delta'$$

First, we claim that if the auxiliary judgment

$$\Gamma \vdash \text{ocs} : B! \Delta' / \Delta_c$$

is derivable then $\llbracket \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, a, \kappa) \in \llbracket B! \Delta' \rrbracket$ for all $\iota \# \text{op} \in \Delta' \cup \Delta_c$, $\eta \in \llbracket \Gamma \rrbracket$, $a \in \llbracket A^{\text{op}} \rrbracket$, and $\kappa : \llbracket B^{\text{op}} \rightarrow B! \Delta' \rrbracket$. Assuming the claim has been established, the above handler rule follows immediately.

The proof of the claim proceeds by induction on the derivation of the auxiliary judgment. For *nil*, we have $\Delta_c = \emptyset$ and the claim obviously holds. The other possibility is

$$\Gamma \vdash (e \# \text{op}' x k \mapsto c \mid \text{ocs}) : B! \Delta' / \Delta'_c$$

for some $\Gamma \vdash \text{ocs} : B! \Delta' / \Delta_c$ and $\Delta'_c \subseteq (\Delta_c \cup R \# \text{op})$. Define $\iota' = \llbracket \Gamma \vdash e : E^R \rrbracket \eta$ and consider two cases. First, if $\iota' \# \text{op}' = \iota \# \text{op}$ then

$$\llbracket e \# \text{op}' x k \mapsto c \mid \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, a, \kappa) = \llbracket \Gamma, x : A^{\text{op}}, k : B^{\text{op}} \rightarrow B! \Delta' \vdash c : B! \Delta' \rrbracket(\eta, a, \kappa),$$

and we may apply the induction hypothesis (for the whole theorem) to c . Second, suppose $\iota' \# \text{op}' \neq \iota \# \text{op}$. Then the assumption $\iota \# \text{op} \in \Delta' \cup \Delta'_c$ implies $\iota \# \text{op} \in \Delta' \cup \Delta_c$. Indeed, if $\iota \# \text{op} \in \Delta'$ there is nothing to prove, and if $\iota \# \text{op} \in \Delta'_c \subseteq (\Delta_c \cup R \# \text{op})$, then either R is not a singleton and $\Delta_c \cup R \# \text{op} = \Delta_c$, or $R = \{\iota' \# \text{op}'\}$ and $\iota \# \text{op} \notin R$. In any case, it follows that

$$\llbracket e \# \text{op}' x k \mapsto c \mid \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, a, \kappa) = \llbracket \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, a, \kappa)$$

and $\Gamma \vdash \text{ocs} : B! \Delta' / \Delta_c$. Thus we may apply the induction hypothesis. \square

5.5. Soundness and adequacy. The soundness and adequacy theorems state that operational semantics and denotational semantics fit together. One is an easy induction, while the other is proved using standard technique of formal approximation relations, e.g., see [1, Theorem 6.3.6].

Theorem 5.4 (Soundness). *If $\vdash c : \underline{C}$ and $c \rightsquigarrow c'$ then $\llbracket \vdash c : \underline{C} \rrbracket = \llbracket \vdash c' : \underline{C} \rrbracket$.*

Proof. We just have to walk through all the defining rules for \rightsquigarrow and verify that they do indeed preserve the meaning of c . \square

To tackle adequacy, see Corollary 5.8 below, we define formal approximation relations \triangleleft_A and $\triangleleft_{\underline{C}}$ whose intuitive meaning is that an element of $\llbracket A \rrbracket$ or $\llbracket \underline{C} \rrbracket$ approximates a closed term of type A or \underline{C} . Given $d \in \llbracket A \rrbracket$ and a closed expression $e : A$, we define $d \triangleleft_A e$ by:

$$\begin{aligned} d \triangleleft_{\text{bool}} e &\iff (d = \text{tt} \wedge e = \text{true}) \vee (d = \text{ff} \wedge e = \text{false}) \\ d \triangleleft_{\text{nat}} e &\iff d = n \wedge e = \text{succ}^n 0 \\ d \triangleleft_{\text{unit}} e &\iff d = \star \wedge e = () \\ d \triangleleft_{A \rightarrow \underline{C}} e &\iff \forall d', e'. (d' \triangleleft_A e' \Rightarrow d(d') \triangleleft_{\underline{C}} e e') \\ d \triangleleft_{ER} e &\iff d = e \\ d \triangleleft_{\underline{C} \Rightarrow \underline{D}} e &\iff \forall d', c. (d' \triangleleft_{\underline{C}} c \Rightarrow d(d') \triangleleft_{\underline{D}} (\text{with } e \text{ handle } c)) \end{aligned}$$

Simultaneously we define when $d \in \llbracket A! \Delta \rrbracket$ approximates a closed computation $c : A! \Delta$, where $d \triangleleft_{A! \Delta} c$ holds when

- $d = \perp$, or
- $d = \text{in}_{\text{val}}(d')$, $c \Downarrow \text{val } e$ and $d' \triangleleft_A e$, or
- $d = \text{in}_{\iota \# \text{op}}(d', \kappa)$, $c \Downarrow \iota \# \text{op } e(y. c')$, $d' \triangleleft_{A^{\text{op}}} e$, and if $d'' \triangleleft_{B^{\text{op}}} e'$ then $\kappa(d'') \triangleleft_{A! \Delta} c'[e''/y]$.

The definition of $\triangleleft_{A! \Delta}$ refers to types A^{op} and B^{op} which are possibly larger than A , thus we again use [14, Theorem 4.16] to establish existence of minimal such \triangleleft_A and $\triangleleft_{\underline{C}}$, much like for the recursive types considered in [14, Section 5].

Lemma 5.5. *If $d \triangleleft_{A! \Delta} c'$ and $c \rightsquigarrow c'$ then $d \triangleleft_{A! \Delta} c$.*

Proof. There is nothing to prove if $d = \perp$. The other two cases follow because $c \rightsquigarrow c'$ and $c' \Downarrow r$ imply $c \Downarrow r$. For instance, if $d \triangleleft_{A! \Delta} c'$ holds because $d = \text{in}_{\text{val}}(d')$, $c \Downarrow \text{val } e$ and $d' \triangleleft_A e$ for some d' and e , then $d \triangleleft_{A! \Delta} c$ holds because $c \rightsquigarrow c'$ implies $c \Downarrow \text{val } e$ and so we may reuse d' and e . \square

Lemma 5.6. *The relations \triangleleft_A and $\triangleleft_{\underline{C}}$ are closed under suprema of chains in the first argument. The relations $\triangleleft_{\underline{C}}$ relate \perp to every computation.*

Proof. The second statement holds by the definition of $\triangleleft_{\underline{C}}$. For the first statement we proceed by induction on the type. The base cases hold because the predomains for ground types are flat. For $A \rightarrow \underline{C}$, suppose $(d_n)_n$ is a chain in $\llbracket A \rightarrow \underline{C} \rrbracket$ and $d_n \triangleleft_{A \rightarrow \underline{C}} e$ for all n . Consider any d', e' such that $d' \triangleleft_A e'$. Then $d_n(d') \triangleleft_{\underline{C}} e e'$ for all n . Suprema in $\llbracket A \rightarrow \underline{C} \rrbracket$ are defined pointwise, so we can use the induction hypothesis for \underline{C} and get

$$(\bigvee_n d_n)(d') = \bigvee_n d_n(d') \triangleleft_{\underline{C}} e e',$$

hence $\bigvee_n d_n \triangleleft_{\underline{C}} e$ as desired. Handler types are treated similarly. Consider a computation type $A! \Delta$, a closed computation $c : A! \Delta$ and a chain $(d_n)_n$ in $\llbracket A! \Delta \rrbracket$ such that $d_n \triangleleft_{\underline{C}} c$ for all n . There are three kinds of chains in $\llbracket A! \Delta \rrbracket$:

- $(d_n)_n$ is constantly \perp : then $\bigvee_n d_n = \perp \triangleleft_{\underline{C}} c$.
- for large enough n there are d'_n such that $d_n = \text{in}_{\text{val}}(d'_n)$: then $(d'_n)_n$ form a chain in $\llbracket A \rrbracket$. Because operational semantics is deterministic, there exists a single e such that $c \Downarrow \text{val } e$, and $d'_n \triangleleft_A e$. By induction hypothesis for A we have $\bigvee_n d'_n \triangleleft_A e$, from which $\bigvee_n d_n \triangleleft_{A! \Delta} c$ follows because $\bigvee_n d_n = \text{in}_{\text{val}}(\bigvee_n d'_n)$.
- there are ι and op such that for large enough n there are d'_n and κ'_n such that $d_n = \text{in}_{\iota \# \text{op}}(d'_n, \kappa'_n)$: this case is treated analogously to the previous one. \square

Lemma 5.7. *Let Γ be the context $x_1 : A_1, \dots, x_n : A_n$. Suppose that for each $1 \leq i \leq n$ we have $d_i \in \llbracket A_i \rrbracket$ and a closed expression $\vdash e_i : A_i$ such that $d_i \triangleleft_{A_i} e_i$.*

- (1) *If $\Gamma \vdash e : A$ then $\llbracket \Gamma \vdash e : A \rrbracket(d_1, \dots, d_n) \triangleleft_A e[e_1/x_1, \dots, e_n/x_n]$.*
- (2) *If $\Gamma \vdash c : \underline{C}$ then $\llbracket \Gamma \vdash c : \underline{C} \rrbracket(d_1, \dots, d_n) \triangleleft_{\underline{C}} c[e_1/x_1, \dots, e_n/x_n]$.*

Proof. We prove the statements by induction on the derivation of the typing judgments. Let $\eta = (d_1, \dots, d_n)$ and $\sigma = [e_1/x_1, \dots, e_n/x_n]$. We write $[\sigma, e/x]$ for the substitution $[e_1/x_1, \dots, e_n/x_n, e/x]$. Throughout we assume that bound variables occurring in various terms do not appear in σ :

Cases VAR, TRUE, FALSE, UNIT, ZERO, INST: these are all trivial.

Case SUCC: $\Gamma \vdash (\text{succ } e) : \text{nat}$. By the induction hypothesis, we have $\llbracket e \rrbracket \eta \triangleleft_{\text{nat}} e\sigma$. Next, notice that the closed expression $e\sigma : \text{nat}$ must be $\text{succ}^k 0$ for some k , hence $\llbracket e \rrbracket \eta = k$, and so

$$\llbracket \text{succ } e \rrbracket \eta = (k + 1) \triangleleft_{\text{nat}} \text{succ } (e\sigma) = (\text{succ } e)\sigma.$$

Case FUN: $\Gamma \vdash (\text{fun } x \mapsto c) : A \rightarrow \underline{C}$. If $d' \triangleleft_A e'$ then by the induction hypothesis for c

$$\llbracket \Gamma, x : A \vdash c \rrbracket(\eta, d') \triangleleft_{\underline{C}} c[\sigma, e'/x].$$

Because $\llbracket \Gamma, x : A \vdash c \rrbracket(\eta, d') = (\llbracket \Gamma \vdash \text{fun } x \mapsto c \rrbracket \eta) d'$ and

$$((\text{fun } x \mapsto c)\sigma) e' = (\text{fun } x \mapsto c\sigma) e' \rightsquigarrow c[\sigma, e'/x],$$

we may use Lemma 5.5 to get the desired conclusion

$$(\llbracket \Gamma \vdash \text{fun } x \mapsto c \rrbracket \eta) d' \triangleleft_{\underline{C}} ((\text{fun } x \mapsto c)\sigma) e'.$$

Most other cases in the proof follow the same pattern, so we shall not explicitly mention uses of Lemma 5.5 anymore.

Case HAND: $\Gamma \vdash (\text{handler val } x \mapsto c_v \mid \text{ocs}) : A! \Delta \Rightarrow B! \Delta'$. We abbreviate the handler as h . Assuming $d \triangleleft_{A! \Delta} c$ we need to show that

$$(\llbracket h \rrbracket \eta)(d) \triangleleft_{B! \Delta'} (\text{with } h\sigma \text{ handle } c).$$

We proceed by an induction on d :

- If $d = \perp$ then the conclusion follows because $\llbracket h \rrbracket \eta$ is strict.
- If $d = \text{in}_{\text{val}}(d')$ then $d \triangleleft_{A! \Delta} c$ implies $c \Downarrow \text{val } e'$ and $d' \triangleleft_A e'$ for some e' and d' . In this case, by induction hypothesis for c_v ,

$$(\llbracket h \rrbracket \eta)(d) = \llbracket \Gamma, x : A \vdash c_v \rrbracket(\eta, d') \triangleleft_{b! \Delta} c_v[\sigma, e'/x],$$

and

$$(\text{with } h\sigma \text{ handle } c) \rightsquigarrow \dots \rightsquigarrow (c_v\sigma)[e'/x] = c_v[\sigma, e'/x].$$

- If $d = \text{in}_{\# \text{op}}(d', \kappa)$ then $d \triangleleft_{A! \Delta} c$ implies $c \Downarrow \iota \# \text{op } e' (y.c')$, $d' \triangleleft_{A^{\text{op}}} e'$ and $\kappa \triangleleft_{B^{\text{op}} \rightarrow A! \Delta} (\text{fun } y \mapsto c')$. Now

$$(\llbracket h \rrbracket \eta)(\text{in}_{\# \text{op}}(d', \kappa)) = \llbracket \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, d', \llbracket h \rrbracket \eta \circ \kappa)$$

and

$$(\text{with } h\sigma \text{ handle } c) \rightsquigarrow \dots \rightsquigarrow (\text{ocs}\sigma)_{\iota \# \text{op}}(e', (\text{fun } y \mapsto \text{with } h\sigma \text{ handle } c')),$$

therefore it suffices to prove

$$\llbracket \text{ocs} \rrbracket_{\iota \# \text{op}}(\eta, d', \llbracket h \rrbracket \eta \circ \kappa) \triangleleft_{B! \Delta'} (\text{ocs}\sigma)_{\iota \# \text{op}}(e', (\text{fun } y \mapsto \text{with } h\sigma \text{ handle } c')),$$

which we do by induction on the length of ocs . When ocs is nil the statement becomes

$$\text{in}_{\# \text{op}}(d', \llbracket h \rrbracket \eta \circ \kappa) \triangleleft_{B! \Delta'} \iota \# \text{op } e' (y. \text{with } h\sigma \text{ handle } c').$$

We already know $d' \triangleleft_{A^{\text{op}}} e'$, and still need

$$\llbracket h \rrbracket \eta(\kappa(d'')) \triangleleft_{B! \Delta'} (\text{with } h\sigma \text{ handle } c'[e''//y])$$

assuming $d'' \triangleleft_{B^{\text{op}}} e''$. This follows from the available induction hypotheses.

When ocs is $(l'\#\text{op}' x k \mapsto c'' \mid ocs')$ there are two further subcases:

– if $l'\#\text{op}' = l'\#\text{op}'$ then we get

$$\begin{aligned} & \llbracket \Gamma, x, k \vdash c'' \rrbracket (\eta, d, \llbracket h \rrbracket \eta \circ \kappa) \triangleleft_{B! \Delta'} \\ & c''[\sigma, e'/x, (\text{fun } y \mapsto \text{with } h\sigma \text{ handle } c')/k] \end{aligned}$$

which holds by induction on c'' ,

– if $l'\#\text{op}' \neq l'\#\text{op}'$ then we are left with

$$\begin{aligned} & \llbracket ocs' \rrbracket_{l'\#\text{op}'} (\eta, d', \llbracket h \rrbracket \eta \circ \kappa) \triangleleft_{B! \Delta'} \\ & (ocs' \sigma)_{l'\#\text{op}'} (e, (\text{fun } y \mapsto \text{with } h\sigma \text{ handle } c')), \end{aligned}$$

which is just the induction hypothesis for ocs' .

Cases IFTHENELSE, MATCH, ABSURD: We consider only

$$\Gamma \vdash (\text{if } e \text{ then } c_1 \text{ else } c_2) : \underline{C},$$

as MATCH is similar and ABSURD is vacuous. Because $e\sigma$ is a closed expression of type `bool` it is either `true` or `false`. Let us take a look at the first possibility. By induction hypothesis for c_1 we have $\llbracket c_1 \rrbracket \eta \triangleleft_{\underline{C}} c_1\sigma$. Again, because $\llbracket \text{if true then } c_1 \text{ else } c_2 \rrbracket \eta = \llbracket c_1 \rrbracket \eta$ and

$$(\text{if true then } c_1 \text{ else } c_2)\sigma \rightsquigarrow c_1\sigma$$

it follows that

$$\llbracket \text{if true then } c_1 \text{ else } c_2 \rrbracket \eta \triangleleft_{\underline{C}} (\text{if true then } c_1 \text{ else } c_2)\sigma,$$

as required.

Case APP: $\Gamma \vdash e'_1 e'_2 : \underline{C}$ where $\Gamma \vdash e'_1 : A \rightarrow \underline{C}$ and $\Gamma \vdash e'_2 : A$. By induction hypotheses for e'_1 and e'_2 we have $\llbracket e'_1 \rrbracket \eta \triangleleft_{A \rightarrow \underline{C}} e'_1\sigma$ and $\llbracket e'_2 \rrbracket \eta \triangleleft_A e'_2\sigma$, therefore by the definition of $\triangleleft_{A \rightarrow \underline{C}}$,

$$\llbracket e'_1 e'_2 \rrbracket \eta = (\llbracket e'_1 \rrbracket \eta)(\llbracket e'_2 \rrbracket \eta) \triangleleft_{\underline{C}} (e'_1\sigma)(e'_2\sigma) = (e'_1 e'_2)\sigma.$$

Case VAL: $\Gamma \vdash \text{val } e : A! \Delta$. By induction on e we have $\llbracket e \rrbracket \eta \triangleleft_A e\sigma$, therefore by the second clause in the definition of $\triangleleft_{A! \Delta}$

$$\llbracket \text{val } e \rrbracket \eta = \text{in}_{\text{val}}(\llbracket e \rrbracket \eta) \triangleleft_{A! \Delta} \text{val } (e\sigma) = (\text{val } e)\sigma.$$

Case OP: $\Gamma \vdash l'\#\text{op}' e(y.c) : A! \Delta$. This case works much like a combination of VAL and APP, so we omit the details.

Case LET: $\Gamma \vdash (\text{let } x = c_1 \text{ in } c_2) : B! \Delta$. This case is treated like a handler which only has a `val` case.

Case WITH: $\Gamma \vdash \text{with } e \text{ handle } c : \underline{D}$ where $\Gamma \vdash e : \underline{C} \Rightarrow \underline{D}$ and $\Gamma \vdash c : \underline{C}$. By induction hypotheses we have $\llbracket e \rrbracket \eta \triangleleft_{\underline{C} \Rightarrow \underline{D}} e\sigma$ and $\llbracket c \rrbracket \eta \triangleleft_{\underline{C}} c\sigma$, therefore by the definition of $\triangleleft_{\underline{C} \Rightarrow \underline{D}}$

$$(\llbracket e \rrbracket \eta)(\llbracket c \rrbracket \eta) \triangleleft_{\underline{D}} (\text{with } e\sigma \text{ handle } c\sigma).$$

The left-hand side equals $\llbracket \text{with } e \text{ handle } c : \underline{D} \rrbracket \eta$ and the right $(\text{with } e \text{ handle } c)\sigma$, so we are done.

Case LETREC: $\Gamma \vdash \text{let rec } fx = c_1 \text{ in } c_2 : \underline{D}$. After a short calculation the problem reduces to showing that

$$g \triangleleft_{A \rightarrow \underline{C}} (\text{fun } x \mapsto \text{let rec } fx = c_1 \sigma \text{ in } c_1 \sigma)$$

where g is the least fixed-point of the operator $\Phi : \llbracket A \rightarrow \underline{C} \rrbracket \rightarrow \llbracket A \rightarrow \underline{C} \rrbracket$, defined by

$$\Phi(h) = \lambda a \in \llbracket A \rrbracket . \llbracket \Gamma, f, x \vdash c_1 \rrbracket (\eta, h, a).$$

By Lemma 5.6 it suffices to show that

$$h \triangleleft_{A \rightarrow \underline{C}} (\text{fun } x \mapsto \text{let rec } fx = c_1 \sigma \text{ in } c_1 \sigma)$$

implies

$$\Phi(h) \triangleleft_{A \rightarrow \underline{C}} (\text{fun } x \mapsto \text{let rec } fx = c_1 \sigma \text{ in } c_1 \sigma).$$

This amounts to proving that if $d \triangleleft_A e$ then

$$\begin{aligned} \Phi(h)(d) &= \llbracket \Gamma, f, x \vdash c_1 \rrbracket (\eta, h, d) \\ &\triangleleft_{\underline{C}} c_1[\sigma, (\text{fun } x \mapsto \text{let rec } fx = c_1 \sigma \text{ in } c_1 \sigma) / f, e/x], \end{aligned}$$

which holds by the induction hypothesis for c_1 .

Case SUBEXPR, SUBCOMP: For SUBEXPR, take $\Gamma \vdash e : A'$ where $\Gamma \vdash e : A$ and $A \leq A'$. From induction hypothesis, we get $\llbracket \Gamma \vdash e : A \rrbracket \eta \triangleleft_A e \sigma$. Since $\llbracket A \rrbracket = \llbracket A' \rrbracket$, we have $\triangleleft_A = \triangleleft_{A'}$. Additionally, $\llbracket \Gamma \vdash e : A' \rrbracket = \llbracket \Gamma \vdash e : A \rrbracket$, hence $\llbracket e \rrbracket \eta \triangleleft_{A'} e \sigma$. For SUBCOMP, the proof is similar. \square

Corollary 5.8 (Adequacy). *If $\vdash c : \text{unit}! \Delta$ and $\llbracket c \rrbracket = \text{in}_{\text{val}}(\star)$ then $c \Downarrow \text{val } ()$.*

Proof. By the previous lemma $\llbracket c \rrbracket \triangleleft_{\text{unit}! \Delta} c$. Therefore, if $\llbracket c \rrbracket = \text{in}_{\text{val}}(\star)$ then $c \Downarrow \text{val } ()$ by the definition of $\triangleleft_{\text{unit}! \Delta}$. \square

The stated adequacy suffices for our purposes, but of course similar statements holds for other ground types. Regarding operations, if $\llbracket c \rrbracket = \text{in}_{\iota \# \text{op}}(d, \kappa)$, then $\iota \# \text{op} \in \Delta$ and so $c \Downarrow \iota \# \text{op } e k$ for some e and k such that $\llbracket e \rrbracket = d$ and $\llbracket k \rrbracket = \kappa$. Therefore, if $\llbracket c \rrbracket \neq \perp$ then $c \Downarrow r$ for some result r .

6. EQUATIONAL REASONING

6.1. Contextual and denotational equivalence. In this section we provide principles that allow us to reason about programs. Let us first recall how *contextual equivalence* is defined. An *expression context* \mathcal{E} is a computation with several occurrences of a *hole* $[]$ in positions where an expression is expected. When the hole is plugged with an expression e we get a computation $\mathcal{E}[e]$. A *computation context* \mathcal{E} is defined analogously, except that the holes appear where computations are expected.

We say that expressions e and e' are *contextually equivalent*, written $e \approx e'$, when for all expression contexts \mathcal{E} such that $\mathcal{E}[e]$ and $\mathcal{E}[e']$ are both of type $\text{unit}! \Delta$, we have $\mathcal{E}[e] \Downarrow \text{val } ()$ if and only if $\mathcal{E}[e'] \Downarrow \text{val } ()$. Contextual equivalence of computations is defined analogously.

Contextually equivalent expressions or computations may be interchanged anywhere in the code. Therefore, it is quite useful to know that a certain contextual equivalence holds, but unfortunately it is difficult to work directly with contextual equivalence. Luckily, denotational equivalence is more easily handled and is related to contextual equivalence by the adequacy theorem.

We write $e \equiv e'$ and $c \equiv c'$ when the denotations of two expressions or computations are the same. More precisely, if $\Gamma \vdash e : A$ and $\Gamma \vdash e' : A$ then $\Gamma \vdash e \equiv e' : A$, or just $e \equiv e'$, means $\llbracket \Gamma \vdash e : A \rrbracket = \llbracket \Gamma \vdash e' : A \rrbracket$, and similarly for computations.

Proposition 6.1. *Denotationally equal expressions are contextually equivalent, and likewise for computations.*

Proof. Suppose $e \equiv e'$ and consider an expression context \mathcal{E} such that $\mathcal{E}[e]$ and $\mathcal{E}[e']$ both have type $\text{unit}! \Delta$. Assume $\mathcal{E}[e] \Downarrow \text{val } ()$. By soundness of denotational semantics $\llbracket \mathcal{E}[e] \rrbracket = \text{in}_{\text{val}}(\star)$. By assumption $\llbracket e \rrbracket = \llbracket e' \rrbracket$ and because denotational semantics is compositional it follows that $\llbracket \mathcal{E}[e] \rrbracket = \llbracket \mathcal{E}[e'] \rrbracket$. Now by adequacy $\mathcal{E}[e'] \Downarrow \text{val } ()$. The proof for computations is the same. \square

Denotational equivalence is a congruence and is preserved by well-typed substitutions. It validates the following β -rules, where the various expressions and computations have suitable types, and h stands for **handler** $\text{val } x \mapsto c_v \mid \text{ocs}$:

$$\begin{aligned}
& \text{if true then } c_1 \text{ else } c_2 \equiv c_1 \\
& \text{if false then } c_1 \text{ else } c_2 \equiv c_2 \\
& \text{match } 0 \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2 \equiv c_1 \\
& \text{match } (\text{succ } e) \text{ with } 0 \mapsto c_1 \mid \text{succ } x \mapsto c_2 \equiv c_2[e/x] \\
& \quad (\text{fun } x \mapsto c) e \equiv c[e/x] \\
& \quad \text{let } x = \text{val } e \text{ in } c \equiv c[e/x] \\
& \text{let } x = e_1 \#_{\text{op}} e_2 (y. c_1) \text{ in } c_2 \equiv e_1 \#_{\text{op}} e_2 (y. \text{let } x = c_1 \text{ in } c_2) \\
& \quad \text{let rec } f x = c_1 \text{ in } c_2 \equiv c_2[(\text{fun } x \mapsto \text{let rec } f x = c_1 \text{ in } c_1)/f] \\
& \quad \text{with } h \text{ handle } (\text{val } e) \equiv c_v[e/x] \\
& \quad \text{with } h \text{ handle } (\iota \#_{\text{op}} e (y. c)) \equiv \text{ocs}_{\iota \#_{\text{op}}}(e, (\text{fun } y \mapsto \text{with } h \text{ handle } c))
\end{aligned}$$

We also have the following η -rules, provided the expressions and computations have easily guessed types:

$$\begin{aligned}
& e \equiv () \\
& \text{fun } x \mapsto e x \equiv e \\
& \text{let } x = c \text{ in val } x \equiv c \\
& \quad \text{if } e \text{ then } c[\text{true}/x] \text{ else } c[\text{false}/x] \equiv c[e/x] \\
& \text{match } e \text{ with } 0 \mapsto c[0/x] \mid \text{succ } y \mapsto c[\text{succ } y/x] \equiv c[e/x] \\
& \quad \text{absurd } e \equiv c[e/x]
\end{aligned}$$

We omit the proofs because they just involve unfolding of semantic definitions. An exception is the η -rule for **let** binding, which is proved by induction in the next section.

A variety of other equivalences is readily validated, for example

$$\text{let } x = c_1 \text{ in } c_2 \equiv \text{with } (\text{handler val } x \mapsto c_2 \mid \text{nil}) \text{ handle } c_1$$

and the ‘‘associativity’’ of **let** binding [12]

$$\text{let } x = (\text{let } y = c_1 \text{ in } c_2) \text{ in } c_3 \equiv \text{let } y = c_1 \text{ in } (\text{let } x = c_2 \text{ in } c_3),$$

where y must not occur freely in c_3 , see [20, 21] for other examples.

6.2. An induction principle for effects. The induction principle for the computation domains from Section 5.1 is useful for deriving general laws that do not depend on a particular choice of effects. It is less useful for specific examples which involve a carefully chosen set of effects and handlers, because it forces us to consider operations that have nothing to do with the situation at hand. The induction principle Theorem 5.2 remedies the drawback.

A typical application arises when we prove an equivalence of the form $\mathcal{E}[c] \equiv \mathcal{E}'[c] : \underline{C}$ for all computations c of a suitable type $A! \Delta$. The computation contexts \mathcal{E} and \mathcal{E}' are interpreted as continuous maps $\llbracket \mathcal{E} \rrbracket, \llbracket \mathcal{E}' \rrbracket : \llbracket A! \Delta \rrbracket \rightarrow \llbracket \underline{C} \rrbracket$, and the equivalence may be phrased as

$$\forall t \in \llbracket A! \Delta \rrbracket. \llbracket \mathcal{E} \rrbracket(t) = \llbracket \mathcal{E}' \rrbracket(t).$$

The equality inside the quantifier is an admissible predicate on $\llbracket A! \Delta \rrbracket$, so the induction principle applies. It is a bit cumbersome to perform the proof using the semantic brackets $\llbracket - \rrbracket$ all over the place. Instead, with a bit of flexibility in notation, we can write the proof in a syntactic manner as follows:

- (1) We write \perp for a non-terminating computation, e.g., `let rec fx = fx in f()`, and check that $\mathcal{E}[\perp] \equiv \mathcal{E}'[\perp]$.
- (2) We verify that $\mathcal{E}[\text{val } e] \equiv \mathcal{E}'[\text{val } e]$ where e is a meta-variable of type A .
- (3) We verify, for each $\iota\#\text{op} \in \Delta$,

$$\mathcal{E}[\iota\#\text{op } e(y.\kappa y)] \equiv \mathcal{E}'[\iota\#\text{op } e(y.\kappa y)],$$

where e is a meta-variable of type A^{op} and κ is a meta-variable of type $B^{\text{op}} \rightarrow A! \Delta$.

The induction hypothesis is that $\mathcal{E}[\kappa e'] \equiv \mathcal{E}'[\kappa e']$ for all expressions e' of type B^{op} .

We apply the method to prove the η -rule

$$\text{let } x = c \text{ in val } x \equiv c$$

by induction:

- (1) $(\text{let } x = \perp \text{ in val } x) \equiv \perp$ because `let` is strict in both arguments,
- (2) $(\text{let } x = (\text{val } a) \text{ in val } x) \equiv (\text{val } x)[a/x] \equiv \text{val } a$ by a β -rule for `let`,
- (3) The induction step is proved by

$$\begin{aligned} \text{let } x = (\iota\#\text{op } e(y.\kappa y)) \text{ in val } x \\ &\equiv \iota\#\text{op } e(y.\text{let } x = \kappa y \text{ in val } x) \\ &\equiv \iota\#\text{op } e(y.\kappa y) \end{aligned}$$

where we used a β -rule in the first step and the induction hypothesis for κy in the second.

A non-trivial application of the induction principle is presented in Section 7.2.

7. EXAMPLE: MUTABLE REFERENCES

As a more elaborate example we consider mutable references. In core *Eff* they are implemented by the effect `ref` with operations

$$\text{lookup} : \text{unit} \rightarrow A \quad \text{and} \quad \text{update} : A \rightarrow \text{unit},$$

where A is a fixed pure type (in full Eff we could use a parameter). The handler which handles the reference given by an expression $r : \mathbf{ref}^R$ is defined by (the underscore $_$ indicates an ignored parameter):

$$\begin{aligned} \mathbf{state}_r &\stackrel{\text{def}}{=} \mathbf{handler} \mathbf{val} \ x \mapsto \mathbf{val} \ (\mathbf{fun} \ s \mapsto \mathbf{val} \ x) \\ &\quad | \ r\#\mathbf{lookup} \ _ \ k \mapsto \mathbf{val} \ (\mathbf{fun} \ s \mapsto \mathbf{let} \ f = k \ s \ \mathbf{in} \ f \ s) \\ &\quad | \ r\#\mathbf{update} \ s' \ k \mapsto \mathbf{val} \ (\mathbf{fun} \ s \mapsto \mathbf{let} \ f = k \ () \ \mathbf{in} \ f \ s') \end{aligned}$$

This is just the usual monadic-style treatment of state which wraps a computation into a state-carrying function. For any instance $\iota \in \mathcal{I}_{\mathbf{ref}}$, pure type B , and dirt Δ , the computation \mathbf{state}_ι has the type

$$B! (\{\iota\#\mathbf{lookup}, \iota\#\mathbf{update}\} \cup \Delta) \Rightarrow (A \rightarrow B!\Delta)!\Delta.$$

The type says that the handler erases lookups and updates of ι from a computation. The return type $(A \rightarrow B!\Delta)!\Delta$ has an outer dirt Δ because effects could happen before the first lookup or update. Note that the double dirt would not arise if we took the call-by-push-value approach to handlers [19].

The handler wraps a handled computation c of type $B!\Delta$ into a function expecting the current state, which explains the type $A \rightarrow B!\Delta$. In practice such a function is immediately applied to an initial state e of type A (such a final transformation of handled computations is so common that in full Eff handlers have a special `finally` case just for this purpose):

$$\mathbf{let} \ f = (\mathbf{with} \ h \ \mathbf{handle} \ c) \ \mathbf{in} \ f \ e$$

The type of this computation is simply $B!\Delta$. In particular, if the only effects in c are lookups and updates of ι , we get a pure computation.

If the type of r is weaker, for example $\mathbf{ref}^{\{\iota_1, \iota_2\}}$, we are not able to deduce anything useful. The best we can do is

$$\mathbf{state}_r : B!\Delta \Rightarrow (A \rightarrow B!\Delta)!\Delta$$

for any dirt Δ .

We may handle several references at once by wrapping a computation into several handlers. For example, let c be the computation which swaps the contents of two references:

$$\begin{aligned} &\mathbf{let} \ y_1 = \iota_1\#\mathbf{lookup} \ () \ \mathbf{in} \\ &\mathbf{let} \ y_2 = \iota_2\#\mathbf{lookup} \ () \ \mathbf{in} \\ &\mathbf{let} \ _ = \iota_1\#\mathbf{update} \ y_2 \ \mathbf{in} \\ &\mathbf{let} \ _ = \iota_2\#\mathbf{update} \ y_1 \ \mathbf{in} \ (\mathbf{val} \ ()) \end{aligned}$$

By itself, c has the type

$$\mathbf{unit}! \{\iota_1\#\mathbf{lookup}, \iota_1\#\mathbf{update}, \iota_2\#\mathbf{lookup}, \iota_2\#\mathbf{update}\},$$

When c is wrapped by the handler $h_2 = \mathbf{state}_{\iota_2}$,

$$\mathbf{let} \ f_2 = (\mathbf{with} \ h_2 \ \mathbf{handle} \ c) \ \mathbf{in} \ f_2 \ e_2,$$

the type becomes $\mathbf{unit}! \{\iota_1\#\mathbf{lookup}, \iota_1\#\mathbf{update}\}$. When the handler $h_1 = \mathbf{state}_{\iota_1}$ is used on top of that,

$$\mathbf{let} \ f_1 = (\mathbf{with} \ h_1 \ \mathbf{handle} \ (\mathbf{let} \ f_2 = (\mathbf{with} \ h_2 \ \mathbf{handle} \ c) \ \mathbf{in} \ f_2 \ e_2)) \ \mathbf{in} \ f_1 \ e_1$$

we get the pure type $\mathbf{unit!}\emptyset$. Beware, it is important that the state is initialized at the correct point in the computation. For instance,

$$\mathbf{let } f_1 = (\mathbf{with } h_1 \mathbf{ handle } (\mathbf{with } h_2 \mathbf{ handle } c)) \mathbf{ in } (\mathbf{let } f_2 = f_1 e_1 \mathbf{ in } f_2 e_2)$$

does not do the right thing. We are warned about possible trouble by the effect system which gives the computation the type $\mathbf{unit!}\{\iota_1\#\mathbf{lookup}, \iota_1\#\mathbf{update}\}$ — the operations for ι_1 are escaping the handlers! A less modular way of handling two instances is to create a new handler with four operation cases, two for each of the instances.

7.1. Reasoning about references. With handlers the workings of a computation may be inspected in a highly intensional way. Consequently, there are few generally valid observational equivalences. However, when known handlers are used to handle operations, we may derive equivalences that describe the behavior of operations. The situation is opposite to that of [19], where we start with an equational theory for operations and require that the handlers respect it.

We demonstrate the technique for mutable state. Let $h = \mathbf{state}_\iota$ and abbreviate

$$\mathbf{let } f = (\mathbf{with } h \mathbf{ handle } c) \mathbf{ in } f e$$

as $\mathcal{H}[c, e]$. Straightforward calculations give us the equivalences

$$\begin{aligned} \mathcal{H}[(\iota\#\mathbf{lookup } () (y. c)), e] &\equiv \mathcal{H}[c[e/y], e] \\ \mathcal{H}[(\iota\#\mathbf{update } e' (-. c)), e] &\equiv \mathcal{H}[c, e'] \\ \mathcal{H}[\mathbf{val } e', e] &\equiv \mathbf{val } e', \end{aligned}$$

for instance,

$$\begin{aligned} &\mathcal{H}[(\iota\#\mathbf{update } e' (-. c)), e] \\ &\equiv \mathbf{let } f = \mathbf{val } (\mathbf{fun } s \mapsto \mathbf{let } f' = (\mathbf{fun } _ \mapsto \mathbf{with } h \mathbf{ handle } c) () \mathbf{ in } f' e') \mathbf{ in } f e \\ &\equiv \mathbf{let } f = \mathbf{val } (\mathbf{fun } s \mapsto \mathcal{H}[c, e']) \mathbf{ in } f e \\ &\equiv (\mathbf{fun } s \mapsto \mathcal{H}[c, e']) e \\ &\equiv \mathcal{H}[c, e']. \end{aligned}$$

These suffice for simple equational reasoning about state. If we read them as rewrite rules they allow us to progressively transform a computation to a simpler form. In fact, the transformations mimic the usual coalgebraic operational semantics for state [18].

Of course, a realistic computation will contain several handlers. As long as they do not interfere with each other, we can still use equivalences to usefully manipulate them. For example, if h' is a handler with no operation case for $\iota\#\mathbf{lookup}$ then

$$\begin{aligned} &\mathcal{H}[(\mathbf{with } h' \mathbf{ handle } \iota\#\mathbf{lookup } () (y. c)), e] \\ &\equiv \mathcal{H}[(\iota\#\mathbf{lookup } () (y. \mathbf{with } h' \mathbf{ handle } c)), e] \\ &\equiv \mathcal{H}[(\mathbf{with } h' \mathbf{ handle } c[e/y]), e]. \end{aligned}$$

It may happen that a lookup or an update is nested deeply inside several handlers. The above transformation allows us to hoist the operation out of the inner handlers so that it is handled by the outer handler, as long as the inner handlers do not attempt to handle ι . The transformation applies to \mathbf{let} bindings too, as they are like handlers without operation cases.

We may validate the seven standard equations governing state [16]. There are four combinations of lookup and update (in the first equation y does not occur free in c):

$$\begin{aligned} \mathcal{H}[\iota\#\text{lookup } () (y.\iota\#\text{update } y(_ . c)), e] &\equiv \mathcal{H}[c, e] \\ \mathcal{H}[\iota\#\text{lookup } () (y.\iota\#\text{lookup } () (z. c)), e] &\equiv \mathcal{H}[\iota\#\text{lookup } () (y. c[y/z]), e] \\ \mathcal{H}[\iota\#\text{update } e(_ . \iota\#\text{update } e'(_ . c)), e] &\equiv \mathcal{H}[\iota\#\text{update } e'(_ . c), e] \\ \mathcal{H}[\iota\#\text{update } e(_ . \iota\#\text{lookup } () (y. c)), e] &\equiv \mathcal{H}[\iota\#\text{update } e(_ . c[e/y]), e] \end{aligned}$$

For instance, the first equation is validated as follows:

$$\begin{aligned} \mathcal{H}[\iota\#\text{lookup } () (y.\iota\#\text{update } y(_ . c)), e] \\ &\equiv \mathcal{H}[\iota\#\text{update } e(_ . c), e] \\ &\equiv \mathcal{H}[c, e] \end{aligned}$$

Three more equations describe commutativity of lookups and updates at different distances. Let $\iota_1 \neq \iota_2$, and write \mathcal{H}_1 and \mathcal{H}_2 for the the abbreviation \mathcal{H} with respect to ι_1 and ι_2 , respectively:

$$\begin{aligned} \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{lookup } () (y_1.\iota_2\#\text{lookup } () (y_2. c)), e_2], e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[\iota_2\#\text{lookup } () (y_2.\iota_1\#\text{lookup } () (y_1. c)), e_2], e_1] \\ \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{update } e_1(_ . \iota_2\#\text{update } e_2(_ . c)), e_2], e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[\iota_2\#\text{update } e_2(_ . \iota_1\#\text{update } e_1(_ . c)), e_2], e_1] \\ \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{update } e(_ . \iota_2\#\text{lookup } () (y_2. c)), e_2], e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[\iota_2\#\text{lookup } () (y_2.\iota_1\#\text{update } e(_ . c)), e_2], e_1] \end{aligned}$$

Let us check the last equation. The left-hand side transforms as

$$\begin{aligned} \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{update } e(_ . \iota_2\#\text{lookup } () (y_2. c)), e_2], e_1] \\ &\equiv \mathcal{H}_1[\iota_1\#\text{update } e(_ . \mathcal{H}_2[\iota_2\#\text{lookup } () (y_2. c), e_2]), e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[\iota_2\#\text{lookup } () (y_2. c), e_2], e] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[c[e_2/y_2], e_2], e] \end{aligned}$$

and the right-hand side as

$$\begin{aligned} \mathcal{H}_1[\mathcal{H}_2[\iota_2\#\text{lookup } () (y_2.\iota_1\#\text{update } e(_ . c)), e_2], e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{update } e(_ . c[e_2/y_2]), e_2], e_1] \\ &\equiv \mathcal{H}_1[\iota_1\#\text{update } e(_ . \mathcal{H}_2[c[e_2/y_2], e_2]), e_1] \\ &\equiv \mathcal{H}_1[\mathcal{H}_2[c[e_2/y_2], e_2], e]. \end{aligned}$$

The remaining two equations are proved much the same way. The symmetry in the equations also shows that it does not matter in which order we nest the handlers for ι_1 and ι_2 .

7.2. Commutativity of non-interfering computations. Swapping lookups and updates that act on different instances is only a basic reasoning step. In practice we want to swap whole computations, as long as they do not interfere. To make the idea precise, let $\Delta_1 = \{\iota_1\#\text{lookup}, \iota_1\#\text{update}\}$, $\Delta_2 = \{\iota_2\#\text{lookup}, \iota_2\#\text{update}\}$, let c_1 and c_2 be computations

of types $A_1! \Delta_1$ and $A_2! \Delta_2$, respectively, and c a computation of type \underline{C} in the context $x_1 : A_1, x_2 : A_2$. We would like to show the equivalence

$$\begin{aligned} & \mathcal{H}_1[\mathcal{H}_2[\text{let } x_1 = c_1 \text{ in } (\text{let } x_2 = c_2 \text{ in } c), e_2], e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = c_1 \text{ in } c), e_2], e_1]. \end{aligned}$$

This is a commutativity law which allows us to transpose, or run in parallel, any computations that use only non-interfering references.

First, let us establish a simpler equivalence, which we are going to use in the proof:

$$\begin{aligned} & \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{lookup } () (y_1.\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = c'_1 \text{ in } c)), e_2], e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = \iota_1\#\text{lookup } () (y_1.c'_1) \text{ in } c), e_2], e_1] \end{aligned}$$

We proceed by induction on c_2 . Since handlers and `let` are strict, both sides are \perp when we set c_2 to \perp . Next, if c_2 is `val` e_2 , the two sides are equal already without handlers:

$$\begin{aligned} & \iota_1\#\text{lookup } () (y_1.\text{let } x_2 = \text{val } e_2 \text{ in } (\text{let } x_1 = c'_1 \text{ in } c)) \\ & \equiv \iota_1\#\text{lookup } () (y_1.\text{let } x_1 = c'_1 \text{ in } c[e_2/x_2]) \\ & \equiv \iota_1\#\text{lookup } () (y_1.\text{let } x_1 = c'_1 \text{ in } (\text{let } x_2 = \text{val } e_2 \text{ in } c)) \\ & \equiv \text{let } x_1 = \iota_1\#\text{lookup } () (y_1.c'_1) \text{ in } (\text{let } x_2 = \text{val } e_2 \text{ in } c) \end{aligned}$$

For the induction step, suppose c_2 is a call of $\iota_2\#\text{update}$ (the case `lookup` is similar):

$$\begin{aligned} & \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{lookup } () (y_1.\text{let } x_2 = \iota_2\#\text{update } e'_2 (z.\kappa z) \text{ in } (\text{let } x_1 = c'_1 \text{ in } c)), e_2], e_1] \\ & \equiv \mathcal{H}_1[\iota_1\#\text{lookup } () (y_1.\mathcal{H}_2[\text{let } x_2 = \kappa() \text{ in } (\text{let } x_1 = c'_1 \text{ in } c), e'_2]), e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\iota_1\#\text{lookup } () (y_1.\text{let } x_2 = \kappa() \text{ in } (\text{let } x_1 = c'_1 \text{ in } c)), e'_2], e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = \kappa() \text{ in } (\text{let } x_1 = \iota_1\#\text{lookup } () (y_1.c'_1) \text{ in } c), e'_2], e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = \iota_2\#\text{update } e'_2 (z.\kappa z) \text{ in } (\text{let } x_1 = \iota_1\#\text{lookup } () (y_1.c'_1) \text{ in } c), e_2], e_1]. \end{aligned}$$

In the third step we used the induction hypothesis for $\kappa()$.

We now prove the main statement by induction on c_1 . When c_1 is \perp or a value, the statement is easy. If c_1 is a call of $\iota_1\#\text{lookup}$, we have:

$$\begin{aligned} & \mathcal{H}_1[\mathcal{H}_2[\text{let } x_1 = \iota_1\#\text{lookup } () (y_1.\kappa y_1) \text{ in } (\text{let } x_2 = c_2 \text{ in } c), e_2], e_1] \\ & \equiv \mathcal{H}_1[\iota_1\#\text{lookup } () (y_1.\mathcal{H}_2[\text{let } x_1 = \kappa y_1 \text{ in } (\text{let } x_2 = c_2 \text{ in } c), e_2]), e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_1 = \kappa e_1 \text{ in } (\text{let } x_2 = c_2 \text{ in } c), e_2], e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = \kappa e_1 \text{ in } c), e_2], e_1] \end{aligned}$$

where in the last step, we used the induction hypothesis for κe_1 . The last line is equivalent to the other side of the desired equivalence:

$$\begin{aligned} & \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = \kappa e_1 \text{ in } c), e_2], e_1] \\ & \equiv \mathcal{H}_1[\iota_1\#\text{lookup } () (y_1.\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = \kappa y_1 \text{ in } c), e_2]), e_1] \\ & \equiv \mathcal{H}_1[\mathcal{H}_2[\text{let } x_2 = c_2 \text{ in } (\text{let } x_1 = \iota_1\#\text{lookup } () (y_1.\kappa y_1) \text{ in } c), e_2], e_1]. \end{aligned}$$

The proof for `update` is similar.

8. FORMALIZATION IN TWELF

We formalized core *Eff*, the effect system and the safety theorems in Twelf. The files are enclosed with this paper, or can be found at the GitHub repository [4]. The code is compatible with Twelf version 1.7.1. Further instructions and description of the code can be found in the file `README.md`.

9. DISCUSSION

The only essential feature of *Eff* that is missing from core *Eff* is dynamic creation of instances with the `new E` construct. We omitted it because it leads to significant complications, both in the effect system and in semantics. One possible treatment of `new` would be to upgrade the current setup with nominal logic and nominal domain theory.

Non-termination is a computational effect which is not reflected in our effect system. It would be interesting to add a “divergence” effect. Such an effect would originate from applications of recursive functions. However, since divergence cannot be handled it would never disappear from effect information, and would likely become an uninformative nuisance. A potential remedy would be to separate general recursive definitions, which may diverge, from structural recursive definitions, which always terminate.

A realistic implementation of our effect system would only be useful if it actually *inferred* computational effects. Such a possibility was explored by the second author [22], and an early prototype is available in the latest implementation of *Eff* [5].

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