

MONADIC SECOND-ORDER DEFINABLE GRAPH ORDERINGS

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ABSTRACT. We study the question of whether, for a given class of finite graphs, one can define, for each graph of the class, a linear ordering in monadic second-order logic, possibly with the help of monadic parameters. We consider two variants of monadic second-order logic: one where we can only quantify over sets of vertices and one where we can also quantify over sets of edges. For several special cases, we present combinatorial characterisations of when such a linear ordering is definable. In some cases, for instance for graph classes that omit a fixed graph as a minor, the presented conditions are necessary and sufficient; in other cases, they are only necessary. Other graph classes we consider include complete bipartite graphs, split graphs, chordal graphs, and cographs. We prove that orderability is decidable for the so called HR-equational classes of graphs, which are described by equation systems and generalize the context-free languages.

1. INTRODUCTION

When studying the expressive power of monadic second-order logic (MSO) for finite graphs, often the question arises of whether one can define a linear order on the vertex set. For instance, the property that a set has even cardinality cannot, in general, be expressed in MSO. If, however, the considered set is linearly ordered, we can write down a corresponding MSO-formula. The same holds for every predicate $\text{Card}_q(X)$ expressing that the cardinality of the set X is a multiple of q . It follows that the extension of MSO by the predicates $\text{Card}_q(X)$, called *counting monadic second-order logic* (CMSO), is no more powerful than MSO on every class of structures on which a linear order is MSO-definable.

Another example of a situation where the availability of a linear order facilitates certain logical constructions is the definability of graph decompositions such as the modular decomposition of a graph. It is shown in [4] that the modular decomposition of a graph is

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definable in MSO if the graph is equipped with a linear order. Finally,¹ although we will not address complexity questions in this article, we recall that, over linearly ordered structures, the complexity class PTIME is captured by least fixed-point logic [11, 16].

A formula $\varphi(x, y)$ with two free first-order variables x and y defines a (linear) order on a relational structure \mathfrak{A} if the binary relation consisting of all pairs (a, b) of elements of \mathfrak{A} satisfying $\mathfrak{A} \models \varphi(a, b)$ is a linear order on A . We say that $\varphi(x, y)$ defines an order on a class of structures if it defines a linear order on each structure of that class. Our objective is to provide combinatorial characterisations of classes of finite graphs whose representing structures are MSO-orderable, i.e., on which one can define an order by an MSO-formula. (The question of whether a *partial* order is definable is trivial since equality is a partial order. Therefore, we only consider linear orders in this article.)

As defined above the notion of an MSO-orderable class is too restrictive. To get interesting results, we allow in the above definitions formulae with *parameters*. That is, we take a formula $\varphi(x, y; \bar{Z})$ with additional free set variables $\bar{Z} = \langle Z_0, \dots, Z_{n-1} \rangle$ and, for each structure \mathfrak{A} in the given class, we choose values $P_0, \dots, P_{n-1} \subseteq A$ for these variables such that the binary relation

$$\{ (a, b) \mid \mathfrak{A} \models \varphi(a, b; \bar{P}) \}$$

is a linear order on A .

There is no MSO-formula (even with parameters) that defines a linear order on all finite graphs. An easy way to see this is to observe that every ordered structure is *rigid*, i.e., that it has no non-trivial automorphism. Since we can find graphs that are not rigid, even after labelling them with a fixed number of parameters, it follows that no formula can order all graphs. The same argument shows that the following classes of finite graphs are not MSO-orderable: (1) graphs without edges; (2) cliques; (3) stars; (4) trees of a fixed height; and (5) bipartite graphs. On the other hand, to take an easy example, the class of all finite connected graphs of degree at most d (for fixed d) is MSO-orderable.

If graphs are replaced by their incidence graphs, MSO-formulae become more powerful, because they can use quantifications over sets of edges. In this case we speak of MSO₂-orderable classes. Otherwise, we call the class MSO₁-orderable. Due to the greater expressive power, the family of MSO₂-orderable classes properly includes that of MSO₁-orderable ones. This means that, in the combinatorial characterisations presented below, the conditions for MSO₁-orderability must be stronger than those for MSO₂-orderability. For instance, the class of all cliques is MSO₂-orderable but not MSO₁-orderable.

There are simple combinatorial criteria showing that a class is not MSO-orderable. For instance, a class of trees is not MSO-orderable if the degree of vertices is unbounded. The reason is that an MSO-formula can only distinguish between a bounded number of neighbours of a vertex. If the number of neighbours is too large, we can swap two of the attached subtrees without affecting the truth value of the formula. Generalising this example, we obtain the following criterion for MSO₂-orderability: if a class \mathcal{C} is MSO₂-orderable, there exists a function f such that, whenever we remove k vertices from a graph in \mathcal{C} , the resulting graph has at most $f(k)$ connected components (Proposition 4.4).

¹Yet another example is the construction of (a combinatorial description of) a plane embedding of a connected planar graph. Such embeddings are definable in MSO if we can order the neighbours of each vertex (see [5]). For 3-connected graphs such an ordering is always definable, but for graphs that are not 3-connected this is not always the case.

In many cases, it turns out that this necessary condition is also sufficient. For instance, we will prove in Theorem 4.13 below that a class of graphs omitting some graph as a minor is MSO_2 -orderable if, and only if, it has the above property.

This article is organised as follows. Sections 2 and 3 introduce notation and basic definitions. The main part consists of Sections 4 and 5, which collect our results on, respectively, MSO_2 -orderability and MSO_1 -orderability.

For MSO_2 -orderability, we present a necessary condition in Section 4.1. We prove that this condition is also sufficient for trees (Theorem 4.8) and, more generally, for classes of graphs omitting some graph as a minor (Theorem 4.13). For some classes of bipartite graphs and of split graphs, we obtain a similar result, using a slightly stronger combinatorial condition (Theorems 4.29 and 4.32). Furthermore, we prove that some classes are not MSO_2 -orderable in a very strong sense: they contain no infinite subclass that is MSO_2 -orderable. This is the case for trees of bounded height (Corollary 4.9) and graphs of bounded n -depth tree-width (Proposition 4.15). Finally, we also prove that, for certain effectively presented classes of graphs, MSO_2 -orderability is decidable (Corollary 4.23).

For MSO_1 -orderability the picture we obtain is slightly more sketchy. We present a necessary condition for MSO_1 -orderability in Section 5.1. We prove that it is also sufficient for cographs (Theorem 5.15) and graphs of bounded n -depth \otimes -width (Theorem 5.23).

Finally, we consider reductions between orderability properties in Section 6. We show that, for split graphs and bipartite graphs, the question of MSO_i -orderability is as hard as for arbitrary graphs. This indicates that we are far from having a combinatorial characterisation of orderability for such classes.

2. PRELIMINARIES

Let us fix our notation and terminology. We write $[n] := \{0, \dots, n-1\}$, for $n \in \mathbb{N}$. We denote tuples $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ with a bar. The empty tuple is $\langle \rangle$. We write $A \Delta B$ for the symmetric difference of two sets A and B . We denote partial orders by symbols like \leq and \preceq , and the corresponding strict partial orders by $<$ and \prec , respectively.

2.1. Structures and graphs. In this article we consider only purely relational structures $\mathfrak{A} = \langle A, R_0^{\mathfrak{A}}, \dots, R_{n-1}^{\mathfrak{A}} \rangle$ with finite signatures $\Sigma = \{R_0, \dots, R_{n-1}\}$. The universe A will always be finite, and we allow it to be empty as this convention is common in graph theory. In some places we will also allow relational structures with constants, but when doing so we will always mention it explicitly. For a relation R and a set X , we write $R \upharpoonright X$ for the restriction of R to X . For a tuple \bar{R} of relations, we denote by $\bar{R} \upharpoonright X$ the corresponding tuple of restrictions.

We will mainly consider graphs instead of arbitrary relational structures. For basic notions of graph theory, we refer the reader to the book [10]. In this article, graphs will always be finite, simple, loop-free, and undirected, with the exception of rooted trees and forests, which we consider to be oriented (see below). We will denote the edge between vertices u and v by (u, v) . Note that the same edge can also be written as (v, u) . There are two ways to represent a graph $G = \langle V, E \rangle$ by a structure. Both of them will be used. We can use structures of the form $\lfloor G \rfloor := \langle V, \text{edg} \rangle$ where the universe V consists of the set of vertices and we have a binary edge relation $\text{edg} \subseteq V \times V$, or we can use structures of the form $\lceil G \rceil := \langle V \cup E, \text{inc} \rangle$ where the universe contains both, the vertices and the (undirected) edges of the graph and we have a binary incidence relation $\text{inc} \subseteq V \times E$ telling us which

vertices belong to which edges. If \mathcal{C} is a class of graphs, we denote the corresponding classes of relational structures by $\llbracket \mathcal{C} \rrbracket$ and $\lceil \mathcal{C} \rceil$, respectively.

Forests will always be rooted and directed in such a way that every edge is oriented away from the root. The *tree-order* associated with a forest F is the partial order defined by

$$x \preceq_F y \quad :\iff \quad \text{some path from a root to } y \text{ contains } x.$$

If $x \prec y$, we call x a *predecessor* of y and y a *successor* of x . We speak of *immediate predecessors* and *immediate successors* if there is no vertex in between. The n -th level of a forest F consists of all vertices at distance n from some root. Hence, the roots form level 0. The *height* of F is the maximal level of its vertices.

Definition 2.1. A graph $G = \langle V, E \rangle$ is *r-sparse*² if, for every subset $X \subseteq V$, we have $|E \upharpoonright X| \leq r \cdot |X|$. \diamond

We denote by $\mathfrak{A} \oplus \mathfrak{B}$ the disjoint union of the structures \mathfrak{A} and \mathfrak{B} . For structures $\llbracket G \rrbracket$ and $\llbracket H \rrbracket$ encoding graphs, we also use a dual operation $\llbracket G \rrbracket \otimes \llbracket H \rrbracket$ that, after forming the disjoint union of $\llbracket G \rrbracket$ and $\llbracket H \rrbracket$, adds all possible edges connecting a vertex of G to a vertex of H . For a set $S \subseteq A$ of elements, we write $\mathfrak{A}[S]$ for the substructure of \mathfrak{A} induced by S and $\mathfrak{A} - S$ for $\mathfrak{A}[A - S]$. We use the analogous notation $G[S]$ and $G - S$, for graphs G .

We assume that the reader is familiar with the notion of a tree decomposition and the tree-width of a graph (see, e.g., [10, 7]). At a few places, we will refer to a variant of tree-width, called *n-depth tree-width*, that was introduced in [2]. It is defined in terms of tree decompositions where the height of the index tree is at most n .

Finally, we will employ tools related to the notion of *clique-width*, which is defined for graphs with *ports* in a finite set $[k]$, that is, graphs $G = \langle V, E, \pi \rangle$ equipped with a function $\pi : V \rightarrow [k]$. We say that a vertex $a \in V$ has *port label* a if $\pi(v) = a$. The notion of clique-width is defined in terms of the following operations on graphs³ with ports:

- for each $a \in [k]$, a constant a denoting the graph with a single vertex that has port label a ;
- the disjoint union \oplus of two graphs with ports;
- the edge addition operation $\text{add}_{a,b}$, for $a, b \in [k]$, adding all edges between some vertex with port label a and some vertex with port label b that do not already exist;
- the port relabelling operation relab_h , for $h : [k] \rightarrow [k]$, changing each port label a to the port label $h(a)$.

Each term using these operations defines a graph with ports in $[k]$. The clique-width of a graph $G = \langle V, E \rangle$ is the least number k such that, for some function $\pi : V \rightarrow [k]$, there exists a term denoting $\langle G, \pi \rangle$ (for details cf. [7, 8, 9]). We denote the clique width of G by $\text{cwd}(G)$.

²In [7] such graphs are called *uniformly r-sparse*.

³For a detailed discussion of concrete graphs versus graphs defined up to isomorphism, see Section 2.2 of [7]

2.2. Monadic second-order logic. Monadic second-order logic (MSO)⁴ is the extension of first-order logic by set variables and quantifiers over such variables. The *quantifier-rank* $\text{qr}(\varphi)$ of an MSO-formula φ is the maximal number of nested quantifiers in φ , where we count both, first-order and second-order quantifiers. The *monadic second-order theory* of quantifier rank h of a structure \mathfrak{A} is the set of all MSO-formulae of quantifier rank h satisfied by \mathfrak{A} . We denote it by $\text{MTh}_h(\mathfrak{A})$. Frequently, we are interested not in the theory of the structure \mathfrak{A} itself, but in the theory of an expansion $\langle \mathfrak{A}, \bar{P}, \bar{a} \rangle$ by unary predicates \bar{P} and constants \bar{a} . In this case we write $\text{MTh}_h(\mathfrak{A}, \bar{P}, \bar{a})$ omitting the brackets. Note that such situations are the only ones in which we allow constants in structures.

Let us remark that, for a fixed signature and a given maximal quantifier-rank, there are only finitely many formulae up to logical equivalence. Furthermore, we can effectively compute an upper bound on the number of classes and there exists an effective normal form for formulae. However, since equivalence of formulae is undecidable, this normal form does not represent logical equivalence. Hence, some equivalence classes contain several formulae in normal form. Details can be found, e.g., in Section 5.6 of [7]. In particular, it follows that, for every $h \in \mathbb{N}$, there are only finitely many theories of quantifier-rank h and we can represent each such theory by the finite set of formulae in normal form it contains. A detailed calculation shows that the number of such theories is roughly $\exp_h(n)$ where

$$\exp_0(n) := n \quad \text{and} \quad \exp_{k+1}(n) := 2^{\exp_k(n)}$$

and the number n only depends on the signature, but not on the quantifier-rank h . Recall that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is *elementary* if it is bounded from above by a function of the form \exp_k , for some fixed $k \in \mathbb{N}$. Furthermore, it follows that we can construct, for each theory Θ of quantifier-rank h , a single formula χ_Θ that is equivalent to it, i.e., such that

$$\mathfrak{A} \models \chi_\Theta \iff \text{MTh}_h(\mathfrak{A}) = \Theta.$$

In fact, χ_Θ is just the conjunction of all formulae in normal form contained in Θ . For this reason we will also denote it by $\bigwedge \Theta$.

Let $\varphi(\bar{x}, \bar{Y}; \bar{Z})$ be an MSO-formula with free first-order variables \bar{x} and free second-order variables \bar{Y}, \bar{Z} . Given a structure \mathfrak{A} and sets $P_i \subseteq A$, we can assign the values \bar{P} to the variables \bar{Z} . This way we obtain a formula $\varphi(\bar{x}, \bar{Y}; \bar{P})$ with partially assigned variables. The values \bar{P} are called the *parameters* of this formula. The *relation defined* by a formula $\varphi(\bar{x}; \bar{P})$ in a structure \mathfrak{A} is the set

$$\varphi(\bar{x}; \bar{P})^{\mathfrak{A}} := \{ \bar{a} \mid \mathfrak{A} \models \varphi(\bar{a}; \bar{P}) \}.$$

One important tool to compute monadic theories is the so-called Composition Theorem (see, e.g, [17, 1, 7]), which allows one to compute the theory of a structure composed from smaller parts from the theories of these parts. There are several variants of the Composition Theorem. We will employ the following version.

Definition 2.2. Let $\mathfrak{A}_0, \dots, \mathfrak{A}_{m-1}$ be structures and let $\bar{a}^i = \langle a_0^i, \dots, a_{n-1}^i \rangle \in A_i^n$ be n -tuples, for $i < m$. The *amalgamation* of the structures \mathfrak{A}_i over the parameters \bar{a}^i is the structure $\langle \mathfrak{A}', \bar{a}' \rangle$ obtained from the disjoint union $\mathfrak{A}_0 \oplus \dots \oplus \mathfrak{A}_{m-1}$ by, for every $k < n$,

⁴There is also counting monadic second-order logic (CMSO) which extends MSO by set predicates of the form $\text{Card}_q(X)$ expressing that the cardinality of X is a multiple of q . Although our results are stated and proved for MSO, they also hold for CMSO: the technical core of our proofs is the composition theorem which holds for CMSO as well. We currently do not have an example of a class of structures that is CMSO-orderable but not MSO-orderable, but it seems likely that such classes do exist.

merging the elements a_k^0, \dots, a_k^{m-1} into a single element a'_k . The tuple $\bar{a}' = \langle a'_0, \dots, a'_{n-1} \rangle$ consists of the elements resulting from the merging. \diamond

Theorem 2.3 (Composition Theorem). *Let $\mathfrak{A}_0, \dots, \mathfrak{A}_{m-1}$ be structures and, for $i < m$, let $\bar{a}_i \in A_i^n$ be n -tuples and $\bar{c}_i \in A_i^{l_i}$ l_i -tuples. Let $\langle \mathfrak{A}', \bar{a}' \rangle$ be the amalgamation of the structures \mathfrak{A}_i over \bar{a}_i . Then*

$$\text{MTh}_h(\mathfrak{A}', \bar{a}'\bar{c}_0 \dots \bar{c}_{m-1})$$

is uniquely determined by the theories

$$\text{MTh}_h(\mathfrak{A}_0, \bar{a}_0\bar{c}_0), \dots, \text{MTh}_h(\mathfrak{A}_{m-1}, \bar{a}_{m-1}\bar{c}_{m-1}).$$

Furthermore, the function mapping these theories to the theory of the amalgamation is computable.

Since disjoint unions are particular amalgamations, we obtain the following corollary.

Corollary 2.4. *There exists an computable function mapping $\text{MTh}_h(\mathfrak{A})$ and $\text{MTh}_h(\mathfrak{B})$ to $\text{MTh}_h(\mathfrak{A} \oplus \mathfrak{B})$.*

2.3. Transductions. The notion of a monadic second-order transduction provides a versatile framework to define transformations of structures. To simplify the definition we first introduce three particular types of transductions and we obtain MSO-transductions as compositions of these.

Definition 2.5. (a) Let $k \geq 2$ be a natural number. The operation copy_k maps a structure \mathfrak{A} to the expansion

$$\text{copy}_k(\mathfrak{A}) := \langle \mathfrak{A} \oplus \dots \oplus \mathfrak{A}, \sim, P_0, \dots, P_{k-1} \rangle$$

of the disjoint union of k copies of \mathfrak{A} by the following relations. Denoting the copy of an element $a \in A$ in the i -th component of $\mathfrak{A} \oplus \dots \oplus \mathfrak{A}$ by the pair $\langle a, i \rangle$, we define

$$P_i := \{ \langle a, i \rangle \mid a \in A \} \quad \text{and} \quad \langle a, i \rangle \sim \langle b, j \rangle \iff a = b.$$

For $k = 1$, we set $\text{copy}_1(\mathfrak{A}) := \mathfrak{A}$.

(b) For $m \in \mathbb{N}$, we define the multi-valued operation exp_m that maps a structure \mathfrak{A} to all of its possible expansions by m unary predicates $Q_0, \dots, Q_{m-1} \subseteq A$. Note that exp_0 is just the identity.

(c) A *basic MSO-transduction* is a partial operation τ on relational structures described by a list

$$\langle \chi, \delta(x), \varphi_0(\bar{x}), \dots, \varphi_{s-1}(\bar{x}) \rangle$$

of MSO-formulae called the *definition scheme* of τ . Given a structure \mathfrak{A} that satisfies the sentence χ , the operation τ produces the structure

$$\tau(\mathfrak{A}) := \langle D, R_0, \dots, R_{s-1} \rangle$$

where

$$D := \{ a \in A \mid \mathfrak{A} \models \delta(a) \} \quad \text{and} \quad R_i := \{ \bar{a} \in D^{e_i} \mid \mathfrak{A} \models \varphi_i(\bar{a}) \}.$$

(e_i is the arity of R_i .) If $\mathfrak{A} \not\models \chi$ then $\tau(\mathfrak{A})$ is undefined.

(d) A *quantifier-free transduction* is a basic MSO-transduction, where all formulae in the definition scheme are quantifier free.

(e) A *k-copying MSO-transduction* τ is a (multi-valued) operation on relational structures of the form $\tau_0 \circ \text{copy}_k \circ \text{exp}_m$ where τ_0 is a basic MSO-transduction. When the value of k does not matter, we will simply speak of a *transduction*.

Due to exp_m , a structure can be mapped to several structures by τ . Consequently, we define $\tau(\mathfrak{A})$ as the *set* of possible values $(\tau_0 \circ \text{copy}_k)(\mathfrak{A}, \bar{P})$ where \bar{P} ranges over all m -tuples of subsets of A .

(f) An MSO-transduction τ is *domain-preserving* if, it is 1-copying and, for every structure \mathfrak{A} such that $\tau(\mathfrak{A})$ is defined, the image $\tau(\mathfrak{A})$ has the same universe as \mathfrak{A} . \diamond

Remark 2.6. (a) The expansion by m unary predicates corresponds, in the terminology of [3, 6], to using m *parameters*.

(b) Note that every basic MSO-transduction is a 1-copying MSO-transduction without parameters. \diamond

The most important property of MSO-transductions is the fact that they are compatible with MSO-theories in the following sense (see, e.g., Theorem 5.10 of [7]).

Lemma 2.7 (Backwards Translation). *Let τ be a transduction. For every MSO-sentence φ , there exists an MSO-sentence φ^τ such that, for all structures \mathfrak{A} ,*

$$\mathfrak{A} \models \varphi^\tau \iff \mathfrak{B} \models \varphi \text{ for some } \mathfrak{B} \in \tau(\mathfrak{A}).$$

Furthermore, if τ is *quantifier-free*, then the *quantifier-rank* of φ^τ is no larger than that of φ .

Corollary 2.8. *Let τ be a quantifier-free transduction and \mathfrak{A} and \mathfrak{B} structures.*

$$\text{MTh}_h(\mathfrak{A}) = \text{MTh}_h(\mathfrak{B}) \text{ implies } \text{MTh}_h(\tau(\mathfrak{A})) = \text{MTh}_h(\tau(\mathfrak{B})).$$

2.4. Equational classes and the Semi-Linearity Theorem. We can use monadic second-order transductions to define two important families of graph classes: the *HR-equational* and the *VR-equational* classes of graphs.

The family \mathcal{VR} of *VR-equational* graph classes consists of all classes \mathcal{C} such that $[\mathcal{C}]$ is the image of the class \mathcal{T} of all trees under a monadic second-order transduction. Similarly, the family \mathcal{HR} of *HR-equational* graph classes consists of all classes \mathcal{C} such that $[\mathcal{C}]$ is the image of \mathcal{T} under a monadic second-order transduction.

Both families can alternatively be defined using systems of equations in a corresponding graph algebra: the VR-equational classes are the solutions of systems of equations over the VR-algebra of graphs, i.e., the graph algebra whose operations define clique-width, and the HR-equational classes are the solutions of systems of equations over the HR-algebra of graphs, i.e., the graph algebra whose operations define tree-width. We recall that every HR-equational class (of simple graphs) is VR-equational.

VR-equationality and HR-equationality are two possible generalisations of the notion of a context-free language to graphs. In light of the alternative definition in terms of systems of equations it is not surprising that there is a close connection between VR-equationality and clique-width and between HR-equationality and tree-width. Every class in \mathcal{VR} has bounded clique-width, while classes in \mathcal{HR} have bounded tree-width. Conversely, every MSO_1 -definable class of graphs of bounded clique-width is VR-equational and every MSO_2 -definable class of graphs of bounded tree-width is HR-equational. However, some VR-equational or HR-equational classes are not of this form. This corresponds to the fact that some context-free languages are not regular.

There is a third characterisation of \mathcal{VR} and \mathcal{HR} in terms of graph grammars. VR-equational classes can be generated by *vertex replacement* grammars, while HR-equational classes can be generated by *hyperedge replacement* grammars. We refer the reader to the book [7] for details. In the present article, we will only consider such classes specified, as defined above, as images of trees under transductions. Note that the definition scheme of a class \mathcal{C} provides a finite representation of \mathcal{C} . Consequently, we can process VR-equational and HR-equational classes by algorithms and we can state decision problems in a meaningful way.

One important property of a VR-equational class \mathcal{C} is the fact that the spectrum of every MSO-definable set predicate inside \mathcal{C} is semi-linear. Recall that a set $S \subseteq \mathbb{N}^n$ is *semi-linear* if it is a finite union of sets of the form

$$P = \{ \bar{k} + i_0 \bar{p}_0 + \dots + i_{m-1} \bar{p}_{m-1} \mid i_0, \dots, i_{m-1} \in \mathbb{N} \},$$

for fixed tuples $\bar{k}, \bar{p}_0, \dots, \bar{p}_{m-1} \in \mathbb{N}^n$.

The following result is Theorem 7.42 of [7] (the fact that one can compute a representation of the semi-linear set is not stated explicitly in [7], but it follows from the proof since all of its steps are effective).

Theorem 2.9 (Semi-Linearity Theorem). *Let \mathcal{C} be a VR-equational class of graphs and let $\varphi(X_0, \dots, X_{n-1})$ be an MSO-formula. The set*

$$M_\varphi(\mathcal{C}) := \{ (|P_0|, \dots, |P_{n-1}|) \mid \lfloor G \rfloor \models \varphi(\bar{P}) \text{ for some } G = \langle V, E \rangle \in \mathcal{C} \\ \text{and } P_0, \dots, P_{n-1} \subseteq V \}$$

is semi-linear, and a finite representation of this set can be computed from φ and a representation of \mathcal{C} .

3. DEFINABLE ORDERS

For simplicity, we will use the term *order* for linear orders. When considering non-linear partial orders, we will explicitly speak of *partial orders*.

Definition 3.1. Let Σ be a relational signature and \mathcal{C} a class of Σ -structures.

(a) An MSO-formula $\varphi(x, y; \bar{Z})$ *defines an order* on \mathcal{C} if, for every non-empty structure $\mathfrak{A} \in \mathcal{C}$, there are sets $P_0, \dots, P_{n-1} \subseteq A$ such that the formula $\varphi(x, y; \bar{P})$ defines an order on \mathfrak{A} .

(b) The class \mathcal{C} is *MSO-orderable* if there is an MSO-formula φ defining an order on \mathcal{C} .

(c) A class \mathcal{C} of graphs is *MSO₁-orderable* if the class $\lfloor \mathcal{C} \rfloor$ is MSO-orderable, and we call it *MSO₂-orderable* if $\lceil \mathcal{C} \rceil$ is MSO-orderable. \diamond

Remark 3.2. (a) For orderability by a formula $\varphi(x, y; \bar{Z})$, we only require that there are *some* parameters \bar{P} such that $\varphi(x, y; \bar{P})$ defines an order. We do not care about the behaviour of φ for other values of the parameters. We could require the formula $\varphi(x, y; \bar{P}')$ to be always false for such parameters \bar{P}' . This is no loss of generality, as we can replace $\varphi(x, y; \bar{Z})$ by the formula

$$\varphi(x, y; \bar{Z}) \wedge \text{ord}_\varphi(\bar{Z}),$$

where the formula

$$\begin{aligned} \text{ord}_\varphi(\bar{Z}) := & \forall x \forall y [\varphi(x, y; \bar{Z}) \wedge \varphi(y, x; \bar{Z}) \leftrightarrow x = y] \\ & \wedge \forall x \forall y \forall z [\varphi(x, y; \bar{Z}) \wedge \varphi(y, z; \bar{Z}) \rightarrow \varphi(x, z; \bar{Z})] \end{aligned}$$

states that the relation defined by φ with parameters \bar{Z} is an order.

(b) For every MSO-formula $\varphi(x, y; \bar{Z})$ there exists a largest class \mathcal{C}_φ of Σ -structures that is ordered by φ . This class can be defined by $\exists \bar{Z} \text{ord}_\varphi(\bar{Z})$. Fixing an enumeration $\varphi_0(x, y; \bar{Z}), \dots, \varphi_{n-1}(x, y; \bar{Z})$ of all MSO-formulae of quantifier-rank m with k parameters Z_0, \dots, Z_{k-1} , we obtain the class $\mathcal{C}_{m,k}$ of all Σ -structures ordered by some of these formulae. It is defined by $\exists \bar{Z} \bigvee_{i < n} \text{ord}_{\varphi_i}(\bar{Z})$. This class can be ordered by the formula

$$\psi_{m,k}(x, y; \bar{Z}) := \bigvee_{i < n} \left[\bigwedge_{j < i} \neg \text{ord}_{\varphi_j}(\bar{Z}) \wedge \text{ord}_{\varphi_i}(\bar{Z}) \wedge \varphi_i(x, y; \bar{Z}) \right].$$

It follows that any MSO-orderable class \mathcal{C} can be ordered by $\psi_{m,k}$ for sufficiently large m and k . \diamond

Remark 3.3. By definition, a class is MSO₂-orderable if, in each graph $G = \langle V, E \rangle$, we can define a order on the set $V \cup E$. This is in fact equivalent to requiring just an order on the set V of vertices since, for simple graphs, any such order induces one on $V \cup E$. For instance, we can require that every vertex is smaller than all edges, and that an edge (u, v) is smaller than an edge (u', v') (orienting these pairs such that $u < v$ and $u' < v'$) if either $u < u'$, or $u = u'$ and $v < v'$. \diamond

Proposition 3.4. *Let \mathcal{C} and \mathcal{K} be non-empty classes of Σ -structures.*

- (a) $\mathcal{C} \cup \mathcal{K}$ is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.
- (b) $\mathcal{C} \oplus \mathcal{K} := \{ \mathfrak{A} \oplus \mathfrak{B} \mid \mathfrak{A} \in \mathcal{C}, \mathfrak{B} \in \mathcal{K} \}$ is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.

Proof. (a) Clearly, if φ defines an order on $\mathcal{C} \cup \mathcal{K}$, it also defines orders on \mathcal{C} and on \mathcal{K} . Conversely, let $\varphi(x, y; \bar{Z})$ and $\psi(x, y; \bar{Z}')$ be MSO-formulae defining an order on, respectively, \mathcal{C} and \mathcal{K} . Let $\text{ord}_\varphi(\bar{Z})$ be the formula (of quantifier-rank $\text{qr}(\varphi) + 3$) from Remark 3.2 stating that the relation defined by φ with parameters \bar{Z} is an order. Then we can order $\mathcal{C} \cup \mathcal{K}$ by the formula

$$\vartheta(x, y; \bar{Z}, \bar{Z}') := [\text{ord}_\varphi(\bar{Z}) \wedge \varphi(x, y; \bar{Z})] \vee [\neg \text{ord}_\varphi(\bar{Z}) \wedge \psi(x, y; \bar{Z}')].$$

(b) First, suppose that \mathcal{C} and \mathcal{K} are ordered by the formulae $\varphi(x, y; \bar{Z})$ and $\psi(x, y; \bar{Z}')$, respectively. We order $\mathcal{C} \oplus \mathcal{K}$ as follows. Consider $\mathfrak{A} \oplus \mathfrak{B} \in \mathcal{C} \oplus \mathcal{K}$ and let \bar{P} and \bar{Q} be the parameters used by φ and ψ to order \mathfrak{A} and \mathfrak{B} , respectively. Using the set B as one additional parameter, we can define the order

$$\begin{aligned} x \leq y \quad & :\iff x, y \in A \text{ and } \mathfrak{A} \models \varphi(x, y; \bar{P}) \\ & \text{or } x, y \in B \text{ and } \mathfrak{B} \models \psi(x, y; \bar{Q}) \\ & \text{or } x \in A \text{ and } y \in B. \end{aligned}$$

Conversely, suppose that there is a formula $\varphi(x, y; \bar{Z})$ ordering $\mathcal{C} \oplus \mathcal{K}$. We construct a formula $\psi(x, y; \bar{Z})$ ordering \mathcal{C} . (The orderability of \mathcal{K} follows by symmetry.) By the Composition Theorem, there exist finite lists p_0, \dots, p_{m-1} , q_0, \dots, q_{m-1} , and s_0, \dots, s_{n-1} ,

t_0, \dots, t_{n-1} of MSO-theories of quantifier-rank $h := \text{qr}(\varphi)$ and $h+3 = \text{qr}(\text{ord}_\varphi)$, respectively, such that, for all $\mathfrak{A} \in \mathcal{C}$, $\mathfrak{B} \in \mathcal{K}$, \bar{P} in $\mathfrak{A} \oplus \mathfrak{B}$, and $a, b \in A$,

$$\begin{aligned} \mathfrak{A} \oplus \mathfrak{B} \models \varphi(a, b; \bar{P}) &\iff \text{MTh}_h(\mathfrak{A}, \bar{P} \upharpoonright A, a, b) = p_i \text{ and} \\ &\text{MTh}_h(\mathfrak{B}, \bar{P} \upharpoonright B) = q_i, \text{ for some } i < m, \end{aligned}$$

$$\begin{aligned} \text{and } \mathfrak{A} \oplus \mathfrak{B} \models \text{ord}_\varphi(\bar{P}) &\iff \text{MTh}_{h+3}(\mathfrak{A}, \bar{P} \upharpoonright A) = s_i \text{ and} \\ &\text{MTh}_{h+3}(\mathfrak{B}, \bar{P} \upharpoonright B) = t_i, \text{ for some } i < n. \end{aligned}$$

We fix a structure $\mathfrak{B}_0 \in \mathcal{K}$ and set

$$I := \{i < n \mid \mathfrak{B}_0 \models \exists \bar{Z} \wedge t_i(\bar{Z})\}.$$

For each $i \in I$, we choose parameters \bar{Q}_i in \mathfrak{B}_0 such that $\text{MTh}_{h+3}(\mathfrak{B}_0, \bar{Q}_i) = t_i$, and we set

$$J_i := \{j < m \mid \text{MTh}_h(\mathfrak{B}_0, \bar{Q}_i) = q_j\}.$$

We claim that the formula

$$\psi(x, y; \bar{Z}) := \bigvee_{i \in I} \left[\bigwedge_{\substack{k \in I \\ k < i}} \neg \vartheta_k(\bar{Z}) \wedge \vartheta_i(\bar{Z}) \wedge \bigvee_{j \in J_i} \chi_j(x, y; \bar{Z}) \right]$$

orders \mathcal{C} where $\vartheta_i(\bar{Z}) := \bigwedge s_i$ and $\chi_i(x, y; \bar{Z}) := \bigwedge p_i$. Let $\mathfrak{A} \in \mathcal{C}$ and let $l \in I$ be the minimal index such that $\mathfrak{A} \models \exists \bar{Z} \vartheta_l(\bar{Z})$. We choose sets \bar{P} in \mathfrak{A} such that $\text{MTh}_{h+3}(\mathfrak{A}, \bar{P}) = s_l$. By choice of s_l and t_l it follows that $\varphi(x, y; \bar{P} \cup \bar{Q}_l)$ orders $\mathfrak{A} \oplus \mathfrak{B}_0$. ($\bar{P} \cup \bar{Q}_l$ denotes the tuple where each component is the union of the corresponding components of \bar{P} and \bar{Q}_l .) For $a, b \in A$, it further follows that

$$\begin{aligned} \mathfrak{A} \models \psi(a, b; \bar{P}) &\iff \text{there is some } i \in I \text{ such that} \\ &\text{MTh}_{h+3}(\mathfrak{A}, \bar{P}) = s_i, \\ &\text{MTh}_{h+3}(\mathfrak{A}, \bar{P}) \neq s_k, \text{ for all } k < i, \text{ and} \\ &\text{MTh}_h(\mathfrak{A}, \bar{P}, a, b) = p_j, \text{ for some } j \in J_i, \\ &\iff \text{MTh}_h(\mathfrak{A}, \bar{P}, a, b) = p_j, \text{ for some } j \in J_l, \\ &\iff \text{there is some } j < m \text{ such that} \\ &\text{MTh}_h(\mathfrak{A}, \bar{P}, a, b) = p_j \text{ and } \text{MTh}_h(\mathfrak{B}_0, \bar{Q}_l) = q_j \\ &\iff \mathfrak{A} \oplus \mathfrak{B}_0 \models \varphi(a, b; \bar{P} \cup \bar{Q}_l). \end{aligned}$$

Hence, $\psi(x, y; \bar{P})$ orders \mathfrak{A} . □

Remark 3.5. Every class consisting of a single (finite) structure is obviously MSO-orderable. By Proposition 3.4, it follows that all finite classes are MSO-orderable. ◇

Remark 3.6. Let \mathcal{C} be a class of graphs and let $\varphi(x, y; \bar{Z})$ be an MSO-formula defining an order on $[\mathcal{C}]$. Let \mathcal{C}_+ be the class of all graphs obtained from graphs in \mathcal{C} by adding edges arbitrarily. Then $[\mathcal{C}_+]$ can be ordered by the formula $\varphi_+(x, y; \bar{Z}, \bar{Z}')$ obtained from $\varphi(x, y; \bar{Z})$ by replacing every atomic formula of the form $\text{inc}(u, v)$ by the formula $\text{inc}(u, v) \wedge v \in \bar{Z}'$, and by relativising every quantifier to the set \bar{Z}' . (If \bar{P} are parameters such that $\varphi(x, y; \bar{P})$ orders the graph $G = \langle V, E \rangle$, then $\varphi_+(x, y; \bar{P}, V \cup E)$ orders every supergraph $G_+ = \langle V, E_+ \rangle$ such that $E_+ \supseteq E$.) ◇

Remark 3.7. Definition 3.1 can be formulated in terms of monadic second-order transductions. A class \mathcal{C} of Σ -structures is MSO-orderable if, and only if, there exists a noncopying, domain-preserving transduction σ mapping each structure $\mathfrak{A} \in \mathcal{C}$ to an expansion $\langle \mathfrak{A}, \leq \rangle$ by a linear order \leq . Moreover it is easy to write down a transduction τ mapping any ordered structure $\langle \mathfrak{A}, \leq \rangle$ to a path that connects all elements of \mathfrak{A} . Consequently, if \mathcal{C} is infinite (up to isomorphism) and MSO-orderable, we obtain an MSO-transduction $\tau \circ \sigma$ mapping \mathcal{C} to the class of all finite paths. This implies that, in the transduction hierarchy (cf. [2]), the class \mathcal{C} lies above the class of all paths. \diamond

The opposite of an orderable class is a class of which no infinite subclass can be ordered. We call such classes *hereditarily unorderable*.

Definition 3.8. A class \mathcal{C} of structures is *hereditarily MSO-unorderable*, if it is infinite and no infinite subclass of \mathcal{C} is MSO-orderable. For classes of graphs, we define the terms *hereditarily MSO₁-unorderable* and *hereditarily MSO₂-unorderable* analogously. \diamond

Example 3.9. (a) The class $\mathcal{C} = \{K_n \mid n \in \mathbb{N}, n > 0\}$ of cliques is MSO₂-orderable and hereditarily MSO₁-unorderable. To order K_n , we can choose a set of edges P forming a Hamiltonian path in K_n . Let Q be a singleton set consisting of one end-point of this path. Then we can use P and Q to define a linear order on K_n .

Without using MSO₂-parameters, such a definition is not possible. For each fixed number k of parameters and all sufficiently large n , every expansion of K_n by k parameters P_0, \dots, P_{k-1} admits a nontrivial automorphism. Consequently, no formula can define a linear order on $\langle K_n, \bar{P} \rangle$.

(b) The class \mathcal{T}_n of trees of height at most n is both, hereditarily MSO₁-unorderable and hereditarily MSO₂-unorderable. This follows from Theorem 4.8 below. \diamond

4. MSO₂-DEFINABLE ORDERINGS

In this section we derive characterisations for MSO₂-orderable classes. MSO₁-orderability will be considered in Section 5.

4.1. Necessary conditions. We start by providing a necessary condition for MSO₂-orderability. Below we will then show that, for certain classes of graphs, this condition is also sufficient.

Definition 4.1. Let $\mathfrak{A} = \langle A, \bar{R} \rangle$ be a relational structure.

(a) We call \mathfrak{A} *connected* if it cannot be written as a disjoint union $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$ of two nonempty substructures. A *connected component* of \mathfrak{A} is a maximal substructure that is connected and nonempty.

(b) For a number $k \in \mathbb{N}$, we denote by $\text{Sep}(\mathfrak{A}, k)$ the maximal number of connected components of $\mathfrak{A} - S$, where $S \subseteq A$ ranges over all sets of size at most k . For a graph G , we set $\text{Sep}(G, k) := \text{Sep}(\lfloor G \rfloor, k)$.

(c) For a function $f : \mathbb{N} \rightarrow \mathbb{N}$, we say that a class \mathcal{C} of structures has property $\text{SEP}(f)$ if

$$\text{Sep}(\mathfrak{A}, k) \leq f(k), \quad \text{for all } \mathfrak{A} \in \mathcal{C} \text{ and all } k \in \mathbb{N}.$$

We say that \mathcal{C} has property SEP , if it has property $\text{SEP}(f)$, for some function $f : \mathbb{N} \rightarrow \mathbb{N}$. \diamond

Example 4.2. For complete bipartite graphs $K_{n,m}$ with $n \leq m$ we have

$$\text{Sep}(K_{n,m}, k) = \begin{cases} 1 & \text{if } k < n, \\ m & \text{if } k \geq n. \end{cases}$$

For complete d -partite graphs $K_{m_0, \dots, m_{d-1}}$ with $m_0 \geq \dots \geq m_{d-1}$ and $d \geq 2$, we have

$$\text{Sep}(K_{m_0, \dots, m_{d-1}}, k) = \begin{cases} 1 & \text{if } k < m_1 + \dots + m_{d-1}, \\ m_0 & \text{if } k \geq m_1 + \dots + m_{d-1}. \end{cases}$$

We leave the straightforward verification to the reader. \diamond

Example 4.3. Let $f : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ be a function and let $n \in \mathbb{N}$. We construct a graph $G_n(f)$ such that

$$\text{Sep}(G_n(f), k) \geq f(k), \quad \text{for all } k \leq n.$$

Let T be the tree of height n , where every vertex v on level k has $f(k)$ immediate successors. That is,

$$T := \{w \in \mathbb{N}^{\leq n} \mid w(k) < f(k) \text{ for all } k\}.$$

The desired graph $G_n(f)$ is obtained from this tree by adding all edges (x, y) such that $x \prec y$. For a given $k \leq n$, choose a path v_0, \dots, v_{k-1} of length $k - 1$ from the root v_0 to some vertex v_{k-1} on level $k - 1$. Removing the set $S := \{v_0, \dots, v_{k-1}\}$ we obtain a graph $G_n(f) - S$ with more than $f(k)$ connected components, since each of the $f(k)$ immediate successors of v_{k-1} belongs to a different connected component. \diamond

Let us show that having property **SEP** is a necessary condition for a class to be MSO_2 -orderable.

Proposition 4.4. *There exists a function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that $\text{Sep}(G, k) \leq f(n, m, k)$ for every graph G such that $\lceil G \rceil$ can be ordered by an MSO -formula of the form $\varphi(x, y; \bar{P})$ where $\text{qr}(\varphi) \leq m$ and $\bar{P} = \langle P_0, \dots, P_{n-1} \rangle$ are parameters. Furthermore, the function $f(n, m, k)$ is effectively elementary in the argument k , that is, there exists a computable function g such that $f(n, m, k) \leq \exp_{g(n, m)}(k)$.*

Proof. Fixing $k, m, n \in \mathbb{N}$, we define $f(n, m, k) := d$ where d is an upper bound on the number of MSO -theories of the form $\text{MTh}_m(\lceil H \rceil, P_0, \dots, P_{n-1}, v_0, \dots, v_k)$ where H is a graph, P_0, \dots, P_{n-1} are parameters, and v_0, \dots, v_k are vertices of H . For fixed n and m , we can choose d to be elementary in k .

Let $\varphi(x, y; \bar{Z})$ be an MSO -formula of quantifier-rank at most m , let G be a graph with $\text{Sep}(G, k) > f(n, m, k)$, and let P_0, \dots, P_{n-1} parameters from G . We have to show that $\varphi(x, y; \bar{P})$ does not order $\lceil G \rceil$. Fix a set $S = \{s_0, \dots, s_{k-1}\}$ of vertices such that $G - S$ has more than d connected components. Fix distinct connected components C_0, \dots, C_d of $G - S$ and vertices $a_i \in C_i$. By choice of d , there are indices $i < j$ such that

$$\begin{aligned} & \text{MTh}_m(\lceil G[C_i \cup S] \rceil, \bar{P} \upharpoonright (C_i \cup S), s_0, \dots, s_{k-1}, a_i) \\ &= \text{MTh}_m(\lceil G[C_j \cup S] \rceil, \bar{P} \upharpoonright (C_j \cup S), s_0, \dots, s_{k-1}, a_j). \end{aligned}$$

As the structure $\langle \lceil G \rceil, \bar{P}, s_0, \dots, s_{k-1}, a_i, a_j \rangle$ is the amalgamation of the structures

$$\begin{aligned} & \langle \lceil G[C_i \cup S] \rceil, \bar{P} \upharpoonright (C_i \cup S), s_0, \dots, s_{k-1}, a_i \rangle, \\ & \langle \lceil G[C_j \cup S] \rceil, \bar{P} \upharpoonright (C_j \cup S), s_0, \dots, s_{k-1}, a_j \rangle, \end{aligned}$$

and $\langle [G[C_l \cup S]], \bar{P} \upharpoonright (C_l \cup S), s_0, \dots, s_{k-1} \rangle$, for $l \neq i, j$,

over the tuple $\langle s_0, \dots, s_{k-1} \rangle$, it therefore follows by Theorem 2.3 that

$$\text{MTh}_m([G], \bar{P}, s_0, \dots, s_{k-1}, a_i, a_j) = \text{MTh}_m([G], \bar{P}, s_0, \dots, s_{k-1}, a_j, a_i).$$

In particular,

$$G \models \varphi(a_i, a_j; \bar{P}) \iff G \models \varphi(a_j, a_i; \bar{P}).$$

Hence, $\varphi(x, y; \bar{P})$ does not define an order. \square

Corollary 4.5. *An MSO_2 -orderable class of graphs \mathcal{C} has property $\text{SEP}(f)$, for an elementary function f .*

The converse does not hold. For instance, according to Theorem 4.29 below, the class of bipartite graphs of the form $K_{n, 2^{2^n}}$ is not MSO_2 -orderable, while we have seen in Example 4.2 that it has property $\text{SEP}(f)$ for the elementary function f such that $f(n) = 2^{2^n}$. Our objective therefore is to get converse results for particular classes of graphs satisfying certain combinatorial conditions.

Remark 4.6. We have noted in Remark 3.6 that, if a graph G can be ordered by an MSO_2 -formula φ , we can construct from φ a MSO_2 -formula ψ ordering every graph H obtained from G by adding edges. In this case, we further have $\text{Sep}(H, k) \leq \text{Sep}(G, k)$, for all k . \diamond

Remark 4.7. All results of Section 4 also hold for directed graphs since there is an MSO_2 -formula with two parameters that defines an orientation of every undirected graph (see Proposition 9.46 of [7]). It follows that a class of directed graphs is MSO_2 -orderable if, and only if, the corresponding class of undirected graphs is. This is different for MSO_1 -orderability. \diamond

As a simple introductory example, let us consider classes of trees.

Theorem 4.8. *Let \mathcal{T} be a class of (undirected) trees. The following statements are equivalent:*

- (1) \mathcal{T} is MSO_1 -orderable.
- (2) \mathcal{T} is MSO_2 -orderable.
- (3) \mathcal{T} has property SEP .
- (4) There exists a number $d \in \mathbb{N}$ such that every tree in \mathcal{T} has maximal degree at most d .

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (3) has been shown in Corollary 4.5.

(3) \Rightarrow (4) Suppose that \mathcal{T} has property $\text{SEP}(f)$ and let $T \in \mathcal{T}$. Every vertex $v \in T$ has at most $f(1)$ neighbours since $T - \{v\}$ has at most $f(1)$ connected components. Consequently, the maximal degree of T is bounded by $f(1)$.

(4) \Rightarrow (1) Let T be a tree with maximal degree at most d . We use d parameters P_0, \dots, P_{d-1} to order T . Fixing a vertex $r \in T$ as root, we obtain an injective embedding $g : T \rightarrow d^{< \omega}$, for some number $m \in \mathbb{N}$. We set

$$P_i := \{v \in T \mid g(v) = wi \text{ for some } w\}.$$

Note that r is the only vertex of T that is not contained in any of these sets. Hence, using \bar{P} , we can define the tree-order \preceq on T . We can also define the lexicographic ordering:

$$u \leq v \quad :\iff \quad u \preceq v, \text{ or } u_0 \in P_i, v_0 \in P_k, \text{ for } i < k, \text{ where } u_0, v_0 \text{ are the} \\ \text{immediate successors of the longest common} \\ \text{prefix of } u \text{ and } v \text{ with } u_0 \preceq u \text{ and } v_0 \preceq v. \quad \square$$

Corollary 4.9. *Let $k \in \mathbb{N}$. The class of trees of height at most k is hereditarily MSO_2 -unorderable.*

Proof. For any given height k , there are only finitely many trees (up to isomorphism) satisfying condition (4) of the theorem. \square

4.2. Omitting a minor. We start by presenting a characterisation for classes of graphs omitting a fixed graph as minor (for an introduction to graph minors see, e.g., [10]). For short, we will say that such a class *omits a minor*. Recall that a spanning forest F of a graph G is defined to be directed. A spanning forest F is *normal* if the ends of every edge of G are comparable with respect to the tree-order \preceq_F on F (see, e.g., Section 1.5 of [10]).

Definition 4.10. Let G be a graph and $F \subseteq G$ a normal spanning forest of G .

(a) We denote the set of predecessors of a vertex x by

$$\text{Pred}_F(x) := \{y \mid y \prec_F x\}.$$

(b) For $x \in G$, we define

$$B_F(x) := \{v \prec_F x \mid \text{there is an edge } (u, v) \text{ of } G \text{ such that } x \preceq_F u\}. \quad \diamond$$

Lemma 4.11. *Let G be a graph, F a normal spanning forest of G , $x \in G$, and $B \subseteq \text{Pred}_F(x)$.*

- (a) *If $|B| \geq p$ and there are p immediate successors y of x such that $B_F(y) = B \cup \{x\}$, then $K_{p,p}$ is a minor of G .*
- (b) *If $|B| < p$ and $\text{Sep}(G, p) \leq d$, then there are at most d immediate successors y of x such that $B_F(y) = B \cup \{x\}$.*

Proof. (a) Suppose that there are p distinct immediate successors y_0, \dots, y_{p-1} of x with $B_F(y_i) = B \cup \{x\}$ and fix distinct vertices $b_0, \dots, b_{p-1} \in B$. Let H be the minor of G obtained by contracting the subtrees rooted at y_0, \dots, y_{p-1} to single vertices $\tilde{y}_0, \dots, \tilde{y}_{p-1}$ and by removing all remaining vertices except for $\tilde{y}_0, \dots, \tilde{y}_{p-1}$ and b_0, \dots, b_{p-1} . Then $H \cong K_{p,p}$.

(b) Set $S := B \cup \{x\}$ and let y_0, \dots, y_{n-1} be an enumeration of all immediate successors of x such that $B_F(y_i) = S$. Then y_0, \dots, y_{n-1} lie in different connected components of $G - S$. Hence, $n \leq \text{Sep}(G, p) \leq d$. \square

Theorem 4.12. *For every $p, d \in \mathbb{N}$, the class $\mathcal{C}_{p,d}$ of all graphs G that satisfy $\text{Sep}(G, p) \leq d$ and that do not contain $K_{p,p}$ as a minor is MSO_2 -orderable.*

Proof. Consider a graph $G \in \mathcal{C}_{p,d}$. Let F be a normal spanning forest of G . Since G has $\text{Sep}(G, 0) \leq d$ connected components, the forest F has at most d roots. Recall that a forest is oriented with edges pointing away from the roots. We can encode F by two parameters: its set of edges and its set of roots. (Since the first set consists of edges and the second one of vertices, we could even take their union as a single parameter.) We will use a lexicographic

order on F to order G , based on orderings (i) of the roots of F and (ii) of the immediate successors of every vertex of F .

Consider a vertex $x \in F$ with immediate successors y_0, \dots, y_{m-1} . Since each set $B_F(y_i)$ is linearly ordered by \preceq_F , we can define a preorder on the immediate successors by using the lexicographic ordering of the sets $B_F(y_i)$:

$$y_i \sqsubseteq y_k \quad :\iff \quad B_F(y_i) \leq_{\text{lex}} B_F(y_k).$$

To prove that there is a definable order extending this preorder, it is sufficient to show that the equivalence classes of this preorder have bounded cardinality. Let $k := \max\{p, d\}$. For every set $B \subseteq \text{Pred}_F(x)$, there are at most k immediate successors y_i of x with $B_F(y_i) = B \cup \{x\}$: for $|B| \geq p$, this follows from Lemma 4.11 (a); for $|B| < p$, it follows from Lemma 4.11 (b).

The parameters needed to define the desired linear order consist of the set of edges of the spanning forest F and $d + k$ parameters to distinguish and order the roots of F and to order the immediate successors y of a vertex x that have the same set $B_F(y)$. \square

Theorem 4.13. *Let \mathcal{C} be a class of graphs omitting a minor H . The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) \mathcal{C} has property SEP .
- (3) \mathcal{C} has property $\text{SEP}(f)$ for some elementary function f .

Furthermore, given H we can compute a number k such that we can replace $\text{SEP}(f)$ by $\text{SEP}(\text{exp}_k)$ in (3).

Proof. (1) \Rightarrow (3) follows by Corollary 4.5 and (3) \Rightarrow (2) is trivial.

For (2) \Rightarrow (1), suppose that \mathcal{C} has property $\text{SEP}(f)$. By Theorem 4.12, all classes $\mathcal{C}_{p,d}$ are MSO_2 -orderable. Since every graph with n vertices and m edges is a minor of $K_{n,m}$, we can choose p sufficiently large such that H is a minor of $K_{p,p}$. Set $d := f(p)$. Then $\mathcal{C} \subseteq \mathcal{C}_{p,d}$ and it follows that \mathcal{C} is also MSO_2 -orderable. \square

Remark 4.14. (a) For each $k \in \mathbb{N}$, the class of graphs of tree-width at most k excludes some (planar) graph as a minor and, hence, it satisfies the conditions of Theorem 4.13.

(b) Although this fact is not directly related to our work, we mention that Grohe has proved that every class of graphs excluding a minor is orderable in least fixed-point logic. It follows that least fixed-point logic captures PTIME on these classes [15, 14]. \diamond

In contrast to Remark 4.14 (a), we have the following result for classes of graphs of bounded n -depth tree-width (which is defined as tree-width, but where we only consider tree decompositions with index trees of height at most n). This graph complexity measure was introduced in [2].

Proposition 4.15. *Let $n, k \in \mathbb{N}$. A class of graphs of n -depth tree-width at most k is MSO_2 -orderable if, and only if, it is finite. Hence, the class of all graphs of n -depth tree-width at most k is hereditarily MSO_2 -unorderable.*

Proof. Let \mathcal{C} be an infinite class of graphs of n -depth tree-width at most k . As we have argued in Remark 3.7, if \mathcal{C} were MSO_2 -orderable, we could define an MSO_2 -transduction mapping it to the class of all finite paths. This is not possible by Theorem 6.4 of [2]. \square

In the following we try to compute a better bound on the function f in Theorem 4.13 (3). We can improve the bound from elementary to singly exponential.

Lemma 4.16. *Let G be a graph such that $\text{Sep}(G, p) \leq d$ and $K_{p,p}$ is not a minor of G . Let F be a normal spanning forest of G and S a set of at most k vertices of G . For every vertex $x \in S$, at most $k + 2^k \cdot \max\{p, d\}$ connected components of $G - S$ contain an immediate successor of x (in F).*

Proof. Let $s_0 \prec_F \cdots \prec_F s_{m-1} = x$ be an enumeration of $\text{Pred}_F(x) \cup \{x\}$. For an immediate successor y of x , we define

$$I(y) := \{i < m \mid \text{there is some } z \in B_F(y) \text{ such that } z \prec_F s_i \text{ and } (i = 0 \text{ or } s_{i-1} \prec_F z)\}.$$

If y and y' are immediate successors of x in different connected components of $G - S$, then $I(y) \cap I(y') = \emptyset$. Consequently, there are at most $m \leq k$ connected components of $G - S$ containing an immediate successor y of x such that $I(y) \neq \emptyset$.

It remains to show that there are at most $2^k \cdot \max\{p, d\}$ components of $G - S$ containing an immediate successor y with $I(y) = \emptyset$. Every such immediate successor y satisfies $B(y) \subseteq S$. Hence, $B(y)$ can take at most $2^m \leq 2^k$ values and, according to Lemma 4.11, for each such value $B \subseteq S$ there are at most $\max\{p, d\}$ immediate successors y with $B(y) = B$. \square

Proposition 4.17. *Let G be a graph such that $\text{Sep}(G, p) \leq d$ and $K_{p,p}$ is not a minor of G . Then*

$$\text{Sep}(G, k) \leq d + k^2 + k2^k \cdot \max\{p, d\}, \quad \text{for } k \geq p.$$

Proof. Let F be a normal spanning forest of G and S a set of at most k vertices of G . We have seen in Lemma 4.16 that, for every vertex $x \in S$, at most $k + 2^k \cdot \max\{p, d\}$ connected components of $G - S$ contain an immediate successor of x . Since every connected component of $G - S$ contains a root of F or the immediate successor of some $x \in S$, there are at most $d + k(k + 2^k \cdot \max\{p, d\})$ such components. \square

Every class omitting some minor H also omits $K_{p,p}$ as a minor, for all sufficiently large p . The following corollary states that, in order to determine whether such a class is MSO_2 -orderable, it is sufficient to bound the numbers $\text{Sep}(G, p)$ as opposed to the function $k \mapsto \text{Sep}(G, k)$.

Corollary 4.18. *Let $p \in \mathbb{N}$. A class \mathcal{C} of graphs omitting $K_{p,p}$ as a minor is MSO_2 -orderable if, and only if,*

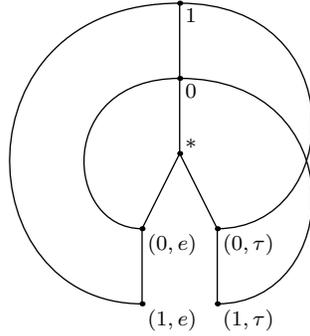
$$\sup \{ \text{Sep}(G, p) \mid G \in \mathcal{C} \} < \infty.$$

Remark 4.19. Graphs omitting a minor H are r -sparse (cf. Definition 2.1), for some number r depending on H . Since, for r -sparse graphs, the expressive powers of MSO_1 and MSO_2 coincide, it follows that the criterion in Corollary 4.18 also characterises MSO_1 -orderability. \diamond

Remark 4.20. The proof technique of Theorem 4.12 can be extended to order certain classes of graphs that do not omit any graph as a minor. We give two examples.

(a) First, let us consider the class of graphs H_p , for $p \geq 1$, defined as follows. The set of vertices of H_p is

$$V := \{*\} \cup [p] \cup [p] \times S_p,$$


 Figure 1: The graph H_2 .

where S_p is the set of permutations of $[p]$. The graph H_p has the following edges:

$$\begin{aligned}
 & (*, 0) \\
 & (*, (0, \sigma)) \quad \text{for } \sigma \in S_p, \\
 & (i, i+1) \quad \text{for } i \in [p], i < p-1, \\
 & ((i, \sigma), (i+1, \sigma)) \quad \text{for } i \in [p], \sigma \in S_p, i < p-1, \\
 & (i, (\sigma(i), \sigma)) \quad \text{for } i \in [p], \sigma \in S_p, i < p.
 \end{aligned}$$

The graph H_2 is shown in Figure 1. (e is the identity and τ is the transposition of 0 and 1.) Note that the vertex $*$ has degree $1+p!$. Clearly, H_p contains $K_{p,p!}$ as a minor. Nevertheless, the class of graphs H_p is MSO_2 -orderable. We can use a spanning tree whose root is the vertex $p-1$ and whose edges consist of the first four of the above types. To compare two immediate successors $(0, \sigma)$ and $(0, \tau)$ of the vertex $*$, we can use a lexicographic order on S_p (where we identify a permutation σ with the sequence $\sigma(0) \dots \sigma(p-1)$). Since each H_p is 2-sparse (as it has an orientation of indegree 2, cf. Proposition 9.40 of [7]), it follows that the class is even MSO_1 -orderable (cf. Theorem 9.37 of [7]).

(b) Another example is the class of cliques. It is MSO_2 -orderable and does not omit a minor. If we replace each edge by a path of length 2, we obtain a class of 2-sparse graphs that is MSO_2 -orderable and that still does not omit a minor. \diamond

Remark 4.21. It is not possible to extend Theorem 4.13 to r -sparse graphs. A counterexample is given by the class \mathcal{C} of all graphs obtained from a bipartite graph of the form $K_{n,f(n)}$ by replacing every edge by a path of length 2, where $f : \mathbb{N} \rightarrow \mathbb{N}$ is a fixed non-elementary function. This is a class of 2-sparse graphs with property SEP that, according to Corollary 4.5, is not MSO_2 -orderable. \diamond

4.3. Deciding MSO_2 -orderability. In Theorem 4.13 above, we have presented a combinatorial property characterising MSO_2 -orderability for classes of graphs omitting a minor. A natural question is whether this property is decidable. Of course, this question does only make sense for classes of graphs that can be described in a finitary way. Therefore, we will concentrate on HR-equational and VR-equational classes.

Proposition 4.22. *It is decidable whether a VR-equational class \mathcal{C} has property SEP.*

Proof. Let \mathcal{C} be a VR-equational class and let $\varphi(X, Y)$ be an MSO-formula expressing, for a graph G , that the set Y contains exactly one vertex of each connected component of $G - X$. The class \mathcal{C} has property **SEP** if, and only if, there exists a function f such that, for all $G = \langle V, E \rangle \in \mathcal{C}$ and $P, Q \subseteq V$,

$$G \models \varphi(P, Q) \quad \text{implies} \quad |Q| \leq f(|P|).$$

According to the Semi-Linearity Theorem, the set

$$M(\mathcal{C}) := \{ (|P|, |Q|) \mid G \models \varphi(P, Q) \text{ for some } G = \langle V, E \rangle \in \mathcal{C} \text{ and } P, Q \subseteq V \}$$

is semi-linear and an effective description of $M(\mathcal{C})$ can be computed from a system of equations for \mathcal{C} . Using this description, we can check whether or not, for every $n \in \mathbb{N}$, the set $\{p \mid (n, p) \in M(\mathcal{C})\}$ is bounded. This is the case if, and only if, \mathcal{C} has property **SEP**. \square

Corollary 4.23. *For an HR-equational class \mathcal{C} , it is decidable whether \mathcal{C} is MSO_2 -orderable.*

Proof. An HR-equational class \mathcal{C} has bounded tree-width (Proposition 4.7 of [7]) and, hence, omits some $K_{p,p}$ as a minor. Since HR-equational classes (of simple graphs) are VR-equational, it follows from Theorem 4.13 that \mathcal{C} is MSO_2 -orderable if, and only if, it has property **SEP**. The latter is decidable by the above proposition. \square

Remark 4.24. An alternative decidability proof can be based on Corollary 4.18. As the tree-width of $K_{p,p}$ is p , every class \mathcal{C} of tree-width at most $p - 1$ omits $K_{p,p}$ as a minor. Furthermore, an upper bound on the tree-width of an HR-equational class \mathcal{C} can be computed from a system of equations for \mathcal{C} (see Proposition 4.7 of [7]). By Corollary 4.18, \mathcal{C} is MSO_2 -orderable if, and only if, the set $\{\text{Sep}(G, p) \mid G \in \mathcal{C}\}$ is bounded. To check this condition, we consider the formula $\varphi(X)$ expressing that there exists a set S of size $|S| \leq p$ such that X contains exactly one vertex of each connected component of $G - S$. By the Semi-Linearity Theorem, we can compute a representation of the semi-linear set

$$M(\mathcal{C}) := \{ |P| \mid G \models \varphi(P) \text{ for some } G = \langle V, E \rangle \in \mathcal{C} \text{ and } P \subseteq V \}.$$

Using this representation we can check whether or not $M(\mathcal{C})$ is finite. \diamond

For VR-equational classes we do not obtain decidability since we cannot apply Theorem 4.13. We conjecture that a corresponding statement holds also for these classes.

Conjecture 4.25. Every VR-equational class that has property **SEP** is MSO_2 -orderable.

Below we will prove this conjecture for the special cases of complete d -partite graphs (Corollary 4.30) and chordal graphs (Corollary 4.37).

4.4. Dense graphs. We have characterised MSO_2 -orderability in Theorem 4.13 for classes excluding a minor. The graphs in such classes are sparse. In this section and the next one, we consider the opposite extreme of certain dense graphs, in particular, multi-partite graphs and chordal graphs.

Lemma 4.26. *Let $s, r \in \mathbb{N}$ and let \mathcal{C} be a class of graphs such that each $G \in \mathcal{C}$ is obtained from some $K_{n,m}$ with $n \leq m \leq 2^{sn+r}$ by possibly adding new edges. Then \mathcal{C} is MSO_2 -orderable.*

Proof. Consider a graph $G = \langle V, E \rangle \in \mathcal{C}$ obtained by adding new edges from a bipartite graph $K_{n,m}$ where $n \leq m \leq 2^{sn+r}$ (see also Remark 3.6). If $n = 0$, then G has $m \leq 2^r$ vertices and we can order G using r parameters. Thus, it remains to consider the case where $n > 0$. Since $m \leq 2^{sn+r} \leq 2^{(s+r)n}$, there exists an injective function $\mu : [m] \rightarrow \mathcal{P}([(s+r)n])$. Fixing enumerations a_0, \dots, a_{n-1} and b_0, \dots, b_{m-1} of the two vertex classes of $K_{n,m}$, we define an ordering of G using the following parameters.

$$\begin{aligned} A &:= \{a_i \mid i < n\} \subseteq V, \\ B &:= \{b_i \mid i < m\} \subseteq V, \\ S &:= \{(a_i, b_j) \mid i \leq j\} \subseteq E, \\ R_k &:= \{(a_i, b_j) \mid kn + i \in \mu(j)\} \subseteq E, \quad \text{for } k < s + r. \end{aligned}$$

First, we define a strict order $<_A$ on A by

$$u <_A v \quad :\iff \quad u \neq v \text{ and, for all } x \in B, (u, x) \in S \Rightarrow (v, x) \in S.$$

By definition of S , this order is linear. We extend it to all vertices of G by defining $u < v$ if, and only if, one of the following conditions holds:

- $u, v \in A$ and $u <_A v$.
- $u \in A$ and $v \in B$.
- $u, v \in B$, $u \neq v$, and, if k is the minimal number such that, for some $x \in A$,

$$(x, u) \in R_k \Leftrightarrow (x, v) \notin R_k,$$

and if $x \in A$ is the $<_A$ -least element with this property, then $(x, u) \in R_k$ and $(x, v) \notin R_k$. \square

The technique employed in this proof will be used several times in this article. Given an already defined order on a set A , we can order the vertices not in A using the lexicographic ordering on their sets of neighbours in A .

Lemma 4.27. *A class \mathcal{C} of complete bipartite graphs is MSO₂-orderable if, and only if, there exists a constant s such that*

$$K_{n,m} \in \mathcal{C} \text{ with } n \leq m \text{ implies } m \leq 2^{s(n+1)}.$$

Proof. (\Leftarrow) is a special case of Lemma 4.26.

(\Rightarrow) Suppose that \mathcal{C} is ordered by an MSO-formula $\varphi(x, y; \bar{Z})$ with s set variables Z_0, \dots, Z_{s-1} . We claim that there is no $K_{n,m} \in \mathcal{C}$ such that $m > 2^{s(n+1)}$.

For a contradiction, suppose that there is such a graph $K_{n,m} \in \mathcal{C}$. Let \bar{P} be the parameters such that $\varphi(x, y; \bar{P})$ orders $[K_{n,m}]$. We enumerate the two vertex sets of $K_{n,m}$ as a_0, \dots, a_{n-1} and b_0, \dots, b_{m-1} . Since $m > 2^{s(n+1)}$ there is a subset $I \subseteq [m]$ of cardinality $|I| > 2^{s(n+1)}/2^s = 2^{sn}$ such that

$$b_i \in P_l \Leftrightarrow b_j \in P_l \quad \text{for all } i, j \in I \text{ and all } l < s.$$

Similarly, there is a subset $J \subseteq I$ of cardinality $|J| > 2^{sn}/2^{sn} = 1$ such that

$$(a_k, b_i) \in P_l \Leftrightarrow (a_k, b_j) \in P_l \quad \text{for all } i, j \in J \text{ and all } l < s \text{ and } k < n.$$

Hence, there are at least two indices $i < j$ in J . The mapping $\pi : K_{n,m} \rightarrow K_{n,m}$ that interchanges b_i and b_j and leaves every other vertex fixed is an automorphism of the structure $\langle [K_{n,m}], \bar{P} \rangle$. Hence,

$$[K_{n,m}] \models \varphi(b_i, b_j; \bar{P}) \quad \iff \quad [K_{n,m}] \models \varphi(b_j, b_i; \bar{P}),$$

and φ does not define an order on $K_{n,m}$. A contradiction. \square

Lemma 4.28. *Let \mathcal{C} be a class of graphs of the form $K_{m_0, \dots, m_{d-1}}$ where*

$$d > 2 \quad \text{and} \quad m_1 + \dots + m_{d-1} \geq m_0 \geq m_1 \geq \dots \geq m_{d-1} \geq 1.$$

Then \mathcal{C} is MSO_2 -orderable.

Proof. Consider $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $m_0 \geq \dots \geq m_{d-1} \geq 1$. Let A_0, \dots, A_{d-1} be the vertex sets of this graph and let $a_0^k, \dots, a_{m_k-1}^k$ be an enumeration of A_k . Using the parameter

$$R := \{ (a_0^k, a_0^{k+1}) \mid 0 \leq k < d-1 \}$$

we can define the preorder

$$u \sqsubseteq v \quad :\iff \quad u \in A_i \text{ and } v \in A_k \text{ for all } i \leq k.$$

As usual, we write

$$\begin{aligned} u \equiv v & \quad :\iff \quad u \sqsubseteq v \text{ and } v \sqsubseteq u, \\ u \sqsubset v & \quad :\iff \quad u \sqsubseteq v \text{ and } v \not\sqsubseteq u. \end{aligned}$$

Using the parameter $S := \{ (a_i^k, a_j^{k+1}) \mid i \leq j \}$ and \sqsubseteq , we can define a linear order \leq_B on $B := A_1 \cup \dots \cup A_{d-1}$ by setting $u \leq_B v$ if, and only if,

- $u \sqsubset v$ or
- $u \equiv v$ and, for all $x \sqsubset u$, $(x, u) \in S$ implies $(x, v) \in S$.

Hence, it remains to define a linear order \leq_A on A_0 . Since $m_0 \leq m_1 + \dots + m_{d-1}$, we can fix an enumeration b_0, \dots, b_{n-1} of B and use the parameter $S_0 := \{ (a_i^0, b_j) \mid i \leq j \}$ to define such an order. \square

Theorem 4.29. *Let \mathcal{C} be a class of graphs that are all complete d -partite for some $d \in \mathbb{N}$. (We do not require the number d to be the same for every graph.) The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) There exists a constant s such that \mathcal{C} has property $\text{SEP}(f)$ where $f(k) = 2^{s(k+1)}$.
- (3) There exists a constant s such that

$$K_{m_0, \dots, m_{d-1}} \in \mathcal{C} \quad \text{implies} \quad M \leq 2^{s(N-M+1)}$$

where $M := \max_{i < d} m_i$ and $N := \sum_{i < d} m_i$.

Proof. (3) \Rightarrow (1) Consider $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $m_0 \geq \dots \geq m_{d-1} \geq 1$. We distinguish several cases.

- If $d \leq 2$, the claim follows by Lemma 4.26.
- If $d > 2$ and $M \geq N - M$, we have $K_{N-M, M} \subseteq K_{m_0, \dots, m_{d-1}}$ and the claim follows by Remark 3.6 and Lemma 4.26.
- If $d > 2$ and $M < N - M$ the claim follows by Lemma 4.28.

(1) \Rightarrow (3) Suppose that $[\mathcal{C}]$ is ordered by an MSO -formula $\varphi(x, y; \bar{Z})$ with s set variables Z_0, \dots, Z_{s-1} . We claim that there is no $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$ with $M > 2^{s(N-M)+s}$.

For a contradiction, suppose that there is such a graph $K_{m_0, \dots, m_{d-1}} \in \mathcal{C}$. Let \bar{P} be parameters such that $\varphi(x, y; \bar{P})$ orders $K_{m_0, \dots, m_{d-1}}$. Let A be a vertex set of $K_{m_0, \dots, m_{d-1}}$ of size M and let B be its complement. We enumerate A and B as a_0, \dots, a_{M-1} and

b_0, \dots, b_{N-M-1} , respectively. Since $M > 2^{s(N-M)+s}$ there is a subset $I \subseteq [M]$ of cardinality $|I| > 2^{s(N-M)+s}/2^s = 2^{s(N-M)}$ such that

$$a_i \in P_l \Leftrightarrow a_j \in P_l \quad \text{for all } i, j \in I \text{ and all } l < s.$$

Similarly, there is a subset $J \subseteq I$ of cardinality $|J| > 2^{s(N-M)}/2^{s(N-M)} = 1$ such that

$$(a_i, b_k) \in P_l \Leftrightarrow (a_j, b_k) \in P_l \quad \text{for all } i, j \in J, l < s, \text{ and } k < N - M.$$

Hence, there are at least two different indices $i, j \in J$. The mapping $\pi : K_{m_0, \dots, m_{d-1}} \rightarrow K_{m_0, \dots, m_{d-1}}$ that interchanges a_i and a_j and leaves every other vertex fixed is an automorphism of the structure $\langle [K_{m_0, \dots, m_{d-1}}], \bar{P} \rangle$. Hence,

$$[K_{m_0, \dots, m_{d-1}}] \models \varphi(a_i, a_j; \bar{P}) \iff [K_{m_0, \dots, m_{d-1}}] \models \varphi(a_j, a_i; \bar{P}),$$

and φ does not define an order on $K_{m_0, \dots, m_{d-1}}$. A contradiction.

(3) \Rightarrow (2) Let $K_{m_0, \dots, m_{d-1}}$ be a complete d -partite graph and set $M := \max_{i < d} m_i$ and $N := \sum_{i < d} m_i$. If $M \leq 2^{s(N-M+1)}$, then

$$\begin{aligned} \text{Sep}(K_{m_0, \dots, m_{d-1}}, k) &= \begin{cases} 1 & \text{if } k < N - M \\ M & \text{if } k \geq N - M \end{cases} \\ &\leq \begin{cases} 2^{s(k+1)} & \text{if } k < N - M \\ 2^{s(N-M+1)} & \text{if } k \geq N - M \end{cases} \\ &\leq 2^{s(k+1)}. \end{aligned}$$

(2) \Rightarrow (3) Suppose that \mathcal{C} has property $\text{SEP}(f)$ where $f(k) = 2^{s(k+1)}$. Note that

$$\text{Sep}(K_{m_0, \dots, m_{d-1}}, k) = \begin{cases} 1 & \text{if } k < N - M, \\ M & \text{if } k \geq N - M, \end{cases}$$

where M and N are as above. It follows that

$$M = \text{Sep}(K_{m_0, \dots, m_{d-1}}, N - M) \leq f(N - M) = 2^{s(N-M+1)}. \quad \square$$

As a corollary we obtain a special case of Conjecture 4.25 for classes of complete d -partite graphs.

Corollary 4.30. *Let \mathcal{C} be a VR-equational class of complete d -partite graphs, for some fixed natural number $d > 1$. Then \mathcal{C} is MSO_2 -orderable if, and only if, it has property SEP . This property is decidable.*

Proof. For every $d \in \mathbb{N}$, there is an MSO-formula $\varphi_d(X_0, \dots, X_{d-1})$ stating that X_0, \dots, X_{d-1} are the vertex sets of a complete d -partite graph. By the Semi-Linearity Theorem, it follows that the set

$$M_d := \{ (m_0, \dots, m_{d-1}) \mid K_{m_0, \dots, m_{d-1}} \in \mathcal{C} \}$$

is semi-linear.

Suppose that \mathcal{C} has property SEP . By Example 4.2, it follows that, for every choice of m_0, \dots, m_{d-2} , there are only finitely many m_{d-1} such that $K_{m_0, \dots, m_{d-2}, m_{d-1}} \in \mathcal{C}$. Semi-linearity of M_d therefore implies that there are numbers $a, b \in \mathbb{N}$ such that

$$m_{d-1} \leq a(m_0 + \dots + m_{d-2}) + b, \quad \text{for all } K_{m_0, \dots, m_{d-1}} \in \mathcal{C}.$$

By Theorem 4.29 it follows that \mathcal{C} is MSO_2 -orderable. \square

4.5. Split graphs and chordal graphs. As the next step towards Conjecture 4.25, the case of a VR-equational class of cographs suggests itself, but, so far, we were unable to find a proof. (See Section 5.2 for the definition of a cograph. Note that Corollary 4.30 contains a solution for complete multi-partite graphs, which are a special kind of cographs.) Instead, we consider split graphs and, more generally, chordal graphs.

Definition 4.31. Let G be a graph.

(a) G is a *split graph* if there exists a partition of its vertex set into two parts A and B such that A induces a clique whereas B is independent, i.e., $G[B]$ contains no edges.

(b) Let F be a spanning forest of G with tree-order \preceq_F . We call F a *perfect spanning forest* if it is normal (cf. Section 4.2) and, for every vertex $v \in F$, the set of all neighbours u of v such that $u \prec_F v$ induces a clique in G .

(c) G is *chordal* if it has a perfect spanning forest. \diamond

Every split graph is chordal. There are many equivalent definitions of chordal graphs. See Proposition 2.72 of [7] for an overview and a proof of their equivalence.

Theorem 4.32. *A class \mathcal{C} of split graphs is MSO_2 -orderable if, and only if, there is some $s \in \mathbb{N}$ such that \mathcal{C} has property $\text{SEP}(f)$ for the function f such that $f(n) = 2^{s(n+1)}$.*

Proof. (\Leftarrow) Given s , we construct an MSO_2 -formula $\varphi(x, y; \bar{Z})$ with $s + 1$ parameters that orders every split graph G such that $\text{Sep}(G, n) \leq 2^{s(n+1)}$, for all n . Let $G = \langle V, E \rangle$ be such a split graph and let $V = A \cup B$ be the partition of V into a clique A and an independent set B . We use one parameter P to define an order on A as follows. Fixing an enumeration a_0, \dots, a_{n-1} of A we set

$$P := \{a_0\} \cup \{(a_i, a_{i+1}) \mid i < n - 1\}.$$

Then we can write down an MSO_2 -formula $\psi(x, y; P)$ stating that every path that connects the unique vertex in P to y and that only uses edges in P contains the vertex x . This defines a linear order \leq_A on A .

We use this order to define an order on B as follows. For $b \in B$ let

$$N(b) := \{a \in A \mid (a, b) \in E\}.$$

We first define a preorder \sqsubseteq on B by

$$b \sqsubseteq b' \quad :\Leftrightarrow \quad N(b) = N(b') \text{ or the } \leq_A\text{-least element of } N(b) \Delta N(b') \text{ belongs to } N(b).$$

Since this preorder is linear, i.e., there are no incomparable elements, it is sufficient to define an order on each class of the equivalence relation associated with \sqsubseteq . Given $b \in B$, we fix an enumeration b_0, \dots, b_{m-1} of all vertices $b_i \in B$ such that $N(b_i) = N(b)$ and a \leq_A -increasing enumeration a_0, \dots, a_{n-1} of $N(b)$. Then

$$m \leq \text{Sep}(G, n) \leq 2^{s(n+1)}.$$

Choosing an injective function $\pi : [m] \rightarrow \mathcal{P}([s(n+1)])$, we set, for $k < s$,

$$Q_k := \{(b_i, a_l) \mid k(n+1) + l \in \pi(i)\} \cup \{b_i \mid k(n+1) + n \in \pi(i)\}.$$

Using the parameters Q_0, \dots, Q_{s-1} , we can order b_0, \dots, b_{m-1} by

$$b_i <_B b_j \quad :\Leftrightarrow \quad \text{the least element of } \pi(i) \Delta \pi(j) \text{ belongs to } \pi(i).$$

Finally, by combining \leq_A , \sqsubseteq , and $<_B$, we can define an order on all vertices of G .

(\Rightarrow) Suppose that a split graph $G = \langle V, E \rangle$ is ordered by a formula $\varphi(x, y; \bar{P})$ with s parameters P_0, \dots, P_{s-1} . We will prove that $\text{Sep}(G, n) \leq 2^{(s+1)(n+1)}$. Let $V = A \cup B$

be the partition of V into a clique A and an independent set B . We start by showing that, for every $b \in B$, there are at most $2^{s(|N(b)|+1)}$ vertices $b' \in B$ with $N(b') = N(b)$, where $N(b)$ is defined as above. Let b_0, \dots, b_{m-1} be a list of distinct vertices of B such that $N(b_0) = \dots = N(b_{m-1})$. For a contradiction, suppose that $m > 2^{s|N(b_0)|+s}$. Then there are indices $i < j$ such that

$$\begin{aligned} b_i \in P_k &\iff b_j \in P_k, && \text{for all } k < s, \\ (b_i, a) \in P_k &\iff (b_j, a) \in P_k, && \text{for all } k < s \text{ and } a \in N(b_0). \end{aligned}$$

It follows that the mapping that interchanges b_i and b_j and that fixes every other vertex of $\langle G, \bar{P} \rangle$ is an automorphism. Hence,

$$\lceil G \rceil \models \varphi(b_i, b_j; \bar{P}) \iff \lceil G \rceil \models \varphi(b_j, b_i; \bar{P}),$$

and φ does not define an order on G . A contradiction.

To compute $\text{Sep}(G, n)$ consider a set $S \subseteq V$ of size $|S| \leq n$. We have seen above that, for every set $X \subseteq S \cap A$, there are at most $2^{s(|X|+1)}$ vertices $b \in B$ such that $N(b) = X$. Setting $k := |S \cap A|$, it follows that there are at most $2^k \cdot 2^{s(k+1)}$ vertices $b \in B$ such that $N(b) \subseteq S \cap A$. Consequently, $G - S$ has at most

$$1 + 2^k \cdot 2^{s(k+1)} \leq 2^{sk+s+k+1} = 2^{(s+1)(k+1)} \leq 2^{(s+1)(n+1)}$$

connected components and the claim follows. \square

Lemma 4.33. *For every increasing and unbounded function $g : \mathbb{N} \rightarrow \mathbb{N}$ there exists a class of split graphs that is not MSO_2 -orderable but has property $\text{SEP}(f)$ for the function f such that $f(n) := 2^{ng(n)}$.*

Proof. For $k \in \mathbb{N}$, let $G_k := K_k \otimes D_{2^{kg(k)}}$ where D_n denotes the graph with n vertices and no edges. We claim that $\mathcal{C} := \{G_k \mid k \in \mathbb{N}\}$ has the desired properties. Note that

$$\text{Sep}(G_k, n) \leq \begin{cases} 1 & \text{if } n < k, \\ 2^{ng(n)} & \text{if } n \geq k. \end{cases}$$

Hence, \mathcal{C} has property SEP , but it does not have property $\text{SEP}(f)$, for any function f such that $f(n) = 2^{s(n+1)}$ for some $s \in \mathbb{N}$. By Theorem 4.32, it follows that \mathcal{C} is not MSO_2 -orderable. \square

Remark 4.34. The class in the preceding lemma is not VR-equational since it does not satisfy the Semi-Linearity Theorem. Hence, it does not provide a counterexample to Conjecture 4.25. \diamond

It would be interesting to extend Theorem 4.32 to classes of chordal graphs. At this point, we are only able to present a sufficient condition for MSO_2 -orderability. But there are examples showing that it is not necessary. We start with a technical lemma.

Lemma 4.35. *Let F be a perfect spanning forest of a chordal graph G with tree-order \preceq_F . If $u \prec_F v \preceq_F w$ are vertices then*

$$(u, w) \in E \text{ implies } (u, v) \in E.$$

Proof. Let $x_n \prec_F \dots \prec_F x_0$ be the path in F from $v = x_n$ to $w = x_0$. We show by induction on i , that $(u, x_i) \in E$. For $i = 0$, there is nothing to do. Hence, suppose that $i > 0$ and that we have already shown that $(u, x_{i-1}) \in E$. Then u and x_i are both neighbours of x_{i-1} . Since $u, x_i \prec_F x_{i-1}$, it follows by definition of a perfect spanning forest that $(u, x_i) \in E$. \square

Proposition 4.36. *Let \mathcal{C} be a class of chordal graphs with property $\text{SEP}(f)$ where $f(n) = 2^{s(n+1)}$, for some $s \in \mathbb{N}$. Then \mathcal{C} is MSO_2 -orderable.*

Proof. Let $G = \langle V, E \rangle$ be a chordal graph such that $\text{Sep}(G, n) \leq 2^{s(n+1)}$. To order G , we fix a perfect spanning forest F of G . It is sufficient to define, for every vertex v , an order on the immediate successors of v in F . Then we can use the lexicographic ordering on F to order G . Fix a vertex v and let u_0, \dots, u_{n-1} be the immediate successors of v in F . For $i < n$, we define

$$B_i := \{ w \preceq_F v \mid (w, u_i) \in E \}.$$

We start by showing that, for every set $B \subseteq V$, there are at most $2^{s(|B|+1)}$ indices i such that $B_i = B$. Given B , let I be the set of all $i < n$ such that $B_i = B$. By Lemma 4.35, it follows that, for every $i \in I$ and every edge $(x, y) \in E$ such that $x \prec_F u_i \preceq_F y$, we have $x \in B_i = B$. Hence,

$$|I| \leq \text{Sep}(G, |B|) \leq 2^{s(|B|+1)}$$

as desired. As in the proof of Theorem 4.32, we can use $s + 1$ parameters Q_0, \dots, Q_s to colour the edges of the subgraphs $B_i \otimes u_i$ such a way that we can define the ordering

$$u_i < u_k \iff i < k, \quad \text{for } i, k \in I.$$

Consequently, we can order all immediate successors of v by

$$u_i \leq u_k \iff B_i = B_k \text{ and } i \leq k, \text{ or} \\ \text{the } \prec_F\text{-least element of } B_i \Delta B_k \text{ belongs to } B_i. \quad \square$$

Corollary 4.37. *Let \mathcal{C} be a VR-equational class of chordal graphs. The following statements are equivalent:*

- (1) \mathcal{C} is MSO_2 -orderable.
- (2) \mathcal{C} has property SEP .
- (3) There are constants $r, s \in \mathbb{N}$ such that \mathcal{C} has property $\text{SEP}(f)$ where f is the function such that $f(n) = rn + s$.

These properties are decidable.

Since we have already proved (3) \Rightarrow (1) and (1) \Rightarrow (2) in Proposition 4.36 and Corollary 4.5, only the implication (2) \Rightarrow (3) remains to be proved. We leave this proof to the reader; it is similar to that of Corollary 4.30.

5. MSO_1 -DEFINABLE ORDERS

After having studied MSO_2 -orderability, we consider MSO_1 -orderability. For classes that are r -sparse, for some r , MSO_1 and MSO_2 have the same expressive power (see Theorem 9.38 of [7]). For these classes we can therefore use the results of Section 4. For general classes, MSO_1 -orderability turns out to be more difficult to characterise than MSO_2 -orderability.

5.1. Necessary conditions. We will employ tools related to the notion of clique-width. Instead of using the exact operations defining clique-width (cf. Section 2.1), we introduce related ones that are more convenient in our context.

Definition 5.1. Let $k \in \mathbb{N}$ and $R \subseteq [k] \times [k]$.

(a) For undirected graphs G and H with ports in $[k]$, we construct the undirected graph $G \otimes_R H$ by adding to the disjoint union $G \oplus H$ all edges (x, y) such that

- either $x \in G$ and $y \in H$, or $x \in H$ and $y \in G$; and
- x has port label a and y has port label b , for some $(a, b) \in R$.

Similarly, we define $G \otimes_R H$ for graphs G and H with ports expanded by additional unary predicates (vertex colours) and constants.

(b) For a graph G with ports, we denote by $\text{Del}(G)$ the graph obtained from G by deleting all port labels. \diamond

Remark 5.2. (a) The operation \otimes_R is associative and commutative with the empty graph as neutral element. Furthermore, $\otimes_R = \otimes_{R \cup R^{-1}}$.

(b) With only one port label, there are two operations of the form \otimes_R : the operations \oplus and \otimes used to build cographs (see Section 5.2 below).

(c) We have $\overline{G \otimes_R H} = \overline{G} \otimes_{R'} \overline{H}$ where $R' := ([k] \times [k]) \setminus R$ and \overline{G} denotes the edge complement of G .

(d) We can express \otimes_R as a combination of the operations defining clique-width in the following way:

$$G \otimes_R H = \text{relab}_{h_-}(\text{add}_{a_0, b_0}(\cdots \text{add}_{a_n, b_n}(G \oplus \text{relab}_{h_+}(H)) \cdots)),$$

for suitable functions $h_+ : [k] \rightarrow [2k]$ and $h_- : [2k] \rightarrow [k]$ and port labels $a_0, b_0, \dots, a_n, b_n \in [2k]$. (h_+ is needed to make the port labels appearing in H distinct from those appearing in G .) \diamond

Remark 5.3. (a) As in Proposition 3.4(b), one can show that

$$\mathcal{C} \otimes_R \mathcal{K} := \{ G \otimes_R H \mid G \in \mathcal{C}, H \in \mathcal{K} \}$$

is MSO-orderable if, and only if, \mathcal{C} and \mathcal{K} are MSO-orderable.

(b) $\overline{\mathcal{C}} := \{ \overline{G} \mid G \in \mathcal{C} \}$ is MSO-orderable if, and only if, \mathcal{C} is MSO-orderable. \diamond

To give a necessary condition for MSO_1 -orderability, we introduce a combinatorial property similar to **SEP**, but based on the operation \otimes_R .

Definition 5.4. Let G be a graph (without port labels) and $k \in \mathbb{N}$.

(a) We denote by $\text{Cut}(G, k)$ the maximal number n such that there exist nonempty graphs H_0, \dots, H_{n-1} with ports in $[k]$ and a relation $R \subseteq [k] \times [k]$ such that

$$G \cong \text{Del}(H_0 \otimes_R \cdots \otimes_R H_{n-1}).$$

(b) We say that a class \mathcal{C} of graphs has property $\text{CUT}(f)$, for a function $f : \mathbb{N} \rightarrow \mathbb{N}$, if

$$\text{Cut}(G, k) \leq f(k), \quad \text{for all } G \in \mathcal{C} \text{ and all } k \in \mathbb{N}.$$

We say that \mathcal{C} has property **CUT**, if it has property $\text{CUT}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. \diamond

Remark 5.5. Note that $\text{Cut}(G, k) = \text{Cut}(\overline{G}, k)$. \diamond

For the proof that **CUT** is a necessary condition for MSO_1 -orderability, we use the following technical lemma.

Lemma 5.6. *Let G, G', H, H' be labelled graphs, $\bar{P}, \bar{P}', \bar{Q}, \bar{Q}'$ tuples of sets of vertices of the respective graphs, and $\bar{a}, \bar{a}', \bar{b}, \bar{b}'$ tuples of vertices. For each port label c , let C_c, C'_c, D_c, D'_c be the sets of all vertices of, respectively, G, G', H, H' that have port label c . Then*

$$\text{MTh}_m(\lfloor G \rfloor, \bar{P}, \bar{C}, \bar{a}) = \text{MTh}_m(\lfloor G' \rfloor, \bar{P}', \bar{C}', \bar{a}')$$

and

$$\text{MTh}_m(\lfloor H \rfloor, \bar{Q}, \bar{D}, \bar{b}) = \text{MTh}_m(\lfloor H' \rfloor, \bar{Q}', \bar{D}', \bar{b}')$$

implies that

$$\text{MTh}_m(\lfloor G \otimes_R H \rfloor, \bar{S}, \bar{a}\bar{b}) = \text{MTh}_m(\lfloor G' \otimes_R H' \rfloor, \bar{S}', \bar{a}'\bar{b}'),$$

where $S_i := P_i \cup Q_i$ and $S'_i = P'_i \cup Q'_i$.

Proof. Let σ be a quantifier-free transduction that maps a structure \mathfrak{A} to its expansion $\langle \mathfrak{A}, I \rangle$ where $I := A \times A$ is the equivalence relation on A with a single class. Given R , we can write down a quantifier-free transduction τ such that

$$\langle \lfloor G \otimes_R H \rfloor, \bar{S}, \bar{a}\bar{b} \rangle = \tau(\sigma(\langle \lfloor G \rfloor, \bar{P}, \bar{C}, \bar{a} \rangle) \oplus \sigma(\langle \lfloor H \rfloor, \bar{Q}, \bar{D}, \bar{b} \rangle))$$

and

$$\langle \lfloor G' \otimes_R H' \rfloor, \bar{S}', \bar{a}'\bar{b}' \rangle = \tau(\sigma(\langle \lfloor G' \rfloor, \bar{P}', \bar{C}', \bar{a}' \rangle) \oplus \sigma(\langle \lfloor H' \rfloor, \bar{Q}', \bar{D}', \bar{b}' \rangle)).$$

This transduction uses the relation I to mark the two components of the disjoint union. The claim now follows from the Composition Theorem and the Backwards Translation Lemma. \square

Proposition 5.7. *There exists a function $f : \mathbb{N}^3 \rightarrow \mathbb{N}$ such that $\text{Cut}(G, k) \leq f(n, m, k)$ for every graph G such that $\lfloor G \rfloor$ can be ordered by an MSO-formula of the form $\varphi(x, y; \bar{P})$ where $\text{qr}(\varphi) \leq m$ and $\bar{P} = \langle P_0, \dots, P_{n-1} \rangle$ are parameters. Furthermore, the function $f(n, m, k)$ is effectively elementary in the argument k , that is, there exists a computable function g such that $f(n, m, k) \leq \exp_{g(n, m)}(k)$.*

Proof. Fixing $k, m, n \in \mathbb{N}$, we choose for $f(n, m, k)$ an upper bound on the number of MSO-theories of the form

$$\text{MTh}_m(\lfloor H \rfloor, v, P_0, \dots, P_{n-1}, Q_0, \dots, Q_{k-1})$$

where H is a graph, v is a vertex of H and P_0, \dots, Q_0, \dots are parameters. For fixed m , we can choose this bound to be elementary in k .

Let $\varphi(x, y; \bar{Z})$ be an MSO-formula of quantifier-rank at most m , let G be a graph with $\text{Cut}(G, k) > f(n, m, k)$, and let P_0, \dots, P_{n-1} be parameters from G . We have to show that $\varphi(x, y; \bar{P})$ does not order G . We choose graphs H_0, \dots, H_{d-1} with $d = \text{Cut}(G, k)$ and a relation $R \subseteq [k] \times [k]$ such that

$$G = \text{Del}(H_0 \otimes_R \dots \otimes_R H_{d-1}).$$

For $c < k$, let

$$C_c := \{x \in G \mid x \in H_i, \text{ for some } i < d, \text{ and } x \text{ has port label } c \text{ in } H_i\}.$$

Since $d > f(n, m, k)$, there are indices $i < j$ such that

$$\text{MTh}_m(\lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i) = \text{MTh}_m(\lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j).$$

As there exists a graph F such that

$$\langle \lfloor G \rfloor, a_i a_j, \bar{P}, \bar{Q} \rangle = \langle \lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i \rangle \otimes_R \langle \lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j \rangle \otimes_R F$$

and $\langle \lfloor G \rfloor, a_j a_i, \bar{P}, \bar{Q} \rangle = \langle \lfloor H_j \rfloor, a_j, \bar{P} \upharpoonright H_j, \bar{C} \upharpoonright H_j \rangle \otimes_R \langle \lfloor H_i \rfloor, a_i, \bar{P} \upharpoonright H_i, \bar{C} \upharpoonright H_i \rangle \otimes_R F,$

it follows by Lemma 5.6 that

$$\text{MTh}_m(\lfloor G \rfloor, a_i a_j, \bar{P}, \bar{C}) = \text{MTh}_m(\lfloor G \rfloor, a_j a_i, \bar{P}, \bar{C}).$$

In particular, we have

$$\lfloor G \rfloor \models \varphi(a_i, a_j; \bar{P}) \iff \lfloor G \rfloor \models \varphi(a_j, a_i; \bar{P}).$$

Hence, $\varphi(x, y; \bar{P})$ does not define an order on G . \square

Corollary 5.8. *An MSO_1 -orderable class of graphs \mathcal{C} has property $\text{CUT}(f)$, for an elementary function f .*

Example 5.9. The following classes are not MSO_1 -orderable:

- the class of all cliques K_n ;
- the class of all complete bipartite graphs $K_{n,m}$;
- any class of graphs of the form $G \otimes (H_0 \oplus \dots \oplus H_n)$ where the number n is unbounded and each H_i is nonempty.

In each case, after fixing a number k of parameters, we can choose a graph G that is sufficiently large such that any colouring with k parameters P_0, \dots, P_{k-1} admits a nontrivial automorphism. Hence, no formula can define an order on $\langle \lfloor G \rfloor, \bar{P} \rangle$. \diamond

As MSO_1 -orderability implies MSO_2 -orderability, we can expect that the property CUT implies SEP . The following lemma proves this fact.

Lemma 5.10. *A class \mathcal{C} of graphs with property $\text{CUT}(f)$ has property $\text{SEP}(g)$ where g is the function such that $g(n) := f(n + 2^n) - 1$.*

Proof. Let $G = \langle V, E \rangle \in \mathcal{C}$ and consider a set $S \subseteq V$ of size $|S| \leq n$. Let C_0, \dots, C_{d-1} be an enumeration of the connected components of $G - S$. We claim that $d \leq g(n)$.

We define colourings $\varrho : S \rightarrow D$ and $\pi_i : C_i \rightarrow D$, for $i < d$, as follows. The set of colours is $D := S \cup \mathcal{P}(S)$. (To be formally correct, we have to take the set $[k]$ where $k := |S \cup \mathcal{P}(S)|$. To simplify notation, we will use $S \cup \mathcal{P}(S)$ instead.) We set

$$\varrho(s) := s \quad \text{and} \quad \pi_i(v) := \{s \in S \mid (v, s) \in E\}.$$

It follows that

$$G = \text{Del}(\langle S, \varrho \rangle \otimes_R \langle C_0, \pi_0 \rangle \otimes_R \dots \otimes_R \langle C_{d-1}, \pi_{d-1} \rangle),$$

where

$$R := \{(s, X) \in S \times \mathcal{P}(S) \mid s \in X\}.$$

Consequently, $\text{Cut}(G, |D|) \geq d + 1$. Since $|D| \leq n + 2^n$, it follows that

$$d + 1 \leq \text{Cut}(G, n + 2^n) \leq f(n + 2^n) = g(n) + 1. \quad \square$$

The converse obviously does not hold. A special case, where it *does* hold is the case of r -sparse graphs (cf. Definition 2.1). This case is of particular interest since, for r -sparse graphs, the expressive powers of MSO_1 and MSO_2 coincide (see Theorem 9.37 of [7]).

Lemma 5.11. *The graph $K_{m,n}$ is r -sparse if, and only if, $r \geq \frac{mn}{m+n}$.*

Proof. Every induced subgraph of $K_{m,n}$ is of the form $K_{m',n'}$ with $m' \leq m$ and $n' \leq n$. Such a subgraph has $m' + n'$ vertices and $m'n'$ edges. The ratio is

$$\frac{m'n'}{m' + n'} = \frac{1}{\frac{1}{m'} + \frac{1}{n'}} \leq \frac{1}{\frac{1}{m} + \frac{1}{n}} = \frac{mn}{m+n}. \quad \square$$

Lemma 5.12. *A class \mathcal{C} of r -sparse graphs with property $\text{SEP}(f)$ has property $\text{CUT}(g)$ where $g(k) := f(2k^2r(2r+1))$.*

Proof. Let $G \in \mathcal{C}$. Suppose that

$$G = \text{Del}((H_0, \pi_0) \otimes_R \cdots \otimes_R (H_{d-1}, \pi_{d-1})) \quad \text{where } R \subseteq [k] \times [k].$$

Without loss of generality, we may assume that R is symmetric. We have to show that $d \leq g(k)$.

Set $I_a := \{i < d \mid \pi_i^{-1}(a) \neq \emptyset\}$. First, let us show that

$$|I_a| \leq 2r+1 \quad \text{or} \quad |I_b| \leq 2r+1, \quad \text{for every } (a, b) \in R.$$

For a contradiction, suppose that there is some $(a, b) \in R$ that $|I_a| \geq 2r+2$ and $|I_b| \geq 2r+2$. Choose subsets $I'_a \subseteq I_a$ and $I'_b \subseteq I_b$ of size $m := 2r+2$ and select vertices $x_i \in \pi_i^{-1}(a)$, for $i \in I'_a$, and $y_i \in \pi_i^{-1}(b)$, for $i \in I'_b$. The subgraph induced by these vertices has $m^2 - |I_a \cap I_b| \geq m^2 - m$ edges and $2m$ vertices. Since

$$\frac{m^2 - m}{2m} = \frac{m-1}{2} = \frac{2r+1}{2} > r,$$

it follows that G is not r -sparse. A contradiction.

For $a, b \in [k]$, we set

$$S_{ab} := \bigcup \{ \pi_i^{-1}(a) \mid i \in I_a, |\pi_i^{-1}(a)| \leq 2r \},$$

$$S := \bigcup \{ S_{ab} \mid (a, b) \in R, |I_a| \leq 2r+1 \}.$$

Note that

$$|S_{ab}| \leq 2r|I_a| \quad \text{and} \quad |S| \leq |R| \cdot (2r+1) \cdot (2r) \leq 2k^2r(2r+1).$$

We claim that every connected component of $G - S$ is contained in $H_i - S$, for some i . For a contradiction, suppose that there is a connected component C of $G - S$ containing vertices from both $H_i - S$ and $H_j - S$. Then there exists an edge (x, y) of G with $x \in H_i - S$ and $y \in H_j - S$. Let $a := \pi_i(x)$ and $b := \pi_j(y)$. Then $(a, b) \in R$. We have shown above that $|I_a| \leq 2r+1$ or $|I_b| \leq 2r+1$. In the first case, we have $x \in \pi_i^{-1}(a) \subseteq S_{ab} \subseteq S$, in the second case, we have $y \in \pi_j^{-1}(b) \subseteq S_{ba} \subseteq S$. Hence, both cases lead to a contradiction.

It follows that $G - S$ has at least d connected components. Consequently,

$$d \leq \text{Sep}(G, |S|) \leq \text{Sep}(G, 2k^2r(2r+1)) \leq f(2k^2r(2r+1)) = g(k). \quad \square$$

5.2. Cographs. A well-known VR-equational class is the class of cographs. A *cograph* is a graph that can be constructed from single vertices using the operations of disjoint union \oplus and complete join \otimes . Each cograph can be denoted by a term over \oplus , \otimes , and a constant 1 that denotes an isolated vertex. For instance, $(1 \oplus 1) \otimes (1 \oplus 1 \oplus 1)$ denotes the graph $K_{2,3}$, and $1 \otimes 1 \otimes \cdots \otimes 1$ denotes a clique. Since \oplus and \otimes are associative and commutative, we consider them as operations of variable arity and we ignore the order of the arguments. The class \mathcal{C} of cographs is VR-equational. It can be defined by the equation

$$\mathcal{C} = \mathcal{C} \oplus \mathcal{C} \cup \mathcal{C} \otimes \mathcal{C} \cup \{1\}.$$

A cograph G with more than one vertex is either disconnected and of the form $G = H_0 \oplus \cdots \oplus H_n$ for connected cographs H_0, \dots, H_n , or it is connected and of the form $G = H_0 \otimes \cdots \otimes H_n$ for cographs H_0, \dots, H_n each of which is either disconnected or a single vertex.

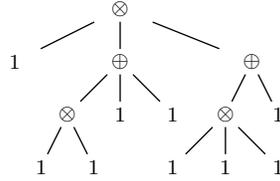
Furthermore, these decompositions of G are unique, up to the ordering of H_0, \dots, H_n . Using this observation, we can associate with every cograph a unique term as follows.

Definition 5.13. A term t over the operations $\oplus, \otimes, 1$ (where we consider \oplus and \otimes as many-ary operations with unordered arguments) is a *cotree* if there is no node that is labelled by the same operation as one of its immediate successors. Every cograph has a unique cotree. The *depth* of a cograph is the height of this cotree. \diamond

Example 5.14. The cograph G defined by the term

$$(1 \otimes (1 \oplus (1 \oplus (1 \otimes 1)))) \otimes ((1 \otimes (1 \otimes 1)) \oplus 1)$$

has the cotree



The leaves of this tree correspond to the vertices of G and every subtree is the cotree of an induced subgraph of G . \diamond

Recall (see, e.g., [4]) that a *module* of a graph $G = \langle V, E \rangle$ is a set M of vertices such that every vertex in $V \setminus M$ is either adjacent to all elements of M , or to none of them. A module M is called *strong* if there is no module N such that $M \setminus N$ and $N \setminus M$ are both nonempty (cf. [18, 4, 12]). Clearly, being a module and being a strong module are expressible in MSO_1 . In a cograph there are two types of strong modules: the connected and the disconnected ones.

Theorem 5.15. *Let \mathcal{C} be a class of cographs. The following statements are equivalent.*

- (1) \mathcal{C} is MSO_1 -orderable.
- (2) \mathcal{C} has property CUT.
- (3) There exists a constant $d \in \mathbb{N}$ such that the cotree of every graph in \mathcal{C} has outdegree at most d .

Proof. (3) \Rightarrow (1) is Corollary 6.12 from [4] and (1) \Rightarrow (2) was shown in Corollary 5.8.

For (2) \Rightarrow (3), suppose that, for every $d \in \mathbb{N}$, there exists a graph $G_d \in \mathcal{C}$ with a cotree of maximal outdegree at least d . It is sufficient to show that $\text{Cut}(G_d, 3) > d$.

By assumption, we can find a strong module A of G_d containing strong submodules B_0, \dots, B_{n-1} , for $n > d$, such that either (i) $A = B_0 \oplus \dots \oplus B_{n-1}$, or (ii) $A = B_0 \otimes \dots \otimes B_{n-1}$. Let $C := G - A$ be the graph induced by the complement of A . Every vertex $v \in C$ is either connected to all vertices of A , or to none of them. We assign the port label 0 to the former vertices and the port label 1 to the latter ones. Each vertex of A gets port label 2. It follows that

$$G_d = C \otimes_R B_0 \otimes_R \dots \otimes_R B_{n-1}$$

where $R = \{(0, 2), (2, 0)\}$ or $R = \{(0, 2), (2, 0), (2, 2)\}$. Consequently, we have $\text{Cut}(G_d, 3) \geq n + 1 > d$. \square

Corollary 5.16. *Let $k \in \mathbb{N}$. The class of cographs of depth at most k is hereditarily MSO_1 -unorderable.*

Proof. For any given depth k , there are only finitely many cographs (up to isomorphism) satisfying condition (3) of Theorem 5.15. \square

Corollary 5.17. *For VR-equational classes of cographs, MSO₁-orderability is decidable.*

Proof. Let \mathcal{C} be a VR-equational class of cographs. By Theorem 5.15, it is sufficient to decide whether there is a constant d such that every cotree of a graph in \mathcal{C} has maximal outdegree at most d . Let $\varphi(X)$ be an MSO₁-formula stating that there exists a strong module Z such that $X \subseteq Z$ and every strong module $Y \subset Z$ contains at most one element of X . Given a cograph G , it follows that the maximal outdegree of the cotree of G is equal to the maximal size of a set X satisfying φ in G . Using the Semi-Linearity Theorem, we can decide whether this size is bounded. \square

Remark 5.18. If a class \mathcal{C} of cographs is MSO₁-orderable, there exists an MSO-transduction mapping each graph in \mathcal{C} to its cotree (see [4]). But, conversely, the existence of such an MSO-transduction is not enough to ensure MSO₁-orderability: there exists an MSO-transduction from the class of all cographs of depth k to their respective cotrees (this is a routine construction). But, as we have just seen, this class is hereditarily MSO₁-unorderable. \diamond

5.3. \otimes -decompositions. Cographs are precisely the graphs of clique-width 2. A natural aim is thus to extend the equivalence (1) \Leftrightarrow (2) of Theorem 5.15 to classes of graphs of bounded clique-width. However, we must leave this as a conjecture. Instead we only consider the special case of graphs where the height of the decomposition (as defined below) is bounded. Such graphs generalise cographs of bounded depth, and we show that they are hereditarily MSO₁-unorderable.

We start by introducing a kind of decomposition associated with the notion of clique-width.

Definition 5.19. Let $G = \langle V, E \rangle$ be a graph.

(a) A \otimes -decomposition of G of width k is a family $(H_v)_{v \in T}$ of labelled graphs $H_v = \langle U_v, F_v, \pi_v \rangle$ with $\pi_v : U_v \rightarrow [k]$ such that

- the index set T is a rooted tree,
- $H_\diamond = \langle V, E, \pi_\diamond \rangle$, for some labelling π_\diamond ,
- $|U_v| = 1$, for every leaf $v \in T$,
- for every internal node $v \in T$ with immediate successors u_0, \dots, u_{d-1} , there is some $R_v \subseteq [k] \times [k]$ such that

$$\text{Del}(H_v) = \text{Del}(H_{u_0} \otimes_{R_v} \cdots \otimes_{R_v} H_{u_{d-1}}).$$

We call \otimes_{R_v} the *operation at v* . Note that the port labels of H_v and $H_{u_0}, \dots, H_{u_{d-1}}$ are unrelated. (Hence, the labelling π_\diamond of the root is arbitrary. We have added it to keep the notation uniform.)

(b) A *strong \otimes -decomposition* of G is a \otimes -decomposition $(H_v)_{v \in T}$ such that, for each internal node $v \in T$ with immediate successors u_0, \dots, u_{d-1} , there is some $R_v \subseteq [k] \times [k]$ and some function $\varrho : [k] \rightarrow [k]$ such that

$$H_v = \text{relab}_\varrho(H_{u_0} \otimes_{R_v} \cdots \otimes_{R_v} H_{u_{d-1}}).$$

(c) The *height* of a \otimes -decomposition $(H_v)_{v \in T}$ is the height of the tree T .

(d) We define $\text{wd}_n^\otimes(G)$ as the least number k such that G has a \otimes -decomposition of width at most k and height at most n . Similarly, we define $\text{swd}_n^\otimes(G)$ as the least number k

such that G has a strong \otimes -decomposition of width at most k and height at most n . We call $\text{wd}_n^\otimes(G)$ the n -depth \otimes -width of G and $\text{swd}_n^\otimes(G)$ is its *strong n -depth \otimes -width*.⁵ \diamond

Remark 5.20. (a) For every graph G and all n, m such that $m < n$, we have

$$\text{wd}_n^\otimes(G) \leq \text{swd}_n^\otimes(G) \leq |V|,$$

$$\text{wd}_n^\otimes(G) \leq \text{wd}_m^\otimes(G),$$

and

$$\text{swd}_n^\otimes(G) \leq \text{swd}_m^\otimes(G).$$

(b) Recall the definition of clique-width in Section 2.1. Since the operation \otimes_R can be expressed by the operations clique-width is based on, but by using twice as many port labels, it follows that the clique-width of a graph is at most twice its strong n -depth \otimes -width (for any n). Since, conversely, for sufficiently large n , the strong n -depth \otimes -width of a graph G is at most its clique-width, it follows that, for every graph G and all sufficiently large n ,

$$\text{swd}_n^\otimes(G) \leq \text{cwd}(G) \leq 2 \cdot \text{swd}_n^\otimes(G).$$

If we define $\text{swd}^\otimes(G)$ as the minimal value of $\text{swd}_n^\otimes(G)$ when n ranges over \mathbb{N} , we therefore obtain a nontrivial width measure that is equivalent to clique-width.

(c) Note that $\text{wd}_n^\otimes(G) \leq 2$, for every graph G with n vertices. Hence, the width $\text{wd}_n^\otimes(G)$ is only of interest if there is a bound on n . \diamond

Because of its relation to clique-width, the strong \otimes -width is of more interest than the \otimes -width (which becomes trivial for large depths). We have introduced the simpler notion of \otimes -width since, in the special case we consider, there exists a bound on the depth of \otimes -decompositions. In this case we can use the following lemma to transform a bound on the \otimes -width of a class into a bound on its strong \otimes -width.

Lemma 5.21. *For every graph G and every $n \in \mathbb{N}$,*

$$\text{wd}_n^\otimes(G) \leq \text{swd}_n^\otimes(G) \leq [\text{wd}_n^\otimes(G)]^{n+1}.$$

Proof. The first inequality being trivial, we only prove the second one. Given a \otimes -decomposition $(H_v)_{v \in T}$ of G of height n and width $k := \text{wd}_n^\otimes(G)$, we construct a strong \otimes -decomposition $(H'_v)_{v \in T}$ of G of the same height and width k^n . Consider $v \in T$ and let v_0, \dots, v_m be the path in T from the root $\langle \rangle = v_0$ to $v = v_m$, where $m < n$. Suppose that $H_v = \langle U_v, F_v, \pi_v \rangle$. We set $H'_v := \langle U_v, F_v, \pi'_v \rangle$ where

$$\pi'_v(x) := \langle \pi_{v_0}(x), \dots, \pi_{v_m}(x) \rangle.$$

This labelling uses $1 + k + k^2 + \dots + k^n \leq k^{n+1}$ port labels. Then

$$H'_v = \text{relab}_\varrho(H'_{u_0} \otimes_{R_v} \dots \otimes_{R_v} H'_{u_{d-1}}),$$

where the function ϱ maps $\langle a_0, \dots, a_m, a_{m+1} \rangle$ to $\langle a_0, \dots, a_m \rangle$. \square

⁵Recently a closely related notion, called *shrub-depth*, was introduced in [13]. Its exact relation to strong n -depth \otimes -width remains to be investigated.

Lemma 5.22. *Let G be a graph and $(H_v)_{v \in T}$ a \otimes -decomposition of G of width at most k . Every vertex of T has less than $\text{Cut}(G, k + 2^k)$ immediate successors.*

Proof. Suppose that $H_v = \langle U_v, F_v, \pi_v \rangle$. Let $v \in T$ be a vertex with immediate successors u_0, \dots, u_{m-1} . Hence,

$$H_v = H_{u_0} \otimes_R \cdots \otimes_R H_{u_{m-1}},$$

where \otimes_R is the operation at v . Let $C := G - H_v$, i.e., the subgraph induced by the complement of the set of vertices of H_v . We claim that

$$G = C \otimes_{R'} H_{u_0} \otimes_{R'} \cdots \otimes_{R'} H_{u_{m-1}},$$

for a suitable labelling $\varrho : C \rightarrow [k + 2^k]$ of C and a suitable relation $R' \subseteq [k + 2^k] \times [k + 2^k]$. This implies that $m + 1 \leq \text{Cut}(G, k + 2^k)$, as desired.

It remains to define ϱ and R' . Fix a bijection $\pi_0 : \mathcal{P}([k]) \rightarrow [2^k]$ and set $\pi(B) := \pi_0(B) + k$, for $B \subseteq [k]$. Defining

$$\varrho(x) := \pi(\{\pi_v(y) \mid y \in U_v, (x, y) \in E\}), \quad \text{for } x \in C,$$

and

$$R' := R \cup \{(a, \pi(B)) \mid a \in [k], B \subseteq [k], a \in B\},$$

we obtain $G = C \otimes_{R'} H_{u_0} \otimes_{R'} \cdots \otimes_{R'} H_{u_{m-1}}$. \square

We obtain the following characterisation of MSO_1 -orderable classes of bounded n -depth \otimes -width.

Theorem 5.23. *Let \mathcal{C} be a class of graphs such that, for some $n, k \in \mathbb{N}$,*

$$\text{wd}_n^\otimes(G) \leq k, \quad \text{for all } G \in \mathcal{C}.$$

The following statements are equivalent:

- (1) \mathcal{C} is MSO_1 -orderable.
- (2) \mathcal{C} has property CUT .
- (3) *There is a constant $d \in \mathbb{N}$ such that every $G \in \mathcal{C}$ has a \otimes -decomposition $(H_v)_{v \in T}$ of height at most n and width at most k where every vertex of T has outdegree at most d .*
- (4) \mathcal{C} is finite.

Proof. (4) \Rightarrow (1) is trivial and (1) \Rightarrow (2) follows from Corollary 5.8.

(2) \Rightarrow (3) Suppose that \mathcal{C} has property $\text{CUT}(f)$. Let $G \in \mathcal{C}$ and let $(H_v)_{v \in T}$ be a \otimes -decomposition of G of height at most n and width at most k . Then it follows by Lemma 5.22 that every vertex of T has less than $d := f(k + 2^k)$ immediate successors.

(3) \Rightarrow (4) Since every tree of height at most n and maximal outdegree at most d has at most $1 + (d - 1) + (d - 1)^2 + \cdots + (d - 1)^{n-1} < d^n$ vertices, it follows that every graph in \mathcal{C} has at most that many elements. \square

We obtain the following extension of Corollary 5.16.

Corollary 5.24. *For every $n, k \in \mathbb{N}$, the class of all graphs of n -depth \otimes -width at most k is hereditarily MSO_1 -unorderable.*

6. REDUCTIONS BETWEEN DIFFICULT CASES

In this section we consider classes of graphs for which the question of orderability is as hard as in the general case.

Definition 6.1. Let $G = \langle V, E \rangle$ be a graph.

(a) The *incidence graph* of G is the graph $\text{Inc}(G) := \langle V \cup E, I, P \rangle$ where the edge relation

$$I := \text{inc} \cup \text{inc}^{-1} = \{ (x, y) \mid x \text{ is an end-vertex of } y \text{ or } y \text{ is an end-vertex of } x \}$$

is the symmetric version of the incidence relation and $P := V$ is a unary relation identifying the vertices of G .

(b) The *incidence split graph* of G is the graph $\text{IS}(G) := \langle V \cup E, J \rangle$ where

$$J := I \cup \{ (x, y) \in V \times V \mid x \neq y \}$$

and I is the symmetric incidence relation from (a). Note that $\text{IS}(G)$ is a split graph.

(c) For a class of graphs \mathcal{C} , we set

$$\text{Inc}(\mathcal{C}) := \{ \text{Inc}(G) \mid G \in \mathcal{C} \} \quad \text{and} \quad \text{IS}(\mathcal{C}) := \{ \text{IS}(G) \mid G \in \mathcal{C} \}. \quad \diamond$$

The proposition below suggests that obtaining a characterisation of MSO_1 -orderability for classes of split graphs is as hard as obtaining one of MSO_2 -orderability for arbitrary classes of graphs. We start with a technical lemma.

Lemma 6.2. *Let \mathcal{C} be a class of graphs.*

- (a) \mathcal{C} has property **SEP** if, and only if, $\text{Inc}(\mathcal{C})$ has property **SEP**.
- (b) $\text{Inc}(\mathcal{C})$ has property **CUT** if, and only if, $\text{IS}(\mathcal{C})$ has property **CUT**.

Proof. (a) (\Leftarrow) Suppose that $\text{Inc}(\mathcal{C})$ has property **SEP**(f), for some $f : \mathbb{N} \rightarrow \mathbb{N}$. We claim that \mathcal{C} also has property **SEP**(f). Let $G = \langle V, E \rangle$ be a graph in \mathcal{C} . To compute $\text{Sep}(G, k)$ consider a set $S \subseteq V$ of cardinality $|S| \leq k$. Let C_0, \dots, C_{m-1} be the connected components of $G - S$. Then the connected components of $\text{Inc}(G) - S$ are $C'_0, \dots, C'_{m-1}, e_0, \dots, e_{n-1}$ where e_0, \dots, e_{n-1} is an enumeration of the edges of $G[S]$ and C'_i is the induced subgraph of $\text{Inc}(G)$ that is obtained from $\text{Inc}(C_i)$ by adding (as vertices) all edges of G connecting a vertex in S to some vertex of C_i . It follows that

$$\text{Sep}(G, k) \leq \text{Sep}(\text{Inc}(G), k) \leq f(k).$$

(\Rightarrow) Suppose that \mathcal{C} has property **SEP**(f), for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $G = \langle V, E \rangle$ be a graph in \mathcal{C} with $\text{Inc}(G) = \langle V \cup E, I, P \rangle$. To compute $\text{Sep}(\text{Inc}(G), k)$ we consider a set $S \subseteq V \cup E$ of size $|S| \leq k$. For each edge $e \in S \cap E$, we select one end-vertex. Let X be the set of these end-vertices and set $S' := (S \setminus E) \cup X$. Then $\text{Inc}(G) - S'$ has at least as many connected components as $\text{Inc}(G) - S$. Since $S' \subseteq V$ it follows by what we have seen above that $\text{Inc}(G) - S'$ has at most $m + \binom{k}{2}$ connected components, where m is the number of connected components of $G - S'$. Consequently,

$$\text{Sep}(\text{Inc}(G), k) \leq \text{Sep}(G, k) + \frac{k}{2}(k-1).$$

It follows that $\text{Inc}(\mathcal{C})$ has property **SEP**(f') for the function f' such that $f'(k) = f(k) + \frac{k}{2}(k-1)$.

(b) (\Rightarrow) Suppose that $\text{Inc}(\mathcal{C})$ has property **CUT**(f), for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $\text{Inc}(G) = \langle V \cup E, I, P \rangle$ be a graph in $\text{Inc}(\mathcal{C})$ and let $\text{IS}(G) = \langle V \cup E, J \rangle$. To compute $\text{Cut}(\text{IS}(G), k)$

suppose that

$$\text{IS}(G) = \text{Del}(H_0 \otimes_R \cdots \otimes_R H_{m-1}),$$

for k -labelled graphs H_0, \dots, H_{m-1} and a relation $R \subseteq [k] \times [k]$. Suppose that $H_i = \langle U_i, J_i \rangle$, for $i < m$, and let π_i be the labelling of H_i . We set $H'_i := \langle U_i, I_i, P_i \rangle$ where $I_i := J_i \setminus (V \times V)$ and $P_i := U_i \cap V$. We label H'_i by

$$\pi'_i(v) := \begin{cases} \pi_i(v) & \text{if } v \notin V, \\ \pi_i(v) + k & \text{if } v \in V. \end{cases}$$

Then $\text{Inc}(G) = \text{Del}(H'_0 \otimes_{R'} \cdots \otimes_{R'} H'_{m-1})$, where

$$R' := \{(x, y), (x + k, y), (x, y + k) \mid (x, y) \in R\}.$$

Consequently, $\text{Cut}(\text{IS}(G), k) \leq \text{Cut}(\text{Inc}(G), 2k) \leq f(2k)$.

(\Leftarrow) Suppose that $\text{IS}(\mathcal{C})$ has property $\text{CUT}(f)$, for some $f : \mathbb{N} \rightarrow \mathbb{N}$. Let $\text{Inc}(G) = \langle V \cup E, I, P \rangle$ be a graph in $\text{Inc}(\mathcal{C})$ and let $\text{IS}(G) = \langle V \cup E, J \rangle$. To compute $\text{Cut}(\text{Inc}(G), k)$ suppose that

$$\text{Inc}(G) = \text{Del}(H_0 \otimes_R \cdots \otimes_R H_{m-1}),$$

for k -labelled graphs H_0, \dots, H_{m-1} and a relation $R \subseteq [k] \times [k]$. Suppose that $H_i = \langle U_i, I_i, P_i \rangle$, for $i < m$, and let π_i be the labelling of H_i . We define the graph $H'_i := \langle U_i, J_i \rangle$ where $J_i := I_i \cup \{(x, y) \mid x, y \in P_i, x \neq y\}$ with labelling

$$\pi'_i(v) := \begin{cases} \pi_i(v) & \text{if } v \in V, \\ \pi_i(v) + k & \text{if } v \notin V. \end{cases}$$

Then $\text{IS}(G) = \text{Del}(H'_0 \otimes_{R'} \cdots \otimes_{R'} H'_{m-1})$, where

$$R' := ([k] \times [k]) \cup \{(x, y), (x + k, y), (x, y + k), (x + k, y + k) \mid (x, y) \in R\}.$$

Consequently, $\text{Cut}(\text{Inc}(G), k) \leq \text{Cut}(\text{IS}(G), 2k) \leq f(2k)$. \square

Proposition 6.3. *Let \mathcal{C} be a class of graphs.*

- (a) \mathcal{C} is MSO_2 -orderable if, and only if, $\text{IS}(\mathcal{C})$ is MSO_1 -orderable.
- (b) \mathcal{C} has property SEP if, and only if, $\text{IS}(\mathcal{C})$ has property CUT .

Proof. (a) is a routine construction. (b) follows by the preceding lemma since $\text{Inc}(\mathcal{C})$ is 2-sparse and, by Lemmas 5.10 and 5.12, such a class has property SEP if, and only if, it has property CUT . \square

Corollary 6.4. *Let \mathcal{P} be a graph property such that a class of split graphs is MSO_1 -orderable if, and only if, it has properties CUT and \mathcal{P} . Then a class of arbitrary graphs is MSO_2 -orderable if, and only if, it has properties SEP and $\text{IS}^{-1}(\mathcal{P})$.*

Remark 6.5. (a) Characterising MSO_2 -orderable classes therefore amounts to characterising MSO_1 -orderable classes of split graphs contained in the image of the function IS .

(b) If \mathcal{C} is a class of graphs with property SEP that is not MSO_2 -orderable, then $\text{IS}(\mathcal{C})$ is a class of split graphs with property CUT that is not MSO_1 -orderable. \diamond

We also present a lemma suggesting that finding a characterisation of MSO_1 -orderability for classes of bipartite graphs is as hard as finding a characterisation of MSO_1 -orderability for arbitrary classes of graphs. We leave the proof – which is similar to the one above – to the reader.

Definition 6.6. For a graph $G = \langle V, E \rangle$ we define

$$\text{BP}(G) := \langle V \times [4], E' \rangle$$

where

$$\begin{aligned} E' := & \{ ((x, 0), (y, 3)) \mid (x, y) \in E \} \\ & \cup \{ ((x, i), (x, i + 1)) \mid x \in V, 0 \leq i < 3 \}. \end{aligned}$$

For classes \mathcal{C} of graphs, we define $\text{BP}(\mathcal{C}) := \{ \text{BP}(G) \mid G \in \mathcal{C} \}$ as usual. \diamond

Lemma 6.7. *Let \mathcal{C} be a class of graphs.*

- (a) \mathcal{C} is MSO_1 -orderable if, and only if, $\text{BP}(\mathcal{C})$ is MSO_1 -orderable.
- (b) \mathcal{C} has property **CUT** if, and only if, $\text{BP}(\mathcal{C})$ has property **CUT**.

7. CONCLUSION

For arbitrary classes of graphs, it is difficult to obtain necessary and sufficient conditions for MSO_i -orderability, as there are many different ways to construct MSO -definable orderings depending on many different structural properties of the considered graphs. General conditions should thus cover simultaneously a large number of possibilities. It is therefore necessary to consider particular graph classes. We have obtained necessary and sufficient conditions in Theorems 4.13, 4.29, 4.32, and 5.15 with corresponding decidability results for the VR-equational classes of graphs.

Concerning future work, we think that the following questions should be fruitfully investigated:

(a) Does Conjecture 4.25 hold? We have already proved several special cases and more cases seem to be within reach. It remains to be seen whether the full conjecture can be solved.

(b) Which condition must be added to the property **SEP** to yield a necessary and sufficient condition for MSO_2 -orderability of a class of cographs? And more generally, for graph classes of bounded clique-width?

(c) What could be an extension of Theorem 5.15, say, for classes of ‘bounded strong \otimes -width’?

(d) Which operations do preserve MSO_i -orderability? Candidates include the operations defining tree-width or clique-width, graph substitutions, and monadic second-order transductions. We presented a few simple results in Proposition 3.4 and Remark 5.3, but it should not be too hard to develop a more comprehensive theory.

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