

PRE-MEASURE SPACES AND PRE-INTEGRATION SPACES IN PREDICATIVE BISHOP-CHENG MEASURE THEORY

IOSIF PETRAKIS ^a AND MAX ZEUNER ^b

^a University of Verona, Department of Computer Science
e-mail address: iosif.petrakis@univr.it

^b Stockholm University, Mathematics Department
e-mail address: zeuner@math.su.se

ABSTRACT. Bishop’s measure theory (BMT), introduced in [Bis67], is an abstraction of the measure theory of a locally compact metric space X , and the use of an informal notion of a set-indexed family of complemented subsets is crucial to its predicative character. The more general Bishop-Cheng measure theory (BCMT), introduced in [BC72] and expanded in [BB85], is a constructive version of the classical Daniell approach to measure and integration, and highly impredicative, as many of its fundamental notions, such as the integration space of p -integrable functions L^p , rely on quantification over proper classes (from the constructive point of view). In this paper we introduce the notions of a pre-measure and pre-integration space, a predicative variation of the Bishop-Cheng notion of a measure space and of an integration space, respectively. Working within Bishop Set Theory (BST), elaborated in [Pet20b], and using the theory of set-indexed families of complemented subsets and set-indexed families of real-valued partial functions within BST, we apply the implicit, predicative spirit of BMT to BCMT. As a first example, we present the pre-measure space of complemented detachable subsets of a set X with the Dirac-measure, concentrated at a single point. Furthermore, we translate in our predicative framework the non-trivial, Bishop-Cheng construction of an integration space from a given measure space, showing that a pre-measure space induces the pre-integration space of simple functions associated to it. Finally, a predicative construction of the canonically integrable functions L^1 , as the completion of an integration space, is included.

1. INTRODUCTION

In the most popular approach to classical measure theory, see e.g., [Hal74], integration is defined through measure. Starting from a measure space (X, \mathcal{A}, μ) , one defines simple and measurable functions, the latter through the Borel sets in \mathbb{R} . As a positive measurable function is the limit of an increasing sequence of positive, simple functions, the obviously

Key words and phrases: constructive measure theory, measure spaces, integration spaces, integrable functions.

The research of the first author was partially supported by LMUexcellent, funded by the Federal Ministry of Education and Research (BMBF) and the Free State of Bavaria under the Excellence Strategy of the Federal Government and the Länder.

defined integral of a simple function is extended to the integral of a positive, measurable function. The integral of a measurable function $f: X \rightarrow \mathbb{R}$ is then defined through the integrals of the positive, measurable functions f_+ and f_- . The highly non-constructive standard approach can be roughly characterised as an approach “from sets to functions”.

In the *Daniell approach* to classical measure theory, see e.g., [Loo53, Tay73], measure is defined through integration. It was introduced by Daniell [Dan18], it was taken further by Weil [Wei40], Kolmogoroff [Kol48], Stone [Sto48], Carathéodory [Car56], and Segal [Seg54, Seg65], and it is incorporated in Bourbaki [Bou04]. The starting point of the Daniell approach is the notion of *Daniell space* (X, L, \int) , where L is a Riesz space of real-valued functions on X and $\int: L \rightarrow \mathbb{R}$ is a positive, linear functional that satisfies the Daniell property, a certain continuity condition. Using the (non-constructive) Bolzano-Weierstrass theorem, one extends L to L^+ , which is the set of functions $f: X \rightarrow \overline{\mathbb{R}}$ that are limits of increasing sequences in L , and \int is extended to $\int^+: L^+ \rightarrow \overline{\mathbb{R}}$ accordingly. The upper $\overline{\int}f$ and lower integral $\underline{\int}f$ of a function $f: X \rightarrow \overline{\mathbb{R}}$ are defined through the (non-constructive) completeness axiom of real numbers, and f is integrable, or an element of L^1 , if $\underline{\int}f = \overline{\int}f \in \mathbb{R}$. A function $f: X \rightarrow [0, +\infty]$ is called *measurable*, if it can be approximated appropriately by integrable functions, and a subset A of X is measurable, if its characteristic function χ_A is measurable, while A is *integrable*, if $\chi_A \in L^1$. If A is integrable, a measure function $A \mapsto \mu(A)$ is defined through the integral of χ_A . A clear advantage of this approach is that “certain properties of the integral already follow from the integrals of the nice functions, which are easier to handle than arbitrary integrable functions” [Wij90]. The Daniell approach can be roughly characterised as an approach “from functions to sets”.

As functions are more appropriate to constructive study than sets, Bishop followed the Daniell approach both in [Bis67], and, in a different and more uniform way, in [BC72, BB85]. We call *Bishop measure theory* (BMT) the measure theory developed by Bishop in [Bis67]. Although the integration theory of locally compact metric spaces within BMT follows the Daniell approach, the treatment of abstract measures within BMT follows the more popular approach to classical measure theory. As the Borel sets are defined inductively in BMT, the set theory required for it must accommodate inductive definitions with rules of countably many premises.

The more general theory of measure introduced in [BC72], and significantly extended in [BB85], is what we call *Bishop-Cheng measure theory* (BCMT), which makes no use of (inductively defined) Borel sets, and hence it is based on a set theory without inductive definitions. Following the tradition of the Daniell approach, Bishop and Cheng consider first the integral on a certain set L of given functions, then extend it to the larger set of functions L^1 , and define the measure at a later stage. Although complemented subsets¹ are first-class citizens both in BMT and in BCMT, their set-indexed families are not employed in BCMT. What we call here the *Bishop-Cheng integration space* is the constructive analogue to Daniell space² that captures all basic examples of the classical Daniell theory. The broadness of results within BCMT presented in [BB85] and in several related publications is striking. Numerous applications of Bishop-Cheng measure theory to probability theory and to the theory of stochastic processes are found in the older work of Chan [Cha72]-[Cha75], and

¹These are pair of subsets that are disjoint in a positive and strong way. Their use in BMT and BCMT is crucial to avoid many negatively defined concepts from their classical counterparts.

²In [Pet24] it is explained why the notion of a Bishop-Cheng integration space is a natural, constructive counterpart to the classical notion of Daniell space.

especially in his recent monograph [Cha21]. The generality of BCMT though, is due to the use of impredicative definitions, which hinder the extraction of efficient computations from proofs.

If $\mathfrak{F}^{se}(X)$ is the totality of strongly extensional, real-valued, partial functions f on a set with a given inequality $(X, =_X, \neq_X)$, the set of integrable functions L^1 is defined in BCMT by the separation scheme as follows:

$$L^1 := \{f \in \mathfrak{F}^{se}(X) \mid f \text{ is integrable}\}.$$

The membership-condition of the totality $\mathfrak{F}^{se}(X)$ involves quantification over the universe of sets, since a partial, real valued function is by definition a set A together with an embedding (or injection) i_A of A into X and a function $f: A \rightarrow \mathbb{R}$. Hence, $\mathfrak{F}^{se}(X)$ is a proper class, and the separation scheme on a proper class does not define a set. Thus, from a predicative point of view, the Bishop-Cheng definition of L^1 does not determine a set. As this impredicativity of L^1 is “dense” in BCMT, the original approach of Bishop and Cheng, as a whole, cannot express successfully the computational content of measure theory. Exactly this computational deficiency of BCMT is also recognised by Spitters in [Spi06a].

Already in the definition of a Bishop-Cheng integration space a similar problem arises. Namely, the integral is supposed to be defined on a subset L of the proper class $\mathfrak{F}^{se}(X)$, without specifying though, how such a subset can be defined i.e., how a subclass of $\mathfrak{F}^{se}(X)$ can be considered to be a set. It seems that both in [BC72] and in [BB85] the totality $\mathfrak{F}^{se}(X)$ is taken to be a set. This fundamental impredicativity built in BCMT directed the subsequent constructive studies of measure theory to different directions³.

Coquand, Palmgren, and later Spitters, also acknowledged that BCMT does not facilitate the extraction of efficient computations. According to Spitters [Spi06a], it is unlikely that BCMT “will be useful when viewing Bishop-style mathematics as a high-level programming language”. As a result, the search of the computational content of measure theory in constructive mathematics was shifted from the Bishop-Cheng theory to more abstract, algebraic, or point-free approaches (see the work of Coquand, Palmgren and Spitters, in [CP02], [Spi06a] and [CS09]). However, in terms of applications⁴, these approaches attain neither the range nor the broadness of BCMT.

Already in BMT though, Bishop avoided impredicativities by using (two) set-indexed families of complemented subsets in his definition of a measure space, in order to quantify over the index-sets only. Discussing in [Bis70], p. 67, the exact definition of a measure space in BMT within his formal system Σ , he writes the following:

³Outside Bishop’s constructivism there are various approaches to measure theory. The theory of measure [Hey56] within Brouwer’s intuitionism contradicts the classical theory, while measure theory [Eda09] within the computability framework of Type-2 Theory of Effectivity is based on classical logic. Measure theory [Šan68], [BD91] within Russian constructivism employs Markov’s principle of unbounded search. In intuitionistic Martin-Löf type theory (MLTT) [ML98] the interest lies mainly in probabilistic programming [BAVG12], while in homotopy type theory [Uni13] univalent techniques, such as higher inductive types, are applied to probabilistic programming too [BFS21].

⁴The applications to probability theory were difficult to explore in Spitters’ approach and postponed in the approach of Coquand and Palmgren. Recently, a decisive step towards a point-free treatment of measure theory has been taken by Simpson [Sim12]. Simpson advocates however in a classical framework, that sublocales, rather than subspaces, be vital. This conceptual move allows one to even circumvent some of the constraints at the outset of measure theory. A fairly constructive development, which had been kept by Simpson for future work, has been proposed by Ciraulo [Cir23]. However, Ciraulo invokes the principle of countable choice. Interestingly, he is also concerned with the status of complemented sets from a point-free perspective.

To formalize in Σ the notion of an abstract measure space, definition 1 of chapter 7 of [Bis67] must be rewritten as follows. A *measure space* is a family $\mathcal{M} \equiv \{A_t\}_{t \in T}$ of complemented subsets of a set X . . . , a map $\mu : T \rightarrow \mathbb{R}^{0+}$, and an additional structure If s and t are in T , there exists an element $s \vee t$ of T such that $A_{s \vee t} < A_s \cup A_t$. Similarly, there exist operations \wedge and \sim on T , corresponding to the set theoretic operations \cap and $-$. The usual algebraic axioms are assumed, such as $\sim(s \vee t) = \sim s \wedge \sim t$ Considerations such as the above indicate that essentially all of the material in [Bis67], appropriately modified, can be comfortably formalised in Σ .

This indexisation method, roughly sketched in [Bis67], is elaborated within Bishop Set Theory (BST) in [Pet20b]. Based on this, we present here the first crucial steps to a predicative reconstruction (PBCMT) of BCMT. Following Bishop’s explanations in [Bis70], we replace a totality of strongly extensional, real-valued, partial functions L in the original definition of a Bishop-Cheng integration space by a set-indexed family Λ of such partial functions. Applying tools and results from [Pet20b], we recover the concept of an integration space in an indexised form. The predicative advantage of the indexisation method within PBCMT is that crucial quantifications are over an index-set and not proper classes. Following [Pet20b], we elaborate the concept of a pre-integration space in which the index-set I is equipped with all necessary operations so that a pre-integral \int can be defined on I . A pre-integration space induces a predicative integration space, the integral \int^* of which on the partial function f_i is given, for every $i \in I$, by

$$\int^* f_i := \int i.$$

We provide a predicative treatment of L^1 by considering only the canonically integrable functions⁵ of a given pre-integration space. Our main result is that the set-indexed family of canonically integrable functions admits the structure of a pre-integration space (Theorem 10.6), which is an appropriate completion of the original pre-integration space (Theorem 10.8). The theory developed in [Pet20b] together with careful arguments that avoid the use of the class of full sets and countable choice (see [Ric01, Sch04] for a critique to the use of countable choice in Bishop-style constructive mathematics, also known as BISH) helped us prove a constructive and predicative version of Lebesgue’s series theorem (Theorem 10.4), which is crucial to the proof of our main result. A predicative definition of L^1 ensures that all concepts defined through quantification over L^1 in BCMT are also predicative. For example, quantification over L^1 is used in the Bishop-Cheng definition of a full set, which is a constructive counterpart to the complement of a null set in classical measure theory. This predicative treatment of L^1 is the first, clear indication that the computational content of measure theory can be grasped by the predicative reconstruction PBCMT of the original BCMT.

2. OVERVIEW OF THIS PAPER

We structure this paper as follows:

- In section 3 we describe the connection between complemented subsets and boolean-valued partial functions, which explains the crucial role of partial functions in BCMT. The constructive way to employ the passage from functions to sets in the classical Daniell approach

⁵This terminology is introduced by Spitters in [Spi02].

through the use of characteristic functions of subsets, is to work with complemented subsets and their (partial) characteristic functions.

- In section 4 we describe the basic properties of set-indexed families of subsets of a given set X . We discuss the set-character within BST of the totality of families of subsets of X indexed by some set I , which will be relevant to our presentation of a pre-measure space.
- In section 5 we define within BST the notions of a family of partial functions and of a family of complemented subsets indexed by some set I . These function-theoretic concepts will be used in PBCMT instead of the abstract sets of partial functions and of subsets, respectively, that are considered in BCMT.
- In section 6 we introduce the notion of a pre-measure space as a predicative counterpart to the notion of Bishop-Cheng measure space in BCMT. The pre-measure space of complemented detachable subsets of a set X with the Dirac-measure concentrated at a single point is studied.
- In section 7 we include the facts on real-valued, partial functions that are necessary to the definition of a pre-integration space within BST (Definition 8.2).
- In section 8 we introduce the notion of a pre-integration space as a predicative counterpart to the notion of an integration space in BCMT. We also briefly describe the pre-integration space $(X, I, \int d\mu)$, where X is a locally compact metric space X with a so-called modulus of local compactness, I is the set of functions with compact support on X , and the integral $\int f d\mu$ of $f \in I$ is the measure $\mu(f)$, where μ is a positive measure on X (Theorem 8.3).
- In section 9 we construct the pre-integration space of simple functions from a given pre-measure space (Theorem 9.9). This is a predicative translation within BST of the construction of a Bishop-Cheng integration space from the simple functions of a measure space (Theorem 10.10 in [BB85]). Although we follow the corresponding construction in section 10 of chapter 6 in [BB85] closely, our approach allows us to not only work completely predicatively, but also to carry out all proofs avoiding the axiom of countable choice.
- In section 10 we first present the canonically integrable functions explicitly as a family of partial functions, in order to avoid the impredicativities of the original Bishop-Cheng definition of L^1 . Based on a predicative version of Lebesgue's series theorem (Theorem 10.4), we then show that this family admits the structure of a pre-integration space (Theorem 10.6) and explain in what sense it can be seen as the completion of our original pre-integration space (Theorem 10.8).
- In section 11 we list some question for future work stemming from the material presented here.

We work within BST, which behaves as a high-level programming language. For all notions and results of Bishop set theory that are used here without definition or proof we refer to [Pet21], in this journal⁶, and to [Pet20b, Pet22]. For all notions and results of constructive real analysis that are used here without definition or proof we refer to [BB85]. The type-theoretic interpretation of Bishop's set theory into the theory of setoids (see especially the work of Palmgren [Pal05]-[PW14]) has become nowadays the standard way to understand Bishop sets⁷. Other suitable, yet different, formal systems for BISH are Myhill's Constructive Set Theory (CST), introduced in [Myh75], and Aczel's system CZF (see [AR10]).

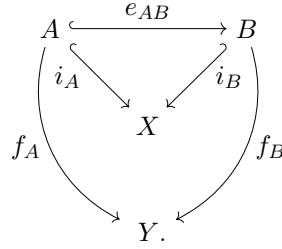
⁶In [Pet21] the theory of spectra of Bishop spaces (see [Pet15]-[Pet19] and [Pet20a]-[Pet23]) is developed within BST.

⁷For an analysis of the relation between intensional MLTT and Bishop's theory of sets see [Pet20b], Chapter 1.

3. PARTIAL FUNCTIONS AND COMPLEMENTED SUBSETS

Bishop set theory (BST), elaborated in [Pet20b], is an informal, constructive theory of totalities and assignment routines that serves as a “completion” of Bishop’s original theory of sets in [Bis67, BB85]. Its first aim is to fill in the “gaps”, or highlight the fundamental notions that were suppressed by Bishop in his account of the set theory underlying Bishop-style constructive mathematics BISH. Its second aim is to serve as an intermediate step between Bishop’s theory of sets and an *adequate* and *faithful* formalisation of BISH in Feferman’s sense [Fef79]. To assure faithfulness, we use concepts or principles that appear, explicitly or implicitly, in BISH. BST “completes” Bishop’s theory of sets in the following ways. It uses explicitly a universe of (predicative) sets \mathbb{V}_0 , which is a proper class. It separates clearly sets from proper classes. Dependent operations, which were barely mentioned in [Bis67, BB85], are first-class citizens in BST. An elaborated theory of set-indexed families of sets is included in BST. As an introduction to the basic concepts of BST is included in [Pet21], in this journal, and in [Pet20b, Pet22], we refer the reader to these sources for all basic concepts and results within BST that are mentioned here without further explanation or proof. Next we present some basic properties of partial functions and complemented subsets within BST, which are necessary to the rest of this paper. A *subset* of a set X is a pair (A, i_A) , where $(A, =_A)$ is a set and $i_A: A \hookrightarrow X$ is an embedding i.e., $i_A(a) =_X i_A(a') \Rightarrow a =_A a'$, for every $a, a' \in A$. The intersection of two subsets is given by the corresponding pullback, and their union is defined in [Bis67], p. 64. We denote the set of functions from A to X by $\mathbb{F}(A, X)$.

Definition 3.1. Let X, Y be sets. A partial function from X to Y is a triplet $\mathbf{f}_A := (A, i_A, f_A)$, where $(A, i_A) \subseteq X$, and $f_A \in \mathbb{F}(A, Y)$. We call f_A total, if $\text{dom}(\mathbf{f}_A) := A =_{\mathcal{P}(X)} X$. Let $\mathbf{f}_A \leq \mathbf{f}_B$, if there is an embedding $e_{AB}: A \hookrightarrow B$ such that the following triangles commute



In this case we write $e_{AB}: \mathbf{f}_A \leq \mathbf{f}_B$. The partial function space $\mathfrak{F}(X, Y)$ is equipped with the equality $\mathbf{f}_A =_{\mathfrak{F}(X, Y)} \mathbf{f}_B := \Leftrightarrow \mathbf{f}_A \leq \mathbf{f}_B \ \& \ \mathbf{f}_B \leq \mathbf{f}_A$. If X, Y are equipped with inequalities \neq_X, \neq_Y , respectively, let $\mathfrak{F}^{\text{se}}(X, Y)$ be the totality⁸ of strongly extensional elements of $\mathfrak{F}(X, Y)$.

Definition 3.2. If $2 := \{0, 1\}$ and $\mathbf{f} = (A, i_A, f_A), \mathbf{g} = (B, i_B, g_B) \in \mathfrak{F}(X, 2)$, let $\mathbf{f} \vee \mathbf{g} := \max\{\mathbf{f}, \mathbf{g}\} = (A \cap B, i_{A \cap B}, f_A \vee g_B)$, $\mathbf{f} \cdot \mathbf{g} := \mathbf{f} \wedge \mathbf{g} := \min\{\mathbf{f}, \mathbf{g}\} := (A \cap B, i_{A \cap B}, f_A \wedge g_B)$, $\sim \mathbf{f} := 1 - \mathbf{f} := (A, i_A, 1 - f_A)$, and $\mathbf{f} \sim \mathbf{g} := \mathbf{f} \wedge (\sim \mathbf{g})$, where 1 also denotes the constant function on A with value 1.

An inequality on a set X induces a positively defined notion of disjointness of subsets of X , which in turn induces the notion of a complemented subset of X . In this way the negatively defined notion of the set-theoretic complement of a subset is avoided.

⁸As the membership condition for $\mathfrak{F}(X, Y)$ requires quantification over the universe of sets \mathbb{V}_0 , the totalities $\mathfrak{F}(X, Y)$ and $\mathfrak{F}^{\text{se}}(X, Y)$ are proper classes.

Definition 3.3. Let $(X, =_X, \neq_X)$ be a set with inequality, and $(A, i_A), (B, i_B) \subseteq X$. We say that A and B are disjoint with respect to \neq_X , in symbols $A \parallel B$, if $\forall a \in A \forall b \in B (i_A(a) \neq_X i_B(b))$. A *complemented subset* of X is a pair $\mathbf{A} := (A^1, A^0)$, where $(A^1, i_{A^1}), (A^0, i_{A^0}) \subseteq X$, such that $A^1 \parallel A^0$. The *characteristic function* of \mathbf{A} is the operation⁹ $\chi_{\mathbf{A}} : A^1 \cup A^0 \rightsquigarrow 2$, defined by

$$\chi_{\mathbf{A}}(x) := \begin{cases} 1 & , x \in A^1 \\ 0 & , x \in A^0. \end{cases}$$

We call \mathbf{A} total, if $\text{dom}(\mathbf{A}) := A^1 \cup A^0 =_{\mathcal{P}(X)} X$, Let $\mathbf{A} \subseteq \mathbf{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0$, and the totality of complemented subsets $\mathcal{P}^{\parallel}(X)$ of X is equipped with the equality $\mathbf{A} =_{\mathcal{P}^{\parallel}(X)} \mathbf{B} :\Leftrightarrow \mathbf{A} \subseteq \mathbf{B} \ \& \ \mathbf{B} \subseteq \mathbf{A}$.

Clearly, the complemented powerset $\mathcal{P}^{\parallel}(X)$ of X is a proper class. If $f_1 : A^1 \subseteq B^1$ and $f_0 : B^0 \subseteq A^0$, then f_1, f_0 are strongly extensional functions. E.g., if $f_1(a_1) \neq_{B^1} f_1(a_1')$, for some $a_1, a_1' \in A^1$, then from the definition of the canonical inequality \neq_{B^1} we get $i_{B^1}(f_1(a_1)) \neq_X i_{B^1}(f_1(a_1'))$. By the extensionality of \neq_X we get $i_{A^1}(a_1) \neq_X i_{A^1}(a_1') :\Leftrightarrow a_1 \neq_{A^1} a_1'$.

Example 3.4. If $(X, =_X)$ is a set, let the following inequality on X :

$$x \neq_{(X, \mathbb{F}(X, 2))} x' :\Leftrightarrow \exists f \in \mathbb{F}(X, 2) (f(x) =_2 1 \ \& \ f(x') =_2 0)$$

If $f \in \mathbb{F}(X, 2)$, the following extensional subsets of X

$$\begin{aligned} \delta_0^1(f) &:= \{x \in X \mid f(x) =_2 1\}, \\ \delta_0^0(f) &:= \{x \in X \mid f(x) =_2 0\}, \end{aligned}$$

are called detachable, or free subsets of X . Clearly, $\boldsymbol{\delta}(f) := (\delta_0^1(f), \delta_0^0(f))$ is a complemented subset of X with respect to the inequality $\neq_X^{\mathbb{F}(X, 2)}$. The characteristic function $\chi_{\boldsymbol{\delta}(f)}$ of $\boldsymbol{\delta}(f)$ is (definitionally equal to) f (recall that $f(x) =_2 1 :\Leftrightarrow f(x) := 1$), and $\delta_0^1(f) \cup \delta_0^0(f) = X$.

Remark 3.5. If $\mathbf{A} \in \mathcal{P}^{\parallel}(X)$, then $\chi_{\mathbf{A}} := (A^1 \cup A^0, i_{A^1 \cup A^0}, \chi_{\mathbf{A}}) \in \mathfrak{F}^{\text{se}}(X, 2)$.

Proof. Let $z, w \in A^1 \cup A^0$ with $\chi_{\mathbf{A}}(z) \neq_2 \chi_{\mathbf{A}}(w)$. If for example $\chi_{\mathbf{A}}(z) := 1$ and $\chi_{\mathbf{A}}(w) := 0$, then $z \in A^1, w \in A^0$. As $A^1 \parallel A^0$, we get $i_{A^1}(z) \neq_X i_{A^0}(w) \Leftrightarrow z \neq_{A^1 \cup A^0} w$. \square

Definition 3.6. If $\mathbf{A}, \mathbf{B} \in \mathcal{P}^{\parallel}(X)$, let the following operations¹⁰ on them:

$$\begin{aligned} \mathbf{A} \vee \mathbf{B} &:= ([A^1 \cap B^1] \cup [A^1 \cap B^0] \cup [A^0 \cap B^1], A^0 \cap B^0), \\ \mathbf{A} \wedge \mathbf{B} &:= (A^1 \cap B^1, [A^1 \cap B^0] \cup [A^0 \cap B^1] \cup [A^0 \cap B^0]), \\ -\mathbf{A} &:= (A^0, A^1), \\ \mathbf{A} - \mathbf{B} &:= \mathbf{A} \wedge (-\mathbf{B}). \end{aligned}$$

⁹A non-dependent assignment routine $f : A \rightsquigarrow B$, where A and B are sets, is called an *operation*. A *function* is an operation that preserves the corresponding equalities. See [Pet21] for a more detailed explanation.

¹⁰BMT and BCMT involve different operations on complemented subsets. We only describe the algebra of complemented subsets given in BCMT, which has a more ‘‘linear’’ behavior (see also [Shu22]). For total complemented subsets the operations given in BMT and BCMT coincide. See [PW22] for an in-depth comparison of the two algebras of complemented subsets.

Proposition 3.7. $(\mathcal{P}\llbracket(X), \wedge, \vee, -)$ satisfies all properties of a distributive lattice except¹¹ for the absorption equalities $(\mathbf{A} \wedge \mathbf{B}) \vee \mathbf{A} = \mathbf{A}$ and $(\mathbf{A} \vee \mathbf{B}) \wedge \mathbf{A} = \mathbf{A}$. Moreover, $-(-\mathbf{A}) = \mathbf{A}$, and $-(\mathbf{A} \vee \mathbf{B}) = (-\mathbf{A}) \wedge (-\mathbf{B})$, for every $\mathbf{A}, \mathbf{B} \in \mathcal{P}\llbracket(X)$.

The classical bijection between $\mathcal{P}(X)$ and 2^X is translated constructively as the existence of “bijective”, proper class-assignment routines between the proper classes $\mathcal{P}\llbracket(X)$ and $\mathfrak{F}^{\text{se}}(X, 2)$. The proof of the following fact is found in [PW22], and it is the only place in this paper that we refer to assignment routines defined on proper classes.

Proposition 3.8. Consider the proper class-assignment routines

$$\begin{aligned} \chi: \mathcal{P}\llbracket(X) &\rightsquigarrow \mathfrak{F}^{\text{se}}(X, 2) \quad \& \quad \delta: \mathfrak{F}^{\text{se}}(X, 2) \rightsquigarrow \mathcal{P}\llbracket(X), \\ \mathbf{A} \mapsto \chi(\mathbf{A}) \quad \mathbf{f}_A &\mapsto \delta(\mathbf{f}_A) := (\delta^1(\mathbf{f}_A), \delta^0(\mathbf{f}_A)), \\ \delta^1(\mathbf{f}_A) &:= \{a \in A \mid \mathbf{f}_A(a) =_2 1\} =: [\mathbf{f}_A =_2 1], \\ \delta^0(\mathbf{f}_A) &:= \{a \in A \mid \mathbf{f}_A(a) =_2 0\} =: [\mathbf{f}_A =_2 0]. \end{aligned}$$

Then χ, δ are well-defined, proper class-functions, which are inverse to each other. Moreover, $\delta(\sim \mathbf{f}) =_{\mathcal{P}\llbracket(X)} -\delta(\mathbf{f})$ and $\chi_{-\mathbf{A}} =_{\mathfrak{F}(X, 2)} \sim \chi_{\mathbf{A}}$, where $\mathbf{f} \in \mathfrak{F}^{\text{se}}(X, 2)$ and $\mathbf{A} \in \mathcal{P}\llbracket(X)$.

Proposition 3.9. Let $\mathbf{A}, \mathbf{B} \in \mathcal{P}\llbracket(X)$ and $\mathbf{f}, \mathbf{g} \in \mathfrak{F}^{\text{se}}(X, 2)$.

- (i) $\chi_{\mathbf{A} \vee \mathbf{B}} =_{\mathfrak{F}(X, 2)} \chi_{\mathbf{A}} \vee \chi_{\mathbf{B}}$, $\chi_{\mathbf{A} \wedge \mathbf{B}} =_{\mathfrak{F}(X, 2)} \chi_{\mathbf{A}} \wedge \chi_{\mathbf{B}}$.
- (ii) $\chi_{-\mathbf{A}} =_{\mathfrak{F}(X, 2)} 1 - \chi_{\mathbf{A}}$ and $\chi_{\mathbf{A} - \mathbf{B}} =_{\mathfrak{F}(X, 2)} \chi_{\mathbf{A}}(1 - \chi_{\mathbf{B}})$.
- (iii) $\delta(\mathbf{f}_A) \vee \delta(\mathbf{f}_B) =_{\mathcal{P}\llbracket(X)} \delta(\mathbf{f}_A \vee \mathbf{f}_B)$ and $\delta(\mathbf{f}_A) \wedge \delta(\mathbf{f}_B) =_{\mathcal{P}\llbracket(X)} \delta(\mathbf{f}_A \wedge \mathbf{f}_B)$.
- (iv) $\delta(\mathbf{f}_A \sim \mathbf{f}_B) =_{\mathcal{P}\llbracket(X)} \delta(\mathbf{f}_A) - \delta(\mathbf{f}_B)$.

4. FAMILIES OF SUBSETS

In this section we present the basic notions and facts on set-indexed families of subsets that are going to be used in the rest of the paper. Roughly speaking, a family of subsets of a set X indexed by some set I is an assignment routine $\lambda_0: I \rightsquigarrow \mathcal{P}(X)$ that behaves like a function i.e., if $i =_I j$, then $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. The following definition is a formulation of this rough description that reveals the witnesses of the equality $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$. This is done “internally”, through the embeddings of the subsets into X . The equality $\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j)$, which is defined “externally” through the transport maps (see [Pet21], Definition 3.1), follows, and a family of subsets is also a family of sets. We start by introducing some notation. For details we refer to [Pet21].

Definition 4.1. Let I be as set and $\lambda_0: I \rightsquigarrow \mathbb{V}_0$. A dependent operation over λ_0

$$\Phi: \bigwedge_{i \in I} \lambda_0(i)$$

assigns to each $i \in I$ an element $\Phi(i) := \Phi_i \in \lambda_0(i)$. We denote by $\mathbb{A}(I, \lambda_0)$ the totality of dependent operations over λ_0 equipped with the equality

$$\Phi =_{\mathbb{A}(I, \lambda_0)} \Psi :\Leftrightarrow \forall i \in I (\Phi_i =_{\lambda_0(i)} \Psi_i).$$

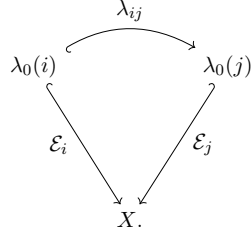
¹¹In [BB85], p. 74, it is mentioned that complemented subsets satisfy “all the usual finite algebraic laws that do not involve the operation of set complementation”. In [CP02], p. 695, it is noticed though, that the absorption equalities are not satisfied.

Definition 4.2. Let X and I be sets and let $D(I) := \{(i, i') \in I \times I \mid i =_I i'\}$ be the diagonal of I . A *family of subsets* of X indexed by I , is a triplet $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$,

$$\begin{aligned} \mathcal{E} &: \bigwedge_{i \in I} \mathbb{F}(\lambda_0(i), X), & \mathcal{E}(i) &:= \mathcal{E}_i; & i \in I, \\ \lambda_1 &: \bigwedge_{(i,j) \in D(I)} \mathbb{F}(\lambda_0(i), \lambda_0(j)), & \lambda_1(i, j) &:= \lambda_{ij}; & (i, j) \in D(I), \end{aligned}$$

such that the following conditions hold:

- (i) For every $i \in I$, the function $\mathcal{E}_i : \lambda_0(i) \rightarrow X$ is an embedding.
- (ii) For every $i \in I$, we have that $\lambda_{ii} =_{\mathbb{F}(\lambda_0(i), \lambda_0(i))} \text{id}_{\lambda_0(i)}$.
- (iii) For every $(i, j) \in D(I)$ we have that $\mathcal{E}_i =_{\mathbb{F}(\lambda_0(i), X)} \mathcal{E}_j \circ \lambda_{ij}$.



We call a pair $A_i := (\lambda_0(i), \mathcal{E}_i)$ an element of $\Lambda(X)$. If $(A, i_A) \in \mathcal{P}(X)$, the *constant I-family of subsets* A is the pair $C^A(X) := (\lambda_0^A, \mathcal{E}^A, \lambda_1^A)$, where $\lambda_0(i) := A$, $\mathcal{E}_i^A := i_A$, and $\lambda_1(i, j) := \text{id}_A$, for every $i \in I$ and $(i, j) \in D(I)$, respectively. If $(A, i_A), (B, i_B) \subseteq X$, the triplet $\Lambda^2(X) := (\lambda_0^2, \mathcal{E}, \lambda_1^2)$, where $\lambda_0^2(0) := A$ and $\lambda_0^2(1) := B$, $\mathcal{E}_0 := i_A$ and $\mathcal{E}_1 := i_B$, $\lambda_1^2(0, 0) := \text{id}_A$ and $\lambda_1^2(1, 1) := \text{id}_B$ is the 2-family of subsets A and B of X . If $\mathbf{Fam}(I, X)$ denotes the totality of I -families of subsets of X , its equality is defined as in [Pet21], Definition 3.2.

Example 4.3. Let $(X, =_X, \neq_{(X, \mathbb{F}(X, 2))})$ be the set with inequality from Example 3.4. The family of subsets $\Delta^1(X) := (\delta_0^1, \mathcal{E}^1, \delta_1^1)$ over the index-set $\mathbb{F}(X, 2)$ is defined by the following rules:

$$\begin{aligned} \delta_0^1 &: \mathbb{F}(X, 2) \rightsquigarrow \mathbb{V}_0, & f &\mapsto \delta_0^1(f), & f \in \mathbb{F}(X, 2), \\ \mathcal{E}^1 &: \bigwedge_{f \in \mathbb{F}(X, 2)} \mathbb{F}(\delta_0^1(f), X), & \mathcal{E}_f^1 &: \delta_0^1(f) \hookrightarrow X & x \mapsto x; & x \in \delta_0^1(f), \\ \delta_1^1 &: \bigwedge_{(f,g) \in D(\mathbb{F}(X, 2))} \mathbb{F}(\delta_0^1(f), \delta_0^1(g)), & \delta_1^1(f, g) &:= \delta_{fg}^1 : \delta_0^1(f) \rightarrow \delta_0^1(g) & x \mapsto x; & x \in \delta_0^1(f). \end{aligned}$$

If $\Delta^0(X) := (\delta_0^0, \mathcal{E}^0, \delta_1^0)$, where $\delta_0^0 : \mathbb{F}(X, 2) \rightsquigarrow \mathbb{V}_0$ is defined by the rule $f \mapsto \delta_0^0(f)$, for every $f \in \mathbb{F}(X, 2)$, and the dependent operations $\mathcal{E}^0, \delta_1^0$ are defined similarly to \mathcal{E}^1 and δ_1^1 , then $\Delta^1(X), \Delta^0(X)$ are *sets* of subsets of X in the following sense.

Definition 4.4. Let $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$. We say that $\Lambda(X)$ is a *set* of subsets of X if

$$\forall_{i,j \in I} (\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j) \Rightarrow i =_I j).$$

In this case we write $\Lambda(X) \in \mathbf{Set}(I, X)$. We can always make $\Lambda(X)$ into a set of subsets $\tilde{\Lambda}(X) := (\lambda_0, \mathcal{E}, \tilde{\lambda}_1)$ indexed by the set $\lambda_0 I(X)$, where $\lambda_0 I(X)$ is the totality I with a new equality given by

$$i =_{\lambda_0 I(X)} j :\Leftrightarrow \lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j),$$

for every $i, j \in I$. The assignment routine $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ and the dependent function $\mathcal{E} : \bigwedge_{i \in I} \mathbb{F}(\lambda_0(i), X)$ are the same as in $\Lambda(X)$. Using the *dependent version of Myhill's axiom of unique choice*¹², one can define the dependent function $\tilde{\lambda}_1 : \bigwedge_{(i,j) \in D(\lambda_0 I(X))} \mathbb{F}(\lambda_0(i), \lambda_0(j))$.

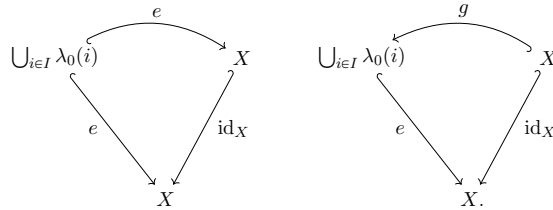
As we explained in [Pet21], the totality $\mathbf{Fam}(I)$ of all I -families of sets cannot be accepted as a set, as the constant I -family with value $\mathbf{Fam}(I)$ would then be defined through a totality in which it belongs to. This does not work as an argument against the set-character of $\mathbf{Fam}(I, X)$. It is not clear how the constant I -family $\mathbf{Fam}(I, X)$ can be seen as a family of subsets of X . If $\nu_0(i) := \mathbf{Fam}(I, X)$, for every $i \in I$, we need to define a modulus of embeddings $\mathcal{N}_i : \mathbf{Fam}(I, X) \hookrightarrow X$, for every $i \in I$. From the given data one could define the assignment routine \mathcal{N}_i by the rule $\mathcal{N}_i(\Lambda(X)) := \mathcal{E}_i(u_i)$, if it is known that $u_i \in \lambda_0(i)$. Even in that case, the assignment routine \mathcal{N}_i cannot be shown to satisfy the expected properties. Clearly, if \mathcal{N}_i was defined by the rule $\mathcal{N}_i(\Lambda(X)) := x_0 \in X$, then it cannot be an embedding. The set-character of the totality $\mathbf{Fam}(I, X)$ is related to the definition of a pre-measure space (see also the discussion after the definition of a Bishop-Cheng measure space in section 6). Next we describe the Sigma- and the Pi-set of a family of subsets.

Definition 4.5. Let $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$. The *interior union*, or the union of $\Lambda(X)$ is the totality $\sum_{i \in I} \lambda_0(i)$, which we denote in this case by $\bigcup_{i \in I} \lambda_0(i)$. Let the non-dependent assignment routine $e : \bigcup_{i \in I} \lambda_0(i) \rightsquigarrow X$ defined by $(i, x) \mapsto \mathcal{E}_i(x)$, for every $(i, x) \in \bigcup_{i \in I} \lambda_0(i)$, and let

$$(i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow e(i, x) =_X e(j, y) :\Leftrightarrow \mathcal{E}_i(x) =_X \mathcal{E}_j(y).$$

If \neq_X is an inequality on X , let $(i, x) \neq_{\bigcup_{i \in I} \lambda_0(i)} (j, y) :\Leftrightarrow \mathcal{E}_i(x) \neq_X \mathcal{E}_j(y)$. The family $\Lambda(X)$ is called a *covering* of X , or $\Lambda(X)$ covers X , if $\bigcup_{i \in I} \lambda_0(i) =_{\mathcal{P}(X)} X$. If \neq_I is an inequality on I , we say that $\Lambda(X)$ is a family of disjoint subsets of X (with respect to \neq_I), if $\forall_{i,j \in I} (i \neq_I j \Rightarrow \lambda_0(i) \parallel \lambda_0(j))$, where by Definition 3.3 $\lambda_0(i) \parallel \lambda_0(j) :\Leftrightarrow \forall_{u \in \lambda_0(i)} \forall_{w \in \lambda_0(j)} (\mathcal{E}_i(u) \neq_X \mathcal{E}_j(w))$.

Clearly, $=_{\bigcup_{i \in I} \lambda_0(i)}$ is an equality on $\bigcup_{i \in I} \lambda_0(i)$, and the operation e is an embedding of $\bigcup_{i \in I} \lambda_0(i)$ into X , hence $(\bigcup_{i \in I} \lambda_0(i), e) \subseteq X$. The inequality $\neq_{\bigcup_{i \in I} \lambda_0(i)}$ is the canonical inequality of the subset $\bigcup_{i \in I} \lambda_0(i)$ of X . Hence, if $(X, =_X, \neq_X)$ is discrete, then $(\bigcup_{i \in I} \lambda_0(i), =_{\bigcup_{i \in I} \lambda_0(i)}, \neq_{\bigcup_{i \in I} \lambda_0(i)})$ is discrete, and if \neq_X is tight, then $\neq_{\bigcup_{i \in I} \lambda_0(i)}$ is tight. As the following left diagram commutes, $\Lambda(X)$ covers X , if and only if the following right diagram commutes i.e., if and only if $X \subseteq \bigcup_{i \in I} \lambda_0(i)$



¹²According to it, if (μ_0, μ_1) is an I -family of sets such that for every $i \in I$ there is a unique (up to equality) $x_i \in \mu_0(i)$, then there is a dependent assignment routine $\Phi : \bigwedge_{i \in I} \mu_0(i)$. The non-dependent version of this axiom is generally accepted by the practitioners of BISH and it is included in Myhill's system CST in [Myh75]. If $i =_{\lambda_0 I(X)} j$, then by the equality $\lambda_0(i) =_{\mathcal{P}(X)} \lambda_0(j)$ there is a *unique* function $\lambda_0(i) \rightarrow \lambda_0(j)$ commuting with the embeddings \mathcal{E}_i and \mathcal{E}_j . To avoid Myhill's axiom, we need to add to our data a dependent assignment routine Δ that corresponds to every element of the diagonal of $\lambda_0 I(X)$ an element of the set of witnesses (e, e') of the corresponding equalities.

If $(i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y)$, it is not necessary that $i =_I j$, hence it is not necessary that $(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y)$ (as we show in the next proposition, the converse implication holds). Consequently, the first projection operation $\mathbf{pr}_1^{\Lambda(X)} := \mathbf{pr}_1^\Lambda$, where Λ is the I -family of sets induced by $\Lambda(X)$, is not necessarily a function! The second projection map on $\Lambda(X)$ is defined by $\mathbf{pr}_2^{\Lambda(X)} := \mathbf{pr}_2^\Lambda$. Notice that $\neq_{\bigcup_{i \in I} \lambda_0(i)}$ is an inequality on $\bigcup_{i \in I} \lambda_0(i)$, without supposing neither an inequality on I , nor an inequality on the sets $\lambda_0(i)$'s. The following remarks are straightforward to show.

Remark 4.6. Let $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$.

- (i) If $(i, x) =_{\sum_{i \in I} \lambda_0(i)} (j, y)$, then $(i, x) =_{\bigcup_{i \in I} \lambda_0(i)} (j, y)$.
- (ii) If $e: \sum_{i \in I} \lambda_0(i) \rightsquigarrow X$ is an embedding, then $(\sum_{i \in I} \lambda_0(i), e) =_{\mathcal{P}(X)} (\bigcup_{i \in I} \lambda_0(i), e)$.
- (iii) If \neq_I is tight and $\Lambda(X)$ is a family of disjoint subsets with respect to \neq_I , then $e: \sum_{i \in I} \lambda_0(i) \hookrightarrow X$.

Remark 4.7. If $i_0 \in I$, $(A, i_A) \subseteq X$, and $C^A(X) := (\lambda_0^A, \mathcal{E}^A, \lambda_1^A) \in \mathbf{Fam}(I, X)$ is the constant family A of subsets of X , then

$$\bigcup_{i \in I} A := \bigcup_{i \in I} \lambda_0^A(i) =_{\mathcal{P}(X)} A.$$

Remark 4.8. If $\Lambda^2(X)$ is the 2-family of subsets A, B of X , $\bigcup_{i \in 2} \lambda_0^2(i) =_{\mathcal{P}(X)} A \cup B$.

Definition 4.9. Let $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$, and $i_0 \in I$. The intersection $\bigcap_{i \in I} \lambda_0(i)$ of $\Lambda(X)$ is the totality defined by

$$\Phi \in \bigcap_{i \in I} \lambda_0(i) :\Leftrightarrow \Phi \in \mathbb{A}(I, \lambda_0) \ \& \ \forall_{i, j \in I} (\mathcal{E}_i(\Phi_i) =_X \mathcal{E}_j(\Phi_j)).$$

Let $e: \bigcap_{i \in I} \lambda_0(i) \rightsquigarrow X$ be defined by $e(\Phi) := \mathcal{E}_{i_0}(\Phi_{i_0})$, for every $\Phi \in \bigcap_{i \in I} \lambda_0(i)$, and

$$\Phi =_{\bigcap_{i \in I} \lambda_0(i)} \Theta :\Leftrightarrow e(\Phi) =_X e(\Theta) :\Leftrightarrow \mathcal{E}_{i_0}(\Phi_{i_0}) =_X \mathcal{E}_{i_0}(\Theta_{i_0}),$$

If \neq_X is a given inequality on X , let $\Phi \neq_{\bigcap_{i \in I} \lambda_0(i)} \Theta :\Leftrightarrow \mathcal{E}_{i_0}(\Phi_{i_0}) \neq_X \mathcal{E}_{i_0}(\Theta_{i_0})$.

The following remarks are straightforward to show.

Remark 4.10. Let $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$.

- (i) $\Phi =_{\bigcap_{i \in I} \lambda_0(i)} \Theta \Leftrightarrow \Phi =_{\mathbb{A}(I, \lambda_0)} \Theta$.
- (ii) If $\Phi \in \bigcap_{i \in I} \lambda_0(i)$, then $\Phi \in \prod_{i \in I} \lambda_0(i)$.
- (iii) If $(X, =_X, \neq_X)$ is discrete, the set $(\bigcap_{i \in I} \lambda_0(i), =_{\bigcap_{i \in I} \lambda_0(i)}, \neq_{\bigcap_{i \in I} \lambda_0(i)})$ is discrete.

Remark 4.11. Let $i_0 \in I$, $(A, i_A) \subseteq X$, and $C^A(X) := (\lambda_0^A, \mathcal{E}^A, \lambda_1^A) \in \mathbf{Fam}(I, X)$ the constant family A of subsets of X . Then

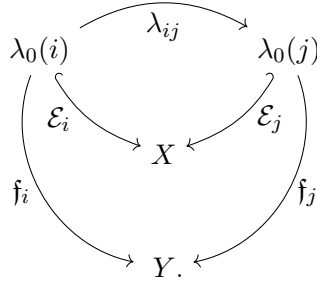
$$\bigcap_{i \in I} A := \bigcap_{i \in I} \lambda_0^A(i) =_{\mathcal{P}(X)} A.$$

Remark 4.12. If $\Lambda^2(X)$ is the 2-family of subsets A, B of X , $\bigcap_{i \in 2} \lambda_0^2(i) =_{\mathcal{P}(X)} A \cap B$.

5. FAMILIES OF PARTIAL FUNCTIONS AND FAMILIES OF COMPLEMENTED SUBSETS

Next we define within BST the notions of a family of partial functions and of a family of complemented subsets indexed by some set I . These function-theoretic concepts will be used in PBCMT instead of the abstract sets of partial functions and of subsets, respectively, that are considered in BCMT.

Definition 5.1. Let X, Y and I be sets. A *family of partial functions* from X to Y indexed by I , or an *I -family of partial functions from X to Y* , is a triplet $\Lambda(X, Y) := (\lambda_0, \mathcal{E}, \lambda_1, \mathbf{f})$, where $\Lambda(X) := (\lambda_0, \mathcal{E}, \lambda_1) \in \mathbf{Fam}(I, X)$ and $\mathbf{f} : \bigwedge_{i \in I} \mathbb{F}(\lambda_0(i), Y)$ with $\mathbf{f}(i) := \mathbf{f}_i$, for every $i \in I$, such that, for every $(i, j) \in D(I)$, the following diagrams commute



If $i \in I$, we call the partial function $\mathbf{f}_i := (\lambda_0(i), \mathcal{E}_i, \mathbf{f}_i) \in \mathfrak{F}(X, Y)$ an element of $\Lambda(X, Y)$.

The equality on the totality $\mathbf{Fam}(I, X, Y)$ of I -families of partial functions from X to Y can be defined in an obvious way, analogously to the equality on $\mathbf{Fam}(I, X)$ given in [Pet21, Def. 3.2].

Clearly, if $\Lambda(X, Y) \in \mathbf{Fam}(I, X, Y)$ and $(i, j) \in D(I)$, then $(\lambda_{ij}, \lambda_{ji}) : \mathbf{f}_i =_{\mathfrak{F}(X, Y)} \mathbf{f}_j$.

Definition 5.2. Let $\Lambda(X, Y) := (\lambda_0, \mathcal{E}, \lambda_1, \mathbf{f}) \in \mathbf{Fam}(I, X, Y)$. We say that $\Lambda(X, Y)$ is a *set of partial functions from X to Y* if

$$\forall_{i, j \in I} (\mathbf{f}_i =_{\mathfrak{F}(X, Y)} \mathbf{f}_j \Rightarrow i =_I j).$$

In this case we write $\Lambda(X, Y) \in \mathbf{Set}(I, X, Y)$ and even $\Lambda(X, Y) \in \mathbf{Set}^{\text{se}}(I, X, Y)$ if $\Lambda(X, Y)$ is a family of strongly extensional partial functions.

As described in Definition 4.4, we can make $\Lambda(X, Y)$ into a $\lambda_0 I(X, Y)$ -set of partial functions $\tilde{\Lambda}(X, Y) := (\lambda_0, \mathcal{E}, \tilde{\lambda}_1, \mathbf{f})$. As in the case for subsets, $\lambda_0 I(X, Y)$ is the totality I equipped with the equality

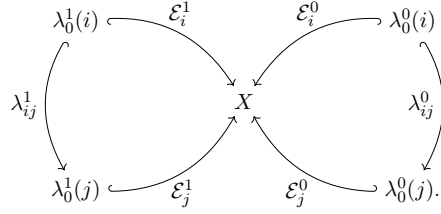
$$i =_{\lambda_0 I(X, Y)} j \Leftrightarrow \mathbf{f}_i =_{\mathfrak{F}(X, Y)} \mathbf{f}_j,$$

or if $\Lambda(X, Y)$ is a family of strongly extensional partial functions,

$$i =_{\lambda_0 I(X, Y)} j \Leftrightarrow \mathbf{f}_i =_{\mathfrak{F}^{\text{se}}(X, Y)} \mathbf{f}_j.$$

The only component changing is the new $\tilde{\lambda}_1$, which is defined using dependent unique choice.

Definition 5.3. Let the sets $(X, =_X, \neq_X)$ and $(I, =_I)$. A *family of complemented subsets of X* indexed by I , or an *I -family of complemented subsets of X* , is a structure $\mathbf{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^1, \lambda_1^1, \lambda_0^0, \mathcal{E}^0, \lambda_1^0)$, such that $\mathbf{\Lambda}^1(X) := (\lambda_0^1, \mathcal{E}^1, \lambda_1^1) \in \mathbf{Fam}(I, X)$ and $\mathbf{\Lambda}^0(X) := (\lambda_0^0, \mathcal{E}^0, \lambda_1^0) \in \mathbf{Fam}(I, X)$ i.e., for every $(i, j) \in D(I)$, the following diagrams commute



Moreover, for every $i \in I$ the element $\lambda_0(i) := (\lambda_0^1(i), \lambda_0^0(i))$ of $\mathbf{A}(X)$ is in $\mathcal{P}^{\parallel}(X)$. Again, the equality on $\mathbf{Fam}(I, \mathbf{X})$, the totality of I -families of complemented subsets of X , is defined in an obvious way, analogously to [Pet21], Definition 3.2.

As in the case of $\mathbf{Fam}(I, X)$, we assume the totality $\mathbf{Fam}(I, \mathbf{X})$ to be a set. The operations \wedge and \vee between complemented subsets in Definition 3.6 are extended to families of complemented subsets. We write $\mathbf{Set}(I, \mathbf{X})$ for the totality of I -sets of complemented subsets of X , which are defined completely analogously to sets of subsets or partial functions.

6. PRE-MEASURE SPACES

In this section we introduce the notion of a pre-measure space as a predicative counterpart to the notion of Bishop-Cheng measure space in BCMT. The pre-measure space of complemented detachable subsets of a set X with the Dirac-measure concentrated at a single point is described. The notion of a Bishop-Cheng measure space is defined in [BB85], p. 282, and appeared first¹³ in [BC72] p. 47.

Definition 6.1 Bishop-Cheng measure space. A (Bishop-Cheng) measure space is a triplet (X, M, μ) consisting of an inhabited set with inequality $(X, =_X, \neq_X)$, a set M of complemented sets in X , and a mapping μ of M into \mathbb{R}^{0+} , such that the following properties hold:

- (BCMS₁) If \mathbf{A} and \mathbf{B} belong to M , then so do $\mathbf{A} \vee \mathbf{B}$ and $\mathbf{A} \wedge \mathbf{B}$, and $\mu(\mathbf{A}) + \mu(\mathbf{B}) = \mu(\mathbf{A} \vee \mathbf{B}) + \mu(\mathbf{A} \wedge \mathbf{B})$.
- (BCMS₂) If \mathbf{A} and $\mathbf{A} \wedge \mathbf{B}$ belong to M , then so does $\mathbf{A} - \mathbf{B}$, and $\mu(\mathbf{A}) = \mu(\mathbf{A} \wedge \mathbf{B}) + \mu(\mathbf{A} - \mathbf{B})$.
- (BCMS₃) There exists \mathbf{A} in M such that $\mu(\mathbf{A}) > 0$.
- (BCMS₄) If (\mathbf{A}_n) is a sequence of elements of M such that $\lim_{k \rightarrow \infty} \mu(\bigwedge_{n=1}^k \mathbf{A}_n)$ exists and is positive, then $\bigcap_n \mathbf{A}_n^1$ is inhabited.

The elements of M are the *integrable sets* of the measure space (X, M, μ) , and for each \mathbf{A} in M the non negative number $\mu(\mathbf{A})$ is the *measure* of \mathbf{A} . In the above definition there is no indication how the set M of complemented sets of X is constructed, and (BCMS₂) requires quantification over the universe \mathbb{V}_0 , as \mathbf{B} is an arbitrary complemented subsets of X . In [Bis67], p. 183, Bishop used two families of complemented subsets, in order to avoid such a quantification in his definition of a measure space within BMT. One set-indexed family which \mathbf{A} and $\mathbf{A} \wedge \mathbf{B}$ belong to, and one which \mathbf{B} belongs to. In Definition 6.2 we predicatively reformulate the Bishop-Cheng definition of a measure space. Especially for condition (BCMS₂) we provide two alternatives. In the first one, condition (PMS₂) in Definition 6.2, we use the fact that within BST the totality $\mathbf{Fam}(\mathbb{1}, X)$ of $\mathbb{1}$ -families of complemented subsets, where

¹³In [BC72], p. 55, condition (BCMS₁) appears in the equivalent form: if $\mathbf{B} \in M$ with $B^1 \subseteq A^1$ and $B^0 \subseteq A^0$, then $\mathbf{A} \in M$.

$\mathbb{1} := \{0\}$, is assumed to be a set¹⁴, hence quantification over it is allowed. In the second alternative, the weaker condition (PMS₂^{*}) in Definition 6.2, only quantification over the index-set is required. If $\mathbf{\Lambda}(X)$ is an I -family of complemented subsets of X , we can define an equality on the index set I , such that the converse implication $\lambda_0(i) =_{\mathcal{P}\mathbb{1}(X)} \lambda_0(j) \Rightarrow i = j$ also holds. The family $\mathbf{\Lambda}(X)$ is then called (as in the case of a family of subsets in [Bis67], p. 65) a *set* of complemented subsets. Consequently, functions on the index-set I are extended to functions on the complemented subsets $\lambda_0(i)$'s.

One could predicatively reformulate the definition of a Bishop-Cheng measure space within BST. We proceed instead directly to define the notion of a pre-measure space, giving an explicit formulation of Bishop's suggestion, expressed in [Bis70], p. 67, with respect to Definition 6.2. The main idea is to define operations on I that correspond to the operations on complemented subsets, and reformulate accordingly the clauses for the measure μ .

Definition 6.2 (Pre-measure space within BST). Let $(X, =_X, \neq_X)$ be an inhabited set, and let $(I, =_I)$ be equipped with operations $\vee : I \times I \rightsquigarrow I$, $\wedge : I \times I \rightsquigarrow I$ (for simplicity we use the same symbols with the ones for the operations on complemented subsets), and $\sim : I \times I \rightsquigarrow I$. If $i, j \in I$, let $i \leq j := \Leftrightarrow i \wedge j = i$. Let $\mathbf{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^1, \lambda_1^1; \lambda_0^0, \mathcal{E}^0, \lambda_1^0) \in \mathbf{Set}(I, \mathbf{X})$, and $\mu : I \rightarrow [0, +\infty)$ such that the following conditions hold:

$$\text{(PMS}_1) \quad \forall_{i,j \in I} \left(\lambda_0(i) \vee \lambda_0(j) = \lambda_0(i \vee j) \ \& \ \lambda_0(i) \wedge \lambda_0(j) = \lambda_0(i \wedge j) \ \& \right. \\ \left. \lambda_0(i) - \lambda_0(j) = \lambda_0(i \sim j) \ \& \ \mu(i) + \mu(j) = \mu(i \vee j) + \mu(i \wedge j) \right).$$

$$\text{(PMS}_2) \quad \forall_{i \in I} \forall_{\mathbf{A}(X) \in \mathbf{Fam}(\mathbb{1}, \mathbf{X})} \left[\exists_{k \in I} \left(\lambda_0(i) \wedge \alpha_0(0) = \lambda_0(k) \right) \Rightarrow \right. \\ \left. \lambda_0(i) - \alpha_0(0) = \lambda_0(i \sim k) \ \& \ \mu(i) = \mu(k) + \mu(i \sim k) \right].$$

$$\text{(PMS}_3) \quad \exists_{i \in I} (\mu(i)) > 0.$$

$$\text{(PMS}_4) \quad \forall_{\alpha \in \mathbf{F}(\mathbb{N}, I)} \left[\exists \lim_{m \rightarrow +\infty} \mu \left(\bigwedge_{n=1}^m \alpha(n) \right) \ \& \ \lim_{m \rightarrow +\infty} \mu \left(\bigwedge_{n=1}^m \alpha(n) \right) > 0 \Rightarrow \right. \\ \left. \exists_{x \in X} \left(x \in \bigcap_{n \in \mathbb{N}} \lambda_0^1(\alpha(n)) \right) \right].$$

We call the triplet $\mathcal{M}(\mathbf{\Lambda}(X)) := (X, I, \mu)$ a *pre-measure space*, the function μ a *pre-measure*, and the index-set I the set of integrable, or measurable indices of $\mathcal{M}(\mathbf{\Lambda}(X))$.

Alternatively to (PMS₂), one could use the following clause:

$$\text{(PMS}_2^*) \quad \forall_{i,j \in I} (\mu(i) = \mu(i \wedge j) + \mu(i \sim j)).$$

Remark 6.3. Condition (PMS₂^{*}) involves quantification over a set and is absolutely safe from a predicative point of view. Actually, it is only (PMS₂^{*}) that is needed to construct the pre-integration space of simple functions.

Corollary 6.4. Let $\mathcal{M}(\mathbf{\Lambda}) := (X, I, \mu)$ be a pre-measure space and $i, j \in I$.

- (i) The operations \vee , \wedge and \sim are functions, and the triplet (I, \vee, \wedge) satisfies the properties of a distributive lattice, except from the absorption equalities.

¹⁴Notice that in order to define an $\mathbb{1}$ -family of complemented subsets we need *first* to construct a complemented subset (A^1, A^0) of X , and *afterwards* to define $\lambda_0^0(0) := A^1$ and $\lambda_0^0(0) := A^0$.

$$(ii) \quad i \leq j \Leftrightarrow \lambda_0(i) \subseteq \lambda_0(j).$$

Proof. (i) We show that \vee is a function, and for \wedge and \sim we proceed similarly. We have that

$$\begin{aligned} i = i' \ \& \ j = j' &\Rightarrow \lambda_0(i) = \lambda_0(i') \ \& \ \lambda_0(j) = \lambda_0(j') \\ &\Rightarrow \lambda_0(i) \vee \lambda_0(j) = \lambda_0(i') \vee \lambda_0(j') \\ &\Rightarrow \lambda_0(i \vee j) = \lambda_0(i' \vee j') \\ &\Rightarrow i \vee j = i' \vee j'. \end{aligned}$$

All properties follow from the corresponding properties of complemented subsets (Proposition 3.7), from (PMS₁), and the fact that $\mathbf{\Lambda}(X) \in \mathbf{Set}(I, \mathbf{X})$. E.g., to show $i \vee j = j \vee i$, we use the equalities $\lambda_0(i \vee j) = \lambda_0(i) \vee \lambda_0(j) = \lambda_0(j) \vee \lambda_0(i) = \lambda_0(j \vee i)$. For the rest of the proof we proceed similarly. \square

Next we give a constructive treatment of the classical Dirac measure as a pre-measure on a set of integrable indices I . First we consider the total case, where $I := \mathbb{F}(X, 2)$ is a Boolean algebra.

Proposition 6.5. *Let $(X, =_X, \neq_{(X, \mathbb{F}(X, 2))})$ be a set inhabited by some $x_0 \in X$, and let the maps $\vee, \wedge, \sim: \mathbb{F}(X, 2) \times \mathbb{F}(X, 2) \rightarrow \mathbb{F}(X, 2)$ and $\sim: \mathbb{F}(X, 2) \rightarrow \mathbb{F}(X, 2)$, defined by the corresponding rules given in Definition 3.2 for partial functions. If $\mathbf{\Delta}(X) := (\delta_0^1, \mathcal{E}^1, \delta_1^1, \delta_0^0, \mathcal{E}^0, \delta_1^0)$ is the set of complemented detachable subsets of X , where by Example 4.3*

$$\delta_0(f) := (\delta_0^1(f), \delta_0^0(f)) := ([f = 1], [f = 0]),$$

and if $\mu_{x_0}: \mathbb{F}(X, 2) \rightsquigarrow [0, +\infty)$ is defined by the rule

$$\mu_{x_0}(f) := f(x_0) =: \chi_{\delta_0(f)}(x_0); \quad f \in \mathbb{F}(X, 2),$$

then the triplet $\mathcal{M}(\mathbf{\Delta}(X)) := (X, \mathbb{F}(X, 2), \mu_{x_0})$ is a pre-measure space.

Proof. Straightforward calculations as in the proof of Proposition 3.9(iii) prove the required equalities between complemented in condition (PMS₁). Clearly, the operation μ_{x_0} is a function, and a simple case-distinction shows the required equality $f(x_0) + g(x_0) = [f(x_0) \vee g(x_0)] + [f(x_0) \wedge g(x_0)]$. Let $f \in \mathbb{F}(X, 2)$ and $\mathbf{B} := (B^1, B^0) \in \mathcal{P}^{\parallel}(X)$ with $\alpha_0(0) := \mathbf{B}$. If $g \in \mathbb{F}(X, 2)$ such that

$$\begin{aligned} \delta_0(f) \wedge \mathbf{B} &:= ([f = 1] \cap B^1, ([f = 1] \cap B^0) \cup ([f = 0] \cap B^1) \cup ([f = 0] \cap B^0)) \\ &= ([g = 1], [g = 0]), \end{aligned}$$

then the equality between the following complemented subsets

$$\delta_0(f) - \mathbf{B} = ([f = 1] \cap B^1, ([f = 1] \cap B^1) \cup ([f = 0] \cap B^0) \cup ([f = 0] \cap B^1)),$$

$$\delta_0(f \sim g) = ([f = 1] \cap [g = 0], [f = 1 = g] \cup [f = 0 = g] \cup [f = 0] \cap [g = 1])$$

follows after considering all necessary cases (the rule *Ex falsum quodlibet* is necessary to this proof). The required equality $f(x_0) = g(x_0) + f(x_0) \wedge (1 - g)(x_0)$ follows after considering all cases. As $\mu_{x_0}(1) = 1 > 0$, (PMS₃) follows. For the proof of (PMS₄) we fix $\alpha: \mathbb{N} \rightarrow \mathbb{F}(X, 2)$,

and we suppose that

$$\begin{aligned} & \exists \lim_{m \rightarrow +\infty} \mu_{x_0} \left(\bigwedge_{n=0}^m \alpha(n) \right) \ \& \ \lim_{m \rightarrow +\infty} \mu_{x_0} \left(\bigwedge_{n=0}^m \alpha(n) \right) > 0 \Leftrightarrow \\ & \exists \lim_{m \rightarrow +\infty} \left(\bigwedge_{n=0}^m \alpha(n) \right) (x_0) \ \& \ \lim_{m \rightarrow +\infty} \left(\bigwedge_{n=0}^m \alpha(n) \right) (x_0) > 0 \Leftrightarrow \\ & \exists \lim_{m \rightarrow +\infty} \prod_{n=0}^m [\alpha(n)](x_0) \ \& \ \lim_{m \rightarrow +\infty} \prod_{n=0}^m [\alpha(n)](x_0) > 0. \end{aligned}$$

Finally, we have that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \prod_{n=0}^m [\alpha(n)](x_0) > 0 & \Rightarrow \lim_{m \rightarrow +\infty} \prod_{n=0}^m [\alpha(n)](x_0) = 1 \\ & \Leftrightarrow \exists m_0 \in \mathbb{N} \forall m \geq m_0 \left(\prod_{n=0}^m [\alpha(n)](x_0) = 1 \right) \\ & \Rightarrow \forall n \in \mathbb{N} ([\alpha(n)](x_0) = 1) \\ & \Leftrightarrow x_0 \in \bigcap_{n \in \mathbb{N}} \delta_0^1(\alpha(n)). \quad \square \end{aligned}$$

Remark 6.6. Although the derivation of (PMS₂) in the above proof requires the Ex falsum quodlibet rule, the derivation of (PMS₂^{*}) rests on trivial calculations. Hence the whole proof in the latter case can be carried out in minimal logic!

If partial functions are considered, then using Proposition 3.9(iii)-(iv) we get similarly the following constructive version of the Dirac measure.

Proposition 6.7. *Let $\Lambda(X, 2) := (\lambda_0, \mathcal{E}, \lambda_1, \mathfrak{f})$ be a family of (strongly extensional) partial functions from X to 2 , with $\mathbf{f}_i := (\lambda_0(i), \mathcal{E}_i, \mathfrak{f}_i) \in \mathfrak{F}^{\text{se}}(X, 2)$, for every $i \in I$. Let \vee, \wedge, \sim be operations on I , such that for every $i, j \in I$ we have that $\mathbf{f}_{i \vee j} = \mathbf{f}_i \vee \mathbf{f}_j$, $\mathbf{f}_{i \wedge j} = \mathbf{f}_i \wedge \mathbf{f}_j$, and $\mathbf{f}_{i \sim j} = \mathbf{f}_i \sim \mathbf{f}_j$. Moreover, let \sim be an operation on I , such that if \mathbf{B} is a given complemented subset of X , then the equality $\delta(\mathbf{f}_i) \wedge \mathbf{B} = \delta(\mathbf{f}_k)$ implies $\delta(\mathbf{f}_i) - \mathbf{B} = \delta(\mathbf{f}_{i \sim k})$, where the assignment routine δ is defined in Proposition 3.8. If $x_0 \in X$ such that $x_0 \in \bigcap_{i \in I} \lambda_0(i)$, and if $\mu_{x_0}: I \rightsquigarrow [0, +\infty)$ is defined by the rule $\mu_{x_0}(i) := \mathfrak{f}_i(x_0)$, for every $i \in I$, then the triplet $\mathcal{M}(\Delta(I, X)) := (X, I, \mu_{x_0})$ is a pre-measure space.*

7. REAL-VALUED, PARTIAL FUNCTIONS

Next we present the facts on real-valued, partial functions that are necessary for the definition of an integration space within BST (Definition 8.2).

Definition 7.1. If $(X, =_X, \neq_X)$ is an inhabited set, let $\mathbf{f}_A := (A, i_A, f_A) \in \mathfrak{F}(X, \mathbb{R})$. We call \mathbf{f}_A strongly extensional, if f_A is strongly extensional, where A is equipped with its canonical inequality as a subset of X i.e., $f_A(a) \neq_{\mathbb{R}} f_A(a') \Rightarrow i_A(a) \neq_X i_A(a')$, for every $a, a' \in A$ (where $a \neq_{\mathbb{R}} b \Leftrightarrow a < b \vee b < a$, for every $a, b \in \mathbb{R}$). Let $\mathfrak{F}(X) := \mathfrak{F}(X, \mathbb{R})$ be the class of partial functions from X to \mathbb{R} , and $\mathfrak{F}^{\text{se}}(X)$ the class of strongly extensional, partial functions from $(X, =_X, \neq_X)$ to $(\mathbb{R}, =_{\mathbb{R}}, \neq_{\mathbb{R}})$. Let $|\mathbf{f}_A| := (A, i_A, |f_A|)$. If $\mathbf{f}_B := (B, i_B, f_B)$ in $\mathfrak{F}(X)$ and $\lambda \in \mathbb{R}$

$$\begin{array}{ccccc}
A & \xrightarrow{i_A} & X & \xleftarrow{i_B} & B \\
& \searrow f_A & & \swarrow f_B & \\
& & \mathbb{R} & &
\end{array}$$

let $\lambda f_A := (A, i_A, \lambda f_A) \in \mathfrak{F}(X)$ and $f_A \square f_B := (A \cap B, i_{A \cap B}, (f_A \square f_B)_{A \cap B})$, where

$$(f_A \square f_B)_{A \cap B}(a, b) := f_A(a) \square f_B(b); \quad (a, b) \in A \cap B, \quad \square \in \{+, \cdot, \wedge, \vee\}.$$

The totality of I -families of strongly extensional, partial functions i.e., of structures $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}, \lambda_1, \mathfrak{f})$, with $\mathbf{f}_i := (\lambda_0(i), \mathcal{E}_i, \mathfrak{f}_i)$ strongly extensional, for every $i \in I$, is denoted by $\mathbf{Fam}^{\text{se}}(I, X, \mathbb{R})$.

The operation $(f_A \square f_B)_{A \cap B}: A \cap B \rightsquigarrow \mathbb{R}$ is a function. If $(a, b) =_{A \cap B} (a', b') \Leftrightarrow i_A =_X i_B(a') \Leftrightarrow a =_A a'$, we get $f_A(a) =_{\mathbb{R}} f_A(a')$. Since $i_B(b) =_X i_A(a)$ and $i_B(b') =_X i_A(a')$, we also get $b =_B b'$ and hence $f_B(b) =_{\mathbb{R}} f_B(b')$. If λ denotes also the constant function $\lambda \in \mathbb{R}$ on X we get as a special case the partial function $f_A \wedge \lambda := (A \cap X, i_{A \cap X}, (f_A \wedge \lambda)_{A \cap X})$, where $A \cap X := \{(a, x) \in A \times X \mid i_A(a) =_X x\}$, $i_{A \cap X}(a, x) := i_A(a)$, and $(f_A \wedge \lambda)_{A \cap X}(a, x) := f_A(a) \wedge \lambda(x) := f_A(a) \wedge \lambda$, for every $(a, x) \in A \cap X$. By Definition 5.1, if $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}, \lambda_1, \mathfrak{f}) \in \mathbf{Fam}(I, X, \mathbb{R})$, then $\mathbf{f}_i := (\lambda_0(i), \mathcal{E}_i, \mathfrak{f}_i) \in \mathfrak{F}(X)$, for every $i \in I$, and if $i =_I j$, the following diagrams commute

$$\begin{array}{ccc}
\lambda_0(i) & \xrightarrow{\lambda_{ij}} & \lambda_0(j) \\
\mathcal{E}_i \searrow & & \swarrow \mathcal{E}_j \\
& X & \\
\mathfrak{f}_i \searrow & & \swarrow \mathfrak{f}_j \\
& \mathbb{R} &
\end{array}$$

If \mathbf{f}_i is strongly extensional, then, for every $u, w \in \lambda_0(i)$, we get $\mathfrak{f}_i(u) \neq_{\mathbb{R}} \mathfrak{f}_i(w) \Rightarrow \mathcal{E}_i(u) \neq_X \mathcal{E}_i(w)$. We may also regard $\Lambda(X, \mathbb{R})$ as a $\lambda_0 I(X, \mathbb{R})$ -set of real valued, strongly extensional, partial functions, following the construction in Definition 5.2.

Definition 7.2. Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}, \lambda_1, \mathfrak{f}) \in \mathbf{Fam}^{\text{se}}(I, X, \mathbb{R})$. We write $g: \lambda_0 I(X, \mathbb{R}) \rightarrow Y$ to denote a function $g: I \rightarrow Y$, where I is equipped with the equality in Definition 5.2, and we may also write $g(\mathbf{f}_i)$ instead of $g(i)$. If $\kappa: \mathbb{N}^+ \rightarrow I$, the family

$$\Lambda(X, \mathbb{R}) \circ \kappa := (\lambda_0 \circ \kappa, \mathcal{E} \circ \kappa, \lambda_1 \circ \kappa, \mathfrak{f} \circ \kappa) \in \mathbf{Fam}(\mathbb{N}^+, X, \mathbb{R})$$

is the κ -subsequence of $\Lambda(X, \mathbb{R})$, where $(\lambda_0 \circ \kappa)(n) := \lambda_0(\kappa(n))$, $(\mathcal{E} \circ \kappa)_n := \mathcal{E}_{\kappa(n)}$, $(\lambda_1 \circ \kappa)(n, n) := \lambda_{\kappa(n)\kappa(n)} := \text{id}_{\lambda_0(\kappa(n))}$ and $(\mathfrak{f} \circ \kappa)_n := \mathfrak{f}_{\kappa(n)}$ for every $n \in \mathbb{N}^+$.

If we consider the intersection $\bigcap_{n \in \mathbb{N}^+} (\lambda_0 \circ \kappa)(n) := \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n))$, by Definition 4.9 we get

$$\Phi \in \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \Leftrightarrow \Phi: \bigwedge_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \ \& \ \forall_{n, m \in \mathbb{N}^+} (\mathcal{E}_{\kappa(n)}(\Phi_n) =_X \mathcal{E}_{\kappa(m)}(\Phi_m)),$$

$$\Phi =_{\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n))} \Theta \Leftrightarrow \mathcal{E}_{\kappa(1)}(\Phi_1) =_X \mathcal{E}_{\kappa(1)}(\Theta_1),$$

$$e^{\Lambda(X, \mathbb{R}) \circ \kappa}: \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \hookrightarrow X, \quad e^{\Lambda(X, \mathbb{R}) \circ \kappa}(\Phi) := (\mathcal{E} \circ \kappa)_1(\Phi_1) := \mathcal{E}_{\kappa(1)}(\Phi_1).$$

Definition 7.3. Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}, \lambda_1, \mathfrak{f}) \in \mathbf{Fam}(I, X, \mathbb{R})$, $\kappa: \mathbb{N}^+ \rightarrow I$, and $\Lambda(X, \mathbb{R}) \circ \kappa$ the κ -subsequence of $\Lambda(X, \mathbb{R})$. If $(A, i_A) \subseteq \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n))$, we define the function

$$\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)}: A \rightarrow \mathbb{R}, \quad \left(\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)} \right)(a) := \sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)} \left([i_A(a)]_n \right); \quad a \in A,$$

under the assumption that the series on the right converges in \mathbb{R} , for every $a \in A$.

In the special case $(\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)), \text{id}_{\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n))}) \subseteq \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n))$, we get the function

$$\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)}: \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \rightarrow \mathbb{R}, \quad \left(\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)} \right)(\Phi) := \sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)}(\Phi_n); \quad \Phi \in \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)),$$

under the same convergence assumption. The following fact is shown in [Pet20b], pp. 212–213, and it is used in Definition 10.2.

Proposition 7.4. *If in Definition 7.3 the partial functions $\mathfrak{f}_{\kappa(n)} := (\lambda_0(\kappa(n)), \mathcal{E}_{\kappa(n)}, \mathfrak{f}_{\kappa(n)})$ are strongly extensional, for every $n \in \mathbb{N}^+$, then the real-valued, partial function*

$$\mathfrak{f}_A := \left(A, e^{\Lambda(X, \mathbb{R}) \circ \kappa} \circ i_A, \sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)} \right)$$

$$A \xleftarrow{i_A} \bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \xleftarrow{e^{\Lambda(X, \mathbb{R}) \circ \kappa}} X$$

$$\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\kappa(n)} \curvearrowright \mathbb{R}$$

is strongly extensional.

8. PRE-INTEGRATION SPACES

In this section, and in accordance to our previous predicative reconstruction of Bishop-Cheng measure space, we introduce the notion of a pre-integration space as a predicative counterpart to the notion of an integration space in BCMT. The notion of a Bishop-Cheng integration space is defined in [BB85], p. 217, and appeared first in [BC72], p. 2. Condition (BCIS₂) is the constructive counterpart to Daniell's classical continuity condition in the definition of a Daniell space. The exact relation of a Bishop-Cheng integration space to that of a Daniell space is explained in [Pet24].

Definition 8.1 Bishop-Cheng integration space. A triplet (X, L, \int) is a (Bishop-Cheng) integration space if $(X, =_X, \neq_X)$ is an inhabited set with inequality, L is a subset of $\mathfrak{F}^{\text{se}}(X)$, and $\int: L \rightarrow \mathbb{R}$, such that the following properties hold.

(BCIS₁) If $\mathfrak{f}, \mathfrak{g} \in L$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha \mathfrak{f} + \beta \mathfrak{g}$, $|\mathfrak{f}|$, and $\mathfrak{f} \wedge \mathbf{1}$ belong to L , and

$$\int(\alpha \mathfrak{f} + \beta \mathfrak{g}) = \alpha \int \mathfrak{f} + \beta \int \mathfrak{g}.$$

- (BCIS₂) If $\mathbf{f} \in L$ and (\mathbf{f}_n) is a sequence of non-negative functions in L such that $\sum_n \int \mathbf{f}_n$ converges and $\sum_n \int (\mathbf{f}_n) < \int \mathbf{f}$, then there exists $x \in X$ such that $\sum_n \mathbf{f}_n(x)$ converges and $\sum_n \mathbf{f}_n(x) < \mathbf{f}(x)$.
- (BCIS₃) There exists a function \mathbf{p} in L with $\int \mathbf{p} = 1$.
- (BCIS₄) For each $\mathbf{f} \in L$, $\lim_{n \rightarrow \infty} \int (\mathbf{f} \wedge \mathbf{n}) = \int \mathbf{f}$ and $\lim_{n \rightarrow \infty} \int (|\mathbf{f}| \wedge \mathbf{n}^{-1}) = 0$.

As already mentioned in the introduction, there is no explanation how the set L is “separated” from the proper class $\mathfrak{F}^{\text{se}}(X)$, so that the integral \int can be defined as a real-valued function on L . The extensional character of L is also not addressed. This impredicative approach to L is behind the simplicity of the Bishop-Cheng integration space. E.g., in condition (BCIS₄) the formulation of the limit is immediate as the terms $\mathbf{f} \wedge \mathbf{n} \in L$ and \int is defined on L . If one predicatively reformulates the Bishop-Cheng definition though, where an I -family of strongly extensional, real-valued, partial functions is going to be used instead of L , then one needs to use an element $\alpha(n)$ of the index-set I such that $\mathbf{f} \wedge \mathbf{n} =_{\mathfrak{F}^{\text{se}}(X)} \mathbf{f}_{\alpha(n)}$, in order to express the corresponding limit. The formulation of the continuity condition (BCIS₂) takes the form

$$\forall_{i \in I} \forall_{\kappa \in \mathbb{F}(\mathbb{N}^+, I)} \left\{ \left[\sum_{n \in \mathbb{N}^+} \int \mathbf{f}_{\kappa(n)} \in \mathbb{R} \ \& \ \sum_{n \in \mathbb{N}^+} \int \mathbf{f}_{\kappa(n)} < \int \mathbf{f}_i \right] \Rightarrow \right. \\ \left. \exists_{(\Phi, u) \in \left(\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \right) \cap \lambda_0(i)} \left(\sum_{n \in \mathbb{N}^+} \mathbf{f}_{\kappa(n)}(\Phi_n) \in \mathbb{R} \ \& \ \sum_{n \in \mathbb{N}^+} \mathbf{f}_{\kappa(n)}(\Phi_n) < \mathbf{f}_i(u) \right) \right\},$$

where

$$\left(\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \right) \cap \lambda_0(i) := \left\{ (\Phi, u) \in \left(\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \right) \times \lambda_0(i) \mid \mathcal{E}_{\kappa(1)}(\Phi) =_X \mathcal{E}_i(u) \right\},$$

$$\begin{array}{ccc} \lambda_0(\kappa(n)) & \xrightarrow{\mathcal{E}_{\kappa(n)}} & X & \xleftarrow{\mathcal{E}_i} & \lambda_0(i) \\ & \searrow \mathbf{f}_{\kappa(n)} & & & \swarrow \mathbf{f}_i \\ & & \mathbb{R} & & \end{array}$$

Next we directly formulate the Bishop-Cheng definition of an integration space using appropriate operations on the index-set of the appropriate family (set) of real-valued partial functions that replaces the original impredicative subset (actually, proper-class) L of $\mathfrak{F}^{\text{se}}(X)$.

Definition 8.2 (Pre-integration space within BST). Let $(X, =_X, \neq_X)$ be an inhabited set with inequality, and let the set $(I, =_I)$ be equipped with operations $\cdot_a : I \rightsquigarrow I$, for every $a \in \mathbb{R}$, $+$: $I \times I \rightsquigarrow I$, $|\cdot|$: $I \rightsquigarrow I$, and $\wedge_1 : I \rightsquigarrow I$, where

$$\cdot_a(i) := a \cdot i, \quad +(i, j) := i + j, \quad |\cdot|(i) := |i|; \quad i \in I, \ a \in \mathbb{R}.$$

Let also the operation $\wedge_a : I \rightsquigarrow I$, defined by the previous operations through the rule

$$\wedge_a := \cdot_a \circ \wedge_1 \circ \cdot_{a^{-1}}; \quad a \in \mathbb{R} \ \& \ a > 0.$$

Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}, \lambda_1, \mathbf{f}) \in \mathbf{Set}^{\text{se}}(I, X, \mathbb{R})$ i.e., $\mathbf{f}_i =_{\mathfrak{F}^{\text{se}}(X)} \mathbf{f}_j \Rightarrow i =_I j$, for every $i, j \in I$, and $\mathbf{f}_i := (\lambda_0(i), \mathcal{E}_i, \mathbf{f}_i)$ is strongly extensional, for every $i \in I$. Let $\int : I \rightarrow \mathbb{R}$ be a function, where $i \mapsto \int i$, for every $i \in I$, such that the following conditions hold:

$$\text{(PIS}_1) \quad \forall_{i \in I} \forall_{a \in \mathbb{R}} \left(a \mathbf{f}_i = \mathbf{f}_{a \cdot i} \ \& \ \int a \cdot i = a \int i \right).$$

$$(PIS_2) \quad \forall_{i,j \in I} \left(\mathbf{f}_i + \mathbf{f}_j = \mathbf{f}_{i+j} \ \& \ \int (i+j) = \int i + \int j \right).$$

$$(PIS_3) \quad \forall_{i \in I} (|\mathbf{f}_i| = \mathbf{f}_{|i|}).$$

$$(PIS_4) \quad \forall_{i \in I} (\mathbf{f}_i \wedge \mathbf{1} = \mathbf{f}_{\wedge_1(i)}).$$

$$(PIS_5) \quad \forall_{i \in I} \forall_{\kappa \in \mathbb{F}(\mathbb{N}^+, I)} \left\{ \left[\sum_{n \in \mathbb{N}^+} \int \kappa(n) \in \mathbb{R} \ \& \ \sum_{n \in \mathbb{N}^+} \int \kappa(n) < \int i \right] \Rightarrow \right. \\ \left. \exists_{(\Phi, u) \in \left(\bigcap_{n \in \mathbb{N}^+} \lambda_0(\kappa(n)) \right) \cap \lambda_0(i)} \left(\sum_{n \in \mathbb{N}^+} \mathbf{f}_{\kappa(n)}(\Phi_n) \in \mathbb{R} \ \& \ \sum_{n \in \mathbb{N}^+} \mathbf{f}_{\kappa(n)}(\Phi_n) < \mathbf{f}_i(u) \right) \right\}$$

$$(PIS_6) \quad \exists_{i \in I} \left(\int i =_{\mathbb{R}} 1 \right).$$

$$(PIS_7) \quad \forall_{i \in I} \left(\lim_{n \rightarrow +\infty} \int \wedge_n(i) \in \mathbb{R} \ \& \ \lim_{n \rightarrow +\infty} \int \wedge_n(i) = \int i \right).$$

$$(PIS_8) \quad \forall_{i \in I} \left(\lim_{n \rightarrow +\infty} \int \wedge_{\frac{1}{n}}(|i|) \in \mathbb{R} \ \& \ \lim_{n \rightarrow +\infty} \int \wedge_{\frac{1}{n}}(|i|) = 0 \right).$$

We call the structure $\mathcal{L}_0 := (X, I, \Lambda(X, \mathbb{R}), \int)$ a pre-integration space.

All the operations on I defined above are functions. E.g., since $\Lambda(X, \mathbb{R}) \in \mathbf{Set}^{\text{se}}(I, X, \mathbb{R})$,

$$i =_I i' \Rightarrow \mathbf{f}_i = \mathbf{f}_{i'} \Rightarrow a\mathbf{f}_i = a\mathbf{f}_{i'} \Rightarrow \mathbf{f}_{a \cdot i} = \mathbf{f}_{a \cdot i'} \Rightarrow a \cdot i =_I a \cdot i'.$$

The most fundamental example of an integration space within BCMT is that induced by a positive measure μ on a locally compact metric space X i.e., a non-zero linear map on the functions with compact support $C^{\text{supp}}(X)$ (see [BB85], pp. 220-221). In [Gru22, GP23] this major example is described as a pre-integration space. For that a notion of a locally compact metric space with a modulus of local compactness is introduced. If (X, d) is an inhabited metric space with $x_0 \in X$, and $(K_n)_{n \in \mathbb{N}}$ is a sequence of compact subsets of X , a *modulus of local compactness* for X is a function $\kappa: \mathbb{N} \rightarrow \mathbb{N}$, $n \mapsto \kappa(n)$, such that $[d_{x_0} \leq n] \subseteq K_{\kappa(n)}$, for every $n \in \mathbb{N}$, where $d_{x_0}: X \rightarrow [0, +\infty)$ is defined by $d_{x_0}(x) := d(x, x_0)$, for every $x \in X$. In this way the initial impredicativity of Bishop's notion of a locally compact metric space (for every bounded subset B of X , there is a compact subset K of X with $B \subseteq K$) is avoided. If $(X, d, (K_n)_{n \in \mathbb{N}}, \kappa)$ is a locally compact metric space with a modulus of local compactness, a uniformly continuous function on every bounded subset of X (this impredicativity can be easily avoided) has compact support if there is $m \in \mathbb{N}$ such that K_m is a support of f i.e., $\forall_{x \in X} (d(x, K_m) > 0 \Rightarrow f(x) = 0)$. If we consider their set $C^{\text{supp}}(X)$ as the index-set of the family $\text{Supp}(X, \mathbb{R}) \in \mathbf{Set}^{\text{se}}(I, X, \mathbb{R})$ of strongly extensional, real-valued, partial functions on X

$$f \mapsto (X, \text{id}_X, f),$$

where X is equipped the canonical inequality induced by its metric ($x \neq_{(X,d)} x' :\Leftrightarrow d(x, x') > 0$), then the following result is shown in [Gru22, GP23] within BST, and it is the starting point of a predicative reconstruction of the integration theory of locally compact metric spaces within BST.

Theorem 8.3 (The pre-integration space of a locally compact metric space with a modulus of local compactness). *Let $(X, d, (K_n)_{n \in \mathbb{N}}, \kappa)$ be a locally compact metric space, $\neq_{(X,d)}$ the canonical inequality on X , and let $I := C^{\text{supp}}(X)$ be equipped with the following operations:*

- (i) *If $a \in \mathbb{R}$, then $\cdot_a: I \rightarrow I$ is defined by $f \mapsto af$.*

- (ii) $+$: $I \times I \rightarrow I$ is the addition of functions on I .
- (iii) $|\cdot|$: $I \rightarrow I$ is defined by $f \mapsto |f|$.
- (iv) \wedge_1 : $I \rightarrow I$ is defined by $f \mapsto f \wedge 1$.
- (v) If $a > 0 \in \mathbb{R}$, then \wedge_a : $I \rightarrow I$ is defined as the composition $\wedge_a := \cdot_a \circ \wedge_1 \circ \cdot_{a^{-1}}$.

Let the obviously defined set $\text{Supp}(X, \mathbb{R})$ of strongly extensional real-valued, partial functions over I . If $\mu: I \rightarrow \mathbb{R}$ is a linear, positive measure on X i.e., there is $f \in I$ with $\mu(f) > 0$, and for every $f \in I$ we have that $f \geq 0 \Rightarrow \mu(f) \geq 0$, let

$$\int_- d\mu: I \rightarrow \mathbb{R}, \quad f \mapsto \int f d\mu := \mu(f); \quad f \in I.$$

Then $(X, I, \text{Supp}(X, \mathbb{R}), \int_- d\mu)$ is a pre-integration space.

9. SIMPLE FUNCTIONS

In this section we construct the pre-integration space of simple functions from a given pre-measure space (Theorem 9.9). This is a predicative translation within BST of the construction of an integration space from the simple functions of a measure space (Theorem 10.10 in [BB85]). Although we follow the corresponding construction in section 10 of chapter 6 in [BB85] closely, our approach allows us to not only work completely predicatively, but also to carry out all proofs avoiding the axiom of countable choice. *For the remainder of this section we fix an inhabited set with inequality $(X, =_X, \neq_X)$, and an I -family of complemented subsets $\mathbf{\Lambda}(X) := (\lambda_0^1, \mathcal{E}^1, \lambda_1^1, \lambda_0^0, \mathcal{E}^0, \lambda_1^0)$ with $i_0 \in I$. For every $i \in I$ let*

$$\chi_i := (\text{dom}_i := \lambda_0^1(i) \cup \lambda_0^0(i), \mathcal{E}_i, \chi_i) \in \mathfrak{F}^{\text{se}}(X),$$

where χ_i is the characteristic function of the complemented subset $\lambda_0(i) := (\lambda_0^1(i), \lambda_0^0(i))$ of X .

Definition 9.1. If $n \in \mathbb{N}^+$, $i_1, \dots, i_n \in I$, and $a_1, \dots, a_n \in \mathbb{R}$, the triplet¹⁵

$$\sum_{k=1}^n a_k \chi_{i_k} := \left(\bigcap_{k=1}^n \text{dom}_{i_k}, i_{\bigcap_{k=1}^n \text{dom}_{i_k}}, \sum_{k=1}^n a_k \chi_{i_k} \right) \in \mathfrak{F}^{\text{se}}(X)$$

is called a simple function. Consider the totality¹⁶ $S(I, \mathbf{\Lambda}(X)) := \sum_{n \in \mathbb{N}^+} (\mathbb{R} \times I)^n$. Let the non-dependent assignment routine $\text{dom}_0: S(I, \mathbf{\Lambda}(X)) \rightsquigarrow \mathbb{V}_0$, defined by $\text{dom}_0(n, u) := \bigcap_{k=1}^n \text{dom}_{i_k}$, for every $n \in \mathbb{N}^+$ and every $u := ((a_1, i_1), \dots, (a_n, i_n))$. Furthermore, let $\mathcal{Z}: \bigwedge_{(n,u) \in S(I, \mathbf{\Lambda}(X))} \mathbb{F}(\text{dom}_0(n, u), X)$ be the dependent assignment routine, where $\mathcal{Z}_{(n,u)}: \text{dom}_0(n, u) \hookrightarrow X$ is the canonical embedding induced by the embeddings $\mathcal{E}_{i_k}^1$ and $\mathcal{E}_{i_k}^0$, where $k \in \{1, \dots, n\}$, and the dependent assignment routine $\mathfrak{f}: \bigwedge_{(n,u) \in S(I, \mathbf{\Lambda}(X))} \mathbb{F}(\text{dom}_0(n, u), \mathbb{R})$

¹⁵The fact that $\sum_{k=1}^n a_k \chi_{i_k}$ is strongly extensional is based on Remark 3.5 and the following properties of $a, b \in \mathbb{R}$: $a + b > 0 \Rightarrow a > 0 \vee b > 0$ and $a \cdot b > 0 \Rightarrow a \neq_{\mathbb{R}} 0 \wedge b \neq_{\mathbb{R}} 0$ (see [BB85], p. 26 and [Pet18], p. 17, respectively).

¹⁶The elements of $S(I, \mathbf{\Lambda}(X))$ are pairs (n, u) , where $n \in \mathbb{N}^+$ and $u = ((a_1, i_1), \dots, (a_n, i_n))$ is an n -tuple of pairs in $\mathbb{R} \times I$. With a bit of abuse of notation we also write the elements of this Sigma-set as $(a_k, i_k)_{k=1}^n$, which is more convenient and contains all the information needed to write down the corresponding element in its proper form.

given by $f_{(n,u)} := \sum_{k=1}^n a_k \chi_{i_k}$, for every $n \in \mathbb{N}^+$ and every $u := ((a_1, i_1), \dots, (a_n, i_n))$. We now take $S(I, \mathbf{\Lambda}(X))$ to be equipped with the equality

$$(a_k, i_k)_{k=1}^n =_{S(I, \mathbf{\Lambda}(X))} (b_\ell, j_\ell)_{\ell=1}^m \Leftrightarrow \sum_{k=1}^n a_k \chi_{i_k} =_{\mathfrak{F}^{\text{se}}(X)} \sum_{\ell=1}^m b_\ell \chi_{j_\ell}$$

and define the set of *simple functions* as the $S(I, \mathbf{\Lambda}(X))$ -set of strongly extensional partial functions $\mathbf{Simple}(\mathbf{\Lambda}(X)) := (\text{dom}_0, \mathcal{Z}, \text{dom}_1, \mathbf{f}) \in \mathbf{Set}^{\text{se}}(S(I, \mathbf{\Lambda}(X)), X, \mathbb{R})$, where dom_1 is defined through dependent unique choice as explained in Definition 5.2.

It is immediate to show that $\mathbf{Simple}(\mathbf{\Lambda}(X))$ is a set of strongly extensional, partial functions over $S(I, \mathbf{\Lambda}(X))$. Next we translate the results from [BB85] needed to prove that the simple functions form a pre-integration space. For the most part the proofs of the many lemmas work exactly analogous to the corresponding ones in [BB85], so we won't give them here. Some of the results can however be sharpened, thus allowing us to avoid the axiom of countable choice altogether, and we present the proofs of those results. First, we state the predicative analogues of lemmas (10.2) - (10.5) of chapter 6 in [BB85]. One of the reasons working always with an inhabited set X is that an element $x_0 \in X$ is needed in Bishop's negativistic definition of the empty subset¹⁷ \emptyset_X of X (see [Bis67], p. 65).

Lemma 9.2. *Let $\mathcal{M}(\mathbf{\Lambda}) := (X, I, \mu)$ be a pre-measure space, $i_1, \dots, i_n \in I$, and $F := \bigcap_{k=1}^n \text{dom}_{i_k}$.*

- (i) *If $i \in I$ such that $\lambda_0^1(i) = \emptyset_X$, then $\mu(i) = 0$.*
- (ii) *There is $j \in I$ such that $\lambda_0(j) = (\emptyset_X, F)$.*
- (iii) *If $j \in I$, then there is $k \in I$ such that $\lambda_0(k) = (\lambda_0^1(j) \cap F, \lambda_0^0(j) \cap F)$ and $\mu(k) = \mu(j)$.*
- (iv) *If $i, j \in I$, $F' := \text{dom}_i \cap \text{dom}_j \cap F$, and $\chi_i(x) \leq \chi_j(x)$, for every $x \in F'$, then $\mu(i) \leq \mu(j)$.*

Remark 9.3. For the proof of Lemma 9.2 it suffices to use condition (PMS_2^*) . It is only here that we rely on (PMS_2) , or (PMS_2^*) .

Many later proofs rely on the fact that we can restrict our attention to disjoint simple functions. The next lemma, which corresponds to lemma (7.8) of chapter 6 in [BB85], makes this fact precise.

Lemma 9.4.

- (i) *If $\bar{n} := \{1, \dots, n\}$, the assignment routine $\mathbf{disjrep}: S(I, \mathbf{\Lambda}(X)) \rightsquigarrow S(I, \mathbf{\Lambda}(X))$*

$$(a_k, i_k)_{k=1}^n \mapsto \left(\sum_{f(k)=1} a_k, j_f := \left(\bigwedge_{f(k)=1} i_k \right) \sim \left(\bigvee_{f(k)=0} i_k \right) \right)_{f \in \mathbb{F}(\bar{n}, 2)}$$

is a function equal to $\text{id}_{S(I, \mathbf{\Lambda}(X))}$. If $v \in S(I, \mathbf{\Lambda}(X))$ and $\mathbf{disjrep}(v) := (b_\ell, j_\ell)_{\ell=1}^m$, then the complemented subsets $\lambda_0(j_\ell)$ are disjoint, i.e., if $\ell \neq k$, then $\chi_{j_k} \cdot \chi_{j_\ell} = 0$ on $\text{dom}_{j_k} \cap \text{dom}_{j_\ell}$.

- (ii) *For every $v := (a_k, i_k)_{k=1}^n \in S(I, \mathbf{\Lambda}(X))$ we have*

$$\sum_{k=1}^n a_k \cdot \mu(i_k) = \sum_{f \in \mathbb{F}(\bar{n}, 2)} \left(\sum_{f(k)=1} a_k \right) \cdot \mu(j_f)$$

where $j_f \in I$ is defined, for every $f \in \mathbb{F}(\bar{n}, 2)$, as in (i).

¹⁷For a positively defined empty subset of X see [PW22].

Remark 9.5. If $v \in S(I, \mathbf{\Lambda}(X))$, we call $\text{disjrep}(v)$ the *disjoint representation* of v . The above lemma allows us to restrict our attention to disjoint simple functions, whenever we want to prove a statement for all simple functions. Combining this with the fact that we can prove statements about simple function by induction on their first component $n \in \mathbb{N}^+$ we can show that some *extensional* property P holds for all $v \in S(I, \mathbf{\Lambda}(X))$ if can show the following:

- P holds for all simple functions of length one $v := (a, i)$, with $a \in \mathbb{R}$ and $i \in I$.
- If P holds for $v = (a_k, i_k)_{k=1}^n$ disjoint and we have $a_{n+1} \in \mathbb{R}$ and $i_{n+1} \in I$ disjoint from any of the indices $i_1, \dots, i_n \in I$, then P also holds for $(a_k, i_k)_{k=1}^{n+1} \in S(I, \mathbf{\Lambda}(X))$.

Lemma 9.6.

- (i) Let $v := (a_k, i_k)_{k=1}^n, w := (b_\ell, j_\ell)_{\ell=1}^m \in S(I, \mathbf{\Lambda}(X))$ such that

$$\sum_{k=1}^n a_k \cdot \chi_{i_k}(x) \leq \sum_{\ell=1}^m b_\ell \cdot \chi_{j_\ell}(x)$$

for all $x \in F := \left(\bigcap_{k=1}^n \text{dom}_{i_k}\right) \cap \left(\bigcap_{\ell=1}^m \text{dom}_{j_\ell}\right)$, then $\sum_{k=1}^n a_k \cdot \mu(i_k) \leq \sum_{\ell=1}^m b_\ell \cdot \mu(j_\ell)$.

- (ii) The assignment-routine $\int_- d\mu: S(I, \mathbf{\Lambda}(X)) \rightsquigarrow \mathbb{R}$, defined by

$$(a_k, i_k)_{k=1}^n \mapsto \sum_{k=1}^n a_k \cdot \mu(i_k),$$

is a function.

The next lemma is a slight improvement of Lemma 10.8 in [BB85] that will allow us to proceed without using the axiom of countable choice in the proof of Theorem 9.9. It is at this point that we use the induction principle for disjoint simple functions as described in Remark 9.5.¹⁸

Lemma 9.7. If $S^+(I, \mathbf{\Lambda}(X)) := \{v \in S(I, \mathbf{\Lambda}(X)) \mid \mathfrak{f}_v \geq 0\}$ is the set of positive simple functions, then there is a function $\phi: \mathbb{N}^+ \times S^+(I, \mathbf{\Lambda}(X)) \rightarrow I$, with $(N, v) \mapsto \phi_N(v) \in I$, such that for every $N \in \mathbb{N}^+$ and every $v \in S^+(I, \mathbf{\Lambda}(X))$, the following conditions hold:

- (i) $\text{dom}_{\phi_N(v)} \subseteq \text{dom}_0(v)$.
- (ii) $\forall_{x \in \lambda_0^0(\phi_N(v))} (\mathfrak{f}_v(x) < N^{-1})$.
- (iii) $\mu(\phi_N(v)) \leq 2N \int v d\mu$.

Proof. Let $N \in \mathbb{N}^+$. For the base case let $a \in \mathbb{R}_{\geq 0}$ and $i \in I$. We construct $\phi_N(a, i) \in I$ satisfying

- $\lambda_0^1(\phi_N(a, i)) \cup \lambda_0^0(\phi_N(a, i)) \subseteq \lambda_0^1(i) \cup \lambda_0^0(i)$,
- $\forall_{x \in \lambda_0^0(\phi_N(a, i))} (a\chi_i(x) < N^{-1})$,
- $\mu(\phi_N(a, i)) \leq 2Na\mu(i)$.

Since $(2N)^{-1} < N^{-1}$ we get that $a < N^{-1}$ or $a > (2N)^{-1}$ and using an algorithm that lets us decide which case obtains (using Corollary 2.17 in [BB85]) we set

$$\phi_N(a, i) := \begin{cases} i \sim i, & \text{if } a < N^{-1} \\ i, & \text{if } a > (2N)^{-1}. \end{cases}$$

¹⁸We owe this alternative proof to a note of the late Erik Palmgren found in the copy of the book [BB85] by Bishop and Bridges that Erik used to own.

Using the fact that $\lambda_0(i \sim i) = (\emptyset, \lambda_0^1(i) \cup \lambda_0^0(i))$, the verifications of the above properties become routine for both possible values of $\phi_N(a, i)$. For the inductive step we assume that we have a disjoint $v = (a_k, i_k)_{k=1}^n \in S^+(I, \mathbf{\Lambda}(X))$ satisfying the above conditions and $a_{n+1} \geq 0$ and $i_{n+1} \in I$ disjoint from all the i_1, \dots, i_n . Let $w = (a_k, i_k)_{k=1}^{n+1}$. Working similarly, we set

$$\phi_N(w) := \begin{cases} \phi_N(v) \vee (i_{n+1} \sim i_{n+1}), & \text{if } a_{n+1} < N^{-1} \\ \phi_N(v) \vee i_{n+1}, & \text{if } a_{n+1} > (2N)^{-1}. \end{cases}$$

First, assume that $a_{n+1} < N^{-1}$. We have that

$$\lambda_0(\phi_N(w)) := (\lambda_0^1(\phi_N(v)) \cap (\lambda_0^1(i_{n+1}) \cup \lambda_0^0(i_{n+1})), \lambda_0^0(\phi_N(v)) \cap (\lambda_0^1(i_{n+1}) \cup \lambda_0^0(i_{n+1}))).$$

The first condition then follows immediately from the inductive hypothesis. Now let $x \in \phi_N(w)$ and observe that this means that either $x \in \lambda_0^0(\phi_N(v)) \cup \lambda_0^1(i_{n+1})$ or $x \in \lambda_0^1(\phi_N(v)) \cup \lambda_0^0(i_{n+1})$. In the first case we get that $\mathfrak{f}_w(x) = a_{n+1} < N^{-1}$ by our disjointness assumption and in the second case we get $\mathfrak{f}_w(x) = \mathfrak{f}_v(x) < N^{-1}$ by the inductive hypothesis. Finally, we get

$$\mu(\phi_N(w)) \leq \mu(\phi_N(v)) + \mu(i_{n+1} \sim i_{n+1}) \leq \mu(\phi_N(v)) \leq 2N \int v \, d\mu \leq 2N \int w \, d\mu.$$

Next we assume that $a_{n+1} > (2N)^{-1}$. We have that

$$\begin{aligned} \lambda_0^1(\phi_N(w)) &:= (\lambda_0^1(\phi_N(v)) \cap \lambda_0^1(i_{n+1})) \cup (\lambda_0^1(\phi_N(v)) \cap \lambda_0^0(i_{n+1})) \cup (\lambda_0^0(\phi_N(v)) \cap \lambda_0^1(i_{n+1})) \\ \lambda_0^0(\phi_N(w)) &:= \lambda_0^0(\phi_N(v)) \cap \lambda_0^0(i_{n+1}) \end{aligned}$$

By the inductive hypothesis the first two conditions follow easily, and for the third we get

$$\mu(\phi_N(w)) \leq \mu(\phi_N(v)) + \mu(i_{n+1}) \leq 2N \int v \, d\mu + 2Na_{n+1}\mu(i_{n+1}) = 2N \int w \, d\mu.$$

It is straightforward to check that the properties of cases (i)-(iii) are extensional. \square

The last lemma needed for the proof of Theorem 9.9 corresponds to Lemma 10.9 in [BB85], and although it reads similar to Lemma 9.7, the way it is used in the proof of Theorem 9.9 does not invoke countable choice and it doesn't allow for an induction proof. Hence, we state it without proof, as a rather direct translation of Lemma 10.9 in [BB85].

Lemma 9.8. *Let $v := (a_k, i_k)_{k=1}^n \in S(I, \mathbf{\Lambda}(X))$ and $c > 0$, such that $\mathfrak{f}_v \leq c$ on $\text{dom}_0(v)$. If $i \in I$, such that $\mathfrak{f}_v \leq 0$ on $\lambda_0^0(i) \cap \text{dom}_0(v)$, then for every $\varepsilon > 0$ there is $j \in I$ satisfying the following conditions:*

- (i) $\text{dom}_j \subseteq \text{dom}_0(v)$.
- (ii) $\forall_{x \in \lambda_1^0(j)} (\mathfrak{f}_v(x) > \varepsilon)$.
- (iii) $\mu(j) \geq c^{-1} (\int v \, d\mu - 2\varepsilon\mu(i))$.

Putting everything together we can prove the main result of this section. The proof follows closely [BB85] but avoids countable choice by using our Lemma 9.7 instead of Lemma 10.8 of [BB85] at the corresponding point in the proof. We refer to [Zeu19] for details.

Theorem 9.9. *The structure $(X, S(I, \mathbf{\Lambda}(X)), \text{Simple}(\mathbf{\Lambda}(X)), \int _ d\mu)$ is a pre-integration space.*

10. CANONICALLY INTEGRABLE FUNCTIONS

One of the most central constructions in BCMT is the completion or, to be more precise, the L^1 -completion of a Bishop-Cheng integration space. To avoid the impredicativities of this definition within BCMT, we first present in this section the *canonically integrable functions* explicitly as a family of partial functions. We then show that this family admits the structure of a pre-integration space and explain in what sense it can be seen as the completion of our original pre-integration space. We follow closely Section 2 of Chapter 6 in [BB85], with the exception that we make almost no mention of *full sets*. This is because quantification over full sets is not allowed, even though the property of being a full set can be defined predicatively. We will discuss this in more detail below. As a result, a few of the key lemmas in [BB85] are missing in our setting, making some of the proofs, like the one of Theorem 10.4, more tedious. We start by giving some basic results on pre-integration spaces, which we will only state without proof, as those work completely analogous to the ones in [BB85], pp. 217–218. *For the remainder of this section we fix a pre-integration space $\mathcal{L}_0 := (X, I, \Lambda(X, \mathbb{R}), \int)$.*

Lemma 10.1.

- (i) *Let $i \in I$ and $\alpha: \mathbb{N}^+ \rightarrow I$, such that for all $n \in \mathbb{N}^+$ we have $\mathfrak{f}_{\alpha_n} \geq 0$ and $\sum_{n \in \mathbb{N}^+} \int \alpha_n \in \mathbb{R}$, and $\int i + \sum_{n \in \mathbb{N}^+} \int \alpha_n > 0$. Then there exists $x \in \lambda_0(i) \cap \bigcap_{n \in \mathbb{N}^+} \lambda_0(\alpha_n)$ such that*

$$\sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\alpha_n}(x) \in \mathbb{R} \quad \& \quad \mathfrak{f}_i(x) + \sum_{n \in \mathbb{N}^+} \mathfrak{f}_{\alpha_n}(x) > 0.$$

- (ii) $\forall i \in I (\mathfrak{f}_i \geq 0 \Rightarrow \int i \geq 0)$.
 (iii) $\forall i \in I (|\int i| \leq \int |i|)$.
 (iv) *If $i, j \in I$ such that $\mathfrak{f}_i(x) \leq \mathfrak{f}_j(x)$, for every $x \in \lambda_0(i) \cap \lambda_0(j)$, then $\int i \leq \int j$.*

In classical measure theory, two functions in L^1 are identified, if they agree almost everywhere. In BCMT, two integrable functions in the L^1 -completion of an integration space are identified, if they agree on a full set. In [BB85] each function \mathbf{f} in L^1 comes with a representing sequence $(\mathbf{f}_n)_n$ of functions from the base integration space.¹⁹ Each representing sequence defines the *canonically integrable function* $\sum_n \mathbf{f}_n$ on a full domain, and the represented function \mathbf{f} agrees with $\sum_n \mathbf{f}_n$ on this domain, i.e. they are identified in L^1 . Classically speaking, each equivalence class of L^1 contains a canonically integrable function given by the representing sequence of an element of the equivalence class. This means that without loss of generality, we can describe L^1 predicatively by focusing only on representing sequences and their associated canonically integrable functions.

Definition 10.2. The set of representations of \mathcal{L}_0 is the totality

$$I_1 := \left\{ \alpha \in \mathbb{F}(\mathbb{N}^+, I) \mid \sum_{n=1}^{\infty} \int |\alpha_n| \in \mathbb{R} \right\},$$

Let $\nu_0: I_1 \rightsquigarrow \mathbb{V}_0$ be given by

$$\nu_0(\alpha) := \left\{ x \in \bigcap_{n=1}^{\infty} \lambda_0(\alpha_n) \mid \sum_{n=1}^{\infty} |\mathfrak{f}_{\alpha_n}(x)| \in \mathbb{R} \right\}.$$

¹⁹This approach to the definition of L^1 was developed by Bishop and Cheng in [BC72] a few years prior to Mikusiński's similar approach to L^1 within the classical Daniell integration theory (see [Mik78, Mik89]).

Furthermore, let $\mathcal{H}: \bigwedge_{\alpha \in I_1} \mathbb{F}(\nu_0(\alpha), X)$ be the dependent assignment routine where $\mathcal{H}_\alpha: \nu_0(\alpha) \hookrightarrow X$ is the canonical embedding induced by the embeddings $\lambda_0(\alpha_n) \hookrightarrow X$ and the dependent assignment routine $\mathfrak{g}: \bigwedge_{\alpha \in I_1} \mathbb{F}(\nu_0(\alpha), \mathbb{R})$ given by $\mathfrak{g}_\alpha(x) := \sum_{n=1}^{\infty} \mathfrak{f}_{\alpha_n}(x)$, for every $x \in \nu_0(\alpha)$. We now take I_1 to be equipped with the equality

$$\alpha =_{I_1} \beta := \Leftrightarrow (\nu_0(\alpha), \mathcal{H}_\alpha, \mathfrak{g}_\alpha) =_{\mathfrak{F}^{\text{se}}(X)} (\nu_0(\beta), \mathcal{H}_\beta, \mathfrak{g}_\beta)$$

and define the set of *canonically integrable functions* as the I_1 -set of strongly extensional²⁰, partial functions $\mathbf{\Lambda}_1 := (\nu_0, \mathcal{H}, \nu_1, \mathfrak{g})$, where ν_1 is defined through dependent unique choice as explained in Definition 5.2.

Let the canonical embedding of I into I_1 be the assignment routine $h: I \rightsquigarrow I_1$, defined by the rule $i \mapsto (i, 0 \cdot i, 0 \cdot i, \dots)$.

Clearly the assignment routine h is an embedding, since

$$\begin{aligned} i =_I j &:= \Leftrightarrow \mathbf{f}_i =_{\mathfrak{F}^{\text{se}}(X)} \mathbf{f}_j \\ &\Leftrightarrow \left(\lambda_0(i), \mathcal{E}_i, \mathfrak{f}_i + \sum_{n=2}^{\infty} 0 \cdot \mathfrak{f}_i \right) =_{\mathfrak{F}^{\text{se}}(X)} \left(\lambda_0(j), \mathcal{E}_j, \mathfrak{f}_j + \sum_{n=2}^{\infty} 0 \cdot \mathfrak{f}_j \right) \\ &\Leftrightarrow h(i) =_{I_1} h(j), \end{aligned}$$

as one can easily verify that $\lambda_0(i) \subseteq \nu_0(h(i))$ and $\lambda_0(j) \subseteq \nu_0(h(j))$.

Following [BB85], p. 224, and with a bit of abuse of notation, we can define basic functions on I_1 such as

$$_ + _ : I_1 \times I_1 \rightarrow I_1, \quad \alpha + \beta := (\alpha_1, \beta_1, \alpha_2, \beta_2, \dots),$$

satisfying

$$\mathfrak{g}_{\alpha+\beta} =_{\mathfrak{F}^{\text{se}}(X)} \mathfrak{g}_\alpha + \mathfrak{g}_\beta \quad \& \quad h(i+j) =_{I_1} h(i) + h(j).$$

Similarly, we obtain functions $_ \cdot _ : \mathbb{R} \times I_1 \rightarrow I_1$ and $|_ |, \wedge_1 : I_1 \rightarrow I_1$ commuting with their counterparts on I and the corresponding operations on $\mathfrak{F}^{\text{se}}(X)$. Note that for construction of these sequences no choice principles are needed. Finally, the integral $\int : I_1 \rightarrow \mathbb{R}$ is given by

$$\int \alpha := \sum_n \int \alpha_n,$$

It is clear that $\int h(i) = \int i$ for all $i \in I$, which justifies our overloaded notation

The proof of the next lemma follows section 2 of chapter 6 in [BB85].

Lemma 10.3.

- (i) $\forall \alpha \in I_1 \left(\left| \int \alpha \right| \leq \int |\alpha| \right)$.
- (ii) If $\alpha \in I_1$, such that $\forall_{x \in \nu_0(\alpha)} (\mathfrak{g}_\alpha(x) \geq 0)$, then $\int \alpha \geq 0$.
- (iii) If $\alpha, \beta \in I_1$, such that $\forall_{x \in \nu_0(\alpha) \cap \nu_0(\beta)} (\mathfrak{g}_\alpha(x) \leq \mathfrak{g}_\beta(x))$, then $\int \alpha \leq \int \beta$.
- (iv) There is a function $\psi: I_1 \times \mathbb{N}^+ \rightarrow I_1$, such that for every $\alpha \in I_1$ and $n \in \mathbb{N}^+$, $\psi(\alpha, n) =_{I_1} \alpha$ and

$$\sum_{k \in \mathbb{N}^+} \int |\psi(\alpha, n)_k| \leq 2^{-n} + \int |\alpha|.$$

²⁰By Proposition 7.4.

Lemma 10.3(iv) is formulated in a way that allows us to avoid countable choice, by explicitly constructing function ψ . Unlike in the previous section, we can however still follow the proof of Lemma 2.14 in [BB85]. We are now able to prove the predicative version of *Lebesgue's series theorem*. The proof generally follows the proof of Theorem 2.15 in [BB85], but we have to be a bit more cautious, since we don't have a set of a full sets at hand. For a subset A we can predicatively define what it means to be full, namely $\exists_{\alpha \in I_1} \nu_0(\alpha) \subseteq A$. However, the totality of full sets is still defined through separation from $\mathcal{P}(X)$ and quantification over full sets is thus not possible.

Theorem 10.4. *Let $\Gamma: \mathbb{N}^+ \rightarrow I_1$, such that $\sum_{n \in \mathbb{N}^+} \int |\Gamma_n| \in \mathbb{R}$, and*

$$A := \left\{ x \in \bigcap_{n=1}^{\infty} \nu_0(\Gamma_n) \mid \sum_{n=1}^{\infty} |\mathfrak{g}_{\Gamma_n}(x)| \in \mathbb{R} \right\}$$

Then there exists $\alpha \in I_1$ such that $\nu_0(\alpha) \subseteq A$ (i.e. A is full) and

$$\forall_{x \in \nu_0(\alpha)} \left(\mathfrak{g}_{\alpha}(x) = \sum_{n=1}^{\infty} \mathfrak{g}_{\Gamma_n}(x) \right).$$

Moreover, if $\alpha \in I_1$ fulfills the above condition, then $\lim_{N \rightarrow \infty} \int |\alpha - \sum_{n=1}^N \Gamma_n| = 0$.

Proof. We only give a proof sketch and refer the reader to the proof of Theorem 4.3.12 in [Zeu19] for details. For each $n \in \mathbb{N}^+$ let $\beta_n := \psi(\Gamma_n, n)$ with ψ as in Lemma 10.3(iv) i.e., $\beta_n \in I_1$, such that for all $n \in \mathbb{N}^+$ we have $\beta_n =_{I_1} \Gamma_n$ and

$$\sum_{k=1}^{\infty} \int |\beta_{nk}| < 2^{-n} + \int |\Gamma_n|.$$

It follows that $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int |f_{\beta_{nk}}| \in \mathbb{R}$. Let

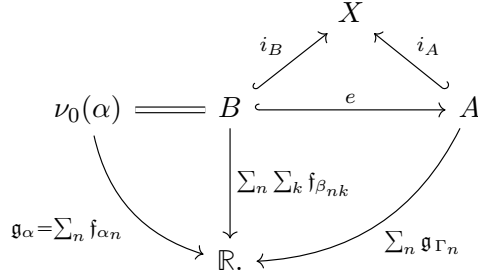
$$B := \left\{ x \in \bigcap_{n \in \mathbb{N}^+} \bigcap_{k \in \mathbb{N}} \lambda_0(\beta_{nk}) \mid \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |f_{\beta_{nk}}(x)| \in \mathbb{R} \right\}$$

and fix a suitable bijection $\varphi: \mathbb{N}^+ \rightarrow \mathbb{N}^+ \times \mathbb{N}^+$ (e.g. as in section 2.3 of [Zeu19]). Let $\alpha: \mathbb{N}^+ \rightarrow I$ be given by $\alpha_n := \beta_{\text{pr}_1(\varphi(n)) \text{pr}_2(\varphi(n))}$, then²¹ $\sum_{n=1}^{\infty} \int |\alpha_n| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int |\beta_{nk}| \in \mathbb{R}$ and hence $\alpha \in I_1$. Using the same argument about double series, we can construct an equality of partial functions:

$$\left(\nu_0(\alpha), \mathcal{H}_{\alpha}, \mathfrak{g}_{\alpha} \right) =_{\mathfrak{F}^{\text{se}}(X)} \left(B, i_B, \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{\beta_{nk}} \right)$$

²¹If $(x_{nk})_{n,k \in \mathbb{N}^+}$ is a sequence of sequences of reals and if $(y_m := x_{\text{pr}_1(\varphi(m)) \text{pr}_2(\varphi(m))})_{m \in \mathbb{N}^+}$, then $\sum_n \sum_k x_{nk}$ converges absolutely if and only if $\sum_m y_m$ converges absolutely, and in this case the two sums are equal. This fact can be proven constructively and without choice principles for a concrete, suitably chosen φ , see Lemma 2.3.2 in [Zeu19].

The moduli of equality $\nu_1(\beta_n, \Gamma_n)$ give inclusions $\nu_0(\beta_n) \hookrightarrow \nu_0(\Gamma_n)$ for $n \in \mathbb{N}^+$ and induce an embedding $e: B \hookrightarrow A$ such that the following diagram commutes



To show the second part of the theorem, let $N \in \mathbb{N}^+$ and $\alpha \in I_1$ such that α satisfies the conditions of the first part of the theorem and set $\gamma := |\alpha - \sum_{n=1}^N \Gamma_n| \in I_1$. If $\delta: \mathbb{N}^+ \rightarrow I$ is an enumeration of the terms

$$\begin{array}{cccc}
 -\gamma_1 & -\gamma_2 & -\gamma_3 & \cdots \\
 |\beta_{(N+1) 1}| & |\beta_{(N+1) 2}| & |\beta_{(N+1) 3}| & \cdots \\
 |\beta_{(N+2) 1}| & |\beta_{(N+2) 2}| & |\beta_{(N+2) 3}| & \cdots \\
 \vdots & \vdots & \vdots & \ddots
 \end{array}$$

into a single sequence using the bijection $\varphi: \mathbb{N}^+ \rightarrow \mathbb{N}^+ \times \mathbb{N}^+$, then $\sum_{n=1}^\infty \int |\delta_n| = \sum_{n=1}^\infty \int |\gamma_n| + \sum_{m=N+1}^\infty \sum_{k=1}^\infty \int |\beta_{nk}| \in \mathbb{R}$, i.e. $\delta \in I_1$. Following the proof in [BB85] (p. 229), for each $x \in \nu_0(\delta)$ we get that

$$\sum_{n=1}^\infty \mathfrak{f}_{\delta_n}(x) = \sum_{n=N+1}^\infty \sum_{k=1}^\infty |\mathfrak{f}_{\beta_{nk}}|(x) - \sum_{m=1}^\infty \mathfrak{f}_{\gamma_m}(x) \geq 0$$

By Lemma 10.3(ii) it follows that $\int \delta \geq 0$. Hence

$$\begin{aligned}
 0 &\leq \int \left(\left| \alpha - \sum_{n=1}^N \Gamma_n \right| \right) = \int \gamma = \sum_{n=1}^\infty \int \gamma_n \\
 &\leq \sum_{n=N+1}^\infty \sum_{k=1}^\infty \int |\beta_{nk}| \leq \sum_{n=N+1}^\infty \left(2^{-n} + \int |\Gamma_n| \right),
 \end{aligned}$$

and the last expression converges to 0 for $N \rightarrow \infty$. □

Corollary 10.5. *If $\alpha \in I_1$, then $\lim_{N \rightarrow \infty} \int |\alpha - \sum_{n=1}^N \alpha_n| = 0$.*

With Lebesgue’s series theorem at hand we can now show that the canonically integrable functions form a pre-integration space, and as such the complete extension of the pre-integration space \mathcal{L}_0 . All these proofs follow closely [BB85] so we will omit them altogether. The final Theorem 2.18 of section 2 of chapter 6 of [BB85] becomes:

Theorem 10.6. *$(X, I_1, \mathbf{\Lambda}_1, \int)$ is a pre-integration space.*

In order to treat L^1 as the completion of \mathcal{L}_0 , we introduce the 1-norm of \mathcal{L}_0 . In classical measure theory one often identifies integrable functions that agree almost everywhere and the normed space L^1 is defined modulo this equivalence relation. The positive, constructive counterpart of this is to identify functions in the complete extension of an integration space that agree on a full set. Proposition 2.12 in [BB85], p. 227, then tells us that we can define

the 1-norm modulo this equality. Since in our predicative setting, we don't have recourse to a set of full set, we need to introduce the 1-norm a bit differently. The following fact is straightforward to show.

Proposition 10.7. *Let $p \in I$, such that $\int p = 1$.*

- (i) *If $i, j \in I$, the relation $i =_{\int} j :\Leftrightarrow \int |i - j| = 0$ is an equivalence relation on I .*
- (ii) *The assignment routine $\int: (I, =_{\int}) \rightsquigarrow \mathbb{R}$, given by the rule $i \mapsto \int i$ is a function.*
- (iii) *The functions \cdot and $+$ turn $(I, =_{\int})$ into an \mathbb{R} -vector space with neutral element $0 \cdot p$.*
- (iv) *The function $\| _ \|_1: I \rightarrow \mathbb{R}_{\geq 0}$, given by the rule*

$$\| i \|_1 := \int |i|,$$

is a norm on $((I, =_{\int}), \cdot, +, 0 \cdot p)$.

Putting everything together, and in correspondence to Corollaries 2.16, 2.17 in [BB85], we get the following.

Theorem 10.8.

- (i) *The canonical embedding $h: I \hookrightarrow I_1$ is norm-preserving.*
- (ii) *$(I, =_{\int}, \| _ \|_1)$ is a dense subspace of $(I_1, =_{\int}, \| _ \|_1)$ through h .*
- (iii) *I_1 is complete with respect to $\| _ \|_1$.*

11. CONCLUDING REMARKS AND FUTURE WORK

We presented here the first steps towards a predicative reconstruction PBCMT of the original impredicative Bishop-Cheng theory of measure and integration BCMT. Based on the theory of set-indexed families of sets within BST, we studied the notions of a pre-measure and pre-integration space, as predicative reformulations of the notions of a measure and integration space in BCMT. As first fundamental examples we presented

- (i) the Dirac measure as a pre-measure,
- (ii) the pre-integration space associated to a locally compact metric space with a modulus of local compactness, and
- (iii) the pre-integration space of simple functions generated by a pre-measure space.

Finally, we gave a predicative treatment of L^1 as an appropriate completion of the pre-integration space of the canonically integrable functions. Using arguments that avoided the use of full sets and the principle of countable choice, we managed to prove a predicative version of the constructive Lebesgue's series theorem.

A predicative definition of L^1 ensures that all concepts defined through quantification over L^1 in BCMT become predicative in PBCMT. For example, quantification over L^1 is used in the Bishop-Cheng definition of a full set²² (see [BB85], p. 224), a constructive counterpart to the complement of a null set in classical measure theory, and in the Bishop-Cheng definition of almost everywhere convergence (see [BB85], p. 265). Our predicative treatment of L^1 is the first, clear indication that the computational content of measure theory can be grasped by PBCMT.

²²The property of being a full set can indeed be defined predicatively by quantification over the set I_1 . However, the totality of full sets is still defined by separation from the class of all subsets and thus itself a proper class.

Many questions arise naturally from our current work. In [BB85], pp. 232–236, the measure space of an integration space is constructed. A complemented subset \mathbf{A} of X is called *integrable*, if its characteristic function $\chi_{\mathbf{A}}$ is in L^1 , and the *measure* $\mu(\mathbf{A})$ is the integral $\int \chi_{\mathbf{A}}$. A predicative treatment of the pre-measure space induced by a pre-integration space is expected to be given by describing the intersection $M = L^1 \cap \mathfrak{F}^{\text{se}}(X, 2)$ as an appropriate set of complemented subsets. The exact relation between the pre-measure space of the pre-integration space of a given pre-measure space with the original pre-measure space needs to be determined. And similarly for the pre-integration space of the pre-measure space of a given pre-integration space. One must also investigate, if the expected (pre-)measure space of the pre-integration space $(X, I_1, \mathbf{A}_1, \int_1)$ is complete, in the sense of a predicative reformulation of the definition of a complete measure space (see [BB85], pp. 288–289 and [Pet20b], p. 209).

The Radon-Nikodym theorem is a core result of classical measure theory, according to which, under appropriate conditions, measures can be expressed as integrals

$$\nu(A) = \int_A f d\mu$$

with respect to other measures. Following the Daniell approach, Shilov and Gurevich offer a classical treatment of the Radon-Nikodym theorem in [SG66]. Although Bishop tackled it already in [Bis67], he humbly admitted that his treatment “follows the classical pattern, except that it is much messier”, partly due to the trade-off requirement of posing stronger hypotheses. In the light of BCMT, Bridges offered an improved and extended constructive version [Bri77], which led to the revised, joint account with Bishop given in [BB85]. The definition of the notion of absolute continuity of one integral over another one, which is central to this constructive proof of the Radon-Nikodym theorem, is impredicative. It requires quantification over all integrable sets, and therefore over the proper class of complemented subsets. As L^1 is here predicatively defined, a predicative treatment of the constructive Radon-Nikodym theorem within PBCMT is expected to be possible.

Bishop and Cheng introduced *profiles* in [BC72] as an auxiliary concept in order to address convergence in the class of integrable functions. The profile theorem expresses positively the classical fact that an increasing function on the reals can have at most countably many discontinuities. At the same time, it is responsible for an abundant supply of integrable sets within BCMT. It also implies the uncountability of reals, and since there are countable sheaf models of reals [Spi06b], there is no hope of proving the profile theorem constructively without employing some choice principle. A proof of a choice-free version of the profile theorem was given by Spitters [Spi06b], using Coquand’s point-free version of the Stone representation theorem. The question whether we can recover the basic applications of the theory of profiles through a choice-free variation of its basic notions and results within PBCMT is an important open problem.

REFERENCES

- [AR10] P. Aczel and M. Rathjen. *Constructive Set Theory*. book draft, 2010.
- [BAVG12] S. Bhat, A. Agarwal, R. Vuduc, and A. Gray. A type theory for probability density functions. *SIGPLAN Not.*, 47(1):545–556, 2012. doi:10.1145/2103621.2103721.
- [BB85] E. Bishop and D. S. Bridges. *Constructive Analysis*. Grundlehren der math. Wissenschaften 279. Springer-Verlag, Heidelberg-Berlin-New York, 1985.
- [BC72] E. Bishop and H. Cheng. *Constructive Measure Theory*, volume 116. Mem. Amer. Math. Soc., 1972.

- [BD91] D. Bridges and O. Demuth. On the Lebesgue measurability of continuous functions in constructive analysis. *Bulletin of the American Mathematical Society*, 24(2):259–276, 1991. doi:10.1090/S0273-0979-1991-16014-3.
- [BFS21] M. E. Bidlingmaier, F. Faissole, and B. Spitters. Synthetic topology in Homotopy Type Theory for probabilistic programming. *Mathematical Structures in Computer Science*, 31(10):1301–1329, 2021. doi:10.1017/S0960129521000165.
- [Bis67] E. Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, 1967.
- [Bis70] E. Bishop. Mathematics as a Numerical Language. In A. Kino, J. Myhill, and R.E. Vesley, editors, *Intuitionism and Proof Theory: Proceedings of the Summer Conference at Buffalo N.Y. 1968*, volume 60 of *Studies in Logic and the Foundations of Mathematics*, pages 53–71. Elsevier, 1970. doi:10.1016/S0049-237X(08)70740-7.
- [Bou04] N. Bourbaki. *Elements of Mathematics Integration I*. Springer, Berlin, Heidelberg, 2004. Chapters 1–6.
- [Bri77] D. Bridges. The constructive Radon-Nikodym theorem. *Pacific J. Math.*, 70(1):51–65, 1977. doi:10.2140/pjm.1977.70.51.
- [Car56] C. Carathéodory. *Mass und Integral und ihre Algebraisierung*. Springer AG, Basel, 1956.
- [Cha72] Y. K. Chan. A constructive approach to the theory of stochastic processes. *Transactions of the American Mathematical Society*, 65:37–44, 1972. doi:10.1090/S0002-9947-1972-0365642-8.
- [Cha75] Y. K. Chan. A short proof of an existence theorem in constructive measure theory. *Proceedings of the American Mathematical Society*, 48(2):435–436, 1975. doi:10.1090/S0002-9939-1975-0364590-5.
- [Cha21] Y. K. Chan. *Foundations of Constructive Probability Theory*. Cambridge University Press, 2021.
- [Cir23] F. Ciraulo. Subspaces in Pointfree Topology: Towards a New Approach to Measure Theory. In D. Bridges, H. Ishihara, M. Rathjen, and H. Schwichtenberg, editors, *Handbook of Constructive Mathematics*, Encyclopedia of Mathematics and its Applications, pages 426–444. Cambridge University Press, 2023. doi:10.1017/9781009039888.017.
- [CP02] T. Coquand and E. Palmgren. Metric boolean algebras and constructive measure theory. *Arch. Math. Logic*, 41:687–704, 2002. doi:10.1007/s001530100123.
- [CS09] T. Coquand and B. Spitters. Integrals and valuations. *Journal of Logic & Analysis*, 1(3):1–22, 2009. doi:10.4115/jla.2009.1.3.
- [Dan18] P. J. Daniell. A General Form of Integral. *Annals of Mathematics, Second Series*, 19(4):279–294, 1918. doi:10.2307/1967495.
- [Eda09] A. Edalat. A computable approach to measure and integration theory. *Information and Computation*, 207:642–659, 2009. doi:10.1016/j.ic.2008.05.003.
- [Fef79] S. Feferman. Constructive Theories of Functions and Classes. In M. Boffa, D. van Dalen, and K. Mcaloon, editors, *Logic Colloquium '78*, pages 159–224. North-Holland, 1979. doi:10.1016/S0049-237X(08)71625-2.
- [GP23] F. L. Grubmüller and I. Petrakis. A Predicative Approach to the Constructive Integration Theory of Locally Compact Metric Spaces, 2023. Submitted.
- [Gru22] F. L. Grubmüller. Towards a Constructive and Predicative Integration Theory of Locally Compact Metric Space. Bachelor’s thesis, LMU, 2022.
- [Hal74] P. R. Halmos. *Measure theory*. Springer-Verlag Inc., New York, 1974.
- [Hey56] A. Heyting. *Intuitionism, An introduction*. North-Holland, 1956.
- [Kol48] A. N. Kolmogoroff. Algèbres de Boole métrique complètes. *Zjazd Matematyków Polskich*, 20:21–30, 1948.
- [Loo53] L. H. Loomis. *An Introduction to Abstract Harmonic Analysis*. D. Van Nostrand Co. Inc., 1953.
- [Mik78] J. Mikusiński. *The Bochner integral*. Birkhäuser, 1978.
- [Mik89] P. Mikusiński. On the Daniell Integral. *Real Analysis Exchange*, 15(1):307–312, 1989.
- [ML98] P. Martin-Löf. An intuitionistic theory of types. In G. Sambin and J. M. Smith, editors, *Twenty-Five Years of Constructive Type Theory*, pages 127–172. Oxford University Press, 1998.
- [Myh75] J. Myhill. Constructive Set Theory. *J. Symbolic Logic*, 40(3):347–382, 1975. doi:10.2307/2272159.
- [Pal05] E. Palmgren. Bishop’s set theory, 2005. Slides from TYPES Summer School 2005, Gothenburg. URL: <https://www.cse.chalmers.se/research/group/logic/TypesSS05/Extra/palmgren.pdf>.
- [Pet15] I. Petrakis. *Constructive Topology of Bishop Spaces*. PhD thesis, LMU, 2015.

- [Pet18] I. Petrakis. Logic, 2018. Lecture notes. URL: <https://www.mathematik.uni-muenchen.de/~petrakis/ml.pdf>.
- [Pet19] I. Petrakis. Constructive uniformities of pseudometrics and Bishop topologies. *Journal of Logic and Analysis*, 11:FT2:1–44, 2019. doi:10.4115/jla.2019.11.FT2.
- [Pet20a] I. Petrakis. Embeddings of Bishop spaces. *Journal of Logic and Computation*, exaa015, 2020. doi:10.1093/logcom/exaa015.
- [Pet20b] I. Petrakis. *Families of Sets in Bishop Set Theory*. Habilitationsschrift, LMU, Munich, 2020. URL: <https://www.mathematik.uni-muenchen.de/~petrakis/Habilitationsschrift.pdf>.
- [Pet21] I. Petrakis. Direct spectra of Bishop spaces and their limits. *Logical Methods in Computer Science*, Volume 17, Issue 2, April 2021. doi:10.23638/LMCS-17(2:4)2021.
- [Pet22] I. Petrakis. Proof-relevance in Bishop-style constructive mathematics. *Mathematical Structures in Computer Science*, 32(1):1–43, 2022. doi:10.1017/S0960129522000159.
- [Pet23] I. Petrakis. Bases of pseudocompact Bishop spaces. In D. Bridges, H. Ishihara, M. Rathjen, and H. Schwichtenberg, editors, *Handbook of Constructive Mathematics*, Encyclopedia of Mathematics and its Applications, pages 359–394. Cambridge University Press, 2023. doi:10.1017/9781009039888.015.
- [Pet24] I. Petrakis. From Daniell spaces to the integration spaces of Bishop and Cheng, 2024. In preparation.
- [PW14] E. Palmgren and O. Wilander. Constructing categories and setoids of setoids in type theory. *Logical Methods in Computer Science*, Volume 10, Issue 3, September 2014. doi:10.2168/LMCS-10(3:25)2014.
- [PW22] I. Petrakis and D. Wessel. Algebras of Complemented Subsets. In U. Berger, J. N. Y. Franklin, F. Manea, and A. Pauly, editors, *Revolutions and Revelations in Computability*, pages 246–258, Cham, 2022. Springer International Publishing. doi:10.1007/978-3-031-08740-0_21.
- [Ric01] F. Richman. Constructive mathematics without choice. In P. Schuster, U. Berger, and H. Osswald, editors, *Reuniting the Antipodes — Constructive and Nonstandard Views of the Continuum: Symposium Proceedings, San Servolo, Venice, Italy, May 16–22, 1999*, pages 199–205, Dordrecht, 2001. Springer Netherlands. doi:10.1007/978-94-015-9757-9_17.
- [Šan68] N. Šanin. Constructive Real Numbers and Function Spaces. *Translations of Mathematical Monographs*, 21, 1968.
- [Sch04] P. Schuster. Countable Choice as a Questionable Uniformity Principle. *Philosophia Mathematica*, 12(2):106–134, 2004. doi:10.1093/philmat/12.2.106.
- [Seg54] I. E. Segal. Abstract Probability Spaces and a Theorem of Kolmogoroff. *American Journal of Mathematics*, 76(3):721–732, 1954. doi:10.2307/2372714.
- [Seg65] I. Segal. Algebraic integration theory. *Bull. Amer. Math. Soc.*, 71:419–489, 1965. doi:10.1090/S0002-9904-1965-11284-8.
- [SG66] G. E. Shilov and B. L. Gurevich. *Integral, Measure and Derivative: A Unified Approach*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966.
- [Shu22] M. Shulman. Affine logic for constructive mathematics. *The Bulletin of Symbolic Logic*, 28(3):327–386, 2022. doi:10.1017/bsl.2022.28.
- [Sim12] A. Simpson. Measure, randomness and sublocales. *Annals of Pure and Applied Logic*, 163(11):1642–1659, 2012. doi:10.1016/j.apal.2011.12.014.
- [Spi02] B. Spitters. *Constructive and intuitionistic integration theory and functional analysis*. PhD thesis, University of Nijmegen, 2002.
- [Spi06a] B. Spitters. Constructive algebraic integration theory. *Annals of Pure and Applied Logic*, 137(1):380–390, 2006. doi:10.1016/j.apal.2005.05.031.
- [Spi06b] B. Spitters. Constructive algebraic integration theory without choice. In T. Coquand, H. Lombardi, and M.-F. Roy, editors, *Mathematics, Algorithms, Proofs*, volume 5021 of *Dagstuhl Seminar Proceedings (DagSemProc)*, pages 1–13, Dagstuhl, Germany, 2006. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/DagSemProc.05021.9.
- [Sto48] M. H. Stone. Notes on Integration: II. *Proceedings of the National Academy of Sciences*, 34(9):447–455, 1948. doi:10.1073/pnas.34.9.447.
- [Tay73] S. J. Taylor. *Introduction to Measure and Integration*. Cambridge University Press, 1973.
- [Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. <https://homotopytypetheory.org/book>, Institute for Advanced Study, 2013.

- [Wei40] A. Weil. Calcul des probabilités, méthode axiomatique, intégration. *Revue Sci. (Rev. Rose Illus.)*, 78:201–208, 1940.
- [Wij90] R. Wijsman. *Invariant Measures on Groups and their Use in Statistics*. Institute of Mathematical Statistics, Hayward, California, 1990.
- [Zeu19] M. Zeuner. Families of Sets in Constructive Measure Theory. Master’s thesis, LMU, arXiv-version, 2019. URL: <https://arxiv.org/abs/2207.04000>.