

REPRESENTATIONS OF MEASURABLE SETS IN COMPUTABLE MEASURE THEORY

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ABSTRACT. This article is a fundamental study in computable measure theory. We use the framework of TTE, the representation approach, where computability on an abstract set X is defined by representing its elements with concrete “names”, possibly countably infinite, over some alphabet Σ . As a basic computability structure we consider a computable measure on a computable σ -algebra. We introduce and compare w.r.t. reducibility several natural representations of measurable sets. They are admissible and generally form four different equivalence classes. We then compare our representations with those introduced by Y. Wu and D. Ding in 2005 and 2006 and claim that one of our representations is the most useful one for studying computability on measurable functions.

1. INTRODUCTION

Measure theory is a fundament of modern analysis. In particular, computable measure theory is a fundament of computable analysis. In recent years a number of articles have been published on computable measure theory, for example [10, 22, 29, 14, 36, 27, 5, 11, 19, 15, 32, 16, 1, 17, 20, 4, 13, 33, 3, 21, 18]. Most of these articles start with a definition of computability concepts in measure theory and then prove, or disprove, a computable version of some classical theorem.

Wu and Ding [34, 35] have defined and compared various definitions of computability on measurable sets. In this article we extend these fundamental studies. We use the representation approach to computable analysis (TTE) [30, 8]. In this approach computability is defined directly on the set Σ^ω of the infinite sequences of symbols, e.g. by Turing machines. Computability is transferred to other sets X by means of representations $\delta : \Sigma^\omega \rightarrow X$ where the elements of Σ^ω are considered as names and computations are performed on names. Obviously, computability on the “abstract” set X depends crucially on the choice

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of the representation δ . Only those representations are of interest which can relate the important structure properties of X with corresponding ones of Σ^ω .

We start from a computable measure on a computable σ -algebra which has proved to be a very useful fundamental concept of computability in measure theory [34, 35, 36]. In addition to the representations studied in these articles we introduce several new representations of the measurable sets and compare all of them w.r.t. reducibility.

In Section 2 we outline very shortly some concepts from the representation approach. In Section 3 we summarize elementary definitions and facts from measure theory which we will need for introducing the new computability concepts.

In Section 4 we define computable σ -algebras $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ where \mathcal{R} is a countable ring which generates the σ -algebra \mathcal{A} in Ω such that $\Omega = \bigcup \mathcal{R}$ and $\alpha : \subseteq \Sigma^* \rightarrow \mathcal{R}$ is a notation of the ring such that set union and difference become computable. A measure μ is computable if $\mu(R)$ is finite for every ring element R and $R \mapsto \mu(R)$ is computable. Then we introduce and study representations ζ_+ , ζ_- and ζ of the measurable sets which exactly allow to compute $\mu(R \cap A)$ for $R \in \mathcal{R}$ and $A \in \mathcal{A}$ from below, from above or from below and above, respectively. We study reducibility and characterize the degree of non-computability for the negative results.

In Section 5 for the sets of finite measure we define a computable metric space and compare its Cauchy representation with the representations defined before.

In Section 6 we partition the set Ω computably by a (majorizing) sequence $(F_i)_{i \in \mathbb{N}}$ of ring elements. For each number i , the measure restricted to F_i is finite and induces a computable metric space, the metric of which can be normalized to a metric d'_i bounded by 1. The weighted sum $\bar{d} = \sum_i 2^{-i} \cdot d'_i$ is a computable metric on the whole σ -algebra the Cauchy representation of which allows to compute the measures of measurable sets from below and above and hence is equivalent to the representation ζ from Section 4.

In Section 7 we show that all the representations are admissible [30]. We compare our representations with those from [34, 35]. It turns out that ζ_+ for which there is no equivalent one in [34, 35] is most interesting.

2. COMPUTABILITY BY MEANS OF REPRESENTATIONS

For studying computability we use the TTE, representation approach to computable analysis [30, 8]. Let Σ be a fixed finite alphabet such that $0, 1 \in \Sigma$. Σ^* denotes the set of finite words over Σ and Σ^ω denotes the set of infinite sequences $p : \mathbb{N} \rightarrow \Sigma$. A partial function $f : \subseteq Y_1 \times \dots \times Y_k \rightarrow Y_0$ (where $Y_i = \Sigma^*$ or $Y_i = \Sigma^\omega$) is computable, iff it can be computed by a Type-2 Turing machine. For encoding pairs and longer tuples of elements from Σ^* and Σ^ω we use tupling functions all of which are denoted by $\langle \rangle$ [30, Definition 2.1.7]. For the wrapping function $\iota : \Sigma^* \rightarrow \Sigma^*$, $\iota(a_1 a_2 \dots a_k) := 110a_1 0a_2 \dots 0a_k 011$, two wrapped words cannot overlap properly. For $w_i \in \Sigma^*$ and $p_i \in \Sigma^\omega$ let $\langle w_1, \dots, w_n \rangle := \iota(w_1)\iota(w_2) \dots \iota(w_n)$, $\langle w_0, p_0 \rangle := \langle p_0, w_0 \rangle := \iota(w_0)p_0 \in \Sigma^\omega$, $\langle p_0, p_1 \rangle := (p_0(0)p_1(0)p_0(1)p_1(1) \dots)$, $\langle p_0, p_1, \dots \rangle \langle i, j \rangle := p_i(j)$ (where $\pi : \mathbb{N}^2 \rightarrow \mathbb{N}$, $\langle i, j \rangle = \pi(i, j)$ is a standard computable bijection), etc. The tupling functions and the projections of their inverses are computable. We will use definitions of the form " p is a list of all pairs $(u, v) \in \Sigma^* \times \Sigma^*$ such that $Q(u, v)$ " meaning: $\iota(\langle u, v \rangle)$ is a subword of p iff $Q(u, v)$.

We use canonical representations $\nu_{\mathbb{N}} : \subseteq \Sigma^* \rightarrow \mathbb{N}$, $\nu_{\mathbb{Q}} : \subseteq \Sigma^* \rightarrow \mathbb{Q}$, of the natural numbers and the rational numbers, respectively. For the real numbers let $\rho_{<}(p) = x$ iff p is a list of all u such that $\nu_{\mathbb{Q}}(u) < x$, $\rho_{>}(p) = x$ iff p is a list of all u such that $\nu_{\mathbb{Q}}(u) > x$ and $\rho\langle p, q \rangle = x$

iff $\rho_{<}(p) = x$ and $\rho_{>}(q) = x$. The representations $\bar{\rho}_{<}, \bar{\rho}_{>}, \bar{\rho}$ of the set $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ are defined accordingly [30, Section 4.1].

A representation of a set X is a partial surjective function $\delta : \subseteq Y \rightarrow X$ where $Y = \Sigma^*$ or $Y = \Sigma^\omega$. For representations $\delta_i : \subseteq Y_i \rightarrow X_i$, ($i = 1, 2$), a function $h : \subseteq Y_1 \rightarrow Y_2$ (operating on names) realizes the (abstract) function $f : \subseteq X_1 \rightarrow X_2$, iff $f \circ \delta_1(p) = \delta_2 \circ h(p)$ for all $p \in \text{dom}(f \circ \delta_1)$. A function f is called (δ_1, δ_2) -continuous (-computable), iff it is realized by a continuous (computable) function. A representation δ_1 is reducible to (translatable to) δ_2 , $\delta_1 \leq \delta_2$, iff the identity function $\text{id} : x \mapsto x$ is (δ_1, δ_2) -computable, that is, there is a computable function h such that $\delta_1(p) = \delta_2 \circ h(p)$ for all $p \in \text{dom}(\delta_1)$. Correspondingly, δ_1 is topologically reducible to δ_2 , $\delta_1 \leq_t \delta_2$, iff there is a continuous function h such that $\delta_1(p) = \delta_2 \circ h(p)$ for all $p \in \text{dom}(\delta_1)$. The two representations are equivalent, $\delta_1 \equiv \delta_2$, iff $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$. Accordingly, they are topologically equivalent, $\delta_1 \equiv_t \delta_2$, iff $\delta_1 \leq_t \delta_2$ and $\delta_2 \leq_t \delta_1$. Equivalent representations induce the same computability on the represented sets. For more details see [30, 8].

3. CONCEPTS FROM CLASSICAL MEASURE THEORY

In this Section we summarize elementary definitions and facts from measure theory which we will need for introducing the new computability concepts.

Let Ω be a set.

- A ring (in Ω) is a set $\mathcal{R} \subseteq 2^\Omega$ such that $\emptyset \in \mathcal{R}$, and $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$ if $A, B \in \mathcal{R}$. Since $A \cap B = A \setminus (A \setminus B)$, every ring is closed under intersection. The ring is called an algebra, if $\Omega \in \mathcal{R}$.
- A σ -algebra (in Ω) is a set $\mathcal{A} \subseteq 2^\Omega$ such that $\Omega \in \mathcal{A}$, $A^c = \Omega \setminus A \in \mathcal{A}$ if $A \in \mathcal{A}$, and $\bigcup_{i=0}^{\infty} A_i \in \mathcal{A}$ if $A_0, A_1, \dots \in \mathcal{A}$. The elements of \mathcal{A} are called the measurable sets. Every σ -algebra is a ring.
- For a set $\mathcal{T} \subseteq 2^\Omega$, $\mathcal{R}(\mathcal{T})$ denotes the smallest ring containing \mathcal{T} and $\mathcal{A}(\mathcal{T})$ denotes the smallest σ -algebra containing \mathcal{T} .
- A measure on a ring \mathcal{R} is a function $\mu : \mathcal{R} \rightarrow \mathbb{R}^\infty (= \mathbb{R} \cup \{\infty\})$ such that $\mu(\emptyset) = 0$, $\mu(A) \geq 0$ for all $A \in \mathcal{R}$, and $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$ for pairwise disjoint sets $A_0, A_1, \dots \in \mathcal{R}$ such that $\bigcup_i A_i \in \mathcal{R}$. (Often μ is called a pre-measure if \mathcal{R} is a ring and a measure only if \mathcal{R} is a σ -algebra.)
- A measure μ on a ring \mathcal{R} is σ -finite, if there is a sequence $E_0, E_1, \dots \in \mathcal{R}$ of sets such that $(\forall i)\mu(E_i) < \infty$ and $\bigcup_i E_i = \Omega$. The sets can be assumed to be pairwise disjoint: for $F_j := E_j \setminus \bigcup_{i < j} E_i$ the F_j are ring elements such that

$$(\forall i \neq j) F_i \cap F_j = \emptyset, \quad (\forall i)\mu(F_i) < \infty \quad \text{and} \quad \bigcup_i F_i = \Omega. \quad (3.1)$$

For two sets A, B let $A \Delta B := (A \setminus B) \cup (B \setminus A)$ be their symmetric difference. Some useful rules for the symmetric difference are listed in the appendix Section 9.

Two sets A and B with $\mu(A \Delta B) = 0$ are essentially identical in measure theory.

Definition 3.1. Let μ be a measure on a ring \mathcal{R} . Define an equivalence relation on \mathcal{R} by $A \sim B \iff \mu(A \Delta B) = 0$. Let $[A] := \{B \in \mathcal{R} \mid A \sim B\}$ be the equivalence class containing A . For $\mathcal{E} \subseteq \mathcal{R}$, let $[\mathcal{E}] := \{[A] \mid A \in \mathcal{E}\}$.

Notice that the following are equivalent: $\mu(A \Delta B) = 0$, $A \sim B$, $A \in [B]$, $B \in [A]$, and $[A] = [B]$.

Lemma 3.2. For $i \in \mathbb{N}$ let $A_i, B_i \in \mathcal{R}$ such that $A_i \sim B_i$.

Then $\mu(A_0) = \mu(B_0)$, $A_0 \text{ op } A_1 \sim B_0 \text{ op } B_1$ for $\text{op} \in \{\cup, \cap, \setminus\}$, and $\bigcup_i A_i \sim \bigcup_i B_i$ if $\bigcup_i A_i \in \mathcal{R}$ and $\bigcup_i B_i \in \mathcal{R}$.

Therefore the following operations are well-defined on equivalence classes:

$\mu([A_0]) := \mu(A_0)$, $[A_0] \text{ op } [A_1] := [A_0 \text{ op } A_1]$ for $\text{op} \in \{\cup, \cap, \setminus\}$, and $\bigcup_i [A_i] := [\bigcup_i A_i]$ if $\bigcup_i A_i \in \mathcal{R}$.

Proof. Straightforward, using in particular (9.6) and (9.7). \square

Our computability concepts in measure theory are based on the following theorem.

Theorem 3.3 (Carathéory extension theorem [2, 12]). *Every σ -finite measure on a ring \mathcal{R} has a unique extension to a measure on the σ -algebra $\mathcal{A}(\mathcal{R})$.*

Therefore, for specifying a measure μ on the σ -algebra $\mathcal{A}(\mathcal{R})$, it suffices to define $\mu(E)$ for every $E \in \mathcal{R}$.

Let μ be a σ -finite measure on a ring \mathcal{R} and let $(F_i)_{i \in \mathbb{N}}$ be a sequence of ring elements which satisfy (3.1). For any set $R \in \mathcal{R}$, we have $\mu(R) = \sum_{i \in \mathbb{N}} \mu(R \cap F_i)$. This implies that the measure μ on the ring \mathcal{R} is completely determined by its restriction to the subring \mathcal{R}^f which consists of all ring elements with finite measure.

In our computable measure theory we will consider only σ -algebras $\mathcal{A}(\mathcal{R})$ spanned by a finite or countable ring \mathcal{R} (which is non-empty since $\emptyset \in \mathcal{R}$) and measures μ such that $\mu(R) < \infty$ for all $R \in \mathcal{R}$ and μ is σ -finite on \mathcal{R} .

Lemma 3.4. *Let μ be a measure on a countable ring $\mathcal{R} \subseteq 2^\Omega$ such that $\mu(R) < \infty$ for all $R \in \mathcal{R}$.*

- (1) *If $\bigcup \mathcal{R} = \Omega$ then the measure μ is σ -finite.*
- (2) *The measure μ is σ -finite in $\Omega' := \bigcup \mathcal{R}$.*

Proof. (1) Since $\mathcal{R} \neq \emptyset$ it has an enumeration (not necessarily injective) $(E_i)_{i \in \mathbb{N}}$. Let $F_0 := E_0$ and $F_{n+1} := E_{n+1} \setminus (E_0 \cup \dots \cup E_n)$. Then the sets F_i are pairwise disjoint elements of \mathcal{R} such that $\mu(F_i) < \infty$ and $\bigcup_i F_i = \bigcup_i E_i = \bigcup \mathcal{R} = \Omega$.

- (2) Since $\mathcal{R} \subseteq 2^{\Omega'}$, μ is σ -finite in Ω' by (1). \square

Therefore, if $\bigcup \mathcal{R} \neq \Omega$, we obtain a σ -finite measure by ignoring $\Omega \setminus \bigcup \mathcal{R}$. We will use the next two theorems for defining representations of the measurable sets. For a measure μ on a σ -algebra \mathcal{A} and a subset $\mathcal{E} \subseteq \mathcal{A}$ let $\mathcal{E}^f := \{A \in \mathcal{E} \mid \mu(A) < \infty\}$ be the set of elements of \mathcal{E} of finite measure.

Special cases of the following theorem are proved in most introductory texts. A complete proof is added in the appendix Section 10.

Theorem 3.5. *Let μ be a measure on a σ -algebra \mathcal{A} . On \mathcal{A}^f the Fréchet metric is defined by $d(A, B) := \mu(A \Delta B)$.*

- (1) *(\mathcal{A}^f, d) , is a complete pseudometric space.*
- (2) *Let $(A_i)_{i \in \mathbb{N}}$ be a sequence in \mathcal{A}^f such that $d(A_i, A_j) \leq 2^{-i}$ for $j > i$.*

For $m \leq k$ let $B_{mk} := \bigcup_{i=m}^k A_i$, let $B_m := \bigcup_{i \geq m} A_i$ and

$B := \bigcap_m B_m = \bigcap_m \bigcup_{i \geq m} A_i$. Then

$$B_{mk} \subseteq B_{m,k+1}, \quad d(B_{mk}, B_{m,k+1}) \leq 2^{-k} \text{ and } d(B_{mk}, B_m) \leq 2 \cdot 2^{-k}, \quad (3.2)$$

$$B_m \supseteq B_{m+1} \in \mathcal{A}^f, \quad d(B_m, B_{m+1}) \leq 2^{-m} \text{ and } d(B_m, B) \leq 2 \cdot 2^{-m}, \quad (3.3)$$

$$B \in \mathcal{A}^f \text{ and } d(A_m, B) \leq 4 \cdot 2^{-m}. \quad (3.4)$$

(3) Let $(A_i)_{i \in \mathbb{N}}$ be a sequence in \mathcal{A}^f such that $d(A_i, A_j) \leq 2^{-i}$ for $j > i$.

For $m \leq k$ let $D_{mk} := \bigcap_{i=m}^k A_i$, let $D_m := \bigcap_{i \geq m} A_i$ and

$D := \bigcup_m D_m = \bigcup_m \bigcap_{i \geq m} A_i$. Then

$$D_{mk} \supseteq D_{m,k+1}, \quad d(D_{mk}, D_{m,k+1}) \leq 2^{-k} \quad \text{and} \quad d(D_{mk}, D_m) \leq 2 \cdot 2^{-k}, \quad (3.5)$$

$$D_m \subseteq D_{m+1} \in \mathcal{A}^f, \quad d(D_m, D_{m+1}) \leq 2^{-m} \quad \text{and} \quad d(D_m, D) \leq 2 \cdot 2^{-m}, \quad (3.6)$$

$$D \in \mathcal{A}^f \quad \text{and} \quad d(A_m, D) \leq 4 \cdot 2^{-m}. \quad (3.7)$$

(4) If \mathcal{R} is a ring such that $\mathcal{A} := \mathcal{A}(\mathcal{R})$ and the measure μ is σ -finite on \mathcal{R} , then \mathcal{R}^f is a dense subset of \mathcal{A}^f .

If $d(A_i, A_j) \leq 2^{-i}$ for $j > i$ then by Theorem 3.5 the sequence $(A_i)_{i \in \mathbb{N}}$ converges to $B = \bigcap_m \bigcup_{i \geq m} A_i \in \mathcal{A}^f$ and to $D = \bigcup_m \bigcap_{i \geq m} A_i \in \mathcal{A}^f$. Notice that $\bigcup_m \bigcap_{i \geq m} A_i \subseteq \bigcap_m \bigcup_{i \geq m} A_i$ since $(x \in A_i \text{ for almost all } i)$ implies $(x \in A_i \text{ infinitely often})$ and that $\mu(\bigcap_m \bigcup_{i \geq m} A_i \setminus \bigcup_m \bigcap_{i \geq m} A_i) = 0$.

A set $A \in \mathcal{A}$ is determined uniquely up to a set of measure 0 by the values $\mu(A \cap E)$ for ring elements E of finite measure. We will use this fact for defining various representations of the set $[A]$.

Lemma 3.6. *Let \mathcal{R} be a ring and let μ be a measure on $\mathcal{A} := \mathcal{A}(\mathcal{R})$ which is σ -finite on \mathcal{R} . Then for $A, B \in \mathcal{A}$,*

$$\mu(A \Delta B) = 0 \iff (\forall E \in \mathcal{R}^f) \mu(A \cap E) = \mu(B \cap E).$$

Proof. \implies : Suppose $\mu(A \Delta B) = 0$ and $E \in \mathcal{R}$. Then

$$\mu((A \cap E) \Delta (B \cap E)) = \mu((A \Delta B) \cap E) = 0, \quad \text{hence} \quad \mu(A \cap E) = \mu(B \cap E) \quad \text{by (9.9).}$$

\impliedby : Suppose $\mu(A \Delta B) > 0$. We may assume, without loss of generality, $\mu(A \setminus B) > 0$. We want to find some $E \in \mathcal{R}^f$ such that $\mu(A \cap E) \neq \mu(B \cap E)$. Since $\Omega = \bigcup_{R \in \mathcal{R}} R$,

$$\mu(A \setminus B) = \mu\left(\bigcup_{R \in \mathcal{R}} R \cap (A \setminus B)\right) = \mu\left(\bigcup_{R \in \mathcal{R}} (R \cap (A \setminus B))\right) \leq \sum_{R \in \mathcal{R}} \mu((R \cap A) \setminus (R \cap B)).$$

Therefore, $c := \mu((R \cap A) \setminus (R \cap B)) > 0$ for some $R \in \mathcal{R}$.

Let $C := R \cap A$ and $D := R \cap B$. Then $\mu(C \setminus D) = c$. By Theorem 3.5(4) there is some $G \in \mathcal{R}^f$ such that $\mu(G \Delta (C \setminus D)) < c/3$. By (9.5),

$$D \cap G \subseteq D \cap ((C \setminus D) \cup (G \Delta (C \setminus D))) \subseteq G \Delta (C \setminus D),$$

hence $\mu(D \cap G) < c/3$. Again by (9.5), $C \setminus D \subseteq G \cup (G \Delta (C \setminus D))$, hence

$$C \setminus D = C \cap (C \setminus D) \subseteq (C \cap G) \cup (G \Delta (C \setminus D)),$$

therefore, $c = \mu(C \setminus D) \leq \mu(C \cap G) + c/3$. We obtain $\mu(B \cap R \cap G) = \mu(D \cap G) < c/3$ and $\mu(A \cap R \cap G) = \mu(C \cap G) \geq 2c/3$, hence for $E := R \cap G$, $\mu(A \cap E) \neq \mu(B \cap E)$. \square

4. THE BASIC REPRESENTATIONS

In computable analysis computability on an uncountable structure is usually introduced by selecting a countable substructure which ‘‘generates’’ it and defining the meaning of ‘‘computable’’ on this substructure (example: computability on the field \mathbb{Q} , completion to \mathbb{R}). The results from the last section suggest that a countable ring with a σ -finite measure should be a good substructure. Then ring operations should become computable as well as the measure restricted to the ring.

Definition 4.1 (Computable σ -algebra, computable measure).

- (1) A *computable σ -algebra* is a tuple $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ such that \mathcal{R} is a countable ring in Ω , $\Omega = \bigcup \mathcal{R}$, $\mathcal{A} = \mathcal{A}(\mathcal{R})$, and $\alpha : \subseteq \Sigma^* \rightarrow \mathcal{R}$ is a notation of \mathcal{R} such that $\text{dom}(\alpha)$ is recursive and the functions $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are computable (w.r.t. α).
- (2) A measure μ on a computable σ -algebra $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ is computable, if it is finite on \mathcal{R} and $R \mapsto \mu(R)$, the restriction of μ to \mathcal{R} , is (α, ρ) -computable.

For a computable σ -algebra the intersection operation on the ring is also computable because $A \cap B = A \setminus (A \setminus B)$. Sometimes it is more convenient to use a numbering $E : \mathbb{N} \rightarrow \mathcal{R}$ of the ring \mathcal{R} where $E_i := E(i) := \alpha \circ h(i)$ for some computable (more precisely, $(\nu_{\mathbb{N}}, \text{id}_{\Sigma^*})$ -computable) bijection $h : \mathbb{N} \rightarrow \text{dom}(\alpha)$. Obviously the functions $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are also (E, E, E) -computable.

Since \mathcal{R} is countable, $\Omega = \bigcup \mathcal{R}$ and the measure μ is finite on \mathcal{R} , the measure is σ -finite. Since $\mu(\Omega) = \sup_{R \in \mathcal{R}} \mu(R)$, $\mu(\Omega)$ is a finite ρ -computable number or $= \infty$. The measure μ is computable on \mathcal{R} , iff $\{(u, v, w) \mid \nu_{\mathbb{Q}}(u) < \mu(\alpha(v)) < \nu_{\mathbb{Q}}(w)\}$ is r.e.

From the notation α of the ring a representation δ of the σ -algebra $\mathcal{A}(\mathcal{R})$ can be defined inductively as follows:

$$\begin{aligned} \delta(0\langle w \rangle 000\dots) &:= \alpha(w), \\ \delta(1p) &:= \Omega \setminus \delta(p), \\ \delta(2\langle p_0, p_1, \dots \rangle) &:= \bigcup_{i \in \mathbb{N}} \delta(p_i). \end{aligned}$$

In this case, if $\delta(p) = B$ then p encodes a finite-path tree (a term) which protocols the generation of the set B from ring elements by repeated application of the unary operation “complement” and the ω -ary operation “countable union”. The tremendous amount of information contained in a δ -name is not really necessary if we are only interested in computing the measure of the set. Instead, for given measure μ the σ -algebra \mathcal{A} is factorized by the equivalence relation $A \sim_{\mu} B \iff \mu(A \Delta B) = 0$.

In the following let μ be a computable measure on the computable σ -algebra $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$.

We define various representations of the class $[\mathcal{A}]$. By Lemma 3.6 and Definition 4.1, $[\mathcal{A}]$ is defined uniquely by the set of all $\mu(A \cap E)$ for $E \in \mathcal{R}$, see Lemma 4.3.

Definition 4.2. Define representations ζ_+ , ζ_- and ζ of $[\mathcal{A}]$ as follows:

- (1) $\zeta_+(p) = [A]$ iff p is (encodes) a list of all $\langle u, v \rangle$ such that

$$\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A),$$

- (2) $\zeta_-(p) = [A]$ iff p is (encodes) a list of all $\langle v, w \rangle$ such that

$$\mu(\alpha(v) \cap A) < \nu_{\mathbb{Q}}(w),$$

- (3) $\zeta(p) = [A]$ iff p is (encodes) a list of all $\langle u, v, w \rangle$ such that

$$\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A) < \nu_{\mathbb{Q}}(w).$$

A ζ_+ -name of a set A consists of all rational lower bounds of the $\mu(R \cap A)$ ($A \in \mathcal{R}$). Since the numbers $\mu(R)$ are ρ -computable, a ζ_- -name of A , yields a list of all rational lower bounds of $\mu(R \setminus A)$ ($A \in \mathcal{R}$) (Definition 4.6, Lemma 4.7). In [35] rational lower bounds of $\mu(A \setminus R)$

instead of $\mu(R \setminus A)$ are used for defining representations which then differ significantly from the ones defined here.

We must show that the definitions do not depend on the representative A of the class $[A]$.

Lemma 4.3. *The representations in Definition 4.2 are well-defined.*

Proof. Suppose $\zeta_+(p) = [A]$ and $\zeta_+(p) = [B]$ according to Definition 4.2. Then for all $r \in \mathbb{Q}$ and for all $R \in \mathcal{R}$,

$$r < \mu(R \cap A) \iff r < \mu(R \cap B), \quad (4.1)$$

hence $\mu(R \cap A) = \mu(R \cap B)$ for all $R \in \mathcal{R}$. By Lemma 3.6 $\mu(A \Delta B) = 0$ and hence $[A] = [B]$.

The argument is the same for ζ . For the case ζ_- replace “ $<$ ” in (4.1) by “ $>$ ”. \square

The representation ζ_+ (ζ_- , ζ) is the poorest representation that allows to compute $\rho_{<}$ -names ($\rho_{>}$ -names, ρ -names) of all $\mu(\alpha(v) \cap A)$.

Lemma 4.4. *For every representation γ of a subset of $[\mathcal{A}]$,*

- (1) $\gamma \leq \zeta_+ \iff ([A], R) \mapsto \mu(R \cap A)$ is $(\gamma, \alpha, \rho_{<})$ -computable,
- (2) $\gamma \leq \zeta_- \iff ([A], R) \mapsto \mu(R \cap A)$ is $(\gamma, \alpha, \rho_{>})$ -computable,
- (3) $\gamma \leq \zeta \iff ([A], R) \mapsto \mu(R \cap A)$ is (γ, α, ρ) -computable.

Proof. The statements can be derived from a general theorem [31, Theorem 13.1]. We give a direct proof here.

(1) There is a Type-2 machine M that on input $(p, v) \in \Sigma^\omega \times \Sigma^*$ computes a list of all $u \in \Sigma^*$ such that (u, v) is listed in p . If $\zeta_+(p) = [A]$ and $\alpha(v) = R$ then $f_M(p, v)$ is a list of all u such that $\nu_{\mathbb{Q}}(u) < \mu(R \cap A)$, hence $\rho_{<} \circ f_M(p, v) = \mu(R \cap A)$. Therefore, the function $([A], R) \mapsto \mu(R \cap A)$ is $(\zeta_+, \alpha, \rho_{<})$ -computable. Consequently, this function is $(\gamma, \alpha, \rho_{<})$ -computable if $\gamma \leq \zeta_+$.

On the other hand, suppose that the function $([A], R) \mapsto \mu(R \cap A)$ is $(\gamma, \alpha, \rho_{<})$ -computable. Then there is a Type-2 machine M which on input $(p, v) \in \text{dom}(\gamma) \times \text{dom}(\alpha)$ writes a list of all $u \in \text{dom}(\nu_{\mathbb{Q}})$, such that $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap \gamma(p))$. From M we can construct a Type-2 machine N which on input p writes a list of all $(u, v) \in \text{dom}(\nu_{\mathbb{Q}}) \times \text{dom}(\alpha)$ such that the machine M on input (p, v) writes u in finitely many steps of computation. Therefore, $f_N(p)$ is a list of all (u, v) such that $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap \gamma(p))$ hence $\zeta_+ \circ f_N(p) = \gamma(p)$. We obtain $\gamma \leq \zeta_+$.

(2) and (3) can be proved accordingly. \square

Therefore, ζ_+ is (up to equivalence) the poorest representation γ of $[\mathcal{A}]$ such that $([A], R) \mapsto \mu(R \cap A)$ is $(\gamma, \alpha, \rho_{<})$ -computable etc.

For representations γ and δ , $\gamma \wedge \delta$ is the greatest lower bound of γ and δ for the reducibility \leq , where $(\gamma \wedge \delta)(p, q) = x \iff \gamma(p) = \delta(q) = x$ [30, Section 3.3]. Remember that for the well-known representations of the real numbers, $\rho \equiv \rho_{<} \wedge \rho_{>}$ [30, Lemma 4.1.9].

Lemma 4.5.

- (1) $\zeta \equiv \zeta_+ \wedge \zeta_-$, in particular, $\zeta \leq \zeta_+$, $\zeta \leq \zeta_-$ and $\zeta_+ \wedge \zeta_- \leq \zeta$.
- (2) The function $[A] \mapsto [A^c]$ is (ζ_+, ζ_-) -computable and (ζ_-, ζ_+) -computable.

Proof. (1) By Lemma 4.4(3), the function $h : ([A], R) \mapsto \mu(R \cap A)$ is (ζ, α, ρ) -computable. Since $\rho \leq \rho_{<}$, the function h is $(\zeta, \alpha, \rho_{<})$ -computable, hence $\zeta \leq \zeta_+$ by Lemma 4.4(1). Accordingly, $\zeta \leq \zeta_-$. Therefore, $\zeta \leq \zeta_+ \wedge \zeta_-$.

On the other hand, since $\zeta_+ \wedge \zeta_- \leq \zeta_+$, by Lemma 4.4(1) the function h is $(\zeta_+ \wedge \zeta_-, \alpha, \rho_<)$ -computable. Accordingly, the function h is $(\zeta_+ \wedge \zeta_-, \alpha, \rho_>)$ -computable. Since $\rho_< \wedge \rho_> \leq \rho$, the function h is $(\zeta_+ \wedge \zeta_-, \alpha, \rho)$ -computable. Finally, $\zeta_+ \wedge \zeta_- \leq \zeta$ by Lemma 4.4(3).

(2) Let $c[A] := [\Omega \setminus A] = [A^c]$. By Lemma 4.4(1) the function $G : ([A], R) \mapsto \mu(R \cap A)$ is $(\zeta_+, \alpha, \rho_<)$ -computable. Since $R \mapsto \mu(R)$ is (α, ρ) -computable, the function $([A], R) \mapsto \mu(R) - \mu(R \cap A)$ is $(\zeta_+, \alpha, \rho_>)$ -computable, hence $\mu(\alpha(w)) - \mu(\alpha(w) \cap \zeta_+(p)) = \rho_> \circ d(p, w)$ for some computable function d . We obtain $G(c \circ \zeta_+(p), \alpha(w)) = \mu(\alpha(w) \cap c \circ \zeta_+(p)) = \mu(\alpha(w)) - \mu(\alpha(w) \cap \zeta_+(p)) = \rho_> \circ d(p, w)$. By Lemma 4.4(2), $c \circ \zeta_+ \leq \zeta_-$, hence c is (ζ_+, ζ_-) -computable.

(ζ_-, ζ_+) -computability of complementation can be proved accordingly. \square

For the representations ζ_+ , ζ_- and ζ a name of a class $[A]$ allows to compute $\mu(\alpha(v) \cap A)$ w.r.t. $\rho_<$, $\rho_>$ and ρ , respectively. Since $(\alpha(v) \cap A) \cup (\alpha(v) \setminus A) = \alpha(v)$ and a ρ -name of $\mu(\alpha(v))$ is known for the computable measure space, from a $\rho_<$ -name ($\rho_>$ -name, ρ -name) of $\mu(\alpha(v) \cap A)$ we can compute a $\rho_>$ -name ($\rho_<$ -name, ρ -name) of $\mu(\alpha(v) \setminus A)$ and vice versa. Therefore, we can define representations such that names allow to compute all $\mu(\alpha(v) \setminus A)$ which are equivalent to the former ones.

Definition 4.6.

(1) $\zeta'_+(p) = [A]$ iff p is (encodes) a list of all (u, v) such that

$$\mu(\alpha(v) \setminus A) < \nu_{\mathbb{Q}}(u),$$

(2) $\zeta'_-(p) = [A]$ iff p is (encodes) a list of all (u, v) such that

$$\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \setminus A).$$

(3) $\zeta'(p) = [A]$ iff p is (encodes) a list of all (u, v, w) such that

$$\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \setminus A) < \nu_{\mathbb{Q}}(w).$$

Notice that

$$\zeta'_-(p) = [A] \iff \zeta_+(p) = [A^c] \quad \text{and} \quad \zeta'_+(p) = [A] \iff \zeta_-(p) = [A^c]. \quad (4.2)$$

Lemma 4.7. $\zeta_+ \equiv \zeta'_+$, $\zeta_- \equiv \zeta'_-$, $\zeta \equiv \zeta'$

Proof. Straightforward. \square

There is a computable measure on a computable σ -algebra such that $\zeta_+ \not\leq \zeta$ (see the proof of Theorem 4.8 (2) below). As usual already translation by a continuous function is impossible, $\zeta_+ \not\leq_t \zeta$. We determine the degree of unsolvability of the translations from ζ_+ to ζ and the other similar ones.

Let $\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2, \mathcal{Y}_2$ be represented sets and let $f_1 : \mathcal{X}_1 \rightrightarrows \mathcal{Y}_1$ and $f_2 : \mathcal{X}_2 \rightrightarrows \mathcal{Y}_2$ be multifunctions. Then $f_1 \leq_W f_2$ iff there are computable functions G, H on Σ^ω such that for all realizations F_2 of f_2 , $F_1 : p \mapsto H(p, F_2 \circ G(p))$ realizes f_1 ([7, 6], where \leq_W is called Weihrauch reducibility). This means that composition with G and H in this manner transforms every realization of f_2 to a realization of f_1 . The multi-functions f_1, f_2 are called W-equivalent, $f_1 \equiv_W f_2$, iff $f_1 \leq_W f_2$ and $f_2 \leq_W f_1$. A stronger reducibility is defined by $f_1 \leq_{sW} f_2$ iff there are computable functions G, H on Σ^ω such that for all realizations F_2 of f_2 , $F_1 : p \mapsto H \circ F_2 \circ G(p)$ realizes f_1 [7, 6]. Obviously, $f_1 \leq_{sW} f_2$ implies $f_1 \leq_W f_2$.

It is known that $\rho_n \not\leq \rho_c$, $\rho_< \not\leq \rho_c$, $\rho_> \not\leq \rho_c$, $\rho_< \not\leq \rho_>$ and $\rho_> \not\leq \rho_<$, where ρ_n is the representation of the real numbers by (not necessarily fast) converging sequences of rational numbers [30]. These five translation problems are of the same sW-degree of

unsolvability. Furthermore, the identity $\text{ECf} : (2^{\mathbb{N}}, \text{En}) \rightarrow (2^{\mathbb{N}}, \text{Cf})$ and complementation of enumeration $\text{CE} : (2^{\mathbb{N}}, \text{En}) \rightarrow (2^{\mathbb{N}}, \text{En})$, $K \mapsto \mathbb{N} \setminus K$, are in this sW-degree [28], where Cf is the characteristic function representation and En is the enumeration representation of the subsets of \mathbb{N} [30].

Let $\text{En}^* : \Sigma^\omega \rightarrow 2^{\Sigma^*}$ be the canonical enumeration representation of the set of subsets of Σ^* , that is, $\iota(w)$ is a subword of $p \in \Sigma^\omega$ iff $w \in \text{En}^*(p)$. Then also complementation $\text{CE}^* : (2^{\Sigma^*}, \text{En}^*) \rightarrow (2^{\Sigma^*}, \text{En}^*)$, $W \mapsto \Sigma^* \setminus W$, is in the sW-degree of CE. Finally, it is known that

$$f \leq_W \text{CE} \iff f \leq_{sW} \text{CE} \quad \text{for all functions } f \text{ on represented sets.} \quad (4.3)$$

Theorem 4.8. *For a given computable measure on a computable σ -algebra \mathcal{A} define $\mathcal{A}_+ := ([\mathcal{A}], \zeta_+)$, $\mathcal{A}_- := ([\mathcal{A}], \zeta_-)$, $\mathcal{A}_0 := ([\mathcal{A}], \zeta)$, and for $a, b \in \{+, -, 0\}$ define the translation $\text{id}_{ab} : \mathcal{A}_a \rightarrow \mathcal{A}_b$ by $\text{id}_{ab}([A]) := [A]$. Then:*

(1) *For every computable measure on a computable σ -algebra,*

$$\text{id}_{+-} \leq_{sW} \text{id}_{+0} \leq_{sW} \text{CE}, \quad \text{id}_{+0} \leq_W \text{id}_{+-}, \quad (4.4)$$

$$\text{id}_{-+} \leq_{sW} \text{id}_{-0} \leq_{sW} \text{CE}, \quad \text{id}_{-0} \leq_W \text{id}_{-+}. \quad (4.5)$$

(2) *There is a computable probability measure on a computable σ -algebra such that*

$$\text{id}_{+-} \equiv_{sW} \text{id}_{+0} \equiv_{sW} \text{id}_{-+} \equiv_{sW} \text{id}_{-0} \equiv_{sW} \text{CE}.$$

Proof. (1) We prove $\text{id}_{+-} \leq_{sW} \text{id}_{+0}$. By Lemma 4.5 there is a computable function H such that $\zeta(q) = \zeta_- \circ H(p)$. Define $G(p) := p$. Suppose F_2 realizes id_{+0} , that is $\zeta_+(p) = \zeta \circ F_2(p)$. Then

$$\zeta_+(p) = \zeta \circ F_2(p) = \zeta_- \circ H \circ F_2 \circ G(p),$$

hence $p \mapsto H \circ F_2 \circ G(p)$ realizes id_{+-} . Since H and G are computable, $\text{id}_{+-} \leq_{sW} \text{id}_{+0}$.

We prove $\text{id}_{+0} \leq_W \text{id}_{+-}$. By Lemma 4.5 there is a computable function h such that $(\zeta_+ \wedge \zeta_-)(q) = \zeta \circ h(p)$. Define H and G by $H(p, q) := h(\langle p, q \rangle)$ and $G(p) := p$. Suppose F_2 realizes id_{+-} , that is $\zeta_+(p) = \zeta_- \circ F_2(p)$. Then

$$\zeta_+(p) = \zeta_- \circ F_2(p) = (\zeta_+ \wedge \zeta_-)(p, F_2(p)) = \zeta \circ h(\langle p, F_2(p) \rangle) = \zeta \circ H(p, F_2 \circ G(p)),$$

hence $p \mapsto H(p, F_2 \circ G(p))$ realizes id_{+0} . Since H and G are computable, $\text{id}_{+0} \leq_W \text{id}_{+-}$.

We prove $\text{id}_{+-} \leq_{sW} \text{CE}$. By Definition 4.2, $\zeta_+(p) = [A]$ means: for all $x \in \Sigma^*$, $\iota(x)$ is a subword of p iff

$$(\exists u \in \text{dom}(\nu_{\mathbb{Q}}))(\exists v \in \text{dom}(\alpha))(x = \langle u, v \rangle \text{ and } \nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A)).$$

There is a Type-2 machine M that on input $q \in \Sigma^\omega$ writes a list of all $\langle v, w \rangle$ such that $v \in \text{dom}(\alpha)$ and for some $u \in \text{dom}(\nu_{\mathbb{Q}})$, $\langle u, v \rangle$ is listed in q and $\nu_{\mathbb{Q}}(w) > \nu_{\mathbb{Q}}(u)$.

Let F_2 be a realization of CE^* . Suppose $\zeta_+(p) = [A]$. Let $q := F_2(p)$. By the definition of M , $f_M(q)$ is a list of words $\langle v, w \rangle$ such that $v \in \text{dom}(\alpha)$ and $w \in \text{dom}(\nu_{\mathbb{Q}})$. Suppose $\mu(\alpha(v) \cap A) < \nu_{\mathbb{Q}}(w)$. Then for some u , $\mu(\alpha(v) \cap A) \leq \nu_{\mathbb{Q}}(u) < \nu_{\mathbb{Q}}(w)$. By the definition of ζ_+ , $\langle u, v \rangle$ is not listed in p and hence listed in $F_2(p)$. Therefore, $\langle v, w \rangle$ is listed in $f_M \circ F_2(p)$. On the other hand suppose $\langle v, w \rangle$ is listed in $f_M \circ F_2(p)$. Then for some u , $\langle u, v \rangle$ is listed in $F_2(p)$ and $\nu_{\mathbb{Q}}(u) < \nu_{\mathbb{Q}}(w)$. Therefore, $\langle u, v \rangle$ is not listed in p , hence $\mu(\alpha(v) \cap A) \leq \nu_{\mathbb{Q}}(u) < \nu_{\mathbb{Q}}(w)$.

Combining the two cases we obtain, $\zeta_+(p) = \zeta_- \circ f_M \circ F_2(p)$. Therefore the function $f_M \circ F_2$ realizes the function id_{+-} , hence $\text{id}_{+-} \leq_{sW} \text{CE}^* \leq_{sW} \text{CE}$.

In summary $\text{id}_{+-} \leq_{sW} \text{id}_{+0} \leq_W \text{id}_{+-} \leq_{sW} \text{CE}$. Applying (4.3) we obtain (4.4).

(4.5) can be proved accordingly.

(2) Let $\Omega := \mathbb{N}$, $\mathcal{A} := 2^{\mathbb{N}}$, $\mathcal{R} :=$ the set of finite subsets of \mathbb{N} with canonical notation α , and for $A \in \mathcal{A}$ let $\mu(A) := \sum_{i \in A} 3^{-i}$. Then μ is a computable measure on the computable σ -algebra $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ such that $\mu(\Omega) = 3/2$.

First, we prove $\text{CE} \leq_{sW} \text{id}_{+-}$.

We show that the function $h : 2^{\mathbb{N}} \rightarrow [\mathcal{A}]$, $h(A) := [A]$ is (En, ζ_+) -computable. Suppose $\text{En}(p) = A$. There is a Type-2 machine M that on input $p \in \text{dom}(\text{En})$ produces a list of all $\langle u, v \rangle$ such that $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A_k)$ for some k , where A_k is the set of all n such that $01^{n+1}0$ is a subword of the first k symbols of p . If $\langle u, v \rangle$ is in this list then $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A_k) \leq \mu(\alpha(v) \cap \text{En}(p))$. If $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap \text{En}(p))$, then $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A_k)$ for some k , hence $\langle u, v \rangle$ is in the list. Therefore, $\zeta_+ \circ f_M(p) = [\text{En}(p)]$, hence f_M realizes h .

We show that the function $h' : [\mathcal{A}] \rightarrow 2^{\mathbb{N}}$, $h'([A]) := \mathbb{N} \setminus A$ is (ζ_-, En) -computable. Since $[A] = [B]$ implies $A = B$, the function h' is well-defined. There is a computable function $g : \mathbb{N} \rightarrow \Sigma^*$ such that $(\forall i) \alpha \circ g(i) = \{i\}$. And there is a computable function $d : \mathbb{N} \rightarrow \Sigma^*$ such that $\nu_{\mathbb{Q}} \circ d(i) = 3^{-i}$. There is a Type-2 machine N that on input $q \in \text{dom}(\zeta_-)$ lists all $01^{i+1}0$ such that $(g(i), d(i))$ is listed in q . Suppose, $[A] = \zeta_-(q)$. Then q is a list of all (v, w) such that $\mu(A \cap \alpha(v)) < \nu_{\mathbb{Q}}(w)$. Since

$$\begin{aligned} i \notin A &\iff \mu(A \cap \alpha \circ g(i)) < \nu_{\mathbb{Q}} \circ d(i) \\ &\iff (g(i), d(i)) \text{ is listed in } q \iff 01^{i+1}0 \text{ is a subword of } f_N(q), \end{aligned}$$

$h' \circ \zeta_-(q) = h'([A]) = \mathbb{N} \setminus A = \text{En} \circ f_N(q)$. Therefore, f_N realizes h' .

Suppose F_2 realizes id_{+-} . Then $f_N \circ F_2 \circ f_M$ realizes $h' \circ \text{id}_{+-} \circ h = \text{CE}$. Therefore, $\text{CE} \leq_{sW} \text{id}_{+-}$. By (4.4), $\text{CE} \leq_{sW} \text{id}_{+-} \leq_{sW} \text{id}_{+0} \leq_{sW} \text{CE}$, hence $\text{id}_{+-} \equiv_{sW} \text{id}_{+0} \equiv_{sW} \text{CE}$. $\text{id}_{-+} \equiv_{sW} \text{id}_{-0} \equiv_{sW} \text{CE}$ can be proved accordingly.

Let $\mu' := 2/3 \cdot \mu$. Then $\mu'(\Omega) = 1$, hence μ' is a probability measure and the results hold as well for μ' . \square

Lemma 4.9. *The function $([A], G) \mapsto [A \cap G]$ for $A \in \mathcal{A}$ and $G \in \mathcal{R}$ is $(\zeta_+, \alpha, \zeta_+)$ -computable, $(\zeta_-, \alpha, \zeta_-)$ -computable and (ζ, α, ζ) -computable.*

Proof. $(\zeta_-, \alpha, \zeta_-)$: There is a computable word function h such that $\alpha(v) \cap \alpha(u) = \alpha \circ h(v, u)$ (see the remark after Definition 4.1).

Suppose $\zeta_-(p) = [A]$, $\alpha(u) = G$ and $\zeta_-(q) = [A \cap G]$. Then p is a list of all (v, w) such that $\mu(\alpha(v) \cap A) < \nu_{\mathbb{Q}}(w)$. Correspondingly, q is a list of all (v, w) such that $\mu(\alpha(v) \cap G \cap A) < \nu_{\mathbb{Q}}(w)$, that is, q is a list of all (v, w) such that $\mu(\alpha \circ h(v, u) \cap A) < \nu_{\mathbb{Q}}(w)$, hence q is a list of all (v, w) such that $(h(v, u), w)$ is in the list p . There is a machine that on input (p, u) writes a list of all (v, w) such that $(h(v, u), w)$ is listed in p . Therefore, $([A], G) \mapsto [A \cap G]$ is $(\zeta_-, \alpha, \zeta_-)$ -computable.

The other two statements can be proved accordingly. \square

Let $\bar{p}_{<}$ and $\bar{p}_{>}$ be the lower and upper representation of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, respectively, and let $\bar{p} = \bar{p}_{<} \wedge \bar{p}_{>}$ [30, Section 4.1]. Informally, $\bar{p}_{<}(p) = x$ iff p is a list of all $a \in \mathbb{Q}$ such that $a < x$, and $\bar{p}_{>}(p) = x$ iff p is a list of all $a \in \mathbb{Q}$ such that $a > x$.

Lemma 4.10.

- (1) $\mu : [A] \mapsto \mu(A)$ is $(\zeta_+, \bar{p}_{<})$ -computable,
- (2) $\mu : [A] \mapsto \mu(A^c)$ is $(\zeta_-, \bar{p}_{<})$ -computable,
- (3) $\mu(\Omega)$ is $\bar{p}_{<}$ -computable,

(4) For finite measures, $\mu : [A] \mapsto \mu(A)$ is (ζ, ρ) -computable iff $\mu(\Omega)$ is ρ -computable.

Proof. (1) Since $\Omega = \bigcup \mathcal{R}$ and \mathcal{R} is countable and closed under union, $\mu(A) = \mu(A \cap \Omega) = \sup_{R \in \mathcal{R}} \mu(R \cap A)$. There is a Type-2 machine M which on input p writes a list q of all u such that for some v , (u, v) is listed in p . If $\zeta_+(p) = [A]$ then q is a list of all $u \in \text{dom}(\nu_{\mathbb{Q}})$ such that $\nu_{\mathbb{Q}}(u) < \mu(A)$. Therefore f_M is a $(\zeta_+, \bar{\rho}_<)$ -realization of $[A] \mapsto \mu(A)$.

(2) Suppose $\zeta'_-(p) = [A]$. By (4.2), $\zeta_+(p) = [A^c]$, hence by (1), $\mu(A^c) = \bar{\rho}_< \circ f(p)$ for some computable function f . Therefore $[A] \mapsto \mu(A^c)$ is $(\zeta'_-, \bar{\rho}_<)$ -computable and hence $(\zeta_-, \bar{\rho}_<)$ -computable by Lemma 4.7.

(3) This follows from (1) since $[\Omega]$ is ζ_+ -computable.

(4) Suppose $\mu(\Omega)$ is ρ -computable. Since by Lemma 4.5 $\zeta \leq \zeta_+$ and $\zeta \leq \zeta_-$, $[A] \mapsto \mu(A)$ and $[A] \mapsto \mu(A^c)$ are $(\zeta, \bar{\rho}_<)$ -computable by (1) and (2) above. Since $\mu(A) + \mu(A^c) = \mu(\Omega)$ and $\mu(\Omega)$ is ρ -computable, $[A] \mapsto \mu(A)$ is (ζ, ρ) -computable.

Suppose $[A] \mapsto \mu(A)$ is (ζ, ρ) -computable. Since $[\Omega] = \zeta(p)$ for some computable $p \in \Sigma^\omega$, $\mu(\Omega)$ must be ρ -computable. \square

Example 4.11 (non-computable $\mu(\Omega)$). Let $\Omega := \mathbb{N}$, $\mathcal{A} := 2^{\mathbb{N}}$, $\mathcal{R} :=$ the set of finite subsets of \mathbb{N} with canonical notation α and $\mu(A) := \sum_{i \in A} 2^{-h(i)}$ where $h : \mathbb{N} \rightarrow \mathbb{N}$ is an injective computable numbering of some r.e. set $K \subseteq \mathbb{N}$ that is not recursive. Then μ is a computable measure on the computable σ -algebra $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$. There is a computable $p \in \Sigma^\omega$ such that $\zeta(p) = [\Omega] = [\mathbb{N}]$. Since $\mu(\Omega) = \sum_{i \in \mathbb{N}} 2^{-h(i)} = \sum_{n \in K} 2^{-n}$, $\mu(\Omega)$ is ρ -computable but not ρ -computable [30, Example 4.2.4]. \square

5. REPRESENTATIONS OF THE SETS OF FINITE MEASURE

In this section we introduce and study representations of the set $[\mathcal{A}^f]$ for the set \mathcal{A}^f of measurable sets of finite measure. $\mu(\Omega)$ may be finite or infinite. By Theorem 3.5, (\mathcal{A}^f, d) with $d(A, B) = \mu(A \Delta B)$ is a complete pseudometric space with \mathcal{R}^f as a dense subset. Remember that for our computable measure μ on the computable σ -algebra, $\mathcal{R}^f = \mathcal{R}$. Then $d([A], [B]) := d(A, B) = \mu(A \Delta B)$ defines a metric on the equivalence classes $[\mathcal{A}^f] = \{[A] \mid A \in \mathcal{A}^f\}$. (As usual, we use the same symbol d for the pseudometric and its factorization.)

A computable metric space is a quadruple (M, d, A, ν) such that (M, d) is a metric space, $A \subseteq M$ is dense and $\nu : \subseteq \Sigma^* \rightarrow A$ is a notation of A such that $\text{dom}(\nu)$ is recursive and the metric d restricted to A is (ν, ν, ρ) -computable (equivalently, the set of all (t, u, v, w) such that $\nu_{\mathbb{Q}}(t) < d(\nu(u), \nu(v)) < \nu_{\mathbb{Q}}(w)$ is r.e.). The Cauchy representation of a computable metric space is defined by $\delta_C(p) = x$ iff p is (encodes) a sequence $v_0, v_1, \dots \in \Sigma^*$ such that $d(\nu(v_i), \nu(v_j)) \leq 2^{-i}$ if $i < j$ and $x = \lim \nu(v_i)$ [30, Section 8.1] [9]. Notice that $d(x, \nu(v_i)) \leq 2^{-i}$. The metric $d : M \times M \rightarrow \mathbb{R}$ is $(\delta_C, \delta_C, \rho)$ -computable.

Lemma 5.1. Let $\overline{\mathcal{A}^f} := ([\mathcal{A}^f], d, [\mathcal{R}], \beta)$ where $d([A], [B]) := \mu(A \Delta B)$ and $\beta(u) := [\alpha(u)]$.

(1) $\overline{\mathcal{A}^f}$ is a complete computable metric space.

(2) For the Cauchy representation ξ_C of $\overline{\mathcal{A}^f}$ the measure $\mu : [A] \mapsto \mu([A]) = \mu(A)$ is (ξ_C, ρ) -computable.

Proof. By Theorem 3.5, (\mathcal{A}^f, d) with $d(A, B) = \mu(A \Delta B)$ is a complete pseudometric space with $\mathcal{R}^f = \mathcal{R}$ as a dense set. Since by Definition 3.1, $d([A], [B]) = 0 \iff [A] = [B]$, $([\mathcal{A}^f], d)$ is a complete metric space with $[\mathcal{R}]$ as a dense subset. Obviously β is a notation

of $[\mathcal{R}]$ with recursive domain. Since $d([A], [B]) = \mu(A \Delta B)$ and by Definition 4.1, the symmetric difference on \mathcal{R} is computable and μ is (α, ρ) -computable, the metric on $[\mathcal{R}]$ is (β, β, ρ) -computable. Therefore, $\overline{\mathcal{A}^f}$ is a computable metric space.

Since d is (ξ_C, ξ_C, ρ) -computable, $\xi_C(q) = [\emptyset]$ for some computable $q \in \Sigma^\omega$ and $\mu(A) = \mu(\emptyset \Delta A) = d([\emptyset], [A])$, the measure μ is (ξ_C, ρ) -computable. \square

We introduce two further representations of the set $[\mathcal{A}^f]$ of measurable sets of finite measure by adding the measure of A to the ζ -names of $[A] \in [\mathcal{A}^f]$.

Definition 5.2. For the space $\overline{\mathcal{A}^f} := ([\mathcal{A}^f], d, [\mathcal{R}], \beta)$ let ξ_C be the Cauchy representation and define representations ξ_+, ξ_- and ξ by

$$\begin{aligned} \xi_+ \langle p, q \rangle = [A] & : \iff \zeta_+(p) = [A] \text{ and } \rho_>(q) = \mu(A), \\ \xi_- \langle p, q \rangle = [A] & : \iff \zeta_-(p) = [A] \text{ and } \rho_<(q) = \mu(A), \\ \xi \langle p, q \rangle = [A] & : \iff \zeta(p) = [A] \text{ and } \rho(q) = \mu(A). \end{aligned}$$

Theorem 5.3. *On the space $\overline{\mathcal{A}^f}$,*

- (1) $[A] \mapsto \mu(A)$ is $(\xi_+, \rho_>)$ -computable, $(\xi_-, \rho_<)$ -computable and (ξ, ρ) -computable.
- (2) $\xi_+ \leq \zeta_+$, $\xi_- \leq \zeta_-$ and $\xi \leq \zeta$,
- (3) $\xi_C \equiv \xi_+ \equiv \xi_- \equiv \xi$,
- (4) $\xi \equiv \zeta$ iff $\mu(\Omega)$ is ρ -computable.

Proof. (1), (2) Obvious.

(3) $\xi \leq \xi_+$: Form a ξ -name of $[A]$ we can compute a ζ -name of $[A]$ and a ρ -name of $\mu(A)$. Since $\zeta \leq \zeta_+$ and $\rho \leq \rho_>$ we can compute a ζ_+ -name p' of $[A]$ and a $\rho_>$ -name q' of $\mu(A)$. Then $\langle p', q' \rangle$ is a ξ_+ -name of $[A]$.

$\xi \leq \xi_-$: Accordingly.

$\xi_+ \leq \xi_C$: Since \mathcal{R} is dense in \mathcal{A}^f (Theorem 3.5) for every $A \in \mathcal{A}^f$ and $\varepsilon > 0$ there is some $R \in \mathcal{R}$ such that $\mu(A \Delta R) < \varepsilon$. Notice that

$$\mu(A \Delta R) = \mu(A \setminus R) + \mu(R \setminus A) = \mu(A) - \mu(A \cap R) + \mu(R) - \mu(A \cap R).$$

The function $[A] \mapsto \mu(A)$ is $(\xi_+, \rho_>)$ -computable by Definition 5.2, the function $R \mapsto \mu(R)$ is $(\alpha, \rho_>)$ -computable, and by $\xi_+ \leq \zeta_+$ and Lemma 4.4(1) the function $([A], R) \mapsto -\mu(A \cap R)$ is $(\xi_+, \alpha, \rho_>)$ -computable. Therefore, $G : ([A], R) \mapsto \mu(A \Delta R)$ is $(\xi_+, \alpha, \rho_>)$ -computable. There is a machine M which on input (p, v) writes a sequence of all (codes of) $a \in \mathbb{Q}$ such that $G(\xi_+(p), \alpha(v)) < a$.

There is a machine N which on input p writes a sequence v_0, v_1, \dots of words where v_i is computed as follows: N runs M as a subprogram and searches some (v_i, k) such that M on input (p, v_i) writes the rational number 2^{-i-1} in at most k steps of computation. If $\xi(p) = [A]$ then for every i the search for v_i is successful. Since $\mu(\xi(p) \Delta \alpha(v_i)) < 2^{-i-1}$, $\xi_C(v_0, v_1, \dots) = [A]$. Therefore, $\xi_+ \leq \xi_C$. $\xi_- \leq \xi_C$ can be proved accordingly..

$\xi_C \leq \xi$: Suppose $\xi_C(r) = [A]$. Then r is (encodes) a sequence R_0, R_1, \dots of ring elements such that $d(R_i, A) = \mu(R_i \Delta A) \leq 2^{-i}$. We must compute $\mu(A)$ and furthermore prove $\xi_C \leq \zeta$ (that is, we must compute a ζ -name of $[A]$).

Since $\mu(R_i \Delta A) \leq 2^{-i}$, for every $R \in \mathcal{R}$, $\mu((R \cap R_i) \Delta (R \cap A)) = \mu(R \cap (R_i \Delta A)) \leq 2^{-i}$, hence

$$|\mu(R \cap R_i) - \mu(R \cap A)| \leq 2^{-i}. \quad (5.1)$$

by (9.10). Since intersection on \mathcal{R} is (α, α, α) -computable, from an α -name of R and $r \in \text{dom}(\xi_C)$ encoding the sequence R_0, R_1, \dots we can compute a sequence $s \in \Sigma^\omega$ encoding the sequence $R \cap R_0, R \cap R_1, \dots$ which, by (5.1) is a ξ_C -name of $[R \cap A]$. By Lemma 5.1 from s we can compute a ρ -name of $\mu([R \cap A])$. Therefore, $([A], R) \mapsto \mu(A \cap R)$ is (ξ_C, α, ρ) -computable. By Lemma 4.4, $\xi_C \leq \zeta$. Since $[A] \mapsto \mu(A)$ is (ξ_C, ρ) -computable by Lemma 5.1, $\xi_C \leq \xi$.

(4) By lemma 4.10, $\xi \equiv \zeta$ iff $[A] \mapsto \mu(A)$ is (ζ, ρ) -computable iff $\mu(\Omega)$ is ρ -computable. \square

By Lemma 4.10(3), $\mu(\Omega) \in \mathbb{R}^\infty$ is $\bar{\rho}_<$ -computable, hence $\mu(\Omega)$ is the limit of an increasing computable sequence of rational numbers which may be finite or ∞ . By Example 4.11 there is a computable finite measure with finite non-computable $\mu(\Omega)$.

If $\mu(\Omega) \in \mathbb{R}$ is a computable real number, then by Lemma 4.10(4), $[A] \mapsto \mu(A)$ is (ζ, ρ) -computable and by Theorem 5.3(4), $\zeta \equiv \xi \equiv \xi_C$.

If $\mu(\Omega) \in \mathbb{R}$ is a computable real number and $\mu(\Omega) > 0$ then $\mu' := \mu/\mu(\Omega)$ is a probability measure with the same computability properties.

6. REPRESENTATIONS BY MEANS OF A PARTITION

We still assume that μ is a computable measure on the computable σ -algebra $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$. As we have mentioned there are ring elements F_0, F_1, \dots such that $(\forall i \neq j) F_i \cap F_j = \emptyset$, $(\forall i) \mu(F_i) < \infty$ and $\bigcup_i F_i = \Omega$ (see (3.1)). Such a sequence $(F_i)_{i \in \mathbb{N}}$ can be computed. For $i \in \mathbb{N}$ define $\mu_i(A) := \mu(A \cap F_i)$. Then every μ_i is a finite measure and $\mu(A) = \sum_{i \in \mathbb{N}} \mu_i(A)$.

By Lemma 3.6 for every $A \in \mathcal{A}$, $[A]$ is defined by the family $(\mu(A \cap R))_{R \in \mathcal{R}}$. The representations ζ_+ , ζ_- and ζ from Definition 4.2 are defined by means of this family (“a ζ_+ -name of $[A]$ is a list of all ...” etc.). Correspondingly, for every i and A , $[A \cap F_i]$ is defined by the family $(\mu_i(A \cap R))_{R \in \mathcal{R}}$. Therefore $[A]$ is defined also by the family $(\mu(A \cap F_i \cap R))_{(i \in \mathbb{N}, R \in \mathcal{R})}$ which is a subfamily of $(\mu(A \cap R))_{R \in \mathcal{R}}$. We introduce representations $\bar{\zeta}_+$, $\bar{\zeta}_-$ and $\bar{\zeta}$ of $[\mathcal{A}]$ by means of this smaller family and compare them with ζ_+ , ζ_- and ζ .

Definition 6.1. A numbering $F : \mathbb{N} \rightarrow \mathcal{R}$ is a *partition for α* iff there is a computable function $g : \mathbb{N} \rightarrow \Sigma^*$ such that $F(i) = \alpha \circ g(i)$ (that is, $F \leq \alpha$) and

$$(\forall i \neq j) F_i \cap F_j = \emptyset, \quad (\forall i) \mu(F_i) < \infty \quad (6.1)$$

and it is *majorising* if there is a computable function $g' : \subseteq \Sigma^* \rightarrow \mathbb{N}$ such that

$$(\forall w \in \text{dom}(\alpha)) \quad \alpha(w) \subseteq \bigcup_{i \leq g'(w)} F_i. \quad (6.2)$$

Lemma 6.2. *There is a majorising partition for α .*

Proof. There is a bijective computable function $h : \mathbb{N} \rightarrow \text{dom}(\alpha)$. For the numbering $E = \alpha \circ h$ define $F(n) := F_n := E_n \setminus \bigcup_{i < n} E_i$. Then F satisfies (6.1). Since union and set difference are (E, E, E) -computable, there is some computable function $g_1 : \mathbb{N} \rightarrow \mathbb{N}$ such that $F_n = E \circ g_1(n) = \alpha \circ h \circ g_1$. Then $g := h \circ g_1$ is computable and $F = \alpha \circ g$.

Let $g' := h^{-1}$. From the definition of F , $E_n = \bigcup_{i \leq n} F_i$, hence $\alpha(w) = E_{g'(w)} = \bigcup_{i \leq g'(w)} F_i$. Then g' is computable and satisfies (6.2). \square

For a given partition for α we introduce three further representations of $[A]$.

Definition 6.3. For a fixed partition F for α define representations $\bar{\zeta}_+$, $\bar{\zeta}_-$ and $\bar{\zeta}$ of $[A]$ as follows:

(1) $\bar{\zeta}_+(p) = [A]$ iff p is (encodes) a list of all (u, i, v) such that

$$\nu_{\mathbb{Q}}(u) < \mu(F_i \cap \alpha(v) \cap A),$$

(2) $\bar{\zeta}_-(p) = [A]$ iff p is (encodes) a list of all (i, v, w) such that

$$\mu(F_i \cap \alpha(v) \cap A) < \nu_{\mathbb{Q}}(w),$$

(3) $\bar{\zeta}(p) = [A]$ iff p is (encodes) a list of all (u, i, v, w) such that

$$\nu_{\mathbb{Q}}(u) < \mu(F_i \cap \alpha(v) \cap A) < \nu_{\mathbb{Q}}(w).$$

The three representations are well-defined (see Definition 4.2 and Lemma 4.3).

Theorem 6.4. $\zeta_+ \leq \bar{\zeta}_+$, $\zeta_- \leq \bar{\zeta}_-$ and $\zeta \leq \bar{\zeta}$. Furthermore, $\zeta_+ \equiv \bar{\zeta}_+$, $\zeta_- \equiv \bar{\zeta}_-$ and $\zeta \equiv \bar{\zeta}$ if F is majorising.

Proof. Since intersection is computable on \mathcal{R} there is a computable function d such that $\alpha(v) \cap F_i = \alpha \circ d(v, i)$. There is a Type-2 machine N which on input $p \in \text{dom}(\zeta_+)$ enumerates all (u, i, v) such that $(u, d(v, i))$ is listed by p . Then f_N translates ζ_+ to $\bar{\zeta}_+$, hence $\zeta_+ \leq \bar{\zeta}_+$.

For proving the other direction let g' be the computable function from (6.2). Then

$$\begin{aligned} & \nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap A) \\ \iff & \nu_{\mathbb{Q}}(u) < \sum_i \mu(F_i \cap \alpha(v) \cap A) \\ \iff & \nu_{\mathbb{Q}}(u) < \sum_{i \leq g'(v)} \mu(F_i \cap \alpha(v) \cap A) \\ \iff & (\exists u_0, \dots, u_{g'(v)}) \\ & (\nu_{\mathbb{Q}}(u) < \sum_{i \leq g'(v)} \nu_{\mathbb{Q}}(u_i) \text{ and } (\forall i \leq g'(v)) \nu_{\mathbb{Q}}(u_i) < \mu(F_i \cap \alpha(v) \cap A)). \end{aligned}$$

There is a Type-2 machine N that on input $p \in \text{dom}(\bar{\zeta}_+)$ enumerates all (u, v) such that for $k := g'(v)$ there are u_0, \dots, u_k with $\nu_{\mathbb{Q}}(u) < \sum_{i \leq k} \nu_{\mathbb{Q}}(u_i)$ and (u_i, i, v) can be found in the list p for all $0 \leq i \leq k$. The function f_N translates $\bar{\zeta}_+$ to ζ_+ , hence $\bar{\zeta}_+ \leq \zeta_+$.

The other statements can be proved accordingly. \square

We introduce a metric \bar{d} on the σ -algebra $[A]$ and prove that its Cauchy representation is equivalent to $\bar{\zeta}$. This metric is similar to the metric d_1 in [34, Section 5]. We discuss their relation in Section 7 below.

Theorem 6.5. Let F be a partition for α . Then $([A], \bar{d}, [\mathcal{R}], \beta)$ where $\beta(w) := [\alpha(w)]$ and

$$\bar{d}([A], [B]) := \bar{d}(A, B) := \sum_{i \in \mathbb{N}} \frac{\mu(F_i \cap (A \Delta B))}{1 + \mu(F_i \cap (A \Delta B))} \cdot 2^{-i}$$

is a computable metric space such that $\bar{\zeta} \equiv \bar{\xi}_C$ for its Cauchy representation $\bar{\xi}_C$.

For $i \in \mathbb{N}$ and $A \in \mathcal{A}$ let $\mu_i(A) := \mu(F_i \cap A)$. Then μ_i is a computable measure on $(\Omega, \mathcal{A}, \mathcal{R}, \alpha)$ such that $\mu_i(\Omega) = \mu(F_i)$ is (finite and) ρ -computable (see Lemma 4.10 and Theorem 5.3). For every i , d_i defined by $d_i(A, B) := \mu_i(A \Delta B) = \mu(F_i \cap (A \Delta B)) = d(F_i \cap A, F_i \cap B)$ is a computable pseudometric on \mathcal{A} (not only on \mathcal{A}^f). Notice that μ_i is the restriction of the measure μ to F_i and $d_i(A, B)$ is the finite distance of A and B restricted to F_i .

Define $e : [0; \infty) \rightarrow [0; 1)$ by $e(x) := x/(1+x)$. Then $e^{-1}(y) = y/(1-y)$ and e and e^{-1} are (ρ, ρ) -computable increasing functions such that

$$e(x) \leq x \text{ and } e^{-1}(y) \leq 2 \cdot y \text{ for } y \leq 1/2. \quad (6.3)$$

It is known that for a pseudometric d , $d' := e \circ d = d/(1+d)$ is a pseudometric bounded by 1 with the same induced topology. Furthermore, for a sequence $(d_i)_{i \in \mathbb{N}}$ of pseudometrics bounded by 1, $d(x, y) := \sum_{i \in \mathbb{N}} d_i(x, y) \cdot 2^{-i}$ is a pseudometric [23]. The statements hold accordingly for metrics.

Proof. By the above remarks \bar{d} is a pseudometric on \mathcal{A} , and since

$$\bar{d}([A], [B]) = 0 \text{ iff } (\forall i) \mu(F_i \cap (A \Delta B)) = 0 \text{ iff } \mu(A \Delta B) = 0 \text{ iff } [A] = [B],$$

\bar{d} is a metric on $[\mathcal{A}]$. Since union, intersection and difference on \mathcal{R} are (α, α, α) -computable, the restriction of \bar{d} to \mathcal{R} is (α, α, ρ) -computable. Below, we show that \mathcal{R} is dense in (\mathcal{A}, \bar{d}) .

$\bar{\zeta} \leq \bar{\xi}_C$: Suppose $\bar{\zeta}(p) = [A]$. Then p is (encodes) a list of all (u, i, v, w) such that $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap F_i \cap A) < \nu_{\mathbb{Q}}(w)$. From (p, i) we can compute a list p' of all (u, v, w) such that $\nu_{\mathbb{Q}}(u) < \mu(\alpha(v) \cap F_i \cap A) < \nu_{\mathbb{Q}}(w)$, hence $\zeta(p') = F_i \cap A$. Since $\zeta \leq \zeta_+$, by Lemma 4.10 we can compute a $\bar{\rho}_<$ -name of $\mu(F_i \cap A)$, hence a $\rho_<$ -name of $\mu(F_i \cap A)$ since $\mu(F_i \cap A)$ is finite. Since $\zeta \leq \zeta_-$, by Definition 5.2 we can compute a ξ_- -name q of $[F_i \cap A]$. Therefore, by Theorem 5.3 from (p, i) we can compute a ξ_C -name r of $[F_i \cap A]$. Then r is (encodes) a sequence v_0, v_1, \dots such that $d(\alpha(v_k), F_i \cap A) \leq 2^{-k}$.

Let $k \in \mathbb{N}$. Since the metric d is (ξ, α, ρ) -computable and \mathcal{R} is dense in (\mathcal{A}^f, d) , for every i we can find some u_i such that for $S_i := \alpha(u_i)$, $\mu((F_i \cap A) \Delta S_i) = d(F_i \cap A, S_i) < 2^{-k-1}/(k+2)$. Let $R := \bigcup_{i \leq k+1} (F_i \cap S_i)$. Since $D \Delta (F \cap S) \subseteq D \Delta S$,

$$\begin{aligned} \mu(F_i \cap (A \Delta R)) &= \mu((F_i \cap A) \Delta (F_i \cap R)) = \mu((F_i \cap A) \Delta (F_i \cap S_i)) \\ &\leq \mu((F_i \cap A) \Delta S_i) < 2^{-k-1}/(k+2) \end{aligned}$$

and hence

$$\begin{aligned} \bar{d}(A, R) &\leq \sum_{i \leq k+1} \frac{\mu(F_i \cap (A \Delta R))}{1 + \mu(F_i \cap (A \Delta R))} \cdot 2^{-i} + 2^{-k-1} \\ &< (k+2) \cdot \frac{2^{-k-1}}{k+2} + 2^{-k-1} \leq 2^{-k}. \end{aligned}$$

This implies that \mathcal{R} is dense in (\mathcal{A}, \bar{d}) . Let $\bar{\zeta}(p) = [A]$. By Definition 6.1, for any i , an α -name of F_i can be computed. So an α -name of R can be computed from p . Hence a sequence (v_0, v_1, \dots) can be computed such that $\bar{d}(A, \alpha(v_k)) \leq 2^{-k-1}$, which by definition constitutes a $\bar{\xi}_C$ -name of $[A]$. Therefore $\bar{\zeta} \leq \bar{\xi}_C$.

By density of \mathcal{R} , $(\mathcal{A}, \bar{d}, \mathcal{R}, \alpha)$ is a computable pseudometric space and $([\mathcal{A}], \bar{d}, [\mathcal{R}], \beta)$ is a computable metric space.

$\bar{\xi}_C \leq \bar{\zeta}$: We apply the following characterization which is similar to Lemma 4.4(3):

$$\gamma \leq \bar{\zeta} \iff ([A], R, H) \mapsto \mu(A \cap R \cap H) \text{ is } (\gamma, \alpha, F, \rho)\text{-computable} \quad (6.4)$$

Suppose $\bar{\xi}_C(p) = [A]$, $R = \alpha(v)$ and $H = F_i$. Let $k \in \mathbb{N}$. Since \mathcal{R} is dense in (\mathcal{A}, \bar{d}) , there is some $S \in \mathcal{R}$ such that $\bar{d}(A, S) \leq 2^{-k-i-1}$. Some u such that $\bar{d}(A, S) \leq 2^{-k-i-1}$ for $S = \alpha(u)$ can be computed from p, i and k .

Since $\mu(F_i \cap R \cap (A \Delta S)) \leq \mu(F_i \cap (A \Delta S))$ and the function $e : x \mapsto x/(1+x)$ is increasing,

$$\sum_j 2^{-j} \cdot \frac{\mu(F_j \cap R \cap (A \Delta S))}{1 + \mu(F_j \cap R \cap (A \Delta S))} \leq \sum_j 2^{-j} \cdot \frac{\mu(F_j \cap (A \Delta S))}{1 + \mu(F_j \cap (A \Delta S))} \leq \bar{d}(A, S),$$

hence by (6.3), $\mu(F_i \cap R \cap (A \Delta S)) \leq 2^{-k}$. It follows that for every i , $\mu((F_i \cap R \cap A) \Delta (F_i \cap R \cap S)) \leq 2^{-k}$, hence $|\mu(F_i \cap R \cap A) - \mu(F_i \cap R \cap S)| \leq 2^{-k}$ by (9.10). Since intersection on \mathcal{R} is computable, from $A = \bar{\xi}_C(p)$, i, k and $R = \alpha(v)$ we can compute some $a := \mu(F_i \cap R \cap S)$ such that $|\mu(F_i \cap R \cap A) - a| \leq 2^{-k}$. Therefore, $([A], R, H) \mapsto \mu(A \cap R \cap H)$ is $(\bar{\xi}_C, \alpha, F, \rho)$ -computable. By (6.4), $\bar{\xi}_C \leq \bar{\zeta}$. \square

Corollary 6.6. *Define $([A], \bar{d})$ and the Cauchy representation $\bar{\xi}_C$ as in Theorem 6.5 by a majorising partition F for the notation α of the ring \mathcal{R} . Then $\zeta \equiv \bar{\xi}_C$.*

Proof. This follows from Theorems 6.4 and 6.5. \square

In the proof of Lemma 6.2 we have constructed a majorizing partition F for α . Although the metric \bar{d} on $[A]$ and the representation $\bar{\xi}_C$ introduced in Definition 6.4 depend on F , the equivalence class of $\bar{\xi}_C$ is the same for all such partitions.

7. SUMMARY AND FINAL REMARKS

Up to equivalence we have the four new representations ζ_+ , ζ_- , ζ and ξ_C . The representations $\bar{\zeta}_+$, $\bar{\zeta}_-$ and $\bar{\zeta}$ are equivalent to the first three ones if they are defined by means of a majorising partition which always exists. For the Cauchy representation ξ_C of the sets of finite measure, $\xi_C \equiv \zeta$, if $\mu(\Omega)$ is (finite and) ρ -computable. If the Cauchy representation $\bar{\xi}_C$ is defined by means of a majorising partition, then $\bar{\xi}_C \equiv \zeta$.

In [34, 35] Wu and Ding have introduced several other representations of the measurable sets. First, we consider [35]. The representation $\delta_{\mathbb{T}_1}$ [35, Theorem 4.1] can be expressed informally as follows: $\delta_{\mathbb{T}_1}(p) = [A]$ iff p consists of a list of all pairs (E, r) such that $\mu(E \setminus A) < r$ and a list of all pairs (E, r) such that $\mu(A \setminus E) < r$ (where $E \in \mathcal{R}$ and $r \in \mathbb{Q}$). Since $\mu(E) = \mu(E \setminus A) + \mu(E \cap A)$ and $\mu(E)$ can be computed, the first list can be replaced by a list of all pairs (E, r) such that $r < \mu(E \cap A)$.

Define $\delta_1(p, q) = [A]$ iff $\zeta_+(p) = [A]$ and $\bar{\rho}_>(q) = \mu(A)$. Then $\delta_1 \equiv \delta_{\mathbb{T}_1}$ (without proof). Therefore, the restriction of $\delta_{\mathbb{T}_1}$ to the sets of infinite measure is equivalent to ζ_+ and its restriction to the sets of finite measure is equivalent to ξ_+ , hence also equivalent to ξ_- , ξ and ξ_C by Theorem 5.3.

Accordingly, the representation $\delta_{\mathbb{T}_2}$ from Section 4.2 is equivalent to the following representation δ_2 defined by $\delta_2(p, q, r) = [A]$ iff $\zeta(p) = [A]$, $\bar{\rho}_>(q) = \mu(A)$ and $\bar{\rho}_>(r) = \mu(A^c)$.

The third representation $\delta_{\mathbb{T}_3}$ from [35, Section 4.3] uses a computable sequence $(C_i)_{i \in \mathbb{N}}$ where $C_n = \bigcup_{i < n} D_i$ for some partition $(D_i)_{i \in \mathbb{N}}$ for α such that $\mu(D_i) > 0$. The condition $\mu(D_i) > 0$ excludes some spaces from consideration. It is irrelevant for the representaion

$\delta_{\mathbb{T}_3}$ but important for the representation $\delta_{\mathbb{D}_1}$ below. The representation $\delta_{\mathbb{T}_3}$ can be defined informally as follows: $\delta_{\mathbb{T}_3}(p) = [A]$ iff p is a list of all (E, i) ($E \in \mathcal{R}$, $i \in \mathbb{N}$) such that $\mu((A \Delta E) \cap C_i) < 2^{-i}$.

From p we can compute a list of all (E, k, r) (r rational) such that $\mu((A \Delta E) \cap D_k) < r$. Using arguments similar to those in the proof of Theorem 6.5 we can prove $\delta_{\mathbb{T}_3} \equiv \bar{\zeta}$. The additional condition $\mu(D_i) > 0$ in [35, Theorem 3.3] is not used in this proof. If the partition D is majorising then $\delta_{\mathbb{T}_3} \equiv \zeta$ (without proof).

In [34, Definiton 5.1] a metric on $[\mathcal{A}]$ is defined by

$$d_1([A], [B]) := \sum_{i \in \mathbb{N}} \frac{\mu(D_i \cap (A \Delta B))}{\mu(D_i)} \cdot 2^{-i}.$$

This definition is only meaningful if $\mu(D_i) > 0$ for all i . Therefore, for the metric \bar{d} in (6.5) we use the denominators $1 + \mu(D_i \cap (A \Delta B))$ instead of $\mu(D_i)$. The Cauchy representation for the computable metric space $\mathbb{D}_1 := ([\mathcal{A}], d_1, [\mathcal{R}], [\alpha])$ is called $\delta_{\mathbb{D}_1}$. By a proof similar to that of Theorem 6.5 it can be shown that $\delta_{\mathbb{D}_1} \equiv \bar{\zeta}$. By Lemma 6.2 there is a majorising partition D . In this case, $\delta_{\mathbb{D}_1} \equiv \zeta$ by Theorem 6.4. Also for another metric a Cauchy representation $\delta_{\mathbb{D}_2}$ is introduced.

Only for the representations $\delta_{\mathbb{T}_3}$ and $\delta_{\mathbb{D}_1}$, which are equivalent (without proof) union and intersection on the measurable sets are computable. It can be shown that union and intersection are computable also for ζ_+ and ζ_- and that countable union is computable for ζ_+ but not for ζ .

A function $f : \Omega \rightarrow X$ to a topological space X is measurable, if $f^{-1}(U)$ is measurable for every open set U . Since intersection and countable union are computable on the open subsets of a computable topological space [31] these operations should also be computable on the measurable sets (since, for example, $f^{-1}(\bigcup U_i) = \bigcup_i f^{-1}(U_i)$). From all the representations of measurable sets mentioned in this article only for the representation ζ_+ intersection and countable union are computable. Therefore, we claim that ζ_+ is the most useful one for studying computability of measurable functions.

In [35, Sections 4.1 and 4.2] proper supersets of $\sigma := \{\uparrow(E, r) \mid R \in \mathcal{R}, r \in \mathbb{Q}_+\}$ where $\uparrow(E, r) := \{A \in \mathcal{A} \mid \mu(R \setminus A) < r\}$ have been used as subbases of topologies for defining the representations $\delta_{\mathbb{T}_1}$ and $\delta_{\mathbb{T}_2}$ of the measurable sets. The set σ itself would yield a representation which is equivalent to ζ_+ . The authors have not taken this case into consideration.

A representation $\delta : \subseteq \Sigma^\omega \rightarrow X$ of a topological T_0 -space (X, τ) is admissible, iff it is continuous and $\gamma \leq \delta$ for every other continuous representation γ of X [30, 25, 24, 26, 8]. For admissible representations, a function on the represented sets is continuous, iff it can be realized by a continuous function on the names.

The Cauchy representation of a computable metric space is admissible [30]. Therefore, the representations ξ_C (Lemma 5.1), $\bar{\xi}_C$ (Theorem 6.5) and $\delta_{\mathbb{D}_1}$ [34] are admissible.

Let $\lambda : \Sigma^* \rightarrow \sigma$ be a notation of a set of subsets of X such that σ is a subbase of a T_0 -topology (X, τ) . Define a representation $\delta : \subseteq \Sigma^\omega \rightarrow X$ as follows: $\delta(p) = x$ iff p is a list of all w such that $x \in \lambda(w)$. Then δ is an admissible representation of the space (X, τ) where τ is the final topology of δ [31]. All the other representations of measurable sets defined in this article can be written in this way and hence are admissible. In each case a subbase of the final topology can be directly extracted from the definition. For example the final topology of ζ is generated by the subbase consisting of all sets $B(a, R, b) := \{[A] \in [\mathcal{A}] \mid a < \mu(R \cap A) < b\}$ such that $a, b \in \mathbb{Q}$ and $R \in \mathcal{R}$.

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9. APPENDIX: SOME USEFUL RULES FOR THE SYMMETRIC DIFFERENCE

$$A \Delta B = B \Delta A, \quad (9.1)$$

$$(A \Delta B) \Delta C = A \Delta (B \Delta C), \quad (9.2)$$

$$A \Delta B \subseteq A \Delta C \cup C \Delta B, \quad (9.3)$$

$$A \cup B = A \cap B \uplus A \Delta B, \quad (9.4)$$

$$A \subseteq B \cup (A \Delta B), \quad (9.5)$$

$$(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C) = (C \setminus A) \Delta (C \setminus B), \quad (9.6)$$

$$\left(\bigcup_{i \in I} A_i \right) \Delta \left(\bigcup_{i \in I} B_i \right) \subseteq \bigcup_{i \in I} (A_i \Delta B_i). \quad (9.7)$$

Let μ be a measure on a ring \mathcal{R} . From (9.4),

$$\mu(A) \leq \mu(A \cup B) = \mu(A \cap B) + \mu(A \Delta B) \leq \mu(B) + \mu(A \Delta B)$$

and accordingly with A and B interchanged. Therefore,

$$\mu(A) \leq \mu(B) + \mu(A \Delta B), \quad (9.8)$$

$$\mu(A) = \mu(A \cup B) = \mu(A \cap B) = \mu(B) \quad \text{if } \mu(A \Delta B) = 0, \quad (9.9)$$

$$|\mu(A) - \mu(B)| \leq \mu(A \Delta B) \quad \text{if } A \text{ and } B \text{ have finite measure.} \quad (9.10)$$

10. APPENDIX: PROOF OF THEOREM 3.5

By (9.3) the mapping $d : (A, B) \mapsto \mu(A \Delta B)$ is a pseudometric on the set \mathcal{A}^f .

Next we prove (2).

Obviously, $B_{mk} \subseteq B_{m,k+1}$. Since $(X \cup Y \cup Z) \setminus (X \cup Y) \subseteq Y \Delta Z$,

$$\begin{aligned} d(B_{mk}, B_{m,k+1}) &= \mu((A_m \cup \dots \cup A_k \cup A_{k+1}) \setminus (A_m \cup \dots \cup A_k)) \\ &\leq \mu(A_k \Delta A_{k+1}) \leq 2^{-k} \end{aligned}$$

and $d(B_{mk}, B_{mk'}) \leq 2^{-k} + \dots + 2^{-(k'-1)} < 2 \cdot 2^{-k}$ for $k < k'$ by induction. Therefore, $d(B_{mk}, B_m) = \mu(B_m \setminus B_{mk}) = \mu((\bigcup_{k < k'} B_{mk'}) \setminus B_{mk}) = \mu(\bigcup_{k < k'} (B_{mk'} \setminus B_{mk})) = \sup_{k < k'} \mu(B_{mk'} \setminus B_{mk}) \leq 2 \cdot 2^{-k}$. This proves (3.2).

Obviously $B_m \supseteq B_{m+1}$. Since $B_{mm} \in \mathcal{A}^f$ and $d(B_{mm}, B_m)$ is finite, $B_m \in \mathcal{A}^f$. Since $(X \cup Y \cup Z) \setminus (X \cup Y) \subseteq Y \Delta Z$,

$$\begin{aligned} d(B_m, B_{m+1}) &= \mu((A_m \cup A_{m+1} \cup \dots) \setminus (A_{m+1} \cup A_{m+2} \cup \dots)) \\ &\leq \mu(A_m \Delta A_{m+1}) \leq 2^{-m} \end{aligned}$$

and $d(B_m, B_{m'}) \leq 2^{-m} + \dots + 2^{-(m'-1)} < 2 \cdot 2^{-m}$ for $m < m'$ by induction. Therefore, $d(B_m, B) = \mu(B_m \setminus B) = \mu(B_m \setminus \bigcap_{m < m'} B_{m'}) = \mu(\bigcup_{m < m'} (B_m \setminus B_{m'})) = \sup_{m < m'} \mu(B_m \setminus B_{m'}) \leq 2 \cdot 2^{-m}$. This proves (3.3).

$B \in \mathcal{A}^f$ since $B \subseteq B_0$ and $B_0 \subseteq \mathcal{A}^f$. By (3.2, 3.3), $d(A_m, B) = d(B_{mm}, B) \leq d(B_{mm}, B_m) + d(B_m, B) \leq 4 \cdot 2^{-m}$. This proves (3.4).

Next we prove (3). Obviously, $D_{mk} \supseteq D_{m,k+1}$. Since $(X \cap Y) \setminus (X \cap Y \cap Z) \subseteq Y \Delta Z$,

$$\begin{aligned} d(D_{mk}, D_{m,k+1}) &= \mu((A_m \cap \dots \cap A_k) \setminus (A_m \cap \dots \cap A_k \cap A_{k+1})) \\ &\leq \mu(A_k \Delta A_{k+1}) \leq 2^{-k}, \end{aligned}$$

and $d(D_{mk}, D_{mk'}) \leq 2^{-k} + \dots + 2^{-(k'-1)} < 2 \cdot 2^{-k}$ for $k < k'$ by induction. Therefore, $d(D_{mk}, D_m) = \mu(D_{mk} \setminus D_m) = \mu(D_{mk} \setminus (\bigcap_{k < k'} D_{mk'})) = \mu(\bigcup_{k < k'} (D_{mk} \setminus D_{mk'})) = \sup_{k < k'} \mu(D_{mk} \setminus D_{mk'}) \leq 2 \cdot 2^{-k}$. This proves (3.5).

Obviously, $D_m \subseteq D_{m+1}$. $D_m \in \mathcal{A}^f$ since $D_{mm} \in \mathcal{A}^f$ and $D_m \subseteq D_{mm}$. Since $(X \cap Y) \setminus (Z \cap X \cap Y) \in Y \Delta Z$,

$$\begin{aligned} d(D_m, D_{m+1}) &= \mu((A_{m+1} \cap A_{m+2} \cap \dots) \setminus (A_m \cap A_{m+1} \cap \dots)) \\ &\leq \mu(A_m \Delta A_{m+1}) \leq 2^{-m} \end{aligned}$$

and $d(D_m, D_{m'}) \leq 2^{-m} + \dots + 2^{-(m'-1)} < 2 \cdot 2^{-m}$ for $m < m'$ by induction. Therefore, $d(D_m, D) = \mu(D \setminus D_m) = \mu((\bigcup_{m < m'} D_{m'}) \setminus D_m) = \mu(\bigcup_{m < m'} (D_{m'} \setminus D_m)) = \sup_{m < m'} \mu(D_{m'} \setminus D_m) \leq 2 \cdot 2^{-m}$. This proves (3.6).

$D \in \mathcal{A}^f$ since $D_0 \in \mathcal{A}^f$ and $d(D, D_0)$ is finite. By (3.5, 3.6), $d(A_m, D) = d(D_{mm}, D) \leq d(D_{mm}, D_m) + d(D_m, D) \leq 4 \cdot 2^{-m}$. This proves (3.7). Altogether we have proved (3).

From (2) or (3) it follows that (\mathcal{A}^f, d) is a complete pseudometric space.

We prove (4), i.e. density of \mathcal{R}^f . For $C \subseteq \Omega$ let $\mathcal{U}(C)$ be the set of all sequences $(R_i)_{i \in \mathbb{N}}$ of ring elements such that $C \subseteq \bigcup_{i \in \mathbb{N}} R_i$. In the Carathéodory proof of the extension theorem [2] the measure μ is defined on \mathcal{A} by its values on the ring as follows:

$$\mu(C) := \inf \left\{ \sum_{i \in \mathbb{N}} \mu(R_i) \mid (R_i)_{i \in \mathbb{N}} \in \mathcal{U}(C) \right\}.$$

Let $C \in \mathcal{A}^f$ and let $\varepsilon > 0$. There is some sequence $(R_i)_{i \in \mathbb{N}} \in \mathcal{U}(C)$ such that $C \subseteq \bigcup_{i \in \mathbb{N}} R_i$ and $0 \leq \sum_{i \in \mathbb{N}} \mu(R_i) - \mu(C) < \varepsilon/2$. Then $(\forall i) R_i \in \mathcal{R}^f$. Let $S_0 := R_0$ and $S_i := R_i \setminus (R_0 \cup$

$\dots \cup R_{i-1}$) for all $i > 0$. Then the S_i are pairwise disjoint sets of finite measure and $C \subseteq \bigcup_{i \in \mathbb{N}} R_i = \bigcup_{i \in \mathbb{N}} S_i$. Since $S_i \subseteq R_i$ for all i ,

$$\mu(C \Delta \bigcup_{i \in \mathbb{N}} S_i) = \mu(C \Delta \bigcup_{i \in \mathbb{N}} R_i) = \mu(\bigcup_{i \in \mathbb{N}} R_i) - \mu(C) \leq \sum_{i \in \mathbb{N}} \mu(R_i) - \mu(C) \leq \varepsilon/2.$$

Furthermore there is some m such that $0 \leq \sum_{i \in \mathbb{N}} \mu(S_i) - \sum_{i \leq m} \mu(S_i) < \varepsilon/2$. Since the S_i are disjoint, $0 \leq \mu(\bigcup_{i \in \mathbb{N}} S_i) - \mu(\bigcup_{i \leq m} S_i) < \varepsilon/2$, hence

$$\mu(\bigcup_{i \in \mathbb{N}} S_i \Delta \bigcup_{i \leq m} S_i) = \mu(\bigcup_{i \in \mathbb{N}} S_i) - \mu(\bigcup_{i \leq m} S_i) < \varepsilon/2.$$

By (9.3), $\mu(C \Delta \bigcup_{i \leq m} S_i) \leq \mu(C \Delta \bigcup_{i \in \mathbb{N}} S_i) + \mu(\bigcup_{i \in \mathbb{N}} S_i \Delta \bigcup_{i \leq m} S_i) \leq \varepsilon$. Since $\bigcup_{i \leq m} S_i \in \mathcal{R}^f$, \mathcal{R}^f is dense in \mathcal{A}^f . \square