

UNIFORM RELIABILITY OF SELF-JOIN-FREE CONJUNCTIVE QUERIES

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ABSTRACT. The *reliability* of a Boolean Conjunctive Query (CQ) over a tuple-independent probabilistic database is the probability that the CQ is satisfied when the tuples of the database are sampled one by one, independently, with their associated probability. For queries without self-joins (repeated relation symbols), the data complexity of this problem is fully characterized by a known dichotomy: reliability can be computed in polynomial time for *hierarchical* queries, and is #P-hard for non-hierarchical queries.

Inspired by this dichotomy, we investigate a fundamental counting problem for CQs without self-joins: how many sets of facts from the input database satisfy the query? This is equivalent to the *uniform* case of the query reliability problem, where the probability of every tuple is required to be $\frac{1}{2}$. Of course, for hierarchical queries, uniform reliability is solvable in polynomial time, like the reliability problem. We show that being hierarchical is also necessary for this tractability (under conventional complexity assumptions). In fact, we establish a generalization of the dichotomy that covers every restricted case of reliability in which the probabilities of tuples are determined by their relation.

1. INTRODUCTION

Probabilistic databases [SORK11] extend the usual model of relational databases by allowing database facts to be uncertain, in order to model noisy and imprecise data. The evaluation of a Boolean query Q over a probabilistic database D is then the task of computing the probability that Q is true under the probability distribution over possible worlds given by D . This computational task has been considered by Grädel, Gurevich and Hirsch [GGH98] as a special case of computing the *reliability* of a query in a model which is nowadays known as *Tuple-Independent probabilistic Databases* (TIDs) [SORK11, DS07]. In a TID, every fact is associated with a probability of being true, and the truth of every fact is an independent random event. While the TID model is rather weak, query evaluation over TIDs can also be used for probabilistic inference over models with correlations among facts, such as Markov Logic Networks [GS16, JS12]. Hence, studying the complexity of query evaluation on TIDs

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is the first step towards understanding which forms of probabilistic data can be tractably queried.

To this end, Grädel et al. [GGH98] showed the first Boolean Conjunctive Query (referred to simply as a *CQ* hereafter) for which query evaluation is $\#P$ -hard on TIDs. Later, Dalvi and Suciu [DS07] established a dichotomy on the complexity of evaluating CQs without self-joins (i.e., without repeated relation symbols) over TIDs: if the CQ is *safe* (or *hierarchical* [DRS09, SORK11] as we explain next), the problem is solvable in polynomial time; otherwise, the problem is $\#P$ -hard. (This result was later extended to the class of all CQs and unions of CQs [DS12].)

The class of *hierarchical* CQs is defined by requiring that, for every two variables x and y , the sets of query atoms that feature x must contain, be contained in, or be disjoint from, the set of atoms that feature y . Beyond query evaluation on TIDs, this class of hierarchical queries was found to characterize the tractability boundary of other query evaluation tasks for CQs without self-joins, over databases without probabilities (and under conventional complexity assumptions). Olteanu and Huang [OH08] showed that a query is hierarchical if and only if, for every database, the *lineage* of the query is a read-once formula. Livshits, Bertossi, Kimelfeld and Sebag [LBKS20] proved that the hierarchical CQs are precisely the ones that have a tractable *Shapley value* as a measure of responsibility of facts to query answers (a result that was later generalized to CQs with negation [LBKS20]); they also conjecture that this complexity classification also holds for another measure of responsibility, namely the *causal effect* [SBSdB16]. (We discuss these measures again later in this introduction.) Berkholz, Keppeler and Schweikardt [BKS17] showed that the hierarchical CQs¹ are (up to conventional assumptions of fine-grained complexity) precisely the ones (Boolean) CQs for which we can use an auxiliary data structure to update the query answer in constant time in response to the insertion or deletion of a tuple.

In this paper, we show that the property of being hierarchical also captures the complexity of a fundamental counting problem for CQs without self-joins: *how many sets of facts from the input database satisfy the query?* This problem, which we refer to as *uniform reliability*, is equivalent to query evaluation over a TID where the probability of *every* fact is equal to $\frac{1}{2}$. In particular, it follows from the aforementioned dichotomy that this problem can be solved in polynomial time for every self-join-free hierarchical CQ Q . Yet, if Q is not hierarchical, it does not necessarily mean that Q is intractable already in this uniform setting. Indeed, it was not known whether enforcing uniformity makes query evaluation on TIDs easier, and the complexity of uniform reliability was already open for the simplest case of a non-hierarchical CQ: $Q_1 :- R(x), S(x, y), T(y)$. The proofs of $\#P$ -hardness of Dalvi and Suciu [DS07] require TIDs with deterministic facts (probability 1), in addition to $\frac{1}{2}$, already in the case of Q_1 . Here, we address this problem and show that the dichotomy is also true for the uniform reliability problem. In particular, uniform reliability is $\#P$ -complete for every non-hierarchical CQ without self-joins (and solvable in polynomial time for every hierarchical CQ without self-joins). In fact, we establish a more general result for the problem of *weighted uniform reliability*, that we discuss later on.

The uniform reliability problem that we study is a basic combinatorial problem on CQs, and a natural restricted case of query answering on TIDs, but it also has a direct application

¹For clarification, the tractability condition of Berkholz et al. [BKS17] is called “q-hierarchical” and it is a strict restriction of the condition of being hierarchical for non-Boolean conjunctive queries. As they explain, the two properties coincide in the Boolean case (i.e., the case discussed here), that is, a Boolean CQ is q-hierarchical if and only if it is hierarchical.

for quantifying the impact (or responsibility) of a fact f on the result of a CQ Q over ordinary (non-probabilistic) databases. One notion of tuple impact is the aforementioned causal effect, defined as the difference between two quantities: the probability of Q conditioning on the existence of f , minus the probability of Q conditioning on the absence of f [SBSdB16]. This causal effect was recently shown [LBKS20] to be the same as the *Banzhaf power index*, studied in the context of wealth distribution in cooperative game theory [DS79] and applied, for instance, to voting in the New York State Courts [GS79]. One notion of causal effect (with so-called *endogenous* facts) is defined by viewing the ordinary database as a TID where the probability of every fact is $\frac{1}{2}$. Therefore, computing the causal effect amounts to solving two variations of uniform reliability, corresponding to the two quantities. In fact, it is easy to see that all of our results apply to each of these two variations.

Uniform reliability also relates to the aforementioned computation of a tuple’s Shapley value, a measure of wealth distribution in cooperative game theory that has been applied to many use cases [Sha53, Rot88]. Livshits et al. [LBKS20] showed that computing a tuple’s Shapley value can be reduced to a generalized variant of uniform reliability. Specifically, for CQs, computing the Shapley value (again for *endogenous* facts) amounts to calculating the number of subinstances that satisfy Q and have precisely m tuples (for a given number m). This generalization of uniform reliability is tractable for every hierarchical CQ without self-joins [LBKS20]. Clearly, our results here imply that this generalization is intractable for every non-hierarchical CQ without self-joins, allowing us to conclude that the dichotomy in complexity also applies to this generalization.

Some natural generalizations of uniform reliability lie between model counting and probabilistic query answering. These include the case where the probability of each tuple of the database is the same, but not necessarily $\frac{1}{2}$. This problem can arise, for example, in scenarios of network reliability, where all connections are equally important and have the same independent probability of failure. A more general case is the one where the probabilities for every relation are the same, but different relations may be associated with different probabilities. This corresponds to data integration scenarios where every relation is a resource with a different level of trust (e.g., enterprise data vs. Web data vs. noisy sensor data). The latter variation is the one we refer to as *weighted uniform reliability*.

Our result in its full generality (namely Theorem 3.5, and its generalization to deterministic queries, Theorem 8.5) completely determines the complexity of weighted uniform reliability: for every non-hierarchical CQ without self-joins, and for *every* fixed assignment of probabilities to relations, probabilistic query answering is #P-hard when the fixed probabilities are less than 1 (while for every hierarchical CQ without self-joins the problem is solvable in polynomial time, as already known due to Dalvi and Suciu [DS07]). When the fixed probabilities can be 1, we have a more complex classification inspired by [DS07, Theorem 8], shown as Theorem 8.5.

Related work. As explained earlier, our work is closely related to existing literature on query evaluation over probabilistic databases. The dichotomy of Dalvi and Suciu [DS07] for CQs without self-joins requires tuples with probabilities $\frac{1}{2}$ and 1. This is also the case for their generalized dichotomy on CQs without self-joins, which covers deterministic relations (i.e., all tuples have probability 1) but allows tuples in the remaining relations to have arbitrary probabilities (including 1). The later generalization of the dichotomy by Dalvi and Suciu [DS12] to CQs with self-joins and to UCQs required an unbounded class of probabilities, not just $\frac{1}{2}$ and 1. In very recent work, Kenig and Suciu [KS20] have

strengthened the generalized dichotomy and showed that probabilities $\frac{1}{2}$ and 1 suffice for UCQs as well.² In that work, they also investigate uniform reliability (that we study here, i.e., where $\frac{1}{2}$ is the only nonzero probability allowed) and prove #P-hardness for the so-called unsafe “final type-I” queries. As they explain in their discussion on the work of this paper (which was posted as a preprint before theirs), their result on uniform reliability complements ours, and it is not clear if any of these two results can be used to prove the other.

Our work in this paper also relates to rewriting techniques used in the case of DNF formulas to reduce weighted model counting to unweighted model counting [CFMV15]. Nevertheless, the results and techniques for this problem are not directly applicable to ours, since model counting for CQs translates to DNFs of a very specific shape (namely, those that can be obtained as the lineage of the query).

Another superficially related problem is that of *symmetric model counting* [BVdBGS15]. This is a variant of uniform reliability where each relation consists of *all possible tuples* over the corresponding domain, and so each fact carries the same weight: these assumptions are often helpful to make model counting tractable. The assumption that we make on databases is much weaker: we do not deal with symmetric databases, but rather with arbitrary databases where all facts of the database (but *not* necessarily all possible facts over the domain) have the same probability, or have a common probability defined by the relation, when they are present. For this reason, the tractability results of Beame et al. [BVdBGS15] do not carry over to our setting. In terms of hardness results, [BVdBGS15, Theorem 3.1] shows the #P₁-hardness of symmetric model counting (hence of uniform reliability) for a specific FO³ sentence, and [BVdBGS15, Corollary 3.2] shows a #P₁-hardness result for *weighted* symmetric model counting for a specific CQ (without assuming self-join-freeness). Hence, these results do not determine the complexity of uniform reliability for self-join-free CQs as we do here.

There is a closer connection to existing dichotomy results on counting *database repairs* [MW13, MW14]. In this setting, the input database may violate the primary key constraints of the relations, and a repair is obtained by selecting one fact from every collection of conflicting facts (i.e., distinct facts that agree on the key): the *repair counting problem* asks how many such repairs satisfy a given CQ. In particular, it can easily be shown that for a CQ Q , there is a reduction from the uniform reliability of Q to repair counting of another CQ Q' . Yet, this reduction can only explain cases of tractability (namely, where Q is hierarchical) which, as explained earlier, are already known. We do not see how to design a reduction in the other direction, from repair counting to uniform reliability, in order to show our hardness result.

Finally, our work relates to the study of the Constraint Satisfaction Problem (CSP). However, there are two key differences. First, we study query evaluation in terms of homomorphisms *from* a fixed CQ, whereas the standard CSP phrasing talks about homomorphisms *to* a given template. Second, the standard counting variant of CSP (namely, #CSP), for which Bulatov has proved a dichotomy [Bul13], is about counting *the number of homomorphisms*, whereas we count *the number of subinstances* for which a homomorphism exists. For these reasons, it is not clear how results on CSP and #CSP can be helpful towards our main result.

²Kenig and Suciu refer to this case as TID with probabilities from $\{0, \frac{1}{2}, 1\}$; we mean the same thing, as in this paper we assume that tuples with probability zero are simply ignored.

Prior publication. A short version of this manuscript appeared in conference proceedings [AK21]. In terms of the results, the main difference between the versions is that this manuscript establishes a stronger result, that is, a dichotomy for *weighted* uniform reliability (with arbitrary probabilities per relation) rather than unweighted uniform reliability (with the single probability $\frac{1}{2}$). In fact, the question of weighted uniform reliability has been stated as an open problem in the conference publication, and we have posed a conjecture regarding the query Q_1 [AK21, Conjecture 7.4]. We prove this conjecture in the present work. We do so by showing our result on weighted uniform reliability for arbitrary queries when we do not allow deterministic relations (Sections 4–7), and addressing the case of deterministic relations in Section 8. Interestingly, our stronger result allows for a simpler proof structure, since now it is possible to define a reduction from Q_1 to every non-hierarchical CQ (while the probabilities may be different). Besides the stronger results, compared to the conference version, this manuscript includes complete proofs.

Organization. We give preliminaries in Section 2. In Section 3, we formally state the studied problems and main results, that is, the dichotomy on the complexity of uniform reliability, and more generally weighted uniform reliability, for CQs without self-joins. We prove this result in Sections 4–7. We discuss a generalization that allows for deterministic relations in Section 8, and conclude in Section 9.

2. PRELIMINARIES

We begin with some preliminary definitions and notation that we use throughout the paper. We first define databases and conjunctive queries, before introducing the task of probabilistic query evaluation, and the uniform reliability problem that we study.

Databases. A (relational) *schema* \mathbf{S} is a collection of *relation symbols* with each relation symbol U in \mathbf{S} having an associated arity. We assume a countably infinite set Const of *constants* that are used as database values. A *fact* over \mathbf{S} is an expression of the form $U(c_1, \dots, c_k)$ where U is a relation symbol of \mathbf{S} , where k is the arity of U , and where c_1, \dots, c_k are values of Const . An *instance* I over \mathbf{S} is a finite set of facts. In particular, we say that an instance J is a *subinstance* of an instance I if we have $J \subseteq I$.

Conjunctive queries. This paper focuses on queries in the form of a Boolean Conjunctive Query, which we refer to simply as a *CQ*. Intuitively, a CQ Q over the schema \mathbf{S} is a relational query definable as an existentially quantified conjunction of atoms. Formally, a CQ is a first-order formula of the form $Q :- U_1(\vec{a}_1), \dots, U_n(\vec{a}_n)$ where each $U_i(\vec{a}_i)$ is an *atom* of Q , formed of a relation symbol of \mathbf{S} and of a tuple \vec{a}_i of constants and (existentially quantified) variables, with the same arity as U_i . In the context of a CQ Q , we omit the schema \mathbf{S} and implicitly assume that \mathbf{S} consists of the relation symbols that occur in Q (with the arities that they have in Q); in that case, we may also refer to an instance I over \mathbf{S} as an instance *over* Q . We write $I \models Q$ to state that the instance I satisfies Q . We denote the set of all subinstances J of I that satisfy Q by:

$$\text{Mod}(Q, I) := \{J \subseteq I \mid J \models Q\}.$$

A *self-join* in a CQ Q is a pair of distinct atoms over the same relation symbol. For example, in $Q :- R(x, y), S(x), R(y, z)$, the first and third atoms constitute a self-join. Our analysis in this paper is restricted to CQs *without self-joins*, that we also call *self-join-free*.

Let Q be a CQ. For each variable x of Q , we denote by $\text{atoms}(x)$ the set of atoms $U_i(\vec{\tau}_i)$ of Q where x occurs. We say that Q is *hierarchical* [DS07] if for all variables x and x' one of the following three relations hold: $\text{atoms}(x) \subseteq \text{atoms}(x')$, $\text{atoms}(x') \subseteq \text{atoms}(x)$, or $\text{atoms}(x) \cap \text{atoms}(x') = \emptyset$. The simplest non-hierarchical self-join-free CQ is Q_1 , which we already mentioned in the introduction:

$$Q_1 :- R(x), S(x, y), T(y) \tag{2.1}$$

Probabilistic query evaluation. The problem of *probabilistic query evaluation* over tuple-independent databases [SORK11] is defined as follows.

Definition 2.1. The problem of *probabilistic query evaluation* (or PQE) for a CQ Q , denoted $\text{PQE}(Q)$, is that of computing, given an instance I over Q and an assignment $\pi : I \rightarrow [0, 1]$ of a probability $\pi(f)$ to every fact f , the probability that Q is true, namely:

$$\Pr(Q, I, \pi) := \sum_{J \in \text{Mod}(Q, I)} \prod_{f \in J} \pi(f) \times \prod_{f \in I \setminus J} (1 - \pi(f)).$$

We again study the *data complexity* of this problem, and we assume that the probabilities attached to the instance I are rational numbers represented by their integer numerator and denominator.

PQE was first studied by Grädel, Gurevich and Hirsch [GGH98] as *query reliability* (which they also generalize beyond Boolean queries). They identified a Boolean CQ Q with self-joins such that the reliability of Q is $\#P$ -hard to compute. Dalvi and Suciu [DS07, DS12] then studied the PQE problem, culminating in their dichotomy for the complexity of PQE on unions of conjunctive queries with self-joins [DS12]. In this paper, we only consider their earlier study of CQs without self-joins [DS07]. They characterize, under conventional complexity assumptions, the self-join-free CQs where PQE is solvable in PTIME. They state the result in terms of safe query plans (“safe CQs”), but the term “hierarchical” was adopted in later publications [DRS09, SORK11]:

Theorem 2.2 [DS07]. *Let Q be a CQ without self-joins. If Q is hierarchical, then $\text{PQE}(Q)$ is solvable in polynomial time. Otherwise, $\text{PQE}(Q)$ is $\#P$ -hard.*

Recall that $\#P$ is the complexity class of problems that count witnesses of an NP-relation (e.g., satisfying assignments of a logical formula, vertex covers of a graph, etc.). A function F is $\#P$ -hard if every function in $\#P$ has a polynomial-time *Turing reduction* (or *Cook reduction*) to F .

We stress that Theorem 2.2 applies to CQs *without* self-joins. In the presence of self-joins, being hierarchical is still necessary for tractability, but no longer sufficient [SORK11, Theorem 4.23, Proposition 4.25].

3. PROBLEM STATEMENT AND MAIN RESULT

We study the query reliability problem (which we equivalently refer to as PQE). Our main focus is on the uniform variant of this problem, where the probability of every fact is $\frac{1}{2}$. Equivalently, the task is to count the subinstances that satisfy the query (up to division/multiplication by 2^n where n is the number of facts in the instance). Formally:

Definition 3.1. The problem of *uniform reliability* for a CQ Q , denoted $\text{UR}(Q)$, is that of determining, given an instance I over Q , how many subinstances of I satisfy Q . In other words, $\text{UR}(Q)$ is the problem of computing $|\text{Mod}(Q, I)|$ given I . We study the *data complexity* of this problem, i.e., Q is fixed and the complexity is a function of the input I .

Let Q be a CQ without self-joins. It follows from Theorem 2.2 that, if Q is hierarchical, then $\text{UR}(Q)$ is solvable in polynomial time. Indeed, there is a straightforward reduction from $\text{UR}(Q)$ to $\text{PQE}(Q)$: given an instance I for Q , let $\pi : I \rightarrow [0, 1]$ be the function that assigns to every fact f of I the probability $\pi(f) = \frac{1}{2}$. Then we have:

$$|\text{Mod}(Q, I)| = 2^{|I|} \times \Pr(Q, I, \pi)$$

because every subset of I has the same probability, namely $2^{-|I|}$.

However, the other direction is not evident. If Q is non-hierarchical, we know that $\text{PQE}(Q)$ is $\#\text{P}$ -hard, but we do not know whether the same is true of $\text{UR}(Q)$. Indeed, this does not follow from Theorem 2.2 (as uniform reliability is a restriction of PQE), and it does not follow from the proof of the theorem either. Specifically, the reduction that Dalvi and Suciu [DS07] used to show hardness consists of two steps.

- (1) Proving that $\text{PQE}(Q_1)$ is $\#\text{P}$ -hard (where Q_1 is defined in (2.1)).
- (2) Constructing a polynomial-time many-one reduction from $\text{PQE}(Q_1)$ to $\text{PQE}(Q)$ for every non-hierarchical CQ Q without self-joins.

In both steps, the constructed instances I consist of facts with two probabilities: $\frac{1}{2}$ and 1 (i.e., *deterministic facts*). If all facts had probability $\frac{1}{2}$, then we would get a reduction to our $\text{UR}(Q)$ problem. However, the proof crucially relies on deterministic facts, and we do not see how to modify it to give the probability $\frac{1}{2}$ to all facts. This is true for both steps. Even for the first step, the complexity of $\text{UR}(Q_1)$ has been unknown so far. For the second step, it is not at all clear how to reduce from $\text{UR}(Q_1)$ to $\text{UR}(Q)$, even if $\text{UR}(Q_1)$ is proved to be $\#\text{P}$ -hard.

In this paper, we resolve the question and prove that $\text{UR}(Q)$ is $\#\text{P}$ -complete whenever Q is a non-hierarchical CQ without self-joins. Hence, we establish that the dichotomy of Theorem 2.2 also holds for uniform reliability. Our main result is:

Theorem 3.2. *Let Q be a CQ without self-joins. If Q is hierarchical, then $\text{UR}(Q)$ is solvable in polynomial time. Otherwise, $\text{UR}(Q)$ is $\#\text{P}$ -complete.*

As said above, the hardness side of Theorem 3.2 is essentially the statement that when Q is non-hierarchical, $\text{PQE}(Q)$ is $\#\text{P}$ -hard even when every fact has the probability $\frac{1}{2}$, and this is what we need to prove. In fact, we prove something stronger: Fix any CQ Q without self-joins, assign to each relation symbol U of Q an arbitrary probability $\varphi(U)$ (which may be different from $\frac{1}{2}$), and consider $\text{PQE}(Q)$ on instances I where fact probabilities are given by φ . We refer to this problem as the *weighted uniform reliability* problem, and write it $\text{WUR}(Q, \varphi)$. Formally:

Definition 3.3. Let Q be a CQ without self-joins, and let φ be a function mapping each relation symbol U of Q to a rational number $0 < \varphi(U) \leq 1$. The *weighted uniform reliability* problem $\text{WUR}(Q, \varphi)$ is the problem of probabilistic query evaluation $\text{PQE}(Q)$ for Q on input instances whose probability function $\pi : I \rightarrow [0, 1]$ is defined by φ , that is, for every fact $f \in I$, we have $\pi(f) = \varphi(U)$ where U is the relation symbol of f .

The problem $\text{WUR}(Q, \varphi)$ is tractable if Q is hierarchical (because it is a special case of PQE). If Q is non-hierarchical, then $\text{PQE}(Q)$ is $\#P$ -hard. In this paper, we show that, when all relation probabilities are strictly less than 1, the problem $\text{WUR}(Q, \varphi)$ is also intractable.

Comment 3.4. Note that it is not clear, to begin with, that $\text{PQE}(Q)$ remains hard if we restrict the use of deterministic facts (and, in fact, increase the level of uncertainty). To illustrate that, we recall an example from the “Probabilistic Databases” book [SORK11]: consider $\text{PQE}(Q_1)$ for Q_1 the hard query of Equation 2.1, under the restriction that S is the cartesian product of the relations R and T , and that every tuple has probability $\frac{1}{2}$. In other words, this is the problem of computing the probability that a *complete* bipartite graph contains an edge and its two incident vertices, when every vertex and edge can independently disappear with probability $\frac{1}{2}$. This problem is solvable in polynomial time by a fairly simple counting argument [SORK11, page 47]. Nevertheless, when one reduces the uncertainty by allowing some tuples to have probabilities 0 or 1, then the problem becomes hard, as $\text{PQE}(Q_1)$ is $\#P$ -hard when the tuples can have probabilities 0, $\frac{1}{2}$, or 1. Hence, in this situation, a problem can become tractable when the use of deterministic facts is restricted. \square

Theorem 3.5. *Let Q and φ be as in Definition 3.3. If Q is non-hierarchical, and if φ maps each relation symbol to a probability strictly less than 1, then $\text{WUR}(Q, \varphi)$ is $\#P$ -hard.*

Note that, even though uniform reliability is a special case of weighted uniform reliability, Theorem 3.5 is finer than Theorem 3.2, and does not follow from it. Indeed, Theorem 3.5 implies that each (fixed) choice of probability is intractable, in particular the choice giving probability $\frac{1}{2}$ to every relation.

We present the proof of this result in most of this paper (Sections 4–7), before discussing in Section 8 a generalization where probability 1 is allowed.

4. REDUCING FROM Q_1 TO ARBITRARY QUERIES

The structure of our proof of Theorem 3.5 is analogous to (but very different from) the reduction of Dalvi and Suciu [DS07]. We prove the $\#P$ -hardness of $\text{WUR}(Q, \varphi)$ in two steps.

- (1) Prove that $\text{WUR}(Q_1, \varphi)$ is $\#P$ -hard for every φ (where Q_1 is defined in (2.1));
- (2) Prove that for every Q and φ as in Theorem 3.5 there exists φ_1 such that there is a polynomial-time many-one reduction from $\text{WUR}(Q_1, \varphi_1)$ to $\text{WUR}(Q, \varphi)$.

In this section, we present the second step. The first step is much more challenging, and presented in Sections 5–7. Here is the statement of the first step:

Lemma 4.1. *Let Q and φ be as in Definition 3.3, where φ maps to probabilities strictly less than 1. If Q is non-hierarchical then there exists φ_1 , also mapping to probabilities strictly less than 1, such that there is a polynomial-time many-one reduction from $\text{WUR}(Q_1, \varphi_1)$ to $\text{WUR}(Q, \varphi)$.*

We prove Lemma 4.1 using the following, more elaborate lemma.

Lemma 4.2. *Let $Q :- U_1(\vec{a}_1), \dots, U_n(\vec{a}_m)$ be a self-join-free CQ, and let φ be as in Definition 3.3 and mapping each relation to a probability strictly less than 1. Suppose that Q is non-hierarchical, and let x and y be two variables of Q witnessing this, i.e., $\text{atoms}(x)$ and $\text{atoms}(y)$ have a nonempty intersection and none contains the other. Let φ_1 be the mapping defined as follows.*

- $\varphi_1(R)$ is the product of the $\varphi(U_i)$ over all i such that $U_i(\vec{a}_i)$ contains x and not y ;

- $\varphi_1(\mathbf{S})$ is the product of the $\varphi(U_i)$ over all i such that $U_i(\vec{a}_i)$ contains both x and y ;
- $\varphi_1(\mathbf{T})$ is the product of the $\varphi(U_i)$ over all i such that $U_i(\vec{a}_i)$ contains y and not x .

There is a polynomial-time many-one reduction from $\text{WUR}(Q_1, \varphi_1)$ to $\text{WUR}(Q, \varphi)$.

Proof. Let I_1 be an input instance for $\text{WUR}(Q_1, \varphi_1)$. If (a, b) is a pair of constants such that I_1 contains all of $\mathbf{R}(a)$, $\mathbf{S}(a, b)$ and $\mathbf{T}(b)$, then we call (a, b) a *match*. Without loss of generality, we assume that I_1 has no dangling tuples, that is, every fact is a part of one or more matches, as we can remove all other facts without changing the probability of Q_1 .

Fix any constant c . For an atom $U_i(\vec{a}_i)$ of Q , we denote by $f_i(a, b)$ the fact over U_i that is obtained from $U_i(\vec{a}_i)$ by replacing every occurrence of x with a , every occurrence of y with b , and every occurrence of every other variable with c . In particular, if $U_i(\vec{a}_i)$ contains neither x nor y , then $f_i(a, b)$ contains only c and the constants of \vec{a}_i .

We construct the input instance I for $\text{WUR}(Q, \varphi)$ by taking every match (a, b) from I_1 and inserting into I the facts $f_i(a, b)$ for all $i = 1, \dots, n$. Following the construction, let us define the following for a match (a, b) :

- $F_x(a)$ is the set of facts $f_i(a, b)$ inserted due to atoms $U_i(\vec{a}_i)$ that contain x and not y ;
- $F_{x,y}(a, b)$ is the set of facts $f_i(a, b)$ inserted due to atoms $U_i(\vec{a}_i)$ with both x and y ;
- $F_y(b)$ is the set of facts $f_i(a, b)$ inserted due to atoms $U_i(\vec{a}_i)$ that contain y and not x ;
- $F()$ is the set of facts $f_i(a, b)$ inserted due to atoms $U_i(\vec{a}_i)$ that contain neither x nor y .

Note that, by construction, these facts only contain the constants of Q and the constant c ; hence, for each such atom U_i , there is precisely one fact of this relation in I .

We complete the proof by showing the following:

$$\Pr(Q, I, \pi) = \Pr(Q_1, I_1, \pi_1) \times \prod_{f \in F()} \pi(f) \quad (4.1)$$

where π and π_1 are the probability functions defined respectively by φ and φ_1 as in Definition 3.3.

To show this, let us first observe that possible worlds of I that do not contain all facts $F()$ cannot satisfy Q : indeed, each of these facts is the only fact for its relation symbol. So this covers the multiplicative factor $\prod_{f \in F()} \pi(f)$, and it suffices to consider the possible worlds of I where all facts of $F()$ are present.

Let us study these possible worlds of I by partitioning them based on possible worlds of I_1 . More precisely, we choose a possible world J of I containing all facts of $F()$ by first choosing a possible world $J_1 \subseteq I_1$ and:

- for all a in the domain of I_1 :
 - if J_1 contains $\mathbf{R}(a)$, retaining all facts $F_x(a)$
 - if J_1 does not contain $\mathbf{R}(a)$, not retaining at least one of the facts of $F_x(a)$
- for each match (a, b) in I_1 :
 - if J_1 contains $\mathbf{S}(a, b)$, retaining all facts $F_{x,y}(a, b)$
 - if J_1 does not contain $\mathbf{S}(a, b)$, not retaining at least one of the facts of $F_{x,y}(a, b)$
- for all b in the domain of I_1 :
 - if J_1 contains $\mathbf{T}(b)$, retaining all facts $F_y(b)$
 - if J_1 does not contain $\mathbf{T}(b)$, not retaining at least one of the facts of $F_y(b)$
- retaining all facts of $F()$

Now, we note that, if a subset J of I satisfies Q , i.e., there is a homomorphism from Q to J , then the image of the variables x and y determine two elements a and b such that J contains all of $F_x(a)$, $F_{x,y}(a, b)$, $F_y(b)$ and $F()$, implying in particular that (a, b) is a match

of Q_1 in I_1 . Conversely, if J contains all of these facts for some match (a, b) of Q_1 in I_1 , then clearly J satisfies Q . Thus, for every subset J of I , we have $J \in \text{Mod}(Q, I)$ if and only there is a match (a, b) such that J contains all of $F_x(a)$, $F_{x,y}(a, b)$, $F_y(b)$ and $F()$. Put differently, when partitioning the possible worlds of J as explained in the previous paragraph, we have $J \in \text{Mod}(Q, I)$ if and only if our choice of $J_1 \subseteq I_1$ satisfies Q_1 . Hence, we conclude the following.

$$\begin{aligned} \Pr(Q, I, \pi) = & \sum_{J_1 \in \text{Mod}(Q_1, I_1)} \left(\prod_{R(a) \in J_1} \prod_{f \in F_x(a)} \pi(f) \right) \left(\prod_{R(a) \in I_1 \setminus J_1} \left(1 - \prod_{f \in F_x(a)} \pi(f) \right) \right) \\ & \times \left(\prod_{S(a,b) \in J_1} \prod_{f \in F_{x,y}(a,b)} \pi(f) \right) \left(\prod_{S(a,b) \in I_1 \setminus J_1} \left(1 - \prod_{f \in F_{x,y}(a,b)} \pi(f) \right) \right) \\ & \times \left(\prod_{T(b) \in J_1} \prod_{f \in F_y(b)} \pi(f) \right) \left(\prod_{T(b) \in I_1 \setminus J_1} \left(1 - \prod_{f \in F_y(b)} \pi(f) \right) \right) \\ & \times \left(\prod_{f \in F()} \pi(f) \right) \end{aligned}$$

Therefore, from the definition of φ_1 , φ , π_1 , and π we conclude that:

$$\begin{aligned} \Pr(Q, I, \pi) = & \sum_{J_1 \in \text{Mod}(Q_1, I_1)} \left(\prod_{R(a) \in J_1} \pi_1(R(a)) \right) \left(\prod_{R(a) \in I_1 \setminus J_1} (1 - \pi_1(R(a))) \right) \\ & \times \left(\prod_{S(a,b) \in J_1} \pi_1(S(a, b)) \right) \left(\prod_{S(a,b) \in I_1 \setminus J_1} (1 - \pi_1(S(a, b))) \right) \\ & \times \left(\prod_{T(b) \in J_1} \pi_1(T(b)) \right) \left(\prod_{T(b) \in I_1 \setminus J_1} (1 - \pi_1(T(b))) \right) \\ & \times \left(\prod_{f \in F()} \pi(f) \right) \end{aligned}$$

Since the first three lines corresponds to simply the probability of J_1 , we immediately conclude Equation (4.1), as promised. \square

The harder part is the first step, namely that $\text{WUR}(Q_1, \varphi)$ is $\#P$ -hard. We will show it in the next sections. Formally, what remains to complete the proof of Theorem 3.5 is to prove the following:

Theorem 4.3. *Consider the query Q_1 , and let φ be a function mapping the relation symbols R , S , and T to rational values such that $0 < \varphi(R), \varphi(T) < 1$ and $0 < \varphi(S) \leq 1$. Then $\text{WUR}(Q_1, \varphi)$ is $\#P$ -hard.*

We will prove Theorem 4.3 in Sections 5–7. Note that this result does not cover the case where relations have deterministic facts, except that we do not need to assume that $\varphi(S) < 1$: we will come back to this issue in Section 8. For now, we only mention that the

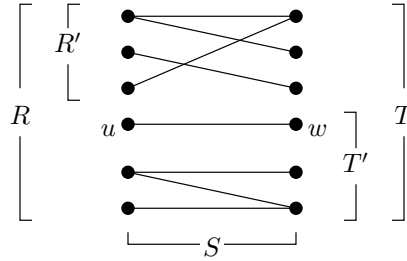


FIGURE 1. Example of the bipartite graph $G = (R \cup T, S)$ and an independent set (R', T')

requirement of being strictly smaller than 1 is necessary for the correctness of the theorem, since the problem is solvable in polynomial time if $\varphi(R) = 1$ or $\varphi(T) = 1$. Lemma 4.2 and Theorem 4.3 also show where in Theorem 3.5 we are using the assumption that φ maps each relation symbol to a probability strictly less than 1: we need the relevant products of probabilities to be smaller than 1 in order to be able to reduce from a hard configuration of the evaluation of Q_1 .

5. DEFINING THE MAIN REDUCTION

In this section and the two next ones, we prove Theorem 4.3, on the CQ $Q_1 : R(x), S(x, y), T(y)$. For simplicity, let us write $\rho := \varphi(R)$, $\sigma := \varphi(S)$, and $\tau := \varphi(T)$ the respective constant probabilities of R, S, and T: we have $0 < \rho, \tau < 1$ and $0 < \sigma \leq 1$. We construct a Turing reduction from the #P-hard problem of counting the independent sets of a bipartite graph. The input to this problem is a bipartite graph $G = (R \cup T, S)$ where $S \subseteq R \times T$, and the goal is to calculate the number P of *independent-set pairs* (R', T') with $R' \subseteq R$ and $T' \subseteq T$, that is, pairs such that $R' \times T'$ is disjoint from S .

Counting the independent sets of a bipartite graph is the same as computing the number of falsifying assignments of a so-called *monotone partitioned 2-DNF formula*, which is a monotone Boolean formula in disjunctive normal form over variables from two disjoint sets where every clause is the conjunction of one variable from one set and one variable from the other. Counting the satisfying assignments of such formulas is #P-hard [PB83], so it is also #P-hard to count falsifying assignments, and thus to count independent-set pairs. To be more precise, the two sides are viewed as the left and right sides, respectively, of the bipartite graph, and every clause $x \wedge y$ corresponds to the edge (x, y) . For example, the formula

$$(x_1 \wedge y_1) \vee (x_1 \wedge y_2) \vee (x_2 \wedge y_2)$$

corresponds to the bipartite graph $G = (R \cup T, S)$ that has the left side $R = \{x_1, x_2\}$, the right side $T = \{y_1, y_2\}$, and the edge set S that consists of the pairs (x_1, y_1) , (x_1, y_2) , and (x_2, y_2) . A pair (R', T') can be viewed as the assignment of truth values that sets to true precisely the variables from R' and T' . In particular, (R', T') corresponds to a falsifying assignment if and only if $R' \cup T'$ is an independent set of the graph (i.e., no edge of S has one endpoint in R' and one in T'). Hence, the number of independent sets is precisely the number of falsifying assignments.

Let us fix $G = (R \cup T, S)$ as the input to the problem. (See Figure 1 for an illustration.) Our proof consists of three steps, which we first sketch before presenting them in detail.

The first step, in the present section, is to describe the reduction, that is, how the input bipartite graph G is used to construct in polynomial time instances $D_{p,\varphi,\psi}$, for various values of p , φ and ψ , on which we invoke our oracle for $\text{WUR}(Q_1, \varphi)$ to obtain probabilities $\Pi_{p,\varphi,\psi}$. The instances $D_{p,\varphi,\psi}$ are constructed from G out of building blocks, called *gadgets*: we first introduce them, before presenting the construction used in the reduction.

The second step, in Section 6, is to show that the oracle answers $\Pi_{p,\varphi,\psi}$ are related to what the reduction needs to compute, that is, the number P of independent-set pairs of G . Specifically, we show that we can express P as a sum of *variables* of the form $X_{i,j,c,d,d'}$. Intuitively, these variables count the subsets of the left and right vertices of the bipartite graph satisfying some conditions given by the *parameters* i , j , c , d and d' . We then show that there is a linear equation system that relates these variables to the oracle answers $\Pi_{p,\varphi,\psi}$. Specifically, we show that there is a vector \vec{Y} defined from these variables which can be expressed by multiplying the vector of the $\Pi_{p,\varphi,\psi}$ by a square matrix A' . We then notice that A' is a Kronecker product of three Vandermonde matrices, two of which are easily seen to be invertible.

The third and last step of the proof is to show that the matrix A' of the equation system is invertible, by showing the invertibility of the last Vandermonde matrix A . This is done in Section 7, and is the most technical part of the proof, where we rely on the specific construction of the instances $D_{p,\varphi,\psi}$ and on the gadgets.

Defining the gadgets. We start our presentation of the first step by describing the gadgets used in the reduction as building blocks for our instances of $\text{WUR}(Q_1, \varphi)$. In our reduction, we will use multiple copies of the gadgets, instantiated with specific elements that will intuitively serve as endpoints to the gadgets. There are two types of gadgets:

- The (a, b) -gadget is an instance with two elements a and b (which are intuitively the endpoints), and the following facts (noting that they satisfy the query):

$$\text{R}(a), \text{S}(a, b), \text{T}(b)$$

We will need to count the possible worlds of this gadget and of subsequent gadgets, because these quantities will be important in the reduction to understand the link between the independent-set pairs of G and the subinstances of our instances of $\text{WUR}(Q_1, \varphi)$ that satisfy the query.

To this end, we denote by λ_{R} the total probability of the possible worlds of the (a, b) -gadget that violate Q_1 when we fix the fact $\text{R}(a)$ to be present. We easily compute: $\lambda_{\text{R}} = 1 - \sigma\tau$. Similarly, we denote by λ_{T} the probability of violating Q_1 when we fix the fact $\text{T}(b)$ to be present. We have: $\lambda_{\text{T}} = 1 - \rho\sigma$. In summary, we have the following notation that we use later in the proof.

$$\lambda_{\text{R}} = 1 - \sigma\tau \quad \lambda_{\text{T}} = 1 - \rho\sigma \quad (5.1)$$

- The (a, b, c, d) -gadget is an instance with elements a , b , c , and d , and the following facts:

$$\text{R}(a), \text{S}(a, b), \text{T}(b), \text{S}(c, b), \text{R}(c), \text{S}(c, d), \text{T}(d)$$

We illustrate the gadget below, where every vertex represents a domain element, every edge represents a pair of elements occurring in a fact, and unary and binary facts are simply written as relation names, respectively above their element and above their edge:

$$\begin{array}{ccccccc} \text{R} & & \text{T} & & \text{R} & & \text{T} \\ & \text{S} & & \text{S} & & \text{S} & \\ a & \longrightarrow & b & \longleftarrow & c & \longrightarrow & d \end{array}$$

We use the following notation for the (a, b, c, d) -gadget.

- γ is the total probability of the possible worlds of the gadget that violate Q_1 where we fix the facts $R(a)$ and $T(d)$ to be present.
- δ_R is the total probability of possible worlds that violate Q_1 when we fix $R(a)$ to be present and $T(d)$ to be absent.
- δ_T is, symmetrically, the total probability of possible worlds that violate Q_1 when we fix $R(a)$ to be absent and $T(d)$ to be present.
- δ_\perp is the total probability of possible worlds that violate Q_1 when we fix $R(a)$ and $T(d)$ to be absent.

We will study the quantities $\gamma, \delta_R, \delta_T, \delta_\perp$ in two lemmas in Section 7.

Defining the reduction. Having defined the various gadgets that we will use, let us describe the instances that we construct from our input bipartite graph $G = (R \cup T, S)$. The vertices of G are the elements of R and T , and its edges are the pairs in S . We write $m := |S|$, the number of edges of G .

Fix $M := (|R|+1) \times (|T|+1) \times (|S|+1)^3$, the number of instances to which we will reduce. Let us define positive integer values B and B' in a somewhat technical way. Specifically, we want B to be large enough so that:

$$B > 2m \frac{|\log \gamma| + |\log \delta_R| + |\log \delta_T| + |\log \delta_\perp|}{|\log \lambda_R|} \quad (5.2)$$

And we then want B' to be large enough so that:

$$B' > 2m \frac{|\log \gamma| + |\log \delta_R| + |\log \delta_T| + |\log \delta_\perp| + 2B |\log \lambda_R|}{|\log \lambda_T|} \quad (5.3)$$

These bounds on B and B' will be used later in the proof in a rounding argument, i.e., arguing that from a value of the form $B'x + By + z$ where x, y , and z are bounded, we can recover the separate values x and y and z . Let us explain why this is possible, and why this can be achieved with values that remain polynomial in the input. For this, notice that $0 < \gamma, \delta_R, \delta_T, \delta_\perp < 1$, by definition and because $0 < \rho, \tau < 1$. Thus, the absolute values of the logarithms of these quantities are positive numbers, which are constants, i.e., they were fixed with φ and do not depend on the instance. This is why we can pick positive integers B and B' to satisfy these conditions, and the *values* of B and B' are polynomial in m , so polynomial in the size of the input G .

Now, for each $0 \leq p < (|S|+1)^3$, $0 \leq \varphi \leq |R|$, $0 \leq \psi \leq |T|$, we construct the instance $D_{p,\varphi,\psi}$ on the schema of Q_1 (i.e., two unary relations R and T and one binary relation S), as follows.

- For each vertex $u \in R$ of G , create:
 - the fact $R(u)$;
 - φ copies of the $(u, *)$ -gadget (using fresh elements for b , as denoted by $*$).
- Similarly, for each vertex $w \in T$ of G create:
 - the fact $T(w)$;
 - ψ copies of the $(*, w)$ -gadget.
- For every edge $(u, w) \in S$ of G , create:
 - p copies of the $(u, *, *, w)$ -gadget connecting u and w (using fresh elements for b and c in each copy);
 - B copies of the $(u, *)$ -gadget

– B' copies of the $(*, w)$ -gadget

Notice that the reduction, i.e., the construction of the instances $D_{p,\varphi,\psi}$ for each $0 \leq p < (|S| + 1)^3$, $0 \leq \varphi \leq |R|$, $0 \leq \psi \leq |T|$ from the input bipartite graph G , is in polynomial time. Indeed, the number of instances that we build is polynomial in G , i.e., it is $(|S| + 1)^3 \times (|R| + 1) \times (|T| + 1)$. Further, for each instance, for each vertex and edge of G , we create copies of gadgets of constant size, and the number of copies is p , φ , ψ , B , or B' : these are polynomial in G , as we pointed out below Equation 5.3.

Further, observe that the construction of $D_{p,\varphi,\psi}$ is designed to ensure that any match of the query Q_1 on a possible world of $D_{p,\varphi,\psi}$ will always be contained in the facts of one of the gadgets (plus the facts $R(u)$ and $T(w)$). This means that we can determine whether the query is true in the possible worlds simply by looking separately at the facts of each gadget (and at the facts on the u and w).

Now, coming back to our reduction, for each choice of p , φ , and ψ , we denote by $\Pi_{p,\varphi,\psi}$ the total probability of subinstances of $D_{p,\varphi,\psi}$ that violate Q_1 . Each of these values can be computed in polynomial time using our oracle for $\text{WUR}(Q_1, \varphi)$: for $0 \leq p < (|S| + 1)^3$, $0 \leq \varphi \leq |R|$ and $0 \leq \psi \leq |T|$, we build $D_{p,\varphi,\psi}$, call the oracle to obtain the total probability of instances that satisfy Q_1 , and define $\Pi_{p,\varphi,\psi}$ as one minus that number.

Hence, in our reduction, given the input bipartite graph G , we have constructed the instances $D_{p,\varphi,\psi}$ and used our oracle to compute the total probability $\Pi_{p,\varphi,\psi}$ of subinstances of each $D_{p,\varphi,\psi}$ that violate Q_1 , for each p, φ, ψ , and this process is in PTIME. We will show in the sequel how these probabilities can be used to recover the answer to our original problem on $G = (R \cup T, S)$, that is, the number P of independent-set pairs of G .

6. OBTAINING THE EQUATION SYSTEM

We now move to the second step of our reduction and explain how the number P of independent-set pairs of G is related to the oracle answers $\Pi_{p,\varphi,\psi}$ by a linear equation system. To define the linear equation system, it will be helpful to introduce some parameters about subsets of vertices of the bipartite graph G . For any $R' \subseteq R$ and $T' \subseteq T$, we write the following:

- $c(R', T')$ denotes the number of edges of S that are *contained* in $R' \times T'$, that is, they have both endpoints in $R' \cup T'$. Formally,

$$c(R', T') := |(R' \times T') \cap S|.$$

- $d(R', T')$ denotes the number of edges of S that are *dangling from* R' , that is, they have one endpoint in R' and the other in $T \setminus T'$. Formally,

$$d(R', T') := |(R' \times (T \setminus T')) \cap S|.$$

- $d'(R', T')$ denotes the number of edges of S that are *dangling from* T' , that is, they have one endpoint in $R \setminus R'$ and the other in T' . Formally,

$$d'(R', T') := |((R \setminus R') \times T') \cap S|.$$

- $e(R', T')$ denotes the number of edges of S that are *excluded* from $R' \cup T'$, that is, they have no endpoint in $R' \cup T'$. Formally,

$$e(R', T') := |S \setminus (R' \times T')|.$$

Clearly, for every R' and T' , each edge of S is either contained in $R' \times T'$, dangling from R' , dangling from T' , or excluded from $R' \cup T'$. Hence,

$$c(R', T') + d(R', T') + d'(R', T') + e(R', T') = m.$$

Observe that a pair (R', T') is an independent-set pair of G iff $c(R', T') = 0$. Thus, given the input G to the reduction, our goal is to compute the following quantity:

$$P = |\{(R', T') \mid R' \subseteq R, T' \subseteq T, c(R', T') = 0\}| = \sum_{\substack{R' \subseteq R, T' \subseteq T, \\ c(R', T') = 0}} 1 \quad (6.1)$$

Let us now define the variables of the equation system for the input graph G . We will use these variables to express P , and we will be able to recover their values from the values $\Pi_{p,\varphi,\psi}$.

Picking variables. Our goal is to construct a linear equation system relating the quantity that we wish to compute, namely P , and the quantities provided by our oracle, namely $\Pi_{p,\varphi,\psi}$ for $0 \leq p < (|S| + 1)^3$, $0 \leq \varphi \leq |R|$, and $0 \leq \psi \leq |T|$. Instead of using P directly, we will construct a system connecting $\Pi_{p,\varphi,\psi}$ to quantities on G that we now define and that will allow us to recover P . We call these quantities *variables* because they are unknown and our goal in the reduction is to compute them from the $\Pi_{p,\varphi,\psi}$ to recover P .

Let us introduce, for each $0 \leq i \leq |R|$, for each $0 \leq j \leq |T|$, for each $c, d, d' \in \{0, \dots, |S|\}$, the variable $X_{i,j,c,d,d'}$, that stands for the number of pairs (R', T') with $|R'| = i$, with $|T'| = j$, and with c - and d - and d' -values exactly as defined earlier in this section. (We do not need e as a parameter here because it is determined from c , d , and d' .) Formally:

$$X_{i,j,c,d,d'} := |\{(R', T') \mid R' \subseteq R, T' \subseteq T, |R'| = i, |T'| = j, \\ c(R', T') = c, d(R', T') = d, d'(R', T') = d'\}|$$

Recall that our fixed query is Q_1 .

To simplify the equations that will follow, let us define, for all i, j, c, d, d' , other variables, which are the ones that we will actually use in the equation system:

$$Y_{i,j,c,d,d'} := \rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j} \times X_{i,j,c,d,d'}$$

The reason why we use these slightly more complicated variables is because they will simplify the equations later.

Getting our answer from the variables. Let us now explain why we can compute our desired value P (the number of independent-set pairs of G) from the variables $Y_{i,j,c,d,d'}$. Refer back to Equation (6.1), and let us split this sum according to the values of the parameters $i = |R'|$, $j = |T'|$, and $d(R', T')$, $d'(R', T')$. Using our variables $X_{i,j,c,d,d'}$, this gives:

$$P = \sum_{0 \leq i \leq |R|} \sum_{0 \leq j \leq |T|} \sum_{0 \leq d, d' \leq m} X_{i,j,0,d,d'}$$

We can insert the variables $Y_{i,j,c,d,d'}$ instead of $X_{i,j,c,d,d'}$ in the above, obtaining:

$$P = \sum_{0 \leq i \leq |R|} \sum_{0 \leq j \leq |T|} \sum_{0 \leq d, d' \leq m} \frac{Y_{i,j,0,d,d'}}{\rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j}} \quad (6.2)$$

This equation justifies that, to compute the quantity P that we are interested in, it suffices to compute the value of the variables $Y_{i,j,0,d,d'}$ for all $0 \leq i \leq |R|$, $0 \leq j \leq |T|$, and $0 \leq d, d' \leq m$.

If we can compute all $Y_{i,j,0,d,d'}$ in polynomial time, then we can use the equation above to compute P in polynomial time, completing the reduction.

Designing the equation system. We will now design a linear equation system that connects the quantities $\Pi_{p,\varphi,\psi}$ for p, φ, ψ computed by our oracle to the quantities $Y_{i,j,c,d,d'}$ for all $0 \leq i \leq |R|$, $0 \leq j \leq |T|$, $0 \leq c, d, d' \leq m$ that we wish to compute. To do so, write the vector $\vec{\Pi} = (\Pi_{0,0,0}, \dots, \Pi_{(|S|+1)^3-1, |R|, |T|})$ and the vector $\vec{Y} = (Y_{0,0,0,0,0}, \dots, Y_{|R|, |T|, m, m, m})$ in some order. We will describe an M -by- M matrix A so that we have the equation $\vec{\Pi} = A\vec{Y}$. We will later justify that the matrix A is invertible, so that we can compute \vec{Y} from $\vec{\Pi}$ and conclude the proof. So, it is left to define A , which we do in the remainder of this section, and to prove that A is invertible, which we do in the next section.

To define the matrix A , let us consider arbitrary subsets $R' \subseteq R$ and $T' \subseteq T$, and an arbitrary $0 \leq p < (|S| + 1)^3$. Let us denote by $\mathcal{D}_{p,\varphi,\psi}(R', T')$ the set of subinstances of $D_{p,\varphi,\psi}$ where R' is the set of vertices of R whose R-fact is kept, and where T' is the set of vertices of T whose T-fact is kept. It is clear that the $\mathcal{D}_{p,\varphi,\psi}(R', T')$ form a partition of the subinstances of $D_{p,\varphi,\psi}$, so that we have the following, where Pr denotes the total probability mass:

$$\Pi_{p,\varphi,\psi} = \sum_{R' \subseteq R, T' \subseteq T} \text{Pr}(\{I' \in \mathcal{D}_{p,\varphi,\psi}(R', T') \mid I' \not\models Q_1\}) \quad (6.3)$$

Let us now study the number in the above sum for each R' and T' , that is, the total probability of the number of instances in $\mathcal{D}_{p,\varphi,\psi}(R', T')$ that violate the query. We can show the following by performing some accounting over all gadgets in the construction. Recall the numbers $\lambda_R, \lambda_T, \delta_R, \delta_T$ and δ_\perp defined in Section 5.

Claim 6.1. For any $0 \leq p < (|S| + 1)^3$, for any choice of R' and T' , writing $i := |R'|$, $j := |T'|$, $c := c(R', T')$, $d := d(R', T')$, $d' := d'(R', T')$, $e := e(R', T') = |S| - c - d - d'$, we have that $\text{Pr}(\{I' \in \mathcal{D}_{p,\varphi,\psi}(R', T') \mid I' \not\models Q_1\})$ is equal to:

$$\rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j} \times (\lambda_R^i)^\varphi \times (\lambda_T^j)^\psi \times \alpha(c, d, d')^p$$

Where $\alpha(c, d, d')$ is defined as the following quantity:

$$\alpha(c, d, d') := \gamma^c \times \delta_R^d \times \delta_T^{d'} \times \delta_\perp^e \times \lambda_R^{B(c+d)} \times \lambda_T^{B'(c+d')} \quad (6.4)$$

Proof. To show this, recall that we can determine whether a possible world in $\mathcal{D}_{p,\varphi,\psi}(R', T')$ satisfies the query simply by looking at each gadget (and at the facts on the elements u and w in the construction of $D_{p,\varphi,\psi}$), as every match of the query can use facts from only a single gadget (and possibly the shared facts of the u and w).

- We have no choice on the R-facts of R' (we must keep them) and on the R-facts of $R \setminus R'$ (we must discard them), inducing a probability of $\rho^{|R'|} \times (1 - \rho)^{|R|-|R'|}$.
- We have no choice on the T-facts of T' (same reasoning), inducing a probability of $\tau^{|T'|} \times (1 - \tau)^{|T|-|T'|}$.
- For each $u \in R'$, we have a probability of λ_R^φ for the $(u, *)$ -gadgets of violating the query. This is true, since λ_R is the total probability that a $(u, *)$ -gadget violates query Q_1 when we fix the R-fact on u to be present, and we consider φ copies of the $(u, *)$ -gadget.
- For each $u \in R \setminus R'$, the $(u, *)$ -gadgets cannot be part of a query match.
- For each $w \in T'$, we have a probability of λ_T^ψ that the $(*, w)$ -gadgets all violate the query (same reasoning).
- For each edge $e = (u, w) \in S$ (and using the same reasoning as above):

- If e is contained in $R' \times T'$:
 - * For the $(u, *)$ -gadgets, we have a probability of $\lambda_R^{B \times p}$ of violating the query.
 - * For the $(*, w)$ -gadgets, we have a probability of $\lambda_T^{B' \times p}$.
 - * For the $(u, *, *, w)$ gadgets connecting u and w , we have a probability of γ^p .
- If e is dangling from R' :
 - * For the $(u, *)$ -gadgets, we have a probability of $\lambda_R^{B' \times p}$.
 - * For the $(*, w)$ -gadgets, there can be no query match, i.e., a probability of 1.
 - * For the $(u, *, *, w)$ gadget connecting u and w , we have a probability of δ_R^p .
- If e is dangling from T' :
 - * For the $(u, *)$ -gadgets, there can be no query match
 - * For the $(*, w)$ -gadgets, we have a probability of $\lambda_T^{B' \times p}$.
 - * For the $(u, *, *, w)$ gadgets connecting u and w , we have a probability of δ_T^p .
- If e is excluded from $R' \cup T'$:
 - * For the $(u, *)$ -gadgets, there can be no query match.
 - * For the $(*, w)$ -gadgets, there can be no query match.
 - * For the $(u, *, *, w)$ gadgets connecting u and w , we have a probability of δ_{\perp}^p .

Therefore, with $i = |R'|$, $j = |T'|$, $c = c(R', T')$, $d = d(R', T')$, $d' = d'(R', T')$, $e = e(R', T')$, we have:

$$\begin{aligned} \Pr(I' \in \mathcal{D}_{p,\varphi,\psi}(R', T') \mid I' \not\models Q_1) &= \rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j} \\ &\times \gamma^{cp} \times \delta_R^{dp} \times \delta_T^{d'p} \times \delta_{\perp}^{ep} \times \lambda_R^{(B(c+d))p} \times \lambda_T^{(B'(c+d'))p} \times \lambda_R^{i\varphi} \times \lambda_T^{j\psi} \end{aligned}$$

This leads directly to the claimed result. \square

Let us use the value of Claim 6.1 in Equation (6.3). Note that this value only depends on the cardinalities of R' and T' and the values of c, d, d', e , but not on the specific choice of R' and T' . Thus, splitting the sum accordingly, we can obtain the following:

Claim 6.2. For any $0 \leq p < (|S| + 1)^3$, we have that:

$$\Pi_{p,\varphi,\psi} = \sum_{\substack{0 \leq i \leq |R| \\ 0 \leq j \leq |T| \\ 0 \leq c, d, d' \leq m}} Y_{i,j,c,d,d'} \times (\lambda_R^i)^{\varphi} \times (\lambda_T^j)^{\psi} \times \alpha(c, d, d')^p.$$

Proof. Substituting the equality from Claim 6.1 into Equation (6.3), and splitting the sum, we get:

$$\Pi_{p,\varphi,\psi} = \sum_{\substack{0 \leq i \leq |R| \\ 0 \leq j \leq |T| \\ 0 \leq c, d, d' \leq m}} \sum_{\substack{R' \subseteq R \\ T' \subseteq T \\ |R'|=i, |T'|=j \\ c(R', T')=c \\ d(R', T')=d, \\ d'(R', T')=d'}} \rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j} \times (\lambda_R^i)^{\varphi} \times (\lambda_T^j)^{\psi} \times \alpha(c, d, d')^p$$

The inner sum does not depend on R' and T' , so let us introduce the variables $X_{i,j,c,d,d'}$:

$$\Pi_{p,\varphi,\psi} = \sum_{\substack{0 \leq i \leq |R| \\ 0 \leq j \leq |T| \\ 0 \leq c, d, d' \leq m}} X_{i,j,c,d,d'} \times \rho^i \times (1 - \rho)^{|R|-i} \times \tau^j \times (1 - \tau)^{|T|-j} \times (\lambda_R^i)^{\varphi} \times (\lambda_T^j)^{\psi} \times \alpha(c, d, d')^p$$

Note that we can now use the variables $Y_{i,j,c,d,d'}$ to eliminate the remaining factors, obtaining the claimed equality. Note that this simplification is the reason why we introduced the variables $Y_{i,j,c,d,d'}$, to make the equality more convenient to work with. \square

The equation of Claim 6.2 can be expressed as a matrix equation $\vec{\Pi} = A' \vec{Y}$, with A' the matrix defined by

$$A'_{(\varphi,\psi,p),(i,j,c,d,d')} := (\lambda_{\mathbf{R}}^i)^\varphi \times (\lambda_{\mathbf{R}}^j)^\psi \times \alpha(c, d, d')^p. \quad (6.5)$$

The matrix A' relates the vector $\vec{\Pi}$ computed from our oracle calls and the variables \vec{Y} that we wish to determine to solve our problem on the graph G . It remains to show that A' is an invertible matrix, so that we can compute its inverse $(A')^{-1}$ in polynomial time, use it to recover \vec{Y} from $\vec{\Pi}$, and from there recover P via Equation (6.2), concluding the reduction.

To study the invertibility of the matrix A' , we will notice that it can be expressed as the *Kronecker product* of three matrices that we will show to be invertible. Recall that the *Kronecker product* of a $\kappa \times \kappa$ matrix C and of a $\nu \times \nu$ matrix C' is the $(\kappa\nu) \times (\kappa\nu)$ -matrix with the following blockwise definition:

$$C \otimes C' = \begin{bmatrix} C_{1,1}C' & C_{1,2}C' & \dots & C_{1,\kappa}C' \\ C_{2,1}C' & C_{2,2}C' & \dots & C_{2,\kappa}C' \\ \vdots & \vdots & \ddots & \vdots \\ C_{\kappa,1}C' & C_{\kappa,2}C' & \dots & C_{\kappa,\kappa}C' \end{bmatrix}$$

We refer the reader to literature such as Henderson, Pukelsheim, and Searle [HPS83] for the details and history of the Kronecker product.

Now, following the definition of A' in Equation (6.5), and numbering the rows lexicographically by (φ, ψ, p) with $0 \leq \varphi \leq |R|$, $0 \leq \psi \leq |T|$, and $0 \leq p < (|S| + 1)^3$, and numbering the columns lexicographically by $(i, j, (c, d, d'))$ with $0 \leq i \leq |R|$, $0 \leq j \leq |T|$, and $c, d, d' \in \{0, \dots, |S|\}$, we see that the matrix A' is by definition the Kronecker product of three matrices:

- (1) The $(|R| + 1) \times (|R| + 1)$ matrix W whose cell (φ, i) contains $(\lambda_{\mathbf{R}}^i)^\varphi$;
- (2) The $(|T| + 1) \times (|T| + 1)$ matrix W' whose cell (ψ, j) contains $(\lambda_{\mathbf{T}}^j)^\psi$;
- (3) The $(|S| + 1)^3 \times (|S| + 1)^3$ matrix A whose cell $p, (c, d, d')$ contains $\alpha(c, d, d')^p$.

We know that the Kronecker product of invertible matrices is invertible, so to show that A' is invertible, it suffices to show that W , W' , and A are invertible. Now, the matrix W is clearly a (transpose of a) Vandermonde matrix,³ and it is invertible: we have $0 < \tau < 1$ from which the definition of $\lambda_{\mathbf{R}}$ implies $0 < \lambda_{\mathbf{R}} < 1$, so the function mapping an integer i to $\lambda_{\mathbf{R}}^i$ is injective. The same reasoning shows that W' is invertible. Hence, it suffices to study if A is invertible. This matrix is clearly also a Vandermonde matrix, so the only remaining point to show that A' is invertible is to show that the coefficients $\alpha(c, d, d')$ of A are different. We do this in the next section.

³Recall that an $m \times m$ matrix is a *Vandermonde matrix* if there are m numbers x_1, \dots, x_m such that each cell (k, ℓ) is $x_k^{\ell-1}$. It is known that such a matrix is invertible if and only if $x_k \neq x_{k'}$ for $k \neq k'$.

7. SHOWING THAT THE MATRIX IS INVERTIBLE

This section presents the third step of the proof of Theorem 4.3 and concludes. Specifically, we show the following:

Claim 7.1. For all $0 \leq c_1, c_2, d_1, d_2, d'_1, d'_2 \leq m$, if $(c_1, d_1, d'_1) \neq (c_2, d_2, d'_2)$, then we have $\alpha(c_1, d_1, d'_1) \neq \alpha(c_2, d_2, d'_2)$, where α is as defined in Claim 6.1.

Claim 7.1 implies that the Vandermonde matrix A is invertible, and concludes the definition of the reduction and the proof of Theorem 4.3.

Let us show the contrapositive of the statement: we take $0 \leq c_1, c_2, d_1, d_2, d'_1, d'_2 \leq m$ such that $\alpha(c_1, d_1, d'_1) = \alpha(c_2, d_2, d'_2)$, and we must show that $c_1 = c_2$, $d_1 = d_2$, and $d'_1 = d'_2$.

We know that the values of α are positive, so let us inject the definition of α from Claim 6.1 and take the logarithm of the equality. We obtain the following, where $e_1 := m - c_1 - d_1 - d'_1$, and likewise $e_2 := m - c_2 - d_2 - d'_2$:

$$\begin{aligned} & c_1 \log \gamma + d_1 \log \delta_R + d'_1 \log \delta_T + e_1 \log \delta_\perp + B(c_1 + d_1) \log \lambda_R + B'(c_1 + d'_1) \log \lambda_T \\ &= c_2 \log \gamma + d_2 \log \delta_R + d'_2 \log \delta_T + e_2 \log \delta_\perp + B(c_2 + d_2) \log \lambda_R + B'(c_2 + d'_2) \log \lambda_T \end{aligned}$$

As $0 < \rho, \tau < 1$, we clearly have $0 < \gamma, \delta_R, \delta_T, \delta_\perp, \lambda_R, \lambda_T < 1$, and their logarithms are negative quantities. Let us divide the equation by $B' \log \lambda_T$, to obtain:

$$\begin{aligned} & \frac{c_1 \log \gamma + d_1 \log \delta_R + d'_1 \log \delta_T + e_1 \log \delta_\perp + B(c_1 + d_1) \log \lambda_R}{B' \log \lambda_T} + c_1 + d'_1 \\ &= \frac{c_2 \log \gamma + d_2 \log \delta_R + d'_2 \log \delta_T + e_2 \log \delta_\perp + B(c_2 + d_2) \log \lambda_R}{B' \log \lambda_T} + c_2 + d'_2 \end{aligned}$$

Recall from Equation (5.3) that our definition of B' makes it large enough to ensure that $(m(\log \gamma + \log \delta_R + \log \delta_T + \log \delta_\perp) + 2mB \log \lambda_R) / B' \log \lambda_T$ has absolute value < 0.5 , using the triangle inequality. Thus, we can bound the absolute value of the first term of the left-hand side of the equation by 0.5. The same applies to the first term of the right-hand side of the equation. By contrast, $c_1 + d'_1$ and $c_2 + d'_2$ are integers. Thus, by a rounding argument, we conclude:

$$c_1 + d'_1 = c_2 + d'_2 \quad (7.1)$$

Simplifying away the common term, multiplying back by $B' \log \lambda_T$, and dividing by $B \log \lambda_R$, we obtain:

$$\begin{aligned} & \frac{c_1 \log \gamma + d_1 \log \delta_R + d'_1 \log \delta_T + e_1 \log \delta_\perp}{B \log \lambda_R} + c_1 + d_1 \\ &= \frac{c_2 \log \gamma + d_2 \log \delta_R + d'_2 \log \delta_T + e_2 \log \delta_\perp}{B \log \lambda_R} + c_2 + d_2 \end{aligned}$$

Recall from Equation (5.2) that our definition of B makes it large enough to ensure that $m(\log \gamma + \log \delta_R + \log \delta_T + \log \delta_\perp) / B \log \lambda_R$ has absolute value $< .5$. Thus we conclude again by a rounding argument that:

$$c_1 + d_1 = c_2 + d_2 \quad (7.2)$$

Finally, simplifying the common term, multiplying back by $B \log \lambda_R$, and exponentiating, we obtain a third equation:

$$\gamma^{c_1} \times \delta_R^{d_1} \times \delta_T^{d'_1} \times \delta_\perp^{e_1} = \gamma^{c_2} \times \delta_R^{d_2} \times \delta_T^{d'_2} \times \delta_\perp^{e_2} \quad (7.3)$$

Let $x = d_1 - d_2$ be the remaining degree of freedom. From Equation (7.2), we know that $x = c_2 - c_1$. From this and Equation (7.1), we know that $x = d'_1 - d'_2$. Combining these three definitions of x , we get that $x - x + x = (d_1 - d_2) - (c_2 - c_1) + (d'_1 - d'_2)$, so that $x = c_1 + d_1 + d'_1 - c_2 - d_2 - d'_2$, i.e., $x = e_2 - e_1$. Thus, we can rewrite Equation (7.3) to:

$$\left(\frac{\delta_R \times \delta_T}{\gamma \times \delta_\perp} \right)^x = 1 \quad (7.4)$$

Our goal is to show that $x = 0$. If true, then from the definitions of x above, it implies $c_1 = c_2$, $d_1 = d_2$, and $d'_1 = d'_2$, what we want to show. For this the key claim is to show that the fraction being exponentiated is not equal to 1. Formally:

Lemma 7.2. $\delta_R \times \delta_T \neq \gamma \times \delta_\perp$.

Proof. To show this result, we will have to study in detail the quantities $\delta_R \times \delta_T$ and $\gamma \times \delta_\perp$. To this end, to simplify the presentation, we name some variants of the (a, b, c, d) -gadget.

- The (a, b, c, d) -*full-gadget* is like the (a, b, c, d) -gadget, but we fix the facts $R(a)$ and $T(d)$ to be present, so that γ is the total probability of possible worlds of such a gadget which violate Q_1 .
- The (a, b, c, d) -*left-gadget* is like the (a, b, c, d) -gadget, but we fix the fact $R(a)$ to be present and the fact $T(d)$ to be absent, so that δ_R is the total probability of possible worlds of such a gadget which violate Q_1 .
- The (a, b, c, d) -*right-gadget* is like the (a, b, c, d) -gadget, but we fix the fact $T(d)$ to be present and the fact $R(a)$ to be absent, so δ_T is the total probability of possible worlds of such a gadget which violate Q_1 .
- The (a, b, c, d) -*trimmed-gadget* is like the (a, b, c, d) -gadget but where we fix the facts $R(a)$ and $T(d)$ to be absent, so δ_\perp is the total probability of possible worlds of such a gadget which violate Q_1 .

Let us consider, on the one hand, a pair of an (a, b, c, d) -left-gadget and of a (a', b', c', d') -right-gadget (giving distinct names to each vertex). As previously we indicate the facts R , S , and T graphically. Further, we will write R in the following way:

- as \underline{R} to mean that the fact was fixed to be present, that is, as we did on the endpoints when defining γ ;
- as \overline{R} to mean that the fact was fixed to be missing (intuitively, the vertex cannot be used for a match of the query);
- as $R?$ to mean that we have not yet fixed the fact

We do the same for the S -facts and T -facts. Under these conventions, the left and right gadgets are, respectively:

$$\begin{array}{ccccccc} \underline{R} & & T? & & R? & & \overline{T} \\ a & \xrightarrow{S?} & b & \xleftarrow{S?} & c & \xrightarrow{S?} & d \\ \overline{R} & & T? & & R? & & \underline{T} \\ a' & \xrightarrow{S?} & b' & \xleftarrow{S?} & c' & \xrightarrow{S?} & d' \end{array}$$

And let us consider, on the other hand, a pair of an (e, f, g, h) -full-gadget and of an (e', f', g', h') -trimmed-gadget, which we represent in the same way:

$$\begin{array}{ccccccc} \underline{R} & & T? & & R? & & \underline{T} \\ e & \xrightarrow{S?} & f & \xleftarrow{S?} & g & \xrightarrow{S?} & h \\ \overline{R} & & T? & & R? & & \overline{T} \\ e' & \xrightarrow{S?} & f' & \xleftarrow{S?} & g' & \xrightarrow{S?} & h' \end{array}$$

Let us compute the difference $\Delta := \delta_R \delta_T - \delta_\perp \gamma$, and show that it is non-zero.

The first term, $\delta_R \delta_T$, is the total probability of possible worlds of the first figure that violate the query. It is clearly unchanged when fixing the fact $S(c, d)$ to be missing, because this fact cannot participate to a match of the query Q_1 because $T(d)$ is missing, and likewise when fixing $S(a', b')$ to be missing because $R(a')$ is missing. Likewise, for $\gamma \delta_\perp$, we can fix $S(e', f')$ and $S(g', h')$ to be missing.

Let us write the remaining parts of the gadgets, where we remove the edges that we have fixed and can no longer intervene in a match of the query:

$$\begin{array}{ccc} \underline{R} & & \underline{T} \\ a \xrightarrow{S?} & b & \xleftarrow{S?} c \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{I} \\ b' \xleftarrow{S?} & c' & \xrightarrow{S?} d' \\ & & \end{array}$$

$$\begin{array}{ccc} \underline{R} & & \underline{T} \\ e \xrightarrow{S?} & f & \xleftarrow{S?} g \xrightarrow{S?} h \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{R} \\ f' \xleftarrow{S?} & g' & \\ & & \end{array}$$

We can now write:

$$\delta_R = (1 - \sigma)\delta_R^- + \delta_R^+$$

where δ_R^+ is the total probability of possible worlds violating Q_1 where we keep the fact $S(a, b)$, and δ_R^- the total probability of possible worlds violating Q_1 where this fact is missing.

Likewise, we write:

$$\gamma = (1 - \sigma)\gamma^- + \gamma^+$$

where γ^+ is the total probability of possible worlds violating Q_1 where we keep the fact $S(e, f)$, and γ^- the total probability of possible worlds violating Q_1 where at least one of these facts is missing.

Thus, we have:

$$\Delta = (1 - \sigma)\Delta^- + \sigma\Delta^+$$

where

$$\Delta^- = \delta_R^- \delta_T - \gamma^- \delta_\perp \qquad \Delta^+ = \delta_R^+ \delta_T - \gamma^+ \delta_\perp.$$

But let us consider the status of the gadgets in the choices made in Δ^- . They are:

$$\begin{array}{ccc} \underline{R} & & \underline{T} \\ a \xrightarrow{\mathcal{S}} & b & \xleftarrow{S?} c \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{I} \\ b' \xleftarrow{S?} & c' & \xrightarrow{S?} d' \\ & & \end{array}$$

$$\begin{array}{ccc} \underline{R} & & \underline{T} \\ e \xrightarrow{\mathcal{S}} & f & \xleftarrow{S?} g \xrightarrow{S?} h \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{R} \\ f' \xleftarrow{S?} & g' & \\ & & \end{array}$$

The absence of the fact $S(a, b)$ means that this edge can no longer be part of a match of the query, and the same holds for (e, f) , yielding:

$$\begin{array}{ccc} \underline{T} & & \underline{R} \\ b \xleftarrow{S?} & c & \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{I} \\ b' \xleftarrow{S?} & c' & \xrightarrow{S?} d' \\ & & \end{array}$$

$$\begin{array}{ccc} \underline{T} & & \underline{I} \\ f \xleftarrow{S?} & g & \xrightarrow{S?} h \\ & & \end{array} \qquad \begin{array}{ccc} \underline{T} & & \underline{R} \\ f' \xleftarrow{S?} & g' & \\ & & \end{array}$$

We note that the gadgets of the first line are isomorphic to that of the second line, so they have the same probability of violating Q_1 . Thus, $\Delta^- = 0$, and $\Delta = \sigma\Delta^+$.

We accordingly study the status of the gadgets in Δ^+ :

$$\begin{array}{ccc}
\frac{\underline{R}}{a} \xrightarrow{\underline{S}} \frac{\text{T?}}{b} \xleftarrow{\text{S?}} \frac{\text{R?}}{c} & & \frac{\text{T?}}{b'} \xleftarrow{\text{S?}} \frac{\text{R?}}{c'} \xrightarrow{\text{S?}} \frac{\underline{\text{I}}}{d'} \\
\frac{\underline{R}}{e} \xrightarrow{\underline{S}} \frac{\text{T?}}{f} \xleftarrow{\text{S?}} \frac{\text{R?}}{g} \xrightarrow{\text{S?}} \frac{\underline{\text{I}}}{h} & & \frac{\text{T?}}{f'} \xleftarrow{\text{S?}} \frac{\text{R?}}{g'}
\end{array}$$

We write in the same way:

$$\delta_{\text{T}} = (1 - \sigma)\delta_{\text{T}}^- + \delta_{\text{T}}^+$$

Where δ_{T}^+ is the total probability of possible worlds violating Q_1 where we keep the fact $S(c, d)$, and δ_{T}^- is the total probability of possible worlds violating Q_1 where at least one of these facts is missing.

We also write:

$$\gamma^+ = (1 - \sigma)\gamma^{+-} + \gamma^{++}$$

Where γ^{++} is the total probability of possible worlds violating Q_1 where we keep the fact $S(g, h)$, and γ^{+-} is the total probability of possible worlds where at least one of these facts is missing.

Thus, we have:

$$\Delta^+ = (1 - \sigma)\Delta^{+-} + \sigma\Delta^{++}$$

where

$$\Delta^{+-} = \delta_{\text{R}}^+\delta_{\text{T}}^- - \gamma^{+-}\delta_{\perp} \quad \Delta^{++} = \delta_{\text{R}}^+\delta_{\text{T}}^+ - \gamma^{++}\delta_{\perp}.$$

And let us consider again the choices made in Δ^{+-} :

$$\begin{array}{ccc}
\frac{\underline{R}}{a} \xrightarrow{\underline{S}} \frac{\text{T?}}{b} \xleftarrow{\text{S?}} \frac{\text{R?}}{c} & & \frac{\text{T?}}{b'} \xleftarrow{\text{S?}} \frac{\text{R?}}{c'} \xrightarrow{\cancel{S}} \frac{\underline{\text{I}}}{d'} \\
\frac{\underline{R}}{e} \xrightarrow{\underline{S}} \frac{\text{T?}}{f} \xleftarrow{\text{S?}} \frac{\text{R?}}{g} \xrightarrow{\cancel{S}} \frac{\underline{\text{I}}}{h} & & \frac{\text{T?}}{f'} \xleftarrow{\text{S?}} \frac{\text{R?}}{g'}
\end{array}$$

Again, we eliminate the fixed parts of the gadget that cannot contribute to a query match, and obtain:

$$\begin{array}{ccc}
\frac{\underline{R}}{a} \xrightarrow{\underline{S}} \frac{\text{T?}}{b} \xleftarrow{\text{S?}} \frac{\text{R?}}{c} & & \frac{\text{T?}}{b'} \xleftarrow{\text{S?}} \frac{\text{R?}}{c'} \\
\frac{\underline{R}}{e} \xrightarrow{\underline{S}} \frac{\text{T?}}{f} \xleftarrow{\text{S?}} \frac{\text{R?}}{g} & & \frac{\text{T?}}{f'} \xleftarrow{\text{S?}} \frac{\text{R?}}{g'}
\end{array}$$

And again these gadgets are isomorphic, so $\Delta^{+-} = 0$ and $\Delta^+ = \sigma\Delta^{++}$.

The gadgets in Δ^{++} are:

$$\begin{array}{ccc}
\frac{\underline{R}}{a} \xrightarrow{\underline{S}} \frac{\text{T?}}{b} \xleftarrow{\text{S?}} \frac{\text{R?}}{c} & & \frac{\text{T?}}{b'} \xleftarrow{\text{S?}} \frac{\text{R?}}{c'} \xrightarrow{\underline{S}} \frac{\underline{\text{I}}}{d'} \\
\frac{\underline{R}}{e} \xrightarrow{\underline{S}} \frac{\text{T?}}{f} \xleftarrow{\text{S?}} \frac{\text{R?}}{g} \xrightarrow{\underline{S}} \frac{\underline{\text{I}}}{h} & & \frac{\text{T?}}{f'} \xleftarrow{\text{S?}} \frac{\text{R?}}{g'}
\end{array}$$

For the query not to be satisfied in the first (top) gadget, it must be the case that the fact $\mathsf{T}(b)$ is missing, as otherwise we have a query match; and the fact $\mathsf{R}(c')$ must be missing for the same reason. This gives us a probability of $(1 - \rho)(1 - \tau)$ choices. Likewise, in the (bottom) second gadget, the facts $\mathsf{T}(f)$ and $\mathsf{R}(g)$ must be missing, which again has a probability of $(1 - \rho)(1 - \tau)$. Thus, we have:

$$\Delta^{++} = (1 - \rho)(1 - \tau)\Delta'$$

Where we have $\Delta' = \delta'_R \delta'_T - \gamma' \delta_\perp$, in which:

- δ'_R is the total probability of possible worlds violating the query of the (a, b, c, d) -left-gadget containing the fact $\mathsf{S}(a, b)$ but not the facts $\mathsf{T}(b)$ and $\mathsf{S}(c, d)$;
- δ'_T is the total probability of possible worlds violating the query of the (a', b', c', d') -right-gadget containing the fact $\mathsf{S}(c', d')$ but not the facts $\mathsf{S}(a', b')$ and $\mathsf{R}(c')$;
- γ' is the total probability of possible worlds violating the query of the (e, f, g, h) -full-gadget containing the facts $\mathsf{S}(e, f)$ and $\mathsf{S}(g, h)$ but not the facts $\mathsf{T}(f)$ and $\mathsf{R}(g)$.

The status of the gadgets in Δ' is:

$$\begin{array}{ccc} \underline{\mathsf{R}} & \underline{\mathsf{S}} & \overline{\mathsf{T}} \\ a \xrightarrow{\quad} b & \xleftarrow{\quad} c & \end{array} \quad \begin{array}{ccc} \mathsf{T}? & \mathsf{S}? & \overline{\mathsf{R}} \\ b' \xleftarrow{\quad} c' & \xrightarrow{\quad} d' & \underline{\mathsf{T}} \end{array}$$

$$\begin{array}{ccc} \underline{\mathsf{R}} & \underline{\mathsf{S}} & \overline{\mathsf{T}} \\ e \xrightarrow{\quad} f & \xleftarrow{\quad} g & \xrightarrow{\quad} h \end{array} \quad \begin{array}{ccc} \mathsf{T}? & \mathsf{S}? & \mathsf{R}? \\ f' \xleftarrow{\quad} g' & & \end{array}$$

It is now clear that Δ' is non-zero: the top gadgets can no longer contain a match of Q_1 , whereas the bottom gadgets have probability $1 - \rho\sigma\tau$ of violating Q_1 . Thus, $\Delta' = \rho\sigma\tau$ and $\Delta = \sigma^2(1 - \rho)(1 - \tau)\rho\sigma\tau$, which is non-zero because $0 < \rho, \tau < 1$ and $0 < \sigma \leq 1$. This concludes the proof of Lemma 7.2. \square

Thus, we conclude from Equation (7.4) that $x = 0$, which as we explained implies $(c_1, d_1, d'_1) = (c_2, d_2, d'_2)$. This establishes Claim 7.1 and shows that all coefficients $\alpha(c, d, d')$ of the Vandermonde matrix A are different, so it is invertible. This concludes the proof of Theorem 4.3, and hence of our hardness result on weighted uniform reliability (Theorem 3.5) and on uniform reliability (Theorem 3.2).

8. DETERMINISTIC RELATIONS

We have shown our main result on uniform reliability (Theorem 3.2). We did so by proving a more general result on weighted uniform model counting (Theorem 3.5), but this result does not cover the case of *deterministic relations*, that is, relations where the fixed probability of every fact is 1. In this section, we address this issue.

Case of the query Q_1 . For the query Q_1 , the weighted uniform model counting problem has three parameters: the probabilities $\varphi(\mathsf{R})$, $\varphi(\mathsf{S})$, and $\varphi(\mathsf{T})$ of the relations R , S , and T . Our intractability result for Q_1 (Theorem 4.3) in fact covered the case where $\varphi(\mathsf{S})$ may be 1. As it turns out, this completely classifies the complexity of the query, because of the following easy fact:

Proposition 8.1. *Let φ be a function mapping R , S , and T to probabilities, and assume that one of $\varphi(\mathsf{R})$ or $\varphi(\mathsf{T})$ is 1. Then $\text{WUR}(Q_1, \varphi)$ is solvable in polynomial time.*

Proof. We only give the argument for when $\varphi(\mathsf{T}) = 1$: the argument for the other case is analogous. Let (I, π) be an input instance. First remove all facts $\mathsf{S}(a, b)$ from I where I does not contain the fact $\mathsf{T}(b)$: this clearly does not change the answer to the problem as such facts can never be part of a match to the query. Let (I', π') be the result of this process, with π' being the restriction of π to I' . Now, observe that every match of the query $Q'_1 : \mathsf{R}(x), \mathsf{S}(x, y)$ in I translates to a match of the query Q_1 on the same choice of x, y ; and conversely it is obvious that any match of Q_1 translates to a match of Q'_1 . Hence, every possible world of (I, π) has a match of Q_1 iff it has a match of Q'_1 . This implies that we can solve $\text{PQE}_{r,s,1}(Q_1)$ on I' , hence on I , by solving $\text{PQE}(Q'_1)$, and this is tractable by the result of Dalvi and Suciu [DS07] (Theorem 2.2) because Q'_1 is a hierarchical self-join-free CQ. \square

Thus, we have the following classification for weighted uniform model counting for the query Q_1 , which was conjectured in the conference version of this paper [AK21, Conjecture 7.4]:

Proposition 8.2. *Let φ be a function mapping R , S , and T to probabilities. If $\varphi(\mathsf{R}) < 1$ and $\varphi(\mathsf{T}) < 1$, then $\text{WUR}(Q_1, \varphi)$ is $\#\text{P-hard}$; otherwise, it is solvable in polynomial time.*

Case of general queries. For arbitrary CQs without self-joins, deterministic relations certainly have an impact on the complexity of weighted uniform reliability, and we cannot hope that Theorem 3.5 generalizes as-is: for instance, if all relations have probability 1, then the weighted uniform reliability problem is equivalent to evaluating a fixed CQ on non-probabilistic data, and hence, in polynomial time. More generally, our reduction from Q_1 (Lemma 4.2) does not work immediately in the same way, as a simple example illustrates:

Example 8.3. Consider the query $Q : \mathsf{R}(x), \mathsf{S}_1(x, y), \mathsf{S}_2(y, z), \mathsf{T}(z)$, and the function φ mapping R and T to $\frac{1}{2}$ and S_1 and S_2 to 1. It is clear that $\text{WUR}(Q, \varphi)$ is $\#\text{P-hard}$ by a simple reduction from weighted uniform reliability for Q_1 , replacing every fact $\mathsf{S}(a, b)$ by two facts $\mathsf{S}_1(a, d), \mathsf{S}_2(d, b)$ for some fresh d . Yet, we cannot find a match of Q_1 in Q by renaming variables as in the proof of Lemma 4.2.

Fortunately, there is a classification of the hardness of probabilistic query evaluation for Boolean CQs without self-joins in [DS07, Theorem 8] in the case where some relations are deterministic, giving a pattern of relations that characterizes intractability. We can re-use this classification and show that the same dichotomy applies for weighted uniform reliability: an intractable query for PQE is also hard under the assumption that all tuples in all probabilistic relations have a common, fixed probability. Here is the criterion used in their dichotomy:

Definition 8.4 (From [DS07], Theorem 8). Let Q be a CQ without self-joins, and φ be a function mapping the relations of Q to probabilities in $(0, 1]$. A *hardness pattern* of Q and φ is a sequence of relations $\mathsf{R}_0, \dots, \mathsf{R}_{k+1}$, with corresponding atoms $U_i(\vec{a}_i)$ in Q , such that:

- $\varphi(\mathsf{R}_0) < 1$ and $\varphi(\mathsf{R}_{k+1}) < 1$.
- There is a variable x occurring in \vec{a}_0 and \vec{a}_1 but not in \vec{a}_{k+1} .
- There is a variable y occurring in \vec{a}_k and \vec{a}_{k+1} but not in \vec{a}_0 .
- For all $i = 1, \dots, k - 1$ we have that $\vec{a}_i \cap \vec{a}_{i+1} \not\subseteq \vec{a}_0 \cup \vec{a}_{k+1}$ (that is, every two internal consecutive atoms have a common variable that is in neither \vec{a}_0 nor \vec{a}_{k+1}), where we abuse notation and see tuples as sets.

We show the following dichotomy result. The result completely classifies the complexity of the weighted uniform reliability problem for CQs without self-joins when deterministic relations are allowed, at the expense of a somewhat more complex classification:

Theorem 8.5. *Let Q be a CQ without self-joins, and φ be a function mapping the relations of Q to probabilities in $(0, 1]$. If Q and φ do not have a hardness pattern, then $\text{WUR}(Q, \varphi)$ is solvable in polynomial time; otherwise, it is $\#P$ -hard.*

Proof. If Q and φ do not have a hardness pattern, then we know by [DS07, Theorem 8] that PQE for Q is PTIME under the restriction that tuples of relations mapped to 1 by φ have probability 1. Hence, as weighted uniform reliability for Q under φ is a special case of PQE for Q under this requirement, it is solvable in polynomial time.

Now, if Q and φ have a hardness pattern, we show that $\text{WUR}(Q, \varphi)$ is $\#P$ -hard. We first note that, if $k = 1$, then the definition of a hardness pattern implies that we can conclude as in Lemma 4.2. Specifically, in this case, we have relations R_0, R_1, R_2 , with $\varphi(R_0) < 1$, $\varphi(R_2) < 1$, and a variable x occurring in R_0 and R_1 but not R_2 , and a variable y occurring in R_1 and R_2 but not R_0 , and we conclude like in Lemma 4.2. Hence, to simplify the presentation, we assume that $k > 1$ from now on.

We show hardness by reducing from $\text{WUR}(Q_1, \varphi_1)$ for some φ_1 that we will define later, analogously to the proof of Lemma 4.2, and with some inspiration from the proof of Dalvi and Suciu [DS07, Theorem 8]. To define φ_1 , we partition the variables of Q into four sets, again abusing set notation to apply to tuples:

- the set $S_x := \vec{a}_0 \setminus \vec{a}_{k+1}$ (so including x);
- the set $S_y := \vec{a}_{k+1} \setminus \vec{a}_0$ (so including y);
- the set $S_C := \bigcup_{1 \leq i \leq k-1} \vec{a}_i \cap \vec{a}_{i+1} \setminus (\vec{a}_0 \cup \vec{a}_{k+1})$ (by the fourth condition in the definition of a hardness pattern, and as $k > 1$, this set is non-empty);
- and the set S_0 of all other variables (including in particular $\vec{a}_0 \cap \vec{a}_{k+1}$, which may be empty).

Note that the sets S_x , S_y , and S_C is non-empty, but S_0 may be empty.

Following this partition of variables, we partition the relations of Q into five kinds depending on the variables that their corresponding atom contain (we ignore the constants that the atoms may also contain):

- 0-relations, which contain only variables of S_0 , including any relations that contain no variables at all; it may be the case that there are no 0-relations.
- x -relations, containing only variables of $S_x \cup S_0$ and at least one variable of S_x ; there is at least one x -relation, namely R_0 .
- y -relations, containing only variables of $S_y \cup S_0$ and at least one variable of S_y ; there is at least one y -relation, namely R_{k+1} .
- xy -relations, containing only variables of $S_x \cup S_y \cup S_0$ and at least one variable of S_x and at least one variable of S_y ; it may be the case that there are no xy -relations.
- C -relations, containing a variable of S_C ; there is at least k C -relations, namely, the relations R_1, \dots, R_{k-1} , so as $k > 1$ there is at least one such relation.

As an example, in the following CQ, assume that $\varphi(R) < 1$ and $\varphi(U) < 1$.

$P(x, y), R(x, y, z), S(x, w, w'), T(w, w', z, x, u), U(u, z), V(x, z, u), V'(x, z, w'), W(t), W'(t, z)$

- The sequence R, S, T, U is a hardness pattern: the variable x occurs in the R - and S -atoms but not in the U -atom, the variable u occurs in the T - and U -atoms but not in the R -atom, the variable w occurs in the S - and T -atoms but not in the R - or U -atoms.

- The set S_x contains $\{x, y\}$, the set S_y contains $\{u\}$, the set S_C contains $\{w, w'\}$, and the set S_0 contains all other variables.
- The relations P and R are x -relations, the relation U is a y -relation, the relation V is an xy -relation, the relations S and T and V' are C -relations, and the relations W and W' are 0-relations.

Next, we show a reduction from $WUR(Q_1, \varphi_1)$ for some hard case of φ_1 . We define φ_1 by mapping the relation R to the product of the φ -values of all x -relations (which is strictly smaller than 1, because $\varphi(R_0) < 1$), mapping the relation T to the product of the φ -values of all y -relations (which is strictly smaller than 1, because $\varphi(R_{k+1}) < 1$), and mapping the relation S to the product of the φ -values of all xy -relations and C -relations (which may be 1 or smaller than 1).

Let I_1 be an input instance for $WUR(Q_1, \varphi_1)$. As in the proof of Lemma 4.2, we call a *match* of Q_1 in I_1 a pair (a, b) of constants such that I_1 contains all of $R(a)$, $S(a, b)$ and $T(b)$, and we assume without loss of generality that every fact of I_1 is part of a match.

We construct from I_1 the instance I for Q as follows: for every match (a, b) in I_1 , we add to I a copy of the query Q where all variables of S_x are replaced by a , all variables of S_y are replaced by b , all variables of S_0 are replaced by a fixed constant c (always the same), and all variables of S_C are replaced by a fresh constant $c_{a,b}$. Note that this may include the same fact multiple times, in which case it is only added once. To be more precise, we can equivalently state the construction as inserting the following facts into I , where we describe the facts in term of the elements that replace variables (any constants used in the CQ are left as-is in the atoms):

- For each 0-relation, one fact of the 0-relation where all variables (if any) are replaced by the fixed constant c ;
- For each element a of I_1 involved in a match (a, b) for some b , and for each x -relation, one fact of that relation where variables of S_0 (if any) are replaced by c and variables of S_x (including x) are replaced by a ;
- For each element b of I_1 involved in a match (a, b) for some a , and for each y -relation, one fact of that relation where variables of S_0 (if any) are replaced by c and variables of S_y (including y) are replaced by b ;
- For each match (a, b) of I_1 , and for each xy -relation, one fact of that relation where variables of S_x are replaced by a , variables of S_y are replaced by b , and variables of S_0 are replaced by c ;
- For each match (a, b) of I_1 , and for each C -relation, one fact of that relation where variables of S_C are replaced by $c_{a,b}$, variables of S_x (if any) are replaced by a , variables of S_y (if any) are replaced by b , and variables of S_0 (if any) are replaced by c .

This construction is clearly in polynomial time.

Continuing from our example, we would produce the facts $W(c)$, $W'(c, c)$, the facts $P(a, a)$ and $R(a, a, c)$ for each a in a match (a, b) , the fact $U(b, c)$ for each b in a match (a, b) , and the facts $S(a, c_{a,b}, c_{a,b})$, $T(c_{a,b}, c_{a,b}, c, a, b)$, $V'(a, c, c_{a,b})$, and $V(a, c, b)$ for each match (a, b) . Hence, we add the following for every match (a, b) :

$$P(a, a) R(a, a, c) S(a, c_{a,b}, c_{a,b}) T(c_{a,b}, c_{a,b}, c, a, b) U(b, c) V(a, c, b) V'(a, c, c_{a,b}) W(c) W'(c, c)$$

Now, as in the proof of Lemma 4.2, we study the possible worlds of I that satisfy Q . We first note that, for 0-relations, we created only one fact for this relation (containing only the constant c and possibly constants used in the CQ), and this fact must be kept to have a query match, so the probability of satisfying Q in I includes a factor $\prod \varphi(R)$ across all the

0-relations to keep their facts, and we can therefore focus on the possible worlds where all these facts are kept. We then define a possible world J of I by choosing a possible world J_1 of I_1 and:

- For each fact $R(a)$ of I_1 :
 - If $R(a)$ is in J_1 , we keep the facts of all x -relations involving a ;
 - Otherwise, we do not keep all these facts, i.e., we discard at least one of these facts.
- For each fact $S(a, b)$ of I_1 :
 - If $S(a, b)$ is in J_1 , we keep the facts of all xy -relations involving a and b , and all facts of C -relations involving c_{ab} ;
 - Otherwise, we do not keep all these facts, i.e., we discard at least one of these facts.
- For each fact $T(b)$ of I_1 :
 - If $T(b)$ is in J_1 , we keep the facts of all y -relations involving b ;
 - Otherwise, we do not keep all these facts, i.e., we discard at least one of these facts.
- We keep all facts of the 0-relations.

Now, if our choice of $J_1 \subseteq I_1$ satisfies Q_1 , then the facts retained in J witness that J satisfies Q . Conversely, if we have a possible world J of I that satisfies Q , we must argue that the corresponding possible world J_1 of I_1 satisfies Q_1 . Intuitively, the challenge is to argue that the facts used to satisfy Q in J can be chosen from one single match of Q_1 in I_1 such that all facts created from this match in the construction were kept in J .

To show this, we first observe that, in the mapping from Q to J witnessing that Q is satisfied in J , the variables of S_0 must be mapped to the constant c , because c is the only element used to create facts at the positions corresponding to variables of S_0 . Now, consider the matches in J of the atoms for relations R_0, \dots, R_{k+1} of Q . The matches of R_0 and R_{k+1} give us a value a for all variables in S_x : by construction of the R_0 -facts in I , all variables of the set S_x must be mapped to the same element, which is some element a for a match (a, b) in I_1 . We obtain a value b for all variables of S_y in the same way. This ensures that all the facts for x -relations for this value of a must be kept in J , and likewise for the facts for y -relations for this value of b , and also for the facts for xy -relations for this value of a and b .

It remains to study to which elements in J the variables at positions in S_C can be mapped by our match of Q , and to show that the facts for C -relations for the match (a, b) of Q_1 in I_1 must be kept in J . We do so by showing that all variables of S_C were in fact mapped to the element $c_{a,b}$ determined from the a and b already defined. To do this, as we know that x is in R_1 , we know that the R_1 -atom of Q is matched to a fact involving a and some $c_{a',b'}$ for the variables of $\vec{a}_1 \cap \vec{a}_2 \setminus (\vec{a}_0 \cup \vec{a}_{k+1})$ which is a non-empty subset of S_C : by construction, it must be $c_{a,b'}$ for some b' . But then, the R_2 -atom is matched to a fact involving this $c_{a,b'}$ and some $c_{a',b''}$ for the variables of $\vec{a}_2 \cap \vec{a}_3 \setminus (\vec{a}_0 \cup \vec{a}_{k+1})$: by construction we must have $a = a'$ and $b' = b''$. Repeating the argument across the path, we show that all variables of S_C are mapped to the same $c_{a,b'}$, and the only way to avoid a contradiction in the end (given that y is in R_k and R_{k+1}) is that $b' = b$. Therefore, it is indeed the case that the match of Q in J maps all variables of S_C to $c_{a,b}$, and it witnesses that all the facts for C -relations for (a, b) must be kept.

To summarize, the match of Q in J must map all variables of S_x to a , all variables of S_y to b , and all variables of S_C to $c_{a,b}$, for some choice of (a, b) which is a match of Q_1 in I_1 . Further, we know that all x -facts involving a , all y -facts involving b , all xy -facts involving a

and b , all C -facts involving $c_{a,b}$, and all 0-facts, were kept in J . Thus, the choice of $J_1 \subseteq I_1$ that corresponds to our choice of $J \subseteq I$ must satisfy Q_1 .

Concluding the proof as in Lemma 4.2, we establish that the probability that Q is satisfied in I is exactly $\prod \varphi(\mathbf{R})$ for all 0-relations \mathbf{R} , times the probability that Q_1 is satisfied in I_1 . This concludes the reduction. Thus, our hardness result follows from our hardness result on weighted uniform reliability for Q_1 (Theorem 3.5). \square

9. CONCLUDING REMARKS

While query evaluation over TIDs has been studied for over a decade, the basic case of a uniform distribution, namely uniform reliability, had been left open. We have settled this open question for the class of CQs without self-joins, and shown a dichotomy on computational complexity of counting satisfying database subsets: this task is tractable for hierarchical queries, and $\#P$ -hard otherwise. Our precise result is more general and applies to weighted uniform reliability, where each relation is associated with a probability that is attached to all of its facts.

One immediate question for future research is whether our results could extend to more general query classes. The obvious challenge is to extend to CQs with self-joins and UCQs with self-joins, and try to match the known dichotomy for non-uniform probabilities [DS12]. Following our work, considerable progress in this direction has been done recently by Kenig and Suciu [KS20], which addresses the case of PQE with probabilities of $\frac{1}{2}$ and 1, and leaves open the case of uniform reliability for arbitrary UCQs. The same study could be undertaken for the general class of queries closed under homomorphisms, following the recent dichotomy on PQE for such queries (on binary signatures) in [AC22].

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REFERENCES

- [AC22] Antoine Amarilli and İsmail İlkan Ceylan. The dichotomy of evaluating homomorphism-closed queries on probabilistic graphs. *LMCS*, 18(1), 2022. doi:10.46298/lmcs-18(1:2)2022.
- [AK21] Antoine Amarilli and Benny Kimelfeld. Uniform reliability of self-join-free conjunctive queries. In *ICDT*, volume 186 of *LIPICs*, pages 17:1–17:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
- [BKS17] Christoph Berkholz, Jens Keppeler, and Nicole Schweikardt. Answering conjunctive queries under updates. In *PODS*, pages 303–318. ACM, 2017.
- [Bul13] Andrei A. Bulatov. The complexity of the counting constraint satisfaction problem. *J. ACM*, 60(5):34:1–34:41, 2013.
- [BVdBGS15] Paul Beame, Guy Van den Broeck, Eric Gribkoff, and Dan Suciu. Symmetric weighted first-order model counting. In *PODS*, pages 313–328. ACM, 2015.
- [CFMV15] Supratik Chakraborty, Dror Fried, Kuldeep S Meel, and Moshe Y Vardi. From weighted to unweighted model counting. In *IJCAI*, 2015.

- [DRS09] Nilesh N. Dalvi, Christopher Ré, and Dan Suciu. Probabilistic databases: Diamonds in the dirt. *Commun. ACM*, 52(7):86–94, 2009.
- [DS79] Pradeep Dubey and Lloyd S. Shapley. Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research*, 4(2):99–131, 1979.
- [DS07] Nilesh Dalvi and Dan Suciu. Efficient query evaluation on probabilistic databases. *VLDB Journal*, 16(4):523–544, 2007.
- [DS12] Nilesh Dalvi and Dan Suciu. The dichotomy of probabilistic inference for unions of conjunctive queries. *J. ACM*, 59(6), 2012.
- [GGH98] Erich Grädel, Yuri Gurevich, and Colin Hirsch. The complexity of query reliability. In *PODS*, pages 227–234. ACM Press, 1998.
- [GS79] B. Grofman and H. Scarrow. *Iannucci and its aftermath: The application of the Banzhaf index to weighted voting in the state of New York*, pages 168–183. Physica-Verlag HD, Heidelberg, 1979. doi:10.1007/978-3-662-41501-6_10.
- [GS16] Eric Gribkoff and Dan Suciu. SlimShot: In-database probabilistic inference for knowledge bases. *PVLDB*, 9(7):552–563, 2016.
- [HPS83] Harold V Henderson, Friedrich Pukelsheim, and Shayle R Searle. On the history of the Kronecker product. *Linear and Multilinear Algebra*, 14(2):113–120, 1983.
- [JS12] Abhay Kumar Jha and Dan Suciu. Probabilistic databases with MarkoViews. *PVLDB*, 5(11):1160–1171, 2012.
- [KS20] Batya Kenig and Dan Suciu. A dichotomy for the generalized model counting problem for unions of conjunctive queries. *CoRR*, abs/2008.00896, 2020.
- [LBKS20] Ester Livshits, Leopoldo E. Bertossi, Benny Kimelfeld, and Moshe Sebag. The Shapley value of tuples in query answering. In *ICDT*, volume 155 of *LIPICs*, pages 20:1–20:19. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020.
- [MW13] Dany Maslowski and Jef Wijsen. A dichotomy in the complexity of counting database repairs. *J. Comput. Syst. Sci.*, 79(6):958–983, 2013.
- [MW14] Dany Maslowski and Jef Wijsen. Counting database repairs that satisfy conjunctive queries with self-joins. In *ICDT*, pages 155–164. OpenProceedings.org, 2014.
- [OH08] Dan Olteanu and Jiewen Huang. Using OBDDs for efficient query evaluation on probabilistic databases. In *SUM*, volume 5291 of *Lecture Notes in Computer Science*, pages 326–340. Springer, 2008.
- [PB83] J. Scott Provan and Michael O. Ball. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM Journal on Computing*, 12(4), 1983.
- [Rot88] Alvin E Roth. *The Shapley value: Essays in honor of Lloyd S. Shapley*. Cambridge University Press, 1988.
- [SBSdB16] Babak Salimi, Leopoldo E. Bertossi, Dan Suciu, and Guy Van den Broeck. Quantifying causal effects on query answering in databases. In *TAPP*, 2016.
- [Sha53] L.S. Shapley. Stochastic games. *Proceedings of the National Academy of Sciences of the United States of America*, 39:1095–1100, 1953.
- [SORK11] Dan Suciu, Dan Olteanu, Christopher Ré, and Christoph Koch. *Probabilistic Databases*. Synthesis Lectures on Data Management. Morgan & Claypool Publishers, 2011.