# CONVEXITY VIA WEAK DISTRIBUTIVE LAWS 

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#### Abstract

We study the canonical weak distributive law $\delta$ of the powerset monad over the semimodule monad for a certain class of semirings containing, in particular, positive semifields. For this subclass we characterise $\delta$ as a convex closure in the free semimodule of a set. Using the abstract theory of weak distributive laws, we compose the powerset and the semimodule monads via $\delta$, obtaining the monad of convex subsets of the free semimodule.


## Introduction

Convexity plays an important role in several areas of computer science ranging from linear programming to program analysis. In the former, the space of solutions of an optimisation problem is convex and this property is crucially exploited to find optimal solutions. In program analysis, the space of stores possibly encountered during the execution of a program is usually not convex but, to make their analysis feasible, one can exploit abstract interpretation [CC77] and consider an abstracted program dealing with convex spaces. Convexity also appears in the study of concurrent probabilistic systems that usually behave both in probabilistic and in non-deterministic ways. Since the very beginning of this field, the behaviour of such systems is often assumed to be convex (see e.g. convex bisimilarity [SL94, Mio14]) and much emphasis has been given in finding algebras to properly deal with both non-determinism and probability [BS01, Mis00, MOW03, Var03].

Such algebras inspire the main intuition of this work, that is convexity arises somewhat naturally when combining non-deterministic and quantitative features. To give a first motivation of this intuition, we show now that convexity can be derived as a law from a few axioms. Consider Table 1: the axioms on the top left, forming the algebraic theory of semilattices, are the rules for non-determinism; those on the top right form the algebraic theory of semimodules for some fixed semiring $S$ that are often exploited to model different quantitative aspects; the four axioms at the bottom of the table distribute the structure of semilattice over the semimodule one. As illustrated in Table 2, from these axioms one can derive the following law:

$$
x \sqcup y=x \sqcup y \sqcup(\alpha \cdot x+\beta \cdot y) \text { for all } \alpha, \beta \in S \text { such that } \alpha+\beta=1 .
$$

The above equality expresses convexity. Take indeed $S$ to be $\mathbb{R}^{+}$, the semiring of non negative reals, and consider the algebra freely generated by the two-element set $\left\{x_{1}, x_{2}\right\}$, i.e. the algebra of terms with variables in $\left\{x_{1}, x_{2}\right\}$ built out of $\perp, \sqcup, 0,+, 1, \lambda$ and quotiented

[^0]

Table 1: The sets of axioms $E_{\mathcal{S L}}$ for semilattices (left), $E_{\mathcal{L S M}}$ for $S$-semimodules (right) and $E_{\mathcal{D}^{\prime}}$ for their distributivity (bottom).
by the axioms in Table 1. Because of the distributivity axioms, the general term is of the form $\left(\lambda_{1} \cdot x_{1}+\mu_{1} \cdot x_{2}\right) \sqcup \cdots \sqcup\left(\lambda_{n} \cdot x_{1}+\mu_{n} \cdot x_{2}\right)$. Now, we can interpret geometrically these terms as follows. The structure of semimodule allows to express single (two-dimensional) vectors, for instance the terms $x_{1}, x_{2}, x_{1}+3 \cdot x_{2}$ denote the vectors $\binom{1}{0},\binom{0}{1},\binom{1}{3}$; the join of two vectors is interpreted by forming a set which, as expressed by the above equality, must contain the two vectors and also all the vectors between them. For instance the term $x_{1} \sqcup x_{2}$ is not just the set of vectors $\left\{\binom{1}{0},\binom{0}{1}\right\}$ but its convex closure $\left\{\left.\alpha \cdot\binom{1}{0}+\beta \cdot\binom{0}{1} \right\rvert\, \alpha+\beta=1\right\}$, i.e. the segment whose vertices are $x_{1}$ and $x_{2}$. Similarly, $x_{1} \sqcup x_{2} \sqcup\left(x_{1}+3 \cdot x_{2}\right)$ is the set of all points enclosed within (and including) the triangle with vertices $\binom{1}{0},\binom{0}{1},\binom{1}{3}$. More generally, as we will show at the end of this paper in Example 6.6, these terms are in one-to-one correspondence with convex polygons in the $\mathbb{R}^{+}$-semimodule $\left(\mathbb{R}^{+}\right)^{2}$.

The aim of this paper is to make this intuition formal by using the language of category theory. Rather than considering the aforementioned algebraic theories [HPP06], we work with the equivalent notion of monad. Monads are nowadays common tools in theoretical computer science, as they embody different notions of computations [Mog91], like nondeterminism, side effects and exceptions.

We show that combining somehow the finite powerset monad, $\mathcal{P}_{f}$, corresponding to the algebraic theory of semilattices, together with the monad $\mathcal{S}$ of semimodules, one obtains the monad of convex subsets that are finitely generated (roughly, that arise as convex combinations of finitely many elements). Moreover, the algebraic theory corresponding to the composed monad is exactly the one depicted in Table 1. The monad for arbitrary-not necessarily finitely generated-convex subsets can instead be achieved by combining the full powerset monad $\mathcal{P}$ with $\mathcal{S}$; the corresponding algebraic theory can be easily obtained from Table 1 by replacing finite joins with arbitrary ones: see Table 3 .

We perform the combination of these monads in a principled way, as in the last decades many works proposed different methods to compose monads and related notions [HPP06, VW06, Böh10, Lac04, Che20]. Indeed, the standard approach of composing monads by means of distributive laws [Bec69] turned out to be somehow unsatisfactory. On the one hand, distributive laws do not exist in many relevant cases: see [KS18, ZM19] for some no-go theorems; on the other hand, proving their existence is error-prone: see [KS18] for

| $\begin{aligned} x \sqcup y \sqcup(\alpha \cdot x+\beta \cdot y) & \stackrel{(*)}{=} x \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup y \sqcup(\alpha \cdot x+\beta \cdot y) \\ & \stackrel{(I)}{=} x \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup y \\ & \stackrel{(*)}{=} x \sqcup y \end{aligned}$ |  |
| :---: | :---: |
| $\begin{aligned} &(x \sqcup y) \stackrel{\left(U_{s}\right)}{=} \\ &(\alpha+\underline{\beta}=1) \\ &\left(\underline{D_{s}} 2\right) \\ &(\stackrel{D}{=} \\ &(\stackrel{D}{=}) \\ &(\stackrel{D}{=}) \\ &\left(\underline{D_{s}} 2\right) \\ &(\alpha+\beta=1) \\ &= \\ &\left(D_{s} 1\right) \end{aligned}$ | $\begin{aligned} & 1 \cdot(x \sqcup y) \\ & (\alpha+\beta) \cdot(x \sqcup y) \\ & \alpha \cdot(x \sqcup y)+\beta \cdot(x \sqcup y) \\ & (\alpha \cdot x \sqcup \alpha \cdot y)+(\beta \cdot x \sqcup \beta \cdot y) \\ & ((\alpha \cdot x \sqcup \alpha \cdot y)+\beta \cdot x) \sqcup((\alpha \cdot x \sqcup \alpha \cdot y)+\beta \cdot y) \\ & (\alpha \cdot x+\beta \cdot x) \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup(\alpha \cdot y+\beta \cdot y) \\ & (\alpha+\beta) \cdot x \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup(\alpha+\beta) \cdot y \\ & 1 \cdot x \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup 1 \cdot y \\ & x \sqcup(\alpha \cdot y+\beta \cdot x) \sqcup(\alpha \cdot x+\beta \cdot y) \sqcup y \end{aligned}$ |

Table 2: Derivation for the convexity law ( $\alpha$ and $\beta$ are assumed to be such that $\alpha+\beta=1$ ). The steps marked with $(*)$ are justified by the derivation in the bottom.
a list of results that were mistakenly assuming the existence of a distributive law of the powerset monad over itself. Nevertheless, some sort of weakening of the notion of distributive law-e.g. distributive laws of functors over monads [Kli11]-proved to be ubiquitous in computer science: they are GSOS specifications [TP97], they are sound coinductive up-to techniques [BPPR14] and complete abstract domains [BGGP18]. In this paper we will exploit weak distributive laws in the sense of [Gar20] that have been recently shown to be successful in composing the monads for nondeterminism and probability [GP20a].

We start with the full powerset monad $\mathcal{P}$, rather than the finite one $\mathcal{P}_{f}$. In this way we can reuse several results in [CHJ14] that provide necessary and sufficient conditions on the semiring $S$ for the existence of a canonical weak [Gar20] distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$. Our first contribution (Theorem 3.5) consists in showing that $\delta$ has a convenient alternative characterisation whenever the underlying semiring is a positive semifield, a condition that is met for example by the semiring of non-negative reals $\mathbb{R}^{+}$and by the semiring of Booleans.

This characterisation allows us to give a handy definition of the canonical weak lifting of $\mathcal{P}$ over $\mathbb{E M}(\mathcal{S})$ (Theorem 4.1) and to observe that this lifting is almost the same as the monad $\mathcal{C}: \mathbb{E M}(\mathcal{S}) \rightarrow \mathbb{E M}(\mathcal{S})$ defined by Jacobs in [Jac08] (Remark 4.2): the only difference is the absence in $\mathcal{C}$ of the empty subset. This divergence becomes crucial when considering the composed monads, named $\mathcal{C M}$ : Set $\rightarrow$ Set in [Jac08] and $\mathcal{P}_{c} \mathcal{S}$ : Set $\rightarrow$ Set in this paper: the latter maps a set $X$ into the set of convex subsets of $\mathcal{S} X$, while the former additionally requires the subsets to be non-empty. It turns out that while the Kleisli category $\mathbb{K l}(\mathcal{C M})$ is not $\mathbb{C P P D}$-enriched, a crucial condition in the setting of [HJS06] (on which [Jac08] is based) to define trace semantics within a coalgebraic framework, $\mathbb{K} l\left(\mathcal{P}_{\mathcal{C}} \mathcal{S}\right)$ indeed is (Theorem 5.3).

Composing monads by means of weak distributive laws is rewarding in many respects: here we exploit the fact that algebras for the composed monad $\mathcal{P}_{c} \mathcal{S}$ coincide with $\delta$-algebras, namely algebras for both $\mathcal{P}$ and $\mathcal{S}$ satisfying a certain pentagonal law. One can extract from this law some distributivity axioms (in the bottom of Table 3) that, together with the axioms

| Complete Semilattice |
| :---: |
| $\begin{array}{cll} \bigsqcup_{i \in\{0\}} x_{i} & \stackrel{(S i)}{=} & x_{0} \\ \bigsqcup_{j \in J} x_{j} & \stackrel{(C I)}{=} & \bigsqcup_{i \in I} x_{f(i)} \text { for all } f: I \rightarrow J \text { surjective } \\ \bigsqcup_{i \in I} x_{i} & \stackrel{(A U)}{=} & \bigsqcup_{j \in J} \bigsqcup_{i \in f^{-1}\{j\}} x_{i} \text { for all } f: I \rightarrow J \\ \hline \hline \end{array}$ |
| $\begin{array}{r} \text { Distributivity } \\ \lambda \cdot \bigsqcup_{i \in I} x_{i} \stackrel{(\stackrel{(D 1)}{=}}{\bigsqcup_{i \in I} \lambda \cdot x_{i} \quad \text { for } \lambda \neq 0} \\ \bigsqcup_{i \in I} x_{i}+\bigsqcup_{j \in J} y_{j} \end{array} \stackrel{(\stackrel{D 2)}{=}}{\bigsqcup_{(i, j) \in I \times J} x_{i}+y_{j}}$ |

Table 3: The sets of axioms $E_{\mathcal{C S L}}$ for complete semilattices (top) and $E_{\mathcal{D}}$ for distributivity with $S$-semimodules (bottom). Note that the axiom ( $C I$ ) of $E_{\mathcal{C S L}}$ generalises the usual idempotency and commutativity properties of finitary $\sqcup$, while $(A U)$ generalises associativity and unity of $\bigsqcup_{i \in \emptyset} x_{i}=\perp$.
for semimodules (algebras for the monad $\mathcal{S}$ ) and those for complete semilattices (algebras for the monad $\mathcal{P}$ ), provide an algebraic theory presenting the monad $\mathcal{P}_{c} \mathcal{S}$ (Theorem 5.5).

We conclude with the finite powerset monad $\mathcal{P}_{f}$. By replacing, in the above theory, complete semilattices with semilattices (algebras for the monad $\mathcal{P}_{f}$ ) one obtains a theory presenting the monad $\mathcal{P}_{f c} \mathcal{S}$ of finitely generated convex subsets (Theorem 6.5), which is formally defined as a restriction of the canonical $\mathcal{P}_{c} \mathcal{S}$. The corresponding algebraic theory is exactly the one in Table 1.

This article is an extend version of [BS21]. Beyond providing all proofs which could not be published in [BS21] because of space limitations, we illustrate here a detailed example (Example 3.8) that gives a sketch of the proof of the fact that the canonical weak distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$ can be expressed in terms of a convex closure when $S$ is a positive semifield and, in Remark 4.4, we explain that even when $S$ only satisfies the minimal assumptions of Theorem 3.1, convexity still plays a crucial role. Moreover, we discuss at the end of Section 6 how a natural candidate for a weak distributive law of $\mathcal{P}_{f}$ over $\mathcal{S}$ fails to work, thus justifying our approach via the full powerset monad. Finally, in Remark 2.4 and Appendix A, we show the existence of a weak extension of the powerset functor to $\mathbb{R e l}$ that is not locally monotone, and thus is not an extension in the sense of Barr [Bar70]. Barr extensions, if they exist, are unique, but this is not true for arbitrary extensions to $\mathbb{R}$ el. Since weak distributive laws are in bijective correspondence with these latter, simpler extensions, this shows that one could have various weak distributive laws of $\mathcal{P}$ over a monad $T$.

Notation. We assume the reader to be familiar with monads and their maps. Given a monad $\left(M, \eta^{M}, \mu^{M}\right)$ on $\mathbb{C}, \mathbb{E M}(M)$ and $\mathbb{K}(M)$ denote, respectively, the Eilenberg-Moore category and the Kleisli category of $M$. The latter is defined as the category whose objects are the same as $\mathbb{C}$ and a morphism $f: X \rightarrow Y$ in $\mathbb{K}(M)$ is a morphism $f: X \rightarrow M(Y)$ in $\mathbb{C}$. We write $U^{M}: \mathbb{E M}(M) \rightarrow \mathbb{C}$ and $U_{M}: \mathbb{K} 1(M) \rightarrow \mathbb{C}$ for the canonical forgetful functors, and
$F^{M}: \mathbb{C} \rightarrow \mathbb{E M}(M), F_{M}: \mathbb{C} \rightarrow \mathbb{K}(M)$ for their respective left adjoints. Recall, in particular, that $F^{M}(X)=\left(X, \mu_{X}^{M}\right)$ and, for $f: X \rightarrow Y, F^{M}(f)=M(f)$.

If $f: X \rightarrow Y$ is in Set, we denote its graph as $\Gamma(f)=\{(x, f(x)) \mid x \in X\}$, which is a morphism from $X$ to $Y$ in the category $\mathbb{R}$ el of sets and relations. If $R \subseteq X \times Y$, then we write $R^{o}$ for its opposite relation given by $R^{o}=\{(y, x) \mid(x, y) \in R\}$.

Given $n$ a natural number, we denote by $\underline{n}$ the set $\{1, \ldots, n\}$.

## 1. (Weak) Distributive LaWs

Given two monads $S$ and $T$ on a category $\mathbb{C}$, is there a way to compose them to form a new monad $S T$ on $\mathbb{C}$ ? This question was answered by Beck [Bec69] and his theory of distributive laws, which are natural transformations $\delta: T S \rightarrow S T$ satisfying four axioms and that provide a canonical way to endow the composite functor $S T$ with a monad structure. We begin by recalling the classic definition. In the following, let $\left(T, \eta^{T}, \mu^{T}\right)$ and $\left(S, \eta^{S}, \mu^{S}\right)$ be two monads on a category $\mathbb{C}$.
Definition 1.1. A distributive law of the monad $S$ over the monad $T$ is a natural transformation $\delta: T S \rightarrow S T$ such that the following diagrams commute.


One important result of Beck's theory is the correspondence between distributive laws, liftings to Eilenberg-Moore algebras and extensions to Kleisli categories, in the following sense.

Definition 1.2. Let $\left(T, \eta^{T}, \mu^{T}\right)$ and $\left(S, \eta^{S}, \mu^{S}\right)$ be two monads on a category $\mathbb{C}$.
(1) A lifting of the functor $S$ to $\mathbb{E M}(T)$ is a functor $\tilde{S}$ on $\mathbb{E M}(T)$ such that

commutes.
(2) An extension of the functor $T$ to $\mathbb{K}(S)$ is a functor $\tilde{T}$ on $\mathbb{K} l(S)$ such that

commutes.
Let also:

- $S_{1}, S_{2}, T_{1}$ and $T_{2}$ be functors on $\mathbb{C}$,
- $\tilde{S}_{1}$ and $\tilde{S}_{2}$ be liftings of $S_{1}$ and $S_{2}$ to $\mathbb{E M}(T)$,
- $\tilde{T}_{1}$ and $\tilde{T}_{2}$ be extensions of $T_{1}$ and $T_{2}$ to $\mathbb{K}(S)$,
- $\varphi: S_{1} \rightarrow S_{2}$ and $\psi: T_{1} \rightarrow T_{2}$ be natural transformations.
(3) A lifting of $\varphi$ to $\mathbb{E M}(T)$ is a natural transformation $\tilde{\varphi}: \tilde{S}_{1} \rightarrow \tilde{S}_{2}$ such that $U^{T} \tilde{\varphi}=\varphi U^{T}$.
(4) An extension of $\psi$ to $\mathbb{K} 1(S)$ is a natural transformation $\tilde{\psi}: \tilde{T}_{1} \rightarrow \tilde{T}_{2}$ such that $\tilde{\psi} F_{S}=F_{S} \psi$.
(5) A lifting of the monad $\left(S, \eta^{S}, \mu^{S}\right)$ to $\mathbb{E M}(T)$ is a monad $\left(\tilde{S}, \eta^{\tilde{S}}, \mu^{\tilde{S}}\right)$ such that $\tilde{S}, \eta^{\tilde{S}}$ and $\mu^{\tilde{S}}$ are lifting of, respectively, $S, \eta^{S}$ and $\mu^{S}$.
(6) An extension of the monad $\left(T, \eta^{T}, \mu^{T}\right)$ to $\mathbb{K} 1(S)$ is a $\operatorname{monad}\left(\tilde{T}, \eta^{\tilde{T}}, \mu^{\tilde{T}}\right)$ such that $\tilde{T}, \eta^{\tilde{T}}$ and $\mu^{\tilde{T}}$ are extensions of, respectively, $T, \eta^{T}$ and $\mu^{T}$.
Theorem 1.3 [Bec69]. Let $S$ and $T$ be two monads on $\mathbb{C}$. There are bijections between distributive laws of the form $T S \rightarrow S T$, liftings of the monad $S$ to $\mathbb{E} \mathbb{M}(T)$ and extensions of the monad $T$ to $\mathbb{K} l(S)$.
Proof (sketch). Given a distributive law $\delta: T S \rightarrow S T$, the corresponding lifting $\tilde{S}$ of $S$ to $\mathbb{E M}(T)$ is given on objects as

$$
\tilde{S}(X, a: T X \rightarrow X)=\left(S X, T S X \xrightarrow{\delta_{X}} S T X \xrightarrow{S a} S X\right)
$$

and on morphisms as $\tilde{S}(f)=S(f)$; while $\eta^{\tilde{S}}=\eta^{S}$ and $\mu^{\tilde{S}}=\mu^{S}$. Vice versa, given a lifting of the monad $\left(S, \eta^{S}, \mu^{S}\right)$ to $\mathbb{E M}(T)$ with $\tilde{S}(X, a)=\left(S X, \sigma_{X, a}\right)$, the corresponding distributive law $\delta: T S \rightarrow S T$ has components:

$$
\begin{equation*}
\delta_{X}=T S X \xrightarrow{T S \eta_{X}^{T}} T S T X \xrightarrow{\sigma_{T X, \mu_{X}^{T}}} S T X . \tag{1.2}
\end{equation*}
$$

Next, given a distributive law $\delta: T S \rightarrow S T$, the corresponding extension $\tilde{T}$ is given on objects by $\tilde{T}(X)=T(X)$ and on a morphism $f: X \rightarrow S(Y)$ in $\mathbb{C}$ as

$$
\tilde{T}(f)=T(X) \xrightarrow{T(f)} T S(Y) \xrightarrow{\delta_{Y}} S T(Y),
$$

while the $X$-th component of $\eta^{\tilde{T}}$ and $\mu^{\tilde{T}}$ are, respectively,

$$
X \xrightarrow{\eta_{X}^{T}} T(X) \xrightarrow{\eta_{T(X)}^{S}} S T(X) \quad \text { and } \quad T T(X) \xrightarrow{\mu_{X}^{T}} T(X) \xrightarrow{\eta_{T(X)}^{S}} S T(X) .
$$

Vice versa, given an extension $\left(\tilde{T}, \eta^{\tilde{T}}, \mu^{\tilde{T}}\right)$ of $T$ to $\mathbb{K} 1(S)$, consider the identity map in $\mathbb{C}$ $i d_{S(X)}^{\mathbb{C}}$, which is a morphism $i d_{S(X)}^{\mathbb{C}}: S(X) \rightarrow X$ in $\mathbb{K} l(S)$. The image along $\tilde{T}$ provides a morphism $\tilde{T}\left(i d_{S(X)}^{\mathbb{C}}\right): T S(X) \rightarrow T(X)$ in $\mathbb{K}(S)$ and therefore a morphism $T S(X) \rightarrow S T(X)$ in $\mathbb{C}$, which is the $X$-th component of the desired distributive law $\delta$ :

$$
\delta_{X}=\tilde{T}\left(i d_{S(X)}^{\mathbb{C}}\right) .
$$

Given a distributive law $\delta: T S \rightarrow S T$, and therefore a lifting of $S$ to $\mathbb{E M}(T)$, the composite monad on $\mathbb{C}$ is the monad induced by the composite adjunction:


Its underlying functor is $S T$, the unit is $\eta^{S} \eta^{T}$ and the multiplication is $\left(\mu^{S} \mu^{T}\right) \circ(S \delta T)$.
It turns out, however, that distributive laws often cannot exist in many practical examples, see for instance [ZM19]. Böhm [Böh10] and Street [Str09] have studied various weaker notions of distributive law, obtained by relaxing in different ways the axioms required by the definition of distributive law. We are interested in one in particular, as done in [Gar20], obtained by dropping the requirement of commutativity of the bottom-right triangle of (1.1), which concerns $\eta^{T}$.

Definition 1.4. Let $\left(T, \eta^{T}, \mu^{T}\right)$ and $\left(S, \eta^{S}, \mu^{S}\right)$ be two monads on $\mathbb{C}$. A weak distributive law of $S$ over $T$ is a natural transformation $\delta: T S \rightarrow S T$ such that the following diagrams commute:


There are now corresponding notions of weak lifting and weak extensions, whose definitions we recall next.

Definition 1.5. Let $\left(S, \eta^{S}, \mu^{S}\right)$ and $\left(T, \eta^{T}, \mu^{T}\right)$ be two monads on a category $\mathbb{C}$.
(1) A weak extension of the monad $T$ to $\mathbb{K} \mathbb{l}(S)$ is a functor $\tilde{T}: \mathbb{K} 1(S) \rightarrow \mathbb{K} 1(S)$ together with a natural transformation $\mu^{\tilde{T}}: \tilde{T} \tilde{T} \rightarrow \tilde{T}$ such that $\tilde{T}$ and $\mu^{\tilde{T}}$ are extensions, respectively, of $T$ and $\mu^{T}$.
(2) A weak lifting of the monad $S$ to $\mathbb{E M}(T)$ consists of a monad $\left(\tilde{S}, \eta^{\tilde{S}}, \mu^{\tilde{S}}\right)$ on $\mathbb{E M}(T)$ and two natural transformations

$$
U^{T} \tilde{S} \xrightarrow{\iota} S U^{T} \xrightarrow{\pi} U^{T} \tilde{S}
$$

such that $\pi \iota=i d_{U^{T} \tilde{S}}$ and such that the following diagrams commute:


Again we have a bijective correspondence between weak distributive laws and weak extensions, whereas the relationship between weak distributive laws and weak liftings is more subtle. We state it in the following Theorem, which is proved in [Gar20].

Theorem 1.6 [Gar20, Propositions 11, 13]. Let $\left(T, \eta^{T}, \mu^{T}\right)$ and $\left(S, \eta^{S}, \mu^{S}\right)$ be two monads on $\mathbb{C}$. Then:
(i) There is a bijective correspondence between weak distributive laws of type $T S \rightarrow S T$ and weak extensions of $T$ to $\mathbb{K}(S)$.
(ii) If idempotents split in $\mathbb{C}$, then there is a bijective correspondence between weak distributive laws of type $T S \rightarrow S T$ and weak lifting of $S$ to $\mathbb{E M}(T)$.
Proof. (i) is proved exactly as in Theorem 1.3.
For (ii): let $\delta: T S \rightarrow S T$ be a weak distributive law. Then the weak lifting of $S$ to $\mathbb{E M}(T)$ is obtained as follows.

- The action of the functor $\tilde{S}: \mathbb{E M}(T) \rightarrow \mathbb{E M}(T)$ on a $T$-algebra $(X, a)$ is given by the $T$-algebra $(Y, b)$ where:
$-Y$ is the result of splitting the idempotent $S a \circ \delta_{X} \circ \eta_{S X}^{T}: S X \rightarrow S X$ in $\mathbb{C}$ :

$$
S X \xrightarrow{\pi_{(X, a)}} Y \xrightarrow{\iota_{(X, a)}} S X,
$$

$-b$ is the composite:

$$
b=T Y \xrightarrow{T \iota_{(X, a)}} T S X \xrightarrow{\delta_{X}} S T X \xrightarrow{S a} S X \xrightarrow{\pi_{(X, a)}} Y .
$$

- Given $f:(X, a) \rightarrow\left(X^{\prime}, a^{\prime}\right)$ in $\mathbb{E M}(T)$ and

$$
S X^{\prime} \xrightarrow{\pi_{\left(X^{\prime}, a^{\prime}\right)}} Y^{\prime} \xrightarrow{\iota_{\left(X^{\prime}, a^{\prime}\right)}} S X^{\prime}
$$

a splitting of the idempotent $S a^{\prime} \circ \delta_{X^{\prime}} \circ \eta_{S X^{\prime}}^{T}$, the action of the functor $\tilde{S}$ on $f$ is the composite:

$$
\tilde{S}(f)=Y \xrightarrow{\iota_{(X, a)}} S X \xrightarrow{S f} S X^{\prime} \xrightarrow{\pi_{\left(X^{\prime}, a^{\prime}\right)}} Y^{\prime} .
$$

- The $(X, a)$-th components of the two natural transformations $\iota$ and $\pi$ are given as above by splitting the idempotent $S a \circ \delta_{X} \circ \eta_{S X}^{T}$.
- The unit $\eta^{\tilde{S}}$ is the unique map rendering commutative the triangles in (1.3) and (1.4).
- The multiplication $\mu^{\tilde{S}}$ is the unique map rendering commutative the rectangles in (1.3) and (1.4).
Vice versa, suppose we have a weak lifting of $S$ to $\mathbb{E M}(T)$. For each $T$-algebra $(X, a)$ with $\tilde{S}(X, a)=(Y, b)$, define the morphism $\sigma_{X, a}$ as

$$
\sigma_{X, a}=T S X \xrightarrow{T \pi_{(X, a)}} T Y \xrightarrow{b} Y \xrightarrow{\iota_{(X, a)}} S X
$$

and then define $\delta: T S \rightarrow S T$ as the family of morphisms whose $X$-th component is (1.2). Then $\delta$ is a weak distributive law.

As in the "strong" case, given a weak distributive law $\delta: T S \rightarrow S T$, and therefore a weak lifting of $S$ to $\mathbb{E M}(T)$ (when idempotents split in $\mathbb{C}$ ), we can define a "composite" monad $\widetilde{S T}$ as the monad arising from the composite adjunction

whose underlying functor this time is not $S T$, but is obtained by splitting the idempotent

$$
S T \xrightarrow{\eta^{T} S T} T S T \xrightarrow{\delta T} S T T \xrightarrow{S \mu^{T}} S T .
$$

## 2. The Powerset and Semimodule Monads

Through this paper we will take as $S$ and $T$ the powerset and semimodule monads, respectively. In this section we recall their definitions as well as some useful known results.
2.1. The Monad $\mathcal{P}$. Let us now consider, as $S$, the powerset monad $\left(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}}\right.$ ), where $\eta_{X}^{\mathcal{P}}(x)=\{x\}$ and $\mu_{X}^{\mathcal{P}}(\mathcal{U})=\bigcup_{U \in \mathcal{U}} U$. Its algebras are precisely the complete semilattices and we have that $\mathbb{K}(\mathcal{P})$ is isomorphic to the category $\mathbb{R e l}$ of sets and relations. Hence, giving a distributive law $T \mathcal{P} \rightarrow \mathcal{P} T$ is the same as giving an extension of $T$ to $\mathbb{R} e l$.

As $\mathbb{R}$ el is enriched over the category of partial orders, and thus is a 2-category, it is quite natural to require $T$ to be a 2 -functor, i.e. to preserve the partial order: if $R \subseteq S$, then $T(R) \subseteq T(S)$. It is not difficult to prove (see, for example, [BdM97, §5.1]) that any 2 -endofunctor $F$ on $\mathbb{R e l}$ preserves functions and opposite relations, in the sense that if $f$ is a function then so is $F(f)$ and if $R^{o}$ is the opposite relation of $R$, then $F\left(R^{o}\right)=F(R)^{o}$. This and the fact that every relation $R: X \rightarrow Y$ with projections $\pi_{X}: R \rightarrow X$ and $\pi_{Y}: R \rightarrow Y$ can be written as a composite in $\mathbb{R}$ el:

$$
R=\Gamma\left(\pi_{Y}\right) \circ \Gamma\left(\pi_{X}\right)^{o}
$$

(recall that $\Gamma$ computes the graph of functions) imply that

$$
F(R)=F\left(\Gamma\left(\pi_{Y}\right) \circ \Gamma\left(\pi_{X}\right)^{o}\right)=F\left(\Gamma\left(\pi_{Y}\right)\right) \circ F\left(\Gamma\left(\pi_{X}\right)\right)^{o} .
$$

Therefore, any functor $T$ : Set $\rightarrow$ Set has at most one extension to a 2 -functor $\tilde{T}$ on $\mathbb{R e l}$, which is necessarily defined as $T$ on objects and as

$$
\tilde{T}(R)=\Gamma\left(T\left(\pi_{Y}\right)\right) \circ \Gamma\left(T\left(\pi_{X}\right)\right)^{o}
$$

on morphisms. $\tilde{T}$ so defined is called the Barr extension of $T$, as it was discussed first in [Bar70]. The problem is that $\tilde{T}$ is not always functorial: for this to happen the notions of weakly cartesian functor and natural transformation are crucial, and we report them next. First, recall that a commutative diagram

is a weak pullback if for every pair of morphisms ( $u: P \rightarrow X, v: P \rightarrow Y$ ) there exists a (not necessarily unique) morphism $t: P \rightarrow W$ such that $p \circ t=u$ and $q \circ t=v$.

Definition 2.1. A functor $T:$ Set $\rightarrow$ Set is said to be weakly cartesian if and only if it preserves weak pullbacks. A natural transformation $\varphi: F \rightarrow G$ is said to be weakly cartesian if and only if its naturality squares are weak pullbacks.

Kurz and Velebil [KV16] proved, using an original argument of Barr [Bar70], that $\tilde{T}$ is functorial if and only if $T$ is weakly cartesian. Moreover, one can show using a similar technique that a natural transformation $\varphi$ between weakly cartesian functors on Set has at most one extension to a 2-natural transformation, which exists precisely when $\varphi$ is weakly cartesian, as mentioned in [Gar20]. These considerations are summarised in the following Theorem.

Theorem 2.2. Any functor $T$ : Set $\rightarrow$ Set has at most one extension to a 2-functor $\tilde{T}$ on $\mathbb{R e l}$; such an extension exists if and only if $T$ is weakly cartesian. $\tilde{T}$ is defined as $T$ on objects and, for every relation $R: X \rightarrow Y$ with projections $\pi_{X}: R \rightarrow X$ and $\pi_{Y}: R \rightarrow Y$, as

$$
\tilde{T}(R)=\Gamma\left(T\left(\pi_{Y}\right)\right) \circ \Gamma\left(T\left(\pi_{X}\right)\right)^{o}
$$

where $\Gamma(f)$, for $f: A \rightarrow B$ a function, is its graph $\Gamma(f)=\{(a, f(a)) \mid a \in A\}$.
Let $T_{1}$ and $T_{2}$ be weakly cartesian functors. Any natural transformation $\varphi: T_{1} \rightarrow T_{2}$ has at most one extension $\tilde{\varphi}: \tilde{T}_{1} \rightarrow \tilde{T}_{2}$; such an extension exists if and only if $\varphi$ is weakly cartesian. Its $X$-th component, for $X$ a set, is given by the graph of $\varphi_{X}$ :

$$
\tilde{\varphi}_{X}=\Gamma\left(\varphi_{X}\right)=\left\{\left(x, \varphi_{X}(x)\right) \mid x \in T_{1}(X)\right\} .
$$

The following result is therefore immediate.
Proposition 2.3 [Gar20, Corollary 16]. For any monad $\left(T, \eta^{T}, \mu^{T}\right)$ on Set:
(1) If $T, \eta^{T}$ and $\mu^{T}$ are weakly cartesian, then there exists a distributive law of $\mathcal{P}$ over $T$.
(2) If $T$ and $\mu^{T}$ are weakly cartesian, then there exists $a$ weak distributive law of $\mathcal{P}$ over $T$.

We follow [Gar20] and call the (weak) distributive law of Proposition 2.3, arising from the unique extension of $T, \mu^{T}$ and possibly $\eta^{T}$ given by Theorem 2.2 , as the canonical (weak) distributive law of $\mathcal{P}$ over $T$.

Remark 2.4 (Kleisli extensions vs Barr extensions). Theorem 2.2 states that any endofunctor $T$ on Set has at most one extension to a 2 -functor on $\mathbb{R e l}$, seen as a poset-enriched category. However, this is asking for more than a Kleisli extension of $T$ to $\mathbb{K} \mathbf{l}(\mathcal{P})=\mathbb{R}$ el in the sense of Definition 1.2.2, since the latter does not require the extension to be locally monotone, but only to agree with $T$ on Set. Kurz and Velebil [KV16] left an open question, in this regard:

Is there a Kleisli extension of a monad $\left(T, \eta^{T}, \mu^{T}\right)$ to $\mathbb{R}$ el that is not a Barr extension?
We can provide a partial answer to this question: there exists a weak extension of a monad that is not a Barr extension. To illustrate this, we reuse from [BMH96] a nonmonotone extension of a functor on $\mathbb{S e t}$ to $\mathbb{R} e l$. Consider the powerset functor $\mathcal{P}$ itself: define $E: \mathbb{R e l} \rightarrow \mathbb{R e l}$ as $E(X)=\mathcal{P}(X)$ and, for $R \subseteq X \times Y$ a relation, $E(R)$ as the function (notice, not just a relation!):

$$
\begin{aligned}
& \mathcal{P}(X) \longrightarrow \mathcal{E ( R )} \mathcal{P}(Y) \\
& \quad A \longmapsto\{y \in Y \mid \exists a \in A . a R y\}
\end{aligned}
$$

This is a somewhat trivial extension of $\mathcal{P}$ to $\mathbb{R e l}$, as for all functions $f: X \rightarrow Y, E(f)$ has exactly the same definition of $\mathcal{P}(f)$. It is clear, therefore, that $E$ and $\mathcal{P}$ agree on $\mathbb{S e t}$ and it is not difficult to see that $E$ is a functor. However, it is not locally monotone: the main reason being that $E(R)$ is a function, and a function in $\mathbb{R e l}$ is included in another if and only
if they are equal. For a specific counterexample, take $X=\{0\}, Y=\{1,2\}, R=\{(0,1)\}$, $S=\{(0,1),(0,2)\}$. Then $R \subseteq S$ but


It turns out that $E$ is not just an extension of the powerset functor, but it is actually a weak extension of the monad: the interested reader can find a detailed explanation in Appendix A.
2.2. The Monad $\mathcal{S}$. Recall that a semiring is a tuple $(S,+, \cdot, 0,1)$ such that $(S,+, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, $\cdot$ distributes over + and 0 is an annihilating element for $\cdot$. In other words, a semiring is a ring where not every element has an additive inverse. Natural numbers $\mathbb{N}$ with the usual operations of addition and multiplication form a semiring. Similarly, integers, rationals and reals form semirings. Also the booleans $\mathbb{B}$ ool $=\{0,1\}$ with $\vee$ and $\wedge$ acting as + and $\cdot$, respectively, form a semiring.

Every semiring $S$ generates a semimodule monad $\mathcal{S}$ on $\mathbb{S e t}$ as follows. Given a set $X$, $\mathcal{S}(X)=\{\varphi: X \rightarrow S \mid \operatorname{supp} \varphi$ finite $\}$, where $\operatorname{supp} \varphi=\{x \in X \mid \varphi(x) \neq 0\}$. For $f: X \rightarrow Y$, define for all $\varphi \in \mathcal{S}(X)$

$$
\mathcal{S}(f)(\varphi)=\left(y \mapsto \sum_{x \in f^{-1}\{y\}} \varphi(x)\right): Y \rightarrow S
$$

This makes $\mathcal{S}$ a functor. The unit $\eta_{X}^{\mathcal{S}}: X \rightarrow \mathcal{S}(X)$ is given by $\eta_{X}^{\mathcal{S}}(x)=\Delta_{x}$, where $\Delta_{x}$ is the Dirac function centred in $x$, while the multiplication $\mu_{X}^{\mathcal{S}}: \mathcal{S}^{2}(X) \rightarrow \mathcal{S}(X)$ is defined for all $\Psi \in \mathcal{S}^{2}(X)$ as

$$
\mu_{X}^{\mathcal{S}}(\Psi)=\left(x \mapsto \sum_{\varphi \in \text { supp } \Psi} \Psi(\varphi) \cdot \varphi(x)\right): X \rightarrow S
$$

An algebra for $\mathcal{S}$ is precisely a left-S-semimodule, namely a set $X$ equipped with a binary operation + , an element 0 and a unary operation $\lambda$. for each $\lambda \in S$, satisfying the equations in Table 1, right. Indeed, if $X$ carries a semimodule structure then one can define a map $a: \mathcal{S} X \rightarrow X$ as, for $\varphi \in \mathcal{S} X$,

$$
\begin{equation*}
a(\varphi)=\sum_{x \in X} \varphi(x) \cdot x \tag{2.1}
\end{equation*}
$$

where the above sum is finite because so is $\operatorname{supp} \varphi$. Vice versa, if $(X, a)$ is an $\mathcal{S}$-algebra, then the corresponding left-semimodule structure on $X$ is obtained by defining for all $\lambda \in S$ and $x, y \in X$

$$
\begin{equation*}
x+{ }^{a} y=a(x \mapsto 1, y \mapsto 1), \quad 0^{a}=a(\varepsilon), \quad \lambda \cdot{ }^{a} x=a(x \mapsto \lambda) . \tag{2.2}
\end{equation*}
$$

Above and in the remainder of the paper, we write the list ( $x_{1} \mapsto s_{1}, \ldots, x_{n} \mapsto s_{n}$ ) for the only function $\varphi: X \rightarrow S$ with support $\left\{x_{1}, \ldots, x_{n}\right\}$ mapping $x_{i}$ to $s_{i}$ and we write the empty list $\varepsilon$ for the function constant to 0 . For instance, for $a=\mu_{X}^{\mathcal{S}}: \mathcal{S S} X \rightarrow \mathcal{S} X$, the left-semimodule structure is defined for all $\varphi_{1}, \varphi_{2} \in \mathcal{S} X$ and $x \in X$ as

$$
\left(\varphi_{1}+\mu_{X}^{\mathcal{S}} \varphi_{2}\right)(x)=\varphi_{1}(x)+\varphi_{2}(x), \quad 0^{\mu_{X}^{\mathcal{S}}}(x)=0, \quad\left(\lambda . \mu_{X}^{\mathcal{S}} \varphi_{1}\right)(x)=\lambda \cdot \varphi_{1}(x)
$$

Proposition 2.3 tells us when a canonical (weak) distributive law of the form $T \mathcal{P} \rightarrow \mathcal{P} T$ exists for an arbitrary monad $T$ on $\operatorname{Set}$. Take then $T=\mathcal{S}$ : when are the functor $\mathcal{S}$ and the

| Positive | $a+b=0 \Longrightarrow a=0=b$ |
| :---: | :--- |
| Semifield | $a \neq 0 \Longrightarrow \exists x \cdot a \cdot x=x \cdot a=1$ |
| Refinable | $a+b=c+d \Longrightarrow \exists x, y, z, t \cdot x+y=a, z+t=b, x+z=c, y+t=d$ |
| (A) | $a+b=1 \Longrightarrow a=0$ or $b=0$ |
| (B) | $a \cdot b=0 \Longrightarrow a=0$ or $b=0$ |
| (C) | $a+c=b+c \Longrightarrow a=b$ |
| (D) | $\forall a, b . \exists x \cdot a+x=b$ or $b+x=a$ |
| (E) | $a+b=c \cdot d \Longrightarrow \exists t:\left\{(x, y) \in S^{2} \mid x+y=d\right\} \rightarrow S$ such that |
|  | $\sum_{x+y=d} t(x, y) x=a, \sum_{x+y=d} t(x, y) y=b, \sum_{x+y=d} t(x, y)=c$. |

Table 4: Definition of some properties of a semiring $S$. Here $a, b, c, d \in S$.
natural transformations $\eta^{\mathcal{S}}$ and $\mu^{\mathcal{S}}$ weakly cartesian? The answer has been given in [CHJ14] (see also [GS01]), where a complete characterisation in purely algebraic properties for $S$ is provided. In Table 4 we recall such properties.
Theorem 2.5 [CHJ14]. Let $S$ be a semiring.
(1) The functor $\mathcal{S}$ is weakly cartesian if and only if $S$ is positive and refinable.
(2) $\eta^{\mathcal{S}}$ is weakly cartesian if and only if $S$ enjoys (A) in Table 4.
(3) If $\mathcal{S}$ is weakly cartesian, then $\mu^{\mathcal{S}}$ is weakly cartesian if and only if $S$ enjoys (B) and (E) in Table 4.

Remark 2.6. In [CHJ14, Proposition 9.1] it is proved that if $S$ enjoys (C) and (D), then $S$ is refinable; if $S$ is a positive semifield, then it enjoys (B) and (E).

In the next Proposition we prove that if $S$ is a positive semifield then it is also refinable, hence $\mathcal{S}$ and $\mu^{\mathcal{S}}$ are weakly cartesian.

Proposition 2.7. If $S$ is a positive semifield, then it is refinable.
Proof. Let $a, b, c$ and $d$ in $S$ be such that $a+b=c+d$. If $a+b=0$, then take $x=y=z=t=0$, otherwise take

$$
x=\frac{a c}{c+d}, \quad y=\frac{a d}{c+d}, \quad z=\frac{b c}{c+d}, \quad t=\frac{b d}{c+d} .
$$

Then $x+y=a, z+t=b, x+z=c, y+t=d$.
Example 2.8. It is known that, for $\mathcal{S}=\mathbb{N}$, a distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$ exists. Indeed one can check that all conditions of Theorem 2.5 are satisfied, therefore we can apply Proposition 2.3.1. In this case, the monad $\mathcal{S} X$ is naturally isomorphic to the commutative monoid monad, which given a set $X$ returns the collection of all finite multisets of elements of $X$. The law $\delta$ is well known (see e.g. [Gar20, HNPR06]): given a finite multiset $\left\langle A_{1}, \ldots, A_{n}\right\rangle$ of subsets of $X$ in $\mathcal{S P} X$, where the $A_{i}$ 's need not be distinct, it returns the set of multisets $\left\{\left\langle a_{1}, \ldots, a_{n}\right\rangle \mid a_{i} \in A_{i}\right\}$.

Theorem 2.5 together with Proposition 2.3.1 tell us that whenever the element 1 of $S$ can be decomposed as a non-trivial sum we do not have a canonical distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$. Semirings with this property abound, for example $\mathbb{Q}, \mathbb{R}, \mathbb{R}^{+}$with the usual operations of sum an multiplication, as well as $\mathbb{B o o l}$ (since $1 \vee 1=1$ ). These semirings are precisely those for which the notion of convex subset of their left-semimodules is non-trivial.

For the existence of the canonical weak distributive law, however, this condition on $1_{S}$ is not required: convexity will indeed play a crucial role in the definition of the weak distributive law.
Definition 2.9. Let $S$ be a semiring, $X$ an $S$-left-semimodule and $A \subseteq X$. The convex closure of $A$ is the set

$$
\bar{A}=\left\{\sum_{i=1}^{n} \lambda_{i} \cdot a_{i} \mid n \in \mathbb{N}, a_{i} \in A, \sum_{i=1}^{n} \lambda_{i}=1\right\} \subseteq X
$$

The set $A$ is said to be convex if and only if $A=\bar{A}$.
Recalling that the category of $S$-left-semimodules is isomorphic to $\mathbb{E M}(\mathcal{S})$, we can use (2.1) to translate Definition 2.9 of convex subset of a semimodule into the following notion of convex subset of a $\mathcal{S}$-algebra $a: \mathcal{S} X \rightarrow X$.
Definition 2.10. Let $S$ be a semiring, $(X, a) \in \mathbb{E M}(\mathcal{S}), A \subseteq X$. The convex closure of $A$ in ( $X, a$ ) is the set

$$
\bar{A}^{a}=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \operatorname{supp} \varphi \subseteq A, \sum_{x \in X} \varphi(x)=1\right\}
$$

$A$ is said to be convex in $(X, a)$ if and only if $A=\bar{A}^{a}$. We denote by $\mathcal{P}_{c}^{a} X$ the set of convex subsets of $X$ with respect to $a$.
Remark 2.11. Observe that $\emptyset$ is convex, because $\bar{\emptyset}^{a}=\emptyset$, since there is no $\varphi \in \mathcal{S} X$ with empty support such that $\sum_{x \in X} \varphi(x)=1$.
Example 2.12. Suppose $S$ is such that $\eta^{\mathcal{S}}$ is weakly cartesian (equivalently (A) holds: $x+y=1 \Longrightarrow x=0$ or $y=0$ ), for example $S=\mathbb{N}$, and let $(X, a) \in \mathbb{E M}(\mathcal{S})$. A $\varphi \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi(x)=1$ and supp $\varphi \subseteq A$ is a function that assigns 1 to exactly one element of $A$ and 0 to all the other elements of $X$. These functions are precisely all the $\Delta_{x}$ for those elements $x \in A$. Since $a: \mathcal{S} X \rightarrow X$ is a structure map for an $\mathcal{S}$-algebra, it maps the function $\Delta_{x}$ into $x$. Therefore $\bar{A}^{a}=\left\{a\left(\Delta_{x}\right) \mid x \in A\right\}=\{x \mid x \in A\}=A$. Thus all $A \in \mathcal{P S} X$ are convex.

Example 2.13. When $S=\mathbb{B}$ ool, we have that $\mathcal{S}$ is naturally isomorphic to $\mathcal{P}_{f}$, the finite powerset monad, whose algebras are idempotent commutative monoids or equivalently semilattices with a bottom element (Table 1, left). So, for $(X, a) \in \mathbb{E M}(\mathcal{S})$, a $\varphi \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi(x)=1$ and $\operatorname{supp} \varphi \subseteq A$ is any finitely supported function from $X$ to $\mathbb{B}$ ool that assigns 1 to at least one element of $A$. Intuitively, such a $\varphi$ selects a non-empty finite subset of $A$, then $a(\varphi)$ takes the join of all the selected elements. Thus, $\bar{A}^{a}$ adds to $A$ all the possible joins of non-empty finite subsets of $A: A$ is convex if and only if it is closed under binary joins.

## 3. The Canonical Weak Distributive Law $\delta: \mathcal{S P} \rightarrow \mathcal{P} \mathcal{S}$

Weak extensions of $\mathcal{S}$ to $\mathbb{K} l(\mathcal{P})=\mathbb{R}$ el only consist of extensions of the functor $\mathcal{S}$ and of the multiplication $\mu^{\mathcal{S}}$, for whose existence it is sufficient that $\mathcal{S}$ and $\mu^{\mathcal{S}}$ be weakly cartesian. Hence for semirings $S$ satisfying the criteria given in Theorems 2.5.1 and 2.5.3 we have a canonical weak distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$, arising from the Barr extension of the functor $\mathcal{S}$ to $\mathbb{R e l}$.

Theorem 3.1. Let $S$ be a positive, refinable semiring satisfying (B) and (E) in Table 4. Then there exists a canonical weak distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$ defined for all sets $X$ and $\Phi \in \mathcal{S P} X$ as:

$$
\delta_{X}(\Phi)=\left\{\varphi \in \mathcal{S} X \left\lvert\, \exists \psi \in \mathcal{S}(\ni X) \cdot\left\{\begin{array}{ll}
\forall A \in \mathcal{P} X . \Phi(A)=\sum_{x \in A} \psi(A, x) & (a)  \tag{3.1}\\
\forall x \in X . \varphi(x)=\sum_{A \ni x} \psi(A, x) & (b)
\end{array}\right\}\right.\right.
$$

where $\ni_{X}$ is the set $\{(A, x) \in \mathcal{P} X \times X \mid x \in A\}$.
Proof. We calculate $\delta$ by first analysing the canonical extension $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ to $\mathbb{R e l}$ and then by following the proof of Theorem 1.3.

Theorem 2.2 gives us the formula to extend the functor $\mathcal{S}$ to $\mathbb{R}$ el: on objects it behaves like $\mathcal{S}$; if $R \subseteq A \times B$ is a relation and $\pi_{A}: R \rightarrow A$ and $\pi_{B}: R \rightarrow B$ are the two projections, we have that for all $\psi \in \mathcal{S}(R)$

$$
\mathcal{S}\left(\pi_{A}\right)(\psi)=\left(a \mapsto \sum_{r \in \pi_{A}^{-1}\{a\}} \psi(r)\right)=\left(a \mapsto \sum_{\substack{b \in B \\ a R b}} \psi(a, b)\right)
$$

and similarly for $\mathcal{S}\left(\pi_{B}\right)(\psi)$. So, since $\widetilde{\mathcal{S}}(R)=\Gamma\left(\mathcal{S}\left(\pi_{B}\right)\right) \circ \Gamma\left(\mathcal{S}\left(\pi_{A}\right)\right)^{o}$, we have that

$$
\widetilde{\mathcal{S}}(R)=\left\{\left(\left(a \mapsto \sum_{\substack{b \in B \\ a R b}} \psi(a, b)\right),\left(b \mapsto \sum_{\substack{a \in A \\ a R b}} \psi(a, b)\right)\right) \mid \psi \in \mathcal{S}(R)\right\} .
$$

We now use the proof of Theorem 1.3 to calculate the canonical weak distributive law $\mathcal{S P} \rightarrow \mathcal{P S}$. Recall the identity-on-objects isomorphism of categories $F: \mathbb{K} \mathbf{l}(\mathcal{P}) \rightarrow \mathbb{R e l}$ where for $f: X \rightarrow \mathcal{P}(Y)$ in Set, $F(f)=\{(x, y) \mid y \in f(x)\} \subseteq X \times Y$ and for $R \subseteq X \times Y$, $F^{-1}(R)=(x \mapsto\{y \in Y \mid x R y\}): X \rightarrow \mathcal{P}(Y)$. Consider $i d_{\mathcal{P} X}: \mathcal{P} X \rightarrow \mathcal{P} X$ in Set. This is a Kleisli map $i d_{\mathcal{P} X}: \mathcal{P} X \rightarrow X$ in $\mathbb{K}(\mathcal{P})$, which corresponds to the relation $\ni_{X}: \mathcal{P} X \rightarrow$ $X=\{(A, x) \mid x \in A\}$. Then we have

$$
\widetilde{\mathcal{S}}(\ni X)=\left\{\left(\left(A \mapsto \sum_{x \in A} \psi(A, x)\right),\left(x \mapsto \sum_{A \ni x} \psi(A, x)\right)\right) \mid \psi \in \mathcal{S}(\ni X)\right\} \subseteq \mathcal{S} \mathcal{P} X \times \mathcal{S} X
$$

Remember that $\widetilde{\mathcal{S}}$ and $\mathcal{S}$ coincide on objects. This relation, seen back as a Kleisli map $\widetilde{\mathcal{S}} \mathcal{P} X \rightarrow \widetilde{\mathcal{S}} X=\mathcal{S} \mathcal{P} X \rightarrow \mathcal{S} X$, thus a function $\mathcal{S P} X \rightarrow \mathcal{P} \mathcal{S} X$, gives us the definition of $\delta_{X}$ :

$$
\begin{aligned}
\delta_{X}(\Phi) & =F^{-1}(\widetilde{\mathcal{S}}(\ni X))(\Phi) \\
& =\{\varphi \in \mathcal{S}(X) \mid(\Phi, \varphi) \in \widetilde{\mathcal{S}}(\ni X)\}
\end{aligned}
$$

which coincides with (3.1). By Theorem 1.6, $\delta$ so defined is indeed a weak distributive law.

Example 3.2. Take $S=\mathbb{R}^{+}$with the usual operations of sum and multiplication. Consider $X=\{x, y, z, a, b\}, A_{1}=\{x, y\}, A_{2}=\{y, z\}$ and $A_{3}=\{a, b\}$. Let $\Phi \in \mathcal{S}(\mathcal{P} X)$ be defined as

$$
\Phi=\left(A_{1} \mapsto 5, \quad A_{2} \mapsto 9, \quad A_{3} \mapsto 13\right)
$$

and $\Phi(A)=0$ for all other sets $A \subseteq X$, so supp $\Phi=\left\{A_{1}, A_{2}, A_{3}\right\}$. In order to find an element $\varphi \in \delta_{X}(\Phi)$, we can first take a $\psi \in \mathcal{S}\left(\ni_{X}\right)$ satisfying condition (a) in (3.1) and then compute the $\varphi \in \mathcal{S} X$ using condition (b).

Among the $\psi \in \mathcal{S}\left(\ni_{X}\right)$, consider for instance the following:

$$
\psi=\left(\begin{array}{lllllllll}
\left(A_{1}, x\right) & \mapsto & 2 & \left(A_{2}, y\right) & \mapsto & 4 & \left(A_{3}, a\right) & \mapsto & 6 \\
\left(A_{1}, y\right) & \mapsto & 3 & \left(A_{2}, z\right) & \mapsto & 5 & \left(A_{3}, b\right) & \mapsto & 7
\end{array}\right) .
$$

Since $\Phi\left(A_{1}\right)=\psi\left(A_{1}, x\right)+\psi\left(A_{1}, y\right), \Phi\left(A_{2}\right)=\psi\left(A_{2}, y\right)+\psi\left(A_{2}, z\right)$ and $\Phi\left(A_{3}\right)=\psi\left(A_{3}, a\right)+$ $\psi\left(A_{3}, b\right)$, we have that $\psi$ satisfies condition (a) in (3.1). Condition (b) forces $\varphi$ to be the following:

$$
\varphi=(x \mapsto 2, \quad y \mapsto 3+4, \quad z \mapsto 5, \quad a \mapsto 6, \quad b \mapsto 7) .
$$

Remark 3.3. If $S$ enjoys (A) in Table 4, then the transformation $\delta$ given in (3.1) is actually a distributive law, and for $S=\mathbb{N}$ we recover the well-known $\delta$ of Example 2.8. Example 3.2 can be repeated with $S=\mathbb{N}$ : then $\Phi$ is the multiset where the set $A_{1}$ occurs five times, $A_{2}$ nine times and $A_{3}$ thirteen times. The elements of $\delta_{X}(\Phi)$ are all those multisets containing one element per copy of $A_{1}, A_{2}$ and $A_{3}$ in supp $\Phi$. The $\varphi$ provided indeed contains five elements of $A_{1}$ (two copies of $x$ and three of $y$ ), nine elements of $A_{2}$ (four copies of $y$ and five of $z$ ), thirteen elements of $A_{3}$ (six copies of $a$ and seven of $b$ ).

As Example 3.2 shows, each element $\varphi$ of $\delta_{X}(\Phi)$ is determined by a function $\psi$ choosing for each set $A \in \operatorname{supp} \Phi$ a finite number of elements $x_{1}^{A}, \ldots, x_{m}^{A}$ in $A$ and $s_{1}^{A}, \ldots, s_{m}^{A}$ in $S$ in such a way that $\sum_{j=1}^{m} s_{j}^{A}=\Phi(A)$. The function $\varphi$ maps each $x_{j}^{A}$ to $s_{j}^{A}$ if the sets in supp $\Phi$ are disjoint; if however there are $x_{j}^{A}$ and $x_{k}^{B}$ such that $x_{j}^{A}=x_{k}^{B}$ (like $y$ in Example 3.2), then $x_{j}^{A}$ is mapped to $s_{j}^{A}+s_{k}^{B}$.

Among those $\psi$ 's, there are some special, minimal ones as it were, that choose for each $A$ in supp $\Phi$ exactly one element of $A$, and assign to it $\Phi(A)$. The induced $\varphi$ in $\delta_{X}(\Phi)$ can be described as $\varphi(x)=\sum_{A \in u^{-1}\{x\}} \Phi(A)$ (equivalently $\left.\varphi=\mathcal{S}(u)(\Phi)^{1}\right)$ where $u: \operatorname{supp} \Phi \rightarrow X$ is a function selecting an element of $A$ for each $A \in \operatorname{supp} \Phi$ (that is $u(A) \in A$ ). We denote the set of such $\varphi$ 's by $\mathfrak{c}(\Phi)$.

$$
\begin{equation*}
\mathfrak{c}(\Phi)=\{\mathcal{S}(u)(\Phi) \mid u: \operatorname{supp} \Phi \rightarrow X \text { such that } \forall A \in \operatorname{supp} \Phi . u(A) \in A\} \tag{3.2}
\end{equation*}
$$

Example 3.4. Take $X, A_{1}$ and $A_{2}$ as in Example 3.2, but a different, smaller, $\Phi \in \mathcal{S}(\mathcal{P} X)$ defined as $\Phi=\left(A_{1} \mapsto 1, \quad A_{2} \mapsto 2\right)$. There are only four functions $u$ : supp $\Phi \rightarrow X$ such that $u(A) \in A$ and thus only four functions $\varphi$ in $\mathfrak{c}(\Phi)$ :

$$
\begin{array}{lll}
u_{1}=\left(A_{1} \mapsto x,\right. & \left.A_{2} \mapsto y\right) & \varphi_{1}=(x \mapsto 1, y \mapsto 2) \\
u_{2}=\left(A_{1} \mapsto x,\right. & \left.A_{2} \mapsto z\right) & \varphi_{2}=(x \mapsto 1, z \mapsto 2) \\
u_{3}=\left(A_{1} \mapsto y,\right. & \left.A_{2} \mapsto y\right) & \varphi_{3}=(y \mapsto 3) \\
u_{4}=\left(A_{1} \mapsto y,\right. & \left.A_{2} \mapsto z\right) & \varphi_{4}=(y \mapsto 1, z \mapsto 2)
\end{array}
$$

Observe that the function $\varphi=(x \mapsto 1, y \mapsto 1, z \mapsto 1)$ belongs to $\delta_{X}(\Phi)$ but not to $\mathfrak{c}(\Phi)$. Nevertheless $\varphi$ can be retrieved as the convex combination $\frac{1}{2} \cdot \varphi_{1}+\frac{1}{2} \cdot \varphi_{2}$.

Our key result of this section, Theorem 3.5 below, states that every $\varphi \in \delta_{X}(\Phi)$ can be written as a convex combination (performed in the $\mathcal{S}$-algebra $\left(\mathcal{S} X, \mu_{X}^{\mathcal{S}}\right)$ ) of functions in $\mathfrak{c}(\Phi)$, at least when $S$ is a positive semifield, which by Remark 2.6 and Proposition 2.7 satisfies all the conditions that make (3.1) a weak distributive law. The proof is rather laborious and will be the subject of the rest of this section.

[^1]Theorem 3.5. Let $S$ be a positive semifield. Then for all sets $X$ and $\Phi \in \mathcal{S P} X$

$$
\begin{equation*}
\delta_{X}(\Phi)=\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathcal{S}^{2} X . \sum_{\varphi \in \mathcal{S} X} \Psi(\varphi)=1, \operatorname{supp} \Psi \subseteq \mathfrak{c}(\Phi)\right\}=\overline{\mathfrak{c}}(\Phi)^{\mu_{X}^{\mathcal{S}}} \tag{3.3}
\end{equation*}
$$

Remark 3.6. If we drop the hypothesis of semifield and only have the minimal assumptions of Theorem 3.1, then (3.3) does not hold any more: $S=\mathbb{N}$ is a counterexample. Indeed, in this case every subset of $\mathcal{S} X$ is convex with respect to $\mu_{X}^{\mathcal{S}}$ (see Example 2.12), therefore we would have $\delta_{X}(\Phi)=\mathfrak{c}(\Phi)$, which is false: the function $\varphi$ of Example 3.2 is an example of an element in $\delta_{X}(\Phi) \backslash \mathfrak{c}(\Phi)$.

Remark 3.7. When $S=\mathbb{B}$ ool (which is a positive semifield), the monad $\mathcal{S}$ coincides with the monad $\mathcal{P}_{f}$. The function $\mathfrak{c}(\cdot)$ in (3.2) can then be described as

$$
\mathfrak{c}(\mathcal{A})=\left\{\mathcal{P}_{f}(u)(\mathcal{A}) \mid u: \mathcal{A} \rightarrow X \text { such that } \forall A \in \mathcal{A} . u(A) \in A\right\}
$$

for all $\mathcal{A} \in \mathcal{P}_{f} \mathcal{P} X$. It is worth remarking that this is the transformation $\chi$ appearing in Example 9 of [KR15] (which is in turn equivalent to the one in Example 2.4.7 of [MM07]). This transformation was erroneously supposed to be a distributive law, as it fails to be natural (see [KS18]). However, by taking its convex closure, as displayed in (3.3), one can turn it into a weak distributive law.

The difficult inclusion of (3.3) will be the left-to-right: before giving the proof, we shall present a "working example" which shows what sort of techniques one has to do to find a convex linear combination $\Psi$ in $\mathcal{S}^{2} X$ of elements in $\mathfrak{c}(\Phi)$ for a given $\varphi \in \delta_{X}(\Phi)$ in such a way that $\mu_{X}^{\mathcal{S}}(\Psi)=\varphi$. In particular we will use the possibility of performing divisions in the semiring, whence the required hypothesis for $S$ to be a semifield in Theorem 3.5.

Example 3.8. Let us continue Example 3.2. We have shown that

$$
\varphi=(x \mapsto 2, \quad y \mapsto 3+4, \quad z \mapsto 5, \quad a \mapsto 6, \quad b \mapsto 7)
$$

is an element of $\delta_{X}(\Phi)$ determined by

$$
\psi=\left(\begin{array}{ccccccccc}
\left(A_{1}, x\right) & \mapsto & 2 & \left(A_{2}, y\right) & \mapsto & 4 & \left(A_{3}, a\right) & \mapsto & 6 \\
\left(A_{1}, y\right) & \mapsto & 3 & \left(A_{2}, z\right) & \mapsto & 5 & \left(A_{3}, b\right) & \mapsto & 7
\end{array}\right) \in \mathcal{S}(\ni X) .
$$

We want to write $\varphi$ as a convex linear combination in the $\mathcal{S}$-algebra $\left(\mathcal{S} X, \mu_{X}^{\mathcal{S}}\right)$ of elements in $\mathfrak{c}(\Phi)$, which means we have to find some functions $\chi \in \mathfrak{c}(\Phi)$ and some numbers $\Psi(\chi)$ for all these $\chi$ 's such that their sum is 1 and $\mu_{X}^{\mathcal{S}}(\Psi)=\varphi$. Now, to give a $\chi \in \mathfrak{c}(\Phi)$ we have to choose one element in each $A_{i}$ (which will have to be one of those that $\psi$ above has already "selected"): we have a choice of $x$ or $y$ for $A_{1}, y$ or $z$ for $A_{2}$ and $a$ or $b$ for $A_{3}$. With the aim of finding a general strategy that does not rely on the peculiarities of this example where, for instance, it happens that the number of elements for each $A_{i}$ we can choose from is the same, we shall consider all the possible combinations of choices, and define one $\chi$ for each. The number of these combinations is $2 \cdot 2 \cdot 2$, that is the cardinality of the set $\underline{2} \times \underline{2} \times \underline{2}$, whose elements are indeed triples of indexes 1,2 . So, write

$$
\begin{array}{lll}
x_{1}^{1}=x & x_{1}^{2}=y & x_{1}^{3}=a \\
x_{2}^{1}=y & x_{2}^{2}=z & x_{2}^{3}=b
\end{array}
$$

Now, for each $w=\left(w_{1}, w_{2}, w_{3}\right) \in \underline{2}^{3}$, we define $\chi_{w}$ as the function choosing the $w_{i}$-th element in $A_{i}$ among those given by $\psi$, namely $x_{w_{i}}^{i}$; all these $\chi_{w}$ will always associate to the $w_{i}$-th element of $A_{i}$ the number $\Phi\left(A_{i}\right)$. For example, for $w^{1}=(1,1,1)$, we have

$$
\chi_{w^{1}}=\left(\begin{array}{cc}
x_{1}^{1} \longmapsto & \Phi\left(A_{1}\right) \\
x_{1}^{2} \longmapsto & \Phi\left(A_{2}\right) \\
x_{1}^{3} \longmapsto & \Phi\left(A_{3}\right)
\end{array}\right)
$$

while for $w^{4}=(1,2,2)$ we have

$$
\chi_{w^{4}}=\left(\begin{array}{cc}
x_{1}^{1} \longmapsto & \Phi\left(A_{1}\right) \\
x_{2}^{2} \longmapsto & \Phi\left(A_{2}\right) \\
x_{2}^{3} \longmapsto & \Phi\left(A_{3}\right)
\end{array}\right)
$$

and so on. All in all, the eight functions $\chi_{w}$ 's are:

$$
\begin{aligned}
& \chi_{w^{1}}=\left(\begin{array}{l}
x \longmapsto 5 \\
y \longmapsto 9 \\
a \longmapsto 13
\end{array}\right) \quad \chi_{w^{2}}=\left(\begin{array}{l}
x \longmapsto 5 \\
y \longmapsto 9 \\
b \longmapsto 13
\end{array}\right) \\
& \chi_{w^{3}}=\left(\begin{array}{l}
x \longmapsto 5 \\
z \longmapsto 9 \\
a \longmapsto 13
\end{array}\right) \quad \chi_{w^{4}}=\left(\begin{array}{l}
x \longmapsto 5 \\
z \longmapsto 9 \\
b \longmapsto 13
\end{array}\right) \\
& \chi_{w^{5}}=\left(\begin{array}{l}
y \longmapsto 5 \\
y \longmapsto 9 \\
a \longmapsto 13
\end{array}\right) \quad \chi_{w^{6}}=\left(\begin{array}{l}
y \longmapsto 5 \\
y \longmapsto 9 \\
b \longmapsto 13
\end{array}\right) \\
& \chi_{w^{7}}=\left(\begin{array}{c}
y \longmapsto 5 \\
z \longmapsto 9 \\
a \longmapsto 13
\end{array}\right) \\
& \chi_{w^{8}}=\left(\begin{array}{l}
y \longmapsto 5 \\
z \longmapsto 9 \\
b \longmapsto 13
\end{array}\right)
\end{aligned}
$$

with the understanding that $\chi_{w^{5}}(y)=\chi_{w^{6}}(y)=5+9$.
Now, how should we define $\Psi\left(\chi_{w}\right)$ in such a way that $\sum_{w} \Psi\left(\chi_{w}\right)=1$ ? We shall use the image along $\psi$ of the elements we chose in each $\chi_{w}$, multiplying them all, and dividing by the product of $\Phi\left(A_{i}\right)$, for example,

$$
\Psi\left(\chi_{w^{1}}\right)=\frac{\psi\left(A_{1}, x\right) \cdot \psi\left(A_{2}, y\right) \cdot \psi\left(A_{3}, a\right)}{\Phi\left(A_{1}\right) \cdot \Phi\left(A_{2}\right) \cdot \Phi\left(A_{3}\right)} .
$$

(Here we crucially rely on the fact that $\mathbb{R}^{+}$is a semifield.) So, we have

$$
\begin{array}{ll}
\Psi\left(\chi_{w^{1}}\right)=\frac{2 \cdot 4 \cdot 6}{5 \cdot 9 \cdot 13} & \Psi\left(\chi_{w^{2}}\right)=\frac{2 \cdot 4 \cdot 7}{5 \cdot 9 \cdot 13} \\
\Psi\left(\chi_{w^{3}}\right)=\frac{2 \cdot 5 \cdot 6}{5 \cdot 9 \cdot 13} & \Psi\left(\chi_{w^{4}}\right)=\frac{2 \cdot 5 \cdot 7}{5 \cdot 9 \cdot 13} \\
\Psi\left(\chi_{w^{5}}\right)=\frac{3 \cdot 4 \cdot 6}{5 \cdot 9 \cdot 13} & \Psi\left(\chi_{w^{6}}\right)=\frac{3 \cdot 4 \cdot 7}{5 \cdot 9 \cdot 13} \\
\Psi\left(\chi_{w^{7}}\right)=\frac{3 \cdot 5 \cdot 6}{5 \cdot 9 \cdot 13} & \Psi\left(\chi_{w^{8}}\right)=\frac{3 \cdot 5 \cdot 7}{5 \cdot 9 \cdot 13}
\end{array}
$$

Now, in order to calculate $\mu_{X}^{\mathcal{S}}(\Psi)(y)$, for example, we have two, equivalent algorithms we can follow. The first one scans each $\chi_{w^{k}}$ (with $1 \leq k \leq 8$ ) one at a time, checks each "line" (indexed by $i \in\{1,2,3\}$ ) to see if $y$ appears there, and if it does, writes down the addend $\Psi\left(\chi_{w^{k}}\right) \cdot \chi_{w^{k}}(y)$ (where the second factor is $\Phi\left(A_{1}\right)$ if $y$ appears in the first line, and $\Phi\left(A_{2}\right)$ if it appears in the second), moves to the next line of $\chi_{w^{k}}$, repeats until it has to change $k$. The other algorithm, instead, scans first each $i \in \underline{3}$ and checks for each $k$ if $y$ appears in the $i$-th line of $\chi_{w^{k}}$. If it does, it writes down the addend $\Psi\left(\chi_{w^{k}}\right) \cdot \chi_{w^{k}}(y)$, then moves on to the next function, until it has to change $i$. Following this second approach, we obtain that $\mu_{X}^{\mathcal{S}}(\Psi)(y)$ is equal to:

$$
\begin{array}{ll}
5 \cdot \frac{3 \cdot 4 \cdot 6}{5 \cdot 9 \cdot 13}+5 \cdot \frac{3 \cdot 4 \cdot 7}{5 \cdot 9 \cdot 13}+5 \cdot \frac{3 \cdot 5 \cdot 6}{5 \cdot 9 \cdot 13}+5 \cdot \frac{3 \cdot 5 \cdot 7}{5 \cdot 9 \cdot 13}+ & (i=1, k=5,6,7,8) \\
+9 \cdot \frac{2 \cdot 4 \cdot 6}{5 \cdot 9 \cdot 13}+9 \cdot \frac{2 \cdot 4 \cdot 7}{5 \cdot 9 \cdot 13}+9 \cdot \frac{3 \cdot 4 \cdot 6}{5 \cdot 9 \cdot 13}+9 \cdot \frac{3 \cdot 4 \cdot 7}{5 \cdot 9 \cdot 13} & (i=2, k=1,2,5,6)
\end{array}
$$

Let us look at the first row, corresponding to index $i=1$. There are a few repeated numbers, which we can isolate on the left: there is $5=\Phi\left(A_{1}\right), \frac{1}{5 \cdot 9 \cdot 13}=\frac{1}{\Phi\left(A_{1}\right) \cdot \Phi\left(A_{2}\right) \cdot \Phi(A 3)}$, and there is also $3=\psi\left(A_{1}, y\right)$. Similarly, in the second row, when $i=2$, we can isolate $\Phi\left(A_{2}\right)$, $\frac{1}{\Phi\left(A_{1}\right) \cdot \Phi\left(A_{2}\right) \cdot \Phi(A 3)}$ and $\psi\left(A_{2}, y\right)$. So, we have that $\mu(\Psi)(y)$ is equal to

$$
\begin{aligned}
& 5 \cdot \frac{1}{5 \cdot 9 \cdot 13} \cdot 3 \cdot(4 \cdot 6+4 \cdot 7+5 \cdot 6+5 \cdot 7)+ \\
& +9 \cdot \frac{1}{5 \cdot 9 \cdot 13} \cdot 4 \cdot(2 \cdot 6+2 \cdot 7+3 \cdot 6+3 \cdot 7)
\end{aligned}
$$

Now, by collecting together the same factors, we have that

$$
\begin{equation*}
4 \cdot 6+4 \cdot 7+5 \cdot 6+5 \cdot 7=(4+5) \cdot(6+7) \tag{3.4}
\end{equation*}
$$

which is precisely, by construction, $\Phi\left(A_{1}\right) \cdot \Phi\left(A_{3}\right)$. Hence, by simplifying everything, we obtain

$$
\mu_{X}^{\mathcal{S}}(\Psi)(y)=3+4=\psi\left(A_{1}, y\right)+\psi\left(A_{2}, y\right)=\varphi(y)
$$

In a similar way, one can also prove that $\mu_{X}^{\mathcal{S}}(\Psi)(t)=\varphi(t)$ for all $t \in X$.
The proof of Theorem 3.5 is a direct generalisation of the calculations taken in the previous example. We need a lemma that generalises the distributivity property in an arbitrary semiring, showing how the observation of equation (3.4) can always be done. The statement of the following Lemma makes sense only for commutative semirings, but it can be adapted for arbitrary semirings by restricting it to sets $L$ only of the form $\underline{n}=\{1, \ldots, n\}$.

Lemma 3.9. Let $n \in \mathbb{N}, L \subset \mathbb{N}$ such that $|L|=n$, $\left(s_{k}\right)_{k \in L} \in \mathbb{N}^{n}$, and $\left(\lambda_{j}^{k}\right)$ a family of elements of $S$, with $k \in L$ and $j \in \underline{s_{k}}$. Then

$$
\begin{equation*}
\sum_{w \in \prod_{k \in L} \underline{s_{k}}} \prod_{k \in L} \lambda_{w_{k}}^{k}=\prod_{k \in L} \sum_{j=1}^{s_{k}} \lambda_{j}^{k} . \tag{3.5}
\end{equation*}
$$

(Mind the overloading use of $\Pi$ as a cartesian product of the sets $\underline{s_{k}}$ and also as a product of the elements $\lambda_{w_{k}}^{k}$ and $\sum_{j=1}^{s_{k}} \lambda_{j}^{k}$ in $S$.)
Proof. By induction on $n$. If $n=0$, both sides of (3.5) are 0 .
Let now $n \geq 0$, suppose that the statement of the Lemma holds for $n$, let us consider a $L \subset \mathbb{N}$ such that $|L|=n+1,\left(s_{k}\right)_{k \in L} \in \mathbb{N}^{n+1},\left(\lambda_{j}^{k}\right)_{\substack{k \in L \\ j \in \underline{s_{k}}}}$. Without loss of generality we can assume that $L=\underline{n+1}$. Then we have:

$$
\begin{aligned}
\sum_{\substack{n+1 \\
w \in \prod_{k=1} s \\
s_{k}}} \prod_{k=1}^{n+1} \lambda_{w_{k}}^{k} & =\sum_{j=1}^{s_{n+1}}\left(\sum_{w^{\prime} \in \prod_{k=1}^{n} \underline{s_{k}}}\left(\left(\prod_{k=1}^{n} \lambda_{w_{k}^{\prime}}^{k}\right) \cdot \lambda_{j}^{n+1}\right)\right) \\
& =\sum_{j=1}^{s_{n+1}}\left(\left(\sum_{w^{\prime} \in \prod_{k=1}^{n} s_{k}} \prod_{k=1}^{n} \lambda_{w_{k}^{\prime}}^{k}\right) \cdot \lambda_{j}^{n+1}\right) \\
& =\sum_{j=1}^{s_{n+1}}\left(\left(\prod_{k=1}^{n} \sum_{u=1}^{s_{k}} \lambda_{u}^{k}\right) \cdot \lambda_{j}^{n+1}\right) \\
& =\left(\prod_{k=1}^{n} \sum_{u=1}^{s_{k}} \lambda_{u}^{k}\right) \cdot\left(\sum_{j=1}^{s_{n+1}} \lambda_{j}^{n+1}\right) \\
& =\prod_{k=1}^{n+1} \sum_{j=1}^{s_{k}} \lambda_{j}^{k} .
\end{aligned}
$$

Proof of Theorem 3.5. We shall discuss first some preliminary cases involving the empty set, excluding each time all the cases previously covered. Recall that

$$
\delta_{X}(\Phi)=\left\{\varphi \in \mathcal{S}(X) \left\lvert\, \exists \psi \in \mathcal{S}\left(\ni_{X}\right) \cdot\left\{\begin{array}{l}
\forall A \in \mathcal{P} X . \Phi(A)=\sum_{x \in A} \psi(A, x) \\
\forall x \in X . \varphi(x)=\sum_{A \ni x} \psi(A, x)
\end{array}\right\}\right.\right.
$$

and write, within the scope of this proof, $\delta_{X}^{\prime}(\Phi)=\overline{\mathfrak{c}(\Phi)}{ }^{\mu_{X}^{S}}$, that is:

$$
\delta_{X}^{\prime}(\Phi)=\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathcal{S}^{2} X . \sum_{\chi \in \mathcal{S} X} \Psi(\chi)=1, \operatorname{supp} \Psi \subseteq \mathfrak{c}(\Phi)\right\}
$$

where

$$
\mathfrak{c}(\Phi)=\left\{\chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in \text { supp } \Phi} A \cdot \forall x \in X \cdot \chi(x)=\sum_{\substack{A \in \text { supp } \Phi \\ x=u_{A}}} \Phi(A)\right\}
$$

(this is of course an equivalent formulation of (3.2)). We need to prove that $\delta_{X}(\Phi)=\delta_{X}^{\prime}(\Phi)$. Since there is no possibility of confusion, we shall write $\ni$ instead of $\ni_{X}$ for the rest of the proof. Recall that we denote with $\varepsilon$ the null function, i.e. the function constantly equal to 0 .

Case 1: $\Phi(\emptyset) \neq 0$. We have that $\delta_{X}(\Phi)=\emptyset$ because there is no $\psi \in \mathcal{S}(\ni)$ such that $\Phi(\emptyset)=$ $\sum_{x \in \emptyset} \psi(A, x)=0$. At the same time, also $\delta_{X}^{\prime}(\Phi)=\emptyset$, because $\mathfrak{c}(\Phi)=\emptyset$-since $\prod_{A \in \text { supp } \Phi} A=$ $\emptyset$-hence there is no $\Psi \in \mathcal{S}^{2} X$ with empty support that can satisfy $\sum_{\chi \in \mathcal{S} X} \Psi(\chi)=1$.

Case 2: $X=\emptyset$. (We also assume that $\Phi(\emptyset)=0$ from now on.) We have that $\mathcal{S P} \emptyset=$ $\{\Omega:\{\emptyset\} \rightarrow S\}$, therefore $\Phi=\varepsilon:\{\emptyset\} \rightarrow S$, the null function. Moreover, $\ni \subseteq \mathcal{P}(\emptyset) \times \emptyset=\emptyset$ and $\mathcal{S}(\emptyset)=\{\emptyset: \emptyset \rightarrow S\}$ is the singleton of the empty map, so

$$
\begin{aligned}
\delta_{\emptyset}(\Phi) & =\left\{\varphi \in \mathcal{S}(\emptyset) \mid \exists \psi \in \mathcal{S}(\emptyset) . \forall A \subseteq \emptyset . \Phi(A)=\sum_{x \in \emptyset} \psi(A, x)\right\} \\
& =\{\varphi \in \mathcal{S}(\emptyset) \mid \Phi(\emptyset)=0\} \\
& =\{\emptyset: \emptyset \rightarrow S\}
\end{aligned}
$$

because, by assumption, $\Phi(\emptyset)=0$. On the other hand, we have that, since supp $\Phi=\emptyset$,

$$
\mathfrak{c}(\Phi)=\left\{\chi \in \mathcal{S}(\emptyset) \mid \exists u \in \prod_{A \in \emptyset} A\right\}=\mathcal{S}(\emptyset)
$$

because the zero-ary product is a choice of a terminal object of Set, a singleton. So,

$$
\begin{aligned}
\delta_{\emptyset}^{\prime}(\Phi) & =\left\{\mu(\Psi) \mid \Psi \in \mathcal{S}^{2} \emptyset, \sum_{\chi \in \mathcal{S} \emptyset} \Psi(\chi)=1\right\} \\
& =\{\mu(\mathcal{S}(\emptyset) \ni \emptyset \mapsto 1)\}=\{\emptyset: \emptyset \rightarrow S\} .
\end{aligned}
$$

Case 3: $\Phi=\varepsilon: \mathcal{P} X \rightarrow S$. (We also assume that $X \neq \emptyset$ from now on.) We have that the only $\psi \in \mathcal{S}(\ni)$ such that for all $\sum_{x \in A} \psi(A, x)=0$ for all $A \subseteq X$ is the null function, therefore $\delta_{X}(\varepsilon: \mathcal{P} X \rightarrow S)=\{\varepsilon: X \rightarrow S\}$. On the other hand, we have that supp $\Phi=\emptyset$, so

$$
\mathfrak{c}(\Phi)=\left\{\chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in \emptyset} A \cdot \forall x \in X \cdot \chi(x)=\sum_{A \in \emptyset} \Phi(A)\right\}=\{\varepsilon: X \rightarrow S\} .
$$

It follows then that

$$
\begin{aligned}
\delta_{X}^{\prime}(\varepsilon: \mathcal{P} S \rightarrow S) & =\left\{\mu(\Psi) \mid \Psi \in \mathcal{S}^{2} X, \sum_{\chi \in \mathcal{S} X} \Psi(\chi)=1, \operatorname{supp} \Psi \subseteq\{0: X \rightarrow S\}\right\} \\
& =\{\mu(\mathcal{S} X \ni 0 \mapsto 1)\} \\
& =\{(S \ni x \mapsto 1 \cdot 0)\}=\{\varepsilon: X \rightarrow S\}
\end{aligned}
$$

We have now discussed all the preliminary cases. For the rest of the proof, we shall assume that $X \neq \emptyset, \Phi \neq \varepsilon: \mathcal{P} X \rightarrow S$ and $\Phi(\emptyset)=0$.

We first prove $\delta_{X}(\Phi) \subseteq \delta_{X}^{\prime}(\Phi)$. To this end, let $\varphi \in \mathcal{S} X$ and $\psi \in \mathcal{S}(\ni)$ such that $\Phi(A)=\sum_{x \in A} \psi(A, x)$ for all $A \in \mathcal{P} X$ and $\varphi(x)=\sum_{A \ni x} \psi(A, x)$ for all $x \in X$. Observe that:

- for all $(A, x) \in \operatorname{supp} \psi$ we have that $A \in \operatorname{supp} \Phi$ and $x \in \operatorname{supp} \varphi \cap A$,
- for all $A \in \operatorname{supp} \Phi$ there is $x \in \operatorname{supp} \varphi \cap A$ such that $(A, x) \in \operatorname{supp} \psi$,
- for all $x \in \operatorname{supp} \varphi$ there exists $A \in \operatorname{supp} \Phi$ such that $A \ni x$ and $(A, x) \in \operatorname{supp} \psi$.

Hence the first elements of pairs in supp $\psi$ range over all and only elements of $\operatorname{supp} \Phi$, and the second entries of pairs in supp $\psi$ range over all and only elements of supp $\varphi$. In other words, say $\operatorname{supp} \Phi=\left\{A_{1}, \ldots, A_{n}\right\}$ : then we have

$$
\operatorname{supp} \psi=\left\{\left(A_{1}, x_{1}^{1}\right), \ldots,\left(A_{1}, x_{s_{1}}^{1}\right), \ldots,\left(A_{n}, x_{1}^{n}\right), \ldots,\left(A_{n}, x_{s_{n}}^{n}\right)\right\}
$$

where $\bigcup_{i=1}^{n}\left\{x_{1}^{i}, \ldots, x_{s_{i}}^{i}\right\}=\operatorname{supp} \varphi$. Notice that for all $i \in \underline{n}, u, v \in \underline{s_{i}}$, if $u \neq v$ then $x_{u}^{i} \neq x_{v}^{i}$, but we may have $x_{a}^{i}=x_{b}^{j}$ if $i \neq j$, because the $A_{i}$ 's are distinct but not disjoint. We can then write:
(I) $\Phi\left(A_{i}\right)=\sum_{j=1}^{s_{i}} \psi\left(A_{i}, x_{j}^{i}\right)$ for all $i \in \underline{n}$
(II) $\varphi(x)=\sum_{\substack{i \in \underline{n} \\ \exists j \in s_{i} \cdot x=x_{j}^{i}}} \psi\left(A_{i}, x\right)$ for all $x \in X$.

We want to find a convex linear combination $\Psi$ of elements of $\mathcal{S} X$ of the form $\sum_{i=1}^{n} \Phi\left(A_{i}\right) \cdot a_{i}$ for some $a_{i} \in A_{i}$ such that $\mu_{X}^{\mathcal{S}}(\Psi)=\varphi$. Now, for every $i \in \underline{n}$, we have many candidates for $a_{i}$, namely $x_{1}^{i}, \ldots, x_{s_{i}}^{i}$. Given that every $\chi \in \operatorname{supp} \Psi$ can only involve one $x_{j}^{i}$ for every $i$, we shall have as many $\chi^{\prime}$ 's as the number of ways to pick one element of $\left\{x_{1}^{i}, \ldots, x_{s_{i}}^{i}\right\}$ for each $i$ : let then $w \in \prod_{i=1}^{n} \underline{s_{i}}$ (it is a tuple of indexes $w_{i}$ which we are going to use as $j$ 's). Define $\chi_{w}$ as

$$
\chi_{w}\left(x_{w_{1}}^{1}\right)=\Phi\left(A_{1}\right), \ldots, \chi_{w}\left(x_{w_{n}}^{n}\right)=\Phi\left(A_{n}\right)
$$

with the understanding that if $x_{w_{i}}^{i}=x_{w_{j}}^{j}$ then $\chi_{w}\left(x_{w_{i}}^{i}\right)=\Phi\left(A_{i}\right)+\Phi\left(A_{j}\right)$. Written more precisely, we define for all $x \in X$

$$
\chi_{w}(x)=\sum_{\substack{i \in n \\ x=x_{w_{i}}^{i}}} \Phi\left(A_{i}\right) .
$$

We now define $\Psi \in \mathcal{S}^{2} X$ with $\operatorname{supp} \Psi=\left\{\chi_{w} \mid w \in \prod_{i=1}^{n} \underline{s_{i}}\right\}$ that assigns to each $\chi_{w}$ a number such that $\sum_{w} \Psi\left(\chi_{w}\right)=1$. We shall use the previous Lemma to show that by defining

$$
\Psi\left(\chi_{w}\right)=\frac{\prod_{i=1}^{n} \psi\left(A_{i}, x_{w_{i}}^{i}\right)}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)}
$$

for all $w$, or more precisely by defining

$$
\Psi(\chi)=\sum_{\substack{w \in \prod_{i=1}^{n}, s_{i} \\ \chi=\chi w}} \frac{\prod_{i=1}^{n} \psi\left(A_{i}, x_{w_{i}}^{i}\right)}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)}
$$

for all $\chi \in \mathcal{S} X$ we indeed have that $\Psi$ satisfies the conditions of $\delta_{X}^{\prime}(\Phi)$ and $\varphi=\mu_{X}^{\mathcal{S}}(\Psi)$. First we prove that $\Psi$ is in fact a convex linear combination:

$$
\begin{aligned}
& \sum_{\chi \in s u p p} \Psi \\
& \Psi(\chi)=\frac{1}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)} \sum_{w \in \prod_{s_{i}}} \prod_{i=1}^{n} \psi\left(A_{i}, x_{w_{i}}^{i}\right) \\
&=\frac{1}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)} \prod_{i=1}^{n} \sum_{j=1}^{s_{i}} \psi\left(A_{i}, x_{j}^{i}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)} \prod_{i=1}^{n} \Phi\left(A_{i}\right) \quad \text { because of }(\mathrm{I}) \\
& =1
\end{aligned}
$$

Next, notice that for every $w$ the vector $u \in \prod_{i=1}^{n} A_{i}$ required by the definition of $\delta_{X}^{\prime}(\Phi)$ is exactly $\left(x_{w_{1}}^{1}, \ldots, x_{w_{n}}^{n}\right)$. Finally, we compute $\mu_{X}^{\mathcal{S}}(\Psi)(x)$ for an arbitrary $x \in X$. The equations marked with ( $*$ ) will be explained later.

$$
\begin{align*}
& \mu_{X}^{\mathcal{S}}(\Psi)(x)=\sum_{\chi \in \text { supp } \Psi} \Psi(\chi) \cdot \chi(x) \\
& =\sum_{w \in \prod_{k=1}^{n} \underline{s_{k}}}\left[\frac{\prod_{i=1}^{n} \psi\left(A_{i}, x_{w_{i}}^{i}\right)}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)} \sum_{\substack{i \in n_{n}^{i} \\
x=x_{w_{i}}}} \Phi\left(A_{i}\right)\right]  \tag{1}\\
& =\sum_{w \in \prod_{k=1}^{n} \underline{s_{k}}} \sum_{\substack{i \in n_{n}^{i} \\
x=x_{w_{i}}}}\left[\frac{\prod_{k=1}^{n} \psi\left(A_{k}, x_{w_{k}}^{k}\right)}{\prod_{k=1}^{n} \Phi\left(A_{k}\right)} \Phi\left(A_{i}\right)\right] \\
& =\sum_{i \in \underline{n}} \sum_{\substack{w \in \prod_{k=1}^{n} \\
x=x_{w_{i}}^{i}}}\left[\frac{\prod_{k=1}^{n} \psi\left(A_{k}, x_{w_{k}}^{k}\right)}{\prod_{k=1}^{n} \Phi\left(A_{k}\right)} \Phi\left(A_{i}\right)\right] \\
& =\sum_{\substack{i \in \underline{n} \\
\exists j \in \underline{s_{i}} . x=x_{j}^{i}}} \sum_{\substack{w \in \prod_{k=1}^{n} \\
x=x_{w_{i}}^{i}}}\left[\Phi\left(A_{i}\right) \frac{s_{k}}{} \frac{\psi\left(A_{i}, x_{w_{i}}^{i}\right)}{\prod_{k=1}^{n} \Phi\left(A_{k}\right)} \prod_{\substack{k \in n \\
k \neq i}} \psi\left(A_{k}, x_{w_{k}}^{k}\right)\right]  \tag{2}\\
& =\sum_{\substack{i \in n \\
\exists j \in \underline{s_{i}} \cdot x=x_{j}^{i}}} \Phi\left(A_{i}\right) \cdot \frac{\psi\left(A_{i}, x\right)}{\prod_{k=1}^{n} \Phi\left(A_{k}\right)} \cdot \sum_{\substack{w \in \prod_{k=1}^{n} \\
x=x_{w_{i}}}} \prod_{\substack { s_{k} \\
\begin{subarray}{c}{k{ s _ { k } \\
\begin{subarray} { c } { k } }\end{subarray}} \psi\left(A_{k}, x_{w_{k}}^{k}\right)
\end{align*}
$$

Now, use Lemma 3.9 with $L=\{1, \ldots, i-1, i+1, \ldots, n\}$ and $\lambda_{j}^{k}=\psi\left(A_{k}, x_{j}^{k}\right)$ in the following chain of equations:

$$
\begin{aligned}
\sum_{\substack{w \in \prod_{k=1}^{n} \\
x=x_{w_{i}}}} \prod_{\substack{s_{k} \\
k}} \psi\left(A_{k}, x_{w_{k}}^{k}\right) & =\sum_{\substack{w^{\prime} \in \prod_{k \in L}}} \prod_{k \in L} \psi\left(A_{k}, x_{w_{k}^{\prime}}^{k}\right) \\
& =\prod_{k \in L} \sum_{j=1}^{s_{k}} \psi\left(A_{k}, x_{j}^{k}\right) \\
& =\prod_{\substack{k \in n \\
k \neq i}} \sum_{j=1}^{s_{k}} \psi\left(A_{k}, x_{j}^{k}\right) \\
& =\prod_{\substack{k \in n \\
k \neq i}} \Phi\left(A_{k}\right)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{array}{rlr}
\mu_{X}^{\mathcal{S}}(\Phi) & =\sum_{\substack{i \in \underline{n} \\
\exists j \in \underline{s i n}^{2} \cdot x=x_{j}^{i}}} \Phi\left(A_{i}\right) \cdot \frac{\psi\left(A_{i}, x\right)}{\prod_{k=1}^{n} \Phi\left(A_{k}\right)} \cdot \prod_{\substack{k \in \underline{n} \\
k \neq i}} \Phi\left(A_{k}\right) & \\
& =\sum_{\substack{i \in \underline{n} \\
\exists j \in \in \underline{s i n}_{i} \cdot x=x_{j}^{i}}} \psi\left(A_{i}, x\right) & \\
& =\varphi(x) & \text { because of (II). }
\end{array}
$$

It remains to explain equations $\left(*_{1}\right)$ and $\left(*_{2}\right)$ above. The latter is simply due to the fact that if $i$ is such that there is no $j$ for which $x=x_{j}^{i}$, then there is not a $w$ such that $x=x_{w_{i}}^{i}$ either, and vice versa. The former is more delicate. It may be the case that $\chi_{w}=\chi_{w^{\prime}}$ for $w \neq w^{\prime}$. So, let us write

$$
\operatorname{supp} \Psi=\left\{\chi_{w^{1}}, \ldots, \chi_{w^{m}}\right\}
$$

where the $\chi_{w^{l}}$ are now all distinct. Then we have

$$
\begin{aligned}
& \sum_{\chi \in s u p p} \Psi \\
& \Psi(\chi) \cdot \chi(x)
\end{aligned}=\sum_{l=1}^{m} \Psi\left(\chi_{w^{l}}\right) \cdot \chi_{w^{l}}(x) \quad\left(\sum_{l=1}^{m} \frac{\prod_{i=1}^{n} \psi\left(A_{i}, x_{w_{i}}^{i}\right)}{\prod_{i=1}^{n} \Phi\left(A_{i}\right)}\right) \chi_{w^{l}}(x) .
$$

which is the right-hand side of $\left(*_{1}\right)$.
The other inclusion, $\delta_{X}^{\prime}(\Phi) \subseteq \delta_{X}(\Phi)$, is easier. Let $\Psi \in \mathcal{S}^{2} X$ be such that $\sum_{\chi \in \mathcal{S} X} \Psi(\chi)=$ 1 and

$$
\operatorname{supp} \Psi \subseteq\left\{\chi \in \mathcal{S} X \mid \exists u \in \prod_{A \in \operatorname{supp} \Phi} A . \forall x \in X \cdot \chi(x)=\sum_{\substack{A \in \operatorname{supp} \Phi \\ x=u_{A}}} \Phi(A)\right\} .
$$

Write again supp $\Phi=\left\{A_{1}, \ldots, A_{n}\right\}$ and let $\operatorname{supp} \Psi=\left\{\chi_{1}, \ldots, \chi_{m}\right\}$. For all $j \in \underline{m}$, let $u^{j} \in \prod_{i=1}^{n} A_{i}$ be such that $\chi_{j}(x)=\sum_{i \in \underline{n}} \Phi\left(A_{i}\right)$. Then we have

$$
\mu_{X}^{\mathcal{S}}(\Psi)(x)=\sum_{j=1}^{m} \Psi\left(\chi_{j}\right) \chi_{j}(x)=\sum_{j=1}^{m} \Psi\left(\chi_{j}\right) \sum_{\substack{i \in \underline{n} \\ x=u_{i}^{j}}} \Phi\left(A_{i}\right)
$$

Define then

$$
\psi(B, x)= \begin{cases}\sum_{x \in \underline{m}} \Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right) & \exists(!) i \in \underline{n} \cdot B=A_{i} \\ 0 & \text { otherwise }\end{cases}
$$

We have $\operatorname{supp} \psi=\left\{\left(A_{i}, u_{i}^{j}\right) \mid i \in \underline{n}, j \in \underline{m}\right\}$ is finite and so $\psi \in \mathcal{S}(\ni)$. We now verify the two conditions required in the definition of $\delta_{X}(\Phi)$. For all $x \in X$ :

$$
\begin{aligned}
\sum_{A \ni x} \psi(A, x) & =\sum_{i=1}^{n} \psi\left(A_{i}, x\right)=\sum_{i=1}^{n} \sum_{\substack{j \in \underline{m} \\
x=u_{i}^{j}}} \Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right)=\sum_{j=1}^{m} \sum_{\substack{i \in n \\
x=u_{i}^{j}}} \Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right) \\
& =\mu_{X}^{\mathcal{S}}(\Psi)(x)
\end{aligned}
$$

while for all $A \subseteq X$, if $A \notin \operatorname{supp} \Phi$, then $\sum_{x \in A}(\psi(A, x))=0=\Phi(A)$ by definition and, for all $i \in \underline{n}$ :

$$
\begin{aligned}
\sum_{x \in A_{i}} \Psi\left(A_{i}, x\right) & =\sum_{x \in A_{i}} \sum_{\substack{j \in \underline{m} \\
x=u_{i}^{j}}} \psi\left(\chi_{j}\right) \Phi\left(A_{i}\right) \stackrel{\left(*_{3}\right)}{=} \sum_{j=1}^{m} \Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right)=\Phi\left(A_{i}\right) \sum_{j=1}^{m} \Psi\left(\chi_{j}\right) \\
& =\Phi\left(A_{i}\right)
\end{aligned}
$$

where equation $\left(*_{3}\right)$ holds because for all $j \in \underline{m}$ the addend $\Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right)$ appears on the left-hand side exactly once, when in the first sum we are using, as $x$, precisely $u_{i}^{j} \in A_{i}$. In other words, given $j$, there is a unique $x \in A_{i}$ such that $x=u_{i}^{j}$, so $\Psi\left(\chi_{j}\right) \Phi\left(A_{i}\right)$ appears in the left-hand side of $\left(*_{3}\right)$ and it does so only once. Therefore $\mu_{X}^{\mathcal{S}}(\Psi) \in \delta_{X}(\Phi)$, and the proof is complete.

## 4. The Canonical Weak Lifting of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$

By exploiting the characterisation of the weak distributive law $\delta$ for positive semifields (Theorem 3.5), we can now describe the weak lifting of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$ generated by $\delta$, by listing all the data required by Definition 1.5.

Let $(X, a: \mathcal{S} X \rightarrow X)$ be a $\mathcal{S}$-algebra. Define $\tilde{\mathcal{P}}(X, a)=\left(\mathcal{P}_{c}^{a} X, \alpha_{a}\right)$, where recall from Definition 2.10 that $\mathcal{P}_{c}^{a} X$ is the set of convex subsets of $X$ with respect to $a$, while the function $\alpha_{a}: \mathcal{S P}_{c}^{a} X \rightarrow \mathcal{P}_{c}^{a} X$ is defined for all $\Phi \in \mathcal{S P}_{c}^{a} X$ as

$$
\begin{equation*}
\alpha_{a}(\Phi)=\{a(\varphi) \mid \varphi \in \mathfrak{c}(\Phi)\} . \tag{4.1}
\end{equation*}
$$

To be completely formal, above we should have written $\mathfrak{c}(\mathcal{S}(\iota)(\Phi))$ in place of $\mathfrak{c}(\Phi)$, but it is immediate to see that the two sets coincide. Proving that $\alpha_{a}: \mathcal{S P}_{c}^{a} X \rightarrow \mathcal{P}_{c}^{a} X$ is well defined (namely, $\alpha_{a}(\Phi)$ is a convex set) and forms an $\mathcal{S}$-algebra requires some ingenuity and will be shown later in Section 4.1. Next, define $\tilde{\mathcal{P}}$ on morphisms $f:(X, a) \rightarrow\left(X^{\prime}, a^{\prime}\right)$ as

$$
\begin{equation*}
\tilde{\mathcal{P}}(f)(A)=\mathcal{P} f(A) \tag{4.2}
\end{equation*}
$$

for all $A \in \mathcal{P}_{c}^{a} X$. This makes $\tilde{\mathcal{P}}$ an endofunctor on $\mathbb{E M}(\mathcal{S})$. For all $(X, a)$ in $\mathbb{E M}(\mathcal{S})$, $\eta_{(X, a)}^{\tilde{\mathcal{P}}}:(X, a) \rightarrow \tilde{\mathcal{P}}(X, a)$ and $\mu_{(X, a)}^{\tilde{\mathcal{P}}}: \tilde{\mathcal{P}} \tilde{\mathcal{P}}(X, a) \rightarrow \tilde{\mathcal{P}}(X, a)$ are defined for $x \in X$ and $\mathcal{A} \in \mathcal{P}_{c}^{\alpha_{a}}\left(\mathcal{P}_{c}^{a} X\right)$ as

$$
\begin{equation*}
\eta_{(X, a)}^{\tilde{\mathcal{P}}}(x)=\{x\} \quad \text { and } \quad \mu_{(X, a)}^{\tilde{\mathcal{P}}}(\mathcal{A})=\bigcup_{A \in \mathcal{A}} A . \tag{4.3}
\end{equation*}
$$

Notice that $\{x\}$ is of course a convex subset of $X$ with respect to the structure map $a$, but it is not at all obvious that $\bigcup_{A \in \mathcal{A}} A$ is. In general the union of convex subsets of $X$ is not again convex, but here $\mathcal{A}$ is not an arbitrary family of convex sets: it is a convex subset of $\mathcal{P}_{c}^{a} X$ with respect to $\alpha_{a}$, and this ensures that its union is convex in $X$ with respect to $a$. We will prove this in Proposition 4.6. Finally, define the functions $\iota_{(X, a)}: \mathcal{P}_{c}^{a} X \rightarrow \mathcal{P} X$ and $\pi_{(X, a)}: \mathcal{P} X \rightarrow \mathcal{P}_{c}^{a} X$ as, for all $A \in \mathcal{P}_{c}^{a} X$ and $B \in \mathcal{P} X$,

$$
\begin{equation*}
\iota_{(X, a)}(A)=A \quad \text { and } \quad \pi_{(X, a)}(B)=\bar{B}^{a} \tag{4.4}
\end{equation*}
$$

that is $\iota_{(X, a)}$ is just the obvious set inclusion and $\pi_{(X, a)}$ performs the convex closure in $a$.
Theorem 4.1. Let $S$ be a positive semifield. Then the canonical weak lifting of the powerset monad $\mathcal{P}$ to $\mathbb{E} \mathbb{M}(\mathcal{S})$, determined by the weak distributive law $\delta$ given in (3.3), consists of the monad $\left(\tilde{\mathcal{P}}, \eta^{\tilde{\mathcal{P}}}, \mu^{\tilde{\mathcal{P}}}\right)$ on $\mathbb{E} \mathbb{M}(\mathcal{S})$ defined as in (4.1), (4.2), (4.3) and the natural transformations $\iota: U^{\mathcal{S}} \tilde{\mathcal{P}} \rightarrow \mathcal{P} U^{\mathcal{S}}$ and $\pi: \mathcal{P} U^{\mathcal{S}} \rightarrow U^{\mathcal{S}} \tilde{\mathcal{P}}$ defined as in (4.4).

It is worth spelling out the left-semimodule structure on $\mathcal{P}_{c}^{a} X$ corresponding to the $\mathcal{S}$-algebra $\alpha_{a}: \mathcal{S P}_{c}^{a} X \rightarrow \mathcal{P}_{c}^{a} X$.

Let us start with $\lambda .{ }^{\alpha_{a}} A$ for some $A \in \mathcal{P}_{c}^{a} X$. By (2.2), $\lambda \cdot{ }^{\alpha_{a}} A=\alpha_{a}(\Phi)$ where $\Phi=(A \mapsto \lambda)$. By (4.1), $\alpha_{a}(\Phi)=\{a(\varphi) \mid \varphi \in \mathfrak{c}(\Phi)\}$. Following the definition of $\mathfrak{c}(\Phi)$ given in (3.2), one has to consider functions $u: \operatorname{supp} \Phi \rightarrow X$ such that $u(B) \in B$ for all $B \in \operatorname{supp} \Phi:$ if $\lambda \neq 0$, then $\operatorname{supp} \Phi=\{A\}$ and thus, for each $x \in A$, there is exactly one function $u_{x}: \operatorname{supp} \Phi \rightarrow X$ mapping $A$ into $x$. It is immediate to see that $\mathcal{S}\left(u_{x}\right)(\Phi)$ is exactly the function $(x \mapsto \lambda)$ and thus $a\left(\mathcal{S}\left(u_{x}\right)(\Phi)\right)$ is, by $(2.2), \lambda^{a} x$. Now if $\lambda=0$, then $\operatorname{supp} \Phi=\emptyset$, so there is exactly one function $u: \operatorname{supp} \Phi \rightarrow X$ and $\mathcal{S}(u)(\Phi)$ is the function mapping all $x \in X$ into 0 and thus, by $(2.2), a(\mathcal{S}(u)(\Phi))=0^{a}$. Summarising,

$$
\lambda \cdot{ }^{\alpha_{a}} A= \begin{cases}\left\{\lambda \cdot^{a} x \mid x \in A\right\} & \text { if } \lambda \neq 0  \tag{4.5}\\ \left\{0^{a}\right\} & \text { if } \lambda=0\end{cases}
$$

Following similar lines of thoughts, one can check that

$$
\begin{equation*}
A+{ }^{\alpha_{a}} B=\left\{x+{ }^{a} y \mid x \in A, y \in B\right\} \quad \text { and } \quad 0^{\alpha_{a}}=\left\{0^{a}\right\} \tag{4.6}
\end{equation*}
$$

Remark 4.2. By comparing (4.6) and (4.5) with (4) and (5) in [Jac08], it is immediate to see that our monad $\tilde{\mathcal{P}}$ coincides with a slight variation of Jacobs's convex powerset monad $\mathcal{C}$, the only difference being that we do allow for $\emptyset$ to be in $\mathcal{P}_{c}^{a} X$. Jacobs insisted on the necessity of $\mathcal{C}(X)$ to be the set of non-empty convex subsets of $X$, because otherwise he was not able to define a semimodule structure on $\mathcal{C}(X)$ such that $0 \cdot \emptyset=\left\{0^{a}\right\}$. However, we do manage to do so, since by (4.5), $0 \cdot A=\left\{0^{a}\right\}$ for all $A$ and in particular for $A=\emptyset$. At first sight, this may look like an ad-hoc solution, but this is not the case: it is intrinsic in the definition of the canonical weak lifting of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$, as stated by Theorem 4.1 and shown next.
4.1. Proof of Theorem 4.1. By Theorem 1.6, the weak distributive law (3.1) corresponds to a weak lifting $\tilde{\mathcal{P}}$ of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$, which we are going to show coincides with the data of (4.1)-(4.4). The image along $\tilde{\mathcal{P}}$ of a $\mathcal{S}$-algebra $(X, a)$ will be a set $Y$ together with a structure map $\alpha_{a}$ that makes it a $\mathcal{S}$-algebra in turn. The proof of Theorem 1.6 gives us the
recipe to build $Y$ and $\alpha_{a}$ appropriately. $Y$ is obtained by splitting the following idempotent in Set:

$$
\begin{equation*}
e_{(X, a)}=\mathcal{P} X \xrightarrow{\eta_{\mathcal{P} X}^{S}} \mathcal{S}(\mathcal{P} X) \xrightarrow{\delta_{X}} \mathcal{P}(\mathcal{S} X) \xrightarrow{\mathcal{P} a} \mathcal{P} X \tag{4.7}
\end{equation*}
$$

as a composite $e_{(X, a)}=\iota_{(X, a)} \circ \pi_{(X, a)}$, where $\pi_{(X, a)}$ is the corestriction of $e_{(X, a)}$ to its image and $\iota_{(X, a)}$ is the set-inclusion of the image of $e_{(X, a)}$ into $\mathcal{P} X$. In other words, $Y$ is the set of fixed points of $e_{(X, a)} . \alpha_{a}$ is obtained as the composite

$$
\alpha_{a}=\mathcal{S} Y \xrightarrow{\mathcal{S} \iota_{(X, a)}} \mathcal{S P} X \xrightarrow{\delta_{X}} \mathcal{P S} X \xrightarrow{\mathcal{P} a} \mathcal{P} X \xrightarrow{\pi_{(X, a)}} Y .
$$

Let us, then, fix an $\mathcal{S}$-algebra $(X, a)$. Given $A \in \mathcal{P} X$, we have $\eta_{\mathcal{P} X}^{\mathcal{S}}(A)=\Delta_{A}: \mathcal{P} X \rightarrow S$, the Dirac-function centred in $A$. The set $\delta_{X}\left(\eta_{\mathcal{P} X}^{\mathcal{S}}(A)\right)$ has a simple description, shown in the next Lemma: it consists of formal convex linear combinations of elements of $A$.

Lemma 4.3. For all $A \in \mathcal{P} X$

$$
\begin{equation*}
\delta_{X}\left(\eta_{\mathcal{P} X}^{\mathcal{S}}(A)\right)=\left\{\varphi \in \mathcal{S} X \mid \operatorname{supp} \varphi \subseteq A, \sum_{x \in X} \varphi(x)=1\right\} . \tag{4.8}
\end{equation*}
$$

Proof. Recall that, by definition,

$$
\delta_{X}\left(\eta_{\mathcal{P} X}^{\mathcal{S}}(A)\right)=\left\{\varphi \in \mathcal{S} X \left\lvert\, \exists \psi \in \mathcal{S}(\ni) \cdot\left\{\begin{array}{l}
\forall B \in \mathcal{P} X . \Delta_{A}(B)=\sum_{x \in B} \psi(B, x) \\
\forall x \in X . \varphi(x)=\sum_{B \ni x} \psi(B, x)
\end{array}\right\}\right.\right.
$$

(In this proof we write $\ni$ instead of $\ni_{X}$.) First of all, if $A=\emptyset$, then the left-hand side of (4.8) is empty, because there is no $\psi \in \mathcal{S}(\ni)$ such that $1=\Delta_{\emptyset}(\emptyset)=\sum_{x \in \emptyset} \psi(B, x)=0$, and so is the right-hand side, as the only function whose support is empty is the null function, which cannot satisfy $\sum_{x \in X} \varphi(x)=1$.

Suppose then that $A \neq \emptyset$. For the left-to-right inclusion of (4.8): observe first of all that $\operatorname{supp} \varphi \neq \emptyset$, because otherwise we would have that $\psi(B, x)=0$ for all $x \in X$ and for all $B \ni x$ due to the fact that $S$ is a positive semiring. This would then lead to the contradiction $1=\Delta_{A}(A)=\sum_{x \in A} \psi(A, x)=0$. Let then $x \in \operatorname{supp} \varphi$ : then $\varphi(x) \neq 0$, hence there exists $B \ni x$ such that $\psi(B, x) \neq 0$. It is necessarily the case, however, that $B=A$, because if $B \neq A$, then $0=\Delta_{A}(B)=\sum_{y \in B} \psi(B, y) \geq \psi(B, x) \neq 0$, which is a contradiction. Thus supp $\varphi \subseteq A$. Moreover,

$$
\sum_{x \in X} \varphi(x)=\sum_{x \in A} \varphi(x)=\sum_{x \in A} \sum_{B \ni x} \psi(B, x) \stackrel{(*)}{=} \sum_{x \in A} \psi(A, x)=\Delta_{A}(A)=1
$$

where equation (*) holds because for all $B \neq A$ and for all $y \in B$ we have $\psi(B, y)=0$, hence the only addend of $\sum_{B \ni x} \psi(B, x)$ which is possibly not-null is the " $A$-th one": $\psi(A, x)$.

Vice versa, let $\varphi: X \rightarrow S$ such that $\operatorname{supp} \varphi$ is finite and contained in $A$ and satisfying $\sum_{x \in X} \varphi(x)=1$. Then the map

$$
\begin{gathered}
\ni \xrightarrow{\psi} \mathbb{R}^{+} \\
(B, x) \longmapsto \begin{cases}\varphi(x) & B=A \\
0 & \text { otherwise }\end{cases}
\end{gathered}
$$

has finite support, as this is in bijective correspondence with supp $\varphi$; for all $B \in \mathcal{P} X$, if $B \neq A$ then $\sum_{x \in B} \psi(B, x)=0$, otherwise

$$
\sum_{x \in B} \psi(B, x)=\sum_{x \in A} \varphi(x)=\sum_{x \in X} \varphi(x)=1 ;
$$

and finally, for all $x \in X, \sum_{B \ni x} \psi(B, x)=\psi(A, x)=\varphi(x)$.
The image along $A$ of the idempotent $e$ is therefore

$$
e(A)=\mathcal{P} a\left(\delta_{X}\left(\eta_{\mathcal{P} X}^{\mathcal{S}}(A)\right)\right)=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \operatorname{supp} \varphi \subseteq A, \sum_{x \in X} \varphi(x)=1\right\}=\bar{A}^{a}
$$

Hence the idempotent $e$ computes the convex closure of elements of $\mathcal{P} X$ and its fixed points are precisely the convex subsets of $X$ with respect to the structure map $a$. Therefore, the carrier set of $\tilde{\mathcal{P}}(X, a)$ is precisely $\mathcal{P}_{c}^{a} X$, the natural transformations $\pi$ and $\iota$ are, respectively, the convex closure operator and the set-inclusion of $\mathcal{P}_{c}^{a} X$ into $\mathcal{P} X$ as in (4.4).
Remark 4.4. Lemma 4.3 holds even for semirings satisfying the minimal assumptions of Theorem 3.1: indeed in its proof we do not rely on the characterisation of $\delta$ given in Theorem 3.5 for positive semifields. This means that although for these less well-behaved semirings we cannot, in general, express the canonical weak distributive law in terms of a convex closure, as noted in Remark 3.6, convexity is still deeply woven into the structure and it emerges explicitly in the canonical weak lifting of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$ associated to $\delta$. This is ultimately due to the fact that the element $1 \in S$ plays a protagonist role in the unit $\eta^{\mathcal{S}}$, and forces us to consider only those formal linear combinations whose coefficients add up to 1 , as we saw in the proof of Lemma 4.3.
$\mathcal{P}_{c}^{a} X$ is then equipped with a structure map $\alpha_{a}: \mathcal{S P}_{c}^{a} X \rightarrow \mathcal{P}_{c}^{a} X$ given by

$$
\alpha_{a}=\mathcal{S P}_{c}^{a} X \xrightarrow{\mathcal{S}_{(X, a)}} \mathcal{S} \mathcal{P} X \xrightarrow{\delta_{X}} \mathcal{P} \mathcal{S} X \xrightarrow{\mathcal{P}_{a}} \mathcal{P} X \xrightarrow{\pi_{(X, a)}} \mathcal{P}_{c}^{a} X
$$

Let us try to calculate $\alpha_{a}$ : given $\Phi: \mathcal{P}_{c}^{a} X \rightarrow S$ with finite support, we have that $\mathcal{S}\left(\iota_{(X, a)}\right)(\Phi)$ is just the extension of $\Phi$ to $\mathcal{P} X$ which assigns 0 to each non-convex subset of $X$. If we write $\iota$ instead of $\iota_{(X, a)}$ for short, we have

$$
\begin{equation*}
\alpha_{a}(\Phi)={\overline{\mathcal{P}} a\left(\delta_{X}(\mathcal{S}(\iota)(\Phi))\right)^{a} . . . . . . . .} \tag{4.9}
\end{equation*}
$$

Next, we can use the following technical result, which will also be useful later on and whose proof is in Appendix B.
Proposition 4.5. Let $(X, a)$ be a $\mathcal{S}$-algebra and let $\mathcal{A} \subseteq \mathcal{S} X$. Then

$$
\mathcal{P} a\left(\overline{\mathcal{A}}^{\mu_{X}^{\mathcal{S}}}\right)=\overline{\mathcal{P} a(\mathcal{A})}^{a} .
$$

Since $\delta_{X}(\mathcal{S}(\iota)(\Phi))$ is the convex closure of $\mathfrak{c}(\mathcal{S}(\iota)(\Phi))$ in $\left(\mathcal{S} X, \mu_{X}^{\mathcal{S}}\right)$, by Proposition 4.5 and equation (4.9) we get:

$$
\alpha_{a}(\Phi)=\mathcal{P} a\left({\left.\overline{\delta_{X}(\mathcal{S}(\iota)(\Phi)}\right)^{\mu}}_{\mu_{X}^{\mathcal{S}}}\right)=\mathcal{P} a\left(\delta_{X}(\mathcal{S}(\iota)(\Phi))\right)=\mathcal{P} a\left(\overline{\mathfrak{c}(\mathcal{S}(\iota)(\Phi))}^{\mu_{X}^{\mathcal{S}}}\right) .
$$

It turns out, however, that also the $\mu_{X}^{\mathcal{S}}$-convex closure is superfluous, due to the fact that $\Phi \in \mathcal{S P}{ }_{c}^{a} X$ (and not simply $\mathcal{S P} X$ ), thus obtaining (4.1). We prove this in Proposition B.1, in Appendix B.

We continue the proof of Theorem 4.1. We have computed the action of the weak lifting $\tilde{\mathcal{P}}$ of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{S})$ on objects: given an $\mathcal{S}$-algebra $(X, a), \tilde{\mathcal{P}}(X, a)=\left(\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}} X, \alpha_{a}\right)$ where $\alpha_{a}$ is defined in (4.1). As for morphisms, technically for $f:(X, a) \rightarrow\left(X^{\prime}, a^{\prime}\right)$ in $\mathbb{E M}(\mathcal{S}), \tilde{\mathcal{P}}(f)$ is defined as

$$
\tilde{\mathcal{P}}(f)(\mathcal{A})=\overline{\mathcal{P}}(f)(\mathcal{A})^{a^{\prime}} \quad \text { for all } \mathcal{A} \in \mathcal{P}_{c}^{a} X
$$

however it is not difficult to see that $\mathcal{P}(f)(\mathcal{A})$ is convex in $\left(X^{\prime}, a^{\prime}\right)$ using the fact that $f$ is a morphism of $\mathcal{S}$-algebras and that $\mathcal{A}$ is convex in $(X, a)$.

The unit of the monad $\tilde{\mathcal{P}}$ is given, for every $(X, a)$ object of $\mathbb{E M}(\mathcal{S})$, as the unique morphism $\eta_{(X, a)}^{\tilde{\mathcal{P}}}:(X, a) \rightarrow \tilde{\mathcal{P}}(X, a)$ that makes the two triangles of (1.3) and (1.4) commute, which are in our case:


By definition of $\iota$ and $\eta^{\mathcal{P}}$, the only arrow that makes the left triangle above commutative is necessarily

$$
\begin{aligned}
(X, a) & \xrightarrow{\eta_{(X, a)}^{\tilde{\tilde{p}_{x}}}\left(\mathcal{P}_{c}^{a} X, \alpha_{a}\right)} \\
x & \longmapsto x\}
\end{aligned}
$$

and this makes also the right triangle commutative, since $\overline{\{x\}}^{a}=\{x\}$. Similarly, the multiplication $\mu^{\tilde{\mathcal{P}}}$ is defined, for every $\mathcal{S}$-algebra $(X, a)$, as the unique morphism making the two rectangles of (1.3) and (1.4) commute. These are in our case:


By definition of $\iota$ and $\mu^{\mathcal{P}}$, the only arrow that makes the left rectangle above commutative is necessarily

$$
\begin{aligned}
& \mathcal{P}_{c}^{\alpha_{a}}\left(\mathcal{P}_{c}^{a} X\right) \xrightarrow{\mu_{(X, a)}^{\tilde{\tilde{}}}} \mathcal{P}_{c}^{a} X \\
& \mathcal{A} \bigcup_{A \in \mathcal{A}} A
\end{aligned}
$$

for all $(X, a) \in \mathbb{E M}(\mathcal{S})$. Next we show why this morphism is well defined, namely why $\mu_{(X, a)}^{\tilde{P}}(\mathcal{A})$ is in $\mathcal{P}_{c}^{a} X$, while in Proposition B. 2 (in Appendix B) we prove how it makes the right rectangle commutative. This will conclude the proof of Theorem 4.1.
Proposition 4.6. Let $(X, a)$ be a $\mathcal{S}$-algebra and $\mathcal{A} \in \mathcal{P}_{c}^{\alpha_{a}}\left(\mathcal{P}_{c}^{a} X\right)$. Then $\bigcup_{A \in \mathcal{A}} A$ is convex in $(X, a)$.

Proof. We want to prove the following equality:

$$
\bigcup_{A \in \mathcal{A}} A=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \sum_{x \in X} \varphi(x)=1, \operatorname{supp} \varphi \subseteq \bigcup_{A \in \mathcal{A}} A\right\} .
$$

The left-to-right inclusion is trivial: for all $A \in \mathcal{A}$ and $x \in A, x=a\left(\Delta_{x}\right)$. Vice versa, let $\varphi \in \mathcal{S} X$ be such that $\sum_{x \in X} \varphi(x)=1$ and $\operatorname{supp} \varphi \subseteq \bigcup_{A \in \mathcal{A}} A$. We want to find an element $A \in \mathcal{A}$ such that $a(\varphi) \in A$. Suppose

$$
\operatorname{supp} \varphi=\left\{x_{1}, \ldots, x_{n}\right\}
$$

Then for all $i \in \underline{n}$ there exists $A_{i} \in \mathcal{A}$ such that $x_{i} \in A_{i}$. Notice that if $i \neq j$ then it is not necessarily the case that $A_{i} \neq A_{j}$. Define $\Phi \in \mathcal{S}\left(\mathcal{P}_{c}^{a} X\right)$ as

$$
\Phi(B)=\sum_{\substack{i \in \bar{n}_{i} \\ B=A_{i}}} \varphi\left(x_{i}\right)
$$

for all $B \in \mathcal{P}_{c}^{a} X$. Then $\operatorname{supp} \Phi=\left\{A_{i} \mid i \in \underline{n}\right\}$ is finite and

$$
\sum_{B \in \mathcal{P}_{c}^{a} X} \Phi(B)=\sum_{B \in \mathcal{P}_{c}^{a} X} \sum_{\substack{i \in \underline{n} \\ B=A_{i}}} \varphi\left(x_{i}\right)=\sum_{i \in \underline{n}} \varphi\left(x_{i}\right)=1 .
$$

Since $\mathcal{A}$ is convex in $\left(\mathcal{P}_{c}^{a} X, \alpha_{a}\right)$, we have that $\alpha_{a}(\Phi) \in \mathcal{A}$. This is going to be our desired $A$ : we shall prove that $a(\varphi) \in \alpha_{a}(\Phi)$.

We have $\alpha_{a}(\Phi)=\mathcal{P} a\left(\delta_{X}(\mathcal{S}(i)(\Phi))\right)$, hence, if we prove that $\varphi \in \delta_{X}(\mathcal{S}(i)(\Phi))$, we have finished. To this end, recall the characterisation of $\delta_{X}$ using elements of $\mathcal{S}\left(\ni_{X}\right)$, for $\ni_{X} \subseteq \mathcal{P} X \times X$ :

$$
\begin{aligned}
& \delta_{X}(\mathcal{S}(i)(\Phi))= \\
& \left\{a(\chi) \left\lvert\, \chi \in \mathcal{S} X . \exists \psi \in \mathcal{S}(\ni X) \cdot\left\{\begin{array}{l}
\forall B \subseteq X . \mathcal{S}(i)(\Phi)(B)=\sum_{x \in B} \psi(B, x) \\
\forall x \in X \cdot \chi(x)=\sum_{B \ni x} \psi(B, x)
\end{array}\right\} .\right.\right.
\end{aligned}
$$

Define, for all $(B, x) \in \mathcal{P} X \times X$ such that $B \ni x$ :

$$
\psi(B, x)=\sum_{\substack{i \in n \\(B, x)=\left(A_{i}, x_{i}\right)}} \varphi\left(x_{i}\right) .
$$

Then supp $\psi=\left\{\left(A_{i}, x_{i}\right) \mid i \in \underline{n}\right\}$ is finite, for all $B \subseteq X$

$$
\sum_{x \in B} \psi(B, x)=\sum_{x \in B} \sum_{\substack{i \in n \\(B, x)=\left(A_{i}, x_{i}\right)}} \varphi\left(x_{i}\right)=\sum_{\substack{i \in n \\ B=A_{i}}} \varphi\left(x_{i}\right)=\mathcal{S}(i)(\Phi)(B)
$$

and, for all $x \in X$,

$$
\sum_{B \ni x} \psi(B, x)=\sum_{B \ni x} \sum_{\substack{i \in \underline{n} \\
(B, x)=\left(A_{i}, x_{i}\right)}} \varphi\left(x_{i}\right)=\left\{\begin{array}{ll}
0 & \forall i \in \underline{n} \cdot x \neq x_{i} \\
\varphi\left(x_{i}\right) & \exists(!) i \in \underline{n} \cdot x=x_{i}
\end{array}=\varphi(x)\right.
$$

where if there is $i$ such that $x=x_{i}$, then such $i$ is unique due to the fact that the $x_{j}$ 's are distinct. This proves that $\varphi \in \delta_{X}(\mathcal{S}(i)(\Phi))$.

## 5. The Composite Monad: an Algebraic Presentation

We can now compose the two monads $\mathcal{P}$ and $\mathcal{S}$ by considering the monad arising from the composition of the following two adjunctions:


Direct calculations show that the resulting endofunctor on $\operatorname{Set}$, which we call $\mathcal{P}_{c} \mathcal{S}$, maps a set $X$ and a function $f: X \rightarrow Y$ into, respectively,

$$
\begin{equation*}
\mathcal{P}_{c} \mathcal{S} X=\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}}(\mathcal{S} X) \quad \text { and } \quad \mathcal{P}_{c} \mathcal{S}(f)(\mathcal{A})=\{\mathcal{S}(f)(\Phi) \mid \Phi \in \mathcal{A}\} \tag{5.1}
\end{equation*}
$$

for all $\mathcal{A} \in \mathcal{P}_{c} \mathcal{S} X$. For all sets $X, \eta_{X}^{\mathcal{P}_{c} \mathcal{S}}: X \rightarrow \mathcal{P}_{c} \mathcal{S} X$ and $\mu_{X}^{\mathcal{P}_{c} \mathcal{S}}: \mathcal{P}_{c} \mathcal{S} \mathcal{P}_{c} \mathcal{S} X \rightarrow \mathcal{P}_{c} \mathcal{S} X$ are defined as

$$
\begin{equation*}
\eta_{X}^{\mathcal{P}_{\mathcal{P}} \mathcal{S}}(x)=\left\{\Delta_{x}\right\} \quad \text { and } \quad \mu_{X}^{\mathcal{P}_{\mathcal{S}} \mathcal{S}}(\mathscr{A})=\bigcup_{\Omega \in \mathscr{A}} \alpha_{\mu_{X}^{\mathcal{S}}}(\Omega) \tag{5.2}
\end{equation*}
$$

for all $x \in X$ and $\mathscr{A} \in \mathcal{P}_{c} \mathcal{S} \mathcal{P}_{c} \mathcal{S} X$.
Theorem 5.1. Let $S$ be a positive semifield. Then the canonical weak distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$ given in Theorem 3.5 induces a monad $\mathcal{P}_{c} \mathcal{S}$ on Set with endofunctor, unit and multiplication defined as in (5.1) and (5.2).
Example 5.2. Let $S=\mathbb{B}$ ool. Then the monad $\mathcal{S}$ is isomorphic to the finite powerset monad $\mathcal{P}_{f}$. In this case, a convex subset of $\mathcal{P}_{f} X$ with respect to the structure map $\mu_{X}^{\mathcal{P}_{f}}$ is a family $\mathcal{A}$ of finite subsets of $X$ that is closed under binary unions (see Example 2.13), in other words, it is a sub-semilattice of $\mathcal{P}_{f} X$ :

$$
\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}}(\mathcal{S} X)=\left\{\mathcal{A} \in \mathcal{P}\left(\mathcal{P}_{f} X\right) \mid \forall A, B \in \mathcal{A} . A \cup B \in \mathcal{A}\right\}
$$

Here a $\Phi \in \mathcal{S}\left(\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}}(\mathcal{S} X)\right)$ is a finite set of the form $\left\{\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}\right\}$ where each $\mathcal{A}_{i}$ is a (possibly infinite) subset of $\mathcal{P}_{f} X$ that is closed under binary unions: if $A, B \in \mathcal{A}_{i}$ then $A \cup B \in \mathcal{A}_{i}$. The set $\mathfrak{c}(\Phi)$ is now $\mathfrak{c}(\Phi)=\left\{\left\{A_{1}, \ldots, A_{n}\right\} \mid A_{i} \in \mathcal{A}_{i}\right\} \subseteq \mathcal{P}_{f}\left(\mathcal{P}_{f} X\right)$ hence

$$
\alpha_{\mu_{X}^{s}}\left(\left\{\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}\right\}\right)=\left\{A_{1} \cup \cdots \cup A_{n} \mid A_{i} \in \mathcal{A}_{i}\right\} \subseteq \mathcal{P}_{f} X .
$$

Thus an $\mathscr{A} \in \mathcal{P}_{c} \mathcal{P}_{f} \mathcal{P}_{c} \mathcal{P}_{f} X$ is a (possibly infinite) collections of $\Phi$ 's as above, closed under binary unions, in the sense that if $\left\{\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}\right\}$ and $\left\{\mathcal{B}_{1}, \ldots, \mathcal{B}_{m}\right\}$ are in $\mathscr{A}$, then $\left\{\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}, \mathcal{B}_{1}, \ldots, \mathcal{B}_{m}\right\} \in \mathscr{A} . \mu(\mathscr{A})$ is then the union of all the $\alpha_{\mu_{X}^{\mathcal{S}}}(\Phi)$ with $\Phi \in \mathscr{A}$ : it is a subset of $\mathcal{P}_{f} X$. The fact that it is actually a convex subset of $\mathcal{P}_{f} X$ (i.e. it is closed under binary unions) is ensured by Theorem 5.1 and it is a consequence of the fact that $\mathscr{A}$ is convex itself.

Recall from Remark 4.2 that the monad $\mathcal{C}: \mathbb{E M}(\mathcal{S}) \rightarrow \mathbb{E M}(\mathcal{S})$ from [Jac08] coincides with our lifting $\tilde{\mathcal{P}}$ modulo the absence of the empty set. The same happens for the composite monad, which is named $\mathcal{C M}$ in [Jac08]. The absence of $\emptyset$ in $\mathcal{C M}$ turns out to be rather problematic for Jacobs. Indeed, in order to use the standard framework of coalgebraic trace semantics [HJS06], one would need the Kleisli category $\mathbb{K}(\mathcal{C M})$ to be enriched over $\mathbb{C P P O}$, the category of $\omega$-complete partial orders with bottom and continuous functions. $\mathbb{K l}(\mathcal{C M})$ is not $\mathbb{C P P O}$-enriched since there is no bottom element in $\mathcal{C M}(X)$. Instead, in
$\mathcal{P}_{c} \mathcal{S} X$ the bottom is exactly the empty set; moreover, $\mathbb{K} \mathfrak{l}\left(\mathcal{P}_{c} \mathcal{S}\right)$ enjoys the properties required by [HJS06].

Theorem 5.3. The category $\mathbb{K} 1\left(\mathcal{P}_{c} \mathcal{S}\right)$ is enriched over $\mathbb{C P P O}$ and satisfies the left-strictness condition: for all $f: X \rightarrow \mathcal{P}_{c} \mathcal{S} Y$ and $Z$ set, $\perp_{Y, Z} \circ f=\perp_{X, Z}$.

It is immediate that every homset in $\mathbb{K l}\left(\mathcal{P}_{\mathcal{C}} \mathcal{S}\right)$ carries a complete partial order. Showing that composition of arrows in $\mathbb{K l}\left(\mathcal{P}_{c} \mathcal{S}\right)$ preserves joins (of $\omega$-chains) requires more work: the proof, which can be found in § 5.2, crucially relies on the algebraic theory presenting the monad $\mathcal{P}_{c} \mathcal{S}$, illustrated next.
5.1. An Algebraic Presentation. Recall that an algebraic theory is a pair $\mathcal{T}=(\Sigma, E)$ where $\Sigma$ is a signature, whose elements are called operations, to each of which is assigned a cardinal number called its arity, while $E$ is a class of equations between $\Sigma$-terms. An algebra for the theory $\mathcal{T}$ is a set $A$ together with, for each operation $o$ of arity $\kappa$ in $\Sigma$, a function $o_{A}: A^{\kappa} \rightarrow A$ satisfying the equations of $E$. A homomorphism of algebras is a function $f: A \rightarrow B$ respecting the operations of $\Sigma$ in their realisations in $A$ and $B$. Algebras and homomorphisms of an algebraic theory $\mathcal{T}$ form a category $\operatorname{Alg}(\mathcal{T})$.

Definition 5.4. Let $M$ be a monad on $\operatorname{Set}$, and $\mathcal{T}$ an algebraic theory. We say that $\mathcal{T}$ presents $M$ if and only if $\mathbb{E M}(M)$ and $\operatorname{Alg}(\mathcal{T})$ are isomorphic.

Left $S$-semimodules are algebras for the theory $\mathcal{L S M}=\left(\Sigma_{\mathcal{L S M}}, E_{\mathcal{L S M}}\right)$ where $\Sigma_{\mathcal{L S M}}=$ $\{+, 0\} \cup\{\lambda \cdot \mid \lambda \in S\}$ and $E_{\mathcal{L S}}$ is the set of axioms in Table 1, top right. As already mentioned in Section 2, left $S$-semimodules are exactly $\mathcal{S}$-algebras and morphisms of $S$-semimodules coincide with those of $\mathcal{S}$-algebras. Thus, the theory $\mathcal{L S M}$ presents the monad $\mathcal{S}$.

Similarly, semilattices are algebras for the theory $\mathcal{S} \mathcal{L}=\left(\Sigma_{\mathcal{S L}}, E_{\mathcal{S L}}\right)$ where $\Sigma_{\mathcal{S L}}=\{\sqcup, \perp\}$ and $E_{\mathcal{S L}}$ is the set of axioms in Table 1, top left. It is well known that semilattices are algebras for the finite powerset monad. Actually, this monad is presented by $\mathcal{S L}$. In order to present the full powerset monad $\mathcal{P}$ we need to take joins of arbitrary arity. A complete semilattice is a set $X$ equipped with joins $\bigsqcup_{x \in A} x$ for all-not necessarily finite- $A \subseteq X$. Formally the (infinitary) theory of complete semilattices is given as $\mathcal{C S L}=\left(\Sigma_{\mathcal{C S L}}, E_{\mathcal{C S L}}\right)$ where $\Sigma_{\mathcal{C S L}}=\left\{\bigsqcup_{I} \mid I\right.$ set $\}$ and $E_{\mathcal{C S \mathcal { L }}}$ is the set of axioms displayed in the top of Table 3 (for a detailed treatment of infinitary algebraic theories see, for example, [Man76]).

We can now illustrate the theory $(\Sigma, E)$ presenting the composite monad $\mathcal{P}_{c} \mathcal{S}$ : the operations in $\Sigma$ are exactly those of complete semilattices and $S$-semimodules, while the axioms are those of complete semilattices and $S$-semimodules together with the set $E_{\mathcal{D}}$ of distributivity axioms consisting of (D1) and (D2) in the bottom of Table 3. In short, $\Sigma=\Sigma_{\mathcal{C S L}} \cup \Sigma_{\mathcal{L S M}}$ and $E=E_{\mathcal{C S L}} \cup E_{\mathcal{L S M}} \cup E_{\mathcal{D}}$.

Theorem 5.5. The monad $\mathcal{P}_{c} \mathcal{S}$ is presented by the algebraic theory $(\Sigma, E)$.
The proof of Theorem 5.5 crucially relies on the fact that $\mathcal{P}_{c} \mathcal{S}$ is obtained by composing $\mathcal{P}$ and $\mathcal{S}$ via $\delta$, as we show in the rest of this current Section 5.1. We shall use the notion of $\delta$-algebra below, originally defined by [Bec69] in the context of ordinary distributive laws but that can be adapted with no changes in the weak setting as well. Here we report the definition as phrased in [Gar20].

Definition 5.6. Let $T$ and $S$ be monads on $\mathbb{C}$ and $\delta: T S \rightarrow S T$ be a (weak) distributive law. A $\delta$-algebra is an object of $\mathbb{C}$ endowed with a $T$-algebra structure $t: T X \rightarrow X$ and an $S$-algebra structure $s: S X \rightarrow X$ such that the following pentagon

commutes. The category $\mathbb{E M}(\delta)$ is the full subcategory of $\mathbb{E M}(S) \times_{\mathbb{C}} \mathbb{E M}(T)$, result of the pullback

whose objects are $\delta$-algebras and morphisms are thus maps in $\mathbb{C}$ which are simultaneously morphisms of $S$ - and $T$-algebras.

The following Proposition, relating $\delta$-algebras with the Eilenberg-Moore algebras for $\mathcal{P}_{c} \mathcal{S}$, appears in [Gar20, Lemma 14] and was proved in a more general context in [Böh10, Proposition 3.7].

Proposition 5.7. Given a weak distributive law $\delta: T S \rightarrow S T$, the categories $\mathbb{E M}(\delta)$ of $\delta$-algebras and $\mathbb{E M}(\widetilde{S T})$ of Eilenberg-Moore algebras of the composite monad $\widetilde{S T}$ (arising from $\delta$ ) are canonically isomorphic.

We can therefore use to our advantage Proposition 5.7 to individuate the algebraic theory presenting the monad $\widetilde{\mathcal{P S}}$, which we called $\mathcal{P}_{c} \mathcal{S}$ : rather than studying the Eilenberg-Moore algebras for the quite complicated composite monad, we analyse the isomorphic category of $\delta$-algebras instead. In order to prove Theorem 5.5 , we show that $\mathbb{E M}(\delta)$ is isomorphic to the category $\mathbb{A l g}(\Sigma, E)$ of $(\Sigma, E)$-algebras (Theorem 5.11) and then use Proposition 5.7 to conclude.

First we prove that every $\delta$-algebra gives rise to an algebra for the theory $(\Sigma, E)$, i.e. a complete semilattice and $S$-semimodule satisfying the equations in $E_{\mathcal{D}}$. For $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$ given in (3.3), we have that a $\delta$-algebra is a set $X$ together with a $\mathcal{S}$-algebra structure $a: \mathcal{S} X \rightarrow X$ and a $\mathcal{P}$-algebra-structure $b: \mathcal{P} X \rightarrow X$ such that pentagon

commutes. A morphism of $\delta$-algebras is a morphism of $\mathcal{P}$ - and $\mathcal{S}$-algebras simultaneously by definition. Hence, by defining $+^{a}, \lambda^{a}$ and $0^{a}$ as in (2.2), we have that $X$ is a $S$-leftsemimodule; moreover, if we define for each set $I$

$$
\begin{equation*}
\bigsqcup_{i \in I}^{b} x_{i}=b\left(\left\{x_{i} \mid i \in I\right\}\right) \tag{5.4}
\end{equation*}
$$

then $X$ is also a complete semilattice. This means that $X$ satisfies all the equations in $E_{\mathcal{C S L}}$ and $E_{\mathcal{L S M}} ;$ moreover, every morphism of $\delta$-algebras preserves all the operations in $\Sigma$. The commutativity of pentagon (5.3) will instead ensure that the equations in $E_{\mathcal{D}}$, that is $(D 1)$ and (D2) in Table 3, are satisfied, as we prove in the following. For $A \subseteq X$, we shall sometimes write $\bigsqcup^{b} A$ for $\bigsqcup_{x \in A}^{b} x$.

First we need a technical result that shows how joins and convex closures interact in $\delta$-algebras.

Proposition 5.8. Let $(X, a: \mathcal{S} X \rightarrow X, b: \mathcal{P} X \rightarrow X)$ be a $\delta$-algebra. For each $A \subseteq X$,

$$
\bigsqcup_{x \in A}^{b} x=\bigsqcup_{x \in \bar{A}^{a}}^{b} x
$$

Proof. Let $\Phi=\eta_{\mathcal{P} X}^{\mathcal{S}}(A)=\Delta_{A} \in \mathcal{S P} X$. We show that the two legs of the commutative diagram (5.3), that is $\mathcal{S} b(\Phi)$ and $b\left(\mathcal{P} a\left(\delta_{X}(\Phi)\right)\right)$, coincide with the two sides of the equation in the statement. Indeed we have

$$
a(\mathcal{S} b(\Phi))=a\left(\mathcal{S} b\left(\Delta_{A}\right)\right)=a\left(\Delta_{\bigsqcup^{b} A}\right)=\bigsqcup^{b} A
$$

while

$$
\delta_{X}\left(\Delta_{A}\right)=\left\{\varphi \in \mathcal{S} X \mid \operatorname{supp} \varphi \subseteq A, \sum_{x \in X} \varphi(x)=1\right\}
$$

because of Lemma 4.3, hence

$$
b\left(\mathcal{P} a\left(\delta_{X}\left(\Delta_{A}\right)\right)\right)=b\left(\bar{A}^{a}\right)=\bigsqcup^{b} \bar{A}^{a}
$$

where recall that $\bar{A}^{a}=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \operatorname{supp} \varphi \subseteq A, \sum_{x \in X} \varphi(x)=1\right\}$ from Definition 2.10.
We can then prove that every $\delta$-algebra satisfies the equations in $E_{\mathcal{D}}$.
Theorem 5.9. Let $(X, a: \mathcal{S} X \rightarrow X, b: \mathcal{P} X \rightarrow X)$ be a $\delta$-algebra. Then for all $A, B \subseteq X$ and for all $\lambda \in S \backslash\left\{0_{S}\right\}$ :

$$
\lambda \cdot \cdot^{a} \bigsqcup_{x \in A}^{b} x=\bigsqcup_{x \in A}^{b} \lambda \cdot{ }^{a} x, \quad \bigsqcup_{x \in A}^{b} x+{ }^{a} \bigsqcup_{y \in B}^{b} y=\bigsqcup_{(x, y) \in A \times B}^{b} x+{ }^{a} y
$$

Proof. Let $\Phi=(A \mapsto \lambda) \in \mathcal{S P} X$. We prove that the images along the two legs of pentagon (5.3) of $\Phi$ coincide with the two sides of the first equation in the statement. It
holds that

$$
\begin{array}{rlr}
a(\mathcal{S}(b)(\Phi)) & =a\left(x \mapsto \sum_{U \in b^{-1}\{x\}} \Phi(U)\right) & \\
& =a(b(A) \mapsto \lambda) & \\
& =a\left(\bigsqcup_{x \in A}^{b} x \mapsto \lambda\right) & \\
& =\lambda \cdot{ }^{a} \bigsqcup_{x \in A}^{b} x &
\end{array}
$$

On the other hand we have that

$$
\begin{array}{rlr}
b\left(\mathcal{P} a\left(\delta_{X}(\Phi)\right)\right) & =\bigsqcup^{b} \mathcal{P} a\left(\overline{\mathfrak{c}(\Phi)} \mu^{\mathcal{S}}\right) & \text { Theorem } 3.5 \\
& =\bigsqcup^{b} \overline{\mathcal{P} a(\mathfrak{c}(\Phi))} & \\
& =\bigsqcup^{b} \mathcal{P} a(\mathfrak{c}(\Phi)) & \\
\text { Proposition } 4.5
\end{array}
$$

Following the discussion after Theorem 4.1, $\mathfrak{c}(\Phi)=\{(x \mapsto \lambda) \mid x \in A\}$ if $\lambda \neq 0$. Therefore, if $\lambda \neq 0$, then $\mathcal{P} a(\mathfrak{c}(\Phi))=\{a(x \mapsto \lambda) \mid x \in A\}$ which by (2.2) is exactly $\left\{\lambda \cdot{ }^{a} x \mid x \in A\right\}$.

Therefore

$$
b\left(\mathcal{P} a\left(\delta_{X}(\Phi)\right)\right)=\bigsqcup_{x \in A}^{b} \lambda \cdot^{a} x
$$

We then conclude by using the commutativity of diagram (5.3). Using the function $\Phi^{\prime}=$ $(A \mapsto 1, B \mapsto 1) \in \mathcal{S P} X$ and a similar argument, one shows the second equation as well.

Vice versa, we want to prove that every algebra for the theory $(\Sigma, E)$ is also a $\delta$ algebra. To this end, let $\left(X,+, \lambda \cdot, 0_{X}, \bigsqcup_{I}\right)$ be a $(\Sigma, E)$-algebra. Then $\left(X,+, \lambda \cdot, 0_{X}\right)$ is a $S$-left-semimodule, $\left(X, \bigsqcup_{I}\right)$ is a complete sup-semilattice. By defining

we have that $(X, a) \in \mathbb{E} \mathbb{M}(\mathcal{S})$ and $(X, b) \in \mathbb{E} \mathbb{M}(\mathcal{P})$. We now have to check that pentagon (5.3) commutes. Given $A \subseteq X$, we define its convex closure $\bar{A}$ in the usual way:

$$
\bar{A}=\left\{\sum_{i=1}^{n} \lambda_{i} \cdot a_{i} \mid n \in \mathbb{N}, a_{i} \in A, \sum_{i=1}^{n} \lambda_{i}=1\right\} \subseteq X
$$

Also, we shall denote by $\bigsqcup A$ the element $b(A) \in X$ for any $A \subseteq X$. Then pentagon (5.3) commutes if and only if for all $\Phi \in \mathcal{S P} X$ :

$$
\sum_{A \in \operatorname{supp} \Phi} \Phi(A) \cdot \bigsqcup A=\bigsqcup \overline{\left\{\sum_{A \in \operatorname{supp} \Phi} \Phi(A) \cdot u(A) \mid u: \operatorname{supp} \Phi \rightarrow X . \forall A \cdot u(A) \in A\right\}}
$$

where the left-hand side is $a(\mathcal{S}(b)(\Phi))$ and the right-hand side is $b\left(\mathcal{P}(a)\left(\delta_{X}(\Phi)\right)\right)$.

In the following lemma we prove that if $X$ is a $(\Sigma, E)$-algebra and $A \subseteq X$, then $\bigsqcup A=\bigsqcup \bar{A}$. (This is a generalisation of the derivation in Table 2 for arbitrary joins.) Using this fact and the axioms $(D 1),(D 2)$ of Table 3 we will have shown the commutativity of pentagon (5.3).
Lemma 5.10. Let $\left(X,+, \lambda \cdot, 0_{X}, \bigsqcup_{I}\right)$ be a $(\Sigma, E)$-algebra. Then for all $A \subseteq X$

$$
\bigsqcup A=\bigsqcup \bar{A}
$$

Proof. Let $\lambda_{1}, \ldots, \lambda_{n} \in S$ with $\sum_{i=1}^{n} \lambda_{i}=1$. Then

$$
\bigsqcup A=1 \cdot \bigsqcup A=\left(\sum_{i=1}^{n} \lambda_{i}\right) \cdot \bigsqcup A=\sum_{i=1}^{n}\left(\lambda_{i} \cdot \bigsqcup A\right)=\bigsqcup\left\{\sum_{i=1}^{n} \lambda_{i} \cdot a_{i} \mid\left(a_{1}, \ldots, a_{n}\right) \in A^{n}\right\}
$$

because of the axioms of $S$-semimodule and the distributivity $(D 1),(D 2)$ in Table 3 . Let $\leq$ be the partial order determined by the complete semilattice structure of $X$. Then we have that for all $n \in \mathbb{N}$, for all $\lambda_{1}, \ldots, \lambda_{n} \in S$ such that $\sum_{i=1}^{n} \lambda_{i}=1$ and for all $\left(a_{1}, \ldots, a_{n}\right) \in A^{n}$ :

$$
\sum_{i=1}^{n} \lambda_{i} \cdot a_{i} \leq \bigsqcup\left\{\sum_{i=1}^{n} \lambda_{i} \cdot b_{i} \mid\left(b_{1}, \ldots, b_{n}\right) \in A^{n}\right\}=\bigsqcup A
$$

hence

$$
\bigsqcup \bar{A}=\bigsqcup\left\{\sum_{i=1}^{n} \lambda_{i} \cdot a_{i} \mid n \in \mathbb{N}, a_{i} \in A, \sum_{i=1}^{n} \lambda_{i}=1\right\} \leq \bigsqcup A
$$

while the other inequality is trivial because $A \subseteq \bar{A}$.
Finally, again a morphism of $(\Sigma, E)$-algebras is a morphism respecting all the operations of $\Sigma$, which means of $\Sigma_{\mathcal{L S M}}$ (thus is a morphism of $\mathbb{E M}(\mathcal{S})$ ) and of $\Sigma_{\mathcal{C S L}}$ (thus is a morphism of $\mathbb{E M}(\mathcal{P}))$ at the same time. We have therefore proved the following theorem.
Theorem 5.11. The category $\mathbb{E M}(\delta)$ of $\delta$-algebras is isomorphic to the category $\operatorname{Alg}(\Sigma, E)$ of ( $\Sigma, E)$-algebras.

Since $\mathbb{E M}(\delta)$ is canonically isomorphic to $\mathbb{E M}\left(\mathcal{P}_{c} \mathcal{S}\right)$ by Proposition 5.7, we have proved Theorem 5.5. We conclude this section with a couple of examples.
Example 5.12 (cf. [Gar20, §4.3]). Let $S=\mathbb{B}$ ool $=\{0,1\}$, where addition is $V$ and multiplication is $\wedge$. Then a $\mathbb{B o o l - s e m i m o d u l e ~ i s ~ a ~ c o m m u t a t i v e ~ m o n o i d ~}(X, *, e)$ where $*$ is idempotent (see Example 2.13). A $\delta$-algebra, in this case, is therefore a complete lattice $X$ with a commutative, associative, unitary and idempotent operation $*$ such that

$$
\bigsqcup A * \bigsqcup B=\bigsqcup\{a * b \mid a \in A, b \in B\} .
$$

Thus $\delta$-algebras are exactly commutative, unitary and idempotent quantales.
Example 5.13. Let $S$ be $\mathbb{R}^{+}$and let $[a, b]$ with $a, b \in \mathbb{R}^{+}$denote the set $\left\{x \in \mathbb{R}^{+} \mid a \leq\right.$ $x \leq b\}$ and $[a, \infty)$ the set $\left\{x \in \mathbb{R}^{+} \mid a \leq x\right\}$. For $1=\{x\}, \mathcal{P}_{c} \mathcal{S}(1)=\{\emptyset\} \cup\{[a, b] \mid a, b \in$ $\left.\mathbb{R}^{+}\right\} \cup\left\{[a,+\infty) \mid a \in \mathbb{R}^{+}\right\}$. The $\mathcal{P}_{c} \mathcal{S}$-algebra $\mu_{1}^{\mathcal{P}_{c} \mathcal{S}}: \mathcal{P}_{c} \mathcal{S} \mathcal{P}_{c} \mathcal{S} 1 \rightarrow \mathcal{P}_{c} \mathcal{S} 1$ induces a $\delta$-algebra where the structure of complete lattice is given as ${ }^{2}$

$$
\bigsqcup_{i \in I} A_{i}= \begin{cases}{\left[\inf _{i \in I}, a_{i}, \sup _{i \in I} b_{i}\right]} & \text { if, for all } i \in I, A_{i}=\left[a_{i}, b_{i}\right] \wedge \sup _{i \in I} b_{i} \in \mathbb{R}^{+} \\ \left.\operatorname{[if}_{i \in I} a_{i}, \infty\right) & \text { otherwise }\end{cases}
$$

[^2]The $\mathbb{R}^{+}$-semimodule is as expected, e.g. $\left[a_{1}, b_{1}\right]+\left[a_{2}, b_{2}\right]=\left[a_{1}+a_{2}, b_{1}+b_{2}\right]$.
5.2. Proof of Theorem 5.3. We need to show that the Kleisli category $\mathcal{P}_{c} \mathcal{S}$ is $\left.\mathbb{C P P (}\right)$ enriched and satisfies the left-strictness condition. The algebraic presentation of the composite monad $\mathcal{P}_{c} \mathcal{S}$ provided by Theorem 5.5 is going to be useful. Let us begin with an analysis of some basic properties of $\mathbb{K l}\left(\mathcal{P}_{c} \mathcal{S}\right)$.

First of all, since $\left(\mathcal{P}_{c} \mathcal{S} X, \mu_{X}^{\mathcal{P}_{\mathcal{C}} \mathcal{S}}\right)$ is a $\mathcal{P}_{c} \mathcal{S}$-algebra, we have that $\mathcal{P}_{c} \mathcal{S} X$ has a structure of semimodule and complete semilattice given, via the canonical isomorphism $\mathbb{E M}\left(\mathcal{P}_{c} \mathcal{S}\right) \rightarrow$ $\mathbb{E M}(\delta)$, by:

$$
\begin{gathered}
\mathcal{A}+\mathcal{B}=\{\varphi+\psi \mid \varphi \in \mathcal{A}, \psi \in \mathcal{B}\} \\
\lambda \cdot \mathcal{A}=\{\lambda \cdot \varphi \mid \varphi \in \mathcal{A}\} \\
\bigsqcup_{i \in I} \mathcal{A}_{i}=\overline{\bigcup_{i \in I} \mathcal{A}_{i}}
\end{gathered}
$$

for all $\mathcal{A}, \mathcal{B} \in \mathcal{P}_{c} \mathcal{S} X$ and $\lambda \in S, \lambda \neq 0$.
This means that each homset $\mathbb{K} 1\left(\mathcal{P}_{c} \mathcal{S}\right)(X, Y)$ of functions $f: X \rightarrow \mathcal{P}_{c} \mathcal{S}(Y)$, which is partially ordered point-wise, inherits the structure of complete semilattice from $\mathcal{P}_{c} \mathcal{S} Y$, in particular its bottom element $\perp_{X, Y}$ is the constant function mapping $x$ to $\emptyset$ for all $x \in X$. Given a function $g: Y \rightarrow \mathcal{P}_{c} \mathcal{S} Z$, its Kleisli extension $g^{\sharp}: \mathcal{P}_{c} \mathcal{S} Y \rightarrow \mathcal{P}_{c} \mathcal{S} Z$ is given by $\mu_{Z}^{\mathcal{P}_{c} \mathcal{S}} \circ \mathcal{P}_{c} \mathcal{S} g$, which for all $\mathcal{A} \in \mathcal{P}_{c} \mathcal{S} Y$ computes:

$$
g^{\sharp}(\mathcal{A})=\bigsqcup_{\varphi \in \mathcal{A}} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot g(y) .
$$

Composition of a function $f: X \rightarrow \mathcal{P}_{c} \mathcal{S} Y$ with $g: Y \rightarrow \mathcal{P}_{c} \mathcal{S} Z$ is therefore given as

$$
(g \circ f)(x)=g^{\sharp}(f(x))=\bigsqcup_{\varphi \in f(x)} \sum_{y \in \text { supp } \varphi} \varphi(y) \cdot g(y) \in \mathcal{P}_{c} \mathcal{S} Z .
$$

We can now prove that $\mathbb{K} l\left(\mathcal{P}_{c} \mathcal{S}\right)$ is actually enriched over directed-complete partial orders and that it satisfies the left-strictness condition, which implies Theorem 5.3.

Theorem 5.14. The category $\mathbb{K l}\left(\mathcal{P}_{c} \mathcal{S}\right)$ is enriched over the category of directed-complete partial orders and satisfies the left-strictness condition:

$$
\perp_{Y, Z} \circ f=\perp_{X, Z}
$$

for all $f: X \rightarrow \mathcal{P}_{c} \mathcal{S} Y$ and $Z$ set.
Proof. In this proof we will use a generalisation of (D2) in Table 3 for $n$-ary sums of arbitrary joins in ( $\Sigma, E)$-algebras, given by:

$$
\begin{equation*}
\sum_{j=1}^{n} \bigsqcup_{i_{j} \in I_{j}} x_{i_{j}}=\bigsqcup_{\left(i_{1}, \ldots, i_{n}\right) \in \prod_{j=1}^{n} I_{j}} \sum_{j=1}^{n} x_{i_{j}} . \tag{5.5}
\end{equation*}
$$

Let $f: X \rightarrow \mathcal{P}_{c} \mathcal{S} Y$ and $\left\{g_{i} \mid i \in I\right\}$ a directed subset of $\mathbb{K}\left(\mathcal{P}_{c} \mathcal{S}\right)(Y, Z)$. Then:

$$
\left(\left(\bigsqcup_{i \in I} g_{i}\right) \circ f\right)(x)=\bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot\left(\bigsqcup_{i \in I} g_{i}\right)(y) \quad \text { definition of composition }
$$

$$
\begin{align*}
& =\bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot \bigsqcup_{i \in I}\left(g_{i}(y)\right) \\
& =\bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} p} \bigsqcup_{\varphi} \varphi(y) \cdot g_{i}(y)  \tag{D1}\\
& =\bigsqcup_{\varphi \in f(x)} \bigsqcup_{\left(i_{y}\right) \in I^{s u p p} \varphi} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot g_{i_{y}}(y)  \tag{5.5}\\
& =\bigsqcup_{\varphi \in f(x)} \bigsqcup_{i \in I} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot g_{i}(y) \quad\left\{g_{i} \mid i \in I\right\} \text { is directed } \\
& =\bigsqcup_{i \in I} \bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot g_{i}(y) \quad \text { joins commute } \\
& =\bigsqcup_{i \in I}\left(g_{i} \circ f\right)(x) \quad \text { definition of composition. }
\end{align*}
$$

Given, instead, an arbitrary subset $\left\{f_{i} \mid i \in I\right\}$ of $\mathbb{K l}\left(\mathcal{P}_{c} \mathcal{S}\right)(X, Y)$ and a $g: Y \rightarrow \mathcal{P}_{c} \mathcal{S} Z$, we have

$$
\begin{array}{rlr}
\left(g \circ\left(\bigsqcup_{i \in I} f_{i}\right)\right)(x) & =g^{\sharp}\left(\bigsqcup_{i \in I} f_{i}(x)\right) & \text { definition of composition } \\
& =\bigsqcup_{i \in I} g^{\sharp}\left(f_{i}(x)\right) & g^{\sharp} \text { is a morphism of } \mathcal{P}_{c} \mathcal{S} \text {-algebras } \\
& =\bigsqcup_{i \in I}(g \circ f)(x) & \text { definition of composition }
\end{array}
$$

( $g^{\sharp}$ is a morphism of $\mathcal{P}_{c} \mathcal{S}$-algebras as it is given by the universal property of the free algebra $\mathcal{P}_{c} \mathcal{S} Y$, hence it preserves arbitrary suprema). Finally,

$$
\left(\perp_{Y, Z} \circ f\right)(x)=\bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} \varphi} \varphi(y) \cdot \emptyset=\bigsqcup_{\varphi \in f(x)} \sum_{y \in \operatorname{supp} \varphi} \emptyset=\bigsqcup_{\varphi \in f(x)} \emptyset=\emptyset
$$

where $\varphi(y) \cdot \emptyset=\emptyset$ because $\varphi(y) \neq 0$ when $y \in \operatorname{supp} \varphi$.

## 6. Finite Joins and Finitely Generated Convex Sets

In the previous section we have used the algebraic theory $(\Sigma, E)$ to show that the category $\mathbb{K} l\left(\mathcal{P}_{c} \mathcal{S}\right)$ is enriched over $\mathbb{C P P O}$. However, in various situations it is convenient to consider semilattices which are not complete, but have just finitary joins. In this section we show that by restricting the theory $(\Sigma, E)$ to finite joins we obtain a presentation for the monad of finitely generated convex sets. We will introduce several monads and maps between them, which all appear in the following diagram, to be used as a road map through this section:


Here we recall the definition of monad map.

Definition 6.1. Let $\left(S, \eta^{S}, \mu^{S}\right)$ and $\left(T, \eta^{T}, \mu^{T}\right)$ be monads on a category $\mathbb{C}$. A monad map $\lambda: T \rightarrow S$ is a natural transformation such that

commutes, where $\lambda \lambda$ is $\lambda S \circ T \lambda=S \lambda \circ \lambda T$.
The algebraic theory $(\Sigma, E)$ described in Theorem 5.5 determines a monad $T_{\Sigma, E}:$ Set $\rightarrow$ Set where, for any set $X, T_{\Sigma, E}(X)$ is the set of all $\Sigma$-terms with variables in $X$ quotiented by the equations in $E$. Recall that a $\Sigma$-term with variables in $X$ is defined inductively as:

- every variable $x$ is a $\Sigma$-term,
- if $o$ is an operation in $\Sigma$ with arity $\kappa$ and $t_{1}, \ldots, t_{\kappa}$ are $\Sigma$-terms, then $o\left(t_{1}, \ldots, t_{\kappa}\right)$ is a $\Sigma$-term.

If $f: X \rightarrow Y$ is a function, $T_{\Sigma, E}(f): T_{\Sigma, E}(X) \rightarrow T_{\Sigma, E}(Y)$ sends a term $t$ in $t[f(x) / x]$, where every variable $x$ is substituted by its image $f(x)$. The unit $\eta^{T}$ is simply defined as $\eta_{X}^{T}(x)=x$, while the multiplication is defined by induction as:

$$
\begin{aligned}
& T_{\Sigma, E}\left(T_{\Sigma, E}(X)\right) \mu_{X}^{T} \\
& t \in T_{\Sigma, E}(X) \longmapsto t \\
& o_{\Sigma, E}(X) \\
& o_{\kappa}\left(t_{1}, \ldots, t_{\kappa}\right) \longmapsto o_{\kappa}\left(\mu_{X}^{T}\left(t_{1}\right), \ldots, \mu_{X}^{T}\left(t_{\kappa}\right)\right)
\end{aligned}
$$

This construction is standard for finitary algebraic theories, where every operation in $\Sigma$ has finite arity. The fact that it makes sense also for our case, where we have an operation for every cardinal, is ensured by the fact that our $(\Sigma, E)$ is tractable in the sense of [Man76, Definition 1.5.44], because we proved in Theorem 5.5 that it presents a monad on Set, namely $\mathcal{P}_{c} \mathcal{S}$. Tractability ensures that the class of $\Sigma$-terms, once quotiented by $E$, is forced to be a set.

Now, the category of Eilenberg-Moore algebras for the monad $T_{\Sigma, E}$ is, in fact, isomorphic to $\operatorname{Alg}(\Sigma, E)$, hence also to $\mathbb{E M}\left(\mathcal{P}_{c} \mathcal{S}\right)$, via a functor $F$ such that

commutes. This generates an isomorphism of monads $\varphi: T_{\Sigma, E} \rightarrow \mathcal{P}_{c} \mathcal{S}$ where, for all sets $X$, $\varphi_{X}=F\left(\mu_{X}^{\mathcal{P}_{c} \mathcal{S}}\right) \circ T_{\Sigma, E}\left(\eta_{X}^{\mathcal{P}_{c} \mathcal{S}}\right)$, thanks to the following general result:

Theorem 6.2. Let $\left(S, \eta^{S}, \mu^{S}\right)$ and $\left(T, \eta^{T}, \mu^{T}\right)$ be monads on a category $\mathbb{C}$. Suppose $\mathbb{E} \mathbb{M}(S)$ and $\mathbb{E M}(T)$ are isomorphic via a functor $F: \mathbb{E M}(S) \rightarrow \mathbb{E M}(T)$ such that $U^{T} F=U^{S}$. Then $T$ and $S$ are isomorphic as monads, that is, there is an isomorphism of monads $\varphi: T \rightarrow S$. Specifically, $\varphi_{X}$ is given as the unique morphism of $T$-algebras (hence, a morphism in $\mathbb{C}$ )
granted by the universal property of $\left(T X, \mu_{X}^{T}\right)$ as the free $T$-algebra on $X$, induced by $\eta_{X}^{S}$ :


Direct calculations show that our $\varphi_{X}: T_{\Sigma, E}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)=\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}} \mathcal{S} X$ acts as follows:

$$
\begin{aligned}
\varphi_{X}(x) & =\left\{\Delta_{x}\right\} \quad \text { for } x \in X \\
\varphi_{X}(0) & =\{0: X \rightarrow S\} \\
\varphi_{X}\left(t_{1}+t_{2}\right) & =\varphi_{X}\left(t_{1}\right)+^{\alpha_{\mu}^{S}} \varphi_{X}\left(t_{2}\right) \\
\varphi_{X}(\lambda \cdot t) & =\lambda \cdot{ }^{\alpha_{X}^{\mathcal{S}}} \varphi_{X}(t) \\
\varphi_{X}\left(\bigsqcup_{I}\left\{t_{i} \mid i \in I\right\}\right) & ={\overline{\bigcup_{i \in I} \varphi\left(t_{i}\right)}}^{\mu_{X}^{\mathcal{S}}}
\end{aligned}
$$

where $\varphi_{X}\left(t_{1}\right)+{ }^{\alpha_{\mu}^{\mathcal{S}}} \varphi_{X}\left(t_{2}\right)$ is the result of adding up in the $\mathcal{S}$-algebra $\left(\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}} \mathcal{S} X, \alpha_{\mu_{X}^{\mathcal{S}}}\right)$ (recall from (4.1) the definition of the $\mathcal{S}$-algebra structure map $\alpha_{a}$ on $\mathcal{P}_{c}^{a} X$ for a given $\mathcal{\mathcal { S }}$-algebra $(X, a)$ ), seen as a $S$-semimodule, the convex subsets $\varphi_{X}\left(t_{1}\right)$ and $\varphi_{X}\left(t_{2}\right)$ of $\mathcal{S} X$, as explained in (4.6). Similarly $\lambda \cdot{ }^{\alpha} \mu_{X}^{\mathcal{S}} \varphi_{X}(t)$ is the scalar-product in $\mathcal{P}_{c}^{\mu_{X}^{\mathcal{S}}} \mathcal{S} X$. Hence, for $\lambda \neq 0_{S}$ :

$$
\begin{aligned}
\varphi_{X}\left(t_{1}+t_{2}\right) & =\left\{\left.\mu_{X}^{\mathcal{S}}\binom{f_{1} \longmapsto 1}{f_{2} \longmapsto 1} \right\rvert\, f_{1} \in \varphi_{X}\left(t_{1}\right), f_{2} \in \varphi_{X}\left(t_{2}\right)\right\} \\
\varphi_{X}(\lambda \cdot t) & =\left\{\lambda \cdot f \mid f \in \varphi_{X}(t)\right\}
\end{aligned}
$$

We now consider the algebraic theory $\left(\Sigma^{\prime}, E^{\prime}\right)$ obtained by restricting $(\Sigma, E)$ to finitary joins. More precisely, we fix

$$
\Sigma^{\prime}=\Sigma_{\mathcal{S L}} \cup \Sigma_{\mathcal{L S M}} \quad E^{\prime}=E_{\mathcal{S L}} \cup E_{\mathcal{L S M}} \cup E_{\mathcal{D}^{\prime}}
$$

where $\left(\Sigma_{\mathcal{S} \mathcal{L}}, E_{\mathcal{S L}}\right)$ is the algebraic theory for semilattices, $\left(\Sigma_{\mathcal{L S M}}, E_{\mathcal{L S M}}\right)$ is the one for $S$-semimodules, and $E_{\mathcal{D}^{\prime}}$ is the set consisting of the four distributivity axioms illustrated in the bottom of Table 1. It is clear, then, that if we restrict the action of $\varphi_{X}$ to all those terms involving only finite suprema, we obtain a function translating $\Sigma^{\prime}$-terms into convex subsets.

Proposition 6.3. Let $T_{\Sigma^{\prime}, E^{\prime}}(X)$ be the set of $\Sigma^{\prime}$-terms with variables in $X$ quotiented by $E^{\prime}$. Let $\llbracket \cdot \rrbracket_{X}: T_{\Sigma^{\prime}, E^{\prime}}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)$ be the function defined as

$$
\begin{array}{rlrl}
\llbracket x \rrbracket & =\left\{\Delta_{x}\right\} \text { for } x \in X & \llbracket \lambda \cdot t \rrbracket & = \begin{cases}\left\{\lambda \cdot \mu_{X}^{\mathcal{S}} f \mid f \in \llbracket t \rrbracket\right\} & \text { if } \lambda \neq 0 \\
\left\{0^{\mu_{X}^{\mathcal{S}}}\right\} & \text { otherwise }\end{cases} \\
\llbracket 0 \rrbracket & =\left\{0^{\mu_{X}^{\mathcal{S}}}\right\} & \boxed{t_{1}+t_{2} \rrbracket} & =\left\{f_{1}+\mu_{X}^{\mathcal{S}} f_{2} \mid f_{1} \in \llbracket t_{1} \rrbracket,\right. \\
\llbracket \perp & \left.f_{2} \in \llbracket t_{2} \rrbracket\right\} \\
\llbracket \perp \rrbracket & =\emptyset & \llbracket t_{1} \sqcup t_{2} \rrbracket & =\frac{\llbracket t_{1} \rrbracket \cup \llbracket t_{2} \rrbracket}{\mu_{X}^{\mathcal{S}}}
\end{array}
$$

Let $\llbracket \rrbracket \rrbracket: T_{\Sigma^{\prime}, E^{\prime}} \rightarrow \mathcal{P}_{c} \mathcal{S}$ be the family $\left\{\llbracket \cdot \rrbracket_{X}\right\}_{X \in \mid \text { Set } \mid}$. Then $\llbracket \cdot \rrbracket: T_{\Sigma^{\prime}, E^{\prime}} \rightarrow \mathcal{P}_{c} \mathcal{S}$ is a map of monads and, moreover, each $\llbracket \rrbracket_{X}: T_{\Sigma^{\prime}, E^{\prime}}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)$ is injective.

Proof. The obvious restriction $\theta_{X}: T_{\Sigma^{\prime}, E^{\prime}}(X) \rightarrow T_{\Sigma, E}(X)$ induces a monad map $\theta: T_{\Sigma^{\prime}, E^{\prime}} \rightarrow$ $T_{\Sigma, E}$. Observing that $\llbracket \cdot \rrbracket$ is exactly $\varphi \circ \theta$ is enough to conclude that $\llbracket \cdot \rrbracket$ is a monad map. Each $\llbracket \cdot \rrbracket_{X}: T_{\Sigma^{\prime}, E^{\prime}}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)$ is injective since $\theta_{X}$ is injective and $\varphi_{X}$ is an isomorphism.

We say that a set $\mathcal{A} \in \mathcal{P}_{c} \mathcal{S}(X)$ is finitely generated if there exists a finite set $\mathcal{B} \subseteq \mathcal{S}(X)$ such that $\overline{\mathcal{B}}=\mathcal{A}$. We write $\mathcal{P}_{f c} \mathcal{S}(X)$ for the set of all $\mathcal{A} \in \mathcal{P}_{c} \mathcal{S}(X)$ that are finitely generated.

Proposition 6.4. The assignment $X \mapsto \mathcal{P}_{f c} \mathcal{S}(X)$ gives rise to a monad $\mathcal{P}_{f c} \mathcal{S}$ : Set $\rightarrow$ Set where the action on functions, the unit and the multiplication are defined as for $\mathcal{P}_{c} \mathcal{S}$.

To prove Proposition 6.4 we only have to show that: for all functions $f: X \rightarrow Y$, $\mathcal{P}_{c} \mathcal{S}(f): \mathcal{P}_{c} \mathcal{S}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(Y)$ restricts and corestricts to $\mathcal{P}_{f c} \mathcal{S}(X) \rightarrow \mathcal{P}_{f c} \mathcal{S}(Y)$; the unit $\eta_{X}^{\mathcal{P}_{c} \mathcal{S}}$ can be corestricted to $\mathcal{P}_{f c} \mathcal{S}(X)$, and finally the multiplication $\mu_{X}^{\mathcal{P}_{\mathcal{C}} \mathcal{S}}$ restricts and corestricts to $\mathcal{P}_{f c} \mathcal{S} \mathcal{P}_{f c} \mathcal{S} X \rightarrow \mathcal{P}_{f c} \mathcal{S} X$. These facts are illustrated in Appendix C.

Next we are going to use Proposition 6.3 to give a presentation to the monad $\mathcal{P}_{f c} \mathcal{S}$.
Theorem 6.5. The monads $T_{\Sigma^{\prime}, E^{\prime}}$ and $\mathcal{P}_{f c} \mathcal{S}$ are isomorphic. Therefore $\left(\Sigma^{\prime}, E^{\prime}\right)$ is a presentation for the monad $\mathcal{P}_{f c} \mathcal{S}$.
Proof. We first observe that the function

$$
\llbracket \cdot \rrbracket_{X}: T_{\Sigma^{\prime}, E^{\prime}}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)
$$

factors as

$$
T_{\Sigma^{\prime}, E^{\prime}}(X) \xrightarrow{\llbracket \cdot \rrbracket_{X}^{\prime}} \mathcal{P}_{f c} \mathcal{S}(X) \xrightarrow{\iota_{X}} \mathcal{P}_{c} \mathcal{S}(X)
$$

where $\iota_{X}: \mathcal{P}_{f c} \mathcal{S}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(X)$ is the obvious set-inclusion. This can be easily checked by induction on $T_{\Sigma^{\prime}, E^{\prime}}(X)$.

Observe that, since $\llbracket \cdot \rrbracket_{X}$ is injective by Proposition 6.3 , then also $\llbracket \cdot \rrbracket_{X}^{\prime}$ is injective. We conclude by showing that it is also surjective.

Let $\mathcal{A} \in \mathcal{P}_{f c} \mathcal{S}(X)$. Since $\mathcal{A}$ is finitely generated there exists a finite set $\mathcal{B} \subseteq \mathcal{S}(X)$ such that $\overline{\mathcal{B}}=\mathcal{A}$. If $\mathcal{A}=\emptyset$, then $\llbracket \perp \rrbracket_{X}^{\prime}=\mathcal{A}$. If $\mathcal{B}=\left\{\varphi_{1}, \ldots, \varphi_{n}\right\}$ with $\varphi_{i} \in \mathcal{S}(X)$ then, for all $i$, we take the term

$$
t_{i}=\varphi_{i}\left(x_{1}\right) \cdot x_{1}+\cdots+\varphi_{i}\left(x_{m}\right) \cdot x_{m}
$$

where $\left\{x_{1}, \ldots, x_{m}\right\}$ is the support of $\varphi_{i}$. It is easy to check that $\llbracket t_{i} \rrbracket_{X}^{\prime}=\left\{\varphi_{i}\right\}$. Then by the inductive definition of $\llbracket \rrbracket_{X}^{\prime}$, one can easily verify that $\llbracket t_{1} \sqcup \cdots \sqcup t_{n} \rrbracket_{X}^{\prime}=\overline{\left\{\varphi_{1}, \ldots \varphi_{n}\right\}}=\mathcal{A}$.

Example 6.6. Recall $\mathcal{P}_{c} \mathcal{S}(1)$ for $S=\mathbb{R}^{+}$from Example 5.13. By restricting to the finitely generated convex sets, one obtains $\mathcal{P}_{f c} \mathcal{S}(1)=\{\emptyset\} \cup\left\{[a, b] \mid a, b \in \mathbb{R}^{+}\right\}$, that is the sets of the form $[a, \infty)$ are not finitely generated. Table 5 illustrates the morphism $\llbracket \rrbracket: T_{\Sigma^{\prime}, E^{\prime}}(1) \rightarrow \mathcal{P}_{c} \mathcal{S}(1)$. It is worth observing that every closed interval $[a, b]$ is denoted by a term in $T_{\Sigma^{\prime}, E^{\prime}}(1)$ for $1=\{x\}$ : indeed, $\llbracket(a \cdot x) \sqcup(b \cdot x) \rrbracket=[a, b]$. For $2=\{x, y\}, \mathcal{P}_{f c} \mathcal{S}(2)$ is the set containing all convex polygons: for instance the term $\left(r_{1} \cdot x+s_{1} \cdot y\right) \sqcup\left(r_{2} \cdot x+s_{2} \cdot y\right) \sqcup\left(r_{3} \cdot x+s_{3} \cdot y\right)$ denote a triangle with vertexes $\left(r_{i}, s_{i}\right)$. For $n=\left\{x_{0}, \ldots x_{n-1}\right\}$, it is easy to see that $\mathcal{P}_{f c} \mathcal{S}(n)$ contains all convex $n$-polytopes.

$$
\begin{aligned}
& \llbracket \lambda \cdot t \rrbracket= \begin{cases}{[\lambda \cdot a, \lambda \cdot b]} & \text { if } \lambda \neq 0, \llbracket t \rrbracket=[a, b] \\
\emptyset & \text { if } \lambda \neq 0, \llbracket t \rrbracket=\emptyset \\
{[0,0]} & \text { otherwise }\end{cases} \\
& \begin{array}{ll}
\llbracket x \rrbracket & =[1,1] \\
\llbracket 0 \rrbracket & =[0,0]
\end{array} \quad \llbracket t_{1}+t_{2} \rrbracket= \begin{cases}{\left[a_{1}+a_{2}, b_{1}+b_{2}\right]} & \text { if } \llbracket t_{i} \rrbracket=\left[a_{i}, b_{i}\right] \\
\emptyset & \text { otherwise }\end{cases} \\
& \llbracket \perp \rrbracket=\emptyset \\
& \llbracket t_{1} \sqcup t_{2} \rrbracket= \begin{cases}{\left[\min a_{i}, \max b_{i}\right]} & \text { if } \llbracket t_{i} \rrbracket=\left[a_{i}, b_{i}\right] \\
{\left[a_{1}, b_{1}\right]} & \text { if } \llbracket t_{1} \rrbracket=\left[a_{1}, b_{1}\right], \llbracket t_{2} \rrbracket=\emptyset \\
{\left[a_{2}, b_{2}\right]} & \text { if } \llbracket t_{2} \rrbracket=\left[a_{2}, b_{2}\right], \llbracket t_{1} \rrbracket=\emptyset \\
\emptyset & \text { otherwise }\end{cases}
\end{aligned}
$$

Table 5: The inductive definition of the function $\llbracket \cdot \rrbracket_{1}: T_{\Sigma^{\prime}, E^{\prime}}(1) \rightarrow \mathcal{P}_{c} \mathcal{S}(1)$ for $1=\{x\}$.

On combining $\mathcal{S}$ with $\mathcal{P}_{f}$ directly. One might wonder whether $\mathcal{P}_{f c} \mathcal{S}$ arises from a weak distributive law of type $\mathcal{S} \mathcal{P}_{f} \rightarrow \mathcal{P}_{f} \mathcal{S}$. By Theorem 1.6, to give a weak distributive law $\delta: \mathcal{S P}_{f} \rightarrow \mathcal{P}_{f} \mathcal{S}$ is equivalent to give a weak lifting of $\mathcal{P}_{f}$ to $\mathbb{E M}(\mathcal{S})$. This requires, at the very least, a functor $\tilde{\mathcal{P}_{f}}: \mathbb{E M}(\mathcal{S}) \rightarrow \mathbb{E M}(\mathcal{S})$, which we can presume would compute, for a $\mathcal{S}$-algebra $(X, a)$, the set of its convex and finitely generated subsets, in an analogous way of $\tilde{\mathcal{P}}$ computing the set of convex subsets. Moreover, we would also need a natural transformation $\iota: U^{\mathcal{S}} \tilde{\mathcal{P}}_{f} \rightarrow \mathcal{P}_{f} U^{\mathcal{S}}$ which, given a $\mathcal{S}$-algebra $(X, a)$, must assign to each convex and finitely generated subset $A$ of $X$ a finite subset of $X$ itself. A natural choice would be the set of generators of $A$, but how does one find them?

In general, when we take the convex closure of $A \subset X$, with $X$ an $S$-semimodule, we are adding to $A$ first all the segments between the elements of $A$ (the generators), then all the segments between all the new points that we have just added, and so on. The generators are exactly the extreme points of its closure $\bar{A}$, to use the terminology of [Grü03]. For an arbitrary $U \subseteq X$, the extreme points of $U$ are defined to be those points that are not properly inside any segment of $U$ :

$$
\operatorname{ext}(U)=\left\{x \in U \mid \forall y, z \in U \cdot \forall \lambda, \mu \in S \cdot\left(\left\{\begin{array}{l}
\lambda+\mu=1 \\
x=\lambda y+\mu z
\end{array}\right) \Longrightarrow y=z\right\}\right.
$$

If in a semimodule $X$ we have $A=\bar{B}$ with $B$ finite, then it is not difficult to see that $\operatorname{ext}(A) \subseteq B$. Hence a natural candidate for $\iota_{(X, a)}(A)$, with $A$ convex and finitely generated, is $\operatorname{ext}(A)$. However, this $\iota$ fails to be natural, and we show here why. The naturality square of $\iota$, for $f:(X, a) \rightarrow(Y, b)$ in $\mathbb{E M}(\mathcal{S})$, computes $f(\operatorname{ext}(A))$ and $\operatorname{ext}(f(A))$, which do not coincide if we take $S=\mathbb{R}^{+}, X=\mathcal{S}(\{x, y, z\}), Y=\mathcal{S}(\{u, v\}), f=\mathcal{S}(x \mapsto u, y \mapsto u, z \mapsto v)$ and finally

$$
A=\overline{\left\{\frac{1}{2} x+\frac{1}{2} y, \frac{1}{2} x+\frac{1}{2} z, z\right\}} .
$$

In this case, $\operatorname{ext}(A)=\left\{\frac{1}{2} x+\frac{1}{2} y, \frac{1}{2} x+\frac{1}{2} z, z\right\}$, so we have

$$
f(\operatorname{ext}(A))=\left\{u, \frac{1}{2} u+\frac{1}{2} v, v\right\} .
$$

On the other hand, since $f$ is a morphism of $\mathcal{S}$-algebras,

$$
f(A)=\overline{f\left(\left\{\frac{1}{2} x+\frac{1}{2} y, \frac{1}{2} x+\frac{1}{2} z, z\right\}\right)}=\overline{\left\{u, \frac{1}{2} u+\frac{1}{2} v, v\right\}}
$$

hence

$$
\operatorname{ext}(f(A))=\{u, v\} \neq f(\operatorname{ext}(A))
$$

This is, of course, saying less than saying that "there does not exist a weak lifting of $\mathcal{P}_{f}$ to $\mathbb{E M}(\mathcal{S})$ ", but it shows that a very natural candidate does not work. It remains an open problem whether a weak distributive law $\mathcal{S} \mathcal{P}_{f} \rightarrow \mathcal{P}_{f} \mathcal{S}$ exists.

## 7. Conclusions: Related and Future Work

Our work was inspired by [GP20a] where Goy and Petrisan compose the monads of powerset and probability distributions by means of a weak distributive law in the sense of Garner [Gar20]. Our results also heavily rely on the work of Clementino et al. [CHJ14] that illustrates sufficient conditions on a semiring $S$ for the existence of a weak distributive law $\delta: \mathcal{S P} \rightarrow \mathcal{P S}$. However, to the best of our knowledge, the alternative characterisation of $\delta$ provided by Theorem 3.5 was never shown.

This characterisation is essential to give a handy description of the lifting $\tilde{\mathcal{P}}: \mathbb{E M}(\mathcal{S}) \rightarrow$ $\mathbb{E M}(\mathcal{S})$ (Theorem 4.1) as well as to observe the strong relationships with the work of Jacobs (Remark 4.2) and of Klin and Rot (Remark 3.7). The weak distributive law $\delta$ also plays a key role in providing the algebraic theories presenting the composed monad $\mathcal{P}_{c} \mathcal{S}$ (Theorem 4.1) and its finitary restriction $\mathcal{P}_{f c} \mathcal{S}$ (Theorem 6.5). These two theories resemble those appearing in, respectively, [GP20a] and [BSV19] where the monad of probability distributions plays the role of the monad $\mathcal{S}$ in our work.

Theorem 5.3 allows to reuse the framework of coalgebraic trace semantics [HJS06] for modelling over $\mathbb{K l}\left(\mathcal{P}_{c} \mathcal{S}\right)$ systems with both nondeterminism and quantitative features. The alternative framework based on coalgebras over $\mathbb{E M}\left(\mathcal{P}_{c} \mathcal{S}\right)$ directly leads to nondeterministic weighted automata. A proper comparison with those in [DKV09] is left as future work. Thanks to the abstract results in [BPPR14], language equivalence for such coalgebras could be checked by means of coinductive up-to techniques. It is worth remarking that, since $\delta$ is a weak distributive law, then thanks to the work in [GP20b] up-to techniques are also sound for "convex-bisimilarity" (in coalgebraic terms, behavioural equivalence for the lifted functor $\tilde{\mathcal{P}}: \mathbb{E M}(\mathcal{S}) \rightarrow \mathbb{E M}(\mathcal{S}))$.

We conclude by recalling that we have two main examples of positive semifields: $\mathbb{B}$ ool and $\mathbb{R}^{+}$. Booleans could lead to a coalgebraic modal logic and trace semantics for alternating automata in the style of [KR15]. For $\mathbb{R}^{+}$, we hope that exploiting the ideas in [Rut05] our monad could shed some light on the behaviour of linear dynamical systems featuring some sort of nondeterminism.

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## Appendix A. Another weak distributive law of $\mathcal{P}$ over $\mathcal{P}$

In Remark 2.4 we have seen a Kleisli extension of the functor $\mathcal{P}$ to $\mathbb{R e l}$ given by $E: \mathbb{R e l} \rightarrow \mathbb{R e l}$, where for $R \subseteq X \times Y$ a relation, $E(R)$ is the function

$$
\begin{aligned}
& \mathcal{P}(X) \xrightarrow{E(R)} \mathcal{P}(Y) \\
& A \longmapsto\{y \in Y \mid \exists a \in A . a R y\} .
\end{aligned}
$$

Now we ask: do the unit $\eta$ and the multiplication $\mu$ of the powerset monad also extend to natural transformations $\tilde{\eta}: i d_{\mathbb{R e l}} \rightarrow E$ and $\tilde{\mu}: E E \rightarrow E$ ?

As observed in [HNPR06], if a natural transformation $\psi: T_{1} \rightarrow T_{2}$ has an extension $\tilde{\psi}: \tilde{T}_{1} \rightarrow \tilde{T}_{2}$ to $\mathbb{K} l(S)$, then it is unique: Definition 1.2 .5 requires that the components of $\tilde{\psi}$ be exactly $\tilde{\psi}_{X}=F_{S}\left(\psi_{X}\right)$, with $F_{S}: \mathbb{C} \rightarrow \mathbb{K} l(S)$ the free functor. Hence all that one has to do to see whether a $\psi$ has an extension to $\mathbb{K} l(S)$ is to check if the family of morphisms $\left(F_{S}\left(\psi_{X}\right)\right)_{X \in \mathbb{C}}$ is natural with respect to the functors $\tilde{T}_{1}, \tilde{T}_{2}$. (In other words, although the functors $T_{1}$ and $T_{2}$ may have many extensions to $\mathbb{K} l(S)$, once we fix an extension we can only have one $\tilde{\psi}$ between them.) In our case then, we have

$$
\tilde{\eta}_{X}=\{(x,\{x\}) \mid x \in X\} \subseteq X \times \mathcal{P}(X), \quad \tilde{\mu}_{X}=\left\{\left(\mathcal{A}, \bigcup_{A \in \mathcal{A}} A\right) \mid \mathcal{A} \in \mathcal{P} \mathcal{P}(X)\right\} \subseteq \mathcal{P} \mathcal{P}(X) \times \mathcal{P}(X)
$$

and the question is: are they natural transformations with respect to the functor $E$ ?
The naturality condition for $\tilde{\eta}$ states that for any $R \subseteq X \times Y$ the following square commutes in $\mathbb{R e l}$ :


It is not difficult to see that the upper leg is

$$
\{(x,\{y \in Y \mid x R y\}) \mid x \in X\}
$$

while the lower leg is

$$
\{(x,\{y\}) \mid(x, y) \in R\}
$$

and the two coincide if and only if $R$ is functional and total, that is if it is a function of sets. Hence $\eta$ does not extend to a natural transformation $\tilde{\eta}: i d_{\mathbb{R e l}} \rightarrow E$.

Regarding the multiplication, instead, we have that $\tilde{\mu}$ is natural as a transformation $E E \rightarrow E$ if and only if for all $R \subseteq X \times Y$ the following square commutes in $\mathbb{R e l}$ :


All the relations involved in the square above are actually functions, hence both legs are functions and, given $\mathcal{A} \in \mathcal{P} \mathcal{P} X$ :

$$
\begin{gathered}
E(R) \circ \tilde{\mu}_{X}(\mathcal{A})=\{y \in Y \mid \exists A \in \mathcal{A} . \exists x \in A \cdot x R y\} \\
\tilde{\mu}_{Y} \circ E E(R)=\bigcup_{A \in \mathcal{A}}\{y \in Y \mid \exists x \in A \cdot x R y\}
\end{gathered}
$$

and the two clearly are the same. Therefore $\mu$ does indeed extend to a natural transformation $\tilde{\mu}: E E \rightarrow E$.

This means the monad $\mathcal{P}$ does not extend to $\mathbb{R}$ el on a whole via the endofunctor $E$, but it does weakly extend. Hence there is an alternative weak distributive law $\mathcal{P} \mathcal{P} \rightarrow \mathcal{P} \mathcal{P}$ induced by $E$ and $\tilde{\mu}$. To find it we use the proof of Theorem 1.6: we need to compute $E\left(\ni_{X}\right)$, which by definition is

$$
\begin{aligned}
& \mathcal{P} \mathcal{P} X \xrightarrow{E(\ni x)} \mathcal{P} X \\
& \mathcal{A} \\
&\{x \in X \mid \exists A \in \mathcal{A} . A \ni x\} .
\end{aligned}
$$

This means that $E(\ni X)(\mathcal{A})=\bigcup_{A \in \mathcal{A}} A=\mu_{X}(\mathcal{A})$ : we have re-obtained $\mu$ ! Hence the weak distributive law $\delta: \mathcal{P} \mathcal{P} \rightarrow \mathcal{P} \mathcal{P}$ is

$$
\delta_{X}(\mathcal{A})=\left\{\mu_{X}(\mathcal{A})\right\}
$$

In other words, $\delta$ is the horizontal composition of $\mu$ and $\eta$, as in the following situation:


Now, what is the induced weak lifting of $\mathcal{P}$ to $\mathbb{E M}(\mathcal{P})$ ? Again the proof of Theorem 1.6 tells us how to find it. Given $(X, a: \mathcal{P} X \rightarrow X)$ a $\mathcal{P}$-algebra (which means that $X$ is a complete semilattice), the weak lifting $\tilde{\mathcal{P}}$ computes a new $\mathcal{P}$-algebra $\tilde{\mathcal{P}}(X, a)=(Y, b)$ where $Y$ is obtained by splitting the idempotent

$$
\begin{array}{r}
\mathcal{P} X \xrightarrow{\eta_{X}} \mathcal{P} \mathcal{P} X \xrightarrow{\delta_{X}} \mathcal{P} \mathcal{P} X \xrightarrow{\mathcal{P} a} \mathcal{P} X \\
A \longmapsto\{A\} \longmapsto \underbrace{\mu_{X}(\{A\})}_{A}\} \longmapsto\{a(A)\}
\end{array}
$$

whose fixed points are the singletons. Hence the carrier set of $\tilde{\mathcal{P}}(X, a)$ is $\mathcal{P}_{1} X=\{\{x\} \mid x \in$ $X\}$. The two natural transformations $\pi$ and $\iota$ required by the definition of weak lifting are

$$
\begin{array}{rlrl}
\mathcal{P} X \xrightarrow{\pi_{(X, a)}} \mathcal{P}_{1} X \\
A & \text { and } & \mathcal{P}_{1} X \xrightarrow{\iota_{(X, a)}} \mathcal{P} X \\
\{a(A)\} & & & \{x\} \longmapsto
\end{array}
$$

while the $\mathcal{P}$-algebra structure map $b: \mathcal{P} \mathcal{P}_{1} X \rightarrow \mathcal{P}_{1} X$ is given by:

$$
\begin{aligned}
\mathcal{P} \mathcal{P}_{1} X \xrightarrow{\mathcal{P}_{(X, a)}} \mathcal{P} \mathcal{P} X \xrightarrow{\delta_{X}} \mathcal{P} \mathcal{P} X \xrightarrow{\mathcal{P} a} \mathcal{P} X \xrightarrow{\pi_{(X, a)}} \mathcal{P}_{1} X \\
\mathcal{A} \longmapsto\{\bigcup \mathcal{A}\} \longmapsto\{a(\bigcup \mathcal{A})\} \longmapsto\{a(\bigcup \mathcal{A})\}
\end{aligned}
$$

In other words, it takes $\mathcal{A} \subseteq \mathcal{P}_{1} X$ which, being a collection of singletons in $X$, is in bijective correspondence with $\bigcup \mathcal{A} \subseteq X$, and returns the singleton of $a(\bigcup \mathcal{A})$. $\mathcal{P}_{1} X$ is therefore a $\mathcal{P}$-algebra essentially in the same way $X$ is.

## Appendix B. Further Details of the Proof of Theorem 4.1

Proof of Proposition 4.5. Let $(X, a)$ be a $\mathcal{S}$-algebra and let $\mathcal{A} \subseteq \mathcal{S} X$. We need to prove that:

$$
\mathcal{P} a\left(\overline{\mathcal{A}}^{\mu_{X}^{\mathcal{S}}}\right)=\overline{\mathcal{P} a(\mathcal{A})}^{a} .
$$

We have

$$
\mathcal{P} a\left(\overline{\mathcal{A}}^{\mu_{X}^{\mathcal{S}}}\right)=\left\{a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right) \mid \Psi \in \mathcal{S}^{2} X, \sum_{\psi \in \mathcal{S} X} \Psi(\psi)=1, \operatorname{supp} \Psi \subseteq \mathcal{A}\right\}
$$

while

$$
\overline{\mathcal{P} a(\mathcal{A})}^{a}=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \sum_{x \in X} \varphi(x)=1, \operatorname{supp} \varphi \subseteq\{a(\psi) \mid \psi \in \mathcal{A}\}\right\} .
$$

For the left-to-right inclusion: let $\Psi \in \mathcal{S}^{2} X$ be such that $\sum_{\psi \in \mathcal{S} X} \Psi(\psi)=1$ and with $\operatorname{supp} \Psi \subseteq \mathcal{A}$. Define $\varphi=\mathcal{S}(a)(\Psi)$. Then $x \in \operatorname{supp} \varphi$ if and only if there exists $\psi \in \operatorname{supp} \Psi$ such that $x=a(\psi)$. Hence, if $x \in \operatorname{supp} \varphi$, then $x=a(\psi)$ for some $\psi \in \mathcal{A}$. Moreover,

$$
\sum_{x \in X} \varphi(x)=\sum_{\substack{x \in X}} \sum_{\substack{\psi \in \operatorname{supp} \Psi \\ a(\psi)=x}} \Psi(\psi)=\sum_{\psi \in \mathcal{S} X} \Psi(\psi)=1
$$

Since $a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=a(\mathcal{S}(a)(\Psi))$ by the properties of $(X, a)$ as $\mathcal{S}$-algebra, and $\mathcal{S}(a)(\Psi)=\varphi$ by definition, we conclude.

Vice versa, given $\varphi \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi(x)=1$ and $\operatorname{supp} \varphi=\left\{a\left(\psi_{1}\right), \ldots, a\left(\psi_{n}\right)\right\}$ where $\psi_{1}, \ldots, \psi_{n} \in \mathcal{A}$, define

$$
\Psi=\left(\begin{array}{cc}
\psi_{1} \longmapsto & \varphi\left(a\left(\psi_{1}\right)\right) \\
\vdots & \vdots \\
\psi_{n} \longmapsto \varphi\left(a\left(\psi_{n}\right)\right)
\end{array}\right)
$$

Then $\sum_{\psi \in \mathcal{S} X} \Psi(\psi)=1$ and

$$
a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=a(\mathcal{S}(a)(\Psi))=a(\varphi)
$$

because $\mathcal{S}(a)(\Psi)=\varphi$.
Proposition B.1. Let $S$ be a positive semifield, $(X, a)$ a $\mathcal{S}$-algebra, $\Phi \in \mathcal{S P}_{c}^{a} X$. Then $\mathcal{P} a\left(\delta_{X}(\mathcal{S}(\iota)(\Phi))\right)=\mathcal{P} a(\mathfrak{c}(\mathcal{S}(\iota)(\Phi)))$.

Proof. In this proof we shall simply write $\Phi$ instead of the more verbose $\mathcal{S}(\iota)(\Phi)$. We want to prove that

$$
\begin{align*}
& \mathcal{P} a\left(\delta_{X}(\Phi)\right)= \\
& \qquad\left\{a(\psi) \mid \psi \in \mathcal{S} X . \exists u: \operatorname{supp} \Phi \rightarrow X . u(A) \in A, \forall x \in X . \psi(x)=\sum_{\substack{A \in \operatorname{supp} \Phi \\
u(A)=x}} \Phi(A)\right\} \tag{B.1}
\end{align*}
$$

where we have, by Theorem 3.5, that

$$
\mathcal{P} a\left(\delta_{X}(\Phi)\right)=\left\{a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right) \mid \Psi \in \mathcal{S}^{2} X, \sum_{\varphi \in \mathcal{S} X} \Psi(\varphi)=1, \operatorname{supp} \Psi \subseteq \mathfrak{c}(\Phi)\right\}
$$

First of all, $\emptyset$ is not a $\mathcal{S}$-algebra, because there is no map $\mathcal{S}(\emptyset) \rightarrow \emptyset$ given that $\mathcal{S}(\emptyset)=$ $\{\emptyset: \emptyset \rightarrow S\}$, hence $X \neq \emptyset$. Next, if $\Phi=\varepsilon: \mathcal{P} X \rightarrow S$, namely the 0 -constant function, then $\mathfrak{c}(\Phi)=\{\varepsilon: X \rightarrow S\}$ therefore one can easily see that the left-hand side of (B.1) is equal to $\{a(\varepsilon: X \rightarrow S)\}$. For the same reason, the right-hand side is also equal to $\{a(\varepsilon: X \rightarrow S)\}$. Moreover, if $\Phi(\emptyset) \neq 0$, then there is no $u: \operatorname{supp} \Phi \rightarrow X$ such that $u(\emptyset) \in \emptyset$, so $\mathfrak{c}(\Phi)=\emptyset$ and so is the left-hand side of (B.1); for the same reason, also the right-hand side is empty.

Suppose then, for the rest of the proof, that $\Phi \neq 0$ and that $\Phi(\emptyset)=0$.
For the right-to-left inclusion in (B.1): given $\psi \in \mathfrak{c}(\Phi)$, consider $\Psi=\eta_{\mathcal{S} X}^{\mathcal{S}}(\psi)=\Delta_{\psi} \in$ $\mathcal{S}^{2} X$. Then $\Psi$ clearly satisfies all the required properties and $\mu_{X}^{\mathcal{S}}(\Psi)=\psi$.

The left-to-right inclusion is more laborious. Let $\Psi \in \mathcal{S}^{2} X$ be such that $\sum_{\chi \in \mathcal{S} X} \Psi(\chi)=1$ and such that supp $\Psi \subseteq \mathfrak{c}(\Phi)$, that is, for all $\varphi \in \operatorname{supp} \Psi$ there is $u^{\varphi}: \operatorname{supp} \Phi \rightarrow X$ such that $u^{\varphi}(A) \in A$ for all $A \in \operatorname{supp} \Phi$ and $\varphi=\mathcal{S}\left(u^{\varphi}\right)(\Phi)$. We have to show that $a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=a(\psi)$ for some $\psi \in \mathcal{S} X$ of the form $\sum_{A \in \text { supp } \Phi} \Phi(A) \cdot u(A)$ for some choice function $u: \operatorname{supp} \Phi \rightarrow X$. Notice that the given $\Psi$ is a convex linear combination of functions $\varphi$ 's in $\mathcal{S} X$ like the one we have to produce: the trick will be to exploit the fact that each $A \in \operatorname{supp} \Phi$ is convex. We first give the idea of the proof, and then write the details of it.

Suppose supp $\Phi=\left\{A_{1}, \ldots, A_{n}\right\}$ and supp $\Psi=\left\{\varphi^{1}, \ldots, \varphi^{m}\right\}$. Call $u^{j}$ the choice function that generates $\varphi^{j}$. Then $\Psi$ is of this form:

$$
\Psi=(\underbrace{\left(\begin{array}{l}
u^{1}\left(A_{1}\right) \mapsto \Phi\left(A_{1}\right) \\
\vdots \\
u^{1}\left(A_{n}\right) \mapsto \Phi\left(A_{n}\right)
\end{array}\right)}_{\varphi^{1}} \mapsto \Psi\left(\varphi^{1}\right), \ldots, \underbrace{\left(\begin{array}{l}
u^{m}\left(A_{1}\right) \mapsto \Phi\left(A_{1}\right) \\
\vdots \\
u^{m}\left(A_{n}\right) \mapsto \Phi\left(A_{n}\right)
\end{array}\right)}_{\varphi^{m}} \mapsto \Psi\left(\varphi^{m}\right))
$$

Define the following element of $\mathcal{S}^{2} X$ :

$$
\Psi^{\prime}=(\underbrace{\left(\begin{array}{l}
u^{1}\left(A_{1}\right) \mapsto \Psi\left(\varphi^{1}\right) \\
\vdots \\
u^{m}\left(A_{1}\right) \mapsto \Psi\left(\varphi^{m}\right)
\end{array}\right)}_{\chi^{1}} \mapsto \Phi\left(A_{1}\right), \ldots, \underbrace{\left(\begin{array}{l}
u^{1}\left(A_{n}\right) \mapsto \Psi\left(\varphi^{1}\right) \\
\vdots \\
u^{m}\left(A_{n}\right) \mapsto \Psi\left(\varphi^{m}\right)
\end{array}\right)}_{\chi^{n}} \mapsto \Phi\left(A_{n}\right))
$$

Observe that $u^{1}\left(A_{i}\right), \ldots, u^{m}\left(A_{i}\right) \in A_{i}$ by definition, and $A_{i}$ is convex by assumption: since $\sum_{j=1}^{m} \Psi\left(\varphi^{j}\right)=1$, we have that $a\left(\chi^{i}\right) \in A_{i}$. Set then $u\left(A_{i}\right)=a\left(\chi^{i}\right)$ and define $\psi=\mathcal{S}(a)\left(\Psi^{\prime}\right)$ : we have $\psi \in \mathfrak{c}(\Phi)$ with $u$ as the generating choice function. We shall prove that $\mu_{X}^{\mathcal{S}}(\Psi)=\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)$, thus obtaining

$$
a(\psi)=a\left(\mathcal{S}(a)\left(\Psi^{\prime}\right)\right)=a\left(\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)\right)=a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)
$$

as desired. We now set this down in detail.
Let $A \in \operatorname{supp} \Phi$. Define for all $x \in X$ :

$$
\chi_{A}(x)=\sum_{\substack{\varphi \in s u p p \\ x=u^{\varphi}(A)}} \Psi(\varphi) .
$$

Then $\operatorname{supp} \chi_{A}=\left\{u^{\varphi}(A) \mid \varphi \in \operatorname{supp} \Psi\right\}$ is finite, hence $\chi_{A} \in \mathcal{S} X$. Let now

$$
u(A)=a\left(\chi_{A}\right)
$$

We have that

$$
\sum_{x \in X} \chi_{A}(x)=\sum_{x \in X} \sum_{\substack{\varphi \in \text { supp } \Psi \\ x \in u^{\varphi}(A)}} \Psi(\varphi)=\sum_{\varphi \in \text { supp } \Psi} \Psi(\varphi)=1,
$$

hence $u(A) \in A$ because $A$ is convex. Define, for all $x \in X$ :

$$
\psi(x)=\sum_{\substack{A \in \operatorname{supp} \varphi \\ x=a\left(\chi_{A}\right)}} \Phi(A) .
$$

Then supp $\psi=\{u(A) \mid A \in \operatorname{supp} \Phi\}$ so $\psi \in \mathcal{S} X$. To conclude, we have to prove that $a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=a(\psi)$. To that end, let for all $\chi \in \mathcal{S} X$

$$
\Psi^{\prime}(\chi)=\sum_{\substack{A \in \text { supp } \\ \chi A=\chi}} \Phi(A)
$$

Then supp $\Psi^{\prime}=\left\{\chi_{A} \mid A \in \operatorname{supp} \Phi\right\}$ is finite. Then we have for all $x \in X$ that

$$
\begin{aligned}
\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)(x) & =\sum_{\chi \in \text { supp } \Psi^{\prime}} \Psi^{\prime}(\chi) \cdot \chi(x) \\
& \stackrel{(*)}{=} \sum_{A \in \text { supp } \Phi} \Phi(A) \cdot \sum_{\substack{\varphi \in \text { supp } \Psi \\
x=u^{\varphi}(A)}} \Psi(\varphi) \\
& =\sum_{\varphi \in \operatorname{supp} \Psi} \Psi(\varphi) \cdot \sum_{\substack{A \in \operatorname{supp} \Phi \\
x=u^{\varphi}(A)}} \Phi(A) \\
& =\sum_{\varphi \in \mathcal{S} X} \Psi(\varphi) \cdot \varphi(x) \\
& =\mu_{X}^{\mathcal{S}}(\Psi)(x)
\end{aligned}
$$

where equation $(*)$ is explained in a similar way than $\left(*_{1}\right)$ in the proof of Theorem 3.5. We also have that

$$
\mathcal{S}(a)\left(\Psi^{\prime}\right)(x)=\sum_{\chi \in a^{-1}\{x\}} \Psi^{\prime}(\chi)=\sum_{\substack{\chi \in \mathcal{S} X \\ a(\chi)=x}} \sum_{\substack{A \in \text { supp } \Phi \\ \chi A=\chi}} \Phi(A)=\sum_{\substack{A \in \operatorname{supp} \Phi \\ a(\chi A)=x}} \Phi(A)=\psi(x) .
$$

Therefore,

$$
a(\psi)=a\left(\mathcal{S}(a)\left(\Psi^{\prime}\right)\right)=a\left(\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)\right)=a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right) .
$$

Proposition B.2. The following diagram commutes.


Proof. In this proof we shall simply write $\bar{A}$ for $\bar{A}^{a}$ for any $A \subseteq X$, because there is no risk of confusion. We have to show that

$$
\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \sum_{x \in X} \varphi(x)=1, \operatorname{supp} \varphi \subseteq \bigcup_{U \in \mathcal{U}} U\right\}=\bigcup_{\substack{\Phi \in \mathcal{S P} \mathcal{P}_{c}^{a} X \\ \sum_{c} \Phi(C)=1 \\ \text { supp } \Phi \subseteq\{\bar{U} \mid U \in \mathcal{U}\}}} \alpha_{a}(\Phi) .
$$

For the left-to-right inclusion: let $\varphi \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi(x)=1$, and suppose supp $\varphi=$ $\left\{x_{1}, \ldots, x_{n}\right\}$. We have that for all $i \in \underline{n}$ there exists $A_{i} \in \mathcal{U}$ such that $x_{i} \in A_{i}$. While the $x_{i}$ 's are distinct, the $A_{i}$ 's need not be. Define, for all $B \in \mathcal{P}_{c}^{a} X$,

$$
\Phi(B)=\sum_{\substack{i \in n \\ B=\overline{A_{i}}}} \varphi\left(x_{i}\right)
$$

Then $\operatorname{supp} \Phi=\left\{\overline{A_{1}}, \ldots, \overline{A_{n}}\right\}$ hence $\Phi \in \mathcal{S} \mathcal{P}_{c}^{a} X$. One can then prove that $a(\varphi) \in \alpha_{a}(\Phi)$ in the same way as showed in the proof of Proposition 4.6.

Vice versa, an element of the right-hand side is of the form $a(\psi)$ for some $\Phi \in \mathcal{S P}_{c}^{a} X$ such that $\sum_{B \in \mathcal{P}_{c}^{a} X} \Phi(B)=1, \operatorname{supp} \Phi=\left\{\overline{A_{1}}, \ldots, \overline{A_{n}}\right\}$ with $A_{i} \in \mathcal{U}$ for all $i \in \underline{n}$ and for some $\psi \in \mathcal{S} X$ and $u \in \prod_{i=1}^{n} \overline{A_{i}}$ such that

$$
\psi(x)=\sum_{\substack{i \in \underline{n} \\ x=u_{i}}} \Phi\left(\overline{A_{i}}\right) .
$$

We want to prove that $a(\psi)=a(\varphi)$ for an appropriate $\varphi \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi(x)=1$ and $\operatorname{supp} \varphi \subseteq \bigcup_{U \in \mathcal{U}} U$.

Since $u_{i} \in \overline{A_{i}}$, we have that $u_{i}=a\left(\varphi_{i}\right)$ for some $\varphi_{i} \in \mathcal{S} X$ such that $\sum_{x \in X} \varphi_{i}(x)=1$ and $\operatorname{supp} \varphi_{i} \subseteq A_{i}$. These $\varphi_{i}$ 's are not necessarily distinct. Let then, for all $\varphi \in \mathcal{S} X$,

$$
\Psi(\varphi)=\sum_{\substack{i \in n \\ \varphi=\overline{\varphi_{i}}}} \Phi\left(\overline{A_{i}}\right)
$$

Then supp $\Psi=\left\{f_{1}, \ldots, f_{n}\right\}$ so $\Psi \in \mathcal{S}^{2} X$ and it is easy to prove, by means of direct calculations, that $\psi=\mathcal{S}(a)(\Phi), \sum_{x \in X} \mu_{X}^{\mathcal{S}}(\Psi)=1$ and that supp $\mu_{X}^{\mathcal{S}}(\Psi) \subseteq \bigcup_{U \in \mathcal{U}} U$. Then we have that $a(\psi)=a(\mathcal{S}(a)(\Psi))=a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)$, and $\mu_{X}^{\mathcal{S}}(\Psi)$ is the $\varphi$ we were looking for.

## Appendix C. Proof of Proposition 6.4

In this appendix, we provide a proof of Proposition 6.4. We need to show that:
(a) for all functions $f: X \rightarrow Y, \mathcal{P}_{c} \mathcal{S}(f): \mathcal{P}_{c} \mathcal{S}(X) \rightarrow \mathcal{P}_{c} \mathcal{S}(Y)$ restricts and corestricts to $\mathcal{P}_{f c} \mathcal{S}(X) \rightarrow \mathcal{P}_{f c} \mathcal{S}(Y) ;$
(b) the unit $\eta_{X}^{\mathcal{P}_{c} \mathcal{S}}$ can be corestricted to $\mathcal{P}_{f c} \mathcal{S}(X)$ (trivial);
(c) the multiplication $\mu_{X}^{\mathcal{P}_{c} \mathcal{S}}$ restricts and corestricts to $\mathcal{P}_{f c} \mathcal{S} \mathcal{P}_{f c} \mathcal{S} X \rightarrow \mathcal{P}_{f c} \mathcal{S} X$.

Hereafter, we shall simply write $\overline{\mathcal{B}}$ in lieu of $\overline{\mathcal{B}}^{\mu_{X}^{\mathcal{S}}}$, for some $\mathcal{B} \subseteq \mathcal{S} X$.
Proposition C.1. Let $f: X \rightarrow Y, \mathcal{A} \subseteq \mathcal{S} X$ such that $\mathcal{A}=\overline{\mathcal{B}}$ for some finite $\mathcal{B} \subseteq \mathcal{A}$. Then

$$
\{\mathcal{S} f(\varphi) \mid \varphi \in \mathcal{A}\}=\overline{\{\mathcal{S} f(\psi) \mid \psi \in \mathcal{B}\}} .
$$

Proof. For the left to right inclusion: let $\varphi \in \mathcal{A}=\overline{\mathcal{B}}$. Then there exists $\Psi \in \mathcal{S}^{2} X$ such that $\sum_{\chi \in \mathcal{S} X} \Psi(\chi)=1$, supp $\Psi \subseteq \mathcal{B}, \varphi=\mu_{X}^{\mathcal{S}}(\Psi)$. We have to prove that there is a $\Psi^{\prime} \in \mathcal{S}^{2} Y$ such that $\sum_{\chi \in \mathcal{S} Y} \Psi^{\prime}(\chi)=1$, supp $\Psi^{\prime} \subseteq\{\mathcal{S} f(\psi) \mid \psi \in \mathcal{B}\}, \mu_{Y}^{\mathcal{S}}\left(\Psi^{\prime}\right)=\varphi$.

Now, because of the naturality of $\mu^{\mathcal{S}}$, we have

$$
\mathcal{S} f(\varphi)=\mathcal{S} f\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=\mu_{Y}^{\mathcal{S}}\left(\mathcal{S}^{2} f(\Psi)\right) .
$$

One can easily see that $\mathcal{S}^{2} f(\Psi)$ works for our desired $\Psi^{\prime}$.
Vice versa, let $\Psi^{\prime} \in \mathcal{S}^{2} Y$ be such that $\sum_{\chi \in \mathcal{S} Y} \Psi^{\prime}(\chi)=1$ and supp $\Psi^{\prime} \subseteq\{\mathcal{S} f(\psi) \mid \psi \in \mathcal{B}\}$. We have to show that there is $\varphi \in \mathcal{A}$ such that $\mu_{Y}^{\mathcal{S}}\left(\Psi^{\prime}\right)=\mathcal{S} f(\varphi)$.

We have that $\Psi^{\prime}$ is of the form

$$
\left(\begin{array}{cc}
\mathcal{S} f\left(\psi_{1}\right) \longmapsto & \Psi^{\prime}\left(\mathcal{S} f\left(\psi_{1}\right)\right) \\
\vdots & \vdots \\
\mathcal{S} f\left(\psi_{n}\right) \longmapsto & \Psi^{\prime}\left(\mathcal{S} f\left(\psi_{n}\right)\right)
\end{array}\right)
$$

Then that means that $\Psi^{\prime}=\mathcal{S}(\mathcal{S} f)(\Psi)$ where $\Psi$ is defined as

$$
\Psi=\left(\begin{array}{cc}
\psi_{1} \longmapsto & \Psi\left(\mathcal{S} f\left(\psi_{1}\right)\right) \\
\vdots & \vdots \\
\psi_{n} \longmapsto & \Psi\left(\mathcal{S} f\left(\psi_{n}\right)\right)
\end{array}\right) \in \mathcal{S}^{2} X
$$

Then, again by naturality of $\mu^{\mathcal{S}}$, we have

$$
\mu_{Y}^{\mathcal{S}}\left(\Psi^{\prime}\right)=\mu_{Y}^{\mathcal{S}}\left(\mathcal{S}^{2} f(\Psi)\right)=\mathcal{S} f\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)
$$

and $\varphi=\mu_{X}^{\mathcal{S}}(\Psi)$ is indeed in $\mathcal{A}$ because $\mathcal{A}=\overline{\mathcal{B}}$.
This tells us that $\mathcal{P}_{f c} \mathcal{S}$ is an endofunctor on Set. Next, $\eta_{X}^{\mathcal{P}_{c} \mathcal{S}}(x)=\left\{\Delta_{x}\right\}$ and $\left\{\Delta_{x}\right\}$ is obviously finitely generated, therefore $\eta^{\mathcal{P}_{c} \mathcal{S}}$ corestricts to $\mathcal{P}_{f c} \mathcal{S}$. How about $\mu^{\mathcal{P}_{c} \mathcal{S}}$ ?

Recall that $\mu_{X}^{\mathcal{P}_{\mathcal{C}} \mathcal{S}}: \mathcal{P}_{c} \mathcal{S} \mathcal{P}_{c} \mathcal{S} X \rightarrow \mathcal{P}_{c} \mathcal{S} X$ is defined for every $\mathscr{A}$ convex subset of $\mathcal{S}\left(\mathcal{P}_{c}(\mathcal{S} X)\right)$ as

$$
\mu_{X}^{\mathcal{P}_{X} \mathcal{S}}(\mathscr{A})=\bigcup_{\Omega \in \mathscr{A}}\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathfrak{c}(\Omega)\right\}
$$

where

$$
\mathfrak{c}(\Omega)=\left\{\Psi \in \mathcal{S}^{2} X \mid \forall \mathcal{A} \in \operatorname{supp} \Omega . \exists u_{\mathcal{A}} \in \mathcal{A} . \forall \varphi \in \mathcal{S} X . \Psi(\varphi)=\sum_{\substack{\mathcal{A} \in \operatorname{supp} \Phi \\ \varphi=u_{\mathcal{A}}}} \Omega(\mathcal{A})\right\}
$$

We aim to prove that $\bigcup_{\Omega \in \mathscr{A}}\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathfrak{c}(\Omega)\right\}$ is, in fact, finitely generated in the hypothesis that $\mathscr{A} \in \mathcal{P}_{f_{c}} \mathcal{S P}_{f c} \mathcal{S} X$. We will achieve this in three steps.

- Step 1: let $\mathscr{B}$ be a finite subset of $\mathscr{A}$ such that $\mathscr{A}=\overline{\mathscr{B}}$. Then we prove that

$$
\bigcup_{\Omega \in \mathscr{A}}\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathfrak{c}(\Omega)\right\}=\bigcup_{\Theta \in \mathscr{B}}\left\{\mu_{X}^{\mathcal{S}}(\Xi) \mid \Xi \in \mathfrak{c}(\Theta)\right\}
$$

showing therefore that we can reduce ourselves to a finite union.

- Step 2: we prove that each $\left\{\mu_{X}^{\mathcal{S}}(\Xi) \mid \Xi \in \mathfrak{c}(\Theta)\right\}$ as of Step 1 is convex and finitely generated.
- Step 3: we prove that the convex closure of a finite union of convex and finitely generated sets is in turn finitely generated.
The next three lemmas will perform each step.
Lemma C.2. Let $\mathscr{A} \in \mathcal{P}_{f c} \mathcal{S P}_{f c} \mathcal{S} X$ and let $\mathscr{B}$ be a finite subset of $\mathscr{A}$ such that $\mathscr{A}=\overline{\mathscr{B}}$. Then

$$
\left.\bigcup_{\Omega \in \mathscr{A}}\left\{\mu_{X}^{\mathcal{S}}(\Psi) \mid \Psi \in \mathfrak{c}(\Omega)\right\}=\overline{\bigcup_{\Theta \in \mathscr{B}}\left\{\mu_{X}^{\mathcal{S}}(\Xi) \mid \Xi \in \mathfrak{c}(\Theta)\right\}}\right\}_{X}^{\mathcal{S}}
$$

Proof. Let $\Omega \in \mathscr{A}=\overline{\mathscr{B}}$. We have that

$$
\Omega=\mu_{\mathcal{P}_{c} \mathcal{S} X}^{\mathcal{S}}\left(\begin{array}{ccc}
\Theta_{1} \longmapsto & \sigma_{1} \\
\vdots & & \vdots \\
\Theta_{t} \longmapsto & \sigma_{t}
\end{array}\right)
$$

where $\Theta_{i} \in \mathscr{B}$ and $\sum_{l=1}^{t} \sigma_{l}=1$. Notice that $\operatorname{supp} \Omega=\bigcup_{l=1}^{t} \operatorname{supp} \Theta_{l}$. Now, if supp $\Omega=$ $\left\{\mathcal{A}_{1}, \ldots, \mathcal{A}_{n}\right\}$ say, we have that any $\Psi \in \mathfrak{c}(\Omega)$ is of the form

$$
\Psi=\left(\begin{array}{cc}
\varphi_{1} \longmapsto \Omega\left(\mathcal{A}_{1}\right)=\sum_{l=1}^{t} \sigma_{l} \cdot \Theta_{l}\left(\mathcal{A}_{1}\right) \\
\vdots & \vdots \\
\varphi_{n} \longmapsto \Omega\left(\mathcal{A}_{n}\right)=\sum_{l=1}^{t} \sigma_{l} \cdot \Theta_{l}\left(\mathcal{A}_{n}\right)
\end{array}\right)
$$

where each $\varphi_{i} \in \mathcal{A}_{i} \subseteq \mathcal{S} X$. This leads us to define, for each $l \in \underline{t}$, a function $\Xi_{l} \in \mathcal{S}^{2} X$ as:

$$
\Xi_{l}=\left(\begin{array}{cc}
\varphi_{1} & \longmapsto \\
\Theta_{l}\left(\mathcal{A}_{1}\right) \\
\vdots & \vdots \\
\varphi_{n} \longmapsto & \Theta_{l}\left(\mathcal{A}_{n}\right)
\end{array}\right)
$$

Notice that, in fact, $\Xi_{l} \in \mathfrak{c}\left(\Theta_{l}\right)$. Indeed, fixed $l$, we have that $\operatorname{supp} \Theta_{l} \subseteq \operatorname{supp} \Omega$, so it can be the case that $\Theta_{l}\left(\mathcal{A}_{i}\right)=0$ for some $i \in \underline{n}$, in which case we have $\Xi_{l}\left(\varphi_{i}\right)=0$. Nonetheless, for each $\mathcal{A}_{i}$ in $\operatorname{supp} \Theta_{l}$, the function $\Xi_{l}$ chooses one of its elements and associates to it $\Theta_{l}\left(\mathscr{A}_{i}\right)$. Define then

$$
\Psi^{\prime}=\left(\begin{array}{cc}
\mu_{X}^{\mathcal{S}}\left(\Xi_{1}\right) \longmapsto & \sigma_{1} \\
\vdots & \vdots \\
\mu_{X}^{\mathcal{S}}\left(\Xi_{t}\right) \longmapsto & \sigma_{t}
\end{array}\right)
$$

Then clearly $\sum_{\chi \in \mathcal{S} X} \Psi^{\prime}(\chi)=1$, and

$$
\operatorname{supp} \chi \subseteq \bigcup_{l=1}^{t}\left\{\mu_{X}^{\mathcal{S}}(\Xi) \mid \Xi \in \mathfrak{c}\left(\Theta_{l}\right)\right\} \subseteq \bigcup_{\Theta \in \mathscr{B}}\left\{\mu_{X}^{\mathcal{S}}(\Xi) \mid \Xi \in \mathfrak{c}(\Theta)\right\}
$$

It is a matter of direct calculations to show that $\mu_{X}^{\mathcal{S}}(\Psi)=\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)$. This proves the left-toright inclusion. The vice versa is immediate to see, recalling that we know that the left-hand side is convex.

Applying the following lemma for the $\mathcal{S}$-algebra $\left(\mathcal{S} X, \mu_{X}^{\mathcal{S}}\right)$, we obtain Step 2.
Lemma C.3. Let $(X, a)$ in $\mathbb{E M}(\mathcal{S})$. Then for all $\Phi \in \mathcal{S P}_{c f}^{a} X$ we have that

$$
\{a(\varphi) \mid \varphi \in \mathfrak{c}(\Phi)\}={\overline{\left\{a(\psi) \mid \psi \in \mathfrak{c}\left(\Phi^{\prime}\right)\right\}}}^{a}
$$

where, if supp $\Phi=\left\{A_{1}, \ldots, A_{n}\right\}$ say, and for all $i A_{i}={\overline{B_{i}}}^{a}$ with $B_{i} \subseteq A_{i}$ finite, then

$$
\Phi^{\prime}=\left(B_{1} \mapsto \Phi\left(A_{1}\right), \ldots, B_{n} \mapsto \Phi\left(A_{n}\right)\right)
$$

Proof. Let $\varphi \in \mathfrak{c}(\Phi)$. Then we can write $\varphi$ as

$$
\varphi=\left(u_{1} \mapsto \Phi\left(A_{1}\right), \ldots, u_{n} \mapsto \Phi\left(A_{n}\right)\right)
$$

where $u_{i} \in A_{i}$ for all $i$. Notice that it is possible for $u_{i}=u_{j}$ for $i \neq j$ : in that case, the notation above implicitly says that $u_{i} \mapsto \Phi\left(A_{i}\right)+\Phi\left(A_{j}\right)$. Now, since $A_{i}={\overline{B_{i}}}^{a}$, we have that $u_{i}=a\left(\chi_{i}\right)$ for some $\chi_{i} \in \mathcal{S} X$ such that $\sum \chi_{i}(x)=1$ and supp $\chi_{i} \subseteq B_{i}$. So:

$$
\begin{aligned}
\varphi & =\left(a\left(\chi_{1}\right) \mapsto \Phi\left(A_{1}\right), \ldots, a\left(\chi_{n}\right) \mapsto \Phi\left(A_{n}\right)\right) \\
& =\mathcal{S}(a)\left(\chi_{1} \mapsto \Phi\left(A_{1}\right), \ldots, \chi_{n} \mapsto \Phi\left(A_{n}\right)\right)
\end{aligned}
$$

Call $\Psi=\left(\chi_{1} \mapsto \Phi\left(A_{1}\right), \ldots, \chi_{n} \mapsto \Phi\left(A_{n}\right)\right)$. Then we just said that $\varphi=\mathcal{S}(a)(\Psi)$. We can write $\Psi$ more explicitly, by listing down the action of each $\chi_{i}$ :

$$
\Psi=\left(\begin{array}{l}
\chi_{1}=\left(\begin{array}{l}
x_{1}^{1} \mapsto \lambda_{1}^{1} \\
\vdots \\
\\
x_{s_{1}}^{1} \mapsto \lambda_{s_{1}}^{1}
\end{array}\right) \mapsto \Phi\left(A_{1}\right), \quad \ldots \quad \chi_{n}=\left(\begin{array}{l}
x_{1}^{n} \mapsto \lambda_{1}^{n} \\
\vdots \\
x_{s_{n}}^{n} \mapsto \lambda_{s_{n}}^{n}
\end{array}\right) \mapsto \Phi\left(A_{n}\right)
\end{array}\right)
$$

where, for each $k=1, \ldots, n, \sum_{j=1}^{s_{k}} \lambda_{j}^{k}=1$ and for all $j=1, \ldots, s_{k}$ we have $x_{j}^{k} \in B_{k}$.
Now, define for all $w \in \prod_{k=1}^{n} \underline{s_{k}}$, a $n$-tuple whose $j$-th entry is a number between 1 and $s_{j}$, the function

$$
\psi_{w}=\left(x_{w_{1}}^{1} \mapsto \Phi\left(A_{1}\right), \ldots, x_{w_{n}}^{n} \mapsto \Phi\left(A_{n}\right)\right) \text {, i.e. } \psi_{w}(x)=\sum_{\substack{i \in n \\ x=x_{w_{i}}^{i}}} \Phi\left(A_{i}\right) .
$$

Define also

$$
\Psi^{\prime}=\left(\psi_{w} \mapsto \prod_{k=1}^{n} \lambda_{w_{k}}^{k}\right)_{w \in \prod_{\underline{s_{k}}}} \text { i.e. } \Psi^{\prime}(\psi)=\sum_{\substack{w \in \prod_{\begin{subarray}{c}{s_{k}} }}} \\
{\prod_{k=1}}\end{subarray}}^{n} \lambda_{w_{k}}^{k} .
$$

Notice that, by Lemma 3.9, we have that

$$
\sum_{w \in \prod_{\underline{s_{k}}}} \prod_{k=1}^{n} \lambda_{w_{k}}^{k}=\prod_{k=1}^{n} \sum_{j=1}^{s_{k}} \lambda_{j}^{k}=\prod_{k=1}^{n} 1=1
$$

This immediately implies that $\sum_{\psi \in \mathcal{S} X} \Psi^{\prime}(\psi)=1$. Next, we show that $\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)=\mu_{X}^{\mathcal{S}}(\Psi)$.

$$
\begin{aligned}
& \mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)(x)=\sum_{\psi \in \mathcal{S} X} \Psi^{\prime}(\psi) \cdot \psi(x) \\
& =\sum_{w}\left[\prod_{k=1}^{n} \lambda_{w_{k}}^{k} \cdot \sum_{\substack{i \in n \\
x=x_{w_{i}}^{i}}} \Phi\left(A_{i}\right)\right] \\
& =\sum_{w} \sum_{\substack{i \in n \\
x=x_{w_{i}}^{i}}}\left[\left(\prod_{k=1}^{n} \lambda_{w_{k}}^{k}\right) \cdot \Phi\left(A_{i}\right)\right] \\
& =\sum_{i \in \underline{n}} \sum_{\substack{w \\
x=x_{w_{i}}^{i}}}[\Phi\left(A_{i}\right) \cdot \underbrace{\lambda_{w_{i}}^{i}}_{=\chi_{i}\left(x_{w_{i}}^{i}\right)=\chi_{i}(x)} \cdot \prod_{\substack{k \in n \\
k \neq i}} \lambda_{w_{k}}^{k}] \\
& =\sum_{i=1}^{n} \Phi\left(A_{i}\right) \cdot \chi_{i}(x) \cdot \underbrace{\sum_{\substack{w \\
x=x_{w_{i}}}} \prod_{k \neq i} \lambda_{w_{k}}^{k}}_{=\prod_{k \neq i} \sum_{j=1}^{s_{k}} \lambda_{j}^{k}=1} \\
& =\mu_{X}^{\mathcal{S}}(\Psi)(x) .
\end{aligned}
$$

Hence:

$$
a(\varphi)=a(\mathcal{S}(a)(\Psi))=a\left(\mu_{X}^{\mathcal{S}}(\Psi)\right)=a\left(\mu_{X}^{\mathcal{S}}\left(\Psi^{\prime}\right)\right)=a\left(\mathcal{S}(a)\left(\Psi^{\prime}\right)\right)
$$

where $a\left(\mathcal{S}(a)\left(\Psi^{\prime}\right)\right)$ indeed belongs to $\overline{\left\{a(\psi) \mid \psi \in \mathfrak{c}\left(\Phi^{\prime}\right)\right\}^{a}}$ because

$$
\operatorname{supp}\left(\mathcal{S}(a)\left(\Psi^{\prime}\right)\right)=\left\{a\left(\psi_{w}\right) \mid w \in \prod_{k=1^{n}} s_{k}\right\} \subseteq\left\{a(\psi) \mid \psi \in \mathfrak{c}\left(\Phi^{\prime}\right)\right\}
$$

and the sum of all its images is $\sum_{w} \prod_{k=1}^{n} \lambda_{w_{k}}^{k}=1$.
Notice that if $\Phi^{\prime} \in \mathcal{S P} X$ is such that $\operatorname{supp} \Phi^{\prime} \subseteq \mathcal{P}_{f}(X)$, then $\mathfrak{c}\left(\Phi^{\prime}\right)$ is finite. Finally, the next lemma, when $A$ and $B$ are finite, proves Step 3.

Lemma C.4. Let $(X, a)$ be in $\mathbb{E M}(\mathcal{S}), A, B \subseteq X$. Then $\overline{\bar{A} \cup \bar{B}}=\overline{A \cup B}$.
Proof. We have:

$$
\begin{aligned}
& \overline{\bar{A} \cup \bar{B}}=\left\{a(\varphi) \mid \varphi \in \mathcal{S} X, \sum_{x} \varphi(x)=1, \operatorname{supp} \varphi \subseteq \bar{A} \cup \bar{B}\right\} \\
& \bar{A} \cup \bar{B}=\left\{a(\psi) \mid \psi \in \mathcal{S} X, \sum_{x} \psi(x)=1, \text { supp } \psi \subseteq A \text { or supp } \psi \subseteq B\right\} \\
& \overline{A \cup B}=\left\{a(\chi) \mid \chi \in \mathcal{S} X, \sum_{x} \chi(x)=1, \operatorname{supp} \chi \subseteq A \cup B\right\} .
\end{aligned}
$$

Let then $x \in \overline{\bar{A} \cup \bar{B}}$. Then

$$
x=a(\varphi)=a\left(\begin{array}{c}
a\left(\psi_{1}\right) \longmapsto \varphi\left(a\left(\psi_{1}\right)\right) \\
\vdots \\
a\left(\psi_{n}\right) \longmapsto \varphi\left(a\left(\psi_{n}\right)\right)
\end{array}\right)=a(\mathcal{S}(a)(\Phi))=a\left(\mu_{X}^{\mathcal{S}}(\Phi)\right)
$$

where $\Phi=\left(\psi_{1} \mapsto \varphi\left(a\left(\psi_{1}\right)\right), \ldots, \psi_{n} \mapsto \varphi\left(a\left(\psi_{n}\right)\right)\right.$. Calling $\chi=\mu_{X}^{\mathcal{S}}(\Phi)$, one can easily check that $\sum_{x} \chi(x)=1$ and that supp $\chi \subseteq A \cup B$, hence $\overline{\bar{A}} \cup \bar{B} \subseteq \overline{A \cup B}$. The other inclusion is obvious, given that $A \cup B \subseteq \bar{A} \cup \bar{B}$.

This concludes the proof of Proposition 6.4.


[^0]:    Key words and phrases: Weak distributive laws, semilattices, semimodules, convexity.
    An extended abstract of this paper has appeared in the proceedings of the FoSSaCS 2021 conference, see [BS21].

[^1]:    ${ }^{1}$ More precisely, we should write $\mathcal{S}(u)\left(\Phi^{\prime}\right)$ where $\Phi^{\prime}$ is the restriction of $\Phi$ to supp $\Phi$.

[^2]:    ${ }^{2}$ For the sake of brevity, we are ignoring the case where some $A_{i}=\emptyset$.

