

ON STREAMS THAT ARE FINITELY RED

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ABSTRACT. Mixing induction and coinduction, we study alternative definitions of streams being finitely red. We organize our definitions into a hierarchy including also some well-known alternatives in intuitionistic analysis. The hierarchy collapses classically, but is intuitionistically of strictly decreasing strength. We characterize the differences in strength in a precise way by weak instances of the Law of Excluded Middle.

1. INTRODUCTION

Finiteness is a concept that seems as intuitive as it is fundamental in all of mathematics. At the same time finiteness is notoriously difficult to capture axiomatically. First, due to compactness, finiteness is not first-order definable. Second, in ZF set theory, there exist several different *approximations* (as ZF is a first-order theory). Tarski's treatise [4] is still a very readable introduction to different definitions of finiteness in set theory without the axioms of infinity and choice. These include the definitions by Dedekind (S is finite if there is no bijection from S to a proper subset of S), Kuratowski (S is finite if S can be obtained from the empty set by adding elements inductively), and Tarski (S is finite if each non-empty set of subsets of S contains a minimal element wrt. set inclusion). Some approximations of finiteness are only equivalent if one assumes additional axioms. And all this already in the realm of classical mathematics.

It will therefore not come as a surprise that in intuitionistic mathematics the situation is even more complicated. In this paper, we will study several classically equivalent definitions of bit-valued functions (binary infinite sequences) that are almost always zero, that is, there are at most finitely many positions where the sequence is one. From the constructive point of view, one has at least the following main variants.

- (1) $\exists n. \forall m \geq n. f m = 0$. This definition expresses that all finitely many m for which $f m = 1$ occur in f before some position n . By the decidability of $=$, they can all be looked up and counted. This is clearly the strongest definition giving all information. By

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the decidability of $=$, this definition is also intuitionistically equivalent to $\exists n. \neg \forall m \geq n. f\ m = 0$, in spite of the double negation prefixing the universal quantifier.

- (2) $\exists n. \forall m. \#\{k \leq m \mid f\ k = 1\} < n$. This definition is weaker than the first one. It only states that there is an upper bound to the number of ones in the sequence, but does not provide information on where to find them.
- (3) $\neg(\forall n. \neg \exists m \geq n. f\ m = 1)$. This definition is equivalent to $\neg \neg(1)$. Note that (3) is stable since it is negative, and therefore does not imply (2). Surprisingly, (3) is also equivalent to $\neg \neg(2)$. The reason is that (1) and (2) are classically equivalent, do not contain disjunction, and have only existential quantification as the main connective of the formula. Therefore their respective double negation translations $\neg \neg(1)$ and $\neg \neg(2)$ are constructively equivalent, so also equally weak.
- (4) $\neg(\forall n. \exists m \geq n. f\ m = 1)$. This definition expresses that the set of positions where the sequence is one is not infinite. It does not give a clue where to find the ones or how many ones there are. Definition (4) is the weakest of all: It negates a strong, positive statement allowing the construction of an infinite subsequence of ones in f .

The variants are listed in decreasing constructive strength. Variants (1) and (2) are positive and therefore strictly stronger than the negative variants (3) and (4). Reversing the implications above requires some form of classical logic. For instance we know that (4) \implies (2) is not constructively valid. We use the occasion to introduce an argument employed more rigorously later in this paper. Let f be an arbitrary bit-valued function. Construct f' starting from $n = 0$ by taking $f'\ n = 0$ as long as $f\ n = 0$. There is no constructive way to find out whether $f\ n$ is always 0 or not, but if $f\ n = 1$ for the first time, we take $f'\ k = 1$ for $n \leq k \leq 2n$ and $f'\ k = 0$ for $k > 2n$. One easily verifies (4) for f' . Now, if (2) would hold for f' we would be able to decide whether f is constant 0 or not. For if there are at most n ones occurring in f' , the first one would occur not later than at n , and this can constructively be tested. In other words, (4) \implies (2) implies an instance of the excluded middle which is not constructively valid.

The paper sets out an expedition to the concept of finiteness from the constructive point of view, with strong assumptions on the set whose finiteness we study. Namely,

- (1) The set is enclosed in another set with decidable equality.
- (2) It is carved out by a decidable predicate (whether a bit-valued function returns 1).
- (3) The enclosing set can be enumerated.

In one word, therefore, we could summarize our setting as “searchable”. As we will see in the paper, even in a searchable setting, there are at least six different notions of “finiteness”.

The remainder of the paper is structured as follows. In the next section, we set up a basis for our development in the paper. Section 3 introduces a spectrum of definitions for sequences being finitely one. In Section 4, we study relative strength of these definitions from the constructive point of view. In Section 5, we relate our analysis to that of finiteness of sets in Bishop’s set theory due to Coquand and Spiwack [2]. We conclude in Section 6.

For methodological uniformity, we prefer to define all datatypes inductively (rules denoted by a single line) or coinductively (rules denoted by a double line).

2. TWO VIEWS OF INFINITE SEQUENCES

We may look at binary infinite sequences in two ways. We may view them as bit-valued *functions on natural numbers* or, which will amount to the same, as *streams* of bits, i.e.,

as elements of a coinductive type. Correspondingly, we will use two different languages to speak about them: arithmetic (as is traditional in logic) for bit-valued functions and the language of inductive and coinductive predicates (as is more customary in functional, in particular, dependently typed, programming) for bitstreams. As a warming-up, in this section, we connect the two views, setting up a basis for our development along the way.

For this paper to have some color, we take a bit to be one of the two colors, red and blue:

$$\overline{R : color} \quad \overline{B : color}$$

In the function-view, an infinite sequence is therefore a function $f : nat \rightarrow color$ mapping natural numbers (positions in the sequence) to colors. Our intended notion of equality of these functions is the *extensional function equality* defined by

$$\frac{\forall n. f \ n = f' \ n}{f \equiv f'}$$

In the stream-view, an infinite sequence is a stream $s : str$ where the stream type is defined coinductively by the following rule:

$$\frac{c : color \quad s : str}{c \ s : str}$$

Two streams s and s' are equal for us, if they are *bisimilar*. This notion of equality is defined coinductively by the rule

$$\frac{s \sim s'}{c \ s \sim c \ s'}$$

The two types are isomorphic. Indeed we can define two functions $s2f : str \rightarrow nat \rightarrow color$ and $f2s : (nat \rightarrow color) \rightarrow str$ mediating between the two types. The function $s2f$ is defined by (structural) recursion by

$$\begin{aligned} s2f \ (c \ s) \ 0 &= c \\ s2f \ (c \ s) \ (n + 1) &= s2f \ s \ n \end{aligned}$$

while the function $f2s$ is defined by (guarded-by-constructors) corecursion by

$$f2s \ f = (f \ 0) \ (f2s \ (\lambda n. f \ (n + 1)))$$

We have that $\forall f, s. f \equiv s2f \ s \Leftrightarrow f2s \ f \sim s$. The \Rightarrow direction is proved by coinduction, the \Leftarrow direction by induction. From this fact it follows immediately that $\forall f. f \equiv s2f \ (f2s \ f)$ and $\forall s. f2s \ (s2f \ s) \sim s$, i.e., that the function and stream types are isomorphic, as well as that $\forall f, f'. f \equiv f' \Rightarrow f2s \ f \sim f2s \ f'$ and $\forall s, s'. s \sim s' \Rightarrow f2s \ s \equiv f2s \ s'$, i.e., that the conversion functions appropriately preserve equality. (In general, we have to ensure that all functions and predicates we define on bit-valued functions and bitstreams respect our notions of equality for them, i.e., extensional function equality and bisimilarity.¹)

Properties of binary infinite sequences can now be defined and analyzed in either one of the two equivalent views. For the stream-view, it is convenient to introduce some operations and predicates as primitives in our language for streams. We define

$$\begin{aligned} (c \ s)@0 &= c & (c \ s)@(n + 1) &= s@n \\ s|_0 &= s & (c \ s)|_{(n+1)} &= s|_n \end{aligned}$$

¹The correspondence between extensional function equality and bisimilarity shows that bisimilarity is the one and only reasonable notion of “extensional stream equality”.

so that $s@n$ denotes the color at the position n in s and $s|_n$ denotes the suffix of s at n . We also define

$$\frac{\overline{red(R\ s)} \quad \overline{blue(B\ s)}}{\frac{X\ s}{\mathcal{F}X\ s} \quad \frac{\mathcal{F}X\ s}{\mathcal{F}X(c\ s)} \quad \frac{X(c\ s) \quad \mathcal{G}X\ s}{\mathcal{G}X(c\ s)}}$$

Here, \mathcal{F} and \mathcal{G} are the “sometime in the future” (“finally”) and “always in the future” (“globally”) modalities of linear-time temporal logic. They are stream predicates parameterized over stream predicates.² Induction and coinduction give us simple proofs of basic facts such as the equivalence

$$\forall s. \mathcal{G}(\lambda t. \neg X\ t)\ s \Leftrightarrow \neg \mathcal{F}X\ s$$

and the implication (converse does not hold)

$$\forall s. \mathcal{F}(\lambda t. \neg X\ t)\ s \Rightarrow \neg \mathcal{G}X\ s$$

Importantly, we can also prove that

$$\begin{aligned} \forall s. \mathcal{F}X\ s &\Leftrightarrow \exists n. X(s|_n) \\ \forall s. \mathcal{G}X\ s &\Leftrightarrow \forall n. X(s|_n) \end{aligned}$$

noticing that $\forall s, n. s2f(s|_n) = \lambda m. s2f\ s(n + m)$.

Both modalities are expressible in the function-view, but the definitions are (perhaps) less elegant, as they involve explicit arithmetical manipulation of positions:

$$\begin{aligned} \forall f. \mathcal{F}(\lambda s. Y(s2f\ s))(f2s\ f) &\Leftrightarrow \exists n. Y(\lambda m. f(n + m)) \\ \forall f. \mathcal{G}(\lambda s. Y(s2f\ s))(f2s\ f) &\Leftrightarrow \forall n. Y(\lambda m. f(n + m)) \end{aligned}$$

In particular,

$$\begin{aligned} \forall s. \mathcal{F}\ red\ s &\Leftrightarrow \exists n. s2f\ s\ n = R \\ \forall s. \mathcal{G}\ blue\ s &\Leftrightarrow \forall n. s2f\ s\ n = B \end{aligned}$$

Accordingly, we have

$$\forall s. (\neg \mathcal{G}\ blue\ s \Rightarrow \mathcal{F}\ red\ s) \Leftrightarrow (\neg(\forall n. s2f\ s\ n = B) \Rightarrow \exists n. s2f\ s\ n = R)$$

and hence

$$[\forall s. \neg \mathcal{G}\ blue\ s \Rightarrow \mathcal{F}\ red\ s] \Leftrightarrow [\forall f. \neg(\forall n. f\ n = B) \Rightarrow \exists n. f\ n = R]$$

We now have arrived at two equivalent formulations of Markov’s Principle (MP). Markov’s Principle is an important principle that is neither valid nor inconsistent constructively, but only classically valid. It is computationally meaningful, however, being realizable by search.

In the function-view (the right-hand side), which is how it is traditionally presented, Markov’s Principle is the statement that

$$\forall f. \neg(\forall n. f\ n = B) \Rightarrow \exists n. f\ n = R$$

(or, equivalently, as $\forall n. \neg A \Leftrightarrow \neg \exists n. A$, the statement $\forall f. \neg(\exists n. f\ n = R) \Rightarrow \exists n. f\ n = R$.)

The computational interpretation is the natural one: if it cannot be that all positions in a given infinite sequence are blue, then we find a red position by exhaustively checking all positions in the natural order $0, 1, 2, \dots$ (Cf. computability theory: this is minimization, not primitive recursion.)

²There is no need to see them as “first-class” predicate transformers, as there is no real impredicativity involved: the argument X in the definition of \mathcal{F} and of \mathcal{G} is constant.

In the stream-view (the left-hand side), Markov's Principle is

$$\forall s. \neg \mathcal{G} \text{ blue } s \Rightarrow \mathcal{F} \text{ red } s$$

stating that if a stream s is not all blue, then it is eventually red. But, in a certain sense, it is more than just any equivalent statement to the function-view counterpart. It is a concise formulation of Markov's Principle based on the stream view of infinite sequences and canonical inductive and coinductive predicates on streams. We would therefore like to think that, for computer scientists, it should be natural to take namely this statement rather than the traditional arithmetical version as the definition of Markov's Principle.

This applies to another important classical axiom of the Lesser Principle of Omniscience which is meaningful as a special case of the Law of Excluded Middle.

The Lesser Principle of Omniscience (LPO) is the assertion of the statement

$$\forall f. (\forall n. f n = B) \vee (\exists n. f n = R)$$

that, in the light of what we already learned, is equivalent to

$$\forall s. \mathcal{G} \text{ blue } s \vee \mathcal{F} \text{ red } s$$

Again, the latter statement is perhaps more basic for a computer scientist than the former: it states that any stream is either all blue or eventually red (which is constructively impossible).

As we have constructively $A \vee B \Rightarrow (\neg A \Rightarrow B)$, LPO implies Markov. LPO is not computationally justified, and therefore strictly stronger than MP.

3. SOME NOTIONS OF "FINITELY RED"

With these preparations done, we can now proceed to possible mathematizations of the informal property of a given infinite sequence (function f or stream s) being "finitely red". We consider six variations. They are all equivalent classically. In Section 4, we will study their relative strength from the constructive point of view.

3.1. Eventually all blue. The simplest mathematization is: "from some position on, the sequence is all blue".

In the function view, this is stated as

$$\exists n. \forall m \geq n. f m = B$$

while the stream-view statement is at least as simple, namely, the stream is "finally" "globally" blue:

$$\mathcal{F} (\mathcal{G} \text{ blue}) s$$

The two statements are equivalent.

$$\forall s. \mathcal{F} (\mathcal{G} \text{ blue}) s \Leftrightarrow \exists n. \forall m \geq n. s 2f s m = B$$

3.2. Boundedly red. This is: “the number of red positions in the sequence is bounded”. In the function view, this is stated as

$$\exists n. \forall m. \#\{k \leq m \mid f k = R\} < n$$

so that for a fixed n , f is red fewer than n up to the m -th position for any m .

The formation of the stream view is similar. We first define a binary predicate $le_n s$, which states that s is fewer than n red, coinductively by

$$\frac{le_{n+1} s}{le_{n+1} (B s)} \quad \frac{le_n s}{le_{n+1} (R s)}$$

Note that there are no clauses for le_0 , reflecting the fact that this unary predicate is everywhere false. Then the stream-view is simply:

$$\exists n. le_n s$$

Again, the two statements are equivalent

$$\forall s, n. le_n s \Leftrightarrow \forall m. \#\{k \leq m \mid s2f s k = R\} < n$$

3.3. Almost always blue. The third definition amounts to the least fixed point of a weak until operator in linear-time temporal logic. It is also found in the thesis of C. Raffalli [5]. We formulate it in the stream view. The weak until operator, \mathcal{W}_X , is parameterized over any predicate X on streams and defined coinductively by

$$\frac{\mathcal{W}_X s}{\mathcal{W}_X (B s)} \quad \frac{X s}{\mathcal{W}_X (R s)}$$

so that $\mathcal{W}_X s$ holds if, whenever the first occurrence of red in s is encountered, X holds on the suffix after the occurrence. Classically it is equivalent to that s is either all blue or it is eventually red and X holds on the suffix after the first occurrence of red (which is guaranteed to exist as s is eventually red). Our definition of \mathcal{W}_X avoids upfront decisions of LPO, i.e., whether a stream is all blue or eventually red.

We then take the least fixed point of \mathcal{W}_X . Define $\mu\mathcal{W}$ inductively in terms of \mathcal{W}_X by the (Park-style) rule:

$$\frac{\mathcal{W}_{\mu\mathcal{W}} s}{\mu\mathcal{W} s}$$

As \mathcal{W}_X is monotone on X , the above definition makes sense. For the purpose of proof, in particular to avoid explicitly invoking monotonicity of the underlying predicate transformer \mathcal{W}_X , it is however convenient to use the Mendler-style rule

$$\frac{\forall s. X s \Rightarrow \mu\mathcal{W} s \quad \mathcal{W}_X s}{\mu\mathcal{W} s}$$

The Park-style rule is derivable from the Mendler-style rule. As \mathcal{W}_X is monotone on X , we can also recover the natural inversion principle for $\mu\mathcal{W}$.

The statement $\mu\mathcal{W} s$ does not give a clue as to where to find the red positions in s or how many of them there are. Nonetheless it refutes that the stream is infinitely often red (to be formulated below). Therefore $\mu\mathcal{W} s$ expresses that s is almost always blue, and in the remainder of the paper we phrase $\mu\mathcal{W}$ as almost always blue.

The function view corresponding to $\mu\mathcal{W}$ could be given by the second-order encoding of induction and coinduction, which is inevitably more verbose and therefore omitted. Instead,

in the following subsections, we will take a closer look at \mathcal{W}_X and $\mu\mathcal{W}$, giving alternative characterizations of streams that are almost always blue.

3.4. Streamless red positions. The fourth definition is inspired by [2]. It states that the set of red positions in the sequence is *streamless*. A set A is streamless if every stream over A has a duplicate. As equality on A is decidable for us, this is equivalent to saying that a set A is streamless if any duplicate-free colist over A is finite.

For any set A , we define duplicate-free colists over A coinductively by

$$\frac{}{\langle \rangle : \text{colist } A} \quad \frac{x : A \quad \ell : \text{colist } (A \setminus \{x\})}{x \ell : \text{colist } A}$$

We define finiteness of colists inductively by

$$\frac{}{\langle \rangle \downarrow} \quad \frac{\ell \downarrow}{x \ell \downarrow}$$

For any sequence, namely function f or stream s , let \mathcal{R}_f (resp. \mathcal{R}_s) denote the set of red positions in f (resp. s). Formally, $n \in \mathcal{R}_f$ (resp. $n \in \mathcal{R}_s$) if $f n = R$ (resp. $s@n = R$).

Then, the fourth definition of streams being finitely red is stated in the stream view as

$$\forall \ell : \text{colist } \mathcal{R}_s. \ell \downarrow$$

or, trivially equivalently in the function view, as

$$\forall \ell : \text{colist } \mathcal{R}_f. \ell \downarrow$$

Finite subsets of a given set S can be characterized using *nat* as for every injection $i : \text{nat} \rightarrow S$ and every finite subset $A \subseteq S$ there exists $n : \text{nat}$ with $i n \notin A$. This naturally leads to a positive formulation of co-finiteness: $A \subseteq S$ is co-finite if for every injection $i : \text{nat} \rightarrow S$ there exists $n : \text{nat}$ with $i n \in A$. In [7], Wim Veldman coined the qualifier *almost full* for a subset A of nat such that for every strictly increasing $i : \text{nat} \rightarrow \text{nat}$ one has $i n \in A$ for some $n : \text{nat}$. Let \mathcal{B}_s denote the set of blue positions in s . It turns out that the notions *almost full* and *streamless* are related in the following precise sense.

Lemma 3.1. $\forall s. \mathcal{B}_s \text{ is almost full} \Leftrightarrow \forall \ell : \text{colist } \mathcal{R}_s. \ell \downarrow$.

Proof. (\Rightarrow): Let s be such that \mathcal{B}_s is almost full and let $\ell : \text{colist } \mathcal{R}_s$. We have to prove $\ell \downarrow$. Define a function $f : \text{nat} \rightarrow \text{colist } \mathcal{R}_s \rightarrow \text{colist } \mathcal{R}_s$ by corecursion by

$$\begin{aligned} f n \langle \rangle &= \langle \rangle \\ f n (m \ell) &= m (f (m+1) \ell) \quad \text{if } n \leq m \\ f n (m \ell) &= f n \ell \quad \text{if } n > m \end{aligned}$$

f is well-defined as the last clause is successively applicable only finitely many times. Moreover, $f n \ell$ is finite precisely when ℓ is. Clearly, $f 0 \ell$ is increasing, and can be turned into an increasing function $i = g 0 (f 0 \ell) : \text{nat} \rightarrow \text{nat}$ by defining $g : \text{nat} \rightarrow \text{colist } \mathcal{R}_s \rightarrow \text{nat} \rightarrow \text{nat}$ recursively by

$$g m \langle \rangle n = n + m \quad g m (x \ell) 0 = x \quad g m (x \ell) (n+1) = g (x+1) \ell n$$

One proves by induction on n that $\forall n, m : \text{nat}. \forall \ell : \text{colist } \mathcal{R}_s. g m \ell n \in \mathcal{B}_s \Rightarrow \ell \downarrow$. Since \mathcal{B}_s is almost full there exist $n : \text{nat}$ such that $i n = g 0 (f 0 \ell) n \in \mathcal{B}_s$, so $f 0 \ell \downarrow$, so $\ell \downarrow$ as required.

(\Leftarrow): Let s be such that $\forall \ell : \text{colist } \mathcal{R}_s. \ell \downarrow$. We have to prove that \mathcal{B}_s is almost full. Let $i : \text{nat} \rightarrow \text{nat}$ be increasing. We define corecursively $hi : \text{nat} \rightarrow \text{colist } \mathcal{R}_s$ by:

$$\begin{aligned} hi n &= \langle \rangle && \text{if } in \in \mathcal{B}_s \\ hi n &= (in) (hi(n+1)) && \text{if } in \in \mathcal{R}_s \end{aligned}$$

One proves by induction on $hi n \downarrow$ that $\forall n : \text{nat}. hi n \downarrow \Rightarrow \exists k : \text{nat}. ik \in \mathcal{B}_s$. Indeed we have $hi 0 : \text{colist } \mathcal{R}_s$, so $hi 0 \downarrow$. It follows that \mathcal{B}_s is almost full. \square

3.5. Not not eventually all blue. In this paper, we are mainly interested in positive variations. However, two negative variations appear natural to consider for us. One of them is the double negation of the first definition of eventually all blue.

Our fifth definition is stated in the function view as,

$$\neg\neg\exists n. \forall m \geq n. fm = B$$

or in the stream view as

$$\neg\neg\mathcal{F}(\mathcal{G} \text{ blue}) s$$

which is equivalent to

$$\neg\mathcal{G}(\neg\mathcal{G} \text{ blue}) s$$

The last formulation, $\neg\mathcal{G}(\neg\mathcal{G} \text{ blue}) s$, turns out handy in proofs and we will use either of them interchangeably.

3.6. Not infinitely often red. The last definition of streams being finitely red is given by streams *not* being infinitely often red. So we first look at definitions of streams being infinitely often red, which admit less variety of definitions.

A well-known definition is given by streams that are “globally” “finally” red, or

$$\mathcal{G}(\mathcal{F} \text{ red}) s$$

This definition is dual to that of eventually-all-blue streams, i.e., $\mathcal{F}(\mathcal{G} \text{ blue}) s$. The modalities \mathcal{G} and \mathcal{F} are flipped, so are the colors *red* and *blue*. The function view of this is stated as

$$\forall n. \exists m \geq n. fm = R$$

The function and stream views are equivalent

$$\forall s. \mathcal{G}(\mathcal{F} \text{ red}) s \Leftrightarrow \forall n. \exists m \geq n. s2f s m = R$$

Similarly, we obtain a definition of streams being infinitely often red, by dualizing the definitions of \mathcal{W}_X and $\mu\mathcal{W}$, yielding

$$\frac{\mathcal{U}_X s}{\mathcal{U}_X(B s)} \quad \frac{X s}{\mathcal{U}_X(R s)} \quad \frac{\mathcal{U}\mathcal{U} s}{\nu\mathcal{U} s}$$

The (strong) until operator \mathcal{U}_X is dual to the weak until operator \mathcal{W}_X : The statement $\mathcal{U}_X s$ says that the suffix of s after the first occurrence of red must satisfy X and the occurrence must exist. Then $\nu\mathcal{U}$ takes the *greatest* fixed point of \mathcal{U}_X , whereas $\mu\mathcal{W}$ is the least fixed point of \mathcal{W}_X .

Interestingly, $\nu\mathcal{U}$ is equivalent to $\mathcal{G}(\mathcal{F} \text{ red})$

$$\forall s. \nu\mathcal{U} s \Leftrightarrow \mathcal{G}(\mathcal{F} \text{ red}) s$$

As we will see in Section 4, $\mu\mathcal{W}$ and \mathcal{F} (\mathcal{G} blue) are *not* equivalent constructively. (Collapsing the two amounts to LPO.)

We conclude this section with the weakest definition in our spectrum of streams being finitely red. Namely,

$$\neg\mathcal{G}(\mathcal{F} \text{ red}) s$$

or in its equivalent function view

$$\neg(\forall n. \exists m \geq n. f m = R)$$

3.7. Accessibility. In this section, we characterize streams that are almost always blue in terms of accessibility of (decidable) relations on natural numbers induced by streams.

We define accessibility of a binary relation \succ on a set U by

$$\frac{\forall m. n \succ m \Rightarrow \text{acc}_\succ m}{\text{acc}_\succ n}$$

For any stream s , we define a decidable relation \succ_s on natural numbers by taking $n \succ_s m$ to mean that m is the position following the first red position from n onward (including n). Formally,

$$\frac{n \leq \ell \quad \forall k. n \leq k < \ell \Rightarrow s@k = B \quad s@\ell = R \quad \ell + 1 = m}{n \succ_s m}$$

An equivalent inductive definition is:

$$\frac{}{0 \succ_{Rs} 1} \quad \frac{0 \succ_s m}{0 \succ_{Bs} m + 1} \quad \frac{n \succ_s m}{n + 1 \succ_{cs} m + 1}$$

The intuition is that $n \succ_s m$ should hold if and only if, whenever $\mathcal{W}_X s|_n$ is true, then this is justified by $X s|_m$. (This means that \succ_s is deterministic, but not functional.) This is what the next lemma proves.

Lemma 3.2. $\forall s, n. \mathcal{W}_X s|_n \Leftrightarrow (\forall m. n \succ_s m \Rightarrow X s|_m)$

Proof. (\Rightarrow): We prove $\forall s, n, m. n \succ_s m \Rightarrow \mathcal{W}_X s|_n \Rightarrow X s|_m$ by induction on the proof of $n \succ_s m$.

The case of $s = R s'$, $n = 0$ and $m = 1$: From the assumption $\mathcal{W}_X s|_0$, i.e., $\mathcal{W}_X s$, we directly learn that $X s'|_1$, i.e., $X s|_1$.

The case of $s = B s'$, $n = 0$ and $m = m' + 1$ and $0 \succ_{s'} m'$: The assumption $\mathcal{W}_X s|_0$, i.e., $\mathcal{W}_X s$, assures us that $\mathcal{W}_X s'$, and by the induction hypothesis we have $\mathcal{W}_X s'|_0 \Rightarrow X s'|_{m'}$. Hence $X s'|_{m'}$, i.e., $X s|_m$.

The case of $s = c s'$, $n = n' + 1$ and $m = m' + 1$ and $n' \succ_{s'} m'$: The assumption $\mathcal{W}_X s|_n$ amounts to $\mathcal{W}_X s'|_{n'}$. By the induction hypothesis, $\mathcal{W}_X s'|_{n'} \Rightarrow X s'|_{m'}$, we get that $X s'|_{m'}$, i.e., $X s|_m$.

(\Leftarrow): We prove $\forall s, n. (\forall m. n \succ_s m \Rightarrow X s|_m) \Rightarrow \mathcal{W}_X s|_n$ by induction on n . In the base case $n = 0$ of the induction, we perform coinduction.

The case of $n = 0$ and $s = R s'$: we know that $0 \succ_s 1$. Hence the assumption $\forall m. 0 \succ_s m \Rightarrow X s|_m$ gives us that $X s'|_1$, i.e., $X s'$, from where it follows that $\mathcal{W}_X s$, i.e., $\mathcal{W}_X s|_0$.

The case of $n = 0$ and $s = B s'$: We know that, if $0 \succ_s m$ for any m , then $m = m' + 1$ for some m' and $0 \succ_{s'} m'$. Hence the assumption $\forall m. 0 \succ_s m \Rightarrow X s|_m$ gives us that $\forall m'. 0 \succ_{s'} m' \Rightarrow X s'|_{m'}$. By the coinduction hypothesis, it follows that $\mathcal{W}_X s'|_0$, i.e., $\mathcal{W}_X s'$, from where we learn $\mathcal{W}_X s$, i.e., $\mathcal{W}_X s|_0$.

The case of $n = n' + 1$ and $s = c s'$: We observe that $n \succ_s m$ if $n' \succ_{s'} m'$ and $m = m' + 1$. Therefore the assumption $\forall m. n \succ_s m \Rightarrow X s|_m$ gives us that $\forall m'. n' \succ_{s'} m' \Rightarrow X s'|_{m'}$. By the induction hypothesis, we get that $\mathcal{W}_X s'|_{n'}$ which is the same as $\mathcal{W}_X s|_n$. \square

It is noteworthy that this lemma, instantiated at $n = 0$, gives us a possible arithmetical definition of the weak until operator \mathcal{W}_X that avoids impredicativity (quantification over predicates). Indeed, it suggests that we could have defined:

$$\mathcal{W}_X s \Leftrightarrow \forall \ell. (\forall k < \ell. s @ k = B) \wedge s @ \ell = R \Rightarrow X s|_{\ell+1}$$

To compare, the impredicative definition is:

$$\mathcal{W}_X s \Leftrightarrow \exists Y. (\forall s'. Y (R s') \Rightarrow X s') \wedge (\forall s'. Y (B s') \Rightarrow Y s) \wedge Y s$$

Further, we have that, for any stream s , s is almost always blue, $\mu\mathcal{W} s$, if and only if 0 is accessible with respect to \succ_s . The claim follows from the following lemma.

Lemma 3.3. $\forall s, n. \mu\mathcal{W} s|_n \Leftrightarrow acc_{\succ_s} n$.

Proof. (\Rightarrow): We prove $\forall s, n. \mu\mathcal{W} s|_n \Rightarrow acc_{\succ_s} n$ by induction on the proof of $\mu\mathcal{W} s|_n$.³ From this proof, we have that, for some stream predicate X , $\forall s'. X s' \Rightarrow \mu\mathcal{W} s'$ and $\mathcal{W}_X s|_n$. By the induction hypothesis, the former gives us $\forall m. X s|_m \Rightarrow acc_{\succ_s} m$ while, by the previous lemma, the latter gives $\forall m. n \succ_s m \Rightarrow X s|_m$. Putting the two together, we get $\forall m. n \succ_s m \Rightarrow acc_{\succ_s} m$, hence $acc_{\succ_s} n$.

(\Leftarrow): By induction on the proof of $acc_{\succ_s} n$. We have $\forall m. n \succ_s m \Rightarrow acc_{\succ_s} m$ and by the induction hypothesis, $\forall m. n \succ_s m \Rightarrow \mu\mathcal{W} s|_m$. The previous lemma therefore gives us $\mathcal{W}_{\mu\mathcal{W} s|_n}$, hence $\mu\mathcal{W} s|_n$, as required. \square

Corollary 3.1. $\forall s. \mu\mathcal{W} s \Leftrightarrow acc_{\succ_s} 0$.

We can in fact rephrase the variant from Section 3.4 (streams for which the sets of red positions are streamless) and the variant from Section 3.6 (streams that are not infinitely often red) in terms of \succ_s , as we will do now.

3.7.1. Strong normalization. Streams whose red positions form streamless sets correspond to streams s for which \succ_s is strongly normalizing at 0.

For any set U and any relation \succ on U , we define (descending) chains in \succ coinductively by

$$\frac{x_0 : U}{\langle \rangle : chain_{\succ} x_0} \quad \frac{x_0 \succ x_1 \quad \ell : chain_{\succ} x_1}{x_1 \ell : chain_{\succ} x_0}$$

so that $x_1 x_2 \dots x_n \langle \rangle : chain_{\succ} x_0$ means that $x_0 \succ x_1 \succ x_2 \succ \dots \succ x_n$. Note that a chain in \succ may be infinite.

We define finiteness of chains inductively by

$$\frac{}{\langle \rangle \downarrow} \quad \frac{\ell \downarrow}{x \ell \downarrow}$$

We use the same notation for finiteness of colists and chains.

A binary relation \succ on a set U is strongly normalizing at $x : U$, $SN \succ x$, if any \succ -chain starting at x is finite, or $\forall \ell : chain_{\succ} x. \ell \downarrow$.

For any stream s , \succ_s is strongly normalizing at 0 if and only if \mathcal{R}_s is streamless.

³To be fully precise, we prove $\forall s'. \mu\mathcal{W} s' \Rightarrow (\forall s, n. s' = s|_n \Rightarrow acc_{\succ_s} n)$ by induction on the proof of $\mu\mathcal{W} s'$. In further proofs we will use these generalizations of coinduction and induction without comments.

Lemma 3.4. $\forall s. \text{SN } \succ_s 0 \Leftrightarrow \mathcal{R}_s \text{ is streamless.}$

Proof. (\Rightarrow): We first notice that $\text{SN } \succ_s 0$ if and only if $\text{SN } \succ_s^+ 0$, where \succ_s^+ is the transitive closure of \succ_s . Define a function $f : \text{nat} \rightarrow \text{colist nat} \rightarrow \text{colist nat}$ by recursion by

$$\begin{aligned} f n \langle \rangle &= \langle \rangle \\ f n (m \ell) &= m \ell \quad \text{if } n < m \\ f n (m \ell) &= f n \ell \quad \text{if } n \geq m \end{aligned}$$

The computation of $f n \ell$ is terminating as ℓ is duplicate-free. (So, $f n \ell$ is welldefined.) Moreover, define a function $g : \text{colist nat} \rightarrow \text{colist nat}$ by corecursion by

$$g \langle \rangle = \langle \rangle \quad g (n \ell) = (n + 1) (g (f n \ell))$$

We have that, for any duplicate-free colist ℓ over \mathcal{R}_s , ℓ is finite if and only if $g \ell$ is finite, and moreover $g \ell$ is a chain in \succ_s^+ starting at 0.

Now, for any given duplicate-free colist $\ell : \text{colist } \mathcal{R}_s$, by our assumption, $g \ell$ is finite, which implies ℓ is finite, as required.

(\Leftarrow): Define a function $f : \text{colist} (\text{nat} \setminus \{0\}) \rightarrow \text{colist nat}$ by corecursion by

$$f \langle \rangle = \langle \rangle \quad f (n \ell) = (n - 1) (f \ell)$$

so that $f \ell$ shifts the elements in ℓ by subtracting one.

For any given $\ell : \text{chain}_{\succ_s} 0$, $f \ell$ is a duplicate-free colist over \mathcal{R}_s , therefore $f \ell$ is finite by our assumption. By construction of f , ℓ is finite, which completes the proof. \square

3.7.2. Antifoundedness. Streams that are infinitely often red correspond to streams s for which \succ_s is antifounded.

We define antifoundedness of binary relation \succ on a set U coinductively by

$$\frac{n \succ m \quad \text{div}_{\succ} m}{\text{div}_{\succ} n}$$

so that $\text{div}_{\succ} n$ means that there is an infinite descending chain in \succ starting from n .

Firstly we rephrase the strong until operator, \mathcal{U}_X , which, unlike the weak until operator \mathcal{W}_X , requires X to hold at some point.

Lemma 3.5. $\forall s, n. \mathcal{U}_X s|_n \Leftrightarrow (\exists m. n \succ_s m \wedge X s|_m).$

Proof. (\Rightarrow): By induction on n and in the base case $n = 0$ also further induction on the proof of $\mathcal{U}_X s$.

The case of $n = 0$ and $s = R s'$: We have that $0 \prec_s 1$ and $X s'$ and can choose $m = 1$.

The case of $n = 0$ and $s = B s'$: We have that $\mathcal{U}_X s'$. The inner induction hypothesis gives us that there is an m' such that $0 \succ_{s'} m' \wedge X s'|_{m'}$. But then we also have that $0 \succ_s m' + 1 \wedge X s|_{m'+1}$, so the desired result is witnessed by $m = m' + 1$.

The case of $n = n' + 1$ and $s = c s'$: The assumption $\mathcal{U}_X s|_n$ amounts to $\mathcal{U}_X s'|_{n'}$. By the outer induction hypothesis, there is an m' such that $n' \succ_s m' \wedge X s'|_{m'}$. But then also $n \succ_s m' + 1 \wedge X s|_{m'+1}$, so we can choose $m = m' + 1$.

(\Leftarrow): We prove $\forall s, n, m. n \succ_s m \wedge X s|_m \Rightarrow \mathcal{U}_X s|_n$ by induction on the proof of $n \succ_s m$.

The case of $s = R s'$, $n = 0$ and $m = 1$: The assumption $X s|_1$, i.e., $X s'$, implies $\mathcal{U}_X s$, i.e., $\mathcal{U}_X s|_0$.

The case of $s = B s'$, $n = 0$, $m = m' + 1$ and $0 \succ_{s'} m'$: The assumption $X s|_m$ amounts to $X s'|_{m'}$. By the induction hypothesis, we have that $\mathcal{U}_X s'|_0$, from where $\mathcal{U}_X s|_0$ follows in turn.

The case of $s = c s'$, $n = n' + 1$, $m = m' + 1$ and $n' \succ_{s'} m'$: The assumption $X s|_m$ amounts to $X s'|_{m'}$. By the induction hypothesis, it holds that $\mathcal{U}_X s'|_{n'}$, which is the same as $\mathcal{U}_X s|_n$. \square

Then we have that, for any stream s , s is infinitely often red, $\nu\mathcal{U} s$, if and only if 0 is antifounded with respect to \succ_s . The claim follows from the following lemma.

Lemma 3.6. $\forall s, n. \nu\mathcal{U} s|_n \Leftrightarrow \text{div}_{\succ_s} n$.

Proof. (\Rightarrow): By coinduction. From the assumption $\nu\mathcal{U} s|_n$, we have that, for some stream predicate X , $\forall s'. X s' \Rightarrow \nu\mathcal{U} s'$ and $\mathcal{U}_X s|_n$. The former and the coinduction hypothesis together give us that, $\forall m'. X s|_{m'} \Rightarrow \text{div}_{\succ_s} m'$. From the latter and the previous lemma, it follows that there exists an m such that $n \succ_s m$ and $X s|_m$. Hence $\text{div}_{\succ_s} m$ and we can also conclude that $\text{div}_{\succ_s} n$.

(\Leftarrow): By coinduction. From the assumption $\text{div}_{\succ_s} n$, we have that there exists some m such that $n \succ_s m$ and $\text{div}_{\succ_s} m$. By the coinduction hypothesis, we have $\nu\mathcal{U} s|_m$. By the previous lemma it follows now that $\mathcal{U}_{\nu\mathcal{U} s}|_n$ whereby we also learn that $\nu\mathcal{U} s|_n$. \square

Corollary 3.2. $\forall s. \nu\mathcal{U} s \Leftrightarrow \text{div}_{\succ_s} 0$.

3.8. Classical fixed point. It turns out that the weak until operator \mathcal{W}_X reaches the fixed point by ω -iteration only classically. In fact, we have a stronger result: closure at ω is equivalent to LPO. Define:

$$\frac{F^n s}{F^\omega s}$$

where $F^0 = \text{False}$ and $F^{n+1} = \mathcal{W}_{F^n}$, so that F^ω is \mathcal{W}_X iterated ω times.

Lemma 3.7. $(\forall s. \mathcal{W}_{F^\omega} s \Rightarrow F^\omega s) \Leftrightarrow (\forall s. \mathcal{F} \text{ red } s \vee \mathcal{G} \text{ blue } s)$.

Proof. (\Rightarrow): Define $f : \text{nat} \rightarrow \text{str} \rightarrow \text{str}$ and $g : \text{nat} \rightarrow \text{str}$ by corecursion

$$\begin{aligned} f n (B s) &= B (f (n+1) s) & f n (R s) &= g n \\ g (n+1) &= R (g n) & g 0 &= B^\infty \end{aligned}$$

where B^∞ denotes a stream of blue, defined by corecursion by $B^\infty = B B^\infty$. The computation of $f 0 s$ looks for the first occurrence of red in s , while keeping track of the number of blue it has seen so far in the second argument. On encountering the first red (if exists), it invokes g , passing n as argument. The stream that $g n$ produces is red up to the n -th position, followed by an all blue stream. The trick is to record the position of the first occurrence of red in s in terms of the number of red in $f 0 s$. If s does not contain red, then $f 0 s$ does not either. This way, if we know the bound on the number of red in $f 0 s$, then we know the bound on the depth of the first occurrence of red in s . We prove $\forall n. F^{n+1} (g n)$ by induction on n , then $\forall n, s. \mathcal{W}_{F^\omega} (f n s)$ by coinduction. We deduce $\forall s. F^\omega (f 0 s)$ by our assumption, therefore $\forall s. \exists n. F^n (f 0 s)$ by definition. For any s , given $F^n (f 0 s)$ for some n , however, it suffices to examine the initial $(n+1)$ -segment of s to know whether s contains red or not, enabling us to decide whether $\mathcal{F} \text{ red } s$ or $\mathcal{G} \text{ blue } s$ holds.

(\Leftarrow): For any given s , suppose $\mathcal{W}_{F^\omega} s$. By our assumption, we have either $\mathcal{G} \text{ blue } s$ or $\mathcal{F} \text{ red } s$. In the case of $\mathcal{G} \text{ blue } s$, we immediately have $F^1 s$, therefore $F^\omega s$. In the case of

$\mathcal{F} \text{ red } s$, let n be the position of the first occurrence of red in s , which is guaranteed to exist by $\mathcal{F} \text{ red } s$. From $\mathcal{W}_{F^\omega} s$, we deduce $\mathcal{F}^\omega s|_{n+1}$, i.e., $\mathcal{F}^m s|_{n+1}$ for some m , which yields $F^{m+1} s$, therefore $F^\omega s$ as required. \square

In fact, F^n is equivalent to le_n . Namely we have that, $\forall n, s. F^n s \Leftrightarrow le_n s$. It is an open question whether there is a constructive closure ordinal.

4. ANALYSIS OF THE SPECTRUM

In this section, we analyze our spectrum of streams being finitely red. We have presented six variants:

- (a) Eventually all blue
- (b) Boundedly red
- (c) Almost always blue
- (d) Streamless red positions
- (e) Double negation of eventually all blue
- (f) Negation of infinitely often red

We have a clear view on relative strength between positive variations. For negative ones, open questions remain. The overall picture is given in Section 6.

We start from downward implications. The six variations above are listed in decreasing order of constructive strength, except that we do not know whether (d) implies (e): we only know that (c) implies (d) and (e), both of which imply (f) (Lemmata 4.3, 4.4, 4.5 and 4.6) and that (e) \Rightarrow (d) amounts to Markov's Principle (Lemma 4.10).

If a stream is eventually all blue, then it is boundedly red.

Lemma 4.1. $\forall s. \mathcal{F}(\mathcal{G} \text{ blue}) s \Rightarrow \exists n. le_n s$.

Proof. By induction on the proof of $\mathcal{F}(\mathcal{G} \text{ blue}) s$. \square

If a stream is boundedly red, then it is almost always blue.

Lemma 4.2. $\forall n, s. le_n s \Rightarrow \mu\mathcal{W} s$.

Proof. By induction on n . The case of $n = 0$ is immediate. The case of $n = n' + 1$: We prove that, $\forall s. le_{n'+1} s \Rightarrow \mathcal{W}_{\mu\mathcal{W}} s$ by coinduction and case analysis on the head color of s . The case of $s@0 = B$ follows from the coinduction hypothesis. The case of $s@0 = R$ follows from the main induction hypothesis. \square

If a stream is almost always blue, then the set of its red positions is streamless.

Lemma 4.3. $\forall s. \mu\mathcal{W} s \Rightarrow \mathcal{R}_s$ is streamless.

Proof. The claim follows from Corollary 3.1 and Lemma 3.4, since accessibility implies strong normalization. \square

If a stream s is almost always blue, then it is not the case that s is not eventually all blue.

Lemma 4.4. $\forall s. \mu\mathcal{W} s \Rightarrow \neg\mathcal{G}(\neg\mathcal{G} \text{ blue}) s$.

Proof. We prove a slightly stronger statement, $\forall s. (\forall n. \neg\mathcal{G} \text{ blue } s|_n) \Rightarrow \forall n. \mu\mathcal{W} s|_n \Rightarrow \text{False}$, from which the claim follows. For a given s , we assume $\forall n. \neg\mathcal{G} \text{ blue } s|_n$. We shall prove $\forall n. \mu\mathcal{W} s|_n \Rightarrow \text{False}$ by induction on the proof of $\mu\mathcal{W} s|_n$. We are given as induction hypothesis that, $\forall n. X s|_n \Rightarrow \text{False}$. We have to prove False , given $\mathcal{W}_X s|_n$. From our assumption, however, it suffices to prove $\mathcal{G} \text{ blue } s|_n$. We do so by proving $\forall n. \mathcal{W}_X s|_n \Rightarrow \mathcal{G} \text{ blue } s|_n$ by coinduction using the main induction hypothesis. \square

If the set of red positions of a stream s is streamless, then s is not infinitely often red.

Lemma 4.5. $\forall s. \mathcal{R}_s \text{ is streamless} \Rightarrow \neg\nu\mathcal{U} s$.

Proof. The claim follows from Lemma 3.4 and Corollary 3.2, since strong normalization contradicts antifoundedness. \square

If it is not the case that a stream s is not eventually all blue, then s is not infinitely often red.

Lemma 4.6. $\forall s. \neg\mathcal{G}(\neg\mathcal{G} \text{ blue}) s \Rightarrow \neg\nu\mathcal{U} s$.

Proof. Noticing $\forall s. \nu\mathcal{U} s \Leftrightarrow \mathcal{G}(\mathcal{F} \text{ red}) s$, the claim follows by contraposition from a tautology $\forall s. \mathcal{G}(\mathcal{F} \text{ red}) s \Rightarrow \mathcal{G}(\neg\mathcal{G} \text{ blue}) s$. \square

We now proceed to study strength of upward implications, which are technically more interesting than downward implications. We know that differences between the first three positive variants amount to LPO (Lemma 4.7 and 4.8). Moreover, (e) \implies (d) amounts to Markov's Principle (Lemma 4.10) and (f) \implies (e) to an instance of Double Negation Shift for a Σ_1^0 -formula (Lemma 4.11). As immediate corollaries from Section 3.7, we have that (d) \implies (c) is equivalent to that SN of \succ_s at 0 implies accessibility of 0 with respect to \succ_s (Corollary 4.1) and that (f) \implies (d) is equivalent to that non-antifoundedness of 0 with respect to \succ_s implies SN of \succ_s at 0 (Corollary 4.3).

Lemma 4.7. $(\forall n, s. le_n s \Rightarrow \mathcal{F}(\mathcal{G} \text{ blue}) s) \Leftrightarrow (\forall s. \mathcal{F} \text{ red } s \vee \mathcal{G} \text{ blue } s)$.

Proof. (\implies): Define $f : str \rightarrow str$ by corecursion

$$f(B s) = B(f s) \quad f(R s) = R B^\infty$$

so that $f s$ contains (exactly) one red if and only if s contains at least one red. We have that, $\forall s. le_2(f s)$, proved by coinduction and case analysis on the head color of $f s$. By our assumption, we have that, $\forall s. \mathcal{F}(\mathcal{G} \text{ blue})(f s)$. The proof of $\mathcal{F}(\mathcal{G} \text{ blue})(f s)$ tells us whether $f s$ contains red or not, deciding whether s is eventually red, $\mathcal{F} \text{ red } s$ or all blue, $\mathcal{G} \text{ blue } s$, as required.

(\impliedby): We prove that, $\forall n, s. le_n s \Rightarrow \mathcal{F}(\mathcal{G} \text{ blue}) s$ by induction on n , assuming $\forall s. \mathcal{F} \text{ red } s \vee \mathcal{G} \text{ blue } s$. The case of $n = 0$ is immediate. The case of $n = n' + 1$: Suppose $le_{n'+1} s$. By our assumption, we have either $\mathcal{F} \text{ red } s$ or $\mathcal{G} \text{ blue } s$. The latter case immediately yields $\mathcal{F}(\mathcal{G} \text{ blue}) s$. For the former case, we prove $\forall s. \mathcal{F} \text{ red } s \rightarrow le_{n'+1} s \rightarrow \mathcal{F}(\mathcal{G} \text{ blue}) s$ by induction on $\mathcal{F} \text{ red } s$ and case analysis on the head color of s , using the main induction hypothesis. \square

Lemma 4.8. $(\forall s. \mu\mathcal{W} s \Rightarrow \exists n. le_n s) \Leftrightarrow (\forall s. \mathcal{F} red s \vee \mathcal{G} blue s)$.

Proof. (\Rightarrow): We prove $\forall s. \mu\mathcal{W} s \Rightarrow \exists n. le_n s$ by induction on the proof of $\mu\mathcal{W} s$, assuming $\forall s. \mathcal{F} red s \vee \mathcal{G} blue s$. We have that, for some stream predicate X , $\forall s'. X s' \Rightarrow \mu\mathcal{W} s'$ and $\forall s'. X s' \Rightarrow \exists n. le_n s'$ and $\mathcal{W}_X s$. We have to prove that there exists some n such that $le_n s$. By our assumption, we have either $\mathcal{G} blue s$ or $\mathcal{F} red s$. The former case follows immediately by coinduction by taking $n = 1$. The latter case is closed by the auxiliary lemma: $\forall s'. \mathcal{F} red s' \Rightarrow \mathcal{W}_X s' \Rightarrow \exists n. le_n s'$ proved by induction on the proof of $\mathcal{F} red s'$ and case analysis on the head color of s' . The case of $s'@0 = B$ follows from the induction hypothesis. The case of $s'@0 = R$ follows from the main induction hypothesis, $\forall s'. X s' \Rightarrow \exists n. le_n s'$.

(\Leftarrow): We prove that, $\forall s. \mathcal{W}_{F^\omega} s \Rightarrow F^\omega s$, assuming $\forall s. \mu\mathcal{W} s \Rightarrow \exists n. le_n s$, where F^ω was defined in Section 3.8. Then the case follows from Lemma 3.7. Suppose $\mathcal{W}_{F^\omega} s$. By Lemma 4.2 and the monotonicity of \mathcal{W}_X on X , we have $\mathcal{W}_{\mu\mathcal{W}} s$, which yields $\mu\mathcal{W} s$ by definition. From our assumption and the equivalence between F^n and le_n , we conclude $\mathcal{F}^\omega s$, as required. \square

The following claim is a corollary from Corollary 3.1 and Lemma 3.4.

Corollary 4.1. $(\forall s. \mathcal{R}_s \text{ is streamless} \Rightarrow \mu\mathcal{W} s) \Leftrightarrow (\forall s. \text{SN } \succ_s 0 \Rightarrow acc_{\succ_s} 0)$

Lemma 4.9. $(\forall s. \mathcal{F} red s \vee \mathcal{G} blue s) \Rightarrow (\forall s. \mathcal{R}_s \text{ is streamless} \Rightarrow \mu\mathcal{W} s)$

Proof. Assume $\forall s. \mathcal{F} red s \vee \mathcal{G} blue s$ (LPO) and let s be given. Based on LPO we can build an increasing colist of all red positions in s . Recall that $s|_n$ denotes the suffix of s at n . Define a function $f : str \rightarrow nat \rightarrow colist \mathcal{R}_s$ corecursively by (m will be justified below):

$$\begin{aligned} f s n &= \langle \rangle && \text{if } \mathcal{G} blue s \\ f s n &= (n + m) (f s|_{m+1} (n + m + 1)) && \text{if } \mathcal{F} red s, \text{ where } m \text{ is the first red position in } s \end{aligned}$$

The computation of f essentially depends on LPO. Observe that, in the second clause, the first red position in s exists by $\mathcal{F} red s$ (cf. $\forall s. \mathcal{F} red s \Leftrightarrow \exists n. s2f s n = R$ in Section 2). Clearly, $f s 0$ is the colist of all red positions in s . If \mathcal{R}_s is streamless, then $f s 0$ is finite: $f s 0 \downarrow$. By induction on the proof of $f s n \downarrow$ one proves that $f s n \downarrow$ implies $\mu\mathcal{W} s$. In the base case $f s n = \langle \rangle$, we have $\mathcal{G} blue s$, and therefore $\mathcal{W}_{\mu\mathcal{W}} s$ by coinduction. In the step case of $f s n$ being a cons-colist, we have $\mathcal{F} red s$. We prove $\mathcal{W}_{\mu\mathcal{W}} s$ by coinduction again, this time using the induction hypothesis that $f s|_{m+1} (n + m + 1) \downarrow$ implies $\mu\mathcal{W} s|_{m+1}$. In both cases, we get that $\mu\mathcal{W} s$ by definition. \square

Lemma 4.10. $(\forall s. \neg\mathcal{G} (\neg\mathcal{G} blue) s \Rightarrow \mathcal{R}_s \text{ is streamless}) \Leftrightarrow (\forall s. \neg\mathcal{G} blue s \Rightarrow \mathcal{F} red s)$

Proof. (\Rightarrow): Define a function $f : str \rightarrow str$ by corecursion by

$$f (R s) = B^\infty \quad f (B s) = R (f s)$$

so that $f s$ is red until the first occurrence of red in s is encountered, from where $f s$ becomes all blue.

For any given s , we assume $\neg\mathcal{G} blue s$. We have to prove $\mathcal{F} red s$. Firstly, we prove $\neg\mathcal{G} (\neg\mathcal{G} blue) (f s)$. It suffices to prove $\forall s. \mathcal{G} (\neg\mathcal{G} blue) (f s) \Rightarrow \mathcal{G} blue s$. We do so by coinduction and case analysis on the head color of s . The case of $s@0 = R$: This is impossible as we then have $\mathcal{G} blue (f s)$, contradicting the assumption $\mathcal{G} (\neg\mathcal{G} blue) (f s)$. The case of $s@0 = B$: From the assumption $\mathcal{G} (\neg\mathcal{G} blue) (f s)$, it follows that, $\mathcal{G} (\neg\mathcal{G} blue) (f s)|_1$. By the coinduction hypothesis, we obtain $\mathcal{G} blue s|_1$, hence $\mathcal{G} blue s$.

Applying our assumption, $\forall s. \neg \mathcal{G}(\neg \mathcal{G} \text{ blue}) s \Rightarrow \mathcal{R}_s$ is streamless, to $\neg \mathcal{G}(\neg \mathcal{G} \text{ blue})(f s)$ yields that $\succ_{(f s)}$ is strongly normalizing at 0 by Lemma 3.4. Below we prove $\mathcal{F} \text{ red } s$, assuming $\text{SN } \succ_{(f s)} 0$, which completes the proof.

Define a function $g : \text{nat} \rightarrow \text{colist nat}$ by recursion by

$$\begin{aligned} g n &= (n + 1) (g (n + 1)) && \text{if } (f s)@n = R \\ g n &= \langle \rangle && \text{if } (f s)@n = B \end{aligned}$$

As $g 0$ is a chain in $\succ_{(f s)}$ starting at 0, i.e., $g 0 : \text{chain}_{\succ_{(f s)}} 0$, by our assumption $g 0$ is finite. By construction of g , we have $(f s)@n = B$, where n is the length of $g 0$. (As $g 0$ is finite, its length is welldefined.) By construction of f , we now have $s@n = R$, which yields $\mathcal{F} \text{ red } s$, as required.

(\Leftarrow): For any given s , we assume $\neg \mathcal{G}(\neg \mathcal{G} \text{ blue}) s$. We have to prove, for any given $\ell : \text{chain}_{\succ_s} 0$, ℓ is finite.

Define a function $f : \text{colist nat} \rightarrow \text{str}$ by corecursion by

$$f \langle \rangle = R^\infty \quad f (n \ell') = B (f \ell')$$

By definition of f , we have that, $\forall \ell' : \text{chain}_{\succ_s} 0. \mathcal{G} \text{ blue}(f \ell') \Rightarrow \mathcal{G}(\neg \mathcal{G} \text{ blue}) s$, proved by coinduction. Hence from the assumption $\neg \mathcal{G}(\neg \mathcal{G} \text{ blue}) s$, we are entitled to conclude $\neg(\mathcal{G} \text{ blue})(f \ell)$. By Markov's Principle it follows that, $\mathcal{F} \text{ red}(f \ell)$. However this means that $\ell \downarrow$, which completes the proof. \square

Lemma 4.11. $(\forall s. \neg \mathcal{G}(\mathcal{F} \text{ red}) s \Rightarrow \neg \mathcal{G}(\neg \mathcal{G} \text{ blue}) s) \Leftrightarrow (\forall s. \mathcal{G}(\neg \neg \mathcal{F} \text{ red}) s \Rightarrow \neg \neg \mathcal{G}(\mathcal{F} \text{ red}) s)$.

Proof. For any given s , we have

$$\neg \neg \mathcal{G}(\neg \neg \mathcal{F} \text{ red}) s \Leftrightarrow \neg \neg \neg \mathcal{F}(\neg \mathcal{F} \text{ red}) s \Leftrightarrow \neg \mathcal{F}(\neg \mathcal{F} \text{ red}) s \Leftrightarrow \mathcal{G}(\neg \neg \mathcal{F} \text{ red}) s$$

Now the claim follows by taking contrapositions of the respective assumptions, noticing $\forall s. \neg \mathcal{G} \text{ blue } s \Leftrightarrow \neg \neg \mathcal{F} \text{ red } s$ and the above equivalence. \square

The corollary below follows from lemmata 4.5, 4.10 and 4.11.

Corollary 4.2. $(\forall s. \neg \nu \mathcal{U} s \Rightarrow \mathcal{R}_s \text{ is streamless}) \Leftrightarrow (\forall s. \neg \mathcal{G} \text{ blues } \Rightarrow \mathcal{F} \text{ red } s)$

The following claim is a corollary from Corollary 3.2 and Lemma 3.4.

Corollary 4.3. $(\forall s. \neg \nu \mathcal{U} s \Rightarrow \mathcal{R}_s \text{ is streamless}) \Leftrightarrow (\forall s. \neg \text{div}_{\succ_s} 0 \Rightarrow \forall \ell : \text{chain}_{\succ_s} 0. \ell \downarrow)$.

5. RELATED WORK: FINITENESS OF SETS OF RED POSITIONS

In [2], Coquand and Spiwack introduce four notions of finiteness of sets in Bishop's set theory [3]. For understanding some of their arguments, for example, on page 222, the 9-th line from below, we had to assume that equality is decidable. Under this assumption their results may be rendered as follows:

- (i) Set A is *enumerated* if it is given by a list.
- (ii) Set A is of *bounded size* if there exists a bound such that any list over A contains duplicates whenever its length exceeds the bound.
- (iii) Set A is *Noetherian* if the root of the tree of duplicate-free lists over A is *accessible* (cf. Section 3.7).
- (iv) Set A is *streamless* if every stream over A has a duplicate.

These four notions are classically equivalent but of decreasing constructive strength. Their hierarchy of finiteness matches pleasantly with our hierarchy of positive variations of streams being finitely red, if we look at sets of red positions in our streams. An important difference is that Coquand and Spiwack consider sets that may not be decidable, whereas we work with decidable sets of natural numbers. As a result, our hierarchy becomes tighter than theirs, allowing us to capture differences in strength of our hierarchy in terms of weak instances of the Law of Excluded Middle.

In this section, we rephrase our hierarchy in terms of Coquand and Spiwack's. Their streamless sets directly correspond to our streams s for which the set of red positions, \mathcal{R}_s , is streamless. We will therefore only consider (i) – (iii). Recall that in this paper we work with decidable sets of natural numbers.

Enumerated sets. A set A is enumerated, $enum\ A$, if all its elements can be listed, or

$$\frac{\forall x : A. \text{false}}{enum\ A} \quad \frac{x : A \quad enum\ (A \setminus \{x\})}{enum\ A}$$

Note that a proof of $enum\ A$ is essentially an exhaustive duplicate-free list of elements of A .

It is easy to see that a stream s is eventually all blue if and only if the set of red positions in s is enumerated.

Lemma 5.1. $\forall s. \mathcal{F}(\mathcal{G}\ \text{blue})\ s \Leftrightarrow enum\ \mathcal{R}_s$.

Proof. (\Rightarrow): Given $\mathcal{F}(\mathcal{G}\ \text{blue})\ s$, we can construct a list of the red positions in s , from which $enum\ \mathcal{R}_s$ follows.

(\Leftarrow): Given $enum\ \mathcal{R}_s$, we know the position of the last occurrence of red in s , which yields $\mathcal{F}(\mathcal{G}\ \text{blue})\ s$. \square

Size-bounded sets. A set A is of bounded size if there exists a natural number n such that any duplicate-free list over A is of length less than n . Specifically, we say A is size-bounded by n if any duplicate-free list over A is of length of less than n . Formally,

$$\frac{\forall x : A. \text{bounded}_n\ (A \setminus \{x\})}{\text{bounded}_{n+1}\ A}$$

Lemma 5.2. $\forall n, s. le_n\ s \Leftrightarrow \text{bounded}_n\ \mathcal{R}_s$.

Proof. For any decidable set A of natural numbers, we define a stream s_A by

$$\begin{aligned} s_A @ k &= R \quad \text{if } k \in A \\ s_A @ k &= B \quad \text{otherwise} \end{aligned}$$

so that s_A is red exactly at the positions in A .

(\Rightarrow): By induction on n . The case of $n = 0$ is immediate. The case of $n = n' + 1$: We are given as induction hypothesis that, $\forall s. le_{n'}\ s \Rightarrow \text{bounded}_{n'}\ \mathcal{R}_s$. We have to prove $\text{bounded}_{n'+1}\ \mathcal{R}_s$, given $le_{n'+1}\ s$. Suppose $x \in \mathcal{R}_s$. It suffices to prove $\text{bounded}_{n'}\ \mathcal{R}_s \setminus \{x\}$. From $le_{n'+1}\ s$, we deduce $le_{n'}\ s_{\mathcal{R}_s \setminus \{x\}}$. By induction hypothesis, we obtain $\text{bounded}_{n'}\ \mathcal{R}_s \setminus \{x\}$, as required.

(\Leftarrow): We prove $\forall n, A. \text{bounded}_n\ A \Rightarrow le_n\ s_A$ by induction on n , from which the case follows. The case of $n = 0$ is immediate. The case of $n = n' + 1$: We are given as induction hypothesis that, $\forall A. \text{bounded}_{n'}\ A \Rightarrow le_{n'}\ s_A$. We have to prove $\forall A. \text{bounded}_{n'+1}\ A \Rightarrow$

$le_{n'+1} s_A$. We do so by coinduction and case analysis on the head color of s_A . The case of $s_A@0 = B$: We have $\text{bounded}_{n'+1} \mathcal{R}_{s_A|_1}$. We close the case by coinduction hypothesis. The case of $s_A@0 = R$: We have $\text{bounded}_{n'} \mathcal{R}_{s_A|_1}$. We close the case by the main induction hypothesis. \square

Noetherian sets. A set A is Noetherian, *Noet* A , if, for all $x \in A$, $A \setminus \{x\}$ is Noetherian. Formally,

$$\frac{\forall x \in A. \text{Noet} (A \setminus \{x\})}{\text{Noet} A}$$

Then, a stream s is almost always blue, $\mu\mathcal{W} s$, if and only if the set of red positions in s is Noetherian. To prove this, it is convenient to reformulate Noetherianness for sets of natural numbers by removing the elements up to n (including n):

$$\frac{\forall n \in A. \text{Noet}' A \setminus \{0, \dots, n\}}{\text{Noet}' A}$$

The two definitions are equivalent.

Lemma 5.3. $\forall A. \text{Noet} A \Leftrightarrow \text{Noet}' A$.

Proof. (\Rightarrow): We prove that, $\forall A. \text{Noet} A \Rightarrow \forall n \in A. \text{Noet}' A \setminus \{0, \dots, n\}$ by induction on the proof of *Noet* A . We are given as induction hypothesis that, $\forall n \in A. \forall m \in A \setminus \{n\}. \text{Noet}' A \setminus \{n\} \setminus \{0, \dots, m\}$. We have to prove that, $\forall n \in A. \text{Noet}' A \setminus \{0, \dots, n\}$, which follows from the induction hypothesis and by case analysis on whether there is $m < n$ such that $m \in A$. (Recall that A is assumed decidable.)

(\Leftarrow): We prove by induction on the proof of *Noet'* A . We are given as induction hypothesis that, $\forall n \in A. \text{Noet} A \setminus \{0, \dots, n\}$. We have to prove $\forall n \in A. \text{Noet} A \setminus \{n\}$, which follows from an auxiliary lemma, $\forall n, A. \text{Noet} A \Rightarrow \text{Noet} (A \cup \{n\})$, proved by induction. \square

Given a set A of natural numbers, we define a relation \succ_A on natural numbers such that $n \succ_A m$ if $m = \ell + 1$ with ℓ being the least natural number such that $n \leq \ell$ and $\ell \in A$. Formally,

$$\frac{n \leq \ell \quad \forall k. n \leq k < \ell \Rightarrow k \notin A \quad \ell \in A \quad \ell + 1 = m}{n \succ_A m}$$

Note that, for any stream s , \succ_s is equivalent to $\succ_{\mathcal{R}_s}$ by definition. So our task is to prove equivalence of A being Noetherian and accessibility of 0 with respect to \succ_A .

For a relation \succ over a set A , \succ^* denotes the reflexive and transitive closure of \succ and \succ^+ denotes the transitive closure.

Lemma 5.4. $\forall \succ. (\forall n. \text{acc}_\succ n) \Leftrightarrow (\forall n. \text{acc}_{\succ^+} n)$.

Proof. (\Rightarrow): We prove a slightly stronger statement, $\forall n. \text{acc}_\succ n \Rightarrow \forall m. n \succ^* m \Rightarrow \text{acc}_{\succ^+} m$ by induction on the proof of $\text{acc}_\succ n$, from which the claim follows.

(\Leftarrow): By induction on the proof of $\text{acc}_{\succ^+} n$. \square

Lemma 5.5. $\forall A. \text{Noet}' A \Leftrightarrow \text{acc}_{\succ_A} 0$.

Proof. (\Rightarrow): By induction on the proof of $\text{Noet}' A$. We are given as induction hypothesis that, $\forall n \in A. \text{acc}_{\succ_{A \setminus \{0, \dots, n\}}} 0$. We have to prove $\forall n. 0 \succ_A n \Rightarrow \text{acc}_{\succ_A} n$, which follows from the induction hypothesis and by observing that, $\forall A, n. \text{acc}_{\succ_{A \setminus \{0, \dots, n-1\}}} 0 \Rightarrow \text{acc}_{\succ_A} n$.

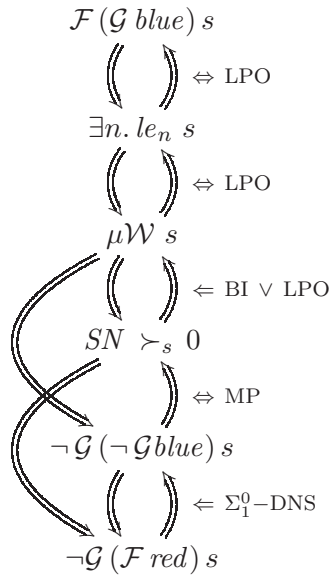
(\Leftarrow): We prove that, $\forall A, n. \text{acc}_{\succ_A^+} n \Rightarrow \text{Noet}' A \setminus \{0, \dots, n-1\}$ by induction on the proof of $\text{acc}_{\succ_A^+} n$. Then the case follows from Lemma 5.4. We are given as induction hypothesis that, $\forall m. n \succ_A^+ m \Rightarrow \text{Noet}' A \setminus \{0, \dots, m-1\}$. We have to prove $\text{Noet}' A \setminus \{0, \dots, n-1\}$, which follows from the induction hypothesis and by case analysis on whether $n \in A$ or not. \square

Combining lemmata 3.3, 5.3 and 5.5, we obtain:

Corollary 5.1. $\forall s. \mu\mathcal{W} s \Leftrightarrow \text{Noet} \mathcal{R}_s$.

6. CONCLUSION

The following diagram summarizes our current understanding of the constructive interrelations between the various notions of finiteness. Implications that are annotated have not been proved constructively; the annotations explain which principle is sufficient and, in some cases, necessary to prove the implication.



We do not know whether the implication $SN \succ_s 0 \Rightarrow \neg \mathcal{G}(\neg \mathcal{G} \text{blue}) s$ holds. As observed by Coquand and Spiwack in [2, Sec. 2.4, p. 225], the implication ‘if A is streamless, then A is Noetherian’ can be proved by Bar Induction (BI). The precise instance of Bar Induction that proves this implication depends on the set A . In our case here, the set A is decidable. Therefore our formulation of the implication, $SN \succ_s 0 \Rightarrow \mu\mathcal{W} s$, can be proved by a weak instance of Bar Induction (BI_D , see [6, Ch. 4, 8.11, p. 229]). We find it remarkable that both BI and LPO prove $SN \succ_s 0 \Rightarrow \mu\mathcal{W} s$, and do not know whether this implication can be proved constructively. Although we consider the latter unlikely, we prefer to consider it as

an open problem. Since \mathcal{R}_s is decidable, constructivity of $SN \succ_s 0 \Rightarrow \mu\mathcal{W} s$ will be more difficult to disprove than the conjecture by Coquand and Spiwack [2, Sec. 2.4, p. 225].

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REFERENCES

- [1] Nakata, K., Uustalu, T.: Resumptions, weak bisimilarity and big-step semantics for While with interactive I/O: an exercise in mixed induction-coinduction. In Aceto, L., Sobocinski, P., eds.: Proceedings of 7th Workshop on Structural Operational Semantics, SOS 2010 (Paris, Aug. 2010), Electron. Proc. in Theor. Comput. Sci., vol. 32, pp. 57–75, 2010.
- [2] Coquand, T., Spiwack, A.: Constructively finite? In Laureano Lambán, L., Romero, A., and Rubio, J., eds.: Scientific contributions in honor of Mirian Andrés Gómez, pp. 217–230, Servicio de Publicaciones, Universidad de La Rioja, Spain, 2010.
- [3] Bishop, E.: Foundations of Constructive Analysis. McGraw-Hill, New York, 1967.
- [4] Tarski, A.: Sur les ensembles finis. Fundam. Math., vol. 6, pp. 45–95, 1924.
- [5] Raffalli, C.: L'Arithmétique Fonctionnelle du Second Ordre avec Points Fixes. Thèse de l'université Paris VII, 1994.
- [6] Troelstra, A.S., van Dalen, D.: Constructivism in Mathematics, Volumes I and II, North-Holland, 1988.
- [7] Veldman, W. and Bezem, M.A.: Ramsey's theorem and the pigeonhole principle in intuitionistic mathematics. J. of London Math. Soc., vol. 47, n. 2, pp. 193–211, 1993.