
STATE OF BÜCHI COMPLEMENTATION *

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ABSTRACT. Complementation of Büchi automata has been studied for over five decades since the formalism was introduced in 1960. Known complementation constructions can be classified into Ramsey-based, determinization-based, rank-based, and slice-based approaches. Regarding the performance of these approaches, there have been several complexity analyses but very few experimental results. What especially lacks is a comparative experiment on all of the four approaches to see how they perform in practice. In this paper, we review the four approaches, propose several optimization heuristics, and perform comparative experimentation on four representative constructions that are considered the most efficient in each approach. The experimental results show that (1) the determinization-based Safra-Piterman construction outperforms the other three in producing smaller complements and finishing more tasks in the allocated time and (2) the proposed heuristics substantially improve the Safra-Piterman and the slice-based constructions.

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1. INTRODUCTION

Büchi automata are nondeterministic finite automata on infinite words. They recognize ω -regular languages and are closed under Boolean operations, namely union, intersection, and complementation. The formalism was first proposed and studied by Büchi in 1960 as part of a decision procedure for second-order logic [5]. Complementation of Büchi automata is significantly more difficult than that of nondeterministic finite automata on finite words. Given a nondeterministic finite automaton on finite words with n states, complementation yields an automaton with 2^n states through the subset construction [24]. Indeed, the subset construction is insufficient for the complementation of nondeterministic Büchi automata. In fact, Michel showed in 1988 that the blow-up of Büchi complementation is at least $n!$ (approximately $(n/e)^n$ or $(0.36n)^n$), which is much higher than 2^n [21]. This lower bound was eventually sharpened by Yan to $(0.76n)^n$ [43], which was matched by an upper bound by Schewe [26].

There are several applications of Büchi complementation in formal verification. For example, whether a system satisfies a property can be verified by checking if the intersection of the system automaton and the complement of the property automaton is empty [38]. Another example is that the correctness of an LTL translation algorithm can be tested with a reference algorithm as done in the development of the GOAL tool [36, 34]¹. Moreover, Büchi complementation also involves in the translation of QPTL [15] and ETL [42] formulae. Both QPTL and ETL are more expressive than LTL. Although recently many works have focused on universality and containment testing without going explicitly through complementation [7, 8, 6], it is still unavoidable in some cases [20].

There have been quite a few complementation constructions, which can be classified into four approaches: Ramsey-based approach [5, 28], determinization-based approach [25, 22, 2, 23], rank-based approach [32, 19, 17], and slice-based approach [13, 41]. The second approach is a deterministic approach while the last two are nondeterministic approaches. The first three approaches were reviewed in [40]. Due to the high complexity of Büchi complementation, optimization heuristics are critical to good performance [11, 9, 26, 14, 18]. However, with much recent emphasis shifted to universality and containment, empirical studies of Büchi complementation have been scarce [18, 11, 14, 35] in contrast with the rich theoretical developments. A comprehensive empirical study would allow us to evaluate the performance of these complementation approaches that has so far been characterized only by theoretical bounds.

In this paper, we review the four complementation approaches and perform comparative experimentation on four selected constructions that we consider the best in each approach. All the four constructions have been implemented in GOAL [36, 34]. Although one might expect that the nondeterministic approaches would be better than the deterministic approach because of better worst-case bounds, our experimental results show that the deterministic construction is the best for complementation in average. At the same time the Ramsey-based approach, which is competitive in universality and containment testing [1, 7, 8], performs rather poorly in our complementation experiments. We also propose optimization heuristics for the determinization-based construction, the rank-based construction, and the slice-based construction. Our experiment shows that the optimization heuristics substantially improve the three constructions. Overall, our work confirms

¹With the help of complementation, implementations of translation algorithms in GOAL have been tested against formulae in Spec Patterns [30] and randomly generated formulae.

the importance of experimentation and heuristics in studying Büchi complementation, as worst-case bounds may not be accurate indicators of performance.

The rest of this paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we review the four complementation approaches. We discuss the results of our comparative experimentation on the four approaches in Section 4. Section 5 describes our optimization heuristics and Section 6 shows the improvement made by our heuristics. We conclude in Section 7.

2. PRELIMINARIES

A (*nondeterministic*) ω -automaton is a five tuple $(\Sigma, Q, q_0, \delta, \mathcal{F})$, where Σ is a nonempty finite alphabet, Q is a nonempty finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times \Sigma \rightarrow 2^Q$ is the transition function, and \mathcal{F} is the acceptance condition, to be described subsequently. The automaton is *deterministic* if $|\delta(q, a)| = 1$ for all $q \in Q$ and $a \in \Sigma$.

Let $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ be an ω -automaton and $w = a_0a_1 \cdots \in \Sigma^\omega$ an infinite word. A *run* of A on w is a sequence $q_0q_1 \cdots \in Q^\omega$ satisfying $\forall i : q_{i+1} \in \delta(q_i, a_i)$. A run is *accepting* if it satisfies the acceptance condition and a word is *accepted* if there is an accepting run on it. The *language* of an ω -automaton A , denoted by $L(A)$, is the set of words accepted by A .

Let ρ be a run and $\text{inf}(\rho)$ be the set of states that occur infinitely often in ρ . Various types of ω -automata can be defined by assigning different acceptance conditions as follows.

- *Büchi condition*: $\mathcal{F} \subseteq Q$. ρ satisfies the condition iff $\text{inf}(\rho) \cap \mathcal{F} \neq \emptyset$, and every $q \in \mathcal{F}$ is called an *accepting state*.
- *Muller condition*: $\mathcal{F} \subseteq 2^Q$. ρ satisfies the condition iff there exists an $F \in \mathcal{F}$ such that $\text{inf}(\rho) = F$.
- *Rabin condition*: $\mathcal{F} \subseteq 2^Q \times 2^Q$. ρ satisfies the condition iff there exists a pair $(E, F) \in \mathcal{F}$ such that $\text{inf}(\rho) \cap E = \emptyset$ and $\text{inf}(\rho) \cap F \neq \emptyset$.
- *Streett condition*: $\mathcal{F} \subseteq 2^Q \times 2^Q$. ρ satisfies the condition iff for all pairs $(E, F) \in \mathcal{F}$, $\text{inf}(\rho) \cap F \neq \emptyset$ implies $\text{inf}(\rho) \cap E \neq \emptyset$.
- *parity condition*: $\mathcal{F} : Q \rightarrow \{0, 1, \dots, 2r\}$. ρ satisfies the condition iff $\min\{\mathcal{F}(q) \mid q \in \text{inf}(\rho)\}$ is even and $\mathcal{F}(q)$ is called the parity of the state q .

The parity condition $\mathcal{F} : Q \rightarrow \{0, 1, \dots, 2r\}$ is a special case of the Rabin condition, referred to as *Rabin chain*, $\{(E_0, F_0), \dots, (E_r, F_r)\}$ where $E_0 \subset F_0 \subset E_1 \subset \dots \subset E_r \subset F_r$, $E_i = \{q \in Q : \mathcal{F}(q) < 2i\}$, and $F_i = \{q \in Q : \mathcal{F}(q) \leq 2i\}$.

We use a system of three-letter acronyms to denote these ω -automata. The first letter indicates whether the automaton is **n**ondeterministic or **d**eterministic. The second letter indicates whether the acceptance condition is **B**üchi, **M**uller, **R**abin, **S**treett, or **p**arity. The third letter is always a “**W**” indicating that the automaton accepts words. For example, NBW stands for a nondeterministic Büchi automaton and DPW stands for a deterministic parity automaton.

Let A be an ω -automaton with an alphabet Σ . A is *universal* iff $L(A) = \Sigma^\omega$. A complement of A is defined as an automaton that accepts exactly the language $\Sigma^\omega - L(A)$, denoted by $\overline{L(A)}$ when the alphabet Σ is clear in the context. A state is *live* if it occurs in an accepting run on some word, and is *dead* otherwise. Dead states can be discovered using a nonemptiness algorithm, cf. [39], and can be pruned off without affecting the language of the automaton. As a complement of a universal automaton has no accepting run, only the initial state will remain in the complement after pruning dead states.

When focusing on Büchi conditions, the acceptance of an infinite word w by an NBW A can be determined not only by the sequential runs of A on w but also by an aggregated tree structure of those runs. Depending on different ways of aggregation, different tree structures can be defined.

The *run tree* of A on w is a tree where a (full) branch corresponds to a run of A on w and there is a corresponding branch for each run of A on w . To determine whether w is accepted by A , one needs only pay attention to the distinction between accepting and non-accepting states, and a run tree may be simplified or abstracted to leave just this much detail. The *split tree* of A on w is a binary tree that abstracts the run tree by grouping accepting children and nonaccepting children of a tree node respectively into a left child and a right child of the node. The word w is accepted by A if there is a *left-recurring branch* in the split tree of A on w while a branch is left-recurring if the branch goes left infinitely many times. Define tree width as the maximal number of nodes that are on the same level. Both run trees and split trees suffer the problem of unbounded tree width, which motivates the next tree structure. The *reduced split tree* of A on w is a binary tree obtained from the split tree of A on w by removing a state from a node if it also occurs in a node to the left on the same level; a node is removed if it becomes empty. Similarly, w is accepted by A if there is a left-recurring branch in the reduced split tree of A on w . Each a split tree or a reduced split tree can be represented by a sequence of *slices* where a slice is a sequence of sets of states representing all nodes on a same level of the tree from left to right.

Consider the NBW in Figure 1 where the alphabet is $\{p, \neg p\}$, the initial state is q_0 , and the acceptance condition is $\{q_1\}$. The split tree and the reduced split tree of the NBW on the accepted word $p\neg pp^\omega$ are shown in Figure 2.

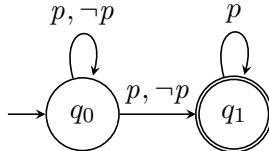


FIGURE 1. An NBW where the alphabet is $\{p, \neg p\}$, the initial state is q_0 , and the acceptance condition is $\{q_1\}$

3. APPROACHES TO COMPLEMENTATION

We review the four approaches to the complementation of Büchi automata in this section. In each approach, we identify one construction with the best worst-case complexity. The four identified constructions will be taken into account later in our comparative experimentation.

Ramsey-based approach. The very first complementation construction introduced by Büchi in 1960 involves a Ramsey-based combinatorial argument and results in a $2^{2^{O(n)}}$ blow-up in the state size [5]. This construction was later improved by Sistla, Vardi, and Wolper to reach a single-exponential complexity $2^{O(n^2)}$ [28]. The improved construction, referred to as RAMSEY in this paper, constructs a complement as the union of several NBWs. Each NBW accepts a subset of the complement language in the form of UV^ω where U and V are recognized respectively by two classic finite automata on finite words.

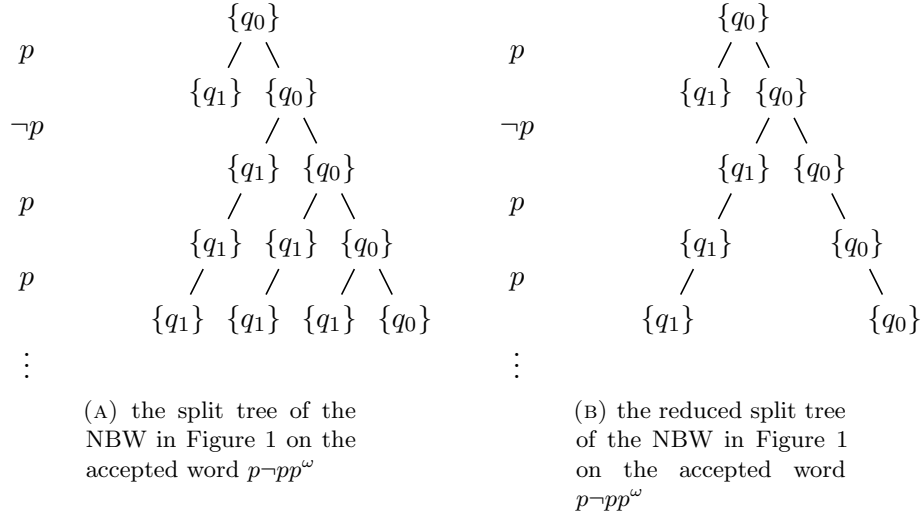


FIGURE 2. Examples of a split tree and a reduced split tree

Various optimization heuristics for the Ramsey-based approach were described in [1, 8], but the focus in these works was on universality and containment. In spite of the quadratic exponent of the Ramsey-based approach, it was shown in [1, 7, 8] to be quite competitive for universality and containment testing.

Determinization-based approach. Safra's $2^{O(n \log n)}$ construction is the first complementation construction that matches the $\Omega(n!)$ lower bound [25]. The main idea is the use of (1) Safra trees to capture the history of all runs on a word and (2) marks to indicate whether a run passes an accepting state again or dies. Later on, Muller and Schupp introduced a similar determinization construction which records more information and yields larger complements in most cases, but can be understood more easily [22, 2].

The determinization-based approach performs complementation in stages: first convert an NBW to a deterministic automaton, then complement the deterministic automaton by modifying only the acceptance condition, and finally convert the complement deterministic automaton to an NBW. Both Safra's construction and the construction by Muller and Schupp use DRWs as the intermediate deterministic automata. In [23], Piterman improved Safra's construction by using a more compact tree structure and using DPWs as the intermediate deterministic automata. The improved construction by Piterman, referred to as SAFRA-PITERMAN (or simply SP) in this paper, yields an upper bound of n^{2^n} . (See also [27].)

As the determinization-based approach performs complementation in stages, different optimization techniques can be applied separately to the different stages. For instance, several optimization heuristics on Safra's determinization and on simplifying the intermediate DRWs were proposed by Klein and Baier [18].

Rank-based approach. The rank-based approach, proposed by Kupferman and Vardi, uses rank functions to measure the progress made by a node of a run tree towards fair

termination [19]. The basic idea of this approach may be traced back to Klarlund’s construction [17]. Both constructions have complexity $2^{O(n \log n)}$. There were also several optimization techniques proposed in [11, 9, 14]. A final improvement was proposed recently by Schewe [26] to the construction in [9]. The construction with Schewe’s improvement, referred to as RANK in this paper, performs a subset construction in the first phase. From some point, it guesses ranks and transits from the first phase to the second phase, where the guesses are verified. Schewe proposed doing this verification in a piecemeal fashion. This yields a complement with $O((0.76n)^n)$ states, which matches the known lower bound modulo an $O(n^2)$ factor.

Unlike the determinization-based approach that collects information from the history, the rank-based approach guesses ranks bounded by $2(n - |\mathcal{F}|)$ and results in many non-deterministic choices. This nondeterminism means that the rank-based construction often creates more useless states because many guesses may be verified later to be incorrect.

Slice-based approach. The slice-based construction was proposed by Kähler and Wilke in 2008 [13]. The blow-up of the construction is $4(3n)^n$ while its preliminary version in [41], referred to as SLICE here, has a $(3n)^n$ blow-up². Unlike the determinization-based and the rank-based approaches that analyze run trees, the slice-based approach analyzes reduced split trees. The construction SLICE uses slices as states of the complement and performs a construction based on the evolution of reduced split trees in the first phase. By decorating nodes in slices at some point, it guesses whether a node belongs to an infinite branch of a reduced split tree or the node has a finite number of descendants. In the second phase, it verifies the guesses and enforces that accepting states will not occur infinitely often.

The first phase of SLICE in general creates more states than the first phase of RANK because of an ordering of nodes in the reduced split trees. Similar to RANK, SLICE also introduces nondeterministic choices in guessing the decorations. While RANK guesses ranks bounded by $2(n - |\mathcal{F}|)$, SLICE guesses only the decorations from a fixed set of size 3.

4. COMPARISON OF COMPLEMENTATION APPROACHES

Based on preliminary experiments [37], we chose four representative complementation constructions, namely RAMSEY [28], SAFRA-PITERMAN [23], RANK [26], and SLICE [41], that we considered the most efficient in each approach. These constructions were implemented in GOAL³ [36, 34]. In the following, we first describe our implementations, the settings of experiments, and then present the experimental results.

4.1. Implementations. All the four implemented constructions use the same automaton structure in GOAL. Unlike modern model checkers that encode transition relations of automata implicitly in BDD [4], GOAL represents automata explicitly, that is, every transition is implemented as a Java object. The implemented constructions also use the same functions to access the alphabet, the states, the transitions, and the acceptance condition of an automaton. An automaton in GOAL is a Java object with a HashSet of atomic propositions (or classical symbols), a HashSet of states, an initial state, a HashSet of transitions, and

²The construction in [13] has a higher complexity than its preliminary version because it aims at treating complementation and disambiguation in a uniform way.

³We use the first generation of GOAL to perform all the experiments.

an acceptance condition. A state in an automaton has a unique ID and a possibly empty label. A transition in an automaton has a unique ID, a reference to the source state of the transition, a reference to the destination state of the transition, and a label. A Büchi condition is a Vector of states. A parity condition is a Vector of Vectors of states where a state in the i -th vector has a parity i . In order to access the successors and predecessors of a state quickly, an automaton contains three HashMaps, namely *fmap*, *tmap*, and *ftmap*, of which *fmap* maps a state to its outgoing transitions, *tmap* maps a state to its incoming transitions, and *ftmap* maps a state s and a state t to the transitions from s to t .

Our implementations of complementation constructions basically construct a complement incrementally from the initial state following the description of the original papers [28, 23, 26, 41]. As a state object in GOAL does not always match the state structure of the complement in theory, we define additional Java classes to model the state structure of the complement. During a construction, two HashMaps are maintained respectively to map a state object created in the complement to the underlying state structure described in the paper and vice versa. Our implementations do not use fancy data structures to represent the underlying state structures of the complements. For example, in SAFRA-PITERMAN, the state structure of the complements is represented by a tree where a tree node contains an ArrayList of states, a reference to its parent, references to its children, and references to its older siblings. In RANK, the state structure is represented by (1) TreeSets of states (for the subset construction) and (2) tuples containing two TreeSets of states, a ranking function represented as a HashMap object, and an integer for the turn-wise cut-point optimization [26].

More details of our implementations can be obtained directly from the source code, which is released with the first generation of GOAL⁴.

4.2. Settings of Experiments. We randomly generated 11,000 automata⁵ with an alphabet of size 2 and state sets of size 15 based on the approach proposed by Tabakov and Vardi [31]. Among the 11,000 automata of state size 15, denoted by \mathcal{A}_{15} , each 100 automata were generated from a combination of 11 transition densities (from 1.0 to 3.0) and 10 acceptance densities (from 0.1 to 1.0). For every generated automaton $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ with n states, symbol $a \in \Sigma$, transition density r , and acceptance density f , we made $q \in \delta(p, a)$ for $\lceil rn \rceil$ pairs of states $(p, q) \in Q^2$ uniformly chosen at random and added $\lceil fn \rceil$ states to \mathcal{F} uniformly at random. Our parameters were chosen to generate a large set of complementation problems, ranging from easy to hard. The experiment was performed on a cluster at Rice University (<http://rcsg.rice.edu/sugar/int/>). For each complementation task, we allocated one 2.83-GHz CPU and 1 GB of memory. The timeout of a complementation task was 10 minutes.

4.3. Experimental Results. The experimental results are summarized on the top of Table 1 where T is the total number of timed-out tasks and M is the total number of tasks that run out of memory. Compared to SAFRA-PITERMAN that has only 5 unfinished tasks, each

⁴The source code of the first generation of GOAL is released per request. Please visit <http://goal.im.ntu.edu.tw/> for more details.

⁵A specialized version of GOAL and all the generated automata used in the experiments can be directly downloaded from <http://goal.im.ntu.edu.tw/> without registration.

TABLE 1. A comparison of the four representative constructions based on \mathcal{A}_{15} without and with preminimization. The preminimization is denoted by +P. We will use SP as a shorthand of SAFRA-PITERMAN in all tables.

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{15} (without preminimization)								
RAMSEY	4,564	36	2,259	513.85	(0)	30.82	(522.50)	0.060
SP	5	0		45.26	(1,843)	2.27	(556.67)	0.050
RANK	5,303	0		260.41	(415)	2.79	(649.17)	0.011
SLICE	3,131	3,213		790.92	(1)	3.03	(530.67)	0.004
\mathcal{A}_{15} (with preminimization)								
RAMSEY+P	4,190	41	3,522	193.72	(247)	26.82	(776.25)	0.218
SP+P	4	0		11.08	(1,712)	2.31	(817.42)	0.572
RANK+P	4,316	0		56.60	(1,546)	2.37	(1,126.42)	0.160
SLICE+P	2,908	2,435		199.52	(17)	2.94	(801.92)	0.092

of RAMSEY⁶, RANK, and SLICE has around 50% of unfinished tasks. Besides the number of unfinished tasks, we also want to compare the sizes of states of the constructed complements. As these constructions may successfully finish different tasks, a construction may be misleadingly considered to be worse in producing more states because it can finish harder tasks that have much larger complements. Therefore, we only collect state-size information from 2,259 *effective samples*, which are tasks finished successfully by all the four constructions. Among the 2,259 effective samples, around 90% of the automata are universal.

The state-size information is shown in the columns S_L and S_R . The column S_R is the average number of reachable states, while S_L is the average number of live states, of the complements. The two columns show that SAFRA-PITERMAN is the best in average state size. The low S_L/S_R ratio shows that RANK and SLICE create more dead states that can be easily pruned off.

In addition to the number of unfinished tasks and the state-size information, the number of smallest complements produced by a construction among the effective samples is another measure of performance. A construction *wins* in an effective sample w.r.t. S_R (resp., S_L) if the complement produced by the construction is the smallest in terms of reachable states (resp., live states). The Win column of a construction in S_R (resp., S_L) is the fractional share of effective samples where the construction wins w.r.t. S_R (resp., S_L). If k constructions win in an effective sample, each gets $1/k$ shares. The Win columns show that although RANK generates more dead states, it produces more complements that are the smallest after pruning dead states.

RANK and SLICE become much closer to SAFRA-PITERMAN in S_L because around 90% of the 2,259 effective samples are universal automata, whose complements have no live states. If we only consider nonuniversal automata, the gaps between SAFRA-PITERMAN and RANK, and SAFRA-PITERMAN and SLICE in S_L become larger as shown on the top of Table 2.

As the heuristic of preminimization applied to the input automata, denoted by +P, is considered to help the nondeterministic constructions more than the deterministic one, we also compare the four constructions with preminimization and summarize the results on the

⁶RAMSEY could not finish all tasks in our previous experiments in [33] because its implementation constructs the full state space. The implementation has been modified to construct only reachable states.

TABLE 2. A comparison of the four representative constructions based on the nonuniversal automata in \mathcal{A}_{15}

Constructions	Eff. Samples	S_R (Win)	S_L (Win)	S_L/S_R
\mathcal{A}_{15} (without preminimization)				
RAMSEY	171	1,892.37 (0)	397.10 (0)	0.210
SP		36.38 (102.5)	17.77 (34.67)	0.488
RANK		156.63 (67.5)	24.61 (127.67)	0.157
SLICE		422.88 (1)	27.75 (8.67)	0.066
\mathcal{A}_{15} (with preminimization)				
RAMSEY+P	418	1,000.89 (0.0)	218.55 (0.25)	0.218
SP+P		21.02 (116.0)	12.02 (41.42)	0.572
RANK+P		77.97 (300.5)	12.51 (350.42)	0.160
SLICE+P		189.90 (1.5)	17.39 (25.92)	0.092

bottom of Tables 1 and 2. We only applied the preminimization implemented in GOAL tool, namely the simplification by simulation in [29]. According to our experimental results, the preminimization does improve RAMSEY, RANK, and SLICE more than SAFRA-PITERMAN in the complementation but does not close too much the gap between them in the comparison, though there are other preminimization techniques that we did not apply in the experiment.

In summary, the experimental results show that (1) SAFRA-PITERMAN is the best in average state size and in the number of finished tasks, (2) RAMSEY is not competitive in complementation even though it is competitive in universality and containment testing as shown in [1, 7, 8], and (3) SLICE has the most unfinished tasks (even more than RAMSEY) and, compared to SAFRA-PITERMAN and RANK, produces many more states. As we will show later, we can improve the performance of SLICE significantly by employing various optimization heuristics.

Besides the 11,000 automata with 15 states, another 11,000 automata with 10 states and another 11,000 automata with 20 states, denoted by \mathcal{A}_{10} and \mathcal{A}_{20} respectively, were also used in our experiments. We include the experimental results based on \mathcal{A}_{10} and \mathcal{A}_{20} in the appendix because \mathcal{A}_{10} contains fewer tough automata especially for SAFRA-PITERMAN while \mathcal{A}_{20} is too hard to get effective samples. Moreover, the comparisons based on \mathcal{A}_{10} and \mathcal{A}_{20} are basically consistent to those based on \mathcal{A}_{15} .

5. OPTIMIZATION TECHNIQUES

The following optimization heuristics are described in this section: simplifying DPWs by simulation (+S) and merging equivalent states (+E) for SAFRA-PITERMAN; maximizing the Büchi acceptance set (+A) for RANK; deterministic decoration (+D), reducing transitions (+R), and merging adjacent nodes (+M) for SLICE. For SAFRA-PITERMAN, the first heuristic +S is applied to an intermediate complement DPW and yields an NPW while the second heuristic +E is applied to the conversion from an NPW to an NBW. The heuristic +A for RANK is applied to the input NBW before complementation and may be also useful for other constructions. The three heuristics +D, +R, and +M for SLICE are applied to the complementation construction.

5.1. For Safra-Piterman. SAFRA-PITERMAN performs complementation via several intermediate stages: starting with the given NBW, it computes first an equivalent DPW, then a complement DPW, and finally a complement NBW. We address (1) the simplification of the complement DPW, which results in an NPW, and (2) the conversion from an NPW to an equivalent NBW.

Simplifying DPWs by simulation (+S). For the simplification of the complement DPW, we borrow from the ideas of Somenzi and Bloem [29]. The direct and reverse simulation relations they introduced are useful in removing transitions and possibly states of an NBW while retaining its language. We define the simulation relations for an NPW in order to apply the same simplification technique. Let $(\Sigma, Q, q_0, \delta, \mathcal{F})$ be an NPW. Define a predecessor function $\delta^{-1}(p, a) = \{q \mid p \in \delta(q, a)\}$ for $p \in Q$ and $a \in \Sigma$. Given $p, q \in Q$, p is *directly simulated* by q iff (1) for all $p' \in \delta(p, a)$, there is $q' \in \delta(q, a)$ such that p' is directly simulated by q' , and (2) $\mathcal{F}(p) = \mathcal{F}(q)$. Similarly, p is *reversely simulated* by q iff (1) for all $p' \in \delta^{-1}(p, a)$, there is $q' \in \delta^{-1}(q, a)$ such that p' is reversely simulated by q' , (2) $\mathcal{F}(p) = \mathcal{F}(q)$, and (3) $p = q_0$ implies $q = q_0$. After simplification using simulation relations, as in [29], a DPW may become nondeterministic. Since the complementation of a DPW is much easier than that of an NPW, the simplification by simulation is applied to the complement DPW (in the second stage) but not to the equivalent DPW (in the first stage).

Merging equivalent states (+E). As for the conversion from an NPW to an NBW, a typical way in the literature is to directly apply the conversion from an NRW to an NBW [16, 10] because the parity condition is a special case of the Rabin condition. Here we first review the conversion from an NRW to an NBW adapted for an NPW.

Intuitively, the conversion nondeterministically guesses the minimal even parity passed infinitely often in a run starting from some state. Once a run is guessed to pass a minimal even parity $2k$ infinitely often starting from a state p , every state q in the run after p should have a parity greater than or equal to $2k$ and q is designated as an accepting state in the resulting NBW if it has parity $2k$.

Given an NPW $P = (\Sigma, Q, q_0, \delta, \mathcal{F})$ where $\mathcal{F} : Q \rightarrow \{0, 1, \dots, 2r\}$, the typical conversion constructs the equivalent NBW $A = (\Sigma, S, s_0, \Delta, \mathcal{G})$ where

- $S = Q \times \{0, 2, \dots, 2r\}$,
- $s_0 = (q_0, 0)$,
- $\Delta : S \times \Sigma \rightarrow 2^S$ is the transition function satisfying the following two conditions:
 - $(q_j, 2k) \in \Delta((q_i, 0), a)$ iff $k > 0$ and $q_j \in \delta(q_i, a)$
 - $(q_j, 2k) \in \Delta((q_i, 2k), a)$ iff $q_j \in \delta(q_i, a)$ and $\mathcal{F}(q_j) \geq 2k$, and
- $\mathcal{G} = \{(q, 2k) \in S \mid \mathcal{F}(q) = 2k\}$.

A run of A will always look like $(q_0, 0) \cdots (q_{i-1}, 0)(q_i, 2k)(q_{i+1}, 2k) \cdots$ where the transitions from $(q_{i-1}, 0)$ to $(q_i, 2k)$ represent the guess of $2k$ to be the minimal even parity passed infinitely often and from there on $2k$ remains unchanged.

Lemma 5.1. *Given an NPW P , the typical conversion constructs an NBW A such that $L(P) = L(A)$.*

Proof. The typical conversion basically follows the conversion in [16] but restricts the Rabin condition to a Rabin chain, which is equivalent to the parity condition of P . Thus, the proof in [16] applies. \square

We propose to perform the conversion with states merged and with the start of guessing the minimal even parity $2k$ delayed to a state that has the even parity. Let $P = (\Sigma, Q, q_0, \delta, \mathcal{F})$ be an NPW where $\mathcal{F} : Q \rightarrow \{0, 1, \dots, 2r\}$. We first define an equivalence relation on states with respect to an even parity in order to merge the states in the conversion. Two states p and q are equivalent with respect to an even parity $2k$, denoted by $p \equiv_{2k} q$, iff $\delta(p, a) = \delta(q, a)$ for all $a \in \Sigma$, and either

- $\mathcal{F}(p) = \mathcal{F}(q) = 2k$,
- $\mathcal{F}(p) > 2k$ and $\mathcal{F}(q) > 2k$, or
- $\mathcal{F}(p) < 2k$ and $\mathcal{F}(q) < 2k$.

Let $[q]_{2k} = \{p \mid p \equiv_{2k} q\}$ be the equivalence class of a state q for a parity $2k$. Let $[Q] = \{[q]_{2k} \mid q \in Q \text{ and } 0 \leq k \leq r\}$ denote the set of equivalence classes of states in Q for all even parities.

Given an NPW $P = (\Sigma, Q, q_0, \delta, \mathcal{F})$ where $\mathcal{F} : Q \rightarrow \{0, 1, \dots, 2r\}$, the improved conversion constructs the equivalent NBW $A' = (\Sigma, S, s_0, \Delta, \mathcal{G})$ where

- $S = [Q] \times \{0, 2, \dots, 2r\}$,
- $s_0 = ([q_0]_0, 0)$,
- $\Delta : S \times \Sigma \rightarrow 2^S$ is the transition function where
 - [TR1]: $([q]_{2k}, 2k) \in \Delta((p]_0, 0), a)$ iff $k > 0$, $\delta(p, a) \cap [q]_{2k} \neq \emptyset$, and $\mathcal{F}(q) = 2k$,
 - [TR2]: $([q]_{2k}, 2k) \in \Delta([p]_{2k}, 2k), a)$ iff $\delta(p, a) \cap [q]_{2k} \neq \emptyset$ and $\mathcal{F}(q) \geq 2k$, and
- $\mathcal{G} = \{([q]_{2k}, 2k) \in S \mid \mathcal{F}(q) = 2k\}$.

Lemma 5.2. *If a word w is accepted by P , then it is accepted by A' .*

Proof. Let w be a word accepted by P and ρ an accepting run $q_0q_1 \dots$ of P on w . Suppose $2k$ is the minimal even parity passed infinitely often in ρ after some state q_i of parity $2k$. By the construction, there is a run $\rho' = [q_0]_0[q_1]_0 \dots [q_{i-1}]_0[q_i]_{2k}[q_{i+1}]_{2k} \dots$ of A' on w . The transitions before and after $[q_i]_{2k}$ follow the transition rules TR1 and TR2 respectively. Since $2k$ is the minimal even parity passed infinitely often in ρ , there are infinitely many $[q_m]_{2k}$'s ($i \leq m$) in ρ' such that $\mathcal{F}(q_m) = 2k$ and $[q_m]_{2k} \in \mathcal{G}$. Thus, ρ' is an accepting run of A' on w . \square

Lemma 5.3. *If a word w is accepted by A' , then it is accepted by P .*

Proof. Let $w = a_0a_1 \dots$ be a word accepted by A' and ρ an accepting run $[q_0]_0[q_1]_0 \dots [q_{i-1}]_0[q_i]_{2k}[q_{i+1}]_{2k} \dots$ of A' on w . Let M be an infinite set of indices such that $m \in M$ iff $[q_m]_{2k} \in \mathcal{G}$. By the construction, we can find a state $q'_1 \in [q_1]_0$ such that $q'_1 \in \delta(q_0, a_0)$. Starting from q'_1 , by the construction and by the definition of equivalence classes, we can find a state $q'_2 \in [q_2]_0$ such that $q'_2 \in \delta(q'_1, a_1)$ and so on. Therefore, there is a run $\rho' = q_0q'_1q'_2 \dots q'_iq'_{i+1} \dots$ of P on w such that $q'_j \in [q_j]_0$ for $0 < j < i$ and $q'_j \in [q_j]_{2k}$ for $j \geq i$. By the transition function and the equivalence relation, $\mathcal{F}(q'_j) \geq 2k$ for $j \geq i$ and $\mathcal{F}(q'_m) = 2k$ for all $m \in M$. Hence, $2k$ is the minimal even parity passed infinitely often in ρ' after q_i and ρ' is an accepting run of P on w . \square

Theorem 5.4. $L(P) = L(A')$.

Proof. The result follows directly from Lemmas 5.2 and 5.3. \square

5.2. For Rank.

Maximizing the Büchi acceptance set (+A). . As stated in Section 3, the ranks for the rank-based approach are bounded by $2(n - |\mathcal{F}|)$. The larger \mathcal{F} is, the fewer the ranks are. Thus, we propose to maximize the acceptance set of the input NBW before complementation based on the following theorem without changing its language, states, or transition function.

Theorem 5.5. *Let $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ and $A' = (\Sigma, Q, q_0, \delta, \mathcal{G})$ be two NBWs where $\mathcal{G} \supseteq \mathcal{F}$. Then, $L(A) = L(A')$ if for all $q \in \mathcal{G}$, every elementary cycle containing q also contains at least one state in \mathcal{F} .*

Proof. Since $\mathcal{G} \supseteq \mathcal{F}$, $L(A) \subseteq L(A')$. To prove $L(A) \supseteq L(A')$, first let $\rho = q_0q_1 \cdots$ be an accepting run of A' on some word w . Since ρ is accepting, there exist some state $q_i \in \rho$ and infinite indices $i_0 < i_1 < i_2 < \cdots$ such that $q_i \in \mathcal{G}$ and $q_i = q_{i_0} = q_{i_1} = q_{i_2} = \cdots$. For all $j \geq 0$, the sequence of states $q_{i_j}q_{i_{j+1}} \cdots q_{i_{j+1}}$ forms a cycle, denoted by C_j . As $q_i \in \mathcal{G}$ and a cycle is formed by elementary cycles, in each C_j , there is some state $q_{k_j} \in \mathcal{F}$. Since there are infinitely many C_j 's but Q is finite, there exists some state $q_k \in \mathcal{F}$ occurring infinitely many times in ρ . Thus, ρ is an accepting of A on w and $L(A) \supseteq L(A')$. \square

The elementary cycles of an automaton can be found by the algorithm in [12] with a time complexity $O((n+e)(c+1))$ and a space complexity $O(n+e)$ where n is the number of states, e the number of transitions, and c the number of elementary cycles in the automaton⁷.

This heuristic can also be applied to other complementation approaches as it maximizes the acceptance set of the input NBW before complementation. We will show the improvement made by this heuristic for SAFRA-PITERMAN, RANK, and SLICE later in Section 6.

5.3. For Slice.

The central idea of SLICE is based on the following lemma [13].

Lemma 5.6. *A word w is rejected by an NBW A iff the reduced split tree of A on w has a cutoff, which is a level i such that after the i -th level in the reduced split tree, all the left children are in finite branches.*

Note that a reduced split tree may have infinitely many cutoffs.

As an example, the reduced split tree of the NBW in Figure 1 on a rejected word $(p-p)^\omega$ is shown in Figure 3 where the superscripts 0, *, and 1 are decorations to explained later. It can be seen that after the first level of the reduced split tree, all the accepting states of the NBW are in finite branches.

Based on Lemma 5.6, SLICE constructs a complement with slices as states to accept all reduced split trees of an input NBW on rejected words by guessing the cutoffs nondeterministically. A decoration scheme is applied to verify whether a slice is on a cutoff. The transition relation of the complement is divided into two phases. In the first phase, the transition relation is based on the evolution of reduced split trees. When the construction nondeterministically chooses a slice as the slice on some cutoff, it guesses the decorations of the nodes in the slice and goes from the first phase to the second phase, where the decorations are verified.

A node in a slice can be decorated by 1, 0, or *. Intuitively, the decoration 1 indicates that a node is in an infinite branch of a reduced split tree. The decoration 0 indicates that the descendants of a node die out eventually before the next *reset slice*, which is a slice

⁷Instead of finding elementary cycles, our implementation makes a state accepting if the state cannot go back to itself without passing an accepting state in \mathcal{F} , which is checked by a depth-first search.

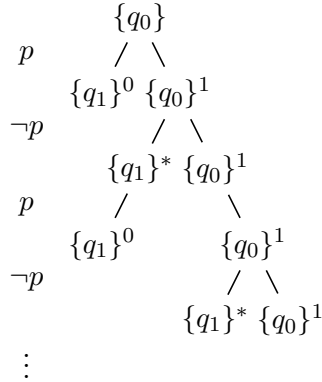


FIGURE 3. A decorated reduced split tree of the NBW in Figure 1 on the rejected word $(p\neg p)^\omega$

with no node decorated by 0. The decoration $*$ has the same meaning as 0 but the check is put on hold after the next reset slice where the children of $*$ -nodes are decorated by 0. Shown in Figure 4 are the decoration rules, which enforce that when reset slices are passed infinitely many times, descendants of the left children after decoration will eventually die out.

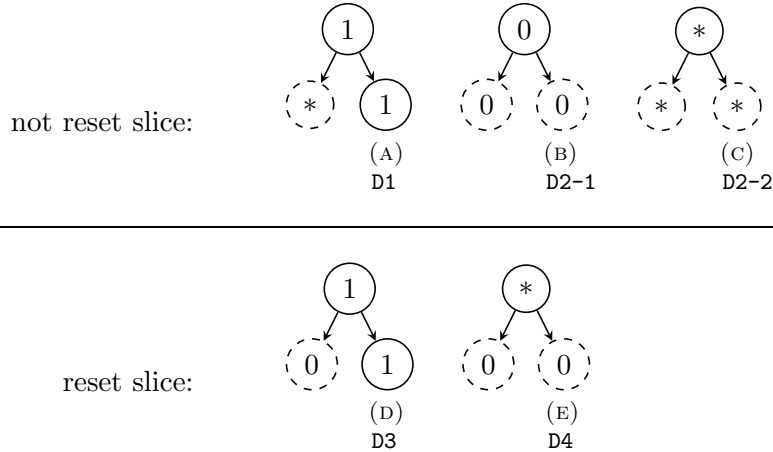


FIGURE 4. The decoration rules D1, D2 (which consists of D2-1 and D2-2), D3, and D4 applied in the basic SLICE construction. The upper three rules are applied only to non-reset slices while the lower two rules only to reset slices. A node in a slice is represented by a circle in which the label denotes the decoration of the node. As an example, when rule D3 is applied to a reset slice, the left child of an 1-node is decorated by 0 while the right child is decorated by 1. A child can be absent when applying a rule if it is dashed and otherwise it must exist.

The reason why both 0 and * are used for nodes on finite branches is that when we focus on the sequence of slices that form a reduced split tree, we want to distinguish a node that just died on the previous level between the same node that is just born on the current level. Otherwise, reset slices may not appear if nodes decorated by 0 are always born on a level immediately after they died out on the previous level. For example, consider the decorated reduced split tree in Figure 3. If the decoration * is replaced by 0, the reduced split tree will no longer contain reset slices because $\{q_1\}^0$ appears on every level after the root, but actually $\{q_1\}^0$ dies on every odd level and is born on every even level.

Before the formal description of the basic SLICE construction, we first introduce some notations. Let $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ be an NBW and D the set $\{0, *, 1\}$. An *undecorated slice* over Q is a finite, pairwise disjoint, sequence $Q_0 \cdots Q_{n-1}$ of non-empty subsets of Q . A *decorated slice* over Q is a finite sequence $(Q_0, d_0) \cdots (Q_{n-1}, d_{n-1})$ where $Q_0 \cdots Q_{n-1}$ form an undecorated slice and $d_i \in D$ for $i < n$. The i -th node of a slice s is denoted by $s(i)$ and the number of nodes of a slice is denoted by $|s|$. The empty sequence, denoted by \perp , is a special slice considered both undecorated and decorated. The set of slices over Q is denoted by $S = S^u \cup S^d$ where S^u is the set of undecorated slices and S^d the set of decorated slices. Let $s = (Q_0, d_0) \cdots (Q_{n-1}, d_{n-1}) \in S^d$. Define $s_{\downarrow Q} = Q_0 \cdots Q_{n-1}$ to be the undecorated version of s and $s_{\downarrow D} = \{d_0, \dots, d_{n-1}\}$. We say s is a *reset slice* iff $0 \notin s_{\downarrow D}$ and s is *doomed* iff $1 \notin s_{\downarrow D}$. In particular, \perp is a reset slice and is doomed.

5.3.1. *The basic SLICE construction.* Let $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ be an NBW. The complement constructed by SLICE is $A' = (\Sigma, S, s_0, \Delta, \mathcal{G})$ where

- $s_0 = \{q_0\}$,
- $\Delta = S \times \Sigma \rightarrow 2^S$ is the transition function described below, and
- $\mathcal{G} = \{s \in S^d \mid s \text{ is a reset slice}\}$.

For all $s \in S$ and $a \in \Sigma$, $\Delta(s, a)$ is defined as follows:

- $\Delta(s, a) = \{\delta_u(s, a)\} \cup \delta_g(s, a)$ if $s \in S^u$.
- $\Delta(s, a) = \{\delta_d(s, a)\}$ if $s \in S^d$.

The functions δ_u , δ_g , and δ_d correspond respectively to the transition functions in the first phase, from the first phase to the second phase, and in the second phase.

- The transition function $\delta_u : S^u \times \Sigma \rightarrow S^u$ represents the first phase of SLICE with $\delta_u(s, a)$ giving the next level of s in a reduced split tree with respect to the symbol a . Let $s = Q_0 \cdots Q_{n-1} \in S^u$ and $a \in \Sigma$. Define $s' = Q'_0 \cdots Q'_{2n-1}$ such that for $i < n$,
 - $Q'_{2i} = (\cup_{q \in Q_i} \delta(q, a) \cap \mathcal{F}) - \bigcup_{j < 2i} Q'_j$, and
 - $Q'_{2i+1} = (\cup_{q \in Q_i} \delta(q, a) - \mathcal{F}) - \bigcup_{j < 2i} Q'_j$.
 By removing \emptyset from s' , we can find $j_0 < \cdots < j_{r-1}$ such that $\{j_0, \dots, j_{r-1}\} = \{j < 2n \mid Q'_j \neq \emptyset\}$. The result $Q_{j_0} \cdots Q_{j_{r-1}}$ is called an *a-successor* of s , denoted by $\delta_u(s, a)$.
- The transition function $\delta_g : S^u \times \Sigma \rightarrow 2^{S^d}$ is applied when SLICE nondeterministically goes from the first phase to the second phase and starts decoration. In this transition, it labels the children of an undecorated slice nondeterministically by 0 or 1. Thus for $s \in S^u$, $a \in \Sigma$, and $s' \in S^d$, $s' \in \delta_g(s, a)$ iff $s'_{\downarrow Q} = \delta_u(s, a)$ and $s'_{\downarrow D} \subseteq \{0, 1\}$.
- The transition function $\delta_d : S^d \times \Sigma \rightarrow S^d$ represents the second phase of SLICE where it verifies the decorations by evolving decorated slices in the following way. Let $s = (Q_0, d_0) \cdots (Q_{n-1}, d_{n-1}) \in S^d$, $a \in \Sigma$, and $s' = (Q'_{j_0}, d'_{j_0}) \cdots (Q'_{j_{r-1}}, d'_{j_{r-1}}) \in S^d$ where j 's and Q'_j 's are defined as in the definition of δ_u , i.e., $s'_{\downarrow Q} = \delta_u(s_{\downarrow Q}, a)$. The decorated slice s' is an *a-successor* of s , denoted by $\delta_d(s, a)$, iff the following two conditions are satisfied:

[C1]: for all $i < n$ with $d_i = 1$, $Q'_{2i+1} \neq \emptyset$,

[C2]: d'_j 's are decorated by the following rules:

[D1]: If s is not a reset slice and $d_i = 1$, then $d'_{2i} = *$ and $d'_{2i+1} = 1$.

[D2]: If s is not a reset slice and $d_i \in \{0, *\}$, then $d'_{2i} = d_i$ and $d'_{2i+1} = d_i$.

[D3]: If s is a reset slice and $d_i = 1$, then $d'_{2i} = 0$ and $d'_{2i+1} = 1$.

[D4]: If s is a reset slice and $d_i = *$, then $d'_{2i} = 0$ and $d'_{2i+1} = 0$.

Theorem 5.7. [41] *Given an NBW A , the basic SLICE construction produces an NBW A' with $L(A') = \overline{L(A)}$.*

5.3.2. *The improved SLICE construction.* We first describe three optimization heuristics applied to the basic SLICE construction and then the resulting improved SLICE construction.

Deterministic decoration (+D). The first heuristic uses 1 to label nodes that *may* (rather than *must*) be in an infinite branch of a reduced split tree and only verifies the condition C2 in the second phase. Thus, all nodes could be decorated by 1 in the guesses. However, since the first evolution of the second phase always labels a left (accepting) child by 0 and a right (nonaccepting) child by 1, we actually decorate accepting nodes by 0 and nonaccepting nodes by 1 in the guesses. Formally, let $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ be an NBW. We define $\delta'_g : S^u \times \Sigma \rightarrow S^d$ and $\delta'_d : S^d \times \Sigma \rightarrow S^d$, which refine respectively δ_g and δ_d based on this heuristic.

- Let $s = Q_0 \cdots Q_{n-1} \in S^u$, $a \in \Sigma$, and $s' = (Q'_{j_0}, d'_{j_0}) \cdots (Q'_{j_{r-1}}, d'_{j_{r-1}}) \in S^d$ where j 's and Q'_j 's are defined as in the basic SLICE construction, i.e., $s' \downarrow_Q = \delta_u(s, a)$. Then $s' = \delta'_g(s, a)$ iff for all $i < n$, $d'_{2i} = 0$ and $d'_{2i+1} = 1$.
- The transition function δ'_d is the same as δ_d in the basic SLICE construction except that the condition C1 of δ_d is not required to be satisfied.

This heuristic results in deterministic decoration. The only nondeterminism comes from choosing when to start decorating.

Reducing transitions (+R). The second heuristic relies on the observation that if a run ends up in the empty sequence \perp , the run will stay in \perp forever and we never need to decorate the run because it can reach \perp without any decoration. Thus we do not allow transitions from decorated slices other than \perp to \perp or from any slice to doomed slices other than \perp ; recall that a slice is doomed if it has no node labeled by 1, i.e., every run through a doomed slice is expected to reach \perp .

Merging adjacent nodes (+M). The third heuristic recursively merges adjacent nodes decorated all by 0 or all by $*$. The observation is that they are all guessed to have a finite number of descendants and their successors will have the same decoration, either 0 or $*$. Let $s = (Q_0, d_0) \cdots (Q_{n-1}, d_{n-1}) \in S^d$. Based on this heuristic, we can recursively merge adjacent nodes (Q_i, d_i) and (Q_{i+1}, d_{i+1}) in s when $d_i = d_{i+1} = 0$ or $d_i = d_{i+1} = *$. We call the result a *merged slice* of s and denote it by $\text{merge}(s)$.

Given an NBW $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$, the improved SLICE with all optimization heuristics in this section constructs the complement $A' = (\Sigma, S, s_0, \Delta, \mathcal{G})$ where

- $s_0 = \{q_0\}$,
- $\Delta = S \times \Sigma \rightarrow 2^S$ is the transition function described below, and
- $\mathcal{G} = \{s \in S^d \mid s \text{ is a reset slice}\}$.

For all $s \in S$ and $a \in \Sigma$, $\Delta(s, a)$ is defined as follows:

- $\Delta(s, a) = \{\delta_u(s, a)\}$ if $s \in S^u$ and $\delta'_g(s, a)$ is doomed.
- $\Delta(s, a) = \{\delta_u(s, a), \text{merge}(\delta'_g(s, a))\}$ if $s \in S^u$ and $\delta'_g(s, a)$ is not doomed.
- $\Delta(s, a) = \{\text{merge}(\delta'_d(s, a))\}$ if $s \in S^d$ and $\delta'_d(s, a)$ is not doomed.

Before proving the correctness of the improved SLICE construction, we define another merge function $\text{merge}_{i,j}$ that will be used in the proof. For a decorated slice s , $\text{merge}_{i,j}(s)$ is a slice obtained from s by merging as many as possible and at most j consecutive mergible nodes starting from the i -th pair of mergible nodes. For example, if $s = (Q_0, 0)(Q_1, 0)(Q_2, 0)(Q_3, 0)(Q_4, 0)$, then $\text{merge}_{1,2}(s) = (Q_0 \cup Q_1, 0)(Q_2, 0)(Q_3, 0)(Q_4, 0)$ and $\text{merge}_{2,3}(s) = (Q_0, 0)(Q_1 \cup Q_2 \cup Q_3, 0)(Q_4, 0)$. By this definition, $\text{merge}(\text{merge}_{i,j}(s)) = \text{merge}(s)$ for any i, j , and decorated slice s .

Lemma 5.8. *For a decorated slice $s \in S^d$ and a symbol $a \in \Sigma$, there exist some i and j such that $\delta'_d(\text{merge}_{1,2}(s), a) = \text{merge}_{i,j}(\delta'_d(s, a))$.*

Proof. Let $s = (Q_0, d_0) \cdots (Q_{n-1}, d_{n-1})$ and $t = (Q'_0, d'_0) \cdots (Q'_{2n-1}, d'_{2n-1})$ be the a -successor of s before removing empty nodes. Assume $d_i = d_{i+1} \in \{0, *\}$, and (Q_i, d_i) and (Q_{i+1}, d_{i+1}) are the first mergible pair of nodes in s . By the decoration rules D2 and D4, $d'_{2i} = d'_{2i+1} = d'_{2i+2} = d'_{2i+3} \in \{0, *\}$. Then, $\text{merge}_{1,2}(s) = (Q_0, d_0) \cdots (Q_{i-1}, d_{i-1})(Q_i \cup Q_{i+1}, d_i)(Q_{i+2}, d_{i+2}) \cdots (Q_{n-1}, d_{n-1})$ and its a -successor before removing empty nodes is $(Q'_0, d'_0) \cdots (Q'_{2i-1}, d'_{2i-1})(Q'_{2i} \cup Q'_{2i+1} \cup Q'_{2i+2} \cup Q'_{2i+3}, d'_{2i})(Q'_{2i+4}, d'_{2i+4}) \cdots (Q'_{2n-1}, d'_{2n-1})$, denote by t' . Since $d'_{2i} = d'_{2i+1} = d'_{2i+2} = d'_{2i+3} \in \{0, *\}$, (Q'_{2i}, d'_{2i}) , (Q'_{2i+1}, d'_{2i+1}) , (Q'_{2i+2}, d'_{2i+2}) , and (Q'_{2i+3}, d'_{2i+3}) are mergible. Suppose (Q'_{2i}, d'_{2i}) and (Q'_{2i+1}, d'_{2i+1}) are the k -th mergible pair of nodes. Then, $\text{merge}_{k,4}(t) = t'$. As $\delta'_d(s, a)$ and $\delta'_d(\text{merge}_{1,2}(s), a)$ are derived respectively from t and t' by removing empty nodes, we can found some i and j ($0 \leq j \leq 4$) such that $\delta'_d(\text{merge}_{1,2}(s), a) = \text{merge}_{i,j}(\delta'_d(s, a))$. \square

Lemma 5.9. *Let $s \in S^d$ be a decorated slice and $a \in \Sigma$ a symbol. If $\delta'_d(s, a)$ is not doomed, then $\Delta(\text{merge}(s), a) = \{\text{merge}(\delta'_d(s, a))\}$.*

Proof. We prove by induction on the number of mergible pairs in s . The base case is that s has no mergible pair, which implies that $\text{merge}(s) = s$. Thus,

$$\begin{aligned} \Delta(\text{merge}(s), a) &= \{\text{merge}(\delta'_d(\text{merge}(s), a))\} \quad (\text{by the definition of } \Delta) \\ &= \{\text{merge}(\delta'_d(s, a))\} \quad (\text{by } \text{merge}(s) = s) \end{aligned}$$

Assume the hypothesis holds for any slice that has n mergible pairs and consider a slice s that has $n + 1$ mergible pairs. Since $\text{merge}_{1,2}(s)$ has n mergible pairs, we know that:

$$\begin{aligned} \Delta(\text{merge}(s), a) &= \Delta(\text{merge}(\text{merge}_{1,2}(s)), a) \quad (\text{by the definition of } \text{merge}_{i,j}) \\ &= \{\text{merge}(\delta'_d(\text{merge}_{1,2}(s), a))\} \quad (\text{by the induction hypothesis}) \\ &= \{\text{merge}(\text{merge}_{i,j}(\delta'_d(s, a)))\} \text{ for some } i \text{ and } j \\ &\quad (\text{by Lemma 5.8}) \\ &= \{\text{merge}(\delta'_d(s, a))\} \quad (\text{by the definition of } \text{merge}_{i,j}) \quad \square \end{aligned}$$

Theorem 5.10. *Given an NBW $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$, the improved SLICE construction produces an NBW $A' = (\Sigma, S, s_0, \Delta, \mathcal{G})$ with $L(A') = \overline{L(A)}$.*

Proof. We first prove that if a word $w = a_0 a_1 \cdots$ is rejected by A , w is accepted by A' . Let $T = s_0 s_1 \cdots$ be the reduced split tree of A on w where $s_{j+1} = \delta_u(s_j, a_j)$ for all j .

- Case 1: There is no run of A on w . Then, there exists some i such that $s_j = \perp$ for all $j \geq i$. By the construction of the improved SLICE, $s_0s_1 \cdots s_{i-1}\perp^\omega$ is an accepting run of w on A' . Thus, w is accepted by A' .
- Case 2: There is at least one run of A on w . Since w is rejected by A , by Lemma 5.6, there exists some cutoff i such that for all $j \geq i$, all accepting states of A in s_j belong to finite branches of T . Then, we can construct a sequence of slices $s_0s_1 \cdots s_{i-1}t_it_{i+1} \cdots$ where $t_j \in S^d$ such that
 - $t_i = \delta'_g(s_{i-1}, a_{i-1})$,
 - $t_{j+1} = \delta'_d(t_j, a_j)$ for $j \geq i$, and
 - $t_{j \downarrow Q} = s_j$ for $j \geq i$.

As there is at least one run of A on w , t_j is not doomed for all $j \geq i$. By Lemma 5.9 and the construction of the improved SLICE, we can find a run $\rho = s_0s_1 \cdots s_{i-1}u_iu_{i+1} \cdots$ of A' on w where $u_i \in \Delta(s_{i-1}, a_{i-1})$, and for $j \geq i$, $u_{j+1} \in \Delta(u_j, a_j)$ and $u_j = \text{merge}(t_j)$. Since all accepting states of A in s_j for $j \geq i$ belong to finite branches of T and these states are decorated by either 0 or * in both t_j and u_j , we can find infinitely many reset slices in ρ by the decoration rules. Thus, ρ is accepting and w is accepted by A' .

We then prove that if a word $w = a_0a_1 \cdots$ is accepted by A' , w is rejected by A . Let $\rho = s_0s_1 \cdots$ be an accepting run of A' on w .

- Case 1: $\perp \in \rho$. In this case, there is some i such that $s_j = \perp$ for all $j \geq i$. Thus, there is no run of A on w and w is rejected by A .
- Case 2: $\perp \notin \rho$. Let $T = t_0t_1 \cdots$ be the reduced split tree of A on w . Assume s_i is the first decorated slice in ρ . Then, $s_j = t_j$ for $j < i$ and s_j is not doomed for $j \geq i$. By Lemma 5.9 and the construction of the improved SLICE, there is a sequence $\rho' = s_0s_1 \cdots s_{i-1}u_iu_{i+1} \cdots$ where $u_j \in S^d$ for $j \geq i$ such that
 - $u_i = \delta'_g(s_{i-1}, a_{i-1})$,
 - $u_{j+1} = \delta'_d(u_j, a_j)$ for $j \geq i$, and
 - $\text{merge}(u_j) = s_j$ and $u_{j \downarrow Q} = t_j$ for $j \geq i$.

Since ρ is accepting, there are infinitely many reset slices in ρ as well as in ρ' . Based on the construction of the improved SLICE, all accepting states of A are decorated by either 0 or * in ρ , the decoration of 0-nodes and *-nodes remains unchanged before the next reset slice, and the decoration of *-nodes becomes 0 after a reset slice. Thus, after u_i in ρ , all these accepting states belong to finite branches. Since the *merge* function does not change any decoration, all these accepting states belong to finite branches after u_i in ρ' . As ρ' and T only differ in decorations, all the accepting states of A belong to finite branches after the i -th level in T . Hence, w is rejected by A according to Lemma 5.6. \square

6. EXPERIMENTAL RESULTS

The heuristics proposed in Section 5 were implemented in GOAL. For RAMSEY, there is a naive optimization which minimizes the classic finite automata before composing the NBWs to construct the complement⁸. This optimization, referred to as **+m**, was also implemented

⁸The optimization heuristics for the Ramsey-based constructions proposed by Breuers *et al.* [3] were published after we had performed the experiments. Although their implementation and ours are not directly comparable, the average size of the complements produced by our SAFRA-PITERMAN+ASE construction without preminimization among the 10,980 finished tasks is 139.18 states (37.55 states after removing dead states). The improved Ramsey-based construction in [3] finished 10,839 tasks with an average size of 361.09

in GOAL. We used the same 11,000 automata as in Section 4 as the test bench. The results showing the improvement made by the heuristics are summarized in Table 3 where the Ratio columns are ratios with respect to the original construction and the other columns have the same meanings as in Section 4.

TABLE 3. A comparison of each construction with its improved versions

Constructions	T	M	Eff. Samples	S_R (Ratio)	S_L (Ratio)	S_L/S_R
RAMSEY	4,564	36	6,388	594.68 (1.00)	22.78 (1.00)	0.04
RAMSEY+A	4,557	33		595.59 (1.00)	22.54 (0.99)	0.04
RAMSEY+m	3,126	2		372.08 (0.63)	11.19 (0.49)	0.03
RAMSEY+Am	3,119	2		371.06 (0.62)	11.02 (0.48)	0.03
SP	5	0	10,977	256.25 (1.00)	58.72 (1.00)	0.23
SP+A	5	0		228.40 (0.89)	54.33 (0.93)	0.24
SP+S	12	9		179.82 (0.70)	47.35 (0.81)	0.26
SP+E	11	0		194.95 (0.76)	45.47 (0.77)	0.23
SP+ASE	13	7		138.97 (0.54)	37.47 (0.64)	0.27
RANK	5,303	0	5,697	569.51 (1.00)	33.96 (1.00)	0.06
RANK+A	3,927	0		181.05 (0.32)	28.41 (0.84)	0.16
SLICE	3,131	3,213	4,514	1,088.72 (1.00)	70.67 (1.00)	0.06
SLICE+A	2,611	2,402		684.07 (0.63)	64.94 (0.92)	0.09
SLICE+D	1,119	0		276.11 (0.25)	117.32 (1.66)	0.42
SLICE+R	3,081	3,250		1,028.42 (0.94)	49.58 (0.70)	0.05
SLICE+M	2,813	3,360		978.01 (0.90)	57.85 (0.82)	0.06
SLICE+ADRM	228	0		102.57 (0.09)	36.11 (0.51)	0.35

The experimental results in Table 3 show that (1) the heuristic **+m** can reduce states down to around one half for RAMSEY, (2) SAFRA-PITERMAN+ASE has 15 more unfinished tasks but creates just around one half of reachable states and live states, (3) the improvement made by **+A** is limited for RAMSEY, SAFRA-PITERMAN, and SLICE but substantial for RANK in helping finish 1,376 more tasks and avoid the creation of around 2/3 dead states, (4) the heuristic **+D** is quite useful in reducing the reachable states down to 1/4 for SLICE but produces more live states, and (5) SLICE+ADRM finishes 6,116 more tasks and significantly reduces the reachable states to 1/10 and live states to one half.

We also compared the four constructions with all optimization heuristics in Section 5 based on 4,851 effective samples and list the results on the top of Table 4. The table shows that SAFRA-PITERMAN+ASE still outperforms the other three in the average state size and in running time. Table 4 also shows the following changes made by our heuristics in the comparison: (1) SAFRA-PITERMAN+ASE outperforms RANK+A in the number of smallest complements after pruning dead states, and (2) SLICE+ADRM creates fewer reachable states than RANK+A in average, and finishes more tasks than RANK+A and RAMSEY+Am.

Same as in Section 4, we also compared the four improved constructions with preminimization. The results are summarized on the bottom of Table 4. Similarly, the preminimization does improve RAMSEY, RANK, and SLICE more than SAFRA-PITERMAN in the complementation but does not close too much the gap between them in the comparison.

states (328.97 states after removing dead states). The maximal size of the complements is 5,238 states by SAFRA-PITERMAN+ASE and is 337,464 by the improved Ramsey-based construction.

TABLE 4. A comparison of the four improved complementation constructions based on \mathcal{A}_{15} without and with preminimization

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{15} (without preminimization)								
RAMSEY+Am	3,119	2	4,851	666.86	(0.0)	251.53	(895.75)	0.38
SP+ASE	13	7		23.88	(4,772.5)	5.77	(1,675.42)	0.24
RANK+A	3,927	0		384.35	(20.0)	11.38	(1,138.42)	0.03
SLICE+ADRM	228	0		185.02	(58.5)	9.65	(1,141.42)	0.05
\mathcal{A}_{15} (with preminimization)								
RAMSEY+PAm	2,825	6	5,618	479.99	(17.50)	225.08	(1,031.25)	0.47
SP+PASE	12	7		19.47	(3,820.67)	5.89	(1,698.75)	0.30
RANK+PA	3,383	0		232.63	(875.67)	10.68	(1,476.75)	0.05
SLICE+PADRM	216	0		135.74	(904.17)	9.12	(1,411.25)	0.07

The comparisons of the four improved constructions based on the nonuniversal automata in \mathcal{A}_{15} without and with preminimization are summarized in Table 5. These comparisons based on the nonuniversal automata are quite consistent to those based all the 11,000 automata. Additional comparisons of the four improved constructions based on \mathcal{A}_{10} and \mathcal{A}_{20} can be found in the appendix.

TABLE 5. A comparison of the four improved complementation constructions based on the nonuniversal automata in \mathcal{A}_{15} without and with preminimization

Constructions	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{15} (without preminimization)						
RAMSEY+Am	1,270	1,310.06	(0.0)	957.96	(0.50)	0.731
SP+ASE		39.90	(1191.5)	19.21	(780.17)	0.482
RANK+A		314.22	(20.0)	40.64	(243.17)	0.129
SLICE+ADRM		186.90	(58.5)	34.04	(246.17)	0.182
\mathcal{A}_{15} (with preminimization)						
RAMSEY+PAm	1,495	1,125.22	(1.0)	843.07	(0.5)	0.749
SP+PASE		38.71	(1110.5)	19.39	(668.0)	0.501
RANK+PA		260.28	(177.0)	37.39	(446.0)	0.144
SLICE+PADRM		163.29	(206.5)	31.47	(380.5)	0.193

7. CONCLUSION

We reviewed the state of Büchi complementation and examined the performance of the four complementation approaches by experiments with our implementations in GOAL and three test sets of 11,000 automata. The experimental results showed that the determinization-based approach performs better than the other three in average. In our implementations, the Ramsey-based approach is not competitive in complementation though it is competitive in universality and containment testing.

We also proposed various optimization heuristics for three of the approaches and performed an experiment with one of the test sets to show the improvement. The experimental results also showed that our heuristics substantially improve SAFRA-PITERMAN and SLICE in creating far fewer states. RANK and especially SLICE can finish more complementation tasks with our heuristics.

As the experimental results showed, the nondeterministic constructions RANK and SLICE produced many more dead states of complements. One reason is that there are many nondeterministic choices (rank functions in the rank-based approach and decorations in the slice-based approach) but only few of them are correct. While Friedgut *et al.* proposed tight ranking in [9] to reduce the number of ranking functions for the rank-based approach, we proposed the heuristic of deterministic decoration to alleviate this problem for the slice-based approach. However, the improved constructions RANK+A and SLICE+ADRM still produced many more dead states compared to SAFRA-PITERMAN+ASE in our experiments. There may be other opportunities to improve the rank-based and the slice-based constructions further.

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APPENDIX A. FULL EXPERIMENTAL RESULTS

This section includes the full experimental results that we performed to compare the four representative complementation constructions and compare their improved versions. In the following, we first recall the settings of our experiments and then describe the comparisons based on the experimental results.

Let the parameter n be 10, 15, or 20, which denotes a size of states. For each n , we randomly generated 11,000 automata with an alphabet of size 2 and states of size n as a test set. Among the 11,000 automata of state size n , denoted by \mathcal{A}_n , 100 automata are generated from each combination of 11 transition densities (from 1.0 to 3.0) and 10 acceptance densities (from 0.1 to 1.0). For every generated automaton $A = (\Sigma, Q, q_0, \delta, \mathcal{F})$ with a given state size n , symbol $a \in \Sigma$, transition density r , and acceptance density f , we made $q \in \delta(p, a)$ for $\lceil rn \rceil$ pairs of states $(p, q) \in Q^2$ uniformly chosen at random and added $\lceil fn \rceil$ states to \mathcal{F} uniformly at random. Our parameters were chosen to generate a large set of complementation problems, ranging from easy to hard.

We chose four representative complementation constructions, namely RAMSEY [28], SAFRA-PITERMAN [23], RANK [26], and SLICE [41], each of which is considered the most efficient construction in its respective approach. These constructions were implemented in the GOAL tool [36, 34]. The experiment was performed on a cluster at Rice University (<http://rcsg.rice.edu/sugar/int/>) with GOAL based on the three generated test sets \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} . For each complementation task, we allocated one 2.83-GHz CPU and 1 GB of memory. The timeout of a complementation task was 10 minutes.

TABLE 6. A comparison of the four representative constructions based on \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20}

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}								
RAMSEY	2,944	81	5,056	406.55	(70.5)	50.06	(1085.75)	0.123
SP	0	0		39.05	(3834.0)	3.50	(1312.92)	0.090
RANK	2,667	0		462.04	(1150.5)	7.63	(1520.92)	0.017
SLICE	1,191	482		913.57	(1.0)	4.92	(1136.42)	0.005
\mathcal{A}_{15}								
RAMSEY	4,564	36	2,259	513.85	(0)	30.82	(522.50)	0.060
SP	5	0		45.26	(1,843)	2.27	(556.67)	0.050
RANK	5,303	0		260.41	(415)	2.79	(649.17)	0.011
SLICE	3,131	3,213		790.92	(1)	3.03	(530.67)	0.004
\mathcal{A}_{20}								
RAMSEY	5,588	240	1,390	549.24	(0)	17.38	(335.5)	0.032
SP	53	0		57.41	(1,101)	1.88	(348.5)	0.033
RANK	6,784	0		290.17	(289)	2.17	(370.5)	0.007
SLICE	3,647	4,224		736.94	(0)	2.42	(335.5)	0.003

A.1. Comparisons of Basic Constructions. The comparisons of the four representative constructions based on the three test sets are summarized in Table 6 where T is the total number of timed-out tasks and M is the total number of tasks that run out of memory.

The column S_R is the average number of reachable states, while S_L is the average number of live states, of the complements. The column Eff. Samples denotes the total number of effective samples where both S_R and S_L are calculated. There are 5056, 2259, and 1390 effective samples respectively in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} . Among the effective samples of each test set, around 90% of the automata are universal. The Win column of a construction in S_R (resp., S_L) denotes the fractional share of effective samples where the construction wins w.r.t. S_R (resp., S_L). More detailed definition of effective samples and win shares can be found in Section 4.

The number of unfinished tasks by a construction is the sum of T and M . Compared to SAFRA-PITERMAN that has less than 0.5% of the unfinished tasks in all the test sets, each of RAMSEY, RANK, and SLICE has more than 15% in \mathcal{A}_{10} , more than 40% in \mathcal{A}_{15} , and more than 50% in \mathcal{A}_{20} .

The columns S_R and S_L show that SAFRA-PITERMAN is the best in average state size. The low S_L/S_R ratio shows that RANK and SLICE create more dead states that can be easily pruned off. Although RANK generates more dead states than SAFRA-PITERMAN, the Win column of S_L shows that RANK produces more complements that are the smallest after pruning dead states.

RANK and SLICE become much closer to SAFRA-PITERMAN in S_L because around 90% of the effective samples are universal automata, whose complements have no live states. If we only consider nonuniversal automata, the gaps between SAFRA-PITERMAN and RANK, and SAFRA-PITERMAN and SLICE in S_L become larger as shown in Table 7. This case also happens in the the Win column of S_L between SAFRA-PITERMAN and RAMSEY, and SAFRA-PITERMAN and SLICE.

TABLE 7. A comparison of the four representative constructions based on the nonuniversal automata in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20}

Constructions	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}						
RAMSEY	713	1,600.64	(0)	348.88	(0)	0.644
SP		47.23	(493.5)	18.74	(227.17)	0.397
RANK		437.58	(218.5)	48.03	(435.17)	0.110
SLICE		686.04	(1)	28.77	(50.67)	0.042
\mathcal{A}_{15}						
RAMSEY	171	1,892.37	(0)	397.10	(0)	0.210
SP		36.38	(102.5)	17.77	(34.67)	0.488
RANK		156.63	(67.5)	24.61	(127.67)	0.157
SLICE		422.88	(1)	27.75	(8.67)	0.066
\mathcal{A}_{20}						
RAMSEY	48	2,052.02	(0)	475.27	(0)	0.232
SP		41.13	(30)	26.58	(13)	0.646
RANK		165.88	(18)	34.75	(35)	0.209
SLICE		206.90	(0)	42.02	(0)	0.203

As the heuristic of preminimization applied to the input automata, denoted by +P, is considered to help the nondeterministic constructions more than the deterministic one, we also compare the four constructions with preminimization and summarize the results in

Table 8. The results based on nonuniversal automata are summarized in Table 9. We only applied the preminimization implemented in GOAL, namely the simplification by simulation in [29]. According to our experimental results, the preminimization does improve RAMSEY, RANK, and SLICE more than SAFRA-PITERMAN in the complementation but does not close too much the gap between them in the comparison, though there are other preminimization techniques that we did not apply in the experiment.

TABLE 8. A comparison of the four representative constructions based on \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} with preminimization

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}								
RAMSEY+P	2,524	85	6,142	273.18	(405.83)	46.75	(1,273.83)	0.171
SP+P	0	0		22.01	(3,452.83)	3.46	(1,502.67)	0.157
RANK+P	2,006	0		278.00	(2,244.33)	8.92	(2,009.33)	0.032
SLICE+P	1,064	300		531.49	(39.00)	4.98	(1,356.17)	0.010
\mathcal{A}_{15}								
RAMSEY+P	4,190	41	3,522	193.72	(247)	26.82	(776.25)	0.218
SP+P	4	0		11.08	(1,712)	2.31	(817.42)	0.572
RANK+P	4,316	0		56.60	(1,546)	2.37	(1,126.42)	0.160
SLICE+P	2,908	2,435		199.52	(17)	2.94	(801.92)	0.092
\mathcal{A}_{20}								
RAMSEY+P	5,334	185	2,623	133.35	(173.67)	17.81	(599.25)	0.134
SP+P	44	0		8.10	(1,235.67)	1.82	(609.25)	0.224
RANK+P	5,758	0		35.90	(1,198.17)	1.66	(808.75)	0.046
SLICE+P	3,343	3,559		100.78	(15.5)	2.18	(605.75)	0.021

TABLE 9. A comparison of the four representative constructions based on the nonuniversal automata in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} with preminimization

Constructions	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}						
RAMSEY+P	1,049	1,160.55	(0.0)	268.88	(0.58)	0.232
SP+P		36.21	(481.5)	15.41	(229.42)	0.426
RANK+P		323.76	(565.5)	47.37	(736.08)	0.146
SLICE+P		450.58	(2.0)	24.27	(82.92)	0.054
\mathcal{A}_{15}						
RAMSEY+P	418	1,000.89	(0.0)	218.55	(0.25)	0.218
SP+P		21.02	(116.0)	12.02	(41.42)	0.572
RANK+P		77.97	(300.5)	12.51	(350.42)	0.160
SLICE+P		189.90	(1.5)	17.39	(25.92)	0.092
\mathcal{A}_{20}						
RAMSEY+P	226	879.32	(0)	196.14	(0)	0.223
SP+P		15.51	(51)	10.50	(11)	0.677
RANK+P		23.81	(179)	8.70	(216)	0.365
SLICE+P		68.17	(0)	14.69	(12)	0.216

In summary, the experimental results show that (1) SAFRA-PITERMAN is the best in average state size and in the number of finished tasks, (2) RAMSEY is not competitive in complementation even though it is competitive in universality and containment testing as shown in [1, 7, 8], and (3) except in \mathcal{A}_{10} , SLICE has the most unfinished tasks (even more than RAMSEY) and, compared to SAFRA-PITERMAN and RANK, produces many more states.

A.2. Comparisons of Improved Constructions. We also compared the four constructions with all optimization heuristics in Section 5 and one for RAMSEY in Section 6 based on 7963, 4851, and 2951 effective samples respectively in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} . Recall that the following heuristics were proposed in this paper: simplifying DPWs by simulation (+S) and merging equivalent states (+E) for SAFRA-PITERMAN; maximizing the Büchi acceptance set (+A) for RANK; deterministic decoration (+D), reducing transitions (+R), and merging adjacent nodes (+M) for SLICE. The heuristic for RAMSEY is the simplification of intermediate classic finite automata on finite words (+m). The comparisons are summarized in Table 10, which shows that SAFRA-PITERMAN+ASE still outperforms the other three in the average state size and in running time. Table 10 also shows the following changes made by our heuristics in the comparison: (1) SAFRA-PITERMAN+ASE outperforms RANK+A in the number of smallest complements after pruning dead states, and (2) SLICE+ADRM creates fewer reachable states than RANK+A in average, and finishes more tasks than RANK+A and RAMSEY+Am.

TABLE 10. A comparison of the four improved constructions based on \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20}

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}								
RAMSEY+Am	924	3	7,963	461.56	(9.50)	257.21	(1,323.75)	0.56
SP+ASE	0	0		26.96	(7,701.83)	7.42	(3,062.75)	0.28
RANK+A	2,285	0		438.66	(71.83)	23.28	(1,818.75)	0.05
SLICE+ADRM	2	0		115.79	(179.83)	12.76	(1,757.75)	0.11
\mathcal{A}_{15}								
RAMSEY+Am	3,119	2	4,851	666.86	(0.0)	251.53	(895.75)	0.38
SP+ASE	13	7		23.88	(4,772.5)	5.77	(1,675.42)	0.24
RANK+A	3,927	0		384.35	(20.0)	11.38	(1,138.42)	0.03
SLICE+ADRM	228	0		185.02	(58.5)	9.65	(1,141.42)	0.05
\mathcal{A}_{20}								
RAMSEY+Am	5,009	14	2,951	618.07	(0.00)	125.76	(618.75)	0.20
SP+ASE	83	133		18.81	(2,929.67)	3.81	(894.75)	0.20
RANK+A	4,955	0		427.75	(5.17)	8.41	(717.25)	0.02
SLICE+ADRM	1,220	0		213.76	(16.17)	5.96	(720.25)	0.03

Same as in Section A.1, we also compared the four improved constructions with preminimization. The results are summarized in Table 11. Similarly, the preminimization does improve RAMSEY, RANK, and SLICE more than SAFRA-PITERMAN in the complementation but does not close too much the gap between them in the comparison.

TABLE 11. A comparison of the four improved constructions based on \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} with preminimization

Constructions	T	M	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}								
RAMSEY+PAm	813	6	8,565	355.43	(53.0)	220.28	(1,453.25)	0.62
SP+PASE	0	0		22.25	(6,032.0)	6.97	(2,794.25)	0.31
RANK+PA	1,765	0		306.92	(1,209.5)	20.54	(2,208.25)	0.07
SLICE+PADRM	2	0		89.70	(1,270.5)	11.40	(2,109.25)	0.13
\mathcal{A}_{15}								
RAMSEY+PAm	2,825	6	5,618	479.99	(17.50)	225.08	(1,031.25)	0.47
SP+PASE	12	7		19.47	(3,820.67)	5.89	(1,698.75)	0.30
RANK+PA	3,383	0		232.63	(875.67)	10.68	(1,476.75)	0.05
SLICE+PADRM	216	0		135.74	(904.17)	9.12	(1,411.25)	0.07
\mathcal{A}_{20}								
RAMSEY+PAm	4,647	15	3,741	390.30	(16.5)	159.04	(762.75)	0.41
SP+PASE	102	110		13.51	(2,335.0)	4.49	(1,059.42)	0.33
RANK+PA	4,422	0		208.18	(685.0)	7.76	(964.92)	0.04
SLICE+PADRM	1,180	0		133.69	(704.5)	6.66	(953.92)	0.05

TABLE 12. A comparison of the four improved constructions based on the nonuniversal automata in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20}

Constructions	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}						
RAMSEY+Am	2,672	944.62	(1)	764.56	(1)	0.809
SP+ASE		41.65	(2,428.17)	20.14	(1,740)	0.484
RANK+A		356.82	(67.67)	67.39	(496)	0.189
SLICE+ADRM		116.78	(175.17)	36.05	(435)	0.309
\mathcal{A}_{15}						
RAMSEY+Am	1,270	1,310.06	(0.0)	957.96	(0.50)	0.731
SP+ASE		39.90	(1191.5)	19.21	(780.17)	0.482
RANK+A		314.22	(20.0)	40.64	(243.17)	0.129
SLICE+ADRM		186.90	(58.5)	34.04	(246.17)	0.182
\mathcal{A}_{20}						
RAMSEY+Am	478	1,117.59	(0.00)	772.57	(0)	0.691
SP+ASE		35.62	(456.67)	18.35	(277)	0.515
RANK+A		350.93	(5.17)	46.74	(99)	0.133
SLICE+ADRM		219.55	(16.17)	31.59	(102)	0.144

The comparisons of the four improved constructions based on the nonuniversal automata without and with preminimization are summarized respectively in Table 12 and in Table 13. These comparisons based on the nonuniversal automata are quite consistent to those based all the automata.

TABLE 13. A comparison of the four improved constructions based on the nonuniversal automata in \mathcal{A}_{10} , \mathcal{A}_{15} , and \mathcal{A}_{20} with preminimization

Constructions	Eff. Samples	S_R	(Win)	S_L	(Win)	S_L/S_R
\mathcal{A}_{10}						
RAMSEY+PA _m	2,752	845.03	(1.5)	682.96	(0.5)	0.808
SP+PASE		39.54	(2,096.5)	19.57	(1,341.5)	0.495
RANK+PA		305.25	(294.5)	61.78	(755.5)	0.202
SLICE+PADRM		107.76	(361.5)	33.35	(656.5)	0.309
\mathcal{A}_{15}						
RAMSEY+PA _m	1,495	1,125.22	(1.0)	843.07	(0.5)	0.749
SP+PASE		38.71	(1110.5)	19.39	(668.0)	0.501
RANK+PA		260.28	(177.0)	37.39	(446.0)	0.144
SLICE+PADRM		163.29	(206.5)	31.47	(380.5)	0.193
\mathcal{A}_{20}						
RAMSEY+PA _m	692	1,084.23	(0.00)	856.38	(0.00)	0.790
SP+PASE		33.57	(478.33)	19.86	(297.67)	0.592
RANK+PA		224.91	(97.33)	37.55	(202.67)	0.167
SLICE+PADRM		165.83	(116.33)	31.61	(191.67)	0.191