

AN EXPLICIT FRAMEWORK FOR INTERACTION NETS

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ABSTRACT. Interaction nets are a graphical formalism inspired by Linear Logic proof-nets often used for studying higher order rewriting e.g. β -reduction. Traditional presentations of interaction nets are based on graph theory and rely on elementary properties of graph theory. We give here a more explicit presentation based on notions borrowed from Girard’s *Geometry of Interaction*: interaction nets are presented as partial permutations and a composition of nets, the *gluing*, is derived from the execution formula. We then define contexts and reduction as the context closure of rules. We prove strong confluence of the reduction within our framework and show how interaction nets can be viewed as the quotient of some generalized proof-nets.

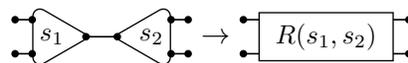
1. INTRODUCTION

Interaction nets were introduced by Yves Lafont in [Laf90] as a way to extract a model of computation from the well-behaved proof-nets of multiplicative linear logic. They have since been widely used as a formalism for the implementation of reduction strategies for the λ -calculus, providing a pictorial¹ way to do explicit substitution [Mac98][MP98][Lip03] and implement optimal reduction [AGL92].

Interaction nets are easy to present: a net is made of cells



with a fixed number of connection ports, depicted as big dots on the picture, one of which is distinguished and called the principal port of the cell, and of free ports, and of wires between those ports such that any port is linked by exactly one wire. Then we define reduction on nets by giving rules of the form



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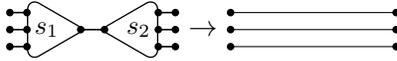
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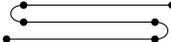
¹By putting a visual emphasis on occurrences of a variable, interaction nets allow a formal reasoning while not being as cumbersome as indices.

where the two cells in the left part are linked by their principal ports and the box in the right part is a net with the same free ports as the left part. Such a rule can be turned into a reduction of nets: as soon as a net contains the left part we replace it with the right part.

Even though this definition is sufficient to work with interaction nets, it is too limited to reason on things like paths or observational equivalence. One of the main issues comes from the fact that we do not really know what a net is. The situation is quite similar for graphs: it is the author belief that we cannot study them relying on drawings only without being deceived by our intuition. Thus, we are inclined to give a precise definition of a graph as a binary relation or as a set of edges.

The main issue to give such a definition for interaction nets is that it should cope with reduction. As an example consider a graph-like construction over ports and a rule



Can it be applied to the interaction net  ? If we are rigorous the left part of the rule is not exactly contained in this net as  is not contained in . Perhaps we could consider this last wire as composed of three smaller ones and two temporary ports like in  and the whole net after reduction would be . But then, to get back a real interaction nets we would have to concatenate all those wires and erase the temporary ports, which would give us the net . We will refer to this process of wire concatenation as *port fusion*.

There are many works giving definitions of interaction nets giving a rigorous description of reduction. Nevertheless, they all share a common point: they deal either implicitly or externally with port fusion. In the seminal article [Laf90] a definition of nets as terms with paired variables is given, it is further refined in [FM99]. In this framework an equivalence relation on variables deals with port fusion. In [Pin00] a concrete machine is given where the computation of the equivalence relation is broken into many steps. A rigorous approach sharing some tools with ours is given in [Vau07], port fusion is done there by an external port rewriting algorithm.

Therefore, we raise the following question: can we give a definition of interaction nets allowing a simple and rigorous description of reduction encompassing port fusion, and upon which we can prove results like strong confluence? This is the aim of this paper.

Our proposition is based on the following observation. When we plug the right part of a rule in a net, new wires are defined based on a back and forth process between the original net and this right part. Such kind of interaction is key to the *geometry of interaction (GoI)* [Gir89] or *game semantics* [AJM94, HO00]. The untyped nature of interaction nets makes the former a possible way to express them. To be able to do so we need to express an interaction net as some kind of partial permutation and use a composition based on the so-called *execution formula*. Such a presentation of multiplicative proof-nets has been made by Jean-Yves Girard in [Gir87]. If we try to think about the fundamental actions one needs to be able to do on interaction nets, it is quite clear that we can distinguish a *wire action* consisting in going from one port to another along a wire and the *cell action* consisting in going from one cell port to another inside the same cell. Those two actions lead to the description of a net as a pair of permutations. One might ask whether it is possible in some

case to faithfully combine this pair in only one permutation, a solution to this question is what one could call a GoI.

The issue of port fusion is not inherent to interaction nets and can be found in other related frameworks. Diagram rewriting [Laf03] uses a compact-closed underlying category allowing mathematically the *straightening* of wires. There are strong links between this approach and ours, for example the characterization of the free compact-closed category over a category given in [KL80] shares a lot of common techniques with our approach. It is not surprising to find such link as compact-closed categories are unavoidable when dealing with *geometry of interaction*. Indeed they provide – through the Int construction of Joyal, Street and Verity [JSV96] – the categorical framework to interpret GoI [AJ92, HS06]. What is different in our work, is that we stay at a syntactical level, thus, providing a rigorous syntax for writing and reducing programs.

This paper is organized as follows. In Section 2 we present the mathematical tools that we are going to use. In Section 3 we define the statics of interaction nets, in Section 4 basic tools for handling them and in Section 5 we present their dynamics. In Section 6 we draw explicit links between interaction nets and proof-nets. In Section 7 we present a categorical double-pushout approach to net rewriting. In Section 8 we briefly discuss implementation of the previous definitions.

2. PERMUTATIONS AND PARTIAL INJECTIONS

We give here the main definitions and constructions that are going to be central to our realization of interaction nets. Those definitions are standard in the partial injections model of *geometry of interaction* [Gir87, DR95] or in the definition of the traced monoidal category Plnj [HS06].

2.1. Permutations. We recall that a permutation of a set E is any bijection acting on E and we write $\mathfrak{S}(E)$ for the set of these permutations. When E is finite, which we will assume from here, for a given $\sigma \in \mathfrak{S}(E)$ we call *order* the least integer n such that $\sigma^n = id_E$, for $x \in E$ we write $\text{Orb}_\sigma(x) = \{\sigma^i(x) \mid i \in \mathbb{N}\}$ and we call it the *orbit* of x , we write $\text{Orbs}(\sigma)$ for the orbits of σ . If o is an orbit we write $|o|$ for its size.

We write (c_1, \dots, c_n) for the permutation sending c_i to c_{i+1} , for $i < n$, c_n to c_1 and being the identity elsewhere, we call it a *cycle of length n* which is also its order. Any permutation is a compound of disjoint cycles.

Let σ be a permutation of E and \mathcal{L} any set, we say that σ is labelled by \mathcal{L} if we have a function $l_\sigma : \text{Orbs}(\sigma) \rightarrow \mathcal{L}$. We say that σ *has pointed orbits* if it is labelled by E and $\forall o \in \text{Orbs}(\sigma)$ we have $l_\sigma(o) \in o$. Remark that an orbit is a sub-cycle and thus, having pointed orbits means that we have chosen a starting point in those sub-cycles.

2.2. Partial injections. A *partial injection (of integers)* f is a bijection from a subset $\text{dom}(f)$ of \mathbb{N} , called its *domain*, to a subset $\text{codom}(f)$ of \mathbb{N} , called its *codomain*. We write $f : A \rightarrow B$ to say that f is any partial injection such that $\text{dom}(f) = A$ and $\text{codom}(f) = B$. We write f^* for the inverse of this bijection viewed as a partial injection.

We call *partial permutation* a partial injection f such that $\text{dom}(f) = \text{codom}(f)$.

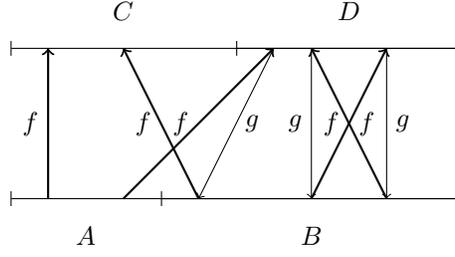


FIGURE 1. Representation of $\text{Ex}(f, g)$ with the notations of proposition 2.1

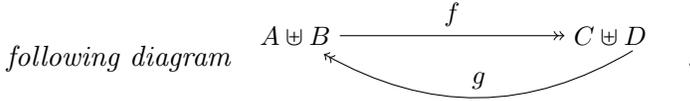
2.3. Execution. Let f be a partial injection and $E', F' \subseteq \mathbb{N}$. We write $f \upharpoonright_{F'}^{E'}$ for the partial injection of domain $\{x \in E' \cap \text{dom} f \mid f(x) \in F'\}$ and such that $f \upharpoonright_{F'}^{E'}(x) = f(x)$ where it is defined. We have

$$f \upharpoonright_{F'}^{E'} : f^{-1}(F') \cap E' \rightarrow f(E') \cap F'$$

If $E' = F'$ we write $f \upharpoonright_{E'} = f \upharpoonright_{E'}^{E'}$.

When $\text{dom}(f) \cap \text{dom}(g) = \emptyset$ and $\text{codom}(f) \cap \text{codom}(g) = \emptyset$, we say that f and g are *disjoint* and we define the sum $f + g$ and the associated refining order \prec as expected. We have $\text{dom}(f + g) = \text{dom}(f) \uplus \text{dom}(g)$ where \uplus is the disjoint union.

Proposition 2.1. *Let $f : A \uplus B \rightarrow C \uplus D$ and $g : D \rightarrow B$ a situation depicted by the*



i) For all $n \in \mathbb{N}$, the partial injection from A to C

$$\text{Ex}_n(f, g) = f \upharpoonright_C^A + (fgf) \upharpoonright_C^A + \cdots + (f(gf)^n) \upharpoonright_C^A$$

is well defined.

- ii) $(\text{Ex}_n(f, g))_{n \in \mathbb{N}}$ is an increasing sequence of partial injections with respect to \prec , whose limit, the increasing union, is noted $\text{Ex}(f, g)$.
- iii) If $\text{dom}(f)$ is finite the sequence $(\text{Ex}_n(f, g))_n$ is stationary and

$$\text{Ex}(f, g) : A \rightarrow C$$

Fig. 1 gives a graphical presentation of execution.

Proof.

i) To assert the validity of the sum all we have to have show is that $\forall i \neq j \in \mathbb{N} :$

$$\begin{aligned} (f(gf)^i)(A) \cap (f(gf)^j)(A) \cap C &= \emptyset \\ (f(gf)^i)^{-1}(C) \cap (f(gf)^j)^{-1}(C) \cap A &= \emptyset \end{aligned}$$

Suppose there is an $x \in (f(gf)^i)(A) \cap (f(gf)^j)(A) \cap C$, we set y and $z \in A$ such that $x = f(gf)^i(y) = f(gf)^j(z)$. We can further suppose that $i < j$, and we have $y = (gf)^{j-i}(z) \in B$, which is contradictory as $y \in A$ and $A \cap B = \emptyset$.

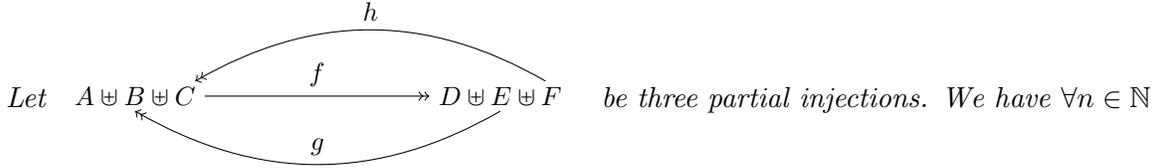
The other equality is proved in the same way.

- ii) Let $n \leq m \in \mathbb{N}$ and $x \in \text{dom}(\text{Ex}_n(f, g))$, by definition of the sum there exists a unique k such that $\text{Ex}_n(f, g)(x) = (f(gf)^k)(x)$. But then $x \in \text{dom}(\text{Ex}_m(f, g))$ and

the uniqueness of k asserts that $\text{Ex}_m(f, g)(x) = (f(gf)^k)(x)$. Thus, $\text{Ex}_m(f, g)$ is a refinement of $\text{Ex}_n(f, g)$.

- iii) Suppose there is a $x \in A - \text{dom}(\text{Ex}(f, g))$, then we should have for all k , $(f(gf)^k)(x) \in D$ or else $\text{Ex}(f, g)(x)$ would be defined. But D being finite, there exists $n \leq m$ such that $(f(gf)^n)(x) = (f(gf)^m)(x)$ and we get $x = (gf)^{m-n}(x) \in B$ which is contradictory. A simple argument on cardinal show then that $\text{codom}(\text{Ex}(f, g)) = C$. \square

Theorem 2.2 (Associativity of execution).



$$\text{Ex}_n(\text{Ex}_n(f, g), h) = \text{Ex}_n(f, g + h) = \text{Ex}_n(\text{Ex}_n(f, h), g)$$

and thus

$$\text{Ex}(\text{Ex}(f, g), h) = \text{Ex}(f, g + h) = \text{Ex}(\text{Ex}(f, h), g)$$

Proof. Let $p \in \text{dom}(\text{Ex}_n(f, g + h))$, there exists $m \leq n \in \mathbb{N}$ such that

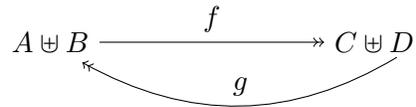
$$\begin{aligned} \text{Ex}_n(f, g + h)(p) &= f((g + h)f)^m(p) \\ &= (f(gf)^{i_1})h \dots h(f(gf)^{i_k})(p) \text{ with } i_1 + \dots + i_k + k - 1 = m \\ &= (\text{Ex}_n(f, g)h\text{Ex}_n(f, g) \dots h\text{Ex}_n(f, g))(p) \\ &= (\text{Ex}_n(f, g)(h\text{Ex}_n(f, g))^{k-1})(p) \\ &= \text{Ex}_n(\text{Ex}_n(f, g), h)(p) \end{aligned}$$

By commutativity of $+$ we get the other equality. These equalities are directly transmitted to Ex . \square

This theorem is of great significance, it is a completely localized version of Church-Rosser property. Indeed, we will see later that confluence results are a corollary of this theorem.

The following proposition states that Ex can always be extended by an independent partial injection.

Proposition 2.3. *Let f, g and h be partial injections such that*



and $\text{dom}(h) \cap \text{dom}(f) = \text{codom}(h) \cap \text{codom}(f) = \emptyset$.

We have $h + \text{Ex}(f, g) = \text{Ex}(f + h, g)$.

Proof. This result directly comes from the relation $(f + h)g(f + h) = fgf + hgf + fgh + hgh = fgf$ as $hg = gh = 0$. \square

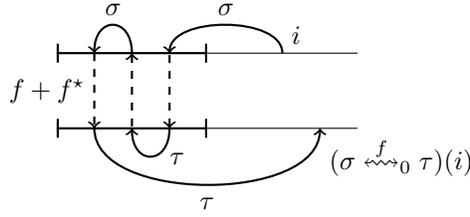


FIGURE 2. Representation of the Ex_0 -composition $\sigma \overset{f}{\rightsquigarrow}_0 \tau$

2.4. w -permutations and Ex-composition. We call w -permutation an involutive partial permutation of finite domain. That means that a w -permutation is a product of disjoint cycles of length at most 2.

Let σ and τ be disjoint w -permutations and let f be a partial injection with $\text{dom}(f) \subseteq \text{dom}(\sigma)$ and $\text{codom}(f) \subseteq \text{dom}(\tau)$. We call the Ex_0 -composition of σ and τ along f the partial permutation

$$\sigma \overset{f}{\rightsquigarrow}_0 \tau = \text{Ex}(\sigma + \tau, f + f^*)$$

Fig. 2 gives a representation of this composition.

Proposition 2.4. $\sigma \overset{f}{\rightsquigarrow}_0 \tau$ is a w -permutation.

Proof. Let x be an element of $\text{dom}(\sigma \overset{f}{\rightsquigarrow}_0 \tau)$, there exists n such that

$$(\sigma \overset{f}{\rightsquigarrow}_0 \tau)(x) = (\sigma + \tau)[(f + f^*)(\sigma + \tau)]^n(x)$$

Note that $(\sigma + \tau)^* = \sigma + \tau$ and $(f + f^*)^* = f + f^*$, and thus, we have $((\sigma + \tau)[(f + f^*)(\sigma + \tau)]^n)^* = [(\sigma + \tau)(f + f^*)]^n(\sigma + \tau) = (\sigma + \tau)[(f + f^*)(\sigma + \tau)]^n$. So $(\sigma \overset{f}{\rightsquigarrow}_0 \tau)^2(x) = x$. \square

We have $\text{dom}(\sigma \overset{f}{\rightsquigarrow}_0 \tau) = (\text{dom}(\sigma) - \text{dom}(f)) \uplus (\text{dom}(\tau) - \text{codom}(f))$. Thus, the Ex computation does not tell anything about elements in $(\text{dom}(\sigma) \cap \text{dom}(f)) \uplus (\text{dom}(\tau) \cap \text{codom}(f))$. For such an element x in $\text{dom}(\sigma) \cap \text{dom}(f)$, either there exists an i in $\text{dom}(\sigma) - \text{dom}(f)$ with x being part of the computation of $(\sigma \overset{f}{\rightsquigarrow}_0 \tau)(i)$ or there exists an n such that $(f^* \tau f \sigma)^n(x) = x$. For those x we get some kind of orbit $O_x = \{(f^* \tau f \sigma)^i(x), i \in \mathbb{N}\}$ to which we get a dual orbit for τ by setting $O'_x = \{(f \sigma f^* \tau)^i((f \sigma)(x)), i \in \mathbb{N}\}$. By applying $f \sigma$ we get a bijective correspondence between O_x and O'_x . We call *double orbit* the set $O_x \cup O'_x$ and we write $\mathbb{O}(\sigma \overset{f}{\rightsquigarrow}_0 \tau)$ for the set of all double orbits. Let $R = \{\min_{x \in O \cup O'} x, (O, O') \in \mathbb{O}(\sigma \overset{f}{\rightsquigarrow}_0 \tau)\}$. We define the full Ex-composition, written $\sigma \overset{f}{\rightsquigarrow} \tau$, of domain $\text{dom}(\sigma \overset{f}{\rightsquigarrow}_0 \tau) \uplus R$ and such that $\forall r \in R, (\sigma \overset{f}{\rightsquigarrow} \tau)(r) = r$.

We can now give a consequence of theorem 2.2, stating some kind of associativity for the Ex-composition.

Proposition 2.5. Let σ, τ, ρ be pairwise disjoint w -permutations with

$$\begin{array}{ccc}
& \text{dom}(\sigma) = A \uplus B \uplus C & \\
& \swarrow f & \downarrow g \\
\text{dom}(\tau) = D \uplus E \uplus F & & \\
& \searrow h & \\
& \text{dom}(\rho) = G \uplus H \uplus I &
\end{array}$$

We have $\sigma \overset{f+g}{\rightsquigarrow} (\tau \overset{h}{\rightsquigarrow} \rho) = (\sigma \overset{f}{\rightsquigarrow} \tau) \overset{g+h}{\rightsquigarrow} \rho = (\sigma \overset{g}{\rightsquigarrow} \rho) \overset{f+h}{\rightsquigarrow} \tau$. When $h = 0$ we get $\sigma \overset{f+g}{\rightsquigarrow} (\tau + \rho) = (\sigma \overset{f}{\rightsquigarrow} \tau) \overset{g}{\rightsquigarrow} \rho = (\sigma \overset{g}{\rightsquigarrow} \rho) \overset{f}{\rightsquigarrow} \tau$.

Proof. This proposition states in fact two separated results, one about Ex_0 -composition and the other about double orbits.

We have $\sigma \overset{f+g}{\rightsquigarrow}_0 (\tau \overset{h}{\rightsquigarrow}_0 \rho) = \text{Ex}(\sigma + \text{Ex}(\tau + \rho, h + h^*), f + f^* + g + g^*) = \text{Ex}(\text{Ex}(\sigma + \tau + \rho, h + h^*), f + f^* + g + g^*)$ by proposition 2.3 and $= \text{Ex}(\sigma + \tau + \rho, f + f^* + g + g^* + h + h^*)$ by theorem 2.2. We get the other equalities in the same way.

For the equalities involving double orbits, we set

$$\mathbb{O}_3 = \mathbb{O}(\sigma \overset{f+g}{\rightsquigarrow}_0 (\tau \overset{h}{\rightsquigarrow}_0 \rho)) - \mathbb{O}(\tau \overset{h}{\rightsquigarrow}_0 \rho) - \mathbb{O}(\sigma \overset{f}{\rightsquigarrow}_0 \tau) - \mathbb{O}(\sigma \overset{g}{\rightsquigarrow}_0 \rho)$$

To conclude, it suffices to show that double orbits in \mathbb{O}_3 do not depend on the order of composition. Indeed, such an orbit is generated by an element x such that

$$\begin{aligned}
x &= ((f^* + g^*)(\tau \overset{h}{\rightsquigarrow}_0 \rho)(f + g)\sigma)^n(x) \\
&= \left(\prod_{i=1}^n (f^* + g^*)(\tau + \rho)((h + h^*)(\tau + \rho))^{k_i} (f + g)\sigma \right)(x) \\
&= \left(\prod_{i=1}^n F_i \right)(x)
\end{aligned}$$

where the F_i s are of the following four shapes:

$$g^* \rho h \tau (h^* \rho h \tau)^{k_i} f \sigma, f^* \tau (h^* \rho h \tau)^{k_i} f \sigma, g^* \rho (h \tau h^* \rho)^{k_i} g \sigma, \text{ and } f^* \tau h^* \rho (h \tau h^* \rho)^{k_i} g \sigma$$

When $k_i = 0$ they can have the shape $g^* \rho g \sigma$ and $f^* \tau f \sigma$. This is enough to be able to group the expression by factoring $\sigma \overset{g}{\rightsquigarrow}_0 \rho$ or $\sigma \overset{f}{\rightsquigarrow}_0 \tau$, and thus, to retrieve the expressions of double orbits in any order of composition. \square

Remark 2.6. The definition of full Ex -composition by means of double orbits could seem like a lot of trouble. We will see in the next section that the recovered fixpoints will allow us to interpret loops in interaction nets. One might argue that loops do not have to be recovered at any cost, and if our framework cannot see them it is for the best. In fact there are real justifications for loops, the main point being that seeing loops is what makes our definition algebraically free. This freeness is really important as it can be seen as a separation of syntax from semantics. A detailed discussion of the need of loops in the context of compact-closed categories can be found in [Abr05].

3. THE STATICS OF INTERACTION NETS

We fix a countable set \mathcal{S} , whose elements are called *symbols*, and a function $\alpha : \mathcal{S} \rightarrow \mathbb{N}$, the *arity*. We will define nets atop \mathbb{N} and in this context an integer will be called a *port*.

Definition 3.1. An interaction net is an ordered pair $R = (\sigma_w, \sigma_c)$ where:

- σ_w is a w -permutation. We write $P_l(R)$ for the fixed points of σ_w and $P(R)$ for the others, called *ports of the net* R .
- σ_c is a partial permutation of $P(R)$ with pointed orbits and labelled by \mathcal{S} in such a way that $\forall o \in \text{Orbs}(\sigma_c), |o| = \alpha(l(o)) + 1$ where l is the labelling function.

The elements of $P_l(R)$ are called *loops* and the other orbits of σ_w , which are necessarily of length 2, are called *wires*. The domain of σ_w is called the *carrier* of the net. We write $P_c(R) = \text{dom}(\sigma_c)$, whose elements are called *cell ports*, and $P_f(R) = P(R) - P_c(R)$, whose elements are called *free ports*.

An orbit of σ_c is called a *cell*. We write pal for the pointing function of σ_w . Let c be a cell, $\text{pal}(c)$ is its *principal port* and for $i < |c|$ the element $(\sigma_c^i \circ \text{pal})(c)$ is its *i th auxiliary port*.

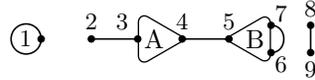
Note that a port of a net is present in exactly one wire and at most one cell.

3.1. Representation. Nets admit a very natural representation. We shall draw a cell of symbol A as a triangle $\triangleleft A \triangleright$ where the principal port is the dot on the apex and auxiliary ports are lined up on the opposing edge. We draw free ports as points. To finish the drawing we add a line between any two ports connected by a wire, and draw circles for loops.

As an example consider the net $R = (\sigma_w, \sigma_c)$ with

$$\sigma_w = (1)(2\ 3)(4\ 5)(6\ 7)(8\ 9) \text{ and } \sigma_c = (\overset{\bullet}{4}\ \overset{\bullet}{3})_A (\overset{\bullet}{5}\ \overset{\bullet}{6}\ \overset{\bullet}{7})_B$$

where permutations are given by cycle decomposition and $(\overset{\bullet}{c}_1\ \overset{\bullet}{c}_2\ \dots\ \overset{\bullet}{c}_n)_S$ is a cell of point c_1 and symbol S . This net will have the representation



3.2. Morphisms of nets and renaming.

Definition 3.2. Let $R = (\sigma_w, \sigma_c)$ and $R' = (\sigma'_w, \sigma'_c)$ be two interaction nets. A function $f : P(R) \mapsto P(R')$ is a *morphism from R to R'* if and only if

$$f \circ \sigma_w = \sigma'_w \circ f, \quad f(P_c(R)) \subseteq P_c(R'),$$

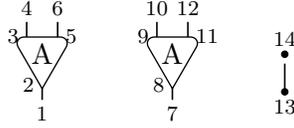
$$\forall p \in P_c(R), (f \circ \sigma_c)(p) = (\sigma'_c \circ f)(p),$$

and $\forall o \in \text{Orbs}(\sigma_c)$ we have $(f \circ \text{pal})(o) = (\text{pal} \circ f)(o)$ and $l(o) = (l \circ f)(o)$. When f is the identity on $P_f(R)$ it is said to be an *internal morphism*.

Example 3.3. Consider the net:

$$R = ((1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)(13\ 14), (\overset{\bullet}{2}\ \overset{\bullet}{3}\ \overset{\bullet}{5})_A (\overset{\bullet}{8}\ \overset{\bullet}{9}\ \overset{\bullet}{11})_A)$$

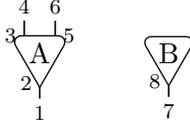
of representation:



and the net:

$$S = ((1\ 2)(3\ 4)(5\ 6)(7\ 8), (\overset{\bullet}{2}\ 3\ 5)_A(\overset{\bullet}{8})_B)$$

of representation:



Let f be the application defined by $f(1) = f(7) = f(13) = 1$, $f(2) = f(8) = f(14) = 2$, $f(3) = f(9) = 3$, $f(5) = f(11) = 5$, $f(4) = f(10) = 4$ and $f(6) = f(12) = 6$. This is a morphism from R to S .

Remark 3.4. The equality $f \circ \sigma_w = \sigma'_w \circ f$ seems quite strong, but could in fact be deduced from a simple inclusion of functional graphs, $f \circ \sigma_w \subseteq \sigma'_w \circ f$. Indeed, let (p, p') be in the graph of $\sigma'_w \circ f$, we can compute $(f \circ \sigma_w)(p)$ which by the inclusion cannot be anything else than p' .

Let us detail a bit more this definition. We note that for any two partial permutations σ and τ , the equation $f \circ \sigma = \sigma' \circ f$ induces that a $o \in \text{Orbs}(\sigma)$ is mapped to an element $f(o) \in \text{Orbs}(\sigma')$ such that $|f(o)|$ is a divisor of $|o|$.

In this case a loop is sent to a loop, a wire to a loop or a wire, and a cell to another cell. The last two equations say that the principal port of a cell is mapped to a principal port, and symbols are preserved. So a cell is mapped to a cell of same arity, and each port is mapped to the same type of port. Moreover only a wire linking free ports can be mapped to a loop or any kind of wire. As soon as the wire is linking one cell port the third condition on the morphism must send it to a wire of the same type.

With those facts, it is natural to call *renaming* (resp. *internal renaming*) an isomorphism (resp. internal isomorphism). An isomorphism class captures interaction nets as they are drawn on paper. On the other hand, an internal isomorphism class corresponds to interaction nets drawn where we have also given distinct names to free ports, hence the name internal. This is an important notion because the drawing  is the same as

Whereas the drawing  is different from .

In fact, as soon as we would like to consider nets as some kind of terms, we will have to consider them up to internal isomorphism. Free ports correspond to free variables, whereas cell ports correspond to bound variables. For example the λ -term $\lambda x.(x)y$ is of course the same as $\lambda z.(z)y$ but it is distinct from $\lambda x.(x)z$.

Remark 3.5. Given the fact that nets have finite carriers we can always consider that two nets have disjoint carriers up to renaming.

4. TOOLS OF THE TRADE

We give here the main tools that are going to be crucial to our definition of reduction.

4.1. Gluing and cutting.

Definition 4.1. Let $R = (\sigma_w, \sigma_c)$ and $R' = (\sigma'_w, \sigma'_c)$ be two nets with disjoint carriers² and let f be a partial injection of domain included in $P_f(R)$ and codomain included in $P_f(R')$.

We call *gluing of R and R' along f* the net $R \overset{f}{\rightsquigarrow} R' = (\sigma_w \overset{f}{\rightsquigarrow} \sigma'_w, \sigma_c + \sigma'_c)$.

From this definitions we get the following obvious facts:

$$\begin{aligned} P(R \overset{f}{\rightsquigarrow} R') &= (P(R) - \text{dom}(f)) \uplus (P(R') - \text{codom}(f)) \\ P_c(R \overset{f}{\rightsquigarrow} R') &= P_c(R) \uplus P_c(R') \\ P_f(R \overset{f}{\rightsquigarrow} R') &= (P_f(R) - \text{dom}(f)) \uplus (P_f(R') - \text{codom}(f)) \\ R \overset{f}{\rightsquigarrow} R' &= R' \overset{f^*}{\rightsquigarrow} R \end{aligned}$$

For the special case of gluing where $f = 0$ we have $R \overset{0}{\rightsquigarrow} R' = (\sigma_w + \sigma'_w, \sigma_c + \sigma'_c)$, we write this special kind of gluing $R + R'$, it is the so-called parallel composition of the two nets.

Fig. 3 gives a representation of gluing.

Proposition 4.2. *If $R = R \overset{f}{\rightsquigarrow} R'$ then $f = 0$ and $R' = \mathbf{0} = (0, 0)$. If $\mathbf{0} = R \overset{f}{\rightsquigarrow} R'$ then $f = 0$ and $R = R' = \mathbf{0}$.*

Proof. We will only prove the first assertion, the second being similar. It is a direct consequence of the previous facts, R' must have no cells, no free ports and no loops. The only net having this property is the empty net $\mathbf{0}$. \square

We can get some kind of associativity property for gluing.

Proposition 4.3. *Let $R = (\sigma_w, \sigma_c)$, $S = (\tau_w, \tau_c)$ and $T = (\rho_w, \rho_c)$ be nets of disjoint carriers and let f, g and h be partial injections satisfying the diagram of proposition 2.5 with respect to σ_w, τ_w and ρ_w .*

$$\text{We have } R \overset{f+g}{\rightsquigarrow} (S \overset{h}{\rightsquigarrow} T) = (R \overset{f}{\rightsquigarrow} S) \overset{g+h}{\rightsquigarrow} T = (R \overset{g}{\rightsquigarrow} T) \overset{f+h}{\rightsquigarrow} S.$$

Proof. The wire part of the equality is a restriction of proposition 2.5 and the cell part is the associativity of $+$. \square

The following corollary will often be sufficient.

Corollary 4.4. *If we have a decomposition $R_0 = R \overset{f}{\rightsquigarrow} (S \overset{g}{\rightsquigarrow} T)$ then there exists f_S, f_T such that $R_0 = (R \overset{f_S}{\rightsquigarrow} S) \overset{g+f_T}{\rightsquigarrow} T$.*

We can use the gluing to define dually the notion of cutting a subnet of an interaction net.

Definition 4.5. Let R be a net, we call *cutting of R* a triple (R_1, f, R_2) such that $R = R_1 \overset{f}{\rightsquigarrow} R_2$. Any net R' appearing in a cutting of R is called a *subnet of R* , noted $R' \subseteq R$.

²Which is not a loss of generality thanks to remark 3.5.



FIGURE 3. Representation of the gluing of two interaction nets

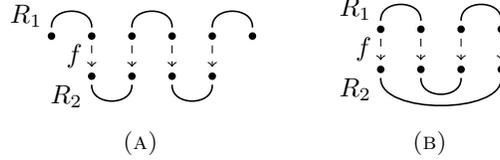


FIGURE 4. Representation of two special cuttings: (a) a cutting of a single wire and (b) a cutting of a loop

Fig. 4 gives an example of cutting. The fact that we can cut many times a wire or that we can divide a loop in many wires hints at the complexity behind these definitions.

Proposition 4.6. *The relation \subseteq is an ordering of nets.*

Proof. The relation \subseteq is **reflexive**: $R = R \overset{0}{\rightsquigarrow} \mathbf{0}$ and thus, $R \subseteq R$.

It is **antisymmetric**: let R_1 and R_2 be nets such that $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$. We have $R_1 = R_2 \overset{f}{\rightsquigarrow} R'_2$ and $R_2 = R_1 \overset{g}{\rightsquigarrow} R'_1$.

So $R_2 = (R_2 \overset{f}{\rightsquigarrow} R'_2) \overset{g}{\rightsquigarrow} R'_1$. By applying the corollary 4.4 we get $R_2 = R_2 \overset{f_1}{\rightsquigarrow} (R'_2 \overset{g+f_2}{\rightsquigarrow} R'_1)$. and by applying the proposition 4.2 twice we get $R'_2 = R'_1 = \mathbf{0}$. So $R_1 = R_2$.

And it is **transitive**: let $R \subseteq S \subseteq T$, then $S = R \overset{f}{\rightsquigarrow} R'$ and $T = S \overset{g}{\rightsquigarrow} S'$, so $T = (R \overset{f}{\rightsquigarrow} R') \overset{g}{\rightsquigarrow} S'$. By applying the corollary 4.4 we have $T = R \overset{f_1}{\rightsquigarrow} (R' \overset{g+f_2}{\rightsquigarrow} S')$, that is to say $R \subseteq T$. \square

4.2. Extending morphisms by gluing.

Proposition 4.7. *If $\alpha : R \rightarrow S$ is a morphism of nets, and $T = S \overset{f}{\rightsquigarrow} S'$, then there exists a morphism $\hat{\alpha} : R \rightarrow T$ extending α .*

Proof. It is obvious how to define the image of a cell in R into T , because α maps it to a cell in S and cells are preserved by gluing. So, the only thing to prove is that we can properly define the image of a wire in R . We consider a wire $(p \ p')$ in R which is mapped to another wire $(\alpha(p) \ \alpha(p'))$ in S (the case where it is a loop is trivial as loops are also preserved by gluing). In T this wire has either become a loop, and thus we send, by $\hat{\alpha}$, p and p' to the loop port, or it has become a wire trough the Ex-composition:

$$q \rightarrow \dots \rightarrow \alpha(p) \xrightarrow{\sigma_w} \alpha(p') \rightarrow \dots \rightarrow q'$$

where $S = (\sigma_w, \sigma_c)$, in which case we define $\hat{\alpha}(p) = q$ and $\hat{\alpha}(p') = q'$.

By construction, $\hat{\alpha}$ is a morphism. \square

With $id_R : R \rightarrow R$ being the identity function on ports, and $T = R \overset{f}{\hookrightarrow} S$, we simply write $R \subseteq T$ for the morphism $\widehat{id_R}$ which we refer to as the inclusion map of R into T . Note that in our setting these maps are not just co-extensions of identity, this is due to our notion of subnets.

Definition 4.8. We say that $\alpha : R \rightarrow S$ is *almost injective* when there exists a decomposition $S = \beta(R) \overset{f}{\hookrightarrow} R'$ with β a renaming and $\widehat{\beta} = \alpha$ where $\widehat{\beta}$ is given by the previous proposition. We also use the notation $\widetilde{\alpha} = \beta$.

Inclusion maps are the archetypal almost injective morphisms. Indeed, every almost injective morphism splits as a renaming followed by an inclusion map.

4.3. Interfaces and contexts. To define reduction by using the subnet relation, it would be easier if we could refer implicitly to the identification function in a gluing. As an intuition, consider terms contexts with multiple holes, to substitute completely such contexts we could give a function from holes to terms and fill them accordingly. But a more natural definition would be to give a distinct number to each hole and to fill based on a list of terms. The substitution would give the first term to the first hole, and so on. The following definition is a direct transposition of this idea in the framework of interaction nets.

Definition 4.9. We call *interface of a net* R a subset $I = \{p_1, \dots, p_n\}$ of $P_f(R)$ together with a linear ordering, the length of the order chain $p_1 < \dots < p_n$ is called the *size*. We say that R contains the interface I , noted $I \subset R$. An interface is *canonical* if it contains all the free ports of a net.

Let I and I' be disjoint interfaces of the same net, we write II' the union of these subsets ordered by the concatenation of the two order chains. Precisely $x \leq_{II'} y \iff x \leq_I y$ or $x \leq_{I'} y$ or $x \in I \wedge y \in I'$.

Let I and I' be two interfaces of same size, there exists one and only order-preserving bijection from I to I' that we write $\rho(I, I')$ and call the *chord between I and I'* .

We call *context* a pair (R, I) where I is an interface contained in the net R , it is written R^I .

Let R^I and $R'^{I'}$ be two contexts with interfaces of same size, we write

$$R^I \overset{f}{\hookrightarrow} R'^{I'} = R \overset{\rho(I, I')}{\hookrightarrow} R'$$

In the following when we write $R^I \overset{f}{\hookrightarrow} R'^{I'}$ we implicitly assume that I and I' are of same size.

We now can state commutativity of gluing directly, the proof being trivial.

Proposition 4.10. $R^I \overset{f}{\hookrightarrow} R'^{I'} = R'^{I'} \overset{f}{\hookrightarrow} R^I$ □

The following trivial fact asserts that any gluing can be seen as a context gluing.

Proposition 4.11. *Let $R \overset{f}{\hookrightarrow} R'$ be a gluing, there exist interfaces $I \subset R$ and $I' \subset R'$ such that $R \overset{f}{\hookrightarrow} R' = R^I \overset{f}{\hookrightarrow} R'^{I'}$.*

Proof. It suffices to take $I = \text{dom}(f)$ with any linear ordering, and to define the only ordering of $I' = \text{codom}(f)$ such that f is strictly increasing. □

Corollary 4.12. $R_1 \subseteq R \iff \exists I_1, R_2, I_2$ such that $R = R_1^{I_1} \rightsquigarrow R_2^{I_2}$.

We can now restate corollary 4.4 with interfaces:

Corollary 4.13. For all nets R, S, T and interfaces I, J, K, L , there exists interfaces I', J', K', L' such that

$$R^I \rightsquigarrow (S^J \rightsquigarrow T^K)^L = (R^{I'} \rightsquigarrow S^{J'})^{L'} \rightsquigarrow T^{K'}.$$

5. DYNAMICS

Given the previous definitions we will now present the dynamics of net. It should be remarked that our definition of dynamics is quite similar to the usual one: it amounts to finding a subnet called a redex and substituting it with another subnet. The main difference lies in our rigorous definition of subnets.

Definition 5.1. Let s_1 and s_2 be symbols. We call *interaction rule* for (s_1, s_2) a couple $(R_r^{I_r}, R_p^{I_p})$ where

$$R_r = \left(\begin{array}{c} (b \ c)(a_1 \ b_1) \dots (a_n \ b_n)(c_1 \ d_1) \dots (c_m \ d_m), \\ (\overset{\bullet}{b} \ b_1 \ \dots \ b_n)_{s_1} (\overset{\bullet}{c} \ c_1 \ \dots \ c_m)_{s_2} \end{array} \right)$$

and I_r and I_p are both canonical – comprised of all free ports – and of same size.

Let $\mathcal{R} = (R_r^{I_r}, R_p^{I_p})$ be a rule. we call *reduction by \mathcal{R}* the binary relation $\xrightarrow{\mathcal{R}}$ on nets such that for all renaming α and β , and for all net S with $S = R^I \rightsquigarrow \alpha(R_r)^{\alpha(I_r)}$ we set $S \xrightarrow{\mathcal{R}} S'$ where $S' = R^I \rightsquigarrow \beta(R_p)^{\beta(I_p)}$.

The net R_r has the representation . Remark that the reduction is defined as soon as a net contains a renaming of the redex R_r . This reduction appears to be non-deterministic but it is only the expansion of a deterministic reduction to cope with all possible renamings.

We recall now the formal definition of the main property of interaction nets and we wish that our definition ensures it.

Definition 5.2. Let $\xrightarrow{\mathcal{R}}$ be a binary relation on a set E , we say that it is *strongly confluent* if and only if for all $x, y, z \in E$ such that $y \neq z$ and $y \xleftarrow{\mathcal{R}} x \xrightarrow{\mathcal{R}} z$ and there exists $t \in E$ with $y \xrightarrow{\mathcal{R}} t \xleftarrow{\mathcal{R}} z$.

Proposition 5.3. Let R be a net and $\mathcal{R}_1, \mathcal{R}_2$ be two interaction rules applicable on R on distinct redexes such that $R_1 \xleftarrow{\mathcal{R}_1} R \xrightarrow{\mathcal{R}_2} R_2$ and all the ports both in R_1 and R_2 are also in R . There exists a net R' such that $R_1 \xrightarrow{\mathcal{R}_2} R' \xleftarrow{\mathcal{R}_1} R_2$.

Proof. For $i = 1, 2$, set $\mathcal{R}_i = (R_{r,i}^{I_{r,i}}, R_{p,i}^{I_{p,i}})$. The shape of redexes allow us to assert that if they are distinct then they are disjoint. As R contains both a redex $\alpha_1(R_{r,1})$ and a redex $\alpha_2(R_{r,2})$, then we can deduce that $\alpha_1(R_{r,1}) + \alpha_2(R_{r,2}) \subseteq R$. More precisely we have

$$R = (\alpha_1(R_{r,1}) + \alpha_2(R_{r,2}))^{\alpha_1(I_{r,1})\alpha_2(I_{r,2})} \rightsquigarrow R_0^I$$

We get

$$R_1 = (\beta_1(R_{p,1}) + \alpha_2(R_{r,2}))^{\beta_1(I_{p,1})\alpha_2(I_{r,2})} \rightsquigarrow R_0^I$$

for a renaming β_1 , and the same kind of expression for R_2 . It is straightforward to check that the net

$$R' = (\beta_1(R_{p,1}) + \beta_2(R_{p,2}))^{\beta_1(I_{p,1})\beta_2(I_{p,2})} \rightsquigarrow R_0^I$$

satisfies the conclusion by applying proposition 4.3. The very existence of this net relies on the disjointness of the $\beta_i(R_{p,i})$ which is ensured by the hypothesis on ports contained in both R_1 and R_2 . \square

Corollary 5.4. *Let \mathcal{L} be a set of rules such that for any pair of symbols there is at most one rule over them. The reduction $\xrightarrow{\mathcal{L}} = \bigcup_{\mathcal{R} \in \mathcal{L}} \xrightarrow{\mathcal{R}}$ is strongly confluent up to a renaming.*

By *up to a renaming* we mean that we might have to rename one of the nets in a critical pair before joining them. This is due to the disjointness condition in proposition 5.3. Remark that we can always substitute one of the branch of the critical pair by another instance of the same rule on the same redex in such a way that this condition is ensured.

5.1. Example. We will now give a thorough example of a net reduction using the Multiplicative Linear Logic symbols and rules. We display representations next to the net definitions.³

Let us consider the rule $\mathcal{R} = (R_r^{I_r}, R_p^{I_p})$ where

$$R_r = \left(\begin{array}{l} (0\ 3)(6\ 1)(7\ 2)(8\ 4)(9\ 5), \\ \begin{array}{c} \bullet \\ (0\ 1\ 2) \wp (3\ 4\ 5) \otimes \end{array} \end{array} \right) \quad \begin{array}{c} \begin{array}{c} 9\ 5 \\ \bullet \bullet \\ \bullet \bullet \end{array} \otimes \begin{array}{c} 3 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 0 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \begin{array}{c} 1\ 6 \\ \bullet \bullet \\ \bullet \bullet \end{array} \\ \wp \end{array} \end{array} \quad I_r = 6 < 7 < 8 < 9$$

$$R_p = \left(\begin{array}{l} (10\ 12)(11\ 13), \\ 0 \end{array} \right) \quad \begin{array}{c} \begin{array}{c} 13 \\ \bullet \end{array} \text{---} \begin{array}{c} 10 \\ \bullet \end{array} \\ \begin{array}{c} \bullet \\ 12 \end{array} \text{---} \begin{array}{c} \bullet \\ 11 \end{array} \end{array} \quad I_p = 10 < 11 < 12 < 13$$

Now let R be the net

$$\left(\begin{array}{l} (0\ 3)(1\ 2)(8\ 4)(7\ 5)(6\ 9), \\ \begin{array}{c} \bullet \\ (0\ 1\ 2) \wp (3\ 4\ 5) \otimes (6\ 7\ 8) \wp \end{array} \end{array} \right) \quad \begin{array}{c} \begin{array}{c} 9\ 6 \\ \bullet \bullet \\ \bullet \bullet \end{array} \quad \begin{array}{c} 7 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 5 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 3 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 0 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} \begin{array}{c} 1 \\ \bullet \bullet \\ \bullet \bullet \end{array} \\ \wp \end{array} \end{array}$$

It can be expressed as

$$\left(\begin{array}{l} (6\ 9)(7\ 10)(8\ 11)(12\ 13), \\ \begin{array}{c} \bullet \\ (6\ 7\ 8) \otimes \end{array} \end{array} \right)^{10 < 11 < 12 < 13} \rightsquigarrow \left(\begin{array}{l} (0\ 3)(16\ 1)(17\ 2)(15\ 4)(14\ 5), \\ \begin{array}{c} \bullet \\ (0\ 1\ 2) \wp (3\ 4\ 5) \otimes \end{array} \end{array} \right)^{14 < 15 < 16 < 17}$$

$$\begin{array}{c} \begin{array}{c} 9\ 6 \\ \bullet \bullet \\ \bullet \bullet \end{array} \quad \begin{array}{c} 7 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 10 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 11 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 12 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 13 \\ \bullet \\ \bullet \end{array} \end{array} \rightsquigarrow \begin{array}{c} \begin{array}{c} 14 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 5 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 3 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 1 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 16 \\ \bullet \\ \bullet \end{array} \quad \begin{array}{c} 17 \\ \bullet \\ \bullet \end{array} \end{array}$$

the latter context being a renaming of $R_r^{I_r}$, which we substitute with the following renaming of $R_p^{I_p}$:

$$\left(\begin{array}{l} (14\ 16)(15\ 17), \\ 0 \end{array} \right)^{14 < 15 < 16 < 17} \quad \begin{array}{c} \begin{array}{c} 17 \\ \bullet \end{array} \text{---} \begin{array}{c} 14 \\ \bullet \end{array} \\ \begin{array}{c} \bullet \\ 16 \end{array} \text{---} \begin{array}{c} \bullet \\ 15 \end{array} \end{array}$$

³Nevertheless, these representations are not required to do the reduction, they are merely here to help the reader.

Thus, we get the net

$$\left(\begin{array}{c} (6\ 9)(7\ 10)(8\ 11)(12\ 13), \\ \bullet \\ (6\ 7\ 8)_{\otimes} \end{array} \right)^{10 < 11 < 12 < 13} \rightsquigarrow \left(\begin{array}{c} (14\ 16)(15\ 17), \\ 0 \end{array} \right)^{14 < 15 < 16 < 17}$$

which simplifies into

$$\left(\begin{array}{c} (6\ 9)(7\ 8), \\ \bullet \\ (6\ 7\ 8)_{\otimes} \end{array} \right) \quad \begin{array}{c} 7 \\ \swarrow \quad \searrow \\ 9 \quad 6 \\ \searrow \quad \swarrow \\ \quad 8 \end{array}$$

6. INTERACTION NETS ARE THE EX-COLLAPSE OF AXIOM/CUT NETS

We introduce now a notion of nets lying between proof-nets of multiplicative linear logic and interaction nets. When we plug directly two interaction nets a complex process of wire simplification occurs. When we plug two proof-nets we only add special wires called *cuts* and we have an external notion of reduction performing such simplification. In this section we define nets with two kinds of wires: *axioms* and *cuts*. Those nets allow us to give a precise account of the folklore assertion that interaction nets are a quotient of multiplicative proof-nets.

6.1. Definition and juxtaposition.

Definition 6.1. An *Axiom/Cut net*, *AC net* for short, is a tuple $R = (\sigma_A, \sigma_C, \sigma_c)$ where:

- σ_A and σ_C are w -permutations of finite domain such that $\text{dom}(\sigma_C) \subseteq \text{dom}(\sigma_A)$, σ_C has no fixed points and if $(a\ b)$ is an orbit of σ_C then there exists $c \neq a$ and $d \neq b$ such that $(c\ a)$ and $(b\ d)$ are orbits of σ_A .

We write $P_l(R)$ for the fixed points of σ_A and $P(R) = \text{dom}(\sigma_A) - \text{dom}(\sigma_C) - P_l(R)$.

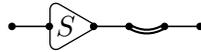
- σ_c is an element of $\mathfrak{S}(P_c(R))$, where $P_c(R) \subseteq P(R)$, has pointed orbits and is labelled by S in such a way that $\forall o \in \text{Orbs}(\sigma_c), |o| = \alpha(l(o))$ where l is the labelling function.

The orbits of σ_C , called *cuts*, are some kind of undirected unary cells linking orbits of σ_A , called *axioms*.

We directly adapt the representation of interaction nets to AC nets by displaying σ_c as double edges. For example the AC net $R = (\sigma_A, \sigma_C, \sigma_c)$ with

$$\sigma_A = (1\ 2)(3\ 4)(5\ 6), \sigma_C = (2\ 3), \sigma_c = (\overset{\bullet}{4}\ 5)_S$$

will be represented by



We can adapt most of the previous definitions for those nets, most importantly free ports, interfaces and contexts. The nice thing about AC nets is that they yield a very simple composition.

Definition 6.2. Let $R^I = (\sigma_A, \sigma_C, \sigma_c)$ and $R^{I'} = (\tau_A, \tau_C, \tau_c)$ be two contexts on AC nets with disjoint carriers, with $I = i_1 > \dots > i_n$ and $I' = i'_1 > \dots > i'_n$.

We call *juxtaposition of R^I and $R^{I'}$* the AC net

$$R^I \leftrightarrow R^{I'} = (\sigma_A + \tau_A, \sigma_C + \tau_C + (i_1\ i'_1) \dots (i_n\ i'_n), \sigma_c + \tau_c)$$

The juxtaposition is from the logical point of view a generalized cut, and its interpretation in terms of permutation is exactly the definition made by Girard in [Gir87].

6.2. Ex-collapse.

Proposition 6.3. *Let $R = (\sigma_A, \sigma_C, \sigma_c)$ be an AC net and f be a partial injection such that $\text{dom}(\sigma_C) = \text{dom}(f)$ and $\text{codom}(f) \cap \text{dom}(\sigma_A) = \emptyset$.*

The couple $(\sigma_A \overset{f}{\rightsquigarrow} f \circ \sigma_C \circ f^, \sigma_c)$, is an interaction net.*

It does not depend on f and we call it the Ex-collapse of R , noted $\text{Ex}(R)$.

For the definition of the Ex-composition to be correct, we have to delocalize σ_C to a domain disjoint from $\text{dom}(\sigma_A)$. The Ex-collapse amounts to replace any maximal chain $a_1 \xrightarrow{\sigma_A} b_1 \xrightarrow{\sigma_C} a_2 \dots b_{n-1} \xrightarrow{\sigma_A} a_n$ by a chain $a_1 \xrightarrow{\sigma_A} b_1 \xrightarrow{f} f(b_1) \xrightarrow{f \circ \sigma_C \circ f^*} f(a_2) \xrightarrow{f^*} \dots b_{n-1} \xrightarrow{\sigma_A} a_n$ and then to compute the Ex-composition to get $a_1 \xrightarrow{\sigma_A \overset{f}{\rightsquigarrow} \sigma_C} a_n$.

Proof. Remark that for this to be an interaction net, the only property to be checked which is not a direct consequence of the definition of AC nets is the fact that $\sigma_A \overset{f}{\rightsquigarrow} f \circ \sigma_C \circ f^*$ is a w -permutation, but this comes directly from proposition 2.4.

This remark asserts that f has only a shallow role in the definition. Indeed, every time f is applied in the Ex-composition, it is followed by an application of its inverse. Moreover, for partial injections $\sigma_1, \tau_1, \dots, \sigma_n, \tau_n$, we have

$$\sigma_n \circ \tau_n \circ \dots \circ \sigma_1 \circ \tau_1 = \sigma_n \circ f_n^* \circ f_n \circ \tau_n \circ g_n^* \circ g_n \circ \dots \circ g_1^* \circ g_1 \circ \sigma_1 \circ f_1^* \circ f_1 \circ \tau_1$$

for every partial injections $f_1, g_1, \dots, f_n, g_n$ such that $\text{dom}(f_i) \subseteq \text{codom}(\tau_i) \cap \text{dom}(\sigma_i)$ and $\text{dom}(g_i) \subseteq \text{codom}(\sigma_i) \cap \text{dom}(\tau_i)$ \square

Proposition 6.4. *For each interaction net R there exists a unique AC net R' of the form $(\sigma_A, 0, \sigma_c)$ such that $\text{Ex}(R') = R$. R' is said to be cutfree.*

Proof. If $R = (\tau_w, \tau_c)$ we only have to take $R' = (\tau_w, 0, \tau_c)$. Uniqueness comes from the fact that $\sigma \overset{0}{\rightsquigarrow} 0 = \sigma$. \square

Definition 6.5. Let R and R' be two AC nets, we say that R and R' are Ex-equivalent, noted $R \overset{\rightsquigarrow}{\sim} R'$ when $\text{Ex}(R) = \text{Ex}(R')$.

We have an obvious correspondence between juxtaposition and gluing.

Proposition 6.6. $\text{Ex}(R^I \leftrightarrow R'^{I'}) = \text{Ex}(R)^I \overset{\rightsquigarrow}{\leftrightarrow} \text{Ex}(R')^{I'}$

Proof. We set $R = (\sigma_A, \sigma_C, \sigma_c)$, $R' = (\tau_A, \tau_C, \tau_c)$.

If we write f (resp. g) the partial injection used in the computation of $\text{Ex}(R)$ (resp. $\text{Ex}(R')$), then we can find a partial injection h such that the partial injection used in the computation of $\text{Ex}(R^I \leftrightarrow R'^{I'})$ is $f + g + h$. Moreover, we can decompose $h = i + i'$ in such a way that $h(\rho(I, I') + \rho(I, I')^*)h^* = i\rho(I, I')i^* + i'\rho(I, I')^*i'^*$.

The main part of the proposition amounts to proving that

$$\begin{aligned} (\sigma_A + \tau_A) \overset{f+g+i+i'}{\rightsquigarrow} (f\sigma_C f^* + g\tau_C g^* + i\rho(I, I')i^* + i'\rho(I, I')^*i'^*) = \\ (\sigma_A \overset{f}{\rightsquigarrow} f\sigma_C f^*) \overset{\rho(I, I')}{\rightsquigarrow} (\tau_A \overset{g}{\rightsquigarrow} g\tau_C g^*) \end{aligned}$$

This equality can be deduced as in the proof of proposition 2.5. The fact that we have extra partial injections f, g, i and i' does not add any new difficulty. \square

Therefore we can claim that

Interaction nets are the quotient of AC nets by $\overset{\sim}{\sim}$.

7. REDUCTION BY MEANS OF DOUBLE PUSHOUT

7.1. Motivation. In this section we briefly recall the double pushout approach of graph rewriting and why we seek such kind of approach in our context.

We consider as rule of graph rewriting a diagram $R \leftarrow I \rightarrow S$ in **Graph**, the category of graphs. The graph I corresponds to some sort of common interface between R and S . As soon as we have a morphism $R \rightarrow G$ we say that G contains the redex of the rule, and we can construct in **Graph** a graph G' such that we have a pushout

$$\begin{array}{ccc} R & \longleftarrow & I \\ \downarrow & \text{po} & \downarrow \\ G & \longleftarrow & G' \end{array}$$

A precise definition of pushout will be given later, but for now let us say that it corresponds to extracting R from the graph G while leaving the common part I . We can construct another pushout in the other direction, thus obtaining the diagram:

$$\begin{array}{ccccc} R & \longleftarrow & I & \longrightarrow & S \\ \downarrow & \text{po} & \downarrow & \text{po} & \downarrow \\ G & \longleftarrow & G' & \longrightarrow & G_r \end{array}$$

The graph G_r is then called the reduct of G by the rule. It is constructed by taking the graph G' and replacing by S the part left empty by the removing of R in G , and then applying some kind of gluing operation along the interface I .

This approach, initiated in the seminal paper [EPS73], leads to a definition of graph reduction which is at the same time intuitive and algebraically rigorous. It is quite natural to try to define it for interaction nets. Indeed, cutting and gluing are explicit operations in our framework.

Note that such kind of approach for interaction nets is defined in the paper [Ban95], but it relies on an embedding of interaction nets in hypergraphs followed by an embedding of hypergraphs in bipartite graphs. In our setting, we can directly state the approach while staying in the realm of interaction nets.

7.2. Pushouts in IN . Let IN be the category whose objects are interaction nets and morphisms are morphisms of interaction nets.

In this section we write $R \xrightarrow{f} S$ to say that f is an almost injective morphism from R to S . We write $R \xrightarrow[\sim]{f} S$ when f is a bijection.

We recall here the definition of pushouts.

Definition 7.1. Let \mathbf{C} be a category. A commutative square $\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ S & & S' \\ g \searrow & T & \swarrow g' \end{array}$ is called a

pushout whenever for any other commutative square $\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ S & & S' \\ h \searrow & T' & \swarrow h' \end{array}$ there exists a unique

$T \xrightarrow{u} T'$ such that $ug = h$ and $ug' = h'$.

We write po in the center of a square to state that it is a pushout.

The following lemma asserts that pushouts are stable under iso of their branches, *i.e.* that we can replace every middle object of the pushout square with an isomorphic one. It will be useful to replace almost injective morphisms by inclusion maps.

Lemma 7.2. Let $\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ S & po & S' \\ g \searrow & T & \swarrow g' \end{array}$ be a pushout square and $S \xrightarrow[\sim]{\tilde{S}}$ be an iso. We also have

the following pushout $\begin{array}{ccc} & R & \\ h \swarrow & & \searrow f' \\ \tilde{S} & po & S' \\ k \searrow & T & \swarrow g' \end{array}$ □

Lemma 7.3. We have the pushout

$$\begin{array}{ccc} & R & \\ \supseteq & & \subseteq \\ R \xrightarrow{f} \underline{S} & po & R \xrightarrow{g} \underline{S}' \\ \subseteq & & \supseteq \\ & R \xrightarrow{f+g} (\underline{S} + \underline{S}') & \end{array}$$

whenever S and S' are disjoint and $\text{dom}(f) \cap \text{dom}(g) = \emptyset$.

Proof. Let

$$\begin{array}{ccc}
 & R & \\
 \supseteq & & \subseteq \\
 R \xleftrightarrow{f} \underline{S} & & R \xleftrightarrow{g} \underline{S}' \\
 \searrow h & T & \swarrow h'
 \end{array}$$

be another commutative square. We will build a morphism u from $R \xleftrightarrow{f+g} (\underline{S} + \underline{S}')$ to T . Let p be a port belonging to $P(R) - \text{dom}(f) - \text{dom}(g)$ or $P(\underline{S}) - \text{codom}(f)$ we just set $u(p) = h(p)$. Similarly we define $u(p) = h'(p)$ when p belongs to $P(\underline{S}') - \text{codom}(g)$. Now, if we take a $p \in \text{dom}(f)$ we can properly define its image p' in $R \xleftrightarrow{f} \underline{S}$. We set $u(p) = h(p') = h'(p)$. We proceed in the same way for a $p \in \text{dom}(g)$.

By construction u is unique and satisfies the required universal property of pushouts. \square

By using the two previous lemmas and the definition of almost injective morphisms, we get the following corollary.

Corollary 7.4. *Let $S \xrightarrow{\alpha} R \xrightarrow{\beta} S'$ be a diagram in IN with S, S' disjoint. By definition of almost injectivity we have $S = \tilde{\alpha}(R) \xleftrightarrow{f} \underline{S}$ and $S' = \tilde{\beta}(R) \xleftrightarrow{g} \underline{S}'$.*

If $\text{dom}(f\tilde{\alpha}) \cap \text{dom}(g\tilde{\beta}) = \emptyset$ then we have the following diagram:

$$\begin{array}{ccc}
 & R & \\
 \alpha \swarrow & & \searrow \beta \\
 S & & S' \\
 \searrow & po & \swarrow \\
 & R \xleftrightarrow{f\tilde{\alpha}+g\tilde{\beta}} (\underline{S} + \underline{S}') &
 \end{array}$$

Remark 7.5. The disjointness of S and S' in the previous lemma is not mandatory as pushouts are only defined up to isomorphism.

Lemma 7.6 (Complement). *If we have $\begin{array}{c} R \\ \alpha \swarrow \\ S \end{array}$ then there exists S' and $R \xrightarrow{\alpha'} S'$ such $\begin{array}{c} \searrow \beta \\ T \end{array}$*

that $\begin{array}{ccc} & R & \\ \alpha \swarrow & & \searrow \alpha' \\ S & & S' \\ \searrow & po & \swarrow \\ & R & \\ \beta \searrow & & \supseteq \\ & T & \end{array}$

Proof. First, we show that we only need to prove the result when all arrows are inclusion maps. Indeed, by applying the definition of almost injectivity we get the following commutative diagram:

$$\begin{array}{ccccc}
& & \tilde{\alpha}(R) & \xrightarrow[\sim]{\tilde{\beta}} & \tilde{\beta}\alpha(R) \\
& \nearrow \tilde{\alpha} & \cap & \tilde{\beta} & \wr \\
R & \xrightarrow[\sim]{\alpha} & S & \xrightarrow[\sim]{\tilde{\beta}} & \tilde{\beta}(S) \\
& & & \searrow \beta & \wr \\
& & & & T
\end{array}$$

If we could complete it with a pushout on the right, as in

$$\begin{array}{ccccccc}
& & \tilde{\alpha}(R) & \xrightarrow[\sim]{\tilde{\beta}} & \tilde{\beta}\alpha(R) & & \\
& \nearrow \tilde{\alpha} & \cap & \tilde{\beta} & \wr & \wr & \\
R & \xrightarrow[\sim]{\alpha} & S & \xrightarrow[\sim]{\tilde{\beta}} & \tilde{\beta}(S) & \text{po} & S' \\
& & & \searrow \beta & \wr & \wr & \\
& & & & T & &
\end{array}$$

we would get the main pushout.

So, let us prove it in the case where $R \subseteq S \subseteq T$. By definition, we have $S = \underline{R} \xrightarrow{f} R$ and $T = S \xrightarrow{g} \underline{S}$. Thus, we have $T = (\underline{R} \xrightarrow{f} R) \xrightarrow{g} \underline{S}$. By corollary 4.13, there exists f_1 and f_2 such that $T = \underline{R} \xrightarrow{f_1} (R \xrightarrow{f_2+g} \underline{S})$. We set $S' = R \xrightarrow{f_2+g} \underline{S}$. We can conclude by applying lemma 7.3. \square

7.3. Generalized reduction.

Definition 7.7. Let $R_r \xleftarrow{\alpha_r} R_i \xrightarrow{\alpha_p} R_p$ be a diagram in \mathbb{IN} . By definition of almost injectivity we have $R_r = \widetilde{\alpha_r}(R_i) \xrightarrow{f_r} \underline{R}_r$ and $R_p = \widetilde{\alpha_p}(R_i) \xrightarrow{f_p} \underline{R}_p$.

We say that this diagram is a *generalized rule* when $\text{dom}(f_r \widetilde{\alpha_r}) = \text{dom}(f_p \widetilde{\alpha_p})$.

Theorem 7.8. If $R_r \xleftarrow{\alpha_r} R_i \xrightarrow{\alpha_p} S_p$ is a generalized rule and we have a morphism $R_r \xrightarrow{\beta} R$ then we can do the following completion

$$\begin{array}{ccccc}
R_r & \xleftarrow{\alpha_r} & R_i & \xrightarrow{\alpha_p} & R_p \\
\beta \downarrow & & \downarrow & & \downarrow \\
R & \supseteq & S & \longrightarrow & T
\end{array}$$

T is called the reduct of R by the generalized rule.

Proof. The proof is just a chaining of the two lemmas 7.6 and 7.4. The condition of equality of domain in the definition of generalized rule ensures that the domain of the gluing function in $R_i \hookrightarrow S$, being disjoint from the domain of the gluing function in $R_i \hookrightarrow R_r$ is also disjoint from the gluing function in $R_i \hookrightarrow R_p$. Thus, the lemma 7.4 is applicable. \square

Proposition 7.9. *This reduction is a generalization of the one defined in section 5.*

Proof. Indeed let $(R_r^{I_r}, R_p^{I_p})$ be an interaction rule and set $I_r = d_1 > \dots > d_m$. We define a net

$$R_i = ((d_1 f_1) \dots (d_m f_m), 0)$$

with m new free ports f_i .

We directly have an inclusion $R_i \subseteq R_r = R_i \xrightarrow{f_r} \underline{R}_r$ and by definition of an interaction rule, we have a bijection between I_r and I_p which can be lifted to an almost injective morphism $R_i \xrightarrow{\alpha_p} R_p = \widetilde{\alpha}_p(R_i) \xrightarrow{f_p} \underline{R}_p$.

The diagram $R_r \supseteq R_i \xrightarrow{\alpha_p} R_p$ is a generalized rule as $\text{dom}(f_r) = \text{dom}(f_p \widetilde{\alpha}_p) = \{f_1, \dots, f_m\}$.

Now let $R_r \xrightarrow{\beta} R$ be an almost injective morphism, we have $R = \widetilde{\beta}(R_r) \xrightarrow{g} \underline{R}$. We are going to consider \underline{R} and \underline{R}_p disjoint, if it is not the case we just need to add an explicit renaming to the following computations. By construction, we get $S = \widetilde{\beta}(R_i) \xrightarrow{g'} \underline{R}$, where g' is the restriction of g to $\widetilde{\beta}(R_i)$, and we have $T = R_i \xrightarrow{g' \widetilde{\beta} + f_p \widetilde{\alpha}_p} (\underline{R} + \underline{R}_p)$ which is the result of the previously defined reduction. \square

8. IMPLEMENTATION

8.1. Introduction. We detail here part of our implementation in OCaml of an interaction net tool. This implementation follows closely the mathematical definitions given earlier. By doing so we hope that we make apparent the idea that this framework, even though involving mathematical objects, can be seen as a natural syntax for implementing interaction nets.

A self-contained net reducer has been extracted from our implementation and is presented in Appendix A. For the sake of brevity we have removed from this code subroutines involving renaming of net.

8.2. Data structures. The easiest way to represent partial permutation is to define them as their list of orbits. The fact that orbits are disjoint and make sense will in fact be ensured by the validity of our operations.

We define two types

```
type 'a lorbit = { cycle : int list; label : 'a }
type 'a lperm = 'a lorbit list
```

for representing labelled permutation, and we only need to set a dummy label to represent an unlabelled permutation.

Therefore, the type for representing a net is

```
type cell_label = { symbol : symbol; pal : int }
type net = { cells : cell_label lperm; wires : unit lperm }
```

Following the previous definitions, we define interface, context and rule

```
type interface = int list
type context = net * interface
type rule = { symbols : symbol * symbol; pattern : context }
```

8.3. Algorithms. To have a full implementation we need to be able to find when a reduction rule could be applied, and then to apply it. Nevertheless the only changing part between this framework and the usual one is the use of **Ex**-composition to define the reduction.

We recall here the standard procedure for reducing nets, next to each step we give the corresponding functions in the code found in Appendix A:

- (1) Extract the list of active wires, *i.e.* wires linking two principal ports
[`net_get_active_wires`]
- (2) Filter out the active wires corresponding to a rule redex
[`net_appliable_rules`]
- (3) For one of these matches, cut out the redex and replace it with the rule pattern
[`net_remove_cell,net_remove_wire,net_apply_rule`]

The main difference here, is that our replacement of the pattern relies on a net gluing [`net_glue`], which in turns relies on an **Ex**-composition [`perm_excomp`].

Algorithm 1 Computation of $\sigma_w \overset{f}{\rightsquigarrow} \tau_w$ for σ_w, τ_w being w -permutations

```

orbits =  $\sigma_w + \tau_w$ 
for  $p \in \text{dom}(f)$  do
   $p' = f(p)$ 
   $w =$  orbit containing  $p$  in orbits
   $w' =$  orbit containing  $p'$  in orbits
  orbits = orbits - [ $w, w'$ ]
  if  $w = w'$  then
    orbits = [ $\min(p, p')$ ] :: orbits
  else
     $(p, q) = w$ 
     $(p', q') = w'$ 
    orbits = [ $q, q'$ ] :: orbits
  end if
end for
return orbits

```

A method for computing $\sigma_w \overset{f}{\rightsquigarrow} \tau_w$ can be found in Algorithm 1. This algorithm amounts to concatenation of orbits from σ_w and τ_w by removing ports that are part of $\text{dom}(f) \cup \text{codom}(f)$. If we consider that every operations used on permutations are linear, as it the case with lists, its complexity is in $\mathcal{O}(|\text{dom}(f)|(|\text{dom}(\sigma_w)| + |\text{dom}(\tau_w)|))$. Note that in most cases $|\text{dom}(f)|$ is small compared to $|\text{dom}(\sigma_w)| + |\text{dom}(\tau_w)|$ because of the local aspect of reduction rules in interaction nets.

8.4. Extensions. Our full interaction net tool⁴ deals with some common extensions of interaction nets.

To be able to handle sharing graphs, in the Abadi, Gonthier and Levy flavour [AGL92] we need to add parameters to cells, the so-called *levels*. These parameters are both used to guard the applicability of a rule and add dependencies on the redex parameters inside the rule pattern. Thus, we extend the previous types with

⁴available in a preliminary version at the address <http://marc.de-falco.fr/mlint>

```

type 'a cell_label = { symbol:symbol; pal:int; parameter:'a}
type 'a rule = { symbols : symbol * symbol;
                pattern  : 'a * 'a -> 'a context;
                guard   : 'a * 'a -> bool }

```

Another common extension is found in differential interaction nets, presented in [ER05], which handles not only nets but formal sum of nets. Concerning the rules it amounts to multiple patterns, therefore we only need to adapt the previous type of pattern to

```

pattern : 'a * 'a -> 'a context list

```

We would like to emphasise on the fact that these extensions do not imply complex changes to the code presented in Appendix A. Indeed, our framework presented here for vanilla interaction net is quite flexible and it could serve as a basis for a rigorous study of extensions of interaction nets.

CONCLUSION

Throughout this paper we have developed a syntactical framework for dealing with interaction nets while still being rigorous. Some specific extensions of this framework – for example the definition of paths in nets, their reduction and its strong confluence – can be found in [dF09].

At this point, it is quite natural to ask about semantics. So far no general notion of denotational semantics for interaction nets can be found in the literature. The closest examples are either based on geometry of interaction [Laf97, dF08] or experiments [Maz07], and all treat of specific cases (interaction combinators or differential interaction nets). Building on this framework, the author has a proposal which will be presented in a further paper.

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APPENDIX A. A LIGHTWEIGHT INTERACTION NET REDUCER IN OCAML

```

type symbol = string
type 'a lorbit = { cycle : int list; label : 'a }
type 'a lperm = 'a lorbit list
type cell_label = { symbol:symbol; pal:int }

```

```

type net = { cells : cell_label lorbit list; wires : unit lperm }
type port = FreePort of int | CellPort of cell_label lorbit * int
type interface = int list
type context = net * interface
type rule = { symbols : symbol * symbol; pattern : context }

( Utility functions for handling orbits )
let cycle l = (List.tl l)@[List.hd l]
let rec cycle_to e l =
  if List.hd l = e then l else cycle_to e (cycle l)
let rec index l p = match l with
  | [] -> raise Not_found
  | hd::tl -> if hd = p then 0 else 1+(index tl p)
let rec filter_opt l =
  match l with
  | [] -> []
  | None::l -> filter_opt l
  | Some a::l -> a::(filter_opt l)
let list_diff l1 l2 = List.filter (fun x -> not (List.mem x l2)) l1

( Get the orbit in p containing an element e
 * This function returns a couple (o, p')
 * where o is the orbit and p' is the remaining permutation *))
let lperm_get_orbit_split e p =
  let rec laux acc e p =
    match p with
    | [] -> raise Not_found
    | o::ol -> if List.mem e o.cycle
      then (o, acc@ol)
      else laux (o::acc) e ol
  in laux [] e p

( Get optionally the orbit containing e in p *))
let lperm_get_orbit e p = try
  let o = fst (lperm_get_orbit_split e p) in Some o
with Failure _ -> None

( Get the element after e along its orbit in the permutation p *))
let lperm_next e p =
  let (o,_) = lperm_get_orbit_split e p in
  List.hd (cycle (cycle_to e o.cycle))

( Get a new permutation p' from permutation p by
 * fusing the orbit oa containing a and the orbit ob
 * containing b. This is done by inserting ob inside oa in
 * such a way that p'(a) = b. In case oa = ob it adds a fixpoint. *))
let lperm_fuse_orbits (p:'a lperm) (a:int) (b:int) =
  let (oa, ola) = lperm_get_orbit_split a p in
  if List.mem b oa.cycle
    then { cycle=[min a b]; label=oa.label }::ola

```

```

    else let (ob, nol) = lperm_get_orbit_split b ola in
      { cycle=[lperm_next a p; lperm_next b p]; label=oa.label }::nol

(* Disjoint sum of permutations p1 and p2 *)
let lperm_sum p1 p2 = p1 @ p2

(* Compute the ex-composition of s and t along f by
 * fusing orbits in the union of s and t along f *)
let perm_excomp s t f =
  let rec fuse_orbits p l = match l with
    [] -> p
  | (a,b)::tl -> fuse_orbits (lperm_fuse_orbits p a b) tl in
  fuse_orbits (s@t) f

(* Net gluing n1 <-f-> n2 *)
let net_glue n1 n2 f = { cells=lperm_sum n1.cells n2.cells;
  wires=perm_excomp n1.wires n2.wires f }
let net_sum n1 n2 = net_glue n1 n2 []
let coord i1 i2 = List.combine i1 i2
(* context gluing n1^i1 <-> n2^i2 *)
let context_glue (n1,i1) (n2,i2) = net_glue n1 n2 (coord i1 i2)

(* Discriminate a given port p in the net n *)
let net_get_port n p =
  match lperm_get_orbit p n.cells with
  | Some c -> let cycle = cycle_to c.label.pal c.cycle in
    CellPort(c, index cycle p)
  | None -> FreePort p

(* Predicate asserting the fact that w is an active wire in n *)
let net_wire_is_active n w = match w.cycle with
  | [ p1; p2 ] -> begin
    match (net_get_port n p1, net_get_port n p2) with
    | (CellPort(c1,-), CellPort(c2,-)) ->
      p1 = c1.label.pal && p2 = c2.label.pal
    | _ -> false
  end
  | _ -> false
let net_get_active_wires n = List.filter (net_wire_is_active n) n.wires

(* Extract the cell containing the port p from the net n *)
let net_remove_cell n p =
  let (c,nc) = lperm_get_orbit_split p n.cells in
  let nw = List.filter
    (fun x -> list_diff c.cycle x.cycle <> []) n.wires in
  (c, { cells=nc; wires=nw })

(* Extract the wire p1-p2 from the net n *)
let net_remove_wire n p1 p2 =
  let nw = List.filter
    (fun w -> list_diff w.cycle [p1;p2] <> []) n.wires in
  { cells=n.cells; wires=nw }

```

```

(* Apply a rule in n by removing the redex containing the active
 * wire p1—p2 and replacing it with ctx *)
let net_apply_rule n (p1,p2,ctx) =
  let (c1,n1) = net_remove_cell n p1 in
  let (c2,n2) = net_remove_cell n1 p2 in
  let n3 = net_remove_wire n2 c1.label.pal c2.label.pal in
  let i = (list_diff c1.cycle [p1])@(list_diff c2.cycle [p2]) in
  context_glue (n3,i) ctx

(* Take a net n and a list of rules rl and return a sublist
 * of rules having a matching redex in n *)
let net_appliable_rules n rl =
  let law = net_get_active_wires n in
  let matching_rule (s1,s2) w =
    match w.cycle with
    | [p1;p2] -> begin
      match (net_get_port n p1, net_get_port n p2) with
      | (CellPort(c1,-), CellPort(c2,-)) -> begin
        match (c1.label.symbol, c2.label.symbol) with
        | cs1, cs2 when cs1 = s1 && cs2 = s2 -> Some (p1, p2)
        | cs1, cs2 when cs1 = s2 && cs2 = s1 -> Some (p2, p1)
        | _ -> None
      end
    | _ -> None
    end
  | _ -> None in
  let res r = List.map
    (fun (p1,p2) -> (p1,p2,r.pattern))
    (filter_opt (List.map (matching_rule r.symbols) law))
  in List.concat (List.map res rl)

(* Take a net n and a list of rules rl and return an optional
 * reduct *)
let net_reduce n rl =
  let res = net_appliable_rules n rl in
  match res with
  | instance::_ -> Some (net_apply_rule n instance)
  | _ -> None

```