

COMPUTABLE JORDAN DECOMPOSITION OF LINEAR CONTINUOUS FUNCTIONALS ON $C[0; 1]$

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ABSTRACT. By the Riesz representation theorem using the Riemann-Stieltjes integral, linear continuous functionals on the set of continuous functions from the unit interval into the reals can either be characterized by functions of bounded variation from the unit interval into the reals, or by signed measures on the Borel-subsets. Each of these objects has an (even minimal) Jordan decomposition into non-negative or non-decreasing objects. Using the representation approach to computable analysis, a computable version of the Riesz representation theorem has been proved by Jafarikhah, Lu and Weihrauch. In this article we extend this result. We study the computable relation between three Banach spaces, the space of linear continuous functionals with operator norm, the space of (normalized) functions of bounded variation with total variation norm, and the space of bounded signed Borel measures with variation norm. We introduce natural representations for defining computability. We prove that the canonical linear bijections between these spaces and their inverses are computable. We also prove that Jordan decomposition is computable on each of these spaces.

1. INTRODUCTION

Let $C[0; 1]$ be the set of continuous functions $h : [0; 1] \rightarrow \mathbb{R}$. By the Riesz representation theorem for every linear continuous function $F : C[0; 1] \rightarrow \mathbb{R}$ there is a function $g : [0; 1] \rightarrow \mathbb{R}$ of bounded variation such that $F(h) = \int h dg$ for every continuous function $h \in C[0; 1]$. For every function $g : [0; 1] \rightarrow \mathbb{R}$ of bounded variation there is a signed Borel measure μ on the unit interval of finite variation norm such that $\int h dg = \int h d\mu$ for every continuous function $h \in C[0; 1]$. Finally for every signed Borel measure μ on the unit interval of finite variation norm the function $h \mapsto \int h d\mu$ for $h \in C[0; 1]$ is linear and continuous.

By the Jordan decomposition theorem for every function $g : [0; 1] \rightarrow \mathbb{R}$ of bounded variation there are non decreasing functions $g^+, g^- : [0; 1] \rightarrow \mathbb{R}$ such that $g = g^+ - g^-$. Similar decomposition theorems have been proved for Functionals F and measures μ : for every linear continuous functional $F : C[0; 1] \rightarrow \mathbb{R}$ there are two non-negative functionals F^+

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and F^- such that $F = F^+ - F^-$, and for every signed Borel measure μ on the unit interval of finite variation norm there are non-negative measures μ^+, μ^- such that $\mu = \mu^+ - \mu^-$. In each case there is a minimal decomposition [8, 9, 10, 12, 14, 17, 7, 1, 2, 16].

In this article we study computability of all of these existence theorems. Computability of the Riesz representation theorem and its converse have been proved in [15] with a revised proof in [11]. Computability of $(\mu, h) \mapsto \int h d\mu$ for continuous h and non-negative bounded Borel measure μ has been proved in [19]. In this article we extend these results.

We study the computable relation between three Banach spaces, the space of linear continuous functionals with operator norm, the space of (normalized) functions of bounded variation with total variation norm, and the space of bounded signed Borel measures with variation norm. We introduce natural representations for defining computability. We prove that the canonical linear bijections $F \mapsto g$, $g \mapsto \mu$ and $\mu \mapsto F$ between these spaces and their inverses are computable. We also prove that (minimal) Jordan decomposition is computable on each of these spaces.

In Section 2 we summarize some definitions and basic facts from classical analysis on linear continuous functionals $F : C[0; 1] \rightarrow \mathbb{R}$, functions of bounded variation and the Riemann-Stieltjes integral, and on signed measures on the Borel sets of the unit interval. We consider only functions $g : [0; 1] \rightarrow \mathbb{R}$ of bounded variation which are normalized in the sense that $g(0) = 0$ and for all $0 < y < 1$, $\lim_{x \nearrow y} g(x) = g(y)$.

In Section 3 we outline very shortly some general concepts from the representation approach to computable analysis [20, 5]. For defining computability we introduce and discuss representations of the functionals, of the functions of bounded variation and of the signed measures and also representations of the subspaces of non-negative or non-decreasing objects, respectively. While in [15, 11] partial functions of bounded variation are considered in this article we use total normalized functions with a representation which is very closely related to the one used for the partial functions.

In Section 4 first we prove for the special case of non-negative functionals F , non-decreasing functions g and non-negative measures μ that the mappings $F \mapsto g$, $g \mapsto \mu$ and $\mu \mapsto F$ such that $F(h) = \int h dg$, $\int h dg = \int h d\mu$ and $\int h d\mu = F(h)$ are computable w.r.t the “non-negative” representations. Then we prove our main results: On the spaces of linear continuous functionals with operator norm, the space of normalized functions of bounded variation with variation norm and the space of signed measures with finite variation norm the operators $F \mapsto g$, $g \mapsto \mu$ and $\mu \mapsto F$ are computable. Furthermore, the Jordan decompositions $F \mapsto (F^+, F^-)$, $g \mapsto (g^+, g^-)$ and $\mu \mapsto (\mu^+, \mu^-)$ are computable. The results can be expressed in such a way that a number of representations of the space of linear continuous functionals are equivalent.

The results can be generalized easily from the unit interval to arbitrary intervals $[a; b]$ with computable endpoints. More generally, the results can be proved computably uniform in a, b , where a and b are given by their standard representation via fast converging Cauchy sequences of rational numbers.

In [13, 22] Jordan decomposition of computable real functions and of polynomial time computable functions on the unit interval has been studied. However, they do not investigate computability of the Jordan decomposition operator but ask whether computability or polynomial computability is preserved under Jordan decomposition. Ko [13] has shown that there is a polynomial time computable function f of bounded variation which is not the difference of two non-decreasing polynomial time computable functions. This has been strengthened by Zheng and Rettinger who have proved that there is a polynomial time

computable function of bounded variation with polynomial modulus of absolute continuity which is not the difference of two non-decreasing computable functions.

2. BASICS FROM THE CLASSICAL THEORY

We summarize some definitions and results about functions of bounded variation and from (non-computable) measure theory which are scattered across many sources [8, 9, 10, 12, 14, 17, 7, 1, 2, 16, 15, 11] or can be derived easily from there. For convenience we consider only the closed unit interval $[0; 1]$ for functions, measures etc.

Let $C[0; 1]$ be the space of continuous functions $h : [0; 1] \rightarrow \mathbb{R}$ with norm $\|h\| = \sup\{|h(x)| \mid x \in [0; 1]\}$. Let $C'[0; 1]$ be the space of linear continuous functionals $F : C[0; 1] \rightarrow \mathbb{R}$ with norm $\|F\| = \sup\{|F(h)| \mid h \in C[0; 1], \|h\| \leq 1\}$. For every non-negative $F \in C'[0; 1]$ (that is, $F(h) \geq 0$ if $h \geq 0$), $\|F\| = F(\mathbb{1})$ (where $\mathbb{1}(x) = 1$ for $0 \leq x \leq 1$).

We shortly introduce functions $g : [0, 1] \rightarrow \mathbb{R}$ of bounded variation and the Riemann-Stieltjes integral $\int h dg$ for continuous functions $h : [0; 1] \rightarrow \mathbb{R}$. A *partition* of a real interval $[a; b]$ ($a < b$) is a sequence $Z = (x_0, x_1, \dots, x_n)$, $n \geq 0$, of real numbers such that $a = x_0 < x_1 \dots < x_n = b$. The partition Z has *precision* k , if $x_i - x_{i-1} < 2^{-k}$ for $1 \leq i \leq n$. A partition $Z' = (x'_0, x'_1, \dots, x'_m)$, is finer than Z , if $\{x_0, x_1, \dots, x_n\} \subseteq \{x'_0, x'_1, \dots, x'_m\}$. For a function $g : [0; 1] \rightarrow \mathbb{R}$, for $0 \leq a < b \leq 1$ and a partition Z of the interval $[a; b]$ define

$$S(g, Z) := \sum_{i=1}^n |g(x_i) - g(x_{i-1})|, \quad (2.1)$$

$$V_a^b(g) := \sup\{S(g, Z) \mid Z \text{ is a partition of } [a; b]\}. \quad (2.2)$$

The function $g : [0; 1] \rightarrow \mathbb{R}$ is of *bounded variation* if its variation $\text{Var}(g) := V_0^1(g)$ is finite. For a function of bounded variation the total variation function $/g/ : [0; 1] \rightarrow \mathbb{R}$ is defined by $/g/(0) := 0$ and $/g/(x) := V_0^x(g)$.

In the following let $h : [0; 1] \rightarrow \mathbb{R}$ be a continuous function and let $g : [0; 1] \rightarrow \mathbb{R}$ be a function of bounded variation. For any partition $Z = (x_0, x_1, \dots, x_n)$ of $[0; 1]$ define

$$S(g, h, Z) := \sum_{i=1}^n h(x_i)(g(x_i) - g(x_{i-1})). \quad (2.3)$$

Since h is continuous and its domain is compact, it has a (uniform) *modulus of continuity*, i.e., a function $m : \mathbb{N} \rightarrow \mathbb{N}$ such that $|h(x) - h(y)| \leq 2^{-k}$ if $|x - y| \leq 2^{-m(k)}$. We may assume that the function m is non-decreasing.

Lemma 2.1 ([15]). *Let $h : [0; 1] \rightarrow \mathbb{R}$ be a continuous function with modulus of continuity $m : \mathbb{N} \rightarrow \mathbb{N}$ and let g be a function of bounded variation. Then there is a unique number $I \in \mathbb{R}$ such that*

$$|I - S(g, h, Z)| \leq 2^{-k} \text{Var}(g)$$

for all $k \in \mathbb{N}$ and for every partition Z of $[0; 1]$ with precision $m(k + 1)$.

The number I from Lemma 2.1 is called the *Riemann-Stieltjes integral* and is denoted by $\int h dg$. The operator $F_g : h \mapsto \int h dg$ is linear and continuous on $C[0; 1]$.

Notice that by Lemma 2.1 the integral $\int h dg$ is determined already by the values of the function g on 0 and 1 and on an arbitrary dense set X , since there are partitions of arbitrary precision that contain points only from the set X . If g is of bounded variation,

then $\lim_{y \nearrow x} g(y)$ and $\lim_{y \searrow x} g(y)$ exist for all $0 \leq x \leq 1$. Functions of bounded variation can be normalized without changing the Riemann-Stieltjes integral over continuous functions.

Let BV be the set of functions $g : [0; 1] \rightarrow \mathbb{R}$ of bounded variation such that

$$g(0) = 0 \text{ and } (\forall 0 < x < 1) g(x) = \lim_{y \nearrow x} g(y). \quad (2.4)$$

Lemma 2.2.

- (1) Every $g \in \text{BV}$ is left-continuous.
- (2) For every $g \in \text{BV}$, $\text{Var}(g) = \|F_g\|$.
- (3) For every function g of bounded variation there is a unique function $g' \in \text{BV}$ such that $\int h dg = \int h dg'$ for all functions $h \in C[0; 1]$.

The function g' can be defined by

$$g'(0) := 0, g'(1) := g(1) - g(0) \text{ and } g'(x) := \lim_{y \nearrow x} g(y) - g(0) \text{ for } 0 < x < 1. \quad (2.5)$$

For every non-decreasing function $g \in \text{BV}$, $\text{Var}(g) = g(1)$.

Let BM be the set of signed measures μ with finite variation norm $\|\mu\|_m$ on the Borel subsets of the unit interval $[0; 1]$ defined by $\|\mu\|_m := \sup_{\pi} \sum_{I \in \pi} |\mu(I)|$ where π runs over all finite partitions of the unit interval into intervals (open, semi-open, closed). If μ is non-negative, then $\|\mu\|_m = \mu([0; 1])$.

The following theorem summarizes the relation between the three spaces introduced above.

Theorem 2.3. *The spaces $(C'[0; 1], \|\cdot\|)$, (BV, Var) and $(\text{BM}, \|\cdot\|_m)$ are Banach spaces.*

- (1) *There is a unique linear homeomorphism $T_{\text{FV}} : C'[0; 1] \rightarrow \text{BV}$ such that $T_{\text{FV}}(F) = g$ implies $(\forall h \in C[0; 1]) F(h) = \int h dg$.*
- (2) *There is a unique linear homeomorphism $T_{\text{VM}} : \text{BV} \rightarrow \text{BM}$ such that $T_{\text{VM}}(g) = \mu$ implies $(\forall h \in C[0; 1]) \int h dg = \int h d\mu$.*
- (3) *There is a unique linear homeomorphism $T_{\text{MF}} : \text{BM} \rightarrow C'[0; 1]$ such that $T_{\text{MF}}(\mu) = F$ implies $(\forall h \in C[0; 1]) \int h d\mu = F(h)$.*

The functions T_{FV} , T_{VM} and T_{MF} preserve the norms. Moreover, if F is non-negative then $T_{\text{FV}}(F)$ is non-decreasing, if g is non-decreasing then $T_{\text{VM}}(g)$ is non-negative, and if μ is non-negative then $T_{\text{MF}}(\mu)$ is non-negative.

The three spaces are not separable. Theorem 2.3(1) includes the Riesz representation theorem [10]. For real numbers x define $x^+ := (|x| + x)/2$ and $x^- := (|x| - x)/2$. Then x^+ and x^- are non-negative numbers such that $x = x^+ - x^-$. Moreover, x^+ and x^- are minimal, that is, $x^+ \leq y^+$ and $x^- \leq y^-$ if y^+, y^- are non-negative such that $x = y^+ - y^-$. By the Jordan decomposition theorem, this kind of decomposition can be generalized to functionals $F \in C'[0; 1]$, to functions $g \in \text{BV}$ and to signed measures $\mu \in \text{BM}$.

Definition 2.4.

- (1) For $F \in C'[0; 1]$ the *Jordan decomposition* is a pair (F^+, F^-) of non-negative functionals in $C'[0; 1]$ such that $F = F^+ - F^-$, and if $G^+, G^- \in C'[0; 1]$ are non-negative functionals such that $F = G^+ - G^-$ then $F^+ \leq G^+$ and $F^- \leq G^-$.
- (2) For $g \in \text{BV}$ the *Jordan decomposition* is a pair (g^+, g^-) of non-decreasing functions in BV such that $g = g^+ - g^-$, and if $t^+, t^- \in \text{BV}$ are non-decreasing functions such that $g = t^+ - t^-$ then $g^+ \leq t^+$ and $g^- \leq t^-$.

- (3) For $\mu \in \text{BM}$ the *Jordan decomposition* is a pair (μ^+, μ^-) of non-negative measures in BM such that $\mu = \mu^+ - \mu^-$, and if $\nu^+, \nu^- \in \text{BM}$ are non-negative measures such that $\mu = \nu^+ - \nu^-$ then $\mu^+ \leq \nu^+$ and $\mu^- \leq \nu^-$.

If a Jordan decomposition exists then it is unique by the minimality condition. Notice that some authors do not require minimality for Jordan decomposition.

Theorem 2.5.

- (1) Every $F \in C'[0; 1]$ has a Jordan decomposition. If $F^+, F^- \in C'[0; 1]$ are non-negative and $F = F^+ - F^-$, then (F^+, F^-) is the Jordan decomposition of F iff $\|F\| = \|F^+\| + \|F^-\|$.
- (2) Every $g \in \text{BV}$ has a Jordan decomposition. If $g^+, g^- \in \text{BV}$ are non-decreasing and $g = g^+ - g^-$, then (g^+, g^-) is the Jordan decomposition of g iff $\text{Var}(g) = \text{Var}(g^+) + \text{Var}(g^-)$.
- (3) Every measure $\mu \in \text{BM}$ has a Jordan decomposition. If $\mu^+, \mu^- \in \text{BM}$ are non-negative measures and $\mu = \mu^+ - \mu^-$, then (μ^+, μ^-) is the Jordan decomposition of μ iff $\|\mu\|_m = \|\mu^+\|_m + \|\mu^-\|_m$.

Corollary 2.6.

- (1) If (F^+, F^-) is the Jordan decomposition of F then $(T_{\text{FV}}(F^+), T_{\text{FV}}(F^-))$ is the Jordan decomposition of $T_{\text{FV}}(F)$.
- (2) If (g^+, g^-) is the Jordan decomposition of g then $(T_{\text{VM}}(g^+), T_{\text{VM}}(g^-))$ is the Jordan decomposition of $T_{\text{VM}}(g)$.
- (3) If (μ^+, μ^-) is the Jordan decomposition of μ then $(T_{\text{MF}}(\mu^+), T_{\text{MF}}(\mu^-))$ is the Jordan decomposition of $T_{\text{MF}}(\mu)$.

Proof. Let (F^+, F^-) be the Jordan decomposition of $F := F^+ - F^-$. Let $g^+ := T_{\text{FV}}(F^+)$ and $g^- := T_{\text{FV}}(F^-)$. Then $g^+ - g^- = T_{\text{FV}}(F^+ - F^-)$. By Theorems 2.3 and 2.5,

$$\text{Var}(g^+ - g^-) = \|F^+ - F^-\| = \|F^+\| + \|F^-\| = \text{Var}(g^+) + \text{Var}(g^-),$$

hence by Theorem 2.5, $(g^+, -g^-)$ is the Jordan decomposition of $(g^+ - g^-)$. Therefore, $(T_{\text{FV}}(F^+), T_{\text{FV}}(F^-))$ is the Jordan decomposition of $T_{\text{FV}}(F)$.

The other statements can be proved accordingly. □

3. THE CONCEPTS OF COMPUTABILITY

In this section we define computability on the three spaces from Theorem 2.3. Since the spaces are not separable, Cauchy representations [20, Chapter 8.1] are not available.

For studying computability we use the representation approach (TTE, Type 2 Theory of Effectivity) for computable analysis [20, 5]. Let Σ be a finite alphabet. Computable functions on Σ^* (the set of finite sequences over Σ) and Σ^ω (the set of infinite sequences over Σ) are defined by Turing machines which map sequences to sequences (finite or infinite). On Σ^* and Σ^ω finite or countable tuplings (injections from cartesian products of Σ^* and Σ^ω to Σ^* or Σ^ω) will be denoted by $\langle \rangle$ [20, Definition 2.1.7]. The tupling functions and the projections of their inverses are computable.

In TTE, sequences from Σ^* or Σ^ω are used as “names” of abstract objects such as rational numbers, real numbers, real functions or points of a metric space. We consider computability of multi-functions w.r.t. representations [20, 5], [21, Sections 3,6,8,9]. A representation of a set X is a function $\delta : \subseteq C \rightarrow X$ where $C = \Sigma^*$ or $C = \Sigma^\omega$. If $\delta(p) = x$ we call p a δ -name of x .

For representations $\gamma : \subseteq Y \rightarrow M$ and $\gamma_0 : \subseteq Y_0 \rightarrow M_0$, a function $h : \subseteq Y \rightarrow Y_0$ is a (γ, γ_0) -realization of a function $f : \subseteq M \rightarrow M_0$, iff for all $p \in Y$ and $x \in M$,

$$\gamma(p) = x \in \text{dom}(f) \implies \gamma_0 \circ h(p) = f(x), \quad (3.1)$$

that is, $h(p)$ is a name of some $f(x)$, if p is a name of $x \in \text{dom}(f)$. The function f is called (γ, γ_0) -computable, if it has a computable (γ, γ_0) -realization and (γ, γ_0) -continuous if it has a continuous realization. The definitions can be generalized straightforwardly to multivariate functions $f : \subseteq M_1 \times \dots \times M_n \rightarrow M_0$ for represented sets M_i .

For two representations $\delta_i : \subseteq Y_i \rightarrow M_i$ ($i = 1, 2$), δ_1 is *reducible to* δ_2 , $\delta_1 \leq \delta_2$, iff there is a computable function $h : \subseteq Y_1 \rightarrow Y_2$ such that $(\forall p \in \text{dom}(\delta_1)) \delta_1(p) = \delta_2 h(p)$ (if p is a δ_1 -name of x then $h(p)$ is a δ_2 -name of x). The two representations are *equivalent*, $\delta_1 \equiv \delta_2$, iff $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_1$.

Let $\delta_i : \subseteq \Sigma^\omega \rightarrow M_i$ ($i = 1, 2$) be representations. The canonical representation $[\delta_1, \delta_2]$ of the product $M_1 \times M_2$ is defined by

$$[\delta_1, \delta_2](p_1, p_2) = (\delta_1(p_1), \delta_2(p_2)). \quad (3.2)$$

There is a representation $[\delta_1 \rightarrow \delta_2]$ of the set of (δ_1, δ_2) -continuous functions $f : M_1 \rightarrow M_2$ which is determined uniquely up to equivalence by **(U)** and **(S)**[20].

$$\textbf{(U)} \text{ The apply function } (f, x) \mapsto f(x) \text{ is } ([\delta_1 \rightarrow \delta_2], \delta_1, \delta_2)\text{-computable,} \quad (3.3)$$

$$\textbf{(S)} \text{ If for some representation } \gamma \text{ of a set of } (\delta_1, \delta_2)\text{-continuous functions } (f, x) \mapsto f(x) \text{ is } (\gamma, \delta_1, \delta_2)\text{-computable then } \gamma \leq [\delta_1 \rightarrow \delta_2]. \quad (3.4)$$

(U) corresponds to the ‘‘universal Turing machine theorem’’ and **(S)** to the ‘‘smn-theorem’’ from computability theory. Roughly speaking, $[\delta_1 \rightarrow \delta_2]$ is (up to equivalence) the ‘‘weakest’’ representation of the set of (δ_1, δ_2) -continuous functions for which the apply function is computable. The generalized Turing machines in [18] are useful tools for defining new computable functions on represented sets from given ones.

We use various canonical notations $\nu : \subseteq \Sigma^* \rightarrow X$: $\nu_{\mathbb{N}}$ for the natural numbers, $\nu_{\mathbb{Q}}$ for the rational numbers, ν_{Pg} for the polygon functions on $[0; 1]$ whose graphs have rational vertices, and ν_I for the set RI of open intervals $(a; b) \subseteq (0; 1)$ with rational endpoints. For functions $m : \mathbb{N} \rightarrow \mathbb{N}$ we use the canonical representation $\delta_{\mathbb{B}} : \subseteq \Sigma^\omega \rightarrow \mathbb{B} = \{m \mid m : \mathbb{N} \rightarrow \mathbb{N}\}$ defined by $\delta_{\mathbb{B}}(p) = m$ if $p = 1^{m(0)}01^{m(1)}01^{m(2)}0\dots$. For the real numbers we use the Cauchy representation $\rho : \subseteq \Sigma^\omega \rightarrow \mathbb{R}$, $\rho(p) = x$ if p is (encodes) a sequence $(a_i)_{i \in \mathbb{N}}$ of rational numbers such that for all i , $|x - a_i| \leq 2^{-i}$, and the lower representation $\rho_{<}$, $\rho_{<}(p) = x$ iff p is (encodes) a sequence $(a_i)_{i \in \mathbb{N}}$ of rational numbers such that $x = \sup_i a_i$. By the Weierstraß approximation theorem the countable set Pg of polygon functions with rational vertices is dense in $C[0; 1]$. Therefore, $C[0; 1]$ with notation ν_{Pg} of the set Pg is a computable metric space [20] for which we use the Cauchy representation δ_C defined as follows: $\delta_C(p) = h$ if p is (encodes) a sequence $(h_i)_{i \in \mathbb{N}}$ of polygons $h_i \in \text{Pg}$ such that for all i , $\|h - h_i\| \leq 2^{-i}$ [20].

Since the representations ρ and δ_C are admissible, a functional $G : C[0; 1] \rightarrow \mathbb{R}$ is continuous iff it is (δ_C, ρ) -continuous [20]. Therefore, $[\delta_C \rightarrow \rho]$ is a representation of the continuous functionals $G : C[0; 1] \rightarrow \mathbb{R}$. This representation is tailored for evaluation $(G, h) \mapsto G(h)$ (3.3) (3.4). We use it for the subspace $C'[0; 1]$ of the linear continuous functionals. The norm on $C'[0; 1]$ is $([\delta_C \rightarrow \rho], \rho_{<})$ -computable but not $([\delta_C \rightarrow \rho], \rho)$ -computable. Since for computations we will need the ρ -name of the norm we include it in the name.

Definition 3.1. Define a representation δ_{CF} of $C'[0; 1]$ by

$$\delta_{CF}\langle p, q \rangle = F : \iff [\delta_C \rightarrow \rho](p) = F \text{ and } \rho(q) = \|F\|.$$

This is the representation of the dual of $C[0; 1]$ space as suggested in Section 15 (see also Definition 3.9) of [V.Brattka: "Computability of Banach Space Principles"] in the case that this dual is not separable. It is admissible and admits computability of scalar multiplication, the norm and the rapid Lim-operator, but vector addition is not computable. This yields a good justification for using δ_{CF} .

In [15, 11] a computable version of the Riesz representation theorem is proved. In these articles the concept of bounded variation is generalized straightforwardly to the set BVC with representation δ_{BVC} of partial functions $g : \subseteq [0; 1] \rightarrow \mathbb{R}$ with countable dense domain containing $\{0, 1\}$ which are continuous on $\text{dom}(g) \setminus \{0, 1\}$. Remember that a function of bounded variation has at most countably many points of continuity. The integral $\int h dg$ for continuous h and an arbitrary function g of bounded variation is defined already by any restriction of g to a countable dense subset containing $\{0, 1\}$ [11]. Every (partial) function $g \in \text{BVC}$ can be extended uniquely to a normalized (total) function $\text{ext}(g) \in \text{BV}$ by $\text{ext}(g)(x) := \lim_{y \nearrow x, y \in \text{dom}(g)} g(y)$ for $x \notin \text{dom}(g)$. Then $\int h dg = \int h d \text{ext}(g)$ for all $h \in C[0; 1]$ and $\text{Var}(g) = \text{Var}(\text{ext}(g))$. In this article instead of δ_{BVC} we use the representation $\delta_V := \text{ext} \circ \delta_{BVC}$ of the normalized functions. The variation is not (δ_V, ρ) -computable but only $(\delta_V, \rho_<)$ -computable. Since for computations we will need the ρ -name of the variation we include it in the name. Notice that for computing the Riemann-Stieltjes integral $\int h dg$ a δ_V -name and an upper bound of $\text{Var}(g)$ suffice [15, 11].

Definition 3.2. Define representations δ_V and δ_{BV} of BV as follows:

- (1) $\delta_V(p) = g$ iff there are $p_0, q_0, p_1, q_1, \dots \in \Sigma^\omega$ such that $p = \langle \langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle, \dots \rangle$, $\rho(p_0) = \rho(q_0) = 0$, $\rho(p_1) = 1$, $g \circ \rho(p_i) = \rho(q_i)$ for all $i \in \mathbb{N}$, $A_p := \{\rho(p_i) \mid i \geq 2\}$ is a dense subset of $(0; 1)$ and g is continuous on A_p .
- (2) $\delta_{BV}\langle p, q \rangle = g : \iff \delta_V(p) = g \text{ and } \rho(q) = \text{Var}(g)$.

A computable version of the Riesz representation theorem and its converse have been proved in [15, 11]. The results can be formulated as follows.

Theorem 3.3 (Computable Riesz representation [15, 11]).

The function $(F, z) \mapsto g$ mapping every functional $F \in C'[0; 1]$ and its norm z to the (unique) function $g \in \text{BV}$ such that $F(h) = \int h dg$ (for all $h \in C[0; 1]$) is $([\delta_C \rightarrow \rho], \rho, \delta_V)$ -computable.

Theorem 3.4 ([15, 11]). *The operator $(g, l) \mapsto F$, mapping every $g \in \text{BV}$ and every $l \in \mathbb{N}$ with $\text{Var}(g) \leq 2^l$ to the functional F defined by $F(h) = \int h dg$ for all $h \in C[0; 1]$, is $(\delta_V, \nu_{\mathbb{N}}, [\delta_C \rightarrow \rho])$ -computable.*

By a slight generalization of the representation δ_m of the probability measures on the Borel sets of the unit interval defined and studied in [19] we obtain a representation of the bounded non-negative Borel measures on the unit interval. Let $\text{Int} := \{(a, b), [0; b), (a, 1], [0; 1] \mid a, b \in \mathbb{Q}, 0 \leq a < b \leq 1\}$ be the set of all rational open subintervals of $[0; 1]$.

Definition 3.5. Let BM_+ be the set of non-negative bounded measures.

- (1) Define a representation δ_m of the set BM_+ as follows. For $p, q \in \Sigma^\omega$ and $\mu \in \text{BM}_+$, $\delta_m \langle p, q \rangle = \mu$ iff $\rho(q) = \mu([0; 1])$ and p is (encodes) a list of all $(a, J) \in \mathbb{Q} \times \text{Int}$ such that $a < \mu(J)$.
- (2) Define a representation of BM by $\delta_{\text{BM}} \langle p, q, r \rangle = \mu$ iff $\mu = \delta_m(p) - \delta_m(q)$ and $\|\mu\|_m = \rho(r)$.

Roughly speaking by this definition, δ_m is the greatest (or "poorest") representation γ of the bounded non-negative measures such that $\mu([0; 1])$ can be computed and $a < \mu(J)$ ($a \in \mathbb{Q}$ and $J \in \text{Int}$) can be enumerated. By the next theorem the representation δ_m is the greatest representation of the non-negative bounded measures for which $\mu([0; 1])$ and integration of continuous functions are computable.

Theorem 3.6.

- (1) The function $\mu \mapsto \mu([0; 1])$ is (δ_m, ρ) -computable, and the function $(\mu, h) \mapsto \int h \, d\mu$ is $(\delta_m, \delta_C, \rho)$ -computable.
- (2) If for some representation γ of BM the function $\mu \mapsto \mu([0; 1])$ is (γ, ρ) -computable, and the function $(\mu, h) \mapsto \int h \, d\mu$ is (γ, δ_C, ρ) -computable then $\gamma \leq \delta_m$.

Proof.

- (1) The first statement is obvious, the second one can be derived easily from the special case for measures with $\mu([0; 1]) = 1$ [19, Theorem 3.6].
- (2) This can be deduced from [19, Theorem 4.2]. □

4. COMPUTABLE EQUIVALENCE OF THE THREE CONCEPTS AND COMPUTABLE JORDAN DECOMPOSITION

We will now apply the representations introduced in Section 3:

- δ_{CF} for the space of linear continuous functionals $F : C[0; 1] \rightarrow \mathbb{R}$ and $[\delta_C \rightarrow \rho]$ for the subset of non-negative ones,
- δ_{BV} for the set BV of (normalized) functions of bounded variations and δ_V for the subset of non-decreasing ones,
- δ_{BM} for the set of signed measures and δ_m for the subset of non-negative ones.

For all these representations the norm or the variation can be computed from the names. For the representations δ_{CF} , δ_{BV} and δ_{BM} it is included explicitly in the names, for the other representation norms can be computed from names: $\|F\| = F(\mathbf{1})$, $\text{Var}(g) = g(1)$, $\|\mu\|_m = \mu([0; 1])$.

Let $T_{\text{FV}}, T_{\text{VM}}$ and T_{MF} be the linear homeomorphisms from Theorem 2.3 and let $T_{\text{FV}}^+, T_{\text{VM}}^+$ and T_{MF}^+ be their restrictions to the spaces of non-negative or non-decreasing objects, respectively.

Theorem 4.1.

- (1) The operator T_{FV}^+ is $([\delta_C \rightarrow \rho], \delta_V)$ -computable.
- (2) The operator T_{VM}^+ is (δ_V, δ_m) -computable.
- (3) The operator T_{MF}^+ is $(\delta_m, [\delta_C \rightarrow \rho])$ -computable.

Proof.

- (1) If F is non-decreasing then $\|F\| = F(\mathbf{1})$. By Theorem 3.3, the restriction is $([\delta_C \rightarrow \rho], \delta_V)$ -computable.

- (2) Suppose, $\delta_V(p) = g$ is non-decreasing with dense set A_p (Definition 3.2). From the classical theory we know that for $0 \leq a < b \leq 1$ the measure $\mu := T_{VM}^+(g)$ satisfies

$$\begin{aligned}\mu([0; b]) &= \sup_{b' < b} g(b') \\ \mu((a; b)) &= \sup_{a < a' < b' < b} (g(b') - g(a'))\end{aligned}$$

and

$$\mu((a; 1]) = \sup_{a < a'} (g(1) - g(a')).$$

Since $\lim_{y \nearrow x} g(y)$ and $\lim_{y \searrow x} g(y)$ exist for all $0 < x < 1$ it suffices to choose a' and b' from the dense set A_p . Therefore,

$$\begin{aligned}\mu([0; b]) &= \sup\{g(b') \mid b' < b, b' \in A_p\}, \\ \mu((a; b)) &= \sup\{g(b') - g(a') \mid a < a' < b' < b, a', b' \in A_p\}, \\ \mu((a; 1]) &= \sup\{g(1) - g(a') \mid a < a', a' \in A_p\}.\end{aligned}$$

The name p of g contains a list of all $((\rho, \rho)$ -names of) $(x, g(x))$ with $x \in A_p$. Since $x < y$ is r.e., for rational numbers $a < b$ we can compute a list of all $d \in \mathbb{Q}$ such that $d < g(b') - g(a')$ for some $a', b' \in A_p$ and $a < a' < b' < b$, which is a list of all $d \in \mathbb{Q}$ such that $d < \mu((a; b))$. Correspondingly, for a rational number $b > 0$ we can compute a list of all $d \in \mathbb{Q}$ such that $d < \mu([0; b])$ and for a rational number $a < 1$ we can compute a list of all $d \in \mathbb{Q}$ such that $d < \mu((a; 1])$. Combining these enumerations for p we can enumerate a list of $(d, J) \in \mathbb{Q} \times \text{Int}$ (Int is defined before Definition 3.5) such that $d < \mu(J)$. Furthermore, from $p = \langle \langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle, \dots \rangle$ we can compute $\mu([0; 1]) = g(1) = \rho(q_1)$. Therefore, we can compute a δ_m -name of the measure μ .

- (3) This follows from Theorem 3.6(1). □

By the following theorems the linear homeomorphisms T_{FV} , T_{VM} and T_{MF} from Theorem 2.3 are computable, and Jordan decomposition on the three spaces is computable.

Theorem 4.2.

- (1) *The operator $T_{FV} : C'[0; 1] \rightarrow \text{BV}$ mapping functionals to functions of bounded variation is $(\delta_{CF}, \delta_{BV})$ -computable.*
- (2) *The operator $T_{VM} : \text{BV} \rightarrow \text{BM}$ mapping functions of bounded variation to signed measures is $(\delta_{BV}, \delta_{BM})$ -computable.*
- (3) *The operator $T_{MF} : \text{BM} \rightarrow C'[0; 1]$ mapping signed measures to functionals is $(\delta_{BM}, \delta_{CF})$ -computable.*

Theorem 4.3 (Computable Jordan decomposition).

- (1) *Jordan decomposition $F \mapsto (F^+, F^-)$ on $C'[0; 1]$ is $(\delta_{CF}, [[\delta_C \rightarrow \rho,], [\delta_C \rightarrow \rho]])$ -computable.
Its inverse is $([[\delta_C \rightarrow \rho,], [\delta_C \rightarrow \rho]], \delta_{CF})$ -computable.*
- (2) *Jordan decomposition $g \mapsto (g^+, g^-)$ on BV is $(\delta_{BV}, [\delta_V, \delta_V])$ -computable.
Its inverse is $([\delta_V, \delta_V], \delta_{BV})$ -computable.*
- (3) *Jordan decomposition $\mu \mapsto (\mu^+, \mu^-)$ on BM is $(\delta_{BM}, [\delta_m, \delta_m])$ -computable.
Its inverse is $([\delta_m, \delta_m], \delta_{BM})$ -computable.*

Since $f \mapsto \|f\|$ for non-negative continuous f is $([\delta_C \rightarrow \rho], \rho)$ -computable in (1) of the theorem $[\delta_C \rightarrow \rho]$ can be replaced by δ_{CF} . Correspondingly, in (2) of the theorem δ_V can be replaced by δ_{BV} and in (3) of the theorem δ_m can be replaced by δ_{BM} .

Proof. This is a merged proof of Theorems 4.2 and 4.3. Almost all statements follow easily from what has already been proved. The only non-trivial part is the proof for the Jordan decomposition $g \mapsto (g^+, g^-)$. In the following (F^+, F^-) , (g^+, g^-) and (μ^+, μ^-) will denote Jordan decompositions. By Theorem 4.1 and Corollary 2.6,

$$(F^+, F^-) \mapsto (g^+, g^-) \mapsto (\mu^+, \mu^-) \mapsto (F^+, F^-) \text{ are computable} \quad (4.1)$$

w.r.t the representations $[\delta_C \rightarrow \rho]$, δ_V and δ_m .

$F \mapsto g$ (Theorem 4.2(1)) This follows immediately from Theorem 3.3.

$g \mapsto (g^+, g^-)$ (first part of Theorem 4.3(2)) Let $g \in \text{BV}$ with Jordan decomposition (g^+, g^-) . From the classical theory we know $g^+ = (/g/ + g)/2$ and $g^- = (/g/ - g)/2$ where $/g/ \in \text{BV}$ is the (non-decreasing) total variation function of g (see Section 2 after (2.2)).

Suppose $\delta_{\text{BV}}(\langle p, q \rangle) = g$. Let A_p be the dense set from Definition 3.2. The functions $/g/$, g^+ and g^- are determined uniquely by their restrictions to the dense subset $A_p \cup \{0, 1\}$, hence it suffices to find $/g/(x)$, $g^+(x)$ and $g^-(x)$ for all $x \in A_p \cup \{0, 1\}$.

Call a partition $Z = (a = x_0 < x_1 < \dots < x_n = b)$ of $[a; b]$ a partition “from A_p ”, if $\{x_0, \dots, x_n\} \subseteq \{0, 1\} \cup A_p$. Suppose $x \in A_p$.

Since g is left-continuous by Lemma 2.2, and A_p is dense, for every partition Z of $[0; x]$ and every ε there is some partition Z' of $[0; x]$ from A_p , such that $|S(g, Z) - S(g, Z')| < \varepsilon$. Therefore,

$$V_0^x(g) = \sup\{S(g, Z) \mid Z \text{ is a partition of } [0; x] \text{ from } A_p\}$$

By Definition 3.2, p can be written as $p = \langle \langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle, \dots \rangle$ such that $\rho(p_0) = \rho(q_0) = 0$, $\rho(p_1) = 1$ and $g \circ \rho(p_k) = \rho(q_k)$ for all $k \in \mathbb{N}$.

Let $x_k := \rho(p_k)$ and $y_k := \rho(q_k) = g(x_k)$. We want to compute a sequence $t := \langle \langle p_0, r_0 \rangle, \langle p_1, r_1 \rangle, \dots \rangle$ such that $\rho(r_k) = /g/(x_k) = V_0^{x_k}(g)$. Since $/g/$ is continuous in x if g is continuous in x , then $\delta_V(t) = /g/$.

Since $/g/(0) = 0$ we can choose $r_0 := q_0$. Since $/g/(1) = \text{Var}(g) = \rho(q)$ (remember that $\delta_{\text{BV}}(\langle p, q \rangle) = g$) we can choose $r_1 := q$.

For $k \geq 2$ let $\Pi(k)$ be the set of all sequences $\sigma = (i_0, i_1, \dots, i_m)$ such that $i_0 = 0$ $i_m = k$ and $x_{i_0} < x_{i_1} < \dots < x_{i_m}$. For $\sigma = (i_0, i_1, \dots, i_m)$ let P_σ be the partition $(x_{i_0}, x_{i_1}, \dots, x_{i_m})$ of $[0, x_k]$ from A_p . Then $V_0^{x_k}(g) = \sup_{\sigma \in \Pi(k)} S(g, P_\sigma)$.

Since the relation $x < y$ for real numbers is (ρ, ρ) -enumerable, from p and k the set $\Pi(k)$ can be enumerated, $\Pi(k) = (\sigma_0, \sigma_1, \dots)$. Since $S(g, P_\sigma)$ can be computed from p and σ (2.1), a $\rho_{<}$ -name of $V_0^{x_k}(g) = \sup_k S(g, P_{\sigma_k})$ can be computed from p and k . Correspondingly, a $\rho_{<}$ -name of $V_{x_k}^1(g) = \sup\{S(g, Z) \mid Z \text{ is a partition of } [x_k; 1] \text{ from } A_p\}$ can be computed from p and k .

Since $V_0^x(g) + V_x^1(g) = V_0^1(g) = \text{Var}(g)$ and ρ -name of $\text{Var}(g)$ is given as an input, from p , q with $\rho(q) = \text{Var}(g)$ and k a ρ -name r_k of $/g/(x_k)$ can be computed. Therefore, some computable function $G : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ maps every δ_{BV} -name $\langle p, q \rangle$ of g where $\delta_V(p) = g$ and $p = \langle \langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle, \dots \rangle$ to some δ_V -name $t = \langle \langle p_0, r_0 \rangle, \langle p_1, r_1 \rangle, \dots \rangle$ of $/g/$.

On these names, $g^+ = (/g/ + g)/2$ and $g^- = (/g/ - g)/2$ can be computed: there are computable functions s, d on Σ^ω such that $(\rho(p) + \rho(q))/2 = \rho \circ s(p, q)$ and $(\rho(p) - \rho(q))/2 = \rho \circ d(p, q)$. Then $t^+ := \langle \langle p_0, s(q_0, r_0) \rangle, \langle p_1, s(q_1, r_1) \rangle, \dots \rangle$ is a δ_V -name of g^+ and

$t^- := \langle \langle p_0, d(q_0, r_0) \rangle, \langle p_1, d(q_1, r_1) \rangle, \dots \rangle$ is a δ_V -name of g^- . In summary, t^+ and t^- , hence $\langle t^+, t^- \rangle$ can be computed from $\langle p, q \rangle$. Therefore, Jordan decomposition $g \mapsto (g^+, g^-)$ on BV is $(\delta_{BV}, [\delta_V, \delta_V])$ -computable.

$(\mu^+, \mu^-) \mapsto \mu$ (second part of Theorem 4.3(3)) By Theorem 2.5, from δ_m -names of a Jordan decomposition (μ^+, μ^-) we can compute a δ_{BM} -name of μ .

$g \mapsto \mu$ (Theorem 4.2(2)) Compute as follows: $g \mapsto (g^+, g^-) \mapsto (\mu^+, \mu^-) \mapsto \mu$.

$\mu \mapsto F$ (Theorem 4.2(3)) Suppose $\delta_{BM}\langle p, q, r \rangle = \mu$, hence $\mu = \mu^+ - \mu^-$ where $\mu^+ = \delta_m(p)$ and $\mu^- = \delta_m(q)$ and $\|\mu\|_m = \rho(r)$. By Theorem 4.1(3) we can compute $[\delta_C \rightarrow \rho]$ -names of functionals $G^+ := T_{MF}(\mu^+)$ and $G^- := T_{MF}(\mu^-)$ such that $F := T_{MF}(\mu) = G^+ - G^-$. By a standard argument we can compute a $[\delta_C \rightarrow \rho]$ -name of F . Since $\|F\| = \|\mu\|_m$ by Theorem 2.3, we can compute a δ_{CF} -name of F .

$(g^+, g^-) \mapsto g$ (second part of Theorem 4.3(2)) Compute as follows: $(g^+, g^-) \mapsto (\mu^+, \mu^-) \mapsto \mu \mapsto F \mapsto g$.

$\mu \mapsto (\mu^+, \mu^-)$ (first part of Theorem 4.3(3)) Compute as follows: $\mu \mapsto F \mapsto g \mapsto (g^+, g^-) \mapsto (\mu^+, \mu^-)$.

$F \mapsto (F^+, F^-)$ (first part of Theorem 4.3(1)) Compute as follows: $F \mapsto g \mapsto (g^+, g^-) \mapsto (F^+, F^-)$.

$(F^+, F^-) \mapsto F$ (second part of Theorem 4.3(1)) Compute as follows: $(F^+, F^-) \mapsto (g^+, g^-) \mapsto g \mapsto \mu \mapsto F$. \square

Corollary 4.4.

- (1) The inverses $(T_{FV}^+)^{-1}$, $(T_{VM}^+)^{-1}$ and $(T_{MF}^+)^{-1}$ are computable.
- (2) The inverses T_{FV}^{-1} , T_{VM}^{-1} and T_{MF}^{-1} are computable.
- (3) $\delta_{CF} \equiv T_{FV}^{-1} \circ \delta_{BV} \equiv T_{MF} \circ \delta_{BM}$ (accordingly for δ_{BV} and δ_{BM}).

Proof.

- (1) For all $F \in C'[0; 1]$, $T_{MF}^+ \circ T_{VM}^+ \circ T_{FV}^+(F) = F$, hence $T_{MF}^+ \circ T_{VM}^+ = (T_{FV}^+)^{-1}$ which is computable by Theorem 4.1. The other statements are proved accordingly.
- (2) As above, but with Theorem 4.2.
- (3) Straightforward by 2. and Theorem 4.2. \square

We introduce further representations of our spaces by differences of functions:

- $\gamma_F\langle p, q, r \rangle = F$ iff $F = [\delta_C \rightarrow \rho](p) - [\delta_C \rightarrow \rho](q)$ and $\|F\| = \rho(r)$,
- $\gamma_V\langle p, q, r \rangle = g$ iff $g = \delta_V(p) - \delta_V(q)$ and $\text{Var}(g) = \rho(r)$,
- $\gamma_{FJ}\langle p, q \rangle = F$ iff $([\delta_C \rightarrow \rho](p), [\delta_C \rightarrow \rho](q))$ is the Jordan decomposition of F ,
- $\gamma_{VJ}\langle p, q \rangle = g$ iff $(\delta_V(p), \delta_V(q))$ is the Jordan decomposition of g ,
- $\gamma_{MJ}\langle p, q \rangle = \mu$ iff $(\delta_m(p), \delta_m(q))$ is the Jordan decomposition of μ .

Corollary 4.5. $\delta_{CF} \equiv \gamma_F \equiv \gamma_{FJ}$, $\delta_{BV} \equiv \gamma_V \equiv \gamma_{VJ}$, $\delta_{BM} \equiv \gamma_{MJ}$.

Proof. Straightforward by Theorems 4.2 and 4.3. \square

Notice that for each of these representations a name of a functional F contains a name of $\|F\|$ or allows to compute it easily. On the Banach spaces $(C'[0; 1], \|\cdot\|)$, (BV, Var) and $(\text{BM}, \|\cdot\|_m)$ with representations δ_{CF} , δ_{BV} and δ_{BM} , respectively, addition is not computable since the norm of the sum cannot be computed. Adding the norm in a representation of the dual space is discussed in [3, 4, 6]. But for non-negative functionals F :

Corollary 4.6. *The sum*

- (1) *of non-negative functionals from $C'[0; 1]$ is computable w.r.t. $[\delta_C \rightarrow \rho]$,*
- (2) *of non-decreasing functions from BV is computable w.r.t δ_V ,*
- (3) *of non-negative bounded measures from BM is computable w.r.t δ_m .*

Proof.

- (1) Straightforward [20, Theorem 6.2.1].
- (2) This follows from [19, Theorem 3.1].
- (3) Since for non-decreasing g_1, g_2 , $g_1 + g_2 = (T_{\text{VM}}^+)^{-1}(T_{\text{VM}}^+(g_1) + T_{\text{VM}}^+(g_2))$, by 3. of this corollary, Theorem 4.1 and Corollary 4.4 the sum on non-decreasing functions is computable w.r.t. δ_V . \square

For functions of bounded variation there is no simple proof since for $g_1 = \delta_V(p_1)$ and $g_2 = \delta_V(g_2)$ in general $A_{p_1} \neq A_{p_2}$ (see Definition 3.2).

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