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SMART CHOICES AND THE SELECTION MONAD

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Abstract. Describing systems in terms of choices and their resulting costs and rewards offers the promise of freeing algorithm designers and programmers from specifying how those choices should be made; in implementations, the choices can be realized by optimization techniques and, increasingly, by machine-learning methods. We study this approach from a programming-language perspective. We define two small languages that support decision-making abstractions: one with choices and rewards, and the other additionally with probabilities. We give both operational and denotational semantics.

In the case of the second language we consider three denotational semantics, with varying degrees of correlation between possible program values and expected rewards. The operational semantics combine the usual semantics of standard constructs with optimization over spaces of possible execution strategies. The denotational semantics, which are compositional, rely on the selection monad, to handle choice, augmented with an auxiliary monad to handle other effects, such as rewards or probability.

We establish adequacy theorems that the two semantics coincide in all cases. We also prove full abstraction at base types, with varying notions of observation in the probabilistic case corresponding to the various degrees of correlation. We present axioms for choice combined with rewards and probability, establishing completeness at base types for the case of rewards without probability.

Contents

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1. Introduction

Models and techniques for decision-making, such as Markov Decision Processes (MDPs) and Reinforcement Learning (RL), enable the description of systems in terms of choices and of the resulting costs and rewards. For example, an agent that plays a board game may be defined by its choices in moving pieces and by how many points these yield in the game. An implementation of such a system may aim to make the choices following a strategy that results in attractive costs and rewards, perhaps the best ones. For this purpose it may rely on classic optimization techniques or, increasingly, on forms of machine-learning (ML). Deep RL has been particularly prominent in the last decade, but contextual bandits and ordinary supervised learning can also be useful.

In a programming context, several languages and libraries support choices, rewards, costs, and related notions in a general way (not specific to any application, such as a particular board game). McCarthy's `amb` operator [McC63] may be seen as an early example of a construct for making choices. More recent work includes many libraries for RL (e.g., [BHQK20]), languages for planning such as DTGolog [BRST00] and some descendants (e.g., [S^+10]) of the Planning Domain Definition Language [MGH^+98], a "credit-assignment" compiler for learning to search built on the Vowpal-Rabbit learning library [CHR^+16], and Dyna [VFLF^+17], a programming language for machine-learning applications based on MDPs. It also includes SmartChoices [CCD^+18], an "approach to making machine-learning (ML) a first class citizen in programming languages", one of the main inspirations for our work. SmartChoices and several other recent industry projects in this space (such as Spiral [Byc18]) extend mainstream programming languages and systems with the ability to make data-driven decisions by coding in terms of choices (or predictions) and feedback (in other words, perceived costs or rewards), and thus aim to have widespread impact on programming practice.

The use of decision-making abstractions has the potential to free algorithm designers and programmers from taking care of many details. For example, in an ordinary programming system, a programmer that implements quicksort should consider how to pick pivot elements and when to fall back to a simpler sorting algorithm for short inputs. Heuristic solutions to such questions abound, but they are not always optimal, and they require coding and sometimes maintenance when the characteristics of the input data or the implementation platform change. In contrast, SmartChoices enables the programmer to code in terms of choices and costs—or, equivalently, rewards, which we define as the opposite of costs—, and to let the implementation of decision-making take care of the details [CCD+18]. As another example, consider the program in Figure 1 that does binary search in a sorted array. This

```
let binsearch(x : Int, a : Array[Int], l : Int, r : Int) =
 if l > r then None  //the special value None represents failure
          else choose m:[l,r]  in  //choose an integer in [l,r]
                   if a[m] = x then m
                   else cost(1);  //pay to recurse
                     if a[m] < x then binsearch(x, a, m+1, r)
                     else binsearch(x, a, l, m-1)
```

Figure 1: Smart binary search

pseudocode is a simplified version of the one in [CCD+18, Section 4.2], which also includes a way of recording observations of the context of choices (in this example, `x`, `a[l]`, and `a[r]`) that facilitate machine-learning. Here, a choice determines the index `m` where the array is split. Behind the scenes, a clever implementation can take into account the distribution of the data in order to decide exactly how to select `m`. For example, if `x` is half way between `a[l]` and `a[r]` but the distribution of the values in the array favors smaller values, then the selected `m` may be closer to `r` than to `l`. In order to inform the implementation, the programmer calls `cost`: each call to `cost` adds to the total cost of an execution, for the notion of cost that the programmer would wish to minimize. In this example, the total cost is the number of recursive calls. In other examples, the total cost could correspond, for instance, to memory requirements or to some application-specific metric such as the number of points in a game.

In this paper, which is a full version of [AP21], we study decision-making abstractions from a programming-language perspective. We define two small languages that support such abstractions, one with choices and rewards, and the other one additionally with probabilities. In the spirit of SmartChoices (and in contrast with DTGolog and Dyna, for instance), the languages are mostly mainstream: only the decision-making abstractions are special. We give them both operational and denotational semantics. In the case of the language with probabilities we provide three denotational semantics, modeling varying degrees of correlation between possible program values and expected rewards.

Their operational semantics combine the usual semantics of standard constructs with optimization over possible strategies (thinking of programs as providing one-person games). Despite the global character of optimization, our results include a tractable, more local formulation of their operational semantics (Theorems 4.3 and 5.3). Their denotational semantics are based on the selection monad [EO10, EOP11, EO11, EO12, Esc15, Hed15, EO17, BHZ18], which we explain below.

We establish that operational and denotational semantics coincide, proving adequacy results for both languages (Theorems 4.6 and 5.10). We also investigate questions of full abstraction (at base types) and program equivalences. Our full abstraction results (particularly Theorems 4.10 and 5.14, and Corollary 5.15) provide further evidence of the match between denotational and operational semantics. We prove full abstraction results at base types for each of our denotational semantics, in each case with respect to appropriate notions of observation. Program equivalences can justify program transformations, and we develop proof systems for them. For example, one of our axioms concerns the commutation of choices and rewards. In particular, in the case of the language for rewards we establish (Theorem 4.10) the soundness and completeness of our proof system with respect to concepts of observational equivalence and semantic equivalence (at base types). In the case of the language with probabilities, finding such completeness results is an open problem. However, we show that our proof systems are complete with respect to proving effect-freeness. For the language without probabilities this holds in all circumstances (Corollary 4.11); for the language with probability it holds under reasonable assumptions (Theorem 5.21).

A brief, informal discussion of the semantics of `binsearch` may provide some intuition on the two semantics and on the role of the selection monad.

- If we are given the sequence of values picked by the `choice` construct in an execution of `binsearch`, a standard operational semantics straightforwardly allows us to construct the rest of the execution. We call this semantics the *ordinary operational semantics*. For each such sequence of values, the ordinary operational semantics implies a resulting total cost, and thus a resulting total reward. We define the *selection operational semantics* by requiring that the sequence of values be the one that maximizes this total reward.

  Although they are rather elementary, these operational semantics are not always a convenient basis for reasoning, because (as usual for operational semantics) they are not compositional, and in addition the selection operational semantics is defined in terms of sequences of choices and accumulated rewards in multiple executions. On the other hand, the chosen values are simply plain integers.
- In contrast, in the denotational semantics, we look at each choice of `binsearch` as being made locally, without implicit reference to the rest of the execution or other executions, by a higher-order function of type $(\mathtt{Int} \to \mathrm{R}) \to \mathtt{Int}$ (where `Int` is a finite set of machine integers), whose expected argument is a reward function f that maps each possible value of the choice to the corresponding reward of type R of the program. We may view f as a reward continuation. One possible such higher-order function is the function argmax that picks a value for the argument x for f yielding the largest reward $f(x)$. (There are different versions of argmax, in particular with different ways of breaking ties, but informally one often identifies them all.)

  The type $(\mathtt{Int} \to \mathrm{R}) \to \mathtt{Int}$ of this example is a simple instance of the selection monad, $\mathrm{S}(X) = (X \to \mathrm{R}) \to X$, where X is any type, and argmax is an example of a selection function. More generally, we use $\mathrm{S}(X) = (X \to \mathrm{R}) \to \mathrm{T}(X)$, where T is another, auxiliary, monad, which can be used to model other computational effects, for example, as we do here, rewards and probabilities. For our language with rewards, we employ the writer monad $\mathrm{T}(X) = \mathrm{R} \times X$. For our language with rewards and probabilities we employ three auxiliary monads modeling the various correlations between final values and rewards. Of these, the simplest is $\mathrm{T}(X) = \mathcal{D}_{\mathrm{f}}(\mathrm{R} \times X)$, the combination of the finite probability distribution monad with the writer monad.

  The monadic approach leads to a denotational semantics that is entirely compositional, and therefore facilitates proofs of program equivalences of the kind mentioned above. The denotational semantics may be viewed as an implementation by translation to a language in which there are no primitives for decision-making, and instead one may program with selection functions.

Sections 2 and 3 concern supporting theory for our two decision-making languages. In Section 2, we review the selection monad, with and without an auxiliary monad, and investigate its algebraic operations. We show how algebraic operations for the selection monad with an auxiliary monad can be obtained from algebraic operations of the auxiliary monad (Equation 2.8); we give a general notion of selection operations (Equation 2.11) and characterize them in terms of generic effects for the selection monad $\mathrm{S}(X)$; and we investigate the equations obeyed by binary selection operations (Theorems 2.6 and 2.7). In Section 3, we present a general language with algebraic operations, give a general adequacy theorem (Theorem 3.6), and briefly discuss a calculus for program equivalences. This section is an adaptation of prior work (see [PP01]). While useful for our project, it is not specific to it.

In Section 4, we define and study our first language with decision-making abstractions; it is a simply typed, higher-order λ-calculus, extended with a binary choice operation $-$ `or` $-$ and a construct for adding rewards. Full abstraction for this language is defined in terms of observing both final values and the corresponding rewards obtained. Theorem 4.15 shows that this notion does not change if we observe only the final value; in contrast Corollary 4.16 shows that it does change if we observe only the final reward: in that case we cannot distinguish programs with different final values but the same optimal final reward.

In Section 5, we proceed to our second language, which adds probabilistic choice to the first. Regarding full abstraction, Theorem 5.17 (an analogue of Theorem 4.15) shows that this notion does not change from that associated to our third semantics for probability and rewards if we observe only the distribution of final values. Probabilistic choices are not subject to optimization, but, combined with binary choice, they enable us to imitate the choice capabilities of MDPs. Unlike MDPs, the language does not support infinite computations. We conjecture they can be treated via a metric approach to semantics; at any rate, there is no difficulty in adding a primitive recursion operator to the language without changing the selection monads, permitting MDP runs of arbitrary prescribed lengths.

In sum, we regard the main contributions of this paper as being (1): the connection between programming languages with decision-making abstractions and the selection monad, and (2): the definition and study of operational and denotational semantics for those languages, and the establishment of adequacy and full abstraction theorems for them. The adequacy theorems show that global operationally-defined optimizations can be characterized compositionally using a semantics based on the selection monad.

As described above, the selection operational semantics and the denotational semantics with the argmax selection function both rely on maximizing rewards. In many cases, optimal solutions are expensive. Even in the case of `binsearch`, an optimal solution that, without ever recursing, immediately picks `m` such that `a[m]` equals `x` seems unrealistic. For efficiency, the optimization may be approximate and data-driven. In particular, as an alternative to the use of maximization in the selection operational semantics, we may sometimes be able to make the choices with contextual-bandit techniques, as in [CCD+18, Section 4.2]. In the denotational semantics, with R the type of real numbers, we may use other selection functions than argmax. (Using argmax is convenient, but our approach does not require it.) For example, instead of computing $\mathrm{argmax}(f)$, we may approximate f by a differentiable function over the real numbers, represented by a neural network with learned parameters, and then find a local maximum of this approximation by gradient ascent. We have explored such approximations only informally so far; Section 6 briefly mentions aspects of this and other subjects for further work.

2. The selection monad and algebraic operations

In this section we present material on the basic selection monad, on the selection monad augmented with an auxiliary monad, and on generic effects and algebraic operations for general monads. This material includes a discussion of generic effects and algebraic operations for the selection monad (whether basic or augmented) and of the equations these operations satisfy. Such algebraic operations are either so-called selection operations arising from the basic selection monad or operations arising from the auxiliary monads and then lifted to the augmented selection monad. For a first reading, it suffices to read the definitions of the selection monads and of generic effects and algebraic operations for general monads. (We repeat the definitions of the specific generic effects and algebraic operations for our two languages when discussing their denotational semantics in Sections 4.4 and 5.4.)

2.1. The selection monad.

The selection monad

$$\mathrm{S}(X) = (X \to \mathrm{R}) \to X$$

introduced in [EO10], is a strong monad available in any cartesian closed category, for simplicity discussed here only in the category of sets. One can think of the $F \in \mathrm{S}(X)$ as *selection functions* which, viewing R as a *reward* type, choose an element $x \in X$, given a *reward function* $\gamma : X \to \mathrm{R}$. In a typical example, the choice x optimizes, perhaps maximizing, the reward $\gamma(x)$. Computationally, we may understand $F \in \mathrm{S}(X)$ as producing x given a *reward continuation* γ, a function giving the reward of the remainder of the computation.

The selection monad has strong connections to logic, similar to those of the continuation monad $\mathrm{K}(X) = (X \to \mathrm{R}) \to \mathrm{R}$. For example, as explained in [EO12], whereas logic translations using K, taking R to be $\bot$, verify the double-negation law $\neg\neg P \supset P$, translations using S verify the instance $((P \supset \mathrm{R}) \supset P) \supset P$ of Peirce's law. Again, with R the truth values, elements of $\mathrm{K}(X)$ correspond to quantifiers, and elements of $\mathrm{S}(X)$ correspond to selection operators, such as Hilbert's ε-operator.

The selection monad has unit $(\eta_{\mathrm{S}})_X : X \to \mathrm{S}(X)$, where $\eta_{\mathrm{S}}(x) = \lambda\gamma \in X \to \mathrm{R}.\, x$. (Here, and below, we may drop subscripts when they are evident from the context.) The Kleisli extension is a little involved, so we explain it in stages. First, for any $F \in \mathrm{S}(X)$ and reward continuation $\gamma : Y \to \mathrm{R}$ we write $\mathbf{R}(F|\gamma)$ for the reward given by the (possibly optimal) $x \in X$ chosen by F, i.e.:

$$\mathbf{R}(F|\gamma) =_{\mathrm{def}} \gamma(F\gamma)$$

(Here, and below, we may omit function application parentheses to improve readability.) For the Kleisli extension, given $f : X \to \mathrm{S}(Y)$ we need a function $f^{\dagger_{\mathrm{S}}} : \mathrm{S}(X) \to \mathrm{S}(Y)$. Equivalently, using f, we need to pick an element of Y, given a computation $F \in \mathrm{S}(X)$ and a reward continuation $\gamma : Y \to \mathrm{R}$. We do so as follows:

- For a given $x \in X$, the reward associated to the possibly optimal element of Y picked by $f(x)$ is $\mathbf{R}(f(x)|\gamma)$.
- Thus we have a reward function from X, viz. $\mathit{rew} = \lambda x \in X.\, \mathbf{R}(f(x)|\gamma)$.
- Using this reward function as the reward continuation of F, we can use F to choose the (possibly optimal) element of X for it, viz. $\mathit{opt} = F(\mathit{rew})$.
- Now that we know the best choice of x, we use it to get the desired element of Y, viz. $f(\mathit{opt})(\gamma)$.

Intuitively, F chooses the $x \in X$ which gives the optimal $y \in Y$, and then f uses that x.

Writing all this out, we find:

$$\begin{array}{rcl} f^{\dagger_{\mathrm{S}}} F\gamma & = & f(opt)(\gamma) \\ & = & f(F(rew))(\gamma) \\ & = & fF(\lambda x \in X.\, \mathbf{R}(f(x)|\gamma))\gamma \end{array}$$

The selection monad has strength $(\mathrm{st}_{\mathrm{S}})_{X,Y} : X \times \mathrm{S}(Y) \to \mathrm{S}(X \times Y)$ where:

$$(\mathrm{st}_{\mathrm{S}})_{X,Y}(x, F) = \lambda\gamma \in X \times Y \to \mathrm{R}.\, \langle x, F(\lambda y \in Y.\, \gamma(x, y))\rangle$$

There is a generalization of this basic selection monad obtained by augmenting it with a strong *auxiliary* monad T. This generalization proves useful when combining additional effects with selection. Suppose that R is a T-algebra with algebra map $\alpha_{\mathrm{T}} : \mathrm{T}(\mathrm{R}) \to \mathrm{R}$. Then, as essentially proved in [EO17] for any cartesian closed category, we can define a strong monad S_{T} (which may just be written S, when T is understood) by setting:

$$\mathrm{S}_{\mathrm{T}}(X) = (X \to \mathrm{R}) \to \mathrm{T}(X)$$

It has unit $(\eta_{\mathrm{S}_{\mathrm{T}}})_X : X \to \mathrm{S}_{\mathrm{T}}(X)$ where $(\eta_{\mathrm{S}_{\mathrm{T}}})_X(x) = \lambda\gamma \in X \to \mathrm{R}.\, (\eta_{\mathrm{T}})(x)$. The Kleisli extension $f^{\dagger_{\mathrm{S}_{\mathrm{T}}}} : \mathrm{S}_{\mathrm{T}}(X) \to \mathrm{S}_{\mathrm{T}}(Y)$ of a function $f : X \to \mathrm{S}_{\mathrm{T}}(Y)$ is given, analogously to the above. First, for $F \in \mathrm{S}_{\mathrm{T}}(X)$ and $\gamma : X \to \mathrm{R}$, generalizing that for S, we define the reward associated to the T-computation selected by F using γ by:

$$\mathbf{R}(F|\gamma) = (\alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma))(F(\gamma)) \tag{2.1}$$

Then the Kleisli extension function $f^{\dagger_{\mathrm{S}_{\mathrm{T}}}} : \mathrm{S}_{\mathrm{T}}(X) \to \mathrm{S}_{\mathrm{T}}(Y)$ of a given $f : X \to \mathrm{S}_{\mathrm{T}}(Y)$ is:

$$f^{\dagger_{\mathrm{S}_{\mathrm{T}}}} F\gamma \;\;=_{\mathrm{def}}\;\; (\lambda x \in X.\, f(x)(\gamma))^{\dagger_{\mathrm{T}}}(F(\lambda x \in X.\, \mathbf{R}(fx|\gamma))) \qquad (F \in \mathrm{S}_{\mathrm{T}}(X), \gamma : Y \to \mathrm{R}) \tag{2.2}$$

As an example, suppose that R is a commutative monoid, and T is the writer monad $\mathrm{W}(X) = \mathrm{R} \times X$. Using the monoid operation, we can set $\alpha_{\mathrm{W}}(r, s) = r + s$. We then have:

$$\mathbf{R}(F|\gamma) = \pi_1(F(\gamma)) + \gamma(\pi_2(F(\gamma)))$$

For $(\mu_{\mathrm{T}})_X = \mathrm{id}^{\dagger_{\mathrm{S}_{\mathrm{T}}}}_{\mathrm{S}_{\mathrm{T}}(X)}$ we find:

$$(\mu_{\mathrm{T}})_X F\gamma \;\;=\;\; (\lambda d \in \mathrm{S}_{\mathrm{T}}(X).\, d\gamma)^{\dagger_{\mathrm{T}}}(F(\lambda d \in \mathrm{S}_{\mathrm{T}}(X).\, \mathbf{R}(d|\gamma))) \tag{2.3}$$

The selection monad has strength $(\mathrm{st}_{\mathrm{S}_{\mathrm{T}}})_{X,Y} : X \times \mathrm{S}_{\mathrm{T}}(Y) \to \mathrm{S}_{\mathrm{T}}(X \times Y)$ where:

$$(\mathrm{st}_{\mathrm{S}_{\mathrm{T}}})_{X,Y}(x, F) = \lambda\gamma \in X \times Y \to \mathrm{R}.\, (\mathrm{st}_{\mathrm{T}})_{X,Y}(x, F(\lambda y \in Y.\, \gamma(x, y)))$$

We remark that if T is the free algebra monad for an equational theory Th, the categories of T-algebras and of models of Th (i.e., algebras satisfying the equations) are equivalent. In particular the T algebra $\alpha_A : \mathrm{T}(A) \to A$ corresponding to a model A is the homomorphism $\mathrm{id}_A^{\dagger_{\mathrm{T}}}$, and $h : A \to B$ is a homomorphism between models of the theory iff it is a T-algebra morphism (from α_A to α_B). Note that the Kleisli extension of a map $f : X \to A$ to a model A is the same as the Kleisli extension of f regarded as a map to a T-algebra.

We can define reward functions for general monads M equipped with an M-algebra $\alpha_M : \mathrm{M}(\mathrm{R}) \to \mathrm{R}$. For $u \in \mathrm{M}(X)$ and $\gamma : X \to \mathrm{R}$, set:

$$\mathbf{R}_{\mathrm{M}_X}(u|\gamma) = (\alpha_{\mathrm{M}} \circ \mathrm{M}(\gamma))(u) \quad (= \gamma^{\dagger_{\mathrm{M}}}(u)) \tag{2.4}$$

Note that, for $x \in X$, $\mathbf{R}_{\mathrm{M}}(\eta_{\mathrm{M}}(x)|\gamma) = \gamma(x)$ and that $\mathbf{R}_{\mathrm{M}}(-|\gamma)$ is an M-algebra morphism. Further, in case M is the free algebra monad for an equational theory, and α_M is the M-algebra corresponding to a model on R, $\mathbf{R}_{\mathrm{M}}(-|\gamma)$ is a homomorphism.

We remark (see [Kel80, KP93, HLPP07a]) that, using the reward function, one can define a morphism θ_{M} from M to the continuation monad, by setting

$$\theta_{\mathrm{M}}(u) = \lambda\gamma \in (X \to \mathrm{R}).\, \mathbf{R}_{\mathrm{M}}(u|\gamma) \quad (u \in \mathrm{M}(X))$$

In the case of the selection monad, define $\alpha_{\mathrm{S_T}} : \mathrm{S_T}(\mathrm{R}) \to \mathrm{R}$ by:

$$\alpha_{\mathrm{S_T}}(F) = \alpha_{\mathrm{T}}(F(\mathrm{id_R}))$$

Fact 2.1. $\alpha_{\mathrm{S_T}} : \mathrm{S_T}(\mathrm{R}) \to \mathrm{R}$ is an $\mathrm{S_T}$-algebra.

Proof. We write S for $\mathrm{S_T}$. We have to show that α_{S} satisfies the unit and multiplication requirements to be an $\mathrm{S_T}$-algebra, i.e., that $\alpha_{\mathrm{S}} \circ (\eta_{\mathrm{S}})_{\mathrm{R}} = \mathrm{id_R}$ and $\alpha_{\mathrm{S}} \circ (\mu_S)_{\mathrm{R}} = \alpha_{\mathrm{S}} \circ \mathrm{S}(\alpha_{\mathrm{S}})$.

For the first requirement we have:

$$\alpha_{\mathrm{S_T}}((\eta_{\mathrm{S}})_{\mathrm{R}}(r)) = \alpha_{\mathrm{T}}((\eta_{\mathrm{S}})_{\mathrm{R}}(r)(\mathrm{id_R})) = \alpha_{\mathrm{T}}((\eta_{\mathrm{T}})_{\mathrm{R}}(r)) = r$$

For the second requirement, for $F \in \mathrm{S}(\mathrm{S}(\mathrm{R}))$ we calculate, first, that:

$$\begin{array}{rcl}
\alpha_{\mathrm{S}}((\mu_{\mathrm{T}})_{\mathrm{R}}F) & = & \alpha_{\mathrm{S}}(\lambda\gamma : \mathrm{R} \to \mathrm{R}.\, (\lambda d \in \mathrm{S}(\mathrm{R}).\, d\gamma)^{\dagger_{\mathrm{T}}}(F(\lambda d \in \mathrm{S}(\mathrm{R}).\, \mathbf{R}(d|\gamma)))) \\
& = & \alpha_{\mathrm{T}}((\lambda d \in \mathrm{S}(\mathrm{R}).\, d\,\mathrm{id_R})^{\dagger_{\mathrm{T}}}(F(\lambda d \in \mathrm{S}(\mathrm{R}).\, \mathbf{R}(d|\mathrm{id_R})))) \\
& = & \alpha_{\mathrm{T}}((\lambda d \in \mathrm{S}(\mathrm{R}).\, d\,\mathrm{id_R})^{\dagger_{\mathrm{T}}}(F(\lambda d \in \mathrm{S}(\mathrm{R}).\, \alpha_{\mathrm{T}}(d\,\mathrm{id_R})))) \\
& = & (\lambda d \in \mathrm{S}(\mathrm{R}).\, \alpha_{\mathrm{T}}(d\,\mathrm{id_R}))^{\dagger_{\mathrm{T}}}(F(\alpha_{\mathrm{S}})) \\
& = & \alpha_{\mathrm{S}}^{\dagger_{\mathrm{T}}}(F(\alpha_{\mathrm{S}}))
\end{array}$$

(where the fourth equality uses the fact that for any M-algebra $\alpha : \mathrm{M}(X) \to X$ and any $f : Y \to \mathrm{M}(X)$ we have $\alpha \circ f^{\dagger_{\mathrm{M}}} = (\alpha \circ f)^{\dagger_{\mathrm{M}}}$) and, second, that:

$$\begin{array}{rcl}
\alpha_{\mathrm{S}}(\mathrm{S}(\alpha_{\mathrm{S}})(F)) & = & \alpha_{\mathrm{S}}(\lambda\gamma : \mathrm{R} \to \mathrm{R}.\, \mathrm{T}(\alpha_{\mathrm{S}})(F(\gamma \circ \alpha_S))) \\
& = & \alpha_{\mathrm{T}}(\mathrm{T}(\alpha_S)(F(\alpha_S))) \\
& = & \alpha_{\mathrm{S}}^{\dagger_{\mathrm{T}}}(F(\alpha_{\mathrm{S}}))
\end{array}$$

(where the last equality uses the fact that for any M-algebra $\alpha : \mathrm{M}(X) \to X$, and any $f : Y \to X$ we have $\alpha \circ \mathrm{M}(f) = f^{\dagger_{\mathrm{M}}}$). This concludes the proof. □

Using the general formula for the reward function for monads equipped with an algebra on R, we then calculate for $F \in \mathrm{S_T}(X) = (X \to \mathrm{R}) \to \mathrm{T}(X)$ and $\gamma : X \to \mathrm{R}$ that:

$$\begin{array}{rcl}
\mathbf{R}_{\mathrm{S_T}}(F|\gamma) & = & \alpha_{\mathrm{S_T}}(\mathrm{S_T}(\gamma)(F)) \\
& = & \alpha_{\mathrm{S_T}}(\lambda\gamma' : \mathrm{R} \to \mathrm{R}.\, \mathrm{T}(\gamma)(F(\gamma' \circ \gamma))) \\
& = & \alpha_{\mathrm{T}}(\mathrm{T}(\gamma)(F\gamma))
\end{array}$$

As desired, this is the reward function of Definition 2.1. Note that $\mathbf{R}_{\mathrm{S_T}}(F|\gamma) = \mathbf{R}_{\mathrm{T}}(F\gamma|\gamma)$.

2.2. **Generic effects and algebraic operations.** In order to be able to give semantics to effectual operations such as probabilistic choice, we use the apparatus of generic effects and algebraic operations in the category of sets discussed in [PP03] (in a much more general setting). Suppose that M is a (necessarily strong) monad on the category of sets. A *generic effect* g with arity (I, O) (written $g : (I, O)$) for M is just a Kleisli map:

$$g : O \to \mathrm{M}(I)$$

An *M-algebraic operation op* with arity (I, O) (written $op : (I, O)$) is a family of functions

$$op_X : O \times \mathrm{M}(X)^I \to \mathrm{M}(X)$$

natural with respect to Kleisli maps in the sense that the following diagram commutes for all $e : X \to \mathrm{M}(Y)$:

$$\begin{array}{ccc} O \times \mathrm{M}(X)^I & \xrightarrow{op_X} & \mathrm{M}(X) \\ {\scriptstyle O \times (e^{\dagger_{\mathrm{M}}})^I}\downarrow & & \downarrow{\scriptstyle e^{\dagger_{\mathrm{M}}}} \\ O \times \mathrm{M}(Y)^I & \xrightarrow{op_Y} & \mathrm{M}(Y) \end{array}$$

There is a 1–1 correspondence between (I, O)-ary generic effects and (I, O)-ary algebraic operations. In one direction, given g, one sets

$$op_X(o, a) = a^{\dagger_{\mathrm{M}}}(g(o)) \tag{2.5}$$

In the other direction, given such a family op, one sets

$$g(o) = op_I(o, (\eta_{\mathrm{M}})_I) \tag{2.6}$$

Naturality implies a weaker but useful property, that the above diagram commutes for maps $\mathrm{M}(f)$, for any $f : X \to Y$. In other words, if we regard $\mathrm{M}(X)$ and $\mathrm{M}(Y)$ as algebras equipped with (any) corresponding algebraic operation components, such maps are homomorphisms $\mathrm{M}(f) : \mathrm{M}(X) \to \mathrm{M}(X)$. Naturality also implies that, as monad mutltiplications $(\mu_{\mathrm{T}})_X$ are Kleisli extensions, they too act homomorphically on algebraic operations.

We generally obtain the algebraic operations we need via their generic effects. When O is a product $O_1 \times \cdots \times O_m$, we obtain semantically useful functions

$$op^{\dagger}_X : (\mathrm{M}(O_1) \times \cdots \times \mathrm{M}(O_m)) \times \mathrm{M}(X)^I \to \mathrm{M}(X)$$

from an algebraic operation

$$op_X : (O_1 \times \cdots \times O_m) \times \mathrm{M}(X)^I \to \mathrm{M}(X)$$

This can be done by applying iterated Kleisli extension to the curried version

$$op'_X : O_1 \to \ldots\ O_m \to \mathrm{M}(X)^I \to \mathrm{M}(X)$$

of op, or, equivalently, using Kleisli extension and the monoidal structure

$$(m_{\mathrm{T}})_{X,Y} : \mathrm{M}(X) \times \mathrm{M}(Y) \to \mathrm{M}(X \times Y)$$

induced by the monadic strength (see [Koc72]).

When $O = \mathbb{1}$, we generally ignore it and equivalently write $g : I$ and $g \in \mathrm{M}(I)$ for generics and $op : I$ and $op_X : \mathrm{M}(X)^I \to \mathrm{M}(X)$ for algebraic operations. We adopt similar conventions below for related occurrences of $\mathbb{1}$. Note that (I, O)-ary algebraic operations op are in an evident correspondence with indexed families op_o ($o \in O$) of I-ary algebraic operations; in particular, when $I = [n]$ (as usual, $[n] = \{i \mid i < n\}$), the op_o can be considered to be families $(op_o)_X : \mathrm{M}(X)^n \to \mathrm{M}(X)$ of n-ary functions. (Here, and below, it is convenient to confuse $[n]$ with n.)

The $[n]$-ary algebraic operations include the projections $\pi_{n,i} : \mathrm{M}(X)^n \to \mathrm{M}(X)$, for $i = 0, \ldots, n-1$, and are closed under composition, meaning that if op is an $[n]$-ary algebraic operation, and op_i are $[m]$-ary algebraic operations, then so is $op \circ \langle op_0, \ldots, op_{n-1}\rangle$ where:

$$(op \circ \langle op_0, \ldots, op_{n-1}\rangle)_X(u_0, \ldots, u_{m-1}) = op_X((op_0)_X(u_0), \ldots, (op_{m-1})_X(u_{m-1}))$$

There are natural corresponding generic effects and operations on them. This is part of a much larger picture. The generic effects of a monad M form its Kleisli category, with objects all sets. This category has all small sums, and so its opposite, termed the large Lawvere theory of M (see [Dub06, HLPP07b]), has all small products. The algebraic operations also form a category, again with objects all sets, and with morphisms from I to O the (I, O)-ary algebraic operations (identity and composition are defined componentwise). The correspondence between generic effects and algebraic operations forms an isomorphism between these two categories.

We say that algebraic operations $op_1 : [n_1], \ldots, op_k : [n_k]$ *satisfy* equations over function symbols $f_1 : n_1, \ldots, f_k : n_k$ iff for any X, $(op_1)_X, \ldots, (op_k)_X$ do, in the usual sense, i.e., if the equations hold with the f_i interpreted as $(op_i)_X$ for $i = 1, \ldots, k$. In the case where M is the free-algebra monad for an equational theory Th with function symbols $op : n$ of given arity, the $op_X : \mathrm{M}(X)^n \to \mathrm{M}(X)$ form $[n]$-ary algebraic operations (indeed, in this case all algebraic operations occur as compositions of these ones and the projection algebraic operations). These algebraic operations satisfy all the equations of Th.

Given an M-algebra, $\alpha : \mathrm{M}(X) \to X$, and an M-algebraic operation $op : (I, O)$ we can induce a corresponding map $op_\alpha : O \times X^I \to X$, by setting

$$op_\alpha(o, u) = \alpha(op_X(o, \eta_X \circ u))$$

and α is then a homomorphism between $op_{\mathrm{M}(X)}$ and the induced map. Given a collection of operations $op_1 : [n_1], \ldots, op_k : [n_k]$, the corresponding induced maps satisfy the same equations the operations do. So, in particular, if M is the free-algebra monad for an equational theory Th with function symbols $op_i : n_i$, X becomes a model of the theory via the $(op_i)_{\mathrm{M}(X)}$. Conversely, if X is a model of the theory then we can define a corresponding M-algebra by setting $\alpha = \mathrm{id}_X^{\dagger_\mathrm{M}}$. These two correspondences yield an isomorphism between the categories of M-algebras and models of the theory (the isomorphism is the identity on morphisms).

Given a monad morphism $\theta : \mathrm{M} \to \mathrm{M}'$, any generic effect $g : O \to \mathrm{M}(I)$ yields a generic effect $g' = \theta_I \circ g$ for M′. Then, see [HPP06], θ is a homomorphism of the corresponding algebraic operations, $op_X : O \times \mathrm{M}(X)^I \to \mathrm{M}(X)$ and $op'_X : O \times \mathrm{M}'(X)^I \to \mathrm{M}'(X)$ in the sense that, for all sets X, the following diagram commutes:

$$\begin{array}{ccc} O \times \mathrm{M}(X)^I & \xrightarrow{op_X} & \mathrm{M}(X) \\ {\scriptstyle O \times (\theta_X)^I}\downarrow & & \downarrow{\scriptstyle \theta_X} \\ O \times \mathrm{M}'(X)^I & \xrightarrow{op'_X} & \mathrm{M}'(X) \end{array}$$

We next consider algebraic operations for the selection monad S_T. Modulo currying, $(I, O \times \mathrm{R}^I)$-ary generic effects $g : (O \times \mathrm{R}^I) \to \mathrm{T}(I)$ for T are in bijective correspondence with (I, O)-ary generic effects $\widetilde{g} : O \to \mathrm{T}(I)^{\mathrm{R}^I}$ for $\mathrm{S_T}$. There is therefore a corresponding bijective correspondence between $(I, O \times \mathrm{R}^I)$-ary T-algebraic operations op and (I, O)-ary $\mathrm{S_T}$-algebraic operations $\widetilde{op}$. This correspondence has a pleasing component-wise expression going from T to $\mathrm{S_T}$. An intermediate function family notion is useful. We define the *auxiliary function family* $\mathrm{aux}_X : O \times \mathrm{R}^X \times \mathrm{T}(X)^I \to \mathrm{T}(X)$ associated to a $(I, O \times \mathrm{R}^I)$-ary T-algebraic operation op by:

$$\mathrm{aux}_X(o, \gamma, u) = op_X(\langle o, \lambda i \in I.\, \mathbf{R}_\mathrm{T}(ui|\gamma)\rangle, u) \tag{2.7}$$

Below we write $\mathrm{aux}_{X,o,\gamma}$ for the function $\mathrm{aux}_X(o,\gamma,-)$.

Proposition 2.2. *Let op be an* $(I, O \times \mathrm{R}^I)$*-ary* T*-algebraic operation. In terms of its associated auxiliary function family* aux*, the corresponding* (I,O)*-ary* $\mathrm{S_T}$*-algebraic operation* $\widetilde{op}$ *is given by:*

$$\widetilde{op}_X(o,a) = \lambda\gamma \in \mathrm{R}^X.\, \mathrm{aux}_{X,o,\gamma}(\lambda i \in I.\, ai\gamma)$$

Conversely, we have:

$$\mathrm{aux}_{X,o,\gamma}(u) = \widetilde{op}_X(o, \lambda i \in I. \lambda\gamma \in X \to \mathrm{R}.\, ui)\gamma$$

Proof. The generic effect $g : O \times \mathrm{R}^I \to \mathrm{T}(I)$ corresponding to op is given by Equation 2.6:

$$g(o,\overline{\gamma}) = op_I(\langle o,\overline{\gamma}\rangle, (\eta_\mathrm{T})_I)$$

Currying, we obtain $\widetilde{g} : O \to \mathrm{S}(I)$ where:

$$\widetilde{g}(o) = \lambda\overline{\gamma} \in \mathrm{R}^I.\, op_I(\langle o,\overline{\gamma}\rangle, (\eta_\mathrm{T})_I)$$

and then, using Equation 2.5, we have:

$$\widetilde{op}_X(o,a) = a^{\dagger_{\mathrm{S_T}}}(\widetilde{g}(o))$$

We next choose a reward continuation $\gamma \in \mathrm{R}^X$ and examine $a^{\dagger_{\mathrm{S_T}}}(\widetilde{g}(o))\gamma$. To this end we first obtain a reward continuation in R^I from a and γ, namely:

$$\overline{\gamma} =_{\mathrm{def}} \lambda i \in I.\, \mathbf{R}_\mathrm{T}(ai|\gamma)$$

and, setting

$$u =_{\mathrm{def}} (\lambda i \in I.\, ai\gamma) \in I \to \mathrm{T}(X)$$

we have:

$$\begin{array}{rcll} a^{\dagger_{\mathrm{S_T}}}(\widetilde{g}(o))\gamma & = & u^{\dagger_\mathrm{T}}(\widetilde{g}(o)(\overline{\gamma})) & \\ & = & u^{\dagger_\mathrm{T}}(op_I(\langle o,\overline{\gamma}\rangle, (\eta_\mathrm{T})_I)) & \\ & = & op_X(\langle o,\overline{\gamma}\rangle, u^{\dagger_\mathrm{T}}(\eta_\mathrm{T})_I) & \text{(by the Kleisli naturality} \\ & & & \text{of algebraic operations)} \\ & = & op_X(\langle o,\overline{\gamma}\rangle, u) & \end{array}$$

Putting these facts together, we have:

$$\begin{array}{rcl} (\widetilde{op})_X(o,a)\gamma & = & a^{\dagger_{\mathrm{S_T}}}(g_{\mathrm{S_T}}(o)) \\ & = & op_X(\langle o,\overline{\gamma}\rangle, u) \\ & = & op_X(\langle o, \lambda i \in I.\, \mathbf{R}_\mathrm{T}(ai|\gamma)\rangle, \lambda i \in I.\, ai\gamma) \\ & = & \mathrm{aux}_{X,o,\gamma}(\lambda i \in I.\, ai\gamma) \end{array}$$

as required. That

$$\mathrm{aux}_{X,o,\gamma}(u) = \widetilde{op}_X(o, \lambda i \in I. \lambda\gamma \in X \to \mathrm{R}.\, ui)\gamma$$

is an immediate consequence, for, setting $a = \lambda i \in I.\, \lambda\gamma \in X \to \mathrm{R}.\, u(i)$, we find:

$$\begin{aligned} \mathrm{aux}_{X,o,\gamma}(u) &= \mathrm{aux}_{X,o,\gamma}(\lambda i \in I.\, ai\gamma) \\ &= (\widetilde{op})_X(o,a)\gamma \\ &= (\widetilde{op})_X(o, \lambda i \in I.\, \lambda\gamma \in X \to \mathrm{R}.\, u(i))\gamma \end{aligned}$$

□

Note that, as is natural, the proposition expresses that $(op_{\mathrm{S_T}})_X$ uses the reward function in R^I which assigns to $i \in I$ the reward obtained by following the ith branch.

The correspondences between the two kinds of algebraic operations and auxiliary functions fit well with fixing parameters. Given an $(I, O \times \mathrm{R}^I)$-ary T-algebraic operation op, we obtain an (I, R^I)-ary $\mathrm{S_T}$-algebraic operation op' by fixing an $o \in O$. The corresponding auxiliary function family $\mathrm{aux}'_X : \mathrm{R}^X \times \mathrm{T}(X)^I \to \mathrm{T}(X)$ is, as one would expect, $\lambda\gamma, a.\, \mathrm{aux}_X(a, \gamma, a)$; the corresponding I-ary algebraic operation for $\mathrm{S_T}$ is $\widetilde{op}_o$.

Using Proposition 2.2, we can reduce questions of equational satisfaction by $\mathrm{S_T}$-algebraic operations to corresponding questions about their auxiliary functions, so reducing questions about $\mathrm{S_T}$ to questions about T. We first need a lemma.

Lemma 2.3.

(1) *The auxiliary function family* $\mathrm{aux}_{X,\gamma}$ *corresponding to an* $[n]$*-ary projection* $\mathrm{S_T}$*-algebraic operation* $\pi_{n,i}$ *is the family* $(\pi_{n,i})_{\mathrm{T}(X)} : \mathrm{T}(X)^n \to \mathrm{T}(X)$ *of projections.*

(2) *Let op be an* $[n]$*-ary* $\mathrm{S_T}$*-algebraic operation, and, for* $i = 0, \ldots, n-1$*, let* op_i *be* $[m]$*-ary* $\mathrm{S_T}$*-algebraic operations for* $\mathrm{S_T}$*, and let their corresponding auxiliary function families be* aux_X *and* $(\mathrm{aux}_i)_X$*, respectively. Then the auxiliary function family* aux'_X *corresponding to the composition* op' *of op with the* op_i *is the corresponding composition of auxiliary functions:*

$$\begin{aligned}&\mathrm{aux}'_{X,\gamma}(u_0, \ldots, u_{m-1})\\ &\qquad = \mathrm{aux}_{X,\gamma}((\mathrm{aux}_0)_{X,\gamma}(u_0, \ldots, u_{m-1}), \ldots, (\mathrm{aux}_{n-1})_{X,\gamma}(u_0, \ldots, u_{m-1}))\end{aligned}$$

Proof.

(1) This is immediate from the second part of Proposition 2.2.

(2) Making use of both parts of Proposition 2.2 we calculate:

$$\begin{aligned}&\mathrm{aux}'_{X,\gamma}(u_0, \ldots, u_{m-1})\\ &\quad = op'_X(\lambda\gamma.u_0, \ldots, \lambda\gamma.u_{m-1})\gamma\\ &\quad = op_X((op_0)_X(\lambda\gamma.u_0, \ldots, \lambda\gamma.u_{m-1}), \ldots, (op_{n-1})_X(\lambda\gamma.u_0, \ldots, \lambda\gamma.u_{m-1}))\gamma\\ &\quad = \mathrm{aux}_{X,\gamma}((op_0)_X(\lambda\gamma.u_0, \ldots, \lambda\gamma.u_{m-1})\gamma, \ldots, (op_{n-1})_X(\lambda\gamma.u_0, \ldots, \lambda\gamma.u_{m-1})\gamma)\\ &\quad = \mathrm{aux}_{X,\gamma}((\mathrm{aux}_0)_{X,\gamma}(u_0, \ldots, u_{m-1}), \ldots, (\mathrm{aux}_{n-1})_{X,\gamma}(u_0, \ldots, u_{m-1}))\end{aligned}$$ □

Proposition 2.4. *Let* op_i *be* $([n_i], O_i \times \mathrm{R}^{[n_i]})$*-ary* T*-algebraic operations, for* $i = 1, \ldots, k$*, and choose* $o_i \in O_i$ $(i = 1, \ldots, k)$*. Then an equation is satisfied by* $(\widetilde{op_1})_{o_1}, \ldots, (\widetilde{op_k})_{o_k}$*, if, for all sets* X *and* $\gamma : X \to \mathrm{R}$*, it is satisfied by* $(\mathrm{aux}_1)_{X,o_1,\gamma}, \ldots, (\mathrm{aux}_k)_{X,o_k,\gamma}$*, where, for* $i = 1, \ldots, k$*,* aux_i *is the auxiliary function family obtained from* op_i*.*

Proof. We can assume without loss of generality that the O_i are all $\mathbb{1}$ and so can be ignored. The interpretation of an algebraic term t with m free variables built from function symbols $f_1 : n_1, \ldots, f_k : n_k$ can be considered as an m-ary function, and an equation $t = u$ over m free variables holds in the interpretation if the two such interpretations are equal.

Fixing a term t with m free variables, for any set X, using the $(\widetilde{op}_i)_X$ to interpret the f_i, we obtain functions $\mathcal{O}[\![t]\!]_X : \mathrm{S_T}(X)^m \to \mathrm{S_T}(X)$, say, and for any set X and $\gamma : X \to \mathrm{R}$, using the $(\mathrm{aux}_i)_{X,\gamma}$ we obtain functions $\mathcal{A}[\![t]\!]_{X,\gamma} : \mathrm{T}(X)^m \to \mathrm{T}(X)$, say. As the projections are algebraic operations and as algebraic operations are closed under composition, a straightforward structural induction shows that the family $\mathcal{O}[\![t]\!]$ is an m-ary algebraic operation for $\mathrm{S_T}$. Using Lemma 2.3, a further straightforward structural induction shows that $\mathcal{A}[\![t]\!]_{X,-}$ is the corresponding auxiliary function family.

Now suppose an equation $t = u$ over m variables is satisfied by $(\mathrm{aux}_1)_{X,\gamma}, \ldots, (\mathrm{aux}_k)_{X,\gamma}$ for all sets X and $\gamma : X \to \mathrm{R}$, that is, suppose that $\mathcal{A}[\![t]\!]_{X,\gamma} = \mathcal{A}[\![u]\!]_{X,\gamma}$, for all such X and γ. Then, using Proposition 2.2, we see that:

$$\begin{array}{rcl}\mathcal{O}[\![t]\!]_X(F_0, \ldots, F_{m-1})\gamma & = & \mathcal{A}[\![t]\!]_{X,\gamma}(F_0\gamma, \ldots, F_{m-1}\gamma) \\ & = & \mathcal{A}[\![u]\!]_{X,\gamma}(F_0\gamma, \ldots, F_{m-1}\gamma) \\ & = & \mathcal{O}[\![u]\!]_X(F_0, \ldots, F_{m-1})\gamma\end{array}$$

holds for all sets X and $\gamma : X \to \mathrm{R}$, concluding the proof. □

We next see that, as one would expect, we can use algebraic operations for T-effects to obtain corresponding ones for $\mathrm{S_T}$-effects. If op is an (I, O)-ary T-algebraic operation, it can be considered to be an $(I \times \mathrm{R}^I, O)$-ary algebraic operation which ignores its reward function argument. The auxiliary functions $\mathrm{aux}_{X,o,\gamma}$ are the same as the $(op_o)_X$ and Proposition 2.2 then yields a (I, O)-ary $\mathrm{S_T}$-algebraic operation $\widetilde{op}$, where:

$$\widetilde{op}_X(o, a) = \lambda\gamma \in \mathrm{R}^X.\, op_X(o, \lambda i \in I.\, ai\gamma) \tag{2.8}$$

which is the natural pointwise definition. In the case where $I = [n]$ this can be written as:

$$\widetilde{op}_X(o, F_1, \ldots, F_n) = \lambda\gamma \in \mathrm{R}^X.\, op_X(o, F_1\gamma, \ldots, F_n\gamma) \tag{2.9}$$

From Proposition 2.4 we further have (as is, in any case, evident from a pointwise argument):

Corollary 2.5. *Let op_i be $[n_i]$-ary T-algebraic operations, for $i = 1, \ldots, k$. Then an equation is satisfied by $\widetilde{op_1}, \ldots, \widetilde{op_k}$, if it is satisfied by $op_1, \ldots, op_k$.*

Another way to obtain algebraic operations is to start from the basic selection monad S. Consider an $(I, O \times \mathrm{R}^I)$-ary generic effect $g : O \times \mathrm{R}^I \to I$ for the identity monad (equivalent via currying to an (I, O)-ary generic effect for S). Viewed as a T-generic effect $\eta_I \circ g$, via the unit for T and using Equation 2.5, we obtain an $(I, O \times \mathrm{R}^I)$-ary T-algebraic operation op_g where, for $o \in O, \overline{\gamma} \in \mathrm{R}^I, u \in \mathrm{T}(X)^I$:

$$(op_g)_X(\langle o, \overline{\gamma}\rangle, u) = u^\dagger_{\mathrm{T}}(\eta_I(g(o, \overline{\gamma}))) = u(g(o, \overline{\gamma}))$$

Then the corresponding auxiliary functions $(\mathrm{aux}_g)_X : O \times \mathrm{R}^X \times \mathrm{T}(X)^I \to \mathrm{T}(X)$ are given, using Definition 2.7, by:

$$(\mathrm{aux}_g)_{X,o,\gamma}(u) = (op_g)_X(\langle o, \lambda i \in I.\, \mathbf{R}_{\mathrm{T}}(ui|\gamma)\rangle, u) = u(g(o, \lambda i \in I.\, \mathbf{R}_{\mathrm{T}}(ui|\gamma)))$$

Finally, via Proposition 2.2, we obtain the (I, O)-ary $\mathrm{S_T}$-algebraic operation $\widetilde{op}_g$ corresponding to op_g. For for $o \in O, a \in \mathrm{S_T}(X)^I, \gamma \in \mathrm{R}^X$, we have:

$$(\widetilde{op}_g)_X(o, a)\gamma = (\mathrm{aux}_g)_{X,o,\gamma}(\lambda i \in I.\, ai\gamma) = a(g(o, \lambda i \in I.\, \mathbf{R}_{\mathrm{S_T}}(ai|\gamma)))\gamma \tag{2.10}$$

So each component $(\widetilde{op}_g)_X$ of $\widetilde{op}_g$ uses g to select a branch of a, depending only on the parameter $o \in P$ and the rewards $\mathbf{R}_{\mathrm{S_T}}(ai|\gamma)$ associated to the branches of a relative to the reward continuation γ. We can turn this observation into a definition. Say that a family of functions

$$f_X : O \times \mathrm{S_T}(X)^I \to \mathrm{S_T}(X)$$

is an (I, O)-ary-*selection operation* if there is a function $g : O \times \mathrm{R}^I \to I$ such that

$$f_X(o, a)\gamma = a(g(o, \lambda i \in I.\, \mathbf{R}_{\mathrm{S_T}}(ai|\gamma)))\gamma \tag{2.11}$$

Equation 2.10 then tells us that the selection operations are exactly the algebraic operations of the form $\widetilde{op_g}$ where, modulo currying, g is a basic selection monad generic effect.

We next consider a particular case: binary selection operations. Here $O = \mathbb{1}$ and $I = [2]$. Such operations arise from [2]-ary generics $g : \mathrm{R}^{[2]} \to [2]$ for the basic selection monad. Viewed as a binary algebraic operation on $\mathrm{S_T}$, Equation 2.10 becomes:

$$(\widetilde{op_g})_X(G_0, G_1)\gamma = \begin{cases} G_0\gamma & (g(\lambda i \in I.\, \mathbf{R}_{\mathrm{S_T}}(G_i|\gamma)) = 0) \\ G_1\gamma & (g(\lambda i \in I.\, \mathbf{R}_{\mathrm{S_T}}(G_i|\gamma)) = 1) \end{cases}$$

Note that [2]-ary generics $g : \mathrm{R}^{[2]} \to [2]$ for the basic selection monad are in bijection with binary relations B on R, with relations B corresponding to generics g_B, where:

$$g_B(\overline{\gamma}) = 0 \equiv_{\mathrm{def}} \overline{\gamma}(0) B \overline{\gamma}(1)$$

(read rBs as "r beats s"). Defining op_B to be the binary $\mathrm{S_T}$-algebraic operation $\widetilde{op_{g_B}}$, we have:

$$(op_B)_X(G_0, G_1)\gamma = \begin{cases} G_0\gamma & (\mathbf{R}_{\mathrm{S_T}}(G_0|\gamma)\, B\, \mathbf{R}_{\mathrm{S_T}}(G_1|\gamma)) \\ G_1\gamma & (\mathbf{R}_{\mathrm{S_T}}(G_0|\gamma)\, \overline{B} \mathbf{R}_{\mathrm{S_T}}(G_1|\gamma)) \end{cases}$$

For optimization purposes it is natural to assume B is a total order $\geq$. We define `or` to be the resulting binary algebraic operation on $\mathrm{S_T}$; it is this operation that we use for the semantics of decision-making in our two languages. Explicitly we have:

$$\mathtt{or}_X(G_0, G_1)(\gamma) = \begin{cases} G_0\gamma & (\text{if } \mathbf{R}_{\mathrm{S_T}}(G_0|\gamma) \;\geq\; \mathbf{R}_{\mathrm{S_T}}(G_1|\gamma)) \\ G_1\gamma & (\text{otherwise}) \end{cases}$$

This can be usefully rewritten. For $\gamma : X \to \mathrm{R}$ define $\max_\gamma : X^2 \to X$ (written infix) by:

$$x \max_\gamma y = \begin{cases} x & (\text{if } \gamma(x) \geq \gamma(y)) \\ y & (\text{otherwise}) \end{cases} \tag{2.12}$$

Then:

$$\mathtt{or}_X(G_0, G_1)(\gamma) = G_0\gamma \max_{\mathbf{R}_{\mathrm{T}}(-|\gamma)} G_1\gamma \tag{2.13}$$

Taking B to be a total order is equivalent to using a version of argmax as a generic effect. First, for finite totally ordered sets I, assuming a total order $\geq$ on R, we define $\mathrm{argmax}_I \in \mathrm{S}(I) = (I \to \mathrm{R}) \to I$, by taking $\mathrm{argmax}\gamma$ to be the least $i \in I$ among those maximizing $\gamma(i)$. Then B corresponds to $\mathrm{argmax}_{[2]}$, with [2] ordered by setting $0 < 1$. We could as well have used generics picking from finite totally ordered sets, with resulting choice functions of corresponding arity.

We next investigate the equations that the algebraic operations op_B obey and their relation to properties of the relations B. Define $(\mathrm{aux}_B)_X : \mathrm{R}^X \times \mathrm{T}(X)^2 \to \mathrm{T}(X)$ to be $(\mathrm{aux}_{g_B})_X$. So:

$$(\mathrm{aux}_B)_{X,\gamma}(u_0, u_1) =_{\mathrm{def}} (\mathrm{aux}_g)_{X,\gamma}(\lambda i \in [2].\, u_i) = u_{g(\lambda i \in [2].\, \mathbf{R}_{\mathrm{T}}(u_i|\gamma))}$$

and we have:

$$(\mathrm{aux}_B)_{X,\gamma}(u_0, u_1) = \begin{cases} u_0 & (\text{if } \mathbf{R}_{\mathrm{T}}(u_0|\gamma)\, B\, \mathbf{R}_{\mathrm{T}}(u_1|\gamma)) \\ u_1 & (\text{otherwise}) \end{cases}$$

In particular, for $x_0, x_1 \in X$ we have:

$$(\mathrm{aux}_B)_{X,\gamma}(\eta_{\mathrm{T}}(x_0), \eta_{\mathrm{T}}(x_1))(\gamma) = \begin{cases} \eta_{\mathrm{T}}(x_0) & (\text{if } \gamma(x_0) B \gamma(x_1)) \\ \eta_{\mathrm{T}}(x_1) & (\text{otherwise}) \end{cases} \tag{2.14}$$

We see from Proposition 2.4 that op_B satisfies an equation if, and only if, $(\mathrm{aux}_B)_{X,\gamma}$ does for every X and $\gamma : X \to \mathrm{R}$.

Say that a binary function f is *left-biased* if the following equation holds:

$$f(x, f(y, x)) = f(x, y)$$

and is *right-biased* if the following equation holds:

$$f(f(x, y), x) = f(y, x)$$

and recall that a relation R is *strongly connected* iff, for all x, y, either xRy or yRx.

Theorem 2.6. *For every binary relation B on* R *we have:*

(1) op_B *is idempotent.*
(2) op_B *is associative iff B and its complement is transitive.*
(3) op_B *is left-biased iff B is strongly connected.*
(4) op_B *is right-biased iff the complement of B is strongly connected.*
(5) op_B *is not commutative (assuming* R *non-empty).*

Proof. Throughout the proof, we use the fact that, like any monad unit, all components of $\eta_{\mathrm{S_T}}$ are 1–1.

(1) This is evident.
(2) (a) Suppose op_B is associative, and choose $r_0, r_1, r_2 \in \mathrm{R}$. Define $\gamma : [3] \to \mathrm{R}$ by: $\gamma(i) = r_i$, for $i = 0, 1, 2$, and set $f = (\mathrm{aux}_B)_{\gamma,[3]}$ and $x_i = (\eta_{\mathrm{T}})_{[3]}(i)$, for $i = 0, 1, 2$. Note that the x_i are all different. By Proposition 2.4 f is associative as op_B is. Suppose first that r_0Br_1 and r_1Br_2. Using Equation 2.14 we see that, as r_0Br_1 and r_1Br_2, $f(x_0, f(x_1, x_2)) = f(x_0, x_1) = x_0$, and $f(f(x_0, x_1), x_2) = f(x_0, x_2)$. So, as f is associative $f(x_0, x_2) = x_0$. As $x_0 \neq x_2$, we have x_0Bx_2, as required. Suppose next that $\neg r_0Br_1$ and $\neg r_1Br_2$. Then, using Equation 2.14 again, we see that $f(x_0, f(x_1, x_2)) = f(x_0, x_2)$ and $f(f(x_0, x_1), x_2) = f(x_1, x_2) = x_2$, and, similarly to before, we conclude that $\neg r_0Br_2$.
   (b) For the converse, suppose that B and its complement is transitive. It suffices to prove that every $f = (\mathrm{aux}_B)_{X,\gamma} : \mathrm{T}(X)^2 \to \mathrm{T}(X)$ is associative. Choose $u_i \in \mathrm{T}(X)$ $(i = 0, 2)$ and set $r_i = \mathbf{R}(u_i|\gamma)$. The proof divides into cases according as each of r_0Br_1 and r_1Br_2 does or does not hold:
      (i) Suppose that r_0Br_1 and r_1Br_2 (and so r_0Br_2). By the definition of aux_B, we then have:
      $$f(u_0, f(u_1, u_2)) = f(u_0, u_1) = u_0 = f(u_0, u_2) = f(f(u_0, u_1), u_2)$$
      (ii) Suppose that r_0Br_1 and $\neg r_1Br_2$. Then:
      $$f(u_0, f(u_1, u_2)) = f(u_0, u_2) = f(f(u_0, u_1), u_2)$$
      (iii) Suppose that $\neg r_0Br_1$ and r_1Br_2. Then:
      $$f(u_0, f(u_1, u_2)) = f(u_0, u_1) = u_1 = f(u_1, u_2) = f(f(u_0, u_1), u_2)$$
      (iv) Suppose that $\neg r_0Br_1$ and $\neg r_1Br_2$.Then $\neg r_0Br_2$, and we have:
      $$f(u_0, f(u_1, u_2)) = f(u_0, u_2) = u_2 = f(u_1, u_2) = f(f(u_0, u_1), u_2)$$

   So in all cases we have
   $$f(u_0, f(u_1, u_2)) = f(f(u_0, u_1), u_2)$$
   and so op_B is associative, as required.

(3) (a) Suppose op_B is left-biased and choose $r_0, r_1 \in \mathrm{R}$. Define $\gamma : [2] \to \mathrm{R}$ by: $\gamma(i) = r_i$, for $i = 0, 1$, and set $f = (\mathrm{aux}_B)_{\gamma,[2]}$ and $x_i = (\eta_{\mathrm{T}})_{[2]}(i)$, for $i = 0, 1$. Note that $x_0 \neq x_1$. By Proposition 2.4 f is left-biased as op_B is.
Suppose that $\neg r_1 B r_0$. Then we have:

$$f(x_0, x_1) = f(x_0, f(x_1, x_0)) = f(x_0, x_0) = x_0$$

and so, as $x_0 \neq x_1$, $r_0 B r_1$.

(b) For the converse, suppose the relation B is strongly connected. It suffices to prove that every $f = (\mathrm{aux}_B)_{X,\gamma} : \mathrm{T}(X)^2 \to \mathrm{T}(X)$ is left-biased. Choose u_i in $\mathrm{T}(X)$ $(i = 0, 1)$ and set $r_i = \mathbf{R}_{\mathrm{T}}(u_i|\gamma)$. Suppose first that $r_1 B r_0$ holds. Then $f(u_0, f(u_1, u_0)) = f(u_0, u_1)$. Otherwise, as B is strongly connected, we have $\neg\, r_1 B r_0$ and $r_0 B r_1$, and so, $f(u_0, f(u_1, u_0)) = f(u_0, u_0) = u_0 = f(u_0, u_1)$. So in either case we have $f(u_0, f(u_1, u_0)) = f(u_0, u_1)$ as required.

(4) (a) Suppose op_B is right-biased and choose $r_0, r_1 \in \mathrm{R}$. Define $\gamma : [2] \to \mathrm{R}$ by: $\gamma(i) = r_i$, for $i = 0, 1$, and set $f = (\mathrm{aux}_B)_{\gamma,[2]}$ and $x_i = (\eta_{\mathrm{T}})_{[2]}(i)$, for $i = 0, 1$. Note that $x_0 \neq x_1$. By Proposition 2.4 f is right-biased as op_B is.
Suppose that $\neg(\neg r_0 B r_1)$, i.e., that $r_0 B r_1$. Then we have:

$$f(f(x, y), x) = f(y, x)$$

$$f(x_1, x_0) = f(f(x_0, x_1), x_0) = f(x_0, x_0) = x_0$$

and so, as $x_0 \neq x_1$, $\neg r_1 B r_0$.

(b) For the converse, suppose that $\neg B$ is strongly connected. It suffices to prove that every $f = (\mathrm{aux}_B)_{X,\gamma} : \mathrm{T}(X)^2 \to \mathrm{T}(X)$ is right-biased. Choose $u_i \in \mathrm{T}(X)$ $(i = 0, 1)$ and set $r_i = \mathbf{R}_{\mathrm{T}}(u_i|\gamma)$. Suppose first that $\neg r_0 B r_1$ holds. Then we have that $f(f(x_0, x_1), x_0) = f(x_1, x_0)$. Otherwise, as B is strongly connected, we have $r_0 B r_1$ and $\neg\, r_1 B r_0$, and so, $f(u_0, f(u_1, u_0)) = f(u_0, u_0) = u_0 = f(u_0, u_1)$. So in either case we have $f(u_0, f(u_1, u_0)) = f(u_0, u_1)$ as required.

(5) Choose $r \in \mathrm{R}$. Define $\gamma : [2] \to \mathrm{R}$ by: $\gamma(0) = \gamma(1) = r$, and set $f = (\mathrm{aux}_B)_{\gamma,[2]}$ and $x_i = (\eta_{\mathrm{S_T}})(i)$, for $i = 0, 1$. Note that $x_0 \neq x_1$. By Proposition 2.4 it suffices to prove that f is not commutative.
In case rBr holds, we have:

$$f(x_0, x_1) = x_0 \neq x_1 = f(x_1, x_0)$$

In case rBr does not hold, we have:

$$f(x_0, x_1) = x_1 \neq x_0 = f(x_1, x_0)$$

In either case $(op_B)_{[2]}$ is not commutative. □

Given a binary relation B on R and an $\mathrm{S_T}$-algebraic operation $op : [n]$, we say that op *distributes* over op_B iff for all X, $i \in [n]$, $F_j \in \mathrm{S_T}(X)$ $(j \in [n], j \neq i)$, and $G_0, G_1 \in \mathrm{S_T}(X)$, we have:

$$\begin{aligned} & op_X(F_0, \ldots, F_{i-1}, op_B(G_0, G_1), F_{i+1}, \ldots, F_{n-1}) = \\ & \quad op_B(op_X(F_0, \ldots, F_{i-1}, G_0, F_{i+1}, \ldots, F_{n-1}), op_X(F_0, \ldots, F_{i-1}, G_1, F_{i+1}, \ldots, F_{n-1})) \end{aligned}$$

Also, given a binary relation B on R and a function $f : \mathrm{R}^n$ we say that f *distributes* over B iff for all $0 \leq i < n$, $r_j \in \mathrm{R}$ $(0 \leq j < n, j \neq i)$, and $s_0, s_1 \in \mathrm{R}$ we have:

$$s\,B\,t \iff f(r_0, \ldots, r_{i-1}, s_0, r_{i+1}, \ldots, r_{n-1})\,B\,f(r_0, \ldots, r_{i-1}, s_1, r_{i+1}, \ldots, r_{n-1})$$

We say that an n-ary function $f : \mathrm{R}^n \to \mathrm{R}$, where $n \geq 1$, *distributes* over a binary relation B on R iff it preserves and reflects B in each argument, i.e., iff for $1 \leq i \leq n$ and $x_1, \ldots, x_{i-1}, y_0, y_1, x_{i+1}, \ldots, x_n \in \mathrm{R}$ we have:

$$B(f(x_1, \ldots, x_{i-1}, y_0, x_{i+1}, \ldots, x_n), f(x_1, \ldots, x_{i-1}, y_1, x_{i+1}, \ldots, x_n)) \iff B(y_0, y_1)$$

Theorem 2.7. *Let $op : [n]$ be a* T*-algebraic operation, and let B be a binary relation on* R*. If op_{R} distributes over B then $\widetilde{op}$ distributes over op_B.*

Proof. To keep notation simple we suppose that $\widetilde{op}$ is binary and establish distributivity in its second argument. That is, we prove, for any X, that:

$$\widetilde{op}(F, op_B(G, H)) = op_B(\widetilde{op}(F, G), \widetilde{op}(F, H)) \quad (F, G, H \in \mathrm{S_T}(X))$$

To do so we use Proposition 2.4 and establish the corresponding equation for the auxiliary functions of these operations. The auxiliary function $\mathrm{aux}_{X,\gamma} : \mathrm{T}(X)^2 \to \mathrm{T}(X)$ of $\widetilde{op}$ is op_X. So we need to show for any $\gamma : X \to \mathrm{R}$ that

$$op(u, (\mathrm{aux}_B)_{X,\gamma}(v_0, v_1)) = (\mathrm{aux}_B)_{X,\gamma}(op(u, v_0), op(u, v_1)) \quad (u, v_0, v_1 \in \mathrm{T}(X))$$

From the definition of the auxiliary function of op_B we see that each side of this equation is either $op(u, v_0)$ or $op(u, v_1)$, and that the LHS is $op(u, v_0)$ iff

$$\mathbf{R}_{\mathrm{T}}(v_0|\gamma)\, B\, \mathbf{R}_{\mathrm{T}}(v_1|\gamma) \qquad (*)$$

and that the RHS is $op(u, v_0)$ iff

$$\mathbf{R}_{\mathrm{T}}(op(u, v_0)|\gamma)\, B\, \mathbf{R}_{\mathrm{T}}(op(u, v_1)|\gamma) \qquad (**)$$

As both $\mathrm{T}(\gamma)$ and α_{T} are homomorphisms, so is $\mathbf{R}_{\mathrm{T}}(-|\gamma) = \alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma)$, and so this last condition is equivalent to:

$$op_{\mathrm{R}}(\mathbf{R}_{\mathrm{T}}(u|\gamma), \mathbf{R}_{\mathrm{T}}(v_0|\gamma))\, B\, op_{\mathrm{R}}(\mathbf{R}_{\mathrm{T}}(u|\gamma), \mathbf{R}_{\mathrm{T}}(v_1|\gamma))$$

and we see, using the fact that op_{R} distributes over B, that the conditions $(*)$ and $(**)$ are equivalent. □

3. A general language with algebraic operations

The goal of this section is to give some definitions and results—in particular an adequacy theorem—for a general language with algebraic operations. We treat our two languages of later sections as instances of this language via such algebraic operations.

3.1. Syntax.

We make use of a standard call-by-value λ-calculus equipped with algebraic operations. Our language is a convenient variant of the one in [PP01] (itself building on Moggi's computational λ-calculus [Mog89]). The somewhat minor differences from [PP01] are that we allow a variety of base types, our algebraic operations may have parameters, and we make use of general big-step transition relations as well as small-step ones.

The types $\sigma, \tau, \ldots$ and terms $L, M, N, \ldots$ of our language are built from:

- a *basic vocabulary*, consisting of:
  (1) *base types*, b (including `Bool`);
  (2) *constants*, $c : b$ of given base types b (including `tt`, `ff` : `Bool`); and

(3) first-order *function symbols*, $f : b_1 \dots b_m \to b$, of given arity $b_1 \dots b_m$ and co-arity b (including equality symbols $=_b$: $b \times b \to \texttt{Bool}$),

together with

- *algebraic operation symbols* $op : b_1 \dots b_n; m$, with given parameter base types $b_1, \dots, b_n$ and arity $m \in \mathbb{N}$.

The types are given by:

$$\sigma ::= b \mid \texttt{Unit} \mid \sigma \times \sigma \mid \sigma \to \sigma$$

and the terms are given by:

$$\begin{array}{rcl} M & ::= & x \mid c \mid f(M_1, \dots, M_m) \mid \texttt{if}\ L\ \texttt{then}\ M\ \texttt{else}\ N \mid \\ & & op(N_1, \dots, N_n; M_1, \dots, M_m) \mid \\ & & * \mid \langle M, N \rangle \mid \texttt{fst}(M) \mid \texttt{snd}(M) \mid \lambda x : \sigma.\, M \mid MN \end{array}$$

The languages considered in the next two sections provide examples of this general setup. We write BTypes for the set of base types and Con_b for the set of constants of type b. We define the *order* (or *rank*) of types by:

$$\mathrm{o}(b) = \mathrm{o}(\texttt{Unit}) = 0 \qquad \mathrm{o}(\sigma \times \tau) = \max(\mathrm{o}(\sigma), \mathrm{o}(\tau)) \qquad \mathrm{o}(\sigma \to \tau) = \max(\mathrm{o}(\sigma) + 1, \mathrm{o}(\tau))$$

We work up to α-equivalence, as usual, and free variables and substitution are also defined as usual. The typing rules are standard, and omitted, except for that for the algebraic operation symbols, which, aside from their parameters, are polymorphic:

$$\frac{\Gamma \vdash N_1 : b_1, \dots, \Gamma \vdash N_n : b_n \quad \Gamma \vdash M_1 : \sigma, \dots, \Gamma \vdash M_m : \sigma}{\Gamma \vdash op(N_1, \dots, N_n; M_1, \dots, M_m) : \sigma} \qquad (op : b_1 \dots b_n; m)$$

where $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ is an environment. We write $M : \sigma$ for $\vdash M : \sigma$ and say then that the (closed) term M is *well-typed*; such terms are the *programs* of our language. We employ standard notation, for example for local definitions writing $\mathrm{let}\, x : \sigma\, \mathrm{be}\, M\, \mathrm{in}\, N$ for $(\lambda x : \sigma.\, N)M$. We also use a cases form

$$\texttt{cases}\ M_1 \Rightarrow N_1 \mid \dots \mid M_n \Rightarrow N_n\ \texttt{else}\ N_{n+1}\ (\text{for } n \geq 0)$$

defined by iterated conditionals (where the M_i are boolean).

Moggi's language has local definitions and computational types $T\sigma$ (with associated term syntax) as primitives; these can be viewed as abbreviations in our language, in particular setting $T\sigma = 1 \to \sigma$.

3.2. **Operational semantics.** The operational semantics of programs is given in three parts: a small-step semantics, a big-step semantics, and an evaluation function. We make use of evaluation contexts, following [FF87]. The set of *values* $V, W, \dots$ is given by:

$$V ::= c \mid * \mid \langle V, W \rangle \mid \lambda x : \sigma.\, M$$

where we restrict $\lambda x : \sigma.\, M$ to be closed. We write Val_σ for the set of values of type σ, i.e., the V such that $V : \sigma$.

The *evaluation contexts* are given by:

$$\begin{array}{rcl} \mathcal{E} & ::= & [\,] \mid f(c_1, \dots, c_{k-1}, \mathcal{E}, M_{k+1}, \dots, M_m) \mid \texttt{if}\ \mathcal{E}\ \texttt{then}\ M\ \texttt{else}\ N \mid \\ & & op(c_1, \dots, c_{k-1}, \mathcal{E}, N_{k+1}, \dots, N_n; M_1, \dots, M_m) \\ & & \langle \mathcal{E}, N \rangle \mid \langle V, \mathcal{E} \rangle \mid \texttt{fst}(\mathcal{E}) \mid \texttt{snd}(\mathcal{E}) \mid \mathcal{E}N \mid (\lambda x : \sigma.\, M)\mathcal{E} \end{array}$$

and are restricted to be closed. The *redexes* are defined by:

$$\begin{array}{rcl} R & ::= & f(c_1, \ldots, c_m) \mid \texttt{if tt then } M \texttt{ else } N \mid \texttt{if ff then } M \texttt{ else } N \mid \\ & & op(c_1, \ldots, c_n; M_1, \ldots, M_m) \\ & & \texttt{fst}(\langle V, W\rangle) \mid \texttt{snd}(\langle V, W\rangle) \mid (\lambda x : \sigma. M)V \end{array}$$

and are restricted to be closed. Any program is of one of two mutually exclusive forms: it is either a value V or else has the form $\mathcal{E}[R]$ for a unique evaluation context $\mathcal{E}$ and redex R.

We define two small-step transition relations on redexes, *ordinary* transition relations and algebraic operation symbol transition relations:

$$R \to N \qquad \text{and} \qquad R \xrightarrow[op_i]{c_1,\ldots,c_n} N \;\; (op : b_1 \ldots b_n; m \text{ and } i = 1, m)$$

The idea of the algebraic operation symbol transitions is to indicate with which parameters an operation is being executed, and which of its arguments is then being followed. The definition of the first kind of transition is standard; we just mention that for each function symbol $f : b_1 \ldots b_m \to b$ and constants $c_1 : b_1, \ldots, c_m : b_n$, we assume we are given a constant $\mathrm{val}_f(c_1, \ldots, c_m) : b$, where, in the case of equality, we have:

$$\mathrm{val}_{=_b}(c_1, c_2) = \begin{cases} \texttt{tt} & (\text{if } c_1 = c_2) \\ \texttt{ff} & (\text{otherwise}) \end{cases}$$

We then have the ordinary transitions:

$$f(c_1, \ldots, c_m) \to c \quad (\mathrm{val}_f(c_1, \ldots, c_m) = c)$$

The algebraic operation symbol transition relations are given by the following rule:

$$op(c_1, \ldots, c_n; M_1, \ldots, M_m) \xrightarrow[op_i]{c_1,\ldots,c_n} M_i$$

We next extend these transition relations to corresponding ordinary and algebraic operation symbol transition relations on programs

$$M \to M' \quad \text{and} \quad M \xrightarrow[op_i]{c_1,\ldots,c_n} M'$$

To do so, we use evaluation contexts in a standard way by means of the following rules:

$$\frac{R \to M'}{\mathcal{E}[R] \to \mathcal{E}[M']} \qquad \frac{R \xrightarrow[op_i]{c_1,\ldots,c_n} M'}{\mathcal{E}[R] \xrightarrow[op_i]{c_1,\ldots,c_n} \mathcal{E}[M']}$$

These transition relations are all deterministic.

For any program M which is not a value, exactly one of two mutually exclusive possibilities holds:

- For some program M'

  $$M \to M'$$

  In this case M' is determined and of the same type as M.
- For some $op : b_1 \ldots b_n; m$ and $c_1 : b_1, \ldots, c_n : b_n$

  $$M \xrightarrow[op_i]{c_1,\ldots,c_n} M_i$$

  for all $i = 1, \ldots, n$ and some M_i. In this case *op*, the c_j and the M_i are uniquely determined and the M_i have the same type as M.

We say a program M is *terminating* if there is no infinite chain of (small-step) transitions from M.

Lemma 3.1. *Every program is terminating.*

Proof. This is a standard computability argument; see the proof of Theorem 1 in [PP01] for some detail. One defines a computability predicate on values by induction on types, and then extends it to well-typed terms by taking such a term M to be computable if there is no infinite chain of (small-step) transitions from M, and every terminating sequence of small-step transitions from M ends in a computable value. □

Using the small-step relations one defines big-step ordinary and algebraic operation symbol transition relations by:

$$\frac{M \to^* V}{M \Rightarrow V} \qquad \frac{M \to^* M' \quad M' \xrightarrow[op_i]{c_1,\ldots,c_n} M''}{M \overset{c_1,\ldots,c_n}{\underset{op_i}{\Longrightarrow}} M''}$$

For any program M which is not a value, similarly to the case of the small-step relations, exactly one of two mutually exclusive possibilities holds:

- For some value V
$$M \Rightarrow V$$
In this case V is determined and of the same type as M.
- For some $op : b_1 \ldots b_n; m$ and $c_1 : b_1, \ldots, c_n : b_n$
$$M \overset{c_1,\ldots,c_n}{\underset{op_i}{\Longrightarrow}} M_i$$
for all $i = 1, \ldots, n$ and some M_i. In this case op, the c_j and the M_i are uniquely determined and the M_i have the same type as M.

The big-step transition relations from a given program M form a finite tree with values at the leafs, with all transitions, except for those leading to values, being algebraic operation symbol transitions, and with transitions of algebraic operation symbols of type $(w; n)$ branching n-fold. We write $||M||$ for the height of this tree.

Rather than use trees, we follow [PP01] and use *effect values* E. These give the same information and, conveniently, form a subset of our programs. They are defined as follows:

$$E ::= V \mid op(c_1, \ldots, c_n; E_1, \ldots, E_m)$$

(Our effect values are a finitary version of the interaction trees of [XZH$^+$20]). Every program $M : \sigma$ has an effect value $\mathrm{Op}(M) : \sigma$ defined using the big-step transition relations:

$$\mathrm{Op}(M) = \begin{cases} V & (\text{if } M \Rightarrow V) \\ op(c_1, \ldots, c_n; \mathrm{Op}(M_1), \ldots, \mathrm{Op}(M_m)) & (\text{if } M \overset{c_1,\ldots,c_n}{\underset{op_i}{\Longrightarrow}} M_i \text{ for } i = 1, m) \end{cases}$$

This definition is justified by induction on $||M||$. Note that $\mathrm{Op}(E) = E$, for any effect value $E : \sigma$. Further, program transitions and evaluations closely parallel each other, indeed:

$$M \Rightarrow V \iff \mathrm{Op}(M) = V \tag{3.1}$$

and

$$M \overset{op_i}{\underset{c_1,\ldots,c_n}{\Longrightarrow}} M_i \iff \mathrm{Op}(M) \overset{op_i}{\underset{c_1,\ldots,c_n}{\Longrightarrow}} \mathrm{Op}(M_i) \tag{3.2}$$

We next give a proof-theoretic account of the evaluation function Op to help us prove our general adequacy theorem. There is a natural equational theory for the operational semantics, with evident rules, which establishes judgments of the form $\vdash_o M = N : \sigma$, taken to be well-formed in case $M : \sigma$ and $N : \sigma$. The axioms are the small-step reductions for the redexes together with a commutation schema that algebraic operations commute with evaluation contexts; they are given (omitting type information) in Figure 2.

$$f(c_1,\ldots,c_m) = c \quad (\mathrm{val}_f(c_1,\ldots,c_m) = c)$$

$$\texttt{if}\ \texttt{tt}\ \texttt{then}\ M\ \texttt{else}\ N = M \qquad \texttt{if}\ \texttt{ff}\ \texttt{then}\ M\ \texttt{else}\ N = N$$

$$\texttt{fst}(\langle V,W\rangle) = V \qquad \texttt{snd}(\langle V,W\rangle) = W$$

$$(\lambda x:\sigma.\, M)V = M[V/x]$$

$$\mathcal{E}[op(c_1,\ldots,c_n;M_1,\ldots,M_m)] = op(c_1,\ldots,c_n;\mathcal{E}[M_1],\ldots,\mathcal{E}[M_m])$$

Figure 2: Axioms

Lemma 3.2. *For any well-typed term $M:\sigma$ we have:*
(1) $M \to M' \implies \vdash_o M = M' : \sigma$
(2) $M \xrightarrow[op_i]{c_1,\ldots,c_n} M_i,$ *for* $i = 1,\ldots,m \implies \vdash_o M = op(c_1,\ldots,c_n;M_1,\ldots,M_m) : \sigma$
(3) $M \Rightarrow V \implies \vdash_o M = V : \sigma$
(4) $M \xRightarrow[op_i]{c_1,\ldots,c_n} M_i,$ *for* $i = 1,\ldots,m \implies \vdash_o M = op(c_1,\ldots,c_n;M_1,\ldots,M_m) : \sigma$

The following proposition is an immediate consequence of this lemma:

Proposition 3.3. *For any program $M:\sigma$ we have:*

$$\vdash_o M = \mathrm{Op}(M) : \sigma$$

There is a useful substitution lemma. Given any effect value $E:b$, a nonempty finite set $u \subseteq \mathrm{Con}_b$ that includes all the constants of type b in E, and a function g from u to programs of type b', $E[g]:b'$, the substitution g of programs for constants, is defined homomorphically by:

$$\begin{array}{rcl} c[g] & = & g(c) \\ op(c_1,\ldots,c_n;E_1,\ldots,E_m)[g] & = & op(c_1,\ldots,c_n;E_1[g],\ldots,E_m[g]) \end{array}$$

Let $c_1,\ldots,c_n$ enumerate u (the order does not matter) and define $\mathrm{F}_g : b \to b'$ to be

$$\lambda x:b.\,\texttt{cases}\, x = c_1 \Rightarrow g(c_1) \mid \ldots \mid x = c_{n-1} \Rightarrow g(c_{n-1})\ \texttt{else}\ g(c_n)$$

With this notation we have:

Lemma 3.4.

$$\mathrm{Op}(\mathrm{F}_g E) = \mathrm{Op}(E[g])$$

Proof. The proof is a structural induction on E. For E a constant c we have:

$$\begin{array}{rcl} \mathrm{Op}(\mathrm{F}_g E) & = & \mathrm{Op}(\texttt{cases}\, c = c_1 \Rightarrow g(c_1) \mid \ldots \mid c = c_{n-1} \Rightarrow g(c_{n-1})\ \texttt{else}\ g(c_n)) \\ & = & \mathrm{Op}(g(c)) \\ & = & \mathrm{Op}(c[g]) \end{array}$$

and for E of the form $op(c_1, \dots, c_n; E_1, \dots, E_m)$ we have:

$$\begin{aligned}
\mathrm{Op}(\mathrm{F}_g E) &= op(c_1, \dots, c_n; \mathrm{Op}(\mathrm{F}_g E_1), \dots, \mathrm{Op}(\mathrm{F}_g E_m)) \\
&= op(c_1, \dots, c_n; \mathrm{Op}(E_1[g]), \dots, \mathrm{Op}(E_m[g])) \\
&= \mathrm{Op}(op(c_1, \dots, c_n; E_1[g], \dots, E_m[g]) \\
&= \mathrm{Op}(op(c_1, \dots, c_n; E_1, \dots, E_m)[g]) \\
&= \mathrm{Op}(E[g])
\end{aligned}$$

□

3.3. **Denotational semantics.** The semantics of our language makes use of a given strong monad, following that of Moggi's computational λ-calculus [Mog89]. In order to be able to give semantics to effectual operations we use the apparatus of generic effects and algebraic operations as discussed above. For the sake of simplicity we work in the category of sets, although the results go through much more generally, for example in any cartesian closed category with binary sums.

To give the semantics of our language a number of ingredients are needed. We assume given:

- a (necessarily) strong monad M on the category of sets,
- nonempty sets $[\![b]\!]$ for the base types b (with $[\![\mathtt{Bool}]\!] = \mathbb{B} =_{\mathrm{def}} \{0, 1\}$),
- elements $[\![c]\!]$ of $[\![b]\!]$ for constants $c : b$ (with $[\![\mathtt{tt}]\!] = 1$ and $[\![\mathtt{ff}]\!] = 0$),
- functions $[\![f]\!] : [\![b_1]\!] \times \cdots \times [\![b_m]\!] \to [\![b]\!]$ for function symbols $f : b_1 \dots b_m \to b$, and
- generic effects
$$g_{op} : [\![b_1]\!] \times \cdots \times [\![b_n]\!] \to \mathrm{M}([n])$$
for algebraic operation symbols $op : b_1 \dots b_n; m$.

We further assume that different constants of the same type receive different denotations, i.e., the $[\![\text{-}]\!] : \mathrm{Val}_b \to [\![b]\!]$ are 1–1 (so we can think of constants as just names for their denotations, just as one thinks of numerals), and that the given denotations of function symbols are consistent with their operational semantics in that:

$$\mathrm{val}_f(c_1, \dots, c_m) = c \implies [\![f]\!]([\![c_1]\!], \dots, [\![c_m]\!]) = [\![c]\!] \tag{3.3}$$

With these ingredients, we can give our language its semantics. Types are interpreted by putting:

$$\begin{array}{lcl}
\mathcal{M}[\![b]\!] & = & [\![b]\!] \\
\mathcal{M}[\![\sigma \times \tau]\!] & = & \mathcal{M}[\![\sigma]\!] \times \mathcal{M}[\![\tau]\!] \\
\mathcal{M}[\![\sigma \to \tau]\!] & = & \mathcal{M}[\![\sigma]\!] \to \mathrm{M}(\mathcal{M}[\![\tau]\!])
\end{array}$$

To every term

$$\Gamma \vdash N : \sigma$$

we associate a function

$$\mathcal{M}[\![\Gamma \vdash N : \sigma]\!] : \mathcal{M}[\![\Gamma]\!] \to \mathrm{M}(\mathcal{M}[\![\sigma]\!])$$

where $\mathcal{M}[\![x_1 : \sigma_1, \dots, x_n : \sigma_n]\!] =_{\mathrm{def}} \mathcal{M}[\![\sigma_1]\!] \times \cdots \times \mathcal{M}[\![\sigma_n]\!]$. When the typing $\Gamma \vdash N : \sigma$ is understood, we generally write $\mathcal{M}[\![N]\!]$ rather than $\mathcal{M}[\![\Gamma \vdash N : \sigma]\!]$.

The semantic clauses for conditionals and the product and function space terms are standard, and we omit them. For constants $c : b$ we put:

$$\mathcal{M}[\![c]\!](\rho) \;=\; (\eta_{\mathrm{M}})_{[\![b]\!]}([\![c]\!])$$

For function symbol applications $f(M_1, \ldots, M_m)$, where $f : b_1 \ldots b_m \to b$, we put:

$$\mathcal{M}[\![f(M_1, \ldots, M_m)]\!](\rho) \;=\; [\![f]\!]^{\sim}([\![\mathcal{M}]\!](M_1)(\rho), \ldots, [\![\mathcal{M}]\!](M_m)(\rho))$$

where

$$[\![f]\!]^{\sim} : \mathrm{M}([\![b_1]\!]) \times \cdots \times \mathrm{M}([\![b_m]\!]) \to \mathrm{M}([\![b]\!])$$

is obtained from $[\![f]\!]$ in a standard way e.g., via iterated Kleisli extension. For terms $\Gamma \vdash op(N_1, \ldots, N_n; M_1, \ldots, M_m) : \sigma$, where $op : b_1 \ldots b_n; m$, we make use of the algebraic operation

$$op_X : ([\![b_1]\!] \times \cdots \times [\![b_n]\!]) \times \mathrm{M}(X)^{[m]} \to \mathrm{M}(X)$$

corresponding to the generic effects g_{op} and put:

$$\begin{array}{l}\mathcal{M}[\![op(N_1, \ldots, N_n; M_1, \ldots, M_m)]\!](\rho) = \\ \qquad op^{\dagger}_{[\![\sigma]\!]}(\langle \mathcal{M}[\![N_1]\!](\rho), \ldots, \mathcal{M}[\![N_n]\!](\rho)\rangle, \langle \mathcal{M}[\![M_1]\!](\rho), \ldots, \mathcal{M}[\![M_m]\!](\rho)\rangle)\end{array}$$

where $op^{\dagger}_{[\![\sigma]\!]}$ is again defined in a standard way, as discussed in Section 2.2. We further give values $V\!:\!\sigma$ an *effect-free* (or *pure*) semantics $\mathcal{M}_p[\![V]\!] \in \mathcal{M}[\![\sigma]\!]$:

$$\begin{array}{lcl}\mathcal{M}_p[\![c]\!] & = & [\![c]\!] \\ \mathcal{M}_p[\![\langle V, V'\rangle]\!] & = & \langle \mathcal{M}_p[\![V]\!], \mathcal{M}_p[\![V']\!]\rangle \\ \mathcal{M}_p[\![\lambda x : \tau.\, N]\!] & = & \mathcal{M}[\![x : \tau \vdash N : \tau']\!]\end{array}$$

This effect-free semantics of values $V\!:\!\sigma$ determines their denotational semantics:

$$\mathcal{M}[\![V]\!](\rho) = (\eta_{\mathrm{M}})_{\mathcal{M}[\![\sigma]\!]}(\mathcal{M}_p[\![V]\!])$$

Below, we regard the effect-free semantics as providing functions:

$$\mathcal{M}_p : \mathrm{Val}_\sigma \to \mathcal{M}[\![\sigma]\!]$$

3.4. **Adequacy.** Our proof system is consistent relative to our denotational semantics:

Lemma 3.5. *If* $\vdash_o M = N : \sigma$ *then* $\mathcal{M}[\![M]\!] = \mathcal{M}[\![N]\!]$.

The proof of this lemma uses the naturality condition on algebraic operations to establish the soundness of the commutation schema.

Our general adequacy theorem is an immediate consequence of Proposition 3.3 and Lemma 3.5:

Theorem 3.6. *For any program* N *we have:* $\mathcal{M}[\![N]\!] = \mathcal{M}[\![\mathrm{Op}(N)]\!]$.

This adequacy theorem differs somewhat from the usual ones where the denotational semantics determines termination and the denotation of any final result; further, for base types they generally determine the value produced by the operational semantics. In our case the first part is not relevant as terms always terminate. We do have that the denotational semantics determines the denotation of any final result. For base types (as at any type) it determines the effect values produced up to their denotation, though the extent of that determination depends on the choice of the generic effects.

3.5. **Program equivalences and purity.** The equational system of Section 3.2, helps prove adequacy, but is too weak for our purposes which are to establish completeness results for programs of base type. Moggi gave a suitable consistent and complete system for his computational λ-calculus in [Mog89]. His system has equational assertions $\Gamma \vdash M = N : \sigma$ and purity (meaning effect-free) assertions $\Gamma \vdash M \downarrow_\sigma$; we always assume that the terms are appropriately typed, and may omit types or environments when the context makes them clear. One can substitute a term M for a variable in Moggi's system only if one can prove $M \downarrow_\sigma$.

Our λ-calculus is an extension of Moggi's and we extend his logic correspondingly; an alternate approach, well worth pursuing, would be to use instead the purely equational fine-grained variant of the computational λ-calculus: see [LPT03]. We keep Moggi's axioms and rules, other than those for computational types $T\sigma$, but extended to our language. (If we set $T\sigma = 1 \to \sigma$, then the rules for computational types, extended to our language, are derived.)

For conditionals we add:

$$\texttt{if}\ \texttt{tt}\ \texttt{then}\ M\ \texttt{else}\ N = M \qquad \texttt{if}\ \texttt{ff}\ \texttt{then}\ M\ \texttt{else}\ N = N$$

$$u(x) = \texttt{if}\ x\ \texttt{then}\ u(\texttt{tt})\ \texttt{else}\ u(\texttt{ff})$$

For the algebraic operations we add two equations, one:

$$u(op(y_1, \ldots, y_n; M_1, \ldots, M_n)) = op(y_1, \ldots, y_n; u(M_1), \ldots, u(M_n)) \tag{3.4}$$

expressing their naturality (and generalizing the commutation schema of Figure 2), and the other:

$$op(N_1, \ldots, N_n; M_1, \ldots, M_m) = \begin{array}{l}\texttt{let}\ y_1 : b_1, \ldots, y_n : b_n\ \texttt{be}\ N_1, \ldots, N_n \\ \texttt{in}\ op(y_1, \ldots, y_n; M_1, \ldots, M_m)\end{array} \quad (\text{no } y_j \text{ in any } \mathrm{FV}(M_i)) \tag{3.5}$$

expressing the order of evaluation of the parameter arguments of op. For function symbols and constants we add the purity axiom $c\downarrow$ and the equation in Figure 2. This equation enables us to evaluate function symbol applications to constants within our proof system. One could certainly add further useful axioms and rules (e.g., that some function on base types is commutative or a form of induction if the natural numbers were a base type); indeed it would be natural to extend to a predicate logic. However, such extensions are not needed for our purposes.

We write

$$\Gamma \vdash_{\mathrm{Ax}} M = N : \sigma \ \text{ and } \ \Gamma \vdash_{\mathrm{Ax}} M \downarrow_\sigma$$

to mean $M = N$ (resp. $M \downarrow_\sigma$) is provable from a set of equational or purity axioms Ax (where $\Gamma \vdash M : \sigma$ and $\Gamma \vdash N : \sigma$). In particular all the axioms of Figure 2 are provable. An equational assertion is true (or holds) in $\mathcal{M}$, written $\Gamma \models_{\mathcal{M}} M = N : \sigma$ if $\mathcal{M}[\![M]\!] = \mathcal{M}[\![N]\!]$; similarly, a purity assertion is true (or holds) in $\mathcal{M}$, written $\Gamma \models_{\mathcal{M}} M \downarrow_\sigma$, if $\exists a \in \mathcal{M}[\![\sigma]\!].\, \mathcal{M}[\![M]\!](\rho) = \eta_M(a)$. A *theory*, i.e., a set of axioms, Ax is *valid* in $\mathcal{M}$ if all the assertions in Ax are true in $\mathcal{M}$.

Equational consistency holds, meaning that, if a theory Ax is valid in $\mathcal{M}$ then:

$$\Gamma \vdash_{\mathrm{Ax}} M = N : \sigma \implies \Gamma \models_{\mathcal{M}} M = N : \sigma$$

as does the analogous *purity consistency*.

We can use Ax to give axioms for particular algebraic operations. For example, we consider languages with a binary decision algebraic operation symbol $\texttt{or} : \varepsilon; 2$ with semantics given by the algebraic operation family $\texttt{or}_X$ of Definition 2.13. Here the associative axioms

$$(L \,\texttt{or}\, M) \,\texttt{or}\, N = L \,\texttt{or}\, (M \,\texttt{or}\, N)$$

hold at all types as, by Theorem 2.6, every component $\texttt{or}_X$ is associative. We will do this extensively for our two languages, as in Figures 3 and 4, below.

4. A language of choices and rewards

Building on the framework of Section 3, in this section we define and study a language with constructs for choices and rewards.

4.1. Syntax.

For the basic vocabulary of our language, in addition to the boolean primitives of Section 3.1, we assume available: a base type $\texttt{Rew}$; a constant $0 : \texttt{Rew}$; and function symbols $+ : \texttt{Rew}\,\texttt{Rew} \to \texttt{Rew}$ and $\leq : \texttt{Rew}\texttt{Rew} \to \texttt{Bool}$. There are exactly two algebraic operation symbols: a *choice operation* $\texttt{or} : \varepsilon; 2$ to make binary choices, and a *reward operation* $\texttt{reward} : \texttt{Rew}; 1$, to prescribe rewards. We leave any other base type symbols, constants, or function symbols unspecified.

We may use infix for $+$ and $\leq$. Similarly, we may use infix notations $M_0 \,\texttt{or}\, M_1$ or $N \cdot M$ for the algebraic operation terms $\texttt{or}(; M_0, M_1)$ and $\texttt{reward}(N; M)$. The signature $\texttt{or} : \varepsilon; 2$ means that M_0 and M_1 must have the same type and that is then the type of $M_0 \,\texttt{or}\, M_1$; the signature $\texttt{reward} : \texttt{Rew}; 1$ means $N \cdot M$ has the same type as M and that N must be of type $\texttt{Rew}$. For example, assuming that 5 and 6 are two constants of type $\texttt{Rew}$, we may write the tiny program:

$$(5 \cdot \texttt{tt}) \,\texttt{or}\, (6 \cdot \texttt{ff}) : \texttt{Bool}$$

Intuitively, this program could potentially return either $\texttt{tt}$ or $\texttt{ff}$, with respective rewards 5 and 6. In the intended semantics that maximizes rewards, then, the program returns $\texttt{ff}$ with reward 6.

When designing our language, we could as well have used choice functions of any finite arity, as in the example in Figure 1. However we felt that binary choice was sufficiently illustrative.

4.2. Rewards and additional effects.

For both the operational and denotational semantics of our language we need a set of rewards R with appropriate structure and a monad employing it. So, we assume such a set R is available, and that it is equipped with:

- a commutative monoid structure, written additively, and
- a total order with addition preserving and reflecting the order in its first argument (and so, too, in its second), in that, for all $r, s, t \in \mathrm{R}$:

$$r \leq s \iff r + t \leq s + t$$

For example, R could be the reals (or the nonnegative reals) with addition, or the positive reals with multiplication, in all cases with the usual order. We further assume that there is an element $[\![c]\!]$ of R for each $c : \mathrm{Rew}$ (with, in particular, $[\![0]\!] = 0$), and that R is *expressively non-trivial* in that there is a $c : \mathrm{Rew}$ with $[\![c]\!] \neq 0$.

Our monad is the so-called *writer* monad $\mathrm{W}(X) = \mathrm{R} \times X$, defined using the commutative monoid structure on R. The operational semantics defined below evaluates programs M

of type σ to pairs $\langle r, V\rangle$, with $r \in \mathrm{R}$ and $V : \sigma$, that is to elements of $\mathrm{W}(\mathrm{Val}_\sigma)$. The denotational semantics uses the selection monad augmented with the writer monad, as described in Section 2.

The writer monad is the free-algebra monad for R-actions, i.e., the algebras with an R-indexed family of unary operations, which we write as $\mathtt{reward}(r, -)$ or $r \cdot -$, satisfying the equations

$$0 \cdot x = x \qquad r \cdot s \cdot x = (r + s) \cdot x \tag{4.1}$$

The resulting algebraic operation $(\mathtt{reward}_W)_X : \mathrm{R} \times \mathrm{W}(X) \to \mathrm{W}(X)$ is given by:

$$(\mathtt{reward}_W)_X(r, \langle s, x\rangle) = \langle r + s, x\rangle$$

and is induced by the generic effect $(g_\mathrm{W})_{\mathtt{reward}} : \mathrm{R} \to \mathrm{W}([1])$, where $(g_\mathrm{W})_{\mathtt{reward}}(r) =_{\mathrm{def}} \langle r, *\rangle$. We generally write applications of $(\mathtt{reward}_\mathrm{W})_X$ using an infix operator, $(\cdot_\mathrm{W})_X$, and, in either case, may drop subscripts when they can be understood from the context. As R is itself an R-action (setting $r \cdot s = r + s$), we obtain a W-algebra $\alpha_\mathrm{W} : \mathrm{W}(\mathrm{R}) \to \mathrm{R}$ as described in Section 2.2, finding that $\alpha_\mathrm{W} = +$.

4.3. **Operational semantics.** While the operational semantics of Section 3 is ordinary and does not address optimization, the *selection operational semantics* selects an optimal choice strategy, as suggested in the Introduction. Below we prove an adequacy result relative to a denotational semantics using the selection monad S_W. We thereby give a compositional account of a global quantity: the optimal reward of a program.

For the ordinary operational semantics, we assume available functions val_f for the function symbols f of the basic vocabulary, as discussed in Section 3.2. The global operational semantics selects strategies maximizing the reward they obtain. To define such strategies we employ the version of argmax defined in Section 2: given a finite totally-ordered set S and a reward function $\gamma : S \to \mathrm{R}$, $\mathrm{argmax}_X(\gamma)$ selects the least $s \in S$ maximizing $\gamma(s)$. So, totally ordering S by:

$$s \preceq_\gamma s' \iff \gamma(s) > \gamma(s') \vee (\gamma(s) = \gamma(s') \wedge s \leq s')$$

the selection is of the least element in this total order. It is convenient to use the notation $\mathrm{argmax}\ s : S.\, e$ for $\mathrm{argmax}(\lambda s \in S.\, e)$.

We next define our strategies. The idea is to view an effect value $E : \sigma$ as a one-player game for Player. The subterms E' of E are the positions of the game. In particular:

- if E' is a value, then E is a final position and the reward is 0;
- if $E' = E'_0\ \mathtt{or}\ E'_1$ then Player can choose whether to move to the position E'_0 or the position E'_1; and
- if $E' = c \cdot E'' : \sigma$ then Player moves to E'' and $[\![c]\!]$ is added to the final reward.

The finite set $\mathrm{Str}(E)$ of strategies of an effect value E is defined by the following rules, writing $s : E$ for $s \in \mathrm{Str}(E)$:

$$* : V \qquad \frac{s : E_1}{1s : E_1\ \mathtt{or}\ E_2} \qquad \frac{s : E_2}{2s : E_1\ \mathtt{or}\ E_2} \qquad \frac{s : E}{s : c \cdot E}$$

These strategies can be reformulated as boolean functions on choice subterms; though standard, this is less convenient. Equivalently, one could work with boolean functions on choice nodes (terms) of the tree naturally associated to a term by the big-step reduction relation, noting that this tree is isomorphic to effect values considered as trees (as we see from equivalences 3.1 and 3.2). In this way we would obtain an equivalent optimizing operational semantics which makes no use of effect values. We preferred to work with effect values as they provide a convenient way to work directly with trees formulated as terms. There are also probabilistic strategies, although, as is generally true for MDPs [Bel57], they would not change the optimal expected reward.

For any effect value $E : \sigma$, the outcome $\mathrm{Out}(s, E) \in \mathrm{R} \times \mathrm{Val}_\sigma = \mathrm{W}(\mathrm{Val}_\sigma)$ of a strategy $s : E$ is defined by:

$$\begin{array}{lcl}
\mathrm{Out}(*, V) & = & \langle 0, V\rangle \quad (= \eta_{\mathrm{W}}(V)) \\
\mathrm{Out}(1s, E_1\,\texttt{or}\,E_2) & = & \mathrm{Out}(s, E_1) \\
\mathrm{Out}(2s, E_1\,\texttt{or}\,E_2) & = & \mathrm{Out}(s, E_2) \\
\mathrm{Out}(s, c \cdot E) & = & \texttt{reward}_W([\![c]\!], \mathrm{Out}(s, E))
\end{array}$$

We can then define the reward of such a strategy by:

$$\mathrm{Rew}(s, E) = \pi_1(\mathrm{Out}(s, E))$$

Note that $\pi_1 : \mathrm{R} \times X \to \mathrm{R}$ can be written as $\mathbf{R}_{\mathrm{W}}(-|0) = \alpha_{\mathrm{W}} \circ \mathrm{W}(0_X)$, with $0_X : X \to \mathrm{R}$ the constantly 0 reward function.

As there can be several strategies maximizing the reward of a game, we need a way of choosing between them. We therefore define a total order $\leq_E$ on the strategies of a given game $E : \sigma$:

- Game is V:

$$* \leq_V *$$

- Game is $E_1 + E_2$:

$$(i, s) \leq_{E_1+E_2} (j, s') \iff \begin{array}{l} i < j \quad \vee \\ i = j = 1 \wedge s \leq_{E_1} s' \quad \vee \\ i = j = 2 \wedge s \leq_{E_2} s' \end{array}$$

- Game is $c \cdot E$:

$$s \leq_{c \cdot E} s' \iff s \leq_E s'$$

We can now give our selection operational semantics $\mathrm{Op_s}(M) \in \mathrm{R} \times \mathrm{Val}_\sigma = \mathrm{W}(\mathrm{Val}_\sigma)$ for programs $M : \sigma$. We first find the $\mathrm{Op}(M)$-strategy s_{opt} maximizing the reward; if there is more than one such strategy, we take the least, according to the $\mathrm{Op}(M)$-strategy total order $\leq_{\mathrm{Op}(M)}$. So we set:

$$s_{\mathrm{opt}} =_{\mathrm{def}} \mathrm{argmax}\, s : \mathrm{Op}(M).\, \mathrm{Rew}(s, \mathrm{Op}(M))$$

and then we use that strategy to define $\mathrm{Op_s}(M)$ by setting:

$$\mathrm{Op_s}(M) = \mathrm{Out}(s_{\mathrm{opt}}, \mathrm{Op}(M))$$

With this idea, the definition is:

$$\mathrm{Op_s}(M) =_{\mathrm{def}} \mathrm{Out}(\mathrm{argmax}\, s : \mathrm{Op}(M).\, \mathrm{Rew}(s, \mathrm{Op}(M)), \mathrm{Op}(M))$$

Note that $\mathrm{Op_s}(M) = \mathrm{Op_s}(\mathrm{Op}(M))$. (This follows from the form of the definition of the optimizing operational semantics and the fact that $\mathrm{Op}^2(M) = \mathrm{Op}(M)$.)

While the operational semantics is defined by a global optimization over all strategies, it can be equivalently given locally without reference to any strategies. We first need two lemmas. Their statements use the $\max_\gamma$ infix notation introduced in Definition 2.12. We omit their straightforward proofs.

Lemma 4.1. *Given functions* $X \xrightarrow{g} Y \xrightarrow{\gamma} \mathrm{R}$*, for all* $u, v \in X$ *we have*

$$g(u \max_{\gamma \circ g} v) \quad = \quad g(u) \max_\gamma g(v)$$

Lemma 4.2 (First argmax lemma)**.** *Let $S_1 \cup S_2$ split a finite total order $\langle S, \leq \rangle$ into two with $S_1 < S_2$ (the latter in the sense that $s_1 < s_2$ for all $s_1 \in S_1$ and $s_2 \in S_2$). Then, for all $\gamma : S \to \mathrm{R}$, we have:*

$$\operatorname{argmax} \gamma = \operatorname{argmax} (\gamma | S_1) \max_\gamma \operatorname{argmax} (\gamma | S_2)$$

We now have our local characterization of the operational semantics:

Theorem 4.3. *For well-typed effect values we have:*

(1) $\mathrm{Op_s}(V) = \langle 0, V \rangle \ \ (= \eta_{\mathrm{W}}(V))$
(2) $\mathrm{Op_s}(E_1 \,\texttt{or}\, E_2) = \mathrm{Op_s}(E_1) \max_{\pi_1} \mathrm{Op_s}(E_2) \ \ (= \mathrm{Op_s}(E_1) \max_{\mathbf{R}_{\mathrm{W}}(-|0)} \mathrm{Op_s}(E_2))$.
(3) $\mathrm{Op_s}(c \cdot E) = \texttt{reward}_{\mathrm{W}}([\![c]\!], \mathrm{Op_s}(E))$

Proof. For Part 1 we calculate:

$$\begin{array}{rcl} \mathrm{Op_s}(V) & = & \mathrm{Out}(\operatorname{argmax} s : V. \,\mathrm{Rew}(s, V), V) \\ & = & \mathrm{Out}(*, V) \\ & = & \langle 0, V \rangle \end{array}$$

The first equality is as $\mathrm{Op}(V) = V$; the second is as values V have only one strategy, $*$.

For part 2, we calculate:

$$\begin{array}{rcl} \mathrm{Op_s}(E_1 \,\texttt{or}\, E_2) & = & \mathrm{Out}(\operatorname{argmax} s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(s, E_1 \,\texttt{or}\, E_2), E_1 \,\texttt{or}\, E_2) \\[2ex] & = & \mathrm{Out}\left(\left(\begin{array}{c} \operatorname{argmax} 1s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(1s, E_1 \,\texttt{or}\, E_2) \\ \max_{\mathrm{Rew}(-, E_1 \,\texttt{or}\, E_2)} \\ \operatorname{argmax} 2s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(2s, E_1 \,\texttt{or}\, E_2) \end{array}\right), E_1 \,\texttt{or}\, E_2\right) \\ & & \qquad \text{(by the first argmax lemma (Lemma 4.2))} \\[2ex] & = & \mathrm{Out}\left(\left(\begin{array}{c} \operatorname{argmax} 1s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(s, E_1) \\ \max_{\pi_1 \circ \mathrm{Out}(-, E_1 \,\texttt{or}\, E_2)} \\ \operatorname{argmax} 2s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(s, E_2) \end{array}\right), E_1 \,\texttt{or}\, E_2\right) \\[2ex] & = & \begin{array}{c} \mathrm{Out}(\operatorname{argmax} 1s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(s, E_1), E_1 \,\texttt{or}\, E_2) \\ \max_{\pi_1} \\ \mathrm{Out}(\operatorname{argmax} 2s : E_1 \,\texttt{or}\, E_2. \,\mathrm{Rew}(s, E_2), E_1 \,\texttt{or}\, E_2) \end{array} \\ & & \qquad \text{(by Lemma 4.1)} \\[2ex] & = & \begin{array}{c} \mathrm{Out}(\operatorname{argmax} s : E_1. \,\mathrm{Rew}(s, E_1), E_1) \\ \max_{\pi_1} \\ \mathrm{Out}(\operatorname{argmax} s : E_2. \,\mathrm{Rew}(s, E_2), E_2) \end{array} \\[2ex] & = & \mathrm{Op_s}(E_1) \max_{\pi_1} \mathrm{Op_s}(E_2) \end{array}$$

And for part 3 we calculate:

$$\begin{array}{rcl} \mathrm{Op_s}(c \cdot E) & = & \mathrm{Out}(\operatorname{argmax} s : c \cdot E. \,\mathrm{Rew}(s, c \cdot E), c \cdot E) \\ & = & \texttt{reward}_{\mathrm{W}}([\![c]\!], \mathrm{Out}(\operatorname{argmax} s : c \cdot E. \,[\![c]\!] + \mathrm{Rew}(s, E), E)) \\ & = & \texttt{reward}_{\mathrm{W}}([\![c]\!], \mathrm{Out}(\operatorname{argmax} s : c \cdot E. \,\mathrm{Rew}(s, E), E)) \\ & = & \texttt{reward}_{\mathrm{W}}([\![c]\!], \mathrm{Out}(\operatorname{argmax} s : E. \,\mathrm{Rew}(s, E), E)) \\ & = & \texttt{reward}_{\mathrm{W}}([\![c]\!], \mathrm{Op_s}(E)) \end{array}$$

where the third equality holds as the monoid preserves and reflects the ordering of R. □

Using this theorem we can show that substitutions of constants for constants can equivalently be done via W. This will prove useful for our investigations of observational equivalence in Section 4.6.

Lemma 4.4. *Suppose $E : b$ is an effect value, and that $f : \mathrm{Val}_b \to \mathrm{Val}_{b'}$. Let $g : u \subseteq \mathrm{Con}_b$ be the restriction of f to a finite set that includes all the constants of type b in E. Then:*

$$\mathrm{Op_s}(E[g]) = \mathrm{W}(f)(\mathrm{Op_s}(E))$$

Proof. The proof is by structural induction. In case E is a constant we have:

$$\mathrm{W}(f)(\mathrm{Op_s}(c)) = \mathrm{W}(f)(\langle 0, c\rangle) = \langle 0, f(c)\rangle = \mathrm{Op_s}(c[g])$$

In case E has the form $E_1 \,\texttt{or}\, E_2$ we have:

$$\begin{array}{lll}
\mathrm{W}(f)(\mathrm{Op_s}(E_1 \,\texttt{or}\, E_2)) & & \\
 & = \mathrm{W}(f)(\mathrm{Op_s}(E_1) \max_{\pi_1} \mathrm{Op_s}(E_2)) & \text{(by Theorem 4.3.2)} \\
 & = \mathrm{W}(f)(\mathrm{Op_s}(E_1) \max_{\pi_1 \circ \mathrm{W}(f)} \mathrm{Op_s}(E_2)) & \text{(as } \pi_1 \circ \mathrm{W}(f) = \pi_1) \\
 & = \mathrm{W}(f)(\mathrm{Op_s}(E_1)) \max_{\pi_1} \mathrm{W}(f)(\mathrm{Op_s}(E_2))) & \text{(by Lemma 4.1)} \\
 & = \mathrm{Op_s}(E_1[g]) \max_{\pi_1} \mathrm{Op_s}(E_2[g]) & \\
 & = SOp(E_1[g] \,\texttt{or}\, E_2[g]) & \text{(by Theorem 4.3.2)} \\
 & = \mathrm{Op_s}((E_1 \,\texttt{or}\, E_2)[g]) &
\end{array}$$

In case E has the form $c \cdot E_1$ we have:

$$\begin{array}{rll}
\mathrm{W}(f)(\mathrm{Op_s}(c \cdot E_1)) & = \mathrm{W}(f)([\![c]\!] \cdot \mathrm{Op_s}(E_1)) & \text{(by Theorem 4.3.3)} \\
 & = [\![c]\!] \cdot (\mathrm{W}(f)(\mathrm{Op_s}(E_1)) & \text{(as W}(f) \text{ is homomorphic)} \\
 & = [\![c]\!] \cdot \mathrm{Op_s}(E_1[g]) & \\
 & = \mathrm{Op_s}(c \cdot (E_1[g])) & \text{(by Theorem 4.3.3)} \\
 & = \mathrm{Op_s}((c \cdot E_1)[g]) &
\end{array}$$

□

4.4. **Denotational semantics.** For the denotational semantics, as discussed in Section 2.1, we need an auxiliary monad T, here to handle the reward effect. and we take T to be $\mathrm{W} = \mathrm{R} \times -$, the writer monad, and we have the W-algebra $\alpha_\mathrm{W} : \mathrm{W}(\mathrm{R}) \to \mathrm{R}$ where $\alpha_\mathrm{W}(\langle r, s\rangle) = r + s$ as discussed in Section 4.2. We therefore have a strong monad

$$\mathrm{S}(X) = (X \to \mathrm{R}) \to \mathrm{R} \times X$$

and use this monad to give the denotational semantics

$$\mathcal{S}_\mathrm{T}[\![M]\!] : \mathcal{S}_\mathrm{T}[\![\Gamma]\!] \to \mathrm{S}(\mathcal{S}_\mathrm{T}[\![\sigma]\!]) \qquad (\text{for } \Gamma \vdash M : \sigma)$$

of our language, following the pattern explained in the previous section. (We often drop the subscript on $\mathcal{S}_\mathrm{T}$ below.)

We assume available semantics of base types, constants, and function symbols, as discussed in Section 3.3 with, in particular: $[\![\texttt{Rew}]\!] = \mathrm{R}$; $[\![c]\!]$ as in Section 4.2, for $c : \texttt{Rew}$; and $[\![+]\!]$ and $[\![\leq]\!]$ the monoid operation and ordering on R. Recall that different constants of the same type are required to receive different denotations and that the consistency condition 3.3 is required to be satisfied.

Turning to the algebraic operation symbols, for `or` we use the algebraic operation family $\texttt{or}_X$ given by Equation 2.13, so:

$$(\texttt{or})_X(G_0, G_1)(\gamma) = G_0\gamma \max_{X, \mathbf{R}_\mathrm{W}(-|\gamma)} G_1\gamma \tag{4.2}$$

where, for any $\gamma : X \to \mathrm{R}$, $\max_{X,\gamma} : X^2 \to X$ is defined by:

$$x \max_{X,\gamma} y = \begin{cases} x & (\text{if } \gamma(x) \geq \gamma(y)) \\ y & (\text{otherwise}) \end{cases}$$

For `reward` we take the algebraic operation family $(\mathtt{reward}_{\mathrm{S}})_X$ induced by the $(\mathtt{reward}_{\mathrm{W}})_X$, so, using Equation 2.9:

$$(\mathtt{reward}_{\mathrm{S}})_X(r, G)(\gamma) \quad = \quad (\mathtt{reward}_{\mathrm{W}})_X(r, G\gamma) \quad = \quad \langle r + \pi_1(G\gamma), \pi_2(G\gamma)\rangle \tag{4.3}$$

4.5. **Adequacy.** We next aim to prove that the selection operational semantics essentially coincides with its denotational semantics. This coincidence is our *selection adequacy theorem.*

We need some notation to connect the operational semantics of programs with their denotations. We set $(\llbracket\text{-}\rrbracket_{\mathrm{W}})_\sigma = \mathrm{W}(\mathcal{S}_p) : \mathrm{W}(\mathrm{Val}_\sigma) \to \mathrm{W}(\mathcal{S}\llbracket\sigma\rrbracket)$. So for $u = \langle r, V\rangle$ in $\mathrm{R} \times \mathrm{Val}_\sigma = \mathrm{W}(\mathrm{Val}_\sigma)$ we have

$$(\llbracket\langle r, V\rangle\rrbracket_{\mathrm{W}})_\sigma = \langle r, \mathcal{S}_p\llbracket V\rrbracket\rangle$$

Lemma 4.5. *For any effect value $E : \sigma$ we have:*

$$\mathcal{S}_{\mathrm{W}}\llbracket E\rrbracket(0) = \llbracket\mathrm{Op_s}(E)\rrbracket_{\mathrm{W}}$$

Proof. We proceed by structural induction on E, and cases according to its form.

(1) Suppose E is a value V. Using Theorem 4.3.1, we calculate:

$$\llbracket\mathrm{Op_s}(V)\rrbracket_{\mathrm{W}} \;=\; \llbracket\langle 0, V\rangle\rrbracket_{\mathrm{W}} \;=\; \langle 0, \mathcal{S}_p\llbracket V\rrbracket\rangle \;=\; \eta_{\mathrm{W}}(\mathcal{S}_p\llbracket V\rrbracket) \;=\; \eta_{\mathrm{S}}(\mathcal{S}_p\llbracket V\rrbracket)(0) \;=\; \mathcal{S}\llbracket V\rrbracket(0)$$

(2) Suppose next that $E = E_1$ `or` E_2. Then:

$$\begin{array}{lll}
\llbracket\mathrm{Op_s}(E_1 \;\mathtt{or}\; E_2)\rrbracket_{\mathrm{W}} & & \\
\quad = \llbracket\mathrm{Op_s}(E_1) \max_{\mathbf{R}_{\mathrm{W}}(-|0)} \mathrm{Op_s}(E_2)\rrbracket_{\mathrm{W}} & & (\text{by Theorem 4.3.2}) \\
\quad = \llbracket\mathrm{Op_s}(E_1) \max_{\alpha_{\mathrm{W}} \circ \mathrm{W}(0)} \mathrm{Op_s}(E_2)\rrbracket_{\mathrm{W}} & & \\
\quad = \llbracket\mathrm{Op_s}(E_1) \max_{\alpha_{\mathrm{W}} \circ \mathrm{W}(0) \circ \mathrm{W}(\llbracket\,\rrbracket_\sigma)} \mathrm{Op_s}(E_2)\rrbracket_{\mathrm{W}} & & \\
\quad = \llbracket\mathrm{Op_s}(E_1)\rrbracket_{\mathrm{W}} \max_{\alpha_{\mathrm{W}} \circ \mathrm{W}(0)} \llbracket\mathrm{Op_s}(E_2)\rrbracket_{\mathrm{W}} & & (\text{using Lemma 4.1}) \\
\quad = \mathcal{S}\llbracket E_1\rrbracket(0) \max_{\alpha_{\mathrm{W}} \circ \mathrm{W}(0)} \mathcal{S}\llbracket E_2\rrbracket(0) & & (\text{by induction hypothesis}) \\
\quad = \mathtt{or}_{\llbracket\sigma\rrbracket}(\mathcal{S}\llbracket E_1\rrbracket, \mathcal{S}\llbracket E_2\rrbracket)(0) & & (\text{by Equation 4.2}) \\
\quad = \mathcal{S}\llbracket E_1 \;\mathtt{or}\; E_2\rrbracket(0) & &
\end{array}$$

(3) Suppose instead that $E = c \cdot E'$. Then:

$$\begin{aligned}
\llbracket\mathrm{Op_s}(c \cdot E')\rrbracket_{\mathrm{W}} &= \mathrm{W}(\mathcal{S}_p)(\mathrm{Op_s}(c \cdot E')) & \\
&= \mathrm{W}(\mathcal{S}_p)(\llbracket c\rrbracket \cdot_{\mathrm{W}} \mathrm{Op_s}(E')) & (\text{by Theorem 4.3.3}) \\
&= \llbracket c\rrbracket \cdot_{\mathrm{W}} \mathrm{W}(\mathcal{S}_p)(\mathrm{Op_s}(E')) & (\text{as } \mathrm{W}(\mathcal{S}_p) \text{ is a homomorphism}) \\
&= \llbracket c\rrbracket \cdot_{\mathrm{W}} \mathcal{S}\llbracket E'\rrbracket(0) & (\text{by induction hypothesis}) \\
&= (\llbracket c\rrbracket \cdot_{\mathrm{S}} \mathcal{S}\llbracket E'\rrbracket)(0) & (\text{by Equation 4.3}) \\
&= \mathcal{S}\llbracket c \cdot E'\rrbracket(0) &
\end{aligned}$$

□

Theorem 4.6 (Selection adequacy)**.** *For any program $M : \sigma$ we have:*

$$\mathcal{S}_{\mathrm{W}}\llbracket M\rrbracket(0) = \llbracket\mathrm{Op_s}(M)\rrbracket_{\mathrm{W}}$$

Proof. We have:

$$\begin{aligned}\mathcal{S}_{\mathrm{W}}[\![M]\!](0) &= \mathcal{S}_{\mathrm{W}}[\![\mathrm{Op}(M)]\!](0) && \text{(by Theorem 3.6)}\\ &= [\![\mathrm{Op_s}(\mathrm{Op}(M))]\!]_{\mathrm{W}} && \text{(by Lemma 4.5)}\\ &= [\![\mathrm{Op_s}(M)]\!]_{\mathrm{W}}\end{aligned}$$

□

This theorem relates the compositional denotational semantics to the globally optimizing operational semantics. In particular, the latter determines the former at the zero-reward continuation. Whereas the denotational semantics optimizes only locally, as witnessed by the semantics of `or`, the latter optimizes over all possible Player strategies. The use of the zero-reward continuation is reasonable as the operational semantics of a program does not consider any continuation, and so, as rewards mount up additively, the zero-reward continuation is appropriate at the top level.

In more detail, setting $\langle r, V\rangle = \mathrm{Op_s}(M)$, the theorem states that $\mathcal{S}_{\mathrm{T}}[\![M]\!](0) = \langle r, \mathcal{S}_p[\![V]\!]\rangle$. So the rewards according to both semantics agree, and the denotation of the value returned by the globally optimizing operational semantics is given by the denotational semantics. In the case of base types (or, more generally, products of base types) the globally optimizing operational semantics is determined by the denotational semantics as the denotations of values of base types determine the values (see Section 3.4), and so, in that case, there is complete agreement between the operational semantics and the denotational semantics at the zero-reward continuation.

4.6. **Full abstraction, program equivalences, and purity.** Given a notion of observations $\mathrm{Ob}(M)$ of programs $M : b$ of a base type b, one can define a notion of *observational* or *behavioural* equivalence in a standard contextual manner; such notions are usually syntactical, being derived from operational semantics, though that is not necessary. Observational equivalence is generally robust against variations in the notion of observation, and we explore such variations in the context of our decision-making languages.

So, for such a notion of observations $\mathrm{Ob}(M)$ of programs of base type b, for programs $M, N : \sigma$, define operational equivalence $M(\approx_{b,\mathrm{Ob}})_\sigma N$ between them by:

$$M(\approx_{b,\mathrm{Ob}})_\sigma N \iff \forall C[\ \ ] : \sigma \to b.\, \mathrm{Ob}(C[M]) = \mathrm{Ob}(C[N])$$

(Here $C[\ \ ]$ ranges over contexts with a single hole, defined in a standard way, and by $C[\ \ ] : \sigma \to \tau$ we mean that for any $L : \sigma$ we have $C[L] : \tau$.) We generally drop the type subscript σ below. Observational equivalence is an equivalence relation at any type, and it is closed under contexts, in the sense that for all programs $M : \sigma$, $N : \sigma$ and contexts $C[\ \ ] : \sigma \to \tau$ we have:

$$M \approx_{b,\mathrm{Ob}} N \implies C[M] \approx_{b,\mathrm{Ob}} C[N]$$

Operational adequacy generally yields the implication:

$$\models_{\mathcal{M}} M = N : b \implies \mathrm{Ob}(M) = \mathrm{Ob}(N) \tag{4.4}$$

and it then follows that

$$\models_{\mathcal{M}} M = N : \sigma \implies M \approx_{b,\mathrm{Ob}} N \tag{4.5}$$

As a particular case of this implication we have $M \approx_{b,\mathrm{Ob}} \mathrm{Op}(M)$ for programs $M : \sigma$. The converse of the implication 4.5 is *full abstraction* (of $\mathcal{M}$ with respect to $\approx_{b,\mathrm{Ob}}$) at type σ.

In the case of our language of choice and rewards, we work with observational equivalence at boolean type, and take the notion of observation to be simply the optimizing operational semantics $\mathrm{Op_s}$, and write $\approx_b$ for $\approx_{b,\mathrm{Op_s}}$, and $\approx$ for $\approx_{\texttt{Bool}}$. Note that the selection adequacy theorem (Theorem 4.6) immediately yields the implication 4.4 (and so also implication 4.5) for $\mathcal{S}_\mathrm{W}$ and $\mathrm{Op_s}$, as expected, and we then also have $M \approx_b \mathrm{Op}(M)$ for base types b and programs $M : \sigma$.

We next see that, with this notion of observation, observational equivalence is robust against changes in choice of base type (Proposition 4.8). We investigate the robustness of observational equivalence against weakenings of the notion of observation later, observing either only values (Theorem 4.13) or only rewards (Corollary 4.16).

Lemma 4.7. *Suppose that b is a base type with at least two constants. Then for any base type b' and programs $M_1, M_2 : b'$ we have:*

$$M_1 \approx_b M_2 \implies \mathrm{Op_s}(M_1) = \mathrm{Op_s}(M_2)$$

Proof. Let E_i be $\mathrm{Op}(M_i)$ for $i = 1, 2$. Then $E_1 \approx_b E_2$ (as $M_i \approx_b \mathrm{Op}(M_i)$ for $i = 1, 2$) and it suffices to prove that $\mathrm{Op_s}(E_1) = \mathrm{Op_s}(E_2)$. Suppose that $\mathrm{Op_s}(E_i) = \langle r_i, c_i\rangle$ for $i = 1, 2$. Let $f : \mathrm{Val}_{b'} \to \mathrm{Val}_b$ be such that $f(c_1)$ and $f(c_2)$ are distinct, in case c_1 and c_2 are, and let g be its restriction to the constants of the E_i of type b'. For $i = 1, 2$, we have:

$$\begin{array}{rcll} \mathrm{Op_s}(\mathrm{F}_g E_i) & = & \mathrm{Op_s}(E_i[g]) & \text{(by Lemma 3.4)}\\ & = & \mathrm{W}(f)(\mathrm{Op_s}(E_i)) & \text{(by Lemma 4.4)}\\ & = & \langle r_i, f(c_i)\rangle & \end{array}$$

As $E_1 \approx_b E_2$, we have $\mathrm{F}_g E_1 \approx_b \mathrm{F}_g E_2$ and so $\mathrm{Op_s}(\mathrm{F}_g E_1) = \mathrm{Op_s}(\mathrm{F}_g E_2)$ and so, from the above equations for the $\mathrm{Op_s}(\mathrm{F}_g E_i)$, that $\langle r_1, f(c_1)\rangle = \langle r_2, f(c_2)\rangle$. So, as f is 1–1 on $\{c_1, c_2\}$, $\langle r_1, c_1\rangle = \langle r_2, c_2\rangle$ as required. □

As an immediate consequence of this lemma we have the following proposition that change of non-trivial base type does not affect observational equivalence:

Proposition 4.8. *For all base types b and programs $M, N : \sigma$, we have*

$$M \approx N \implies M \approx_b N$$

with the converse holding if there are at least two constants of type b.

Because the denotational semantics is compositional, it facilitates proofs of program equivalences, including ones that justify program transformations, and more broadly can be convenient for certain arguments about programs. For this purpose, we rely on the equivalence relation $\Gamma \vdash_{\mathrm{Ax}} M = N : \sigma$ described in Section 3.5. As remarked there, our general semantics is equationally consistent. We interest ourselves in a limited converse, with σ a base type and M and N programs; we call this *base type program completeness.*

Our system of axioms, Ax, is given in Figure 3. As shown in Theorem 2.6, the choice operation is associative and idempotent; from Corollary 2.5 we have that the reward operation is an R-action on the $\mathrm{S}(X)$ since it is on the $\mathrm{W}(X)$; and we see from Theorem 2.7 that the reward operation commutes with the choice operation as the monoid addition preserves and reflects the order. This justifies the first five of our axioms. A pointwise argument then shows that the following equality holds for $r, s \in \mathrm{R}$ and $F, G \in \mathrm{S}(X)$, for any set X:

$$r\cdot F \;\texttt{or}\; s\cdot F = t\cdot F \quad (t = \max(r, s)) \tag{4.6}$$

Using this equality, the left-bias of the choice operation (shown in Theorem 2.6), and associativity, we have:

$$(r\cdot F \;\texttt{or}\; G) \;\texttt{or}\; s\cdot F = r\cdot F \;\texttt{or}\; G \quad (r \geq s) \tag{4.7}$$

and another pointwise argument establishes the equation:

$$(r\cdot F \,\texttt{or}\, G)\,\texttt{or}\, s\cdot F = G\,\texttt{or}\, s\cdot F \quad (r < s) \tag{4.8}$$

These remarks justify our last two axioms.

$$\begin{array}{c}
(L\,\texttt{or}\,M)\,\texttt{or}\,N \;=\; L\,\texttt{or}\,(M\,\texttt{or}\,N) \qquad M\,\texttt{or}\,M \;=\; M \\
0\cdot N \;=\; N \qquad x\cdot(y\cdot N) \;=\; (x+y)\cdot N \\
x\cdot(M\,\texttt{or}\,N) \;=\; (x\cdot M)\,\texttt{or}\,(x\cdot N) \\
\texttt{if}\; x\geq y\;\texttt{then}\; x\cdot M\;\texttt{else}\; y\cdot M \;=\; x\cdot M\,\texttt{or}\,y\cdot M \\
\texttt{if}\; x\geq z\;\texttt{then}\;(x\cdot M\,\texttt{or}\,N)\;\texttt{else}\;(N\,\texttt{or}\,z\cdot M) \;=\; (x\cdot M\,\texttt{or}\,N)\,\texttt{or}\,z\cdot M
\end{array}$$

Figure 3: Equations for choices and rewards

Some useful consequences of these equations, mirroring the equalities 4.6–4.8, are:

$$c\cdot M\,\texttt{or}\,c'\cdot M \;=\; c''\cdot M \quad (\text{where } [\![c'']\!] \;=\; \max([\![c]\!],[\![c']\!])) \tag{R$_1$}$$

$$(c\cdot M\,\texttt{or}\,N)\,\texttt{or}\,c'\cdot M \;=\; c\cdot M\,\texttt{or}\,N \quad (\text{if } [\![c]\!]\geq[\![c']\!])\% \tag{R$_2$}$$

$$(c\cdot M\,\texttt{or}\,N)\,\texttt{or}\,c'\cdot M \;=\; N\,\texttt{or}\,c'\cdot M \quad (\text{if } [\![c]\!]<[\![c']\!])\% \tag{R$_3$}$$

Our equational system allows programs to be put into a canonical form. We say that a *canonical form* (ignoring bracketing of `or`) is an effect value of the form

$$(c_1\cdot V_1)\,\texttt{or}\,\ldots\,\texttt{or}\,(c_n\cdot V_n)$$

with $n>0$ and no V_i occurring twice.

Lemma 4.9. *Every program M is provably equal to a canonical form* $\mathrm{CF}(M)$*.*

Proof. By the ordinary adequacy theorem (Theorem 3.6), M can be proved equal to an effect value E. Using the associativity equations, the fact that `reward` and `or` commute, and the R-action equations, E can be proved equal to a term of the form $c_1\cdot V_1\,\texttt{or}\,\ldots\,\texttt{or}\,V_n\cdot d_n$, possibly with some V's occurring more than once. Such duplications can be removed using equations R_1, R_2, and R_3 and associativity. □

The next theorem shows that, for programs of base type, four equivalence relations coincide, and thereby simultaneously establishes for them: a normal form for provable equality; completeness of our proof system for equations between such programs; and full abstraction.

Theorem 4.10. *For any two programs M and N of base type b, the following equivalences hold:*

$$\mathrm{CF}(M)=\mathrm{CF}(N) \iff \vdash_{\mathrm{Ax}} M=N:b \iff \models_{\mathcal{S}} M=N:b \iff M\approx N$$

Proof. We already know the implications from left-to-right hold. So it suffices to show that:

$$\mathrm{CF}(M)\neq\mathrm{CF}(N) \implies M\not\approx N$$

First fix $l,r:\mathrm{Rew}$ with $l<r$ (possible as R is expressively non-trivial). We remark that, in general, to prove $A\not\approx B$ for $A,B:\sigma$ it suffices to to prove $A'\not\approx B'$ if we have $\vdash_{\mathrm{Ax}} A=A':\sigma$ and $\vdash_{\mathrm{Ax}} B=B':\sigma$. We use this fact freely below. We also find it convenient to confuse sums of Rew constants with their denotations.

Let the canonical forms of M and N be

$$A = (c_1 \cdot d_1)\,\texttt{or}\,\ldots\,\texttt{or}\,(c_n \cdot d_n) \quad \text{and} \quad B = (c'_1 \cdot d'_1)\,\texttt{or}\,\ldots\,\texttt{or}\,(c'_{n'} \cdot d'_{n'})$$

and suppose they are different. It suffices to prove that $A \not\approx B$. Suppose, first, that for some i_0, d_{i_0} is no d'_j. Choose c to be the maximum of the c_i and the c'_j, other than c_{i_0}. Consider the context:

$$C_1[-] =_{\text{def}} \texttt{if}\ [-] = d_{i_0}\ \texttt{then}\ (c+r)\cdot\texttt{tt}\ \texttt{else}\ (c_{i_0}+l)\cdot\texttt{tt}$$

As $c_i + (c_{i_0}+l) < c_{i_0}+(c+r)$, we have $\pi_1(\mathrm{Op_s}(C_1[A])) = c_{i_0}+c+r$. Further $\pi_1(\mathrm{Op_s}(C_1[B]))$ is the maximum of the $c'_j + c_{i_0} + l$ and so $< c_{i_0}+c+r = \mathrm{Op_s}(C_1[A]))$. So we see that $A \not\approx B$ in this case.

Suppose, instead, that for some i_0, d_{i_0} is d'_{j_0} for some j_0 but that $c_{i_0} < c'_{j_0}$. Then we find that $\pi_1(\mathrm{Op_s}(C_1[A])) = c_{i_0}+c+r$, as before, and that $\pi_1(\mathrm{Op_s}(C_1[B]))$ is the maximum of the $c'_j + (c_{i_0}+l)$, for $j \neq j_0$ and $c'_{j_0} + (c+r)$, which is $c'_{j_0}+(c+r)$, and so we have again distinguished A and B.

So, we may assume that for every $1 \le i \le n$ there is a $1 \le j \le n'$ such that $d_i = d'_j$ and $c_i \ge c'_j$. Arguing symmetrically, and recalling that none of the d_i are repeated, and neither are any of the d'_j, we see that we may assume that $n = n'$ and that $(c_1 \cdot d_1), \ldots, (c_n \cdot d_n)$ and $(c'_1 \cdot d'_1), \ldots, (c'_n \cdot d'_n)$ are permutations of each other.

For the last case, suppose there is a first point i_0 at which A and B differ. We can then write them as:

$$A = A_0\,\texttt{or}\,c_{i_0}\cdot d_{i_0}\,\texttt{or}\,A_1\,\texttt{or}\,c_{i_1}\cdot d_{i_1}\,\texttt{or}\,A_2$$

and

$$B = A_0\,\texttt{or}\,c'_{i_0}\cdot d'_{i_0}\,\texttt{or}\,B_1\,\texttt{or}\,c'_{i_2}\cdot d'_{i_2}\,\texttt{or}\,B_2$$

with $d_{i_0} \neq d'_{i_0}$, $c_{i_1}\cdot d_{i_1} = c'_{i_0}\cdot d'_{i_0}$, and $c'_{i_2}\cdot d'_{i_2} = c_{i_0}\cdot d_{i_0}$, and where we allow any of A_0, A_1, A_2, B_1 or B_2 to be either a canonical form or the empty sequence, and, continuing to ignore parentheses, interpret $A\,\texttt{or}\,B$ and $B\,\texttt{or}\,A$ as B when A is empty and B is not.

Let c be the maximum of the c_i and the c'_i, except for c_{i_0} and c'_{i_0}, and consider the context

$$C_2[-] =_{\text{def}} \begin{array}{l} \mathrm{let}\, x : b\, \mathrm{be}\, [-]\, \mathrm{in} \\ \texttt{if}\ x = d_{i_0}\ \texttt{then}\ (c + c'_{i_0} + r)\cdot\texttt{tt}\ \texttt{else} \\ \texttt{if}\ x = d'_{i_0}\ \texttt{then}\ (c + c_{i_0} + r)\cdot\texttt{ff}\ \texttt{else} \\ \quad (c_{i_0} + c'_{i_0} + l)\cdot\texttt{ff} \end{array}$$

Then $C_2[A]$ is provably equal to

$$(\overline{c}_1 \cdot \overline{d}_1)\,\texttt{or}\,\ldots\,\texttt{or}\,(\overline{c}_n \cdot \overline{d}_n)$$

where

$$\begin{array}{lcl} \overline{c}_{i_0}\cdot\overline{d}_{i_0} & = & (c_{i_0} + c + c'_{i_0} + r)\cdot\texttt{tt} \\ \overline{c}_{i_1}\cdot\overline{d}_{i_1} & = & (c'_{i_0} + c + c_{i_0} + r)\cdot\texttt{ff} \\ \overline{c}_{i}\cdot\overline{d}_{i} & = & (c_i + c_{i_0} + c'_{i_0} + l)\cdot\texttt{ff} \quad (i \neq i_0, i_1) \end{array}$$

and we see that $\mathrm{Op_s}(C_2[A]) = \langle c_{i_0} + c + c'_{i_0} + r, \texttt{tt}\rangle$.

Further, $C_2[B]$ is provably equal to

$$(\overline{c}'_1 \cdot \overline{d}'_1)\,\texttt{or}\,\ldots\,\texttt{or}\,(\overline{c}'_n \cdot \overline{d}'_n)$$

where

$$\begin{array}{lcl} \overline{c}'_{i_0} \cdot \overline{d}'_{i_0} & = & (c'_{i_0} + c + c_{i_0} + r) \cdot \texttt{ff} \\ \overline{c}'_{i_2} \cdot \overline{d}'_{i_2} & = & (c_{i_0} + c + c'_{i_0} + r) \cdot \texttt{tt} \\ \overline{c}'_{i} \cdot \overline{d}'_{i} & = & (c'_{i} + c_{i_0} + c'_{i_0} + l) \cdot \texttt{ff} \quad (i \neq i_0, i_2) \end{array}$$

and we see that $\mathrm{Op_s}(C_2[B]) = \langle c'_{i_0} + c + c_{i_0} + r, \texttt{ff}\rangle$. So $C_2[-]$ distinguishes A and B, concluding this final case. □

Theorem 4.10 is in the spirit of [LS18] in giving axiomatic and denotational accounts of observational equivalence at base types, though here at the level of terms rather than, as there, only effect values. (A natural axiomatic account of the observational equivalence of effect values at base types can be given by specializing the above axioms to them, including R_1, R_2, and R_3, but deleting the last two in Figure 3.)

Theorem 4.10 holds a little more generally: for products of base types. The proof remains the same, using the fact that equality at any product of base types can be programmed using equality at base types. It follows that we have full abstraction at products of base types, i.e., for all programs of types of order 0. A standard argument then shows that full abstraction holds for values of types of order 1; whether or not it holds for programs of types of order 1 is, however, open.

As a corollary of Theorem 4.10 we have completeness for purity (i.e., effect-freeness) assertions at base types. Indeed we have it in a strong form:

Corollary 4.11. *For any program $M : b$, we have:*

$$\models_{\mathcal{S}} M \downarrow b \implies \exists\, c\!:\!b \vdash_{\mathrm{Ax}} M = c$$

Proof. Suppose $\models_{\mathcal{S}} M\!\downarrow\! b$. That is, for some $x \in [\![b]\!]$, $\mathcal{S}[\![M]\!] = \eta_{\mathrm{S_W}}(x) = \lambda\gamma.\, \langle 0, x\rangle$. For some $r \in \mathrm{R}$ and $c\!:\!b$, $\mathrm{Op_s}(M) = \langle r, c\rangle$. So, by adequacy we have:

$$\mathcal{S}[\![M]\!](0) = [\![\mathrm{Op_s}(M)]\!]_{\mathrm{W}} = [\![\langle r, c\rangle]\!]_{\mathrm{W}} = \langle r, [\![c]\!]\rangle$$

As $\mathcal{S}[\![M]\!] = \lambda\gamma.\, \langle 0, x\rangle$ we therefore have $r = 0$ and $[\![c]\!] = x$ and so $\models_{\mathrm{W}} M = c$. It then follows from Theorem 4.10 that $\vdash_{\mathrm{Ax}} M = c$. □

As may be expected, more generally we have strong purity completeness for products of base types, i.e., for any $M : \sigma$ where σ is a product of base types we have:

$$\models_{\mathcal{S}} M \downarrow \sigma \implies \exists\, V\!:\!\sigma \vdash_{\mathrm{Ax}} M = V$$

and, indeed, this is a straightforward consequence of the corollary.

A natural question is whether, instead of using the selection monad $\mathrm{S_T}$, we can treat the choice operator at the same level as the reward one, say using a suitable free-algebra monad. This can be done, to some extent, by making use of Theorem 4.10 and the equations we have established for these operations at the term level. Consider an equational system with a binary (infix) operation symbol $- \,\texttt{or}\, -$ and an R-indexed family of unary operation symbols $r \cdot -$ $(r \in \mathrm{R})$, and impose Equations 4.1, associativity and commutativity equations:

$$x \,\texttt{or}\, (y \,\texttt{or}\, z) = (x \,\texttt{or}\, y) \,\texttt{or}\, z \qquad r \cdot (x \,\texttt{or}\, y) = r \cdot x \,\texttt{or}\, r \cdot y$$

and equations corresponding to Equations R_1, R_2, and R_3:

$$\begin{array}{lcl} r \cdot x \,\texttt{or}\, r' \cdot x & = & \max(r, r') \cdot x \\ (r \cdot x \,\texttt{or}\, r' \cdot y) \,\texttt{or}\, r'' \cdot x & = & r \cdot x \,\texttt{or}\, r' \cdot y \quad (\text{if } r \geq r'') \\ (r \cdot x \,\texttt{or}\, r' \cdot y) \,\texttt{or}\, r'' \cdot x & = & r' \cdot y \,\texttt{or}\, r'' \cdot x \quad (\text{if } r < r'') \end{array}$$

Let C be the resulting free-algebra monad, and let $\mathcal{C}$ be the corresponding denotational semantics. One can show that for all effect values $E, E' : b$ of a base type b we have:

$$\models_{\mathcal{C}} E = E' \iff \vdash_{\mathrm{Ax}} E = E' : b$$

Using Theorems 3.6 and 4.10 we then obtain a version of Theorem 4.10 for $\mathcal{C}$, that, for any two programs M and N of base type b:

$$\vdash_{\mathrm{Ax}} M = N : b \iff \models_{\mathcal{C}} M = N : b \iff M \approx N$$

However we do not obtain an adequacy theorem analogous to the adequacy theorem (Theorem 4.6) which relates the operational semantics to the selection monad semantics at the zero-reward continuation. Consider, for example, the two boolean effect values $\mathtt{tt}$ and $\mathtt{tt}\,\mathtt{or}\,\mathtt{ff}$. Operationally they both evaluate to $\mathtt{tt}$. But they have different $\mathcal{S}$-semantics as the second value is sensitive to the choice of reward continuation. They therefore have different $\mathcal{C}$-semantics, i.e., in this sense the $\mathcal{C}$-semantics is not sound. An alternative would be to extend the operational semantics of programs to take a reward continuation into account, as done in [LS18]; however such an extension would be in tension with the idea that programs should be executable without additional information.

Turning to weakening the notion of observation, we may observe either just the reward or just the final value, giving two weakened notions of observation $\mathrm{Ob_r} = \pi_1 \circ \mathrm{Op_s}$, for the first, and $\mathrm{Ob_v} = \pi_2 \circ \mathrm{Op_s}$, for the second. We begin by investigating observing only values.

Lemma 4.12. *For programs $M, N : \mathtt{Bool}$ we have:*

$$M_1 \approx_{\mathrm{Ob_v}} M_2 \implies \mathrm{Op_s}(M_1) = \mathrm{Op_s}(M_2)$$

Proof. As $\mathrm{Ob_v}$ is weaker than $\mathrm{Op_s}$ the implication 4.4 holds for $\mathcal{S}_{\mathrm{W}}$ and it. We can therefore assume without loss of generality that M_1 and M_2 are effect values, E_1 and E_2, say. Suppose $\mathrm{Op_s}(E_i) = \langle r_i, c_i \rangle$ $(i = 1, 2)$.

Assume $E_1 \approx_{\mathrm{Ob_v}} E_2$. We then have $c_1 = c_2 = \mathtt{tt}$, say. Suppose, for the sake of contradiction, that $r_1 \neq r_2$, and then, without loss of generality, that $r_1 < r_2$. Define $f : \mathrm{Val}_{\mathtt{Bool}} \to \mathrm{Val}_{\mathtt{Bool}}$ to be constantly $\mathtt{ff}$. Then we have

$$\begin{array}{rcll} \mathrm{Ob_v}(E_1\,\mathtt{or}\,\mathrm{F}_f E_2) & = & \pi_2(\mathrm{Op_s}(E_1\,\mathtt{or}\,\mathrm{F}_f E_2)) & \\ & = & \pi_2(\mathrm{Op_s}(E_1)\max_{\pi_1}\mathrm{Op_s}(\mathrm{F}_f E_2)) & \text{(by Theorem 4.3.2)} \\ & = & \pi_2(\langle r_1, \mathtt{tt}\rangle \max_{\pi_1} \mathrm{Op_s}(E_2[f])) & \text{(by Lemma 3.4)} \\ & = & \pi_2(\langle r_1, \mathtt{tt}\rangle \max_{\pi_1} \mathrm{W}(f)(\mathrm{Op_s}(E_2))) & \text{(by Lemma 4.4)} \\ & = & \pi_2(\langle r_1, \mathtt{tt}\rangle \max_{\pi_1} \langle r_2, \mathtt{ff}\rangle) & \\ & = & \mathtt{ff} & \end{array}$$

and, similarly,

$$\begin{array}{rcl} \mathrm{Ob_v}(E_2\,\mathtt{or}\,\mathrm{F}_f E_2) & = & \pi_2(\langle r_2, \mathtt{tt}\rangle \max_{\pi_1} \langle r_2, \mathtt{ff}\rangle) \\ & = & \mathtt{tt} \end{array}$$

yielding the required contradiction, as $E_1 \approx_{\mathrm{Ob_v}} E_2$. □

It immediately follows that observing only values does not weaken the notion of observational equivalence.

Theorem 4.13. *For programs $M, N : \mathtt{Bool}$ we have:*

$$M \approx N \iff M \approx_{\mathrm{Ob_v}} N$$

To investigate observing only rewards, we consider another free algebra monad, $\mathrm{M_r}$. It is the free algebra monad for the equational system with a binary (infix) associative, commutative, absorptive binary operation $- \,\texttt{or}\, -$ which forms a module relative to the max-plus structure of R, meaning that there is an R-indexed family of unary operation symbols $r \cdot -$ $(r \in \mathrm{R})$ forming an R-action and with the following two equations holding:

$$r \cdot (x \,\texttt{or}\, y) = r \cdot x \,\texttt{or}\, r \cdot y \qquad r \cdot x \,\texttt{or}\, r' \cdot x = \max(r, r') \cdot x$$

We write $\mathcal{M}_\mathrm{r}$ for the associated denotational semantics of our language with rewards.

Lemma 4.14. *For any programs $M_1, M_2 : b$ of base type, we have:*

$$\mathcal{M}_\mathrm{r}(M_1) = \mathcal{M}_\mathrm{r}(M_2) \implies \mathrm{Ob_r}(M_1) = \mathrm{Ob_r}(M_2)$$

Proof. Assume $\mathcal{M}_\mathrm{r}(M_1) = \mathcal{M}_\mathrm{r}(M_2)$. We can assume M_1 and M_2 are effect values, say E_1 and E_2. These effect values take their denotations in the free algebra $\mathrm{M_r}([\![b]\!])$. They can be considered as algebra terms if we add the constants $c : b$ in E_1 and E_2 to the signature and identify the constants $c : \mathrm{Rew}$ occurring in subterms of the form $c \cdot E$ with their denotations. With that, their denotations are the same as their denotations in the free algebra extended so that the two denotations of the constants agree. So, as their denotations are equal, they can be proved equal in equational logic using closed instances of the axioms. We show by induction on the size of proof that if $E = E'$ is so provable, then $\mathrm{Ob_r}(E) = \mathrm{Ob_r}(E')$.

Other than commutativity, all closed instances $E = E'$ of the axioms hold in $\mathcal{S}$ and so $\mathrm{Op_s}(E) = \mathrm{Op_s}(E')$ for such instances. By Theorem 4.3.2, for any effect values $E, E' : b$ we have $\mathrm{Ob_r}(E \,\texttt{or}\, E') = \mathrm{Ob_r}(E) \max \mathrm{Ob_r}(E')$, and so $\mathrm{Ob_r}(E \,\texttt{or}\, E') = \mathrm{Ob_r}(E \,\texttt{or}\, E')$ for all closed instances $E \,\texttt{or}\, E' = E \,\texttt{or}\, E'$ of commutativity. The only remaining non-trivial cases are the congruence rules. For that for choice we again use Theorem 4.3.2; for that for rewards we use Theorem 4.3.3, which implies $\mathrm{Ob_r}(r \cdot E) = r + \mathrm{Ob_r}(E)$, for any effect value $E : b$. □

Theorem 4.15. *For any programs $M_1, M_2 : b$ of base type, we have:*

$$\mathcal{M}_\mathrm{r}(M_1) = \mathcal{M}_\mathrm{r}(M_2) \iff M_1 \approx_{\mathrm{Ob_r}} M_2$$

Proof. The implication from left to right follows immediately from Lemma 4.14. For the converse, suppose that $M_1 \approx_{\mathrm{Ob_r}} M_2$. We can assume M_1 and M_2 are effect values, say E_1 and E_2. Let $A_1 = (c_1 \cdot d_1) \,\texttt{or}\, \ldots \,\texttt{or}\, (c_n \cdot d_n)$ and $A_2 = (c'_1 \cdot d'_1) \,\texttt{or}\, \ldots \,\texttt{or}\, (c'_{n'} \cdot d'_{n'})$ be their normal forms. As the program equivalences used to put effect values of base type in normal form follow from those true in $\mathcal{M}_\mathrm{r}$, we have $\mathcal{M}_\mathrm{r}(E_i) = \mathcal{M}_\mathrm{r}(A_i)$ $(i = 1, 2)$. So, as $E_1 \approx_{\mathrm{Ob_r}} E_2$ we have $A_1 \approx_{\mathrm{Ob_r}} A_2$, using the implication from left to right. The first part of the proof of Theorem 4.10 that the two normal forms considered there are identical up to a permutation only uses the reward part of the observation notion $\mathrm{Op_s}$. So, reasoning as there, but now with $\mathrm{Ob_r}$ replacing $\mathrm{Op_s}$, we see that $(c_1 \cdot d_1), \ldots, (c_n \cdot d_n)$ is a permutation of $(c'_1 \cdot d'_1), \ldots, (c'_{n'} \cdot d'_{n'})$. As the commutativity program equivalence holds in $\mathcal{M}_\mathrm{r}$, we therefore have $\mathcal{M}_\mathrm{r}(A_1) = \mathcal{M}_\mathrm{r}(A_2)$ and so $\mathcal{M}_\mathrm{r}(E_1) = \mathcal{M}_\mathrm{r}(E_2)$, concluding the proof. □

Corollary 4.16. *The selection monad semantics augmented with auxiliary monad the writer monad* W *is not fully abstract at base types for $\approx_{\mathrm{Ob_r}}$ (and so $\approx_{\mathrm{Ob_r}}$ is strictly weaker than $\approx$). Indeed for programs $M, N : \sigma$ of any type we have:*

$$M \,\texttt{or}\, N \approx_{\mathrm{Ob_r}} N \,\texttt{or}\, M$$

So, if we only care about optimizing rewards, we may even assume that or is commutative.

5. Adding probabilities

We next extend the language of choices and rewards by probabilistic nondeterminism. Thus, we have the three main ingredients of MDPs, though in the setting of a higher-order λ-calculus rather than the more usual state machines. We proceed as in the previous section, often reusing notation.

5.1. **Syntax.** For the syntax of our language, in addition to the basic vocabulary and algebraic operations of the language of Section 4.1, we assume available algebraic operation symbols $+_p : \varepsilon, 2$ ($p \in [0,1]$) and function symbols $\oplus_p : \texttt{Rew}\,\texttt{Rew} \to \texttt{Rew}$ ($p \in [0,1]$). We use infix notation for both the $+_p$ and the $\oplus_p$. The former are for binary probabilistic choice. The latter are for the convex combination of rewards; they prove useful for the equational logics given in Section 5.7. (For example, see Equations 5.8 and 5.9.) As before, we leave any other base type symbols, constants, or function symbols unspecified.

For example (continuing an example from Section 4.1), we may write the tiny program:

$$(5 \cdot \texttt{tt})\;\texttt{or}\;((5 \cdot \texttt{tt}) +_{.5} (6 \cdot \texttt{ff}))$$

Intuitively, like the program of Section 4.1, this program could return either tt or ff, with respective rewards 5 and 6. Both outcomes are possible on the right branch of its choice, each with probability .5. The intended semantics aims to maximize expected rewards, so that branch is selected.

This example illustrates how the language can express MDP-like transitions. In MDPs, at each time step, the decision-maker chooses an action, and the process randomly moves to a new state and yields rewards; the distribution over the new states depends on the current state and the action. In our language, all decisions are binary, but bigger decisions can be programmed from them. Moreover, the decisions are separate from the probabilistic choices and the rewards, but as in this example it is a simple matter of programming to combine them. A more complete encoding of MDPs can be done by adding primitive recursion to the language, as suggested in the Introduction.

5.2. **Rewards and additional effects.** As in Section 4.2 for both the operational and denotational semantics of our language we need a set of rewards R with appropriate structure and a monad employing it. To specify the structure we require on R, we employ the notion of a barycentric commutative monoid. *Barycentric algebras* (also called convex algebras) are equipped with binary *probabilistic choice* functions $+_p : \mathrm{R}^2 \to \mathrm{R}$ ($p \in [0,1]$) such that the following four equations hold:

$$\begin{array}{rcl}
x +_1 y & = & x \\
x +_p x & = & x \\
x +_p y & = & y +_{1-p} x \\
(x +_p y) +_q z & = & x +_{pq} (y +_{\frac{(1-p)q}{1-pq}} z) \quad (p, q < 1)
\end{array}$$

Barycentric commutative monoids are barycentric algebras further equipped with a commutative monoid structure such that the monoid operation distributes over probabilistic choice, i.e., writing additively:

$$r + (s +_p s') = (r + s) +_p (r + s')$$

Barycentric algebras, introduced by Stone in [Sto49], provide a suitable algebraic structure for probability. They are equivalent to *convex spaces* (also called convex algebras), which are algebras equipped with operations $\sum_{i=1}^{n} p_i x_i$ (where the p_i are in $[0,1]$, and $\sum_{i=1}^{n} p_i = 1$), subject to natural axioms [PR95]; we use the two notations interchangeably. Any mathematical expression built up using the operations of convex spaces from mathematical expressions e_i ($n > 0, i = 1, n$) can be rewritten in the form $\sum_{i=1}^{n} p_i e_i$ using the axioms of convex spaces (and uniquely so if the e_i do not involve the operations of convex spaces). For information on the extensive history of these concepts see [SW15, KP17].

Barycentric commutative monoids appear in the semantics of programming languages with probabilistic choice and nondeterminism and in categorical treatments of probability (for example, see [VW06, KP17, DS21, Jac21, DPS18]).

Turning to our assumptions on rewards, we assume a set R of rewards is available, and that it is equipped with:

- a barycentric commutative monoid structure, and
- a total order with probabilistic choice and addition preserving and reflecting the order in their first argument (and so too in their second), in that, for all $r, s, t \in \mathrm{R}$:

$$r \le s \iff r +_p t \le s +_p t \quad (p \in (0,1))$$

  and

$$r \le s \iff r + t \le s + t$$

(Note the restriction on p in the above condition on probabilistic choice.) In the three examples of Section 4.2 (where the domain of R is the set of reals, nonnegative reals, or positive reals, respectively), probabilistic choice can be defined using the usual convex combination of real numbers: $r +_p s = pr + (1-p)s$. As in Section 4.2 we further assume that there is an element $[\![c]\!]$ of R for each $c : \mathrm{Rew}$ (with, in particular, $[\![0]\!] = 0$), and that R is expressively non-trivial.

Our monad is the combination

$$\mathrm{DW}(X) =_{\mathrm{def}} \mathcal{D}_{\mathrm{f}}(\mathrm{R} \times X)$$

of the finite probability distribution monad with the writer monad for both operational and denotational semantics. Our selection operational semantics, defined below, evaluates programs M of type σ to finite distributions of pairs $\langle r, V \rangle$, with $r \in \mathrm{R}$ and $V : \sigma$, that is to elements of $\mathrm{DW}(\mathrm{Val}_\sigma)$. The monad is the free-algebra monad for *barycentric* R*-modules*. These are algebras with: an R-indexed family of unary operations, written as $\texttt{reward}(r, -)$ or $r \cdot -$, forming an R-action (Equation 4.1); and a $[0,1]$-indexed family $- +_p -$ of binary operations forming a barycentric algebra over which the R-action distributes, i.e., with the following equation holding:

$$r \cdot (x +_p y) = r \cdot x +_p r \cdot y \tag{5.1}$$

The resulting monad has unit $(\eta_{\mathrm{DW}})_X(x) = \delta_{\langle 0, x \rangle}$; the extension to $\mathrm{DW}(X)$ of a map $f : X \to A$ to an algebra A is given by

$$f^{\dagger_{\mathrm{DW}}}(\sum_{i=1}^{n} p_i \langle r_i, x_i \rangle) = \sum_{i=1}^{n} p_i (r_i \cdot f(x_i))$$

(We used the Dirac distribution δ_z here; below, as is common, we just write z.) With the assumptions made on R, it forms a barycentric R-module. Viewing R as a DW-algebra, $\alpha_{\mathrm{DW}} : \mathrm{DW}(\mathrm{R}) \to \mathrm{R}$, we have $\alpha_{\mathrm{DW}} = (\mathrm{id}_{\mathrm{R}})^{\dagger_{\mathrm{DW}}}$; explicitly:

$$\alpha_{\mathrm{DW}}(\sum_{i=1}^{n} p_i \langle r_i, s_i \rangle) = \sum_{i=1}^{n} p_i (r_i + s_i)$$

The two DW-algebraic operations are:

$$(\texttt{reward}_{\mathrm{DW}})_X(r, \sum_{i=1}^{n} p_i(r_i, x_i)) = \sum_{i=1}^{n} p_i(r + r_i, x_i) \qquad (+_{p\mathrm{DW}})_X(\mu, \nu) = p\mu + (1-p)\nu$$

They are induced by the generic effects

$$(g_{\mathrm{DW}})_{\texttt{reward}} : \mathrm{R} \to \mathrm{DW}([1]) \qquad (g_{\mathrm{DW}})_{+} : [0,1] \to \mathrm{DW}([2])$$

where $(g_{\mathrm{DW}})_{\texttt{reward}}(r) = \langle r, *\rangle$ and $(g_{\mathrm{DW}})_{+}(p) = p\langle 0,0\rangle + (1-p)\langle 0,1\rangle$. We generally write $(\texttt{reward}_{\mathrm{DW}})_X$ using an infix operator $(\cdot_{\mathrm{DW}})_X$, as in Section 4.2.

5.3. **Operational semantics.** For the ordinary operational semantics, as in Section 4.3 we assume available functions val_f for the function symbols f of the basic vocabulary, as discussed in Section 3.2. For the selection operational semantics, we again take a game-theoretic point of view, with Player now playing a game against Nature, assumed to make probabilistic choices. Player therefore seeks to optimize their expected rewards. Effect values $E : \sigma$ are regarded as games as before, but with one additional clause:

- if $E = E_1 +_p E_2$, it is Nature's turn to move. Nature picks E_1 with probability p, and E_2 with probability $1 - p$.

To account for probabilistic choice we add a rule to the definition of strategies:

$$\frac{s_1 : E_1 \quad s_2 : E_2}{(s_1, s_2) : E_1 +_p E_2}$$

(Player will need a strategy for whichever move Nature chooses) and a case to the definition of the total orders on strategies:

- Game is $E_1 +_p E_2$:

$$(s_1, s_2) \leq_{E_1 +_p E_2} (s_1', s_2') \iff s_1 <_{E_1} s_1' \quad \vee \quad (s_1 = s_1' \ \wedge \ s_2 \leq_{E_2} s_2')$$

For any effect value $E : \sigma$, the outcome $\mathrm{Out}(s, E)$ of a strategy $s : E$ is a finite probability distribution over $\mathrm{R} \times \mathrm{Val}_\sigma$, i.e., an element of $\mathcal{D}_{\mathrm{f}}(\mathrm{R} \times \mathrm{Val}_\sigma)$. It is defined by:

$$\begin{array}{lcl}
\mathrm{Out}(*, V) & = & \langle 0, V\rangle \\
\mathrm{Out}(1s, E_1 \,\texttt{or}\, E_2) & = & \mathrm{Out}(s, E_1) \\
\mathrm{Out}(2s, E_1 \,\texttt{or}\, E_2) & = & \mathrm{Out}(s, E_2) \\
\mathrm{Out}(s, c \cdot E) & = & [\![c]\!] \cdot_{\mathrm{Val}_\sigma} \mathrm{Out}(s, E) \\
\mathrm{Out}((s_1, s_2), E_1 \ +_p \ E_2) & = & p\,\mathrm{Out}(s_1, E_1) \ + \ (1-p)\,\mathrm{Out}(s_2, E_2)
\end{array}$$

The expected reward of a finite probability distribution on $\mathrm{R} \times X$, for a set X, is

$$\mathbf{E}_X \left(\sum_{i=1}^{n} p_i(r_i, x_i) \right) =_{\mathrm{def}} \sum_{i=1}^{n} p_i r_i$$

Note that $\mathbf{E} : \mathcal{D}_{\mathrm{f}}(\mathrm{R} \times X) \to \mathrm{R}$ can be written as $\mathbf{R}_{\mathrm{DW}}(-|0)$ $(= \alpha_{\mathrm{DW}} \circ \mathrm{DW}(0))$, similarly to how $\pi_1 : \mathrm{R} \times X \to \mathrm{R}$ could be in Section 4.3. The expected reward of a strategy is:

$$\mathrm{Rew}(s, E) =_{\mathrm{def}} \mathbf{E}(\mathrm{Out}(s, E))$$

Our selection operational semantics, $\mathrm{Op_s}(M) \in \mathrm{R} \times \mathrm{Val}_\sigma$ for $M : \sigma$, is defined as before by:

$$\mathrm{Op_s}(M) = \mathrm{Out}(\mathrm{argmax}\, s : \mathrm{Op}(M).\, \mathrm{Rew}(s, \mathrm{Op}(M)), \mathrm{Op}(M))$$

where we are now, as anticipated, maximizing expected rewards.

We remark that, now that probabilistic choice is available, we could change our strategies to make a probabilistic choice for effect values $E_1 \, \texttt{or} \, E_2$. However, as with Markov decision processes [Fel08], that would make no change to the optimal expected reward. It would, however, make a difference to the equational logic of choice if we chose with equal probability between effect values with equal expected reward: choice would then be commutative, but not associative.

Much as in Section 4, we now develop a local characterization of the globally optimizing selection operational semantics. We give this characterization in Theorem 5.3, below; it is analogous to Theorem 4.3 in Section 4. Some auxiliary lemmas are required. The first of them is another argmax lemma enabling us to deal with strategies for probabilistic choice.

Lemma 5.1. *(Second argmax lemma) Let P and Q be finite total orders, let $P \times Q$ be given the lexicographic ordering, and suppose $\gamma : P \times Q \to \mathrm{R}$. Define $g : P \to Q$, $\underline{u} \in P$ and $\underline{v} \in Q$ by:*

$$\begin{array}{lcl} g(u) & = & \mathrm{argmax}\, v : Q.\, \gamma(u, v) \\ \underline{u} & = & \mathrm{argmax}\, u : P.\, \gamma(u, g(u)) \\ \underline{v} & = & g(\underline{u}) \end{array}$$

Then:

$$(\underline{u}, \underline{v}) = \mathrm{argmax}\, (u, v) : P \times Q.\, \gamma(u, v)$$

Proof. Consider any pair (u_0, v_0). By the definition of g we have $g(u_0) \preceq v_0$ in the sense that:

$$\gamma(u_0, g(u_0)) > \gamma(u_0, v_0) \vee (\gamma(u_0, g(u_0)) = \gamma(u_0, v_0) \wedge g(u_0) \leq v_0)$$

and it follows that $(u_0, g(u_0)) \preceq_\gamma (u_0, v_0)$.

Next, by the definition of $\underline{u}$ we have $\underline{u} \preceq u_0$ in the sense that:

$$\gamma(\underline{u}, g(\underline{u})) > \gamma(u_0, g(u_0)) \vee (\gamma(\underline{u}, g(\underline{u})) = \gamma(u_0, g(u_0)) \wedge \underline{u} \leq u_0)$$

and it follows that $(\underline{u}, g(\underline{u})) \preceq_\gamma (u_0, g(u_0))$. (The only non-obvious point may be that in the case where $\gamma(\underline{u}, g(\underline{u})) = \gamma(u_0, g(u_0))$, we have $\underline{u} \leq u_0$, so either $\underline{u} < u_0$, when $(\underline{u}, g(\underline{u})) <_\gamma (u_0, g(u_0))$ or else $\underline{u} = u_0$, when $(\underline{u}, g(\underline{u})) = (u_0, g(u_0))$.)

So, as $\underline{v} = g(\underline{u})$, we have

$$(\underline{u}, \underline{v}) = (\underline{u}, g(\underline{u})) \preceq_\gamma (u_0, g(u_0)) \preceq_\gamma (u_0, v_0)$$

establishing the required minimality of $(\underline{u}, \underline{v})$. □

The next lemma concerns expectations for probability distributions constructed by the reward and convex combination operations.

Lemma 5.2. *We have:*

(1) $\mathrm{Rew}(s, c \cdot E) = [\![c]\!] + \mathrm{Rew}(s, E)$
(2) $\mathrm{Rew}(is_i, E_1 \, \texttt{or} \, E_2) = \mathrm{Rew}(s_i, E_i) \quad (i = 1, 2)$
(3) $\mathrm{Rew}((s_1, s_2), E_1 +_p E_2) = p\mathrm{Rew}(s_1, E_1) + (1 - p)\mathrm{Rew}(s_2, E_2)$

Proof. The second part is evident. For the other two, using the fact that $\mathbf{E}$ is a homomorphism, we calculate:

$$\begin{array}{rcl} \mathrm{Rew}(s, c \cdot E) & = & \mathbf{E}(\mathrm{Out}(s, c \cdot E)) \\ & = & \mathbf{E}([\![c]\!] \cdot \mathrm{Out}(s, \cdot E)) \\ & = & [\![c]\!] + \mathbf{E}(\mathrm{Out}(s, \cdot E)) \\ & = & [\![c]\!] + \mathrm{Rew}(s, E) \end{array}$$

and

$$\begin{aligned}\mathrm{Rew}((s_1,s_2),E_1 +_p E_2) &= \mathbf{E}(\mathrm{Out}((s_1,s_2),E_1 +_p E_2))\\ &= \mathbf{E}(p\,\mathrm{Out}(s_1,E_1) \ + \ (1-p)\,\mathrm{Out}(s_2,E_2))\\ &= p\mathbf{E}(\mathrm{Out}(s_1,E_1)) + (1-p)\mathbf{E}(\mathrm{Out}(s_2,E_2))\\ &= p\mathrm{Rew}(s_1,E_1) + (1-p)\mathrm{Rew}(s_2,E_2)\end{aligned}$$

□

Theorem 5.3. *The following hold for well-typed effect values:*

(1) $\mathrm{Op_s}(V) = \langle 0, V\rangle \ (= \eta_{\mathrm{DW}}(V))$
(2) $\mathrm{Op_s}(E_1 \texttt{ or } E_2) = \mathrm{Op_s}(E_1)\max_{\mathbf{E}}\mathrm{Op_s}(E_2) \ \ (= \mathrm{Op_s}(E_1)\max_{\mathbf{R}_{\mathrm{DW}(-|0)}}\mathrm{Op_s}(E_2))$
(3) $\mathrm{Op_s}(c\cdot E) = [\![c]\!]\cdot\mathrm{Op_s}(E)$
(4) $\mathrm{Op_s}(E_1 +_p E_2) = p\mathrm{Op_s}(E_1) + (1-p)\mathrm{Op_s}(E_2)$

Proof. (1) The proof here is the same as the corresponding case of Theorem 4.3.
(2) The proof here is the same as that of the corresponding case of Theorem 4.3, except that π_1 is replaced by $\mathbf{E}$.
(3) The proof here is again the same as that of the corresponding case of Theorem 4.3, except that we use Lemma 5.2 to show that $\mathrm{Rew}(s, c\cdot E) = [\![c]\!] + \mathrm{Rew}(s,E)$.
(4) We just consider the fourth case. We have:

$$\begin{aligned}&\mathrm{Op_s}(E_1 +_p E_2) =\\ &\qquad \mathrm{Out}(\mathrm{argmax}\,(s_1,s_2) : E_1 +_p E_2.\,\mathrm{Rew}((s_1,s_2),E_1 +_p E_2), E_1 +_p E_2)\end{aligned}$$

So, following the second argmax lemma (Lemma 5.1), we first consider the function

$$f(s_1,s_2) \quad =_{\mathrm{def}} \quad \mathrm{Rew}((s_1,s_2),E_1 +_p E_2) \quad = \quad p\mathrm{Rew}(s_1,E_1) + (1-p)\mathrm{Rew}(s_2,E_2)$$

where the second equality holds by Lemma 5.2. We then consider the function:

$$\begin{array}{rcl} g(s_1) & =_{\mathrm{def}} & \mathrm{argmax}\,s_2 : E_2.\,f(s_1,s_2)\\ & = & \mathrm{argmax}\,s_2 : E_2.\,p\mathrm{Rew}(s_1,E_1) + (1-p)\mathrm{Rew}(s_2,E_2)\\ & = & \mathrm{argmax}\,s_2 : E_2.\,\mathrm{Rew}(s_2,E_2)\end{array}$$

where the second equality holds as convex combinations are order-preserving and reflecting in their second argument. Finally we consider

$$\begin{array}{rcl} \underline{s}_1 & =_{\mathrm{def}} & \mathrm{argmax}\,s_1 : E_1.\,f(s_1,g(s_1))\\ & = & \mathrm{argmax}\,s_1 : E_1.\,p\mathrm{Rew}(s_1,E_1) + (1-p)\mathrm{Rew}(g(s_1),E_2)\\ & = & \mathrm{argmax}\,s_1 : E_1.\,\mathrm{Rew}(s_1,E_1)\end{array}$$

where the last equality holds as convex combinations are order-preserving and reflecting in their first argument, and as $g(s_1)$ is independent of s_1.

So setting

$$\underline{s}_2 = g(\underline{s}_1) = \mathrm{argmax}\,s_2 : E_2.\,\mathrm{Rew}(s_2,E_2)$$

by the second argmax lemma (Lemma 5.1) we have:

$$(\underline{s}_1,\underline{s}_2) = \mathrm{argmax}\,(s_1,s_2) : E_1 +_p E_2.\,\mathrm{Rew}((s_1,s_2),E_1 +_p E_2)$$

so we finally have:

$$\begin{aligned}
\mathrm{Op_s}(E_1 +_p E_2) &= \mathrm{Out}((\underline{s}_1, \underline{s}_2), E_1 +_p E_2)\\
&= p\mathrm{Out}(\underline{s}_1, E_1) + (1-p)\mathrm{Out}(\underline{s}_2, E_2)\\
&= p\mathrm{Out}(\mathrm{argmax}\, s_1 : E_1.\, \mathrm{Rew}(s_1, E_1), E_1)\\
&\quad + (1-p)\mathrm{Out}(\mathrm{argmax}\, s_2 : E_2.\, \mathrm{Rew}(s_2, E_2), E_2)\\
&= p\mathrm{Op_s}(E_1) + (1-p)\mathrm{Op_s}(E_2)
\end{aligned}$$

□

There is an analogous lemma to Lemma 4.4, that substitutions of constants for constants can equivalently be done via DW.

Lemma 5.4. *Suppose $E : b$ is an effect value, and that $f : \mathrm{Val}_b \to \mathrm{Val}_{b'}$.*

(1) $\mathbf{E}_{\mathrm{Val}_{b'}} = \mathbf{E}_{\mathrm{Val}_b} \circ \mathrm{DW}(f)$

(2) *Let $g : u \subseteq \mathrm{Con}_b$ be the restriction of f to a finite set that includes all the constants of type b in E. Then:*

$$\mathrm{Op_s}(E[g]) = \mathrm{DW}(f)(\mathrm{Op_s}(E))$$

Proof. The first part is a straightforward calculation. The proof of the second part is by structural induction on E. The only non-trivial case is where $E = E_1 \texttt{ or } E_2$. We calculate:

$$\begin{aligned}
\mathrm{Op_s}((E_1 \texttt{ or } E_2)[g]) &= \mathrm{Op_s}(E_1[g] \texttt{ or } E_2[g])\\
&= \mathrm{Op_s}(E_1[g]) \max\nolimits_{\mathbf{E}_{\mathrm{Val}_{b'}}} \mathrm{Op_s}(E_2[g]) && \text{(By Theorem 5.3.2)}\\
&= \mathrm{DW}(f)(\mathrm{Op_s}(E_1)) \max\nolimits_{\mathbf{E}_{\mathrm{Val}_{b'}}} \mathrm{DW}(f)(\mathrm{Op_s}(E_2)) && \text{(by induction hypothesis)}\\
&= \mathrm{DW}(f)(\mathrm{Op_s}(E_1) \max\nolimits_{\mathbf{E}_{\mathrm{Val}_{b'}} \circ \mathrm{DW}(f)} \mathrm{Op_s}(E_2))\\
&= \mathrm{DW}(f)(\mathrm{Op_s}(E_1) \max\nolimits_{\mathbf{E}_{\mathrm{Val}_b}} \mathrm{Op_s}(E_2)) && \text{(by Lemma 4.1)}\\
&= \mathrm{DW}(f)(\mathrm{Op_s}(E_1 \texttt{ or } E_2)) && \text{(By Theorem 5.3.2)}
\end{aligned}$$

□

5.4. **Denotational semantics.** For the denotational semantics we consider three auxiliary monads $\mathrm{T}_1, \mathrm{T}_2$, and T_3, corresponding to three notions of observation with varying degrees of correlation between possible program values and expected rewards. Consider, for example, the effect value $1 \cdot \texttt{tt} +_{0.5} (2 \cdot \texttt{ff} +_{0.4} 3 \cdot \texttt{tt})$. With probability 0.5 this returns tt with reward 1, with probability 0.2 it returns ff with reward 2, and with probability 0.3 it returns tt with reward 3. This level of detail is recorded using DW as our first monad. At a much coarser grain, we may simply record that tt and ff are returned with respective probabilities 0.8 and 0.2, and that the overall expected reward is 1.8. This level of detail is recorded using our third monad T_3. At an intermediate level we may record the same outcome distribution and the expected reward given a particular outcome (in the example, the expected reward is 1.4, given outcome tt, and 2, given outcome ff). This level of detail is recorded using our second monad T_2.

We work with a general auxiliary monad, and then specialize our results to the T_i. Specifically, we assume we have: a monad T; T-generic effects $(g_\mathrm{T})_{\texttt{reward}} : \mathrm{R} \to \mathrm{T}([1])$ and $(g_\mathrm{T})_{+_p} \in \mathrm{T}([2])$, with corresponding algebraic operations

$$(\texttt{reward}_\mathrm{T})_X : \mathrm{R} \times \mathrm{T}(X) \to \mathrm{T}(X) \qquad (+_{p\mathrm{T}})_X : \mathrm{T}(X)^2 \to \mathrm{T}(X)$$

together with a T-algebra $\alpha_{\mathrm{T}} : \mathrm{T}(\mathrm{R}) \to \mathrm{R}$, such that, using evident infix notations:

(A1) for any set X, $(\mathtt{reward}_{\mathrm{T}})_X : \mathrm{R} \times \mathrm{T}(X) \to \mathrm{T}(X)$ and $(+_{p\mathrm{T}})_X : \mathrm{T}(X)^2 \to \mathrm{T}(X)$ form a barycentric R-module, and

(A2) the algebra map $\alpha_{\mathrm{T}} : \mathrm{T}(\mathrm{R}) \to \mathrm{R}$ is a barycentric R-module homomorphism, i.e., for $x, y \in \mathrm{T}(\mathrm{R})$ we have:

$$\alpha_{\mathrm{T}}(r \cdot x) = r + \alpha_{\mathrm{T}}(x) \qquad \alpha_{\mathrm{T}}(x \, +_p \, y) = \alpha_{\mathrm{T}}(x) \, +_p \, \alpha_{\mathrm{T}}(y)$$

So we have the anticipated strong monad

$$\mathrm{S}(X) = (X \to \mathrm{R}) \to \mathrm{T}(X)$$

We assume available semantics of base types, constants, and function symbols, as discussed for the language without probability in Section 4.4 with, additionally, the function symbols $\oplus_p$ denoting the corresponding convex combination operations on R. As before, different constants of the same type are required to receive different denotations and the consistency condition 3.3 is required to be satisfied.

As regards the algebraic operation symbols, for `or` we use the algebraic operation `or` given by Equation 2.13, so

$$(\mathtt{or})_X(G_0, G_1)(\gamma) = G_0\gamma \max\nolimits_{X, \mathbf{R}_{\mathrm{T}}(-|\gamma)} G_1\gamma \tag{5.2}$$

For `reward` and $+_p$ we take the algebraic operations $(\mathtt{reward}_{\mathrm{S}})_X$ and $(+_{p\mathrm{S}})_X$ induced by the $(\mathtt{reward}_{\mathrm{T}})_X$ and $(+_{p\mathrm{T}})_X$, so:

$$(\mathtt{reward}_{\mathrm{S}})_X(r, G)(\gamma) = (\mathtt{reward}_{\mathrm{T}})_X(r, G\gamma)$$

and

$$(+_{p\mathrm{S}})_X(F, G)(\gamma) = (+_{p\mathrm{T}})_X(F(\gamma), G(\gamma))$$

As mentioned above, our first monad is $\mathrm{T}_1 =_{\mathrm{def}} \mathrm{DW}$. With its associated generics for reward and probabilistic choice and T-algebra it evidently satisfies the two assumptions (A1) and (A2).

Writing $\mathrm{supp}(\nu)$ for the support of a probability distribution ν, our second monad is

$$\mathrm{T}_2(X) = \{\langle \mu, \rho \rangle \mid \mu \in \mathcal{D}_{\mathrm{f}}(X), \rho : \mathrm{supp}(\mu) \to \mathrm{R}\}$$

It is the free-algebra monad for algebras with an R-indexed family of unary operations, written as $\mathtt{reward}(r, -)$ or $r \cdot -$, and a $[0,1]$-indexed family $- +_p -$ of binary operations satisfying the equations for DW together with the equation:

$$r \cdot x \, +_p \, s \cdot x = (r +_p s) \cdot x \tag{5.3}$$

The two $\mathrm{T}_2(X)$-algebraic operations are:

$$(\mathtt{reward}_{\mathrm{T}_2})_X(r, \langle \mu, \rho \rangle) = \langle \mu, x \mapsto r + \rho(x) \rangle$$

and

$$(+_{p\mathrm{T}_2})_X(\langle \mu, \rho \rangle, \langle \mu', \rho' \rangle) = \langle p\mu + (1-p)\nu, \rho'' \rangle$$

where:

$$\rho''(x) = \begin{cases} \rho(x) +_q \rho'(x) & (x \in \mathrm{supp}(\mu) \cap \mathrm{supp}(\mu')) \\ \rho(x) & (x \in \mathrm{supp}(\mu) \backslash \mathrm{supp}(\mu')) \\ \rho'(x) & (x \in \mathrm{supp}(\mu') \backslash \mathrm{supp}(\mu)) \end{cases}$$

where $q = p\mu(x)/(\mu(x) +_p \mu'(x))$. One can then show that:

$$\sum_{i=1}^{n} p_i \langle \mu_i, \rho_i \rangle = \langle \mu, \rho \rangle \tag{5.4}$$

where $\mu = \sum_{i=1}^{n} p_i \mu_i$ and, for $x \in \mathrm{supp}(\mu)$:

$$\rho(x) = \sum_{\mathrm{supp}(\mu_i) \ni x} \frac{p_i \mu_i(x)}{\mu(x)} \rho_i(x)$$

The resulting monad has unit $(\eta_{\mathrm{T}_2})_X(x) = \langle x, x \mapsto 0 \rangle$; the extension to $\mathrm{T}_2(X)$ of a map $f : X \to A$ to an algebra A is given by

$$f^{\dagger_{\mathrm{T}_2}}(\mu, \rho) = \sum_{x \in \mathrm{supp}(\mu)} \mu(x)(\rho(x) \cdot f(x))$$

and for any $f : X \to Y$ we have:

$$\mathrm{T}_2(f)(\mu, \rho) = \langle \mathcal{D}_{\mathrm{f}}(\mu), \rho' \rangle$$

where

$$\rho'(y) = \frac{\sum_{f(x)=y} \mu(x)\rho(x)}{\mathcal{D}_{\mathrm{f}}(\mu)(y)} \quad (y \in \mathrm{supp}(\mathcal{D}_{\mathrm{f}}(\mu)))$$

Equation 5.3 holds for R, using commutativity and homogeneity, so we can take the algebra map α_{T_2} to be $\mathrm{id}_{\mathrm{R}}^{\dagger_{\mathrm{T}_2}}$; explicitly we find:

$$\alpha_{\mathrm{T}_2}\left(\sum_{i=1}^{n} p_i r_i, \rho\right) = \sum_{i=1}^{n} p_i(\rho(r_i) + r_i)$$

Our third monad $\mathrm{T}_3(X) = \mathcal{D}_{\mathrm{f}}(X) \times \mathrm{R}$ is the free-algebra monad for algebras with an R-indexed family of unary operations, written as $\mathtt{reward}(r, -)$ or $r \cdot -$, and a $[0,1]$-indexed family $- +_p -$ of binary operations satisfying the equations for DW and the equation:

$$r \cdot x +_p s \cdot y = (r +_p s) \cdot x +_p (r +_p s) \cdot y \tag{5.5}$$

The two $\mathrm{T}_3(X)$-algebraic operations are:

$$(\mathtt{reward}_{\mathrm{T}_3})_X(r, \langle \mu, s \rangle) = \langle \mu, r + s \rangle$$

and

$$(+_{p\mathrm{T}_3})_X(\langle \mu, r \rangle, \langle \nu, s \rangle) = \langle p\mu + (1-p)\nu, pr + (1-p)s \rangle$$

One can then show that:

$$\sum_{i=1}^{n} p_i \langle \mu_i, r_i \rangle = \langle \sum_{i=1}^{n} p_i, \sum_{i=1}^{n} \mu_i, r_i \rangle \tag{5.6}$$

The resulting monad has unit

$$(\eta_{\mathrm{T}_3})_X(x) = \langle x, 0 \rangle$$

the extension to $\mathrm{T}_3(X)$ of a map $f : X \to A$ to an algebra A is given by

$$f^{\dagger_{\mathrm{T}_3}}(\sum p_i x_i, r) = r \cdot \sum p_i f(x_i)$$

and

$$\mathrm{T}_3(f)(\langle \sum p_i x_i, r \rangle) = \langle \sum p_i f(x_i), r \rangle$$

Unfortunately Equation 5.5 need not hold for R with the assumptions made on it so far; indeed, while it does hold for the two examples with the reals and addition, it does not hold for the example of the positive reals and multiplication. When dealing with T_3 we therefore assume additionally that R satisfies Equation 5.5, and so we can take α_{T_3} to be $\mathrm{id}_{\mathrm{R}}^{\dagger_{\mathrm{T}_3}}$; explicitly we find:

$$\alpha_{\mathrm{T}_3}\left(\sum_{i=1}^{n} p_i r_i, r\right) = r + \left(\sum_{i=1}^{n} p_i r_i\right)$$

Define comparison maps:

$$(\theta_{\mathrm{DW,T}})_X : \mathrm{DW}(X) \to \mathrm{T}(X) \ =_{\mathrm{def}} \ (\eta_{\mathrm{T}})_X^{\dagger_{\mathrm{DW}}}$$

These functions are useful when discussing adequacy and full abstraction. Explicitly we have:

$$(\theta_{\mathrm{DW,T}})_X(\sum_{i=1}^{n} p_i\langle r_i, x_i\rangle) = \sum_{i=1}^{n} p_i(r_i(\cdot_{\mathrm{T}})_{\mathrm{T}(X)}\eta_{\mathrm{T}}(x_i)) \tag{5.7}$$

Lemma 5.5. $\theta_{\mathrm{DW,T}}$ *is a monad morphism.*

Proof. We have to show that θ is natural and preserves the unit and multiplication maps. We make use of Equation 5.7 throughout the proof.

For naturality we need to show that for $f : X \to Y$ we have $\mathrm{T}(f) \circ \theta_X = \theta_Y \circ \mathrm{DW}(f)$. Choosing $\sum_{i=1}^{n} p_i\langle r_i, x_i\rangle \in \mathrm{DW}(X)$ we have:

$$\begin{array}{rcl} \mathrm{T}(f)(\theta_X(\sum_{i=1}^{n} p_i\langle r_i, x_i\rangle)) & = & \mathrm{T}(f)(\sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(x_i))) \\ & = & \sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \mathrm{T}(f)(\eta_{\mathrm{T}}(x_i))) \\ & = & \sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(f(x_i))) \end{array}$$

using the fact that maps of the form $\mathrm{T}(f)$ act homomorphically on algebraic operations, and:

$$\begin{array}{rcl} \theta_Y(\mathrm{DW}(f)(\sum_{i=1}^{n} p_i\langle r_i, x_i\rangle)) & = & \theta_Y(\sum_{i=1}^{n} p_i\langle r_i, f(x_i)\rangle) \\ & = & \sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(f(x_i))) \end{array}$$

For preservation of the unit we have to show that $(\eta_T)_X = \theta_X \circ \eta_{\mathrm{DW}}$. This is immediate from the definition of θ.

For preservation of multiplication we have to show that

$$\theta_X \circ (\mu_{\mathrm{DW}})_X = (\mu_{\mathrm{T}})_X \circ \theta_{\mathrm{T}(X)} \circ \mathrm{DW}(\theta_X)$$

To this end, choose $\sum_{i=1}^{n} p_i\langle r_i, u_i\rangle \in \mathrm{DW}(\mathrm{DW}(X))$ where $u_i = \sum_{j=1}^{m_i} q_{ij}\langle s_{i,j}, x_{i,j}\rangle$, for $i = 1, \ldots, n$. Then, using the fact that monad multiplications act homomorphically on algebraic operations, we have:

$$\begin{array}{rcl} \theta_X((\mu_{\mathrm{DW}})_X(\sum_{i=1}^{n} p_i\langle r_i, u_i\rangle)) & = & \theta_X(\sum_{i=1}^{n} p_i \sum_{j=1}^{m_i} q_{ij}\langle r_i + s_{i,j}, x_{i,j}\rangle) \\ & = & \sum_{i=1}^{n} p_i \sum_{j=1}^{m_i} q_{ij}(r_i + s_{i,j}) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(x_{i,j}) \end{array}$$

and:

$$\begin{aligned}
&(\mu_{\mathrm{T}})_X(\theta_{\mathrm{T}(X)}(\mathrm{DW}(\theta_X)(\textstyle\sum_{i=1}^n p_i\langle r_i, u_i\rangle)))\\
&\qquad = (\mu_{\mathrm{T}})_X(\theta_{\mathrm{T}(X)}(\textstyle\sum_{i=1}^n p_i\langle r_i, \theta_X(u_i)\rangle))\\
&\qquad = (\mu_{\mathrm{T}})_X(\textstyle\sum_{i=1}^n p_i(r_i(\cdot_T)_{\mathrm{T}(\mathrm{T}(X))}(\eta_T)_{\mathrm{T}(X)}\theta_X(u_i)))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(r_i(\cdot_T)_{\mathrm{T}(X)}(\mu_{\mathrm{T}})_X((\eta_T)_{\mathrm{T}(X)}\theta_X(u_i)))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(r_i(\cdot_T)_{\mathrm{T}(X)}\theta_X(u_i))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(r_i(\cdot_T)_{\mathrm{T}(X)}\theta_X(\sum_{j=1}^{m_i} q_{ij}\langle s_{i,j}, x_{i,j}\rangle))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(r_i \cdot_T \sum_{j=1}^{m_i} q_{ij}(s_{i,j} \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(x_{i,j})))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(\sum_{j=1}^{m_i} q_{ij}(s_{i,j} \cdot r_i \cdot_T \cdot_{\mathrm{T}}\eta_{\mathrm{T}}(x_{i,j})))\\
&\qquad = \textstyle\sum_{i=1}^n p_i(\sum_{j=1}^{m_i} q_{ij}((s_{i,j} + r_i) \cdot_T \cdot_{\mathrm{T}}\eta_{\mathrm{T}}(x_{i,j})))
\end{aligned}$$ □

In the case of T_1, θ_{T_1} is the identity. In the case of T_2, first, given a distribution $\mu = \sum_{i=1}^n p_i\langle r_i, x_i\rangle \in \mathrm{T}_1(X)$, define its *value distribution* $\mathrm{VDis}(\mu)$ in $\mathcal{D}_{\mathrm{f}}(X)$, and its *value support* $\mathrm{vsupp}(\mu) \subseteq X$ by:

$$\mathrm{VDis}(\mu) = \sum_{i=1}^n p_i x_i \qquad \mathrm{vsupp}(\mu) = \{x_i\}$$

and then define the *conditional expected reward* of μ given $x \in \mathrm{vsupp}(\mu)$ by:

$$\mathrm{Rew}(\mu|x) = \frac{\sum_{x_i=x} p_i r_i}{\sum_{x_i=x} p_i}$$

We then have:

$$(\theta_{\mathrm{T}_2})_X(\mu) = \langle \mathrm{VDis}(\mu), \mathrm{Rew}(\mu|-)\rangle$$

as, using Equation 5.4, we can calculate:

$$\begin{array}{rcl}
\theta_{\mathrm{DW},\mathrm{T}_2}(\sum_{i=1}^n p_i\langle r_i, x_i\rangle) & = & \sum_{i=1}^n p_i(r_i \cdot_{\mathrm{T}_2} \eta_{\mathrm{T}_2}(x_i))\\
& = & \sum_{i=1}^n p_i\langle x_i, x_i \mapsto r_i\rangle\\
& = & \langle \mu, \rho\rangle
\end{array}$$

where $\mu = \sum_{i=1}^n p_i x_i$ and, for $x \in \mathrm{supp}(\mu)$:

$$\rho(x) = \sum_{x_i=x} \frac{p_i}{\mu(x)} r_i$$

In the case of T_3 we have:

$$(\theta_{\mathrm{T}_3})_X(\mu) = \langle \mathrm{VDis}(\mu), \mathbf{E}(\mu)\rangle$$

as, using Equation 5.6, we can calculate:

$$\sum_{i=1}^n p_i(r_i \cdot_{\mathrm{T}_3} \eta_{\mathrm{T}_3}(x_i)) \;=\; \sum_{i=1}^n p_i\langle x_i, r\rangle \;=\; \langle \sum_{i=1}^n p_i, \sum_{i=1}^n x_i, r\rangle$$

Two properties of the T_i are useful when we consider full abstraction below. For the first property, say that $\mathbb{B}$ is *characteristic* for T if, for any set X and any two $u, v \in \mathrm{T}(X)$ we have:

$$u \neq v \implies \exists f : X \to \mathbb{B}.\, \mathrm{T}(f)(u) \neq \mathrm{T}(f)(v)$$

Lemma 5.6. *$\mathbb{B}$ is characteristic for each of the T_i.*

Proof. Fix X, and, for any $x \in X$ let $f_x : X \to \mathbb{B}$ be the map that sends x to 0 and everything else in X to 1.

For the case of T_1 suppose we have distinct elements of $\mathrm{T}_1(X)$, viz. $\mu = \sum_{i=1}^m p_i\langle r_i, x_i\rangle$ and $\nu = \sum_{j=1}^n q_j\langle s_j, y_j\rangle$. Then there is an $\langle r_i, x_i\rangle$ in the support of (say) μ that is either not in the support of ν or has different probability there. Then $\langle r_i, 0\rangle$ is in the support of $\mathrm{T}_1(f_{x_i})(\mu)$ but is either not in the support of $\mathrm{T}_1(f_{x_i})(\nu)$ or has different probability there.

In the case of T_2 suppose we have distinct elements of $\mathrm{T}_2(X)$, viz. $a = \langle\sum_{i=1}^m p_i x_i, \rho_0\rangle$ and $b = \langle\sum_{j=1}^n q_j y_j, \rho_1\rangle$. If $\sum_{i=1}^m p_i x_i$ and $\sum_{j=1}^n q_j y_j$ are distinct we proceed as in the case of T_1. Otherwise there is an x_i, say x_1, such that $\rho_0(x_1) \neq \rho_1(x_1)$. Let ρ_0' and ρ_1' be the second components of $\mathrm{T}_2(f_{x_1})(a)$ and $\mathrm{T}_2(f_{x_1})(b)$. Then $\rho_0'(0) = \rho_0(x_i)$ and $\rho_1'(0) = \rho_1(x_i)$ and these are different.

In the case of T_3 suppose we have distinct elements of $\mathrm{T}_3(X)$, viz. $a = \langle\sum_{i=1}^m p_i x_i, r\rangle$ and $b = \langle\sum_{j=1}^n q_j y_j, s\rangle$. If $\sum_{i=1}^m p_i x_i$ and $\sum_{j=1}^n q_j y_j$ are distinct we proceed as in the case of T_1. Otherwise, $r \neq s$, and $\mathrm{T}(f)$ distinguishes a and b. ☐

For the second property, for any X and $\gamma : X \to \mathrm{R}$, define the *reward addition* function

$$\mathrm{k}_\gamma : \mathrm{T}(X) \to \mathrm{T}(X)$$

to be $f^{\dagger_\mathrm{T}}$, where $f(x) =_{\mathrm{def}} \gamma(x) \cdot_\mathrm{T} \eta_\mathrm{T}(x)$. Then we say that *reward addition is injective for* T if such functions are always injective.

Lemma 5.7. *Reward addition is injective for each of the* T_i.

Proof. Fix X and $\gamma : X \to \mathrm{R}$. Beginning with T_1 for any $\mu = \sum_{i=1}^m p_i\langle r_i, x_i\rangle$, with no $\langle r_i, x_i\rangle$ repeated, we have

$$\mathrm{k}_\gamma(\mu) = \sum_{i=1}^m p_i\langle\gamma(x_i) + r_i, x_i\rangle$$

with no repeated $\langle\gamma(x_i) + r_i, x_i\rangle$ (since the monoid addition on R reflects the order). So, for any such $\mu = \sum_{i=1}^m p_i\langle r_i, x_i\rangle$ and $\nu = \sum_{j=1}^n q_j\langle s_j, y_j\rangle$, if we have $\mathrm{k}_\gamma(\mu) = \mathrm{k}_\gamma(\nu)$, i.e., if we have $\sum_{i=1}^m p_i\langle r_i, x_i\rangle = \sum_{j=1}^n q_j\langle s_j, y_j\rangle$, then $m = n$ and, for some permutation π of the indices, we have $p_i\langle\gamma(x_i) + r_i, x_i\rangle = q_{\pi(j)}\langle\gamma(y_{\pi(j)}) + s_{\pi(j)}, y_{\pi(j)}\rangle$. So then $x_i = y_{\pi(j)}$, and $\gamma(x_i) = \gamma(y_{\pi(j)})$ follows, and we see that $p_i\langle r_i, x_i\rangle = q_{\pi(j)}\langle s_{\pi(j)}, y_{\pi(j)}\rangle$. So $\mu = \nu$, as required.

The proofs for T_2 and T_3 are similar, using the respective formulas

$$\mathrm{k}_\gamma(\langle\sum_{i=1}^m p_i x_i, \rho\rangle) = \langle\sum_{i=1}^m p_i x_i, x_i \mapsto \rho(x_i) + \gamma(x_i)\rangle$$

and

$$\mathrm{k}_\gamma(\langle\sum_{i=1}^m p_i x_i, r\rangle) = \langle\sum_{i=1}^m p_i x_i, r + \sum_i p_i\gamma(x_i)\rangle$$ ☐

5.5. **Adequacy.** As in Section 4.5, we aim to prove a selection adequacy theorem connecting the globally defined selection operational semantics with the denotational semantics. We again need some notation. Using assumption (A1) of Section 5.4, we set

$$([\![\text{-}]\!]_\mathrm{T})_\sigma = ((\eta_\mathrm{T})_{\mathcal{S}[\![\sigma]\!]} \circ \mathcal{S}_p)^{\dagger_{\mathrm{DW}}} : \mathrm{DW}(\mathrm{Val}_\sigma) \to \mathrm{T}(\mathcal{S}[\![\sigma]\!])$$

So, for $\mu = \sum_{i=1}^{n} p_i\langle r_i, V_i\rangle \in \mathrm{DW}(\mathrm{Val}_\sigma)$ we have:

$$[\![\mu]\!]_{\mathrm{T}} = \sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(\mathcal{S}_p[\![V_i]\!]))$$

Lemma 5.8. *For any $\mu \in \mathrm{DW}(\mathrm{Val}_\sigma)$ we have: $\alpha_{\mathrm{T}}(\mathrm{T}(0_{[\![\sigma]\!]})([\![\mu]\!]_{\mathrm{T}})) = \alpha_{\mathrm{DW}}(\mathrm{DW}(0_{\mathrm{Val}_\sigma})(\mu))$*

Proof. Suppose $\mu = \sum_{i=1}^{n} p_i\langle r_i, V_i\rangle$. We calculate:

$$\begin{array}{ll} \alpha_{\mathrm{T}}(\mathrm{T}(0)([\![\sum_{i=1}^{n} p_i\langle r_i, V_i\rangle]\!]_{\mathrm{T}})) & \\ \quad = \alpha_{\mathrm{T}}(\mathrm{T}(0)(\sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(\mathcal{S}_p[\![V_i]\!])))) & \\ \quad = \alpha_{\mathrm{T}}(\sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \mathrm{T}(0)(\eta_{\mathrm{T}}(\mathcal{S}_p[\![V_i]\!])))) & \\ \quad = \alpha_{\mathrm{T}}(\sum_{i=1}^{n} p_i(r_i \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(0))) & \\ \quad = \sum_{i=1}^{n} p_i(r_i + \alpha_{\mathrm{T}}(\eta_{\mathrm{T}}(0))) & \text{(by assumption (A2) of Section 5.4)} \\ \quad = \sum_{i=1}^{n} p_i r_i & \\ \quad = \alpha_{\mathrm{DW}}(\mathrm{DW}(0)(\sum_{i=1}^{n} p_i\langle r_i, V_i\rangle)) & \end{array}$$

□

Lemma 5.9. *For any effect value $E : \sigma$ we have: $\mathcal{S}[\![E]\!](0) = [\![\mathrm{Op_s}(E)]\!]_{\mathrm{T}}$*

Proof. We proceed by structural induction on E:

(1) Suppose that E is a value V. We calculate:

$$[\![\mathrm{Op_s}(V)]\!]_{\mathrm{T}} \;=\; [\![\langle 0, V\rangle]\!]_{\mathrm{T}} \;=\; \eta_{\mathrm{T}}(\mathcal{S}_p[\![V]\!]) \;=\; \eta_{\mathrm{S}}(\mathcal{S}_p[\![V]\!])(0) \;=\; \mathcal{S}[\![V]\!](0)$$

(2) Suppose that E has the form $E_1 \,\mathtt{or}\, E_2$. We calculate:

$$\begin{array}{ll} [\![\mathrm{Op_s}(E_1 \,\mathtt{or}\, E_2)]\!]_{\mathrm{T}} & \\ \quad = [\![\mathrm{Op_s}(E_1) \max_{\mathbf{R}_{\mathrm{DW}(-|0)}} \mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \text{(by Theorem 5.3.2)} \\ \quad = [\![\mathrm{Op_s}(E_1) \max_{\alpha_{\mathrm{DW}}\circ \mathrm{DW}(0)} \mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \\ \quad = [\![\mathrm{Op_s}(E_1) \max_{\alpha_{\mathrm{T}}\circ \mathrm{T}(0)\circ \mathrm{T}([\![\;]\!]_\sigma)} \mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \text{(by Lemma 5.8)} \\ \quad = [\![\mathrm{Op_s}(E_1)]\!]_{\mathrm{T}} \max_{\alpha_{\mathrm{T}}\circ \mathrm{T}(0)} [\![\mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \text{(using Lemma 4.1)} \\ \quad = \mathcal{S}[\![E_1]\!](0) \max_{\alpha_{\mathrm{T}}\circ \mathrm{T}(0)} \mathcal{S}[\![E_2]\!](0) & \text{(by induction hypothesis)} \\ \quad = \mathtt{or}_{[\![\sigma]\!]}(\mathcal{S}[\![E_1]\!], \mathcal{S}[\![E_2]\!])(0) & \text{(by Equation 5.2)} \\ \quad = \mathcal{S}[\![E_1 \,\mathtt{or}\, E_2]\!](0) & \end{array}$$

(3) Suppose that E has the form $c \cdot E'$. We calculate:

$$\begin{array}{rcll} [\![\mathrm{Op_s}(c \cdot E')]\!]_{\mathrm{T}} & = & [\![[\![c]\!] \cdot_{\mathrm{T}} \mathrm{Op_s}(E')]\!]_{\mathrm{T}} & \text{(by Theorem 5.3.3)} \\ & = & [\![c]\!] \cdot_{\mathrm{T}} [\![\mathrm{Op_s}(E')]\!]_{\mathrm{T}} & \text{(as } [\![\text{-}]\!]_{\mathrm{T}} \text{ is a homomorphism)} \\ & = & [\![c]\!] \cdot_{\mathrm{T}} \mathcal{S}[\![E']\!](0) & \text{(by induction hypothesis)} \\ & = & ([\![c]\!] \cdot_{\mathrm{S}} \mathcal{S}[\![E']\!])(0) & \text{(by Equation 4.3)} \\ & = & \mathcal{S}[\![c \cdot E']\!](0) & \end{array}$$

(4) Suppose that E has the form $E_1 +_p E_2$. We calculate:

$$\begin{array}{rll} [\![\mathrm{Op_s}(E_1 +_p E_2)]\!]_{\mathrm{T}} & = [\![\mathrm{Op_s}(E_1) +_p \mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \text{(by Theorem 5.3.4)} \\ & = [\![\mathrm{Op_s}(E_1)]\!]_{\mathrm{T}} +_p [\![\mathrm{Op_s}(E_2)]\!]_{\mathrm{T}} & \text{(as } [\![\text{-}]\!]_\sigma \text{ is a homomorphism)} \\ & = \mathcal{S}[\![E_1]\!](0) +_p \mathcal{S}[\![E_2]\!](0) & \text{(by induction hypothesis)} \\ & = (+_{p\mathrm{S}})_{[\![\sigma]\!]}(\mathcal{S}[\![E_1]\!], \mathcal{S}[\![E_2]\!])(0) & \\ & = \mathcal{S}[\![E_1 +_p E_2]\!](0) & \end{array}$$

□

We then have selection adequacy for our language with probabilities:

Theorem 5.10 (Selection adequacy)**.** *For any program $M : \sigma$ we have:*

$$\mathcal{S}[\![M]\!](0) = [\![\mathrm{Op_s}(M)]\!]_{\mathrm{T}}$$

The proof of this theorem is the same as that of Theorem 4.6. As before, the adequacy theorem implies that the globally optimizing operational semantics determines the denotational semantics at the zero-reward continuation.

For the converse direction, noting that

$$\begin{array}{rcll} ([\![\text{-}]\!]_{\mathrm{T}})_\sigma & = & ((\eta_{\mathrm{T}})_{\mathcal{S}[\![\sigma]\!]} \circ \mathcal{S}_p)^{\dagger_{\mathrm{DW}}} & \\ & = & (\mathrm{T}(\mathcal{S}_p) \circ (\eta_{\mathrm{T}})_{\mathrm{Val}_\sigma})^{\dagger_{\mathrm{DW}}} & (\eta \text{ is natural}) \\ & = & \mathrm{T}(\mathcal{S}_p) \circ ((\eta_{\mathrm{T}})_{\mathrm{Val}_\sigma})^{\dagger_{\mathrm{DW}}} & \\ & = & \mathrm{T}(\mathcal{S}_p) \circ (\theta_{\mathrm{DW,T}})_{\mathrm{Val}_\sigma} & \end{array}$$

we see from the adequacy theorem that, for $M:\sigma$, the denotational semantics determines $(\theta_{\mathrm{DW,T}})_{\mathrm{Val}_\sigma}(\mathrm{Op_s}(M)) \in \mathrm{T}(\mathrm{Val}_\sigma)$ up to $\mathrm{T}(\mathcal{S}_p)$. We view $(\theta_{\mathrm{DW,T}})_{\mathrm{Val}_\sigma}(\mathrm{Op_s}(M))$ as an observation of the selection operational semantics of M, and so, for $M : \sigma$ we adopt the notation:

$$\mathrm{Ob}_{\sigma,\mathrm{T}}(M) = (\theta_{\mathrm{DW,T}})_{\mathrm{Val}_\sigma}(\mathrm{Op_s}(M)) \in \mathrm{T}(\mathrm{Val}_\sigma)$$

Using this notation, we see that the adequacy theorem determines observations $\mathrm{Ob}_{\sigma,\mathrm{T}}(M)$ up to $\mathrm{T}(\mathcal{S}_p)$. With the aid of the above discussion of the monad morphism $\theta_{\mathrm{DW,T}}$ we find for $M : \sigma$ that:

$$\begin{array}{rcl} \mathrm{Ob}_{\sigma,\mathrm{T}_1}(M) & = & \mathrm{Op_s}(M) \\ \mathrm{Ob}_{\sigma,\mathrm{T}_2}(M) & = & \langle \mathrm{VDis}(\mathrm{Op_s}(M)), \mathrm{Rew}(\mathrm{Op_s}(M)|-)\rangle \\ \mathrm{Ob}_{\sigma,\mathrm{T}_3}(M) & = & \langle \mathrm{VDis}(\mathrm{Op_s}(M)), \mathbf{E}(\mathrm{Op_s}(M))\rangle \end{array}$$

In the case where σ is a product of base types, $\mathrm{T}(\mathcal{S}_p)$ is an injection. (For $\mathcal{S}_p$ is then an injection and T preserves injections with nonempty domain, as do all functors on sets.) So in this case the denotational semantics determines T-observations $\mathrm{Ob}_{\sigma,\mathrm{T}}(M)$ of the selection operational semantics of terms $M : \sigma$.

5.6. **Full abstraction.** We continue to proceed generally, as above, in terms of an auxiliary monad T and algebra $\alpha_{\mathrm{T}} : \mathrm{T}(\mathrm{R}) \to \mathrm{R}$. Having a general notion of observation $\mathrm{Ob}_{b,\mathrm{T}}$ at base types, we have corresponding general observational equivalence relations $\approx_{b,\mathrm{T}}$, and so, instantiating, observational equivalence relations $\approx_{b,\mathrm{T}_i}$ for the T_i. We write $\mathrm{Ob_T}$ and $\approx_{\mathrm{T}}$ for $\mathrm{Ob}_{\mathtt{Bool},\mathrm{T}}$ and $\approx_{\mathtt{Bool},\mathrm{T}}$, respectively, and similarly for the T_i. From the discussion of the selection adequacy theorem (Theorem 5.10) at base types, we see that the implications 4.4 and 4.5 hold for $\mathcal{S}_{\mathrm{T}}$ and all $\mathrm{Ob}_{b,\mathrm{T}}$ and $\approx_{b,\mathrm{T}}$; we then also have $M \approx_{b,\mathrm{T}} \mathrm{Op}(M)$ for base types b and programs $M : \sigma$.

We next consider, as we did for our first language, whether observing at different base types makes a difference to contextual equivalence.

Lemma 5.11. *Suppose that $\mathbb{B}$ is characteristic for* T *and that b is a base type with at least two constants. Then for any base type b' and programs $M_1, M_2 : b'$ we have:*

$$M_1 \approx_{b,\mathrm{T}} M_2 \implies \mathrm{Ob}_{b',\mathrm{T}}(M_1) = \mathrm{Ob}_{b',\mathrm{T}}(M_2)$$

Proof. We can assume without loss of generality that the M_i are effect values, and write E_i for them. We assume $E_1 \approx_{b,\mathrm{T}} E_2$, and suppose, for the sake of contradiction, that $\mathrm{Ob}_{b',\mathrm{T}}(E_1) \neq \mathrm{Ob}_{b',\mathrm{T}}(E_2)$. As $\mathbb{B}$ is characteristic for T, there is a map $f{:}\mathrm{Val}_{b'} \hookrightarrow \mathrm{Val}_{\mathtt{Bool}}$ such that $\mathrm{T}(f)(\mathrm{Ob}_{b',\mathrm{T}}(E_1)) \neq \mathrm{T}(f)(\mathrm{Ob}_{b',\mathrm{T}}(E_2))$. As b has at least two constants, there is an injection $\iota : \mathrm{Val}_{\mathtt{Bool}} \to \mathrm{Val}_b$. Set $f' = f \circ \iota : \mathrm{Val}_{b'} \to \mathrm{Val}_b$. As T preserves injections with nonempty domain we have $\mathrm{T}(f')(\mathrm{Ob}_{b',\mathrm{T}}(E_1)) \neq \mathrm{T}(f')(\mathrm{Ob}_{b',\mathrm{T}}(E_2))$. Let g be the restriction of f' to $g : u \to \mathrm{Val}_b$, where u is the set of constants of type b occurring in E_1 or E_2.

As $E_1 \approx_{b,\mathrm{T}} E_2$ we have $\mathrm{F}_g E_1 \approx_{b,\mathrm{T}} \mathrm{F}_g E_2$, so $\mathrm{Ob}_{b,\mathrm{T}}(\mathrm{F}_g E_1) = \mathrm{Ob}_{b,\mathrm{T}}(\mathrm{F}_g E_2)$, and so, by adequacy, $\mathcal{S}[\![\mathrm{F}_g E_1]\!](0) = \mathcal{S}[\![\mathrm{F}_g E_2]\!](0)$. For $i = 1, 2$, we calculate:

$$
\begin{array}{rcll}
\mathcal{S}[\![\mathrm{F}_g E_i]\!](0) & = & \mathcal{S}[\![E_i[g]]\!](0) & \text{(by Lemma 3.4)} \\
& = & [\![\mathrm{Op_s}(E_i[g])]\!]_{\mathrm{T}} & \text{(by adequacy)} \\
& = & [\![\mathrm{DW}(f')\mathrm{Op_s}(E_i)]\!]_{\mathrm{T}} & \text{(by Lemma 5.4.2)} \\
& = & \mathrm{T}([\![\text{-}]\!])((\theta_{\mathrm{DW},\mathrm{T}})_{\mathrm{Val}_b}(\mathrm{DW}(f')(\mathrm{Op_s}(E_i)))) & \\
& = & \mathrm{T}([\![\text{-}]\!])(\mathrm{T}(f')(\theta_{\mathrm{Val}_{b'}}(\mathrm{Op_s}(E_i)))) & \text{(as } \theta_{\mathrm{DW},\mathrm{T}} \text{ is natural} \\
& & & \text{ by Lemma 5.5)} \\
& = & \mathrm{T}([\![\text{-}]\!])(\mathrm{T}(f')(\mathrm{Ob}_{b',\mathrm{T}}(E_i))) &
\end{array}
$$

So, as $\mathrm{T}([\![\text{-}]\!])$ is injective, $\mathrm{T}(f')(\mathrm{Ob}_{b,\mathrm{T}}(E_1)) = \mathrm{T}(f')(\mathrm{Ob}_{b,\mathrm{T}}(E_2))$, yielding the required contradiction. □

We then have the following analogue of Proposition 4.8:

Proposition 5.12. *Suppose that $\mathbb{B}$ is characteristic for* T. *Then, for all base types b and programs $M, N : \sigma$, we have*

$$M \approx_{\mathrm{T}} N \implies M \approx_{b,\mathrm{T}} N$$

with the converse holding if there are at least two constants of type b.

As $\mathbb{B}$ is characteristic for the T_i (Lemma 5.6), we have invariance of the observational equivalences $\approx_{b,\mathrm{T}_i}$ under changes of base type with at least two constants. Modulo a reasonable definability assumption, each of our three semantics is fully abstract at base types with respect to their corresponding notion of observational equivalence. We establish this via general results for T and $\alpha_{\mathrm{T}} : \mathrm{T}(\mathrm{R}) \to \mathrm{R}$, as above.

We first need a general result on reward continuations. Suppose $u \subseteq \mathrm{Con}_b$ and suppose too that $\gamma : [\![b]\!] \to \mathrm{R}$ is *definable on u* in the sense that there is a (necessarily unique) $g : \mathrm{Con}_b \to \mathrm{Con}_{\mathrm{Rew}}$ such that $\gamma([\![c]\!]) = [\![g(c)]\!]$, for $c \in u$. Set $\mathrm{K}_{u,\gamma} = \mathrm{F}_h : b \to b$ where $h(c) = g(c) \cdot c$ ($c \in u$). We have:

$$\mathcal{S}[\![\mathrm{K}_{u,\gamma} c]\!](0) = \gamma([\![c]\!]) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}([\![c]\!]) \quad (c \in u)$$

This program $\mathrm{K}_{u,\gamma}$ can be used to reduce calling definable reward continuations to calling the zero-reward continuation, modulo reward addition:

Lemma 5.13. *Suppose $E : b$ is an effect value, u a finite set of constants of type b including all those occurring in E, and $\gamma : [\![b]\!] \to \mathrm{R}$ is a reward function definable on u. Then we have:*

$$\mathrm{k}_\gamma(\mathcal{S}[\![E]\!](\gamma)) = \mathcal{S}[\![\mathrm{K}_{u,\gamma} E]\!](0)$$

Proof. The proof is by structural induction on E. If E is a constant c, then

$$\mathrm{k}_\gamma(\mathcal{S}[\![c]\!](\gamma)) = \mathrm{k}_\gamma(\eta_{\mathrm{T}}([\![c]\!])) = \gamma([\![c]\!]) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}([\![c]\!]) = \mathcal{S}[\![\mathrm{K}_{u,\gamma} c]\!](0)$$

Suppose next that E has the form $E_1 \,\texttt{or}\, E_2$. We first show that

$$\alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma) = \alpha_{\mathrm{T}} \circ \mathrm{T}(0) \circ \mathrm{k}_\gamma \qquad (*)$$

We have $\mathrm{k}_\gamma = f^{\dagger_{\mathrm{T}}} : \mathrm{T}([\![b]\!]) \to \mathrm{T}([\![b]\!])$, where $f(x) =_{\mathrm{def}} \gamma(x) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(x)$, for $x \in [\![b]\!]$. Setting $g(x) =_{\mathrm{def}} \mathrm{T}(0)(\gamma(x) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(x))$ $(= \gamma(x) \cdot_{\mathrm{T}} \mathrm{T}(0)(\eta_{\mathrm{T}}(x)))$, for $x \in [\![b]\!]$, we then see that $\mathrm{T}(0) \circ \mathrm{k}_\gamma = g^{\dagger_{\mathrm{T}}} : \mathrm{T}([\![b]\!]) \to \mathrm{T}(\mathrm{R})$. Making use of assumption (A2) of Section 5.4, we next see that $\alpha_{\mathrm{T}}(g(x)) = \alpha_{\mathrm{T}}(\gamma(x) \cdot_{\mathrm{T}} \mathrm{T}(0)(\eta_{\mathrm{T}}(x))) = \alpha_{\mathrm{T}}(\gamma(x) \cdot_{\mathrm{T}} \eta_{\mathrm{T}}(0)) = \gamma(x) + 0 = \gamma(x)$. This, in turn, yields $\alpha_{\mathrm{T}} \circ \mathrm{T}(0) \circ \mathrm{k}_\gamma = \alpha_{\mathrm{T}} \circ g^{\dagger_{\mathrm{T}}} = \alpha_{\mathrm{T}} \circ \mathrm{T}(\alpha_{\mathrm{T}} \circ g) = \alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma)$ as required. (The second equation in this chain holds generally for monad algebras.)

We then calculate:

$$\begin{array}{l}
\mathrm{k}_\gamma(\mathcal{S}[\![E_1 \,\texttt{or}\, E_2]\!](\gamma)) \\
\quad = \mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma) \max_{\mathbf{R}_{\mathrm{T}}(-|\gamma)} \mathcal{S}[\![E_2]\!](\gamma)) \\
\quad = \mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma) \max_{\alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma)} \mathcal{S}[\![E_2]\!](\gamma)) \\
\quad = \left\{ \begin{array}{ll} \mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma)) & (\text{if } (\alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma))\mathcal{S}[\![E_1]\!](\gamma) \geq (\alpha_{\mathrm{T}} \circ \mathrm{T}(\gamma))\mathcal{S}[\![E_2]\!](\gamma)) \\ \mathrm{k}_\gamma(\mathcal{S}[\![E_2]\!](\gamma)) & (\text{otherwise}) \end{array} \right. \\
\quad = \left\{ \begin{array}{ll} \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1)]\!](0) & (\text{if } (\alpha_{\mathrm{T}} \circ \mathrm{T}(0) \circ \mathrm{k}_\gamma)\mathcal{S}[\![E_1]\!](\gamma) \geq (\alpha_{\mathrm{T}} \circ \mathrm{T}(0) \circ \mathrm{k}_\gamma)\mathcal{S}[\![E_2]\!](\gamma)) \\ \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_2)]\!](0) & (\text{otherwise}) \end{array} \right. \\
\hfill (\text{by induction hypothesis and } (*)) \\
\quad = \left\{ \begin{array}{ll} \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1)]\!](0) & (\text{if } (\alpha_{\mathrm{T}} \circ \mathrm{T}(0))\mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma)) \geq (\alpha_{\mathrm{T}} \circ \mathrm{T}(0))\mathrm{k}_\gamma(\mathcal{S}[\![E_2]\!](\gamma))) \\ \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_2)]\!](0) & (\text{otherwise}) \end{array} \right. \\
\quad = \left\{ \begin{array}{ll} \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1)]\!](0) & (\text{if } (\alpha_{\mathrm{T}} \circ \mathrm{T}(0))\mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1)]\!](0) \geq (\alpha_{\mathrm{T}} \circ \mathrm{T}(0))\mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_2)]\!](0)) \\ \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_2)]\!](0) & (\text{otherwise}) \end{array} \right. \\
\hfill (\text{by Lemma 5.13}) \\
\quad = \mathcal{S}[\![(\mathrm{K}_{u,\gamma}E_1) \,\texttt{or}\, (\mathrm{K}_{u,\gamma}E_2)]\!](0) \\
\quad = \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1 \,\texttt{or}\, E_2)]\!](0) \hfill (\text{using Equation 3.4})
\end{array}$$

Suppose next that E has the form $E_1 +_p E_2$. Then we calculate:

$$\begin{array}{rcl}
\mathrm{k}_\gamma(\mathcal{S}[\![E_1 +_p E_2]\!](\gamma)) & = & \mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma) +_p \mathcal{S}[\![E_2]\!](\gamma)) \\
& = & \mathrm{k}_\gamma(\mathcal{S}[\![E_1]\!](\gamma)) +_p \mathrm{k}_\gamma(\mathcal{S}[\![E_2]\!](\gamma)) \\
& = & \mathcal{S}[\![\mathrm{K}_{u,\gamma}E_1]\!](0) +_p \mathcal{S}[\![\mathrm{K}_{u,\gamma}E_2]\!](0) \\
& = & \mathcal{S}[\![\mathrm{K}_{u,\gamma}E_1 +_p \mathrm{K}_{u,\gamma}E_2]\!](0) \\
& = & \mathcal{S}[\![\mathrm{K}_{u,\gamma}(E_1 +_p E_2)]\!](0)
\end{array}$$

Finally, suppose that E has the form $c \cdot E'$. This case is handled similarly to the previous one:

$$\begin{aligned}
\mathrm{k}_\gamma(\mathcal{S}[\![c \cdot E']\!](\gamma)) &= \mathrm{k}_\gamma([\![c]\!] \cdot \mathcal{S}[\![E']\!](\gamma)) \\
&= [\![c]\!] \cdot \mathrm{k}_\gamma(\mathcal{S}[\![E']\!](\gamma)) \\
&= [\![c]\!] \cdot \mathcal{S}[\![\mathrm{K}_{u,\gamma}E']\!](0) \\
&= \mathcal{S}[\![c \cdot \mathrm{K}_{u,\gamma}E']\!](0) \\
&= \mathcal{S}[\![\mathrm{K}_{u,\gamma}(c \cdot E')]\!](0)
\end{aligned}$$

□

We can now demonstrate full abstraction for general T, subject to three assumptions, Say that a type b is *numerable* if all elements of $[\![b]\!]$ are definable by a constant.

Theorem 5.14. *Suppose* $\mathbb{B}$ *is characteristic for* T*, reward addition is injective for* T*, and* Rew *is numerable. Then* $\mathcal{S}$ *is fully abstract with respect to* $\approx_{\mathrm{T}}$ *at* b.

Proof. Suppose $M_1(\approx_{\mathrm{T}})_b M_2$. We wish to show that $\mathcal{S}[\![M_1]\!] = \mathcal{S}[\![M_2]\!]$. By the ordinary adequacy theorem there are effect values E_1, E_2 with $\mathcal{S}[\![M_1]\!] = \mathcal{S}[\![E_1]\!]$ and $\mathcal{S}[\![M_2]\!] = \mathcal{S}[\![E_2]\!]$.

Let u be the set of constants of type b appearing in any one of these effect values. Let $\gamma : [\![b]\!] \to \mathrm{R}$ be a reward function. It is definable on u by the numerability assumption. Using Lemma 5.13 we see that

$$\mathrm{k}_\gamma(\mathcal{S}[\![M_i]\!](\gamma)) = \mathrm{k}_\gamma(\mathcal{S}[\![E_i]\!](\gamma)) = \mathcal{S}[\![\mathrm{K}_{u,\gamma} E_i]\!](0) \qquad (*)$$

As $M_1(\approx_{\mathrm{T}})_b M_2$, we have $E_1(\approx_{\mathrm{T}})_b E_2$ so $\mathrm{K}_{u,\gamma}E_1(\approx_{\mathrm{T}})_b \mathrm{K}_{u,\gamma}E_2$. As $\mathbb{B}$ is characteristic for T, we can then apply Lemma 5.11, finding that $\mathrm{Ob}_{\mathrm{T}}(\mathrm{K}_{u,\gamma}E_1) = \mathrm{Ob}_{\mathrm{T}}(\mathrm{K}_{u,\gamma}E_2)$. So, by adequacy, $\mathcal{S}[\![\mathrm{K}_{u,\gamma}E_1]\!](0) = \mathcal{S}[\![\mathrm{K}_{u,\gamma}E_2]\!](0)$.

With this, we see, using $(*)$, that $\mathrm{k}_\gamma(\mathcal{S}[\![M_1]\!](\gamma)) = \mathrm{k}_\gamma(\mathcal{S}[\![M_2]\!](\gamma))$, so, as reward addition is injective for T, that $\mathcal{S}[\![M_1]\!](\gamma) = \mathcal{S}[\![M_2]\!](\gamma)$. As γ is an arbitrary reward function, we finally have $\mathcal{S}[\![M_1]\!] = \mathcal{S}[\![M_2]\!]$ as required. □

As, by Lemmas 5.6 and 5.7, $\mathbb{B}$ is characteristic for all of three T_i and reward addition is injective for all of them, we immediately obtain:

Corollary 5.15. *Suppose that* Rew *is numerable. Then* $\mathcal{S}_{\mathrm{T}_i}$ *is fully abstract with respect to* $\approx_{\mathrm{T}_i}$ *at base types, for* $i = 1, 2, 3$.

Regarding full abstraction at other types, full abstraction for general T at products of base types and so, too, at values of types of order 1 is a consequence of Theorem 5.14 (under the same assumptions as those of the theorem). We then obtain full abstraction for the T_i at products of base types and at values of types of order 1 (assuming Rew numerable). As in the case of the language of Section 4, the question of full abstraction at other types is open.

There is a "cheap" version of the free-algebra monad C discussed in Section 4.6 for general auxiliary monads T. Take Ax_c to be the set of equations between effect values that hold in $\mathcal{S}_{\mathrm{T}}$, and take D to be the corresponding free algebra monad, yielding a corresponding denotational semantics $\mathcal{D}$. Then we have:

$$\models_{\mathcal{D}} E = E' : b \iff \vdash_{\mathrm{Ax}_c} E = E' : b \iff \models_{\mathcal{S}_{\mathrm{T}}} E = E' : b$$

Assuming $\mathbb{B}$ characteristic for T and reward addition injective for T, using Theorems 3.6 and 5.14 we then obtain a version of Theorem 4.10 for $\mathcal{D}$ for numerable b:

$$\vdash_{\mathrm{Ax}_c} M = N : b \iff \models_{\mathcal{C}} M = N : b \iff M \approx_b N$$

Turning to weakening the notion of observation, analogously to Section 4 we could forget all reward information. We do this by taking our notion of observation $\mathrm{Ob}_{\mathrm{VDis}}$ to be $\mathrm{VDis} \circ \mathrm{Op_s}$, i.e., the distribution of final values. As we next show, the observational equivalence $\approx_{\mathrm{Ob_{VDis}}}$ resulting from this notion coincides with $\approx_{\mathrm{T}_3}$.

Lemma 5.16. *For programs* $M, N :$ `Bool` *we have*

$$M_1 \approx_{\mathrm{Ob_{VDis}}} M_2 \implies \mathrm{Ob}_{\mathrm{T}_3}(M_1) = \mathrm{Ob}_{\mathrm{T}_3}(M_2)$$

Proof. Set $E_i = \mathrm{Op}(M_i)$, for $i = 1, 2$. We have $M_i \approx_{\mathrm{T}_3} E_i$ and so, as $\mathrm{Ob}_{\mathrm{VDis}}$ is weaker than $\mathrm{Ob}_{\mathrm{T}_3}$, we also have $M_i \approx_{\mathrm{Ob}_{\mathrm{VDis}}} E_i$. It therefore suffices to prove that:

$$E_1 \approx_{\mathrm{Ob}_{\mathrm{VDis}}} E_2 \implies \mathrm{Ob}_{\mathrm{T}_3}(E_1) = \mathrm{Ob}_{\mathrm{T}_3}(E_2)$$

So suppose that $E_1 \approx_{\mathrm{Ob}_{\mathrm{VDis}}} E_2$ and, for the sake of contradiction, that, for example, $\mathbf{E}(\mathrm{Op}_\mathrm{s}(E_1)) < \mathbf{E}(\mathrm{Op}_\mathrm{s}(E_2))$.

Since $E_1 \approx_{\mathrm{Ob}_{\mathrm{VDis}}} E_2$, they return the same probability distribution $\mathtt{tt} +_p \mathtt{ff}$ on boolean values. Suppose, without loss of generality, that this distribution is not $\mathtt{ff}$. (If it is, we can work with $\mathtt{tt}$ instead.) Define $f : \mathrm{Val}_{\mathtt{Bool}} \to \mathrm{Val}_{\mathtt{Bool}}$ to be constantly $\mathtt{ff}$. Then we have

$$\begin{array}{rcll}
\mathrm{Ob}_{\mathrm{VDis}}(E_1 \,\mathtt{or}\, \mathrm{F}_f E_2) & = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_1 \,\mathtt{or}\, \mathrm{F}_f E_2)) & \\
& = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_1) \max_{\mathbf{E}} \mathrm{Op}_\mathrm{s}(\mathrm{F}_f E_2)) & \text{(by Theorem 5.3.2)} \\
& = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_1) \max_{\mathbf{E}} \mathrm{Op}_\mathrm{s}(E_2[f])) & \text{(by Lemma 3.4)} \\
& = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_1) \max_{\mathbf{E}} \mathrm{W}(f)(\mathrm{Op}_\mathrm{s}(E_2))) & \text{(by Lemma 5.4.2)} \\
& = & \mathrm{VDis}(\mathrm{W}(f)(\mathrm{Op}_\mathrm{s}(E_2))) & \\
& = & \mathtt{ff} &
\end{array}$$

where the next to last equality holds as, using Lemma 5.4.1, we have:

$$\mathbf{E}(\mathrm{Op}_\mathrm{s}(E_1)) < \mathbf{E}(\mathrm{Op}_\mathrm{s}(E_2)) = \mathbf{E}(\mathrm{W}(f)(\mathrm{Op}_\mathrm{s}(E_1)))$$

Similarly,

$$\begin{array}{rcl}
\mathrm{Ob}_{\mathrm{VDis}}(E_2 \,\mathtt{or}\, \mathrm{F}_f E_2) & = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_2) \max_{\mathbf{E}} \mathrm{W}(f)(\mathrm{Op}_\mathrm{s}(E_2))) \\
& = & \mathrm{VDis}(\mathrm{Op}_\mathrm{s}(E_2)) \\
& = & \mathtt{tt} +_p \mathtt{ff}
\end{array}$$

yielding the required contradiction. □

We then have the following analogue to Theorem 4.13:

Theorem 5.17. *For any programs $M, N : \sigma$, we have*

$$M \approx_{\mathrm{T}_3} N \iff M \approx_{\mathrm{Ob}_{\mathrm{VDis}}} N$$

5.7. **Program equivalences and purity.** We begin by considering the equations holding in $\mathcal{S}_\mathrm{T}$ for a general T as above. We need some terminology and notation. Say that a term $M : \sigma$ is in *expectation PR-form over terms* $L_1, \ldots, L_n$ if it has the form

$$\sum_{i=1}^{m} p_i \sum_{j=1}^{n_i} q_{ij}(M_{ij} \cdot L_i)$$

where the M_{ij} are either variables or constants (and we say M is an *expectation PR-value* if the M_{ij} and L_i are all constants). For such a term we write $\mathbf{E}_s(M) : \mathrm{Rew}$ for the term:

$$\bigoplus_{i=1}^{m} p_i \bigoplus_{j=1}^{n_i} q_{ij} M_{ij}$$

(We write $\bigoplus M_i$ for iterated uses of the $\oplus_p$ to avoid confusion with iterated uses of the $+_p$.) In case the M_{ij} are constants d_{ij}, we set:

$$\left[\!\!\left[\bigoplus_{i=1}^{m} p_i \bigoplus_{j=1}^{n_i} q_{ij} d_{ij} \right]\!\!\right] = \sum_{i=1}^{m} p_i \sum_{j=1}^{n_i} q_{ij} [\![d_{ij}]\!]$$

Our system Ax_1 of equations is given in Figure 4 (where we omit type information). In the last two equations it is assumed that M and N are in expectation PR-form over the same $L_1, \ldots, L_n$. The equations express at the term level, that: choice is idempotent and associative; rewards form an action for the commutative monoid structure on R; probabilistic choice forms a convex algebra; the R-action acts on both forms of choice; probabilistic choice distributes over choice; and, where this can be seen from the syntax, that choice is made according to the highest reward, with priority to the left for ties.

$$
\begin{array}{c}
M \,\texttt{or}\, M \;=\; M \qquad (L \,\texttt{or}\, M) \,\texttt{or}\, N \;=\; L \,\texttt{or}\, (M \,\texttt{or}\, N)\\
0 \cdot N \;=\; N \qquad x \cdot (y \cdot N) \;=\; (x+y) \cdot N\\
M +_1 N = M \qquad M +_p N \;=\; N +_{1-p} M\\
(M +_p N) +_q P \;=\; M +_{pq} (N +_{\frac{(1-p)q}{1-pq}} P) \quad (p,q < 1)\\
x \cdot (M +_p N) \;=\; x \cdot M +_p x \cdot N\\
x \cdot (M \,\texttt{or}\, N) \;=\; (x \cdot M) \,\texttt{or}\, (x \cdot N)\\
L +_p (M \,\texttt{or}\, N) \;=\; (L +_p M) \,\texttt{or}\, (L +_p N)\\
\texttt{if}\; \mathbf{E}_s(M) \geq \mathbf{E}_s(N) \;\texttt{then}\; M \;\texttt{else}\; N \;=\; M \,\texttt{or}\, N\\
\texttt{if}\; \mathbf{E}_s(M) \geq \mathbf{E}_s(N) \;\texttt{then}\; (M \,\texttt{or}\, P) \;\texttt{else}\; (P \,\texttt{or}\, N) \;=\; (M \,\texttt{or}\, P) \,\texttt{or}\, N
\end{array}
$$

Figure 4: Equations for choices, probability, and rewards

Below, for $u \in \mathrm{T}(X)$ and $\gamma : X \to \mathrm{R}$, we set

$$\mathbf{E}(u|\gamma) =_{\mathrm{def}} \mathbf{R}_{\mathrm{T}}(u|\gamma) \; (= \alpha_{\mathrm{T}}(\mathrm{T}(\gamma)(u)))$$

This is the expected reward of u, given γ.

Proposition 5.18. *The axioms hold for general $\mathcal{S}_{\mathrm{T}}$.*

Proof. Other than the last two axiom schemas, this follows from Theorem 2.6, Corollary 2.5, and Theorem 2.7. The last two cases are straightforward pointwise arguments, although we need an observation. We calculate that for a PR-term $M = \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n_i} q_{ij}(d_{ij} \cdot L_i)\right)$ of type σ and a reward function $\gamma : [\![\sigma]\!] \to \mathrm{R}$ we have:

$$
\begin{array}{rcl}
\mathbf{E}(\mathcal{S}[\![\sum_{i=1}^{m} p_i \sum_{j=1}^{n_i} q_{ij}(d_{ij} \cdot L_i)]\!]\gamma \mid \gamma) & = & \sum_{i=1}^{m} p_i(\sum_{j=1}^{n_i} q_{ij}([\![d_{ij}]\!] + \mathbf{E}(\mathcal{S}[\![L_i]\!]\gamma \mid \gamma)))\\
& = & \sum_{i=1}^{m} p_i(\sum_{j=1}^{n_i} q_{ij}[\![d_{ij}]\!]) +\\
& & \quad \sum_{i=1}^{m} p_i(\sum_{j=1}^{n_i} q_{ij}\mathbf{E}(\mathcal{S}[\![L_i]\!]\gamma \mid \gamma))\\
& = & \sum_{i=1}^{m} p_i(\sum_{j=1}^{n_i} q_{ij}([\![d_{ij}]\!])) + \sum_{i=1}^{m} p_i\mathbf{E}(\mathcal{S}[\![L_i]\!]\gamma \mid \gamma)
\end{array}
$$

and we further have:

$$\mathcal{S}\left[\!\!\left[\bigoplus_{i=1}^{m} p_i \bigoplus_{j=1}^{n_i} q_{ij} d_{ij}\right]\!\!\right]\gamma = \eta_{\mathrm{T}}\left(\sum_{i=1}^{m} p_i \sum_{j=1}^{n_i} q_{ij}[\![d_{ij}]\!]\right)$$

So if $M : \sigma$ and $N : \sigma$ are in expectation PR-form over the same $L_1, \ldots, L_n$ then, for $\gamma : [\![\sigma]\!] \to \mathrm{R}$, we have:

$$\mathbf{E}(\mathcal{S}[\![M]\!]\gamma \mid \gamma) \geq \mathbf{E}(\mathcal{S}[\![N]\!]\gamma \mid \gamma) \iff \mathcal{S}[\![\mathbf{E}_s(M) \geq \mathbf{E}_s(N)]\!]\gamma = \eta_{\mathrm{T}}(0)$$

(recall that $[\![\texttt{tt}]\!] = 0$). With this observation, the pointwise argument for the last two equation schemas goes through. ∎

In the case of $\mathcal{S}_{\mathrm{T}_2}$ we inherit Equation 5.3 from T_2 so we additionally have:

$$x \cdot M \ +_p \ y \cdot M = (x \oplus_p y) \cdot M \tag{5.8}$$

Let Ax_2 be Ax_1 extended with this equation. In the case of $\mathcal{S}_{\mathrm{T}_3}$ we inherit Equation 5.5 from T_3 so we have the stronger:

$$x \cdot M \ +_p \ y \cdot N = (x \oplus_p y) \cdot M \ +_p \ (x \oplus_p y) \cdot N \tag{5.9}$$

Let Ax_3 be Ax_1 extended with this equation.

Unfortunately, we do not have any results analogous to Theorem 4.10 for any of the above three axiom systems for the probabilistic case—further axioms may well be needed to obtain completeness for program equivalence at base types. We do, however have a completeness result for purity at base types.

First, some useful consequences of these equations, are the following, where M and N are expectation PR-values over the same $L_1, \ldots, L_n$:

$$M \,\texttt{or}\, N \ = \ M \quad (\text{if } \mathbf{E}_s(M) \geq \mathbf{E}_s(N)) \tag{PR$_1$}$$

$$M \,\texttt{or}\, N \ = \ N \quad (\text{if } \mathbf{E}_s(M) < \mathbf{E}_s(N)) \tag{PR$_2$}$$

$$(M \,\texttt{or}\, L) \,\texttt{or}\, N \ = \ M \,\texttt{or}\, L \quad (\text{if } \mathbf{E}_s(M) \geq \mathbf{E}_s(N))\% \tag{PR$_3$}$$

$$(M \,\texttt{or}\, L) \,\texttt{or}\, N \ = \ L \,\texttt{or}\, N \quad (\text{if } \mathbf{E}_s(M) < \mathbf{E}_s(N))\% \tag{PR$_4$}$$

Next, our equational system Ax_1 allows us to put programs of base type into a weak canonical form. First consider programs which are *PR-effect values*, i.e., programs obtained by probabilistic and reward combinations of constants. Every such term is provably equivalent to one of the form $\sum_{j=1}^{m} p_j(d_j \cdot c_j)$ where $m > 0$, the d_j and the c_j are constants and no $d_j \cdot c_j$ is repeated. We call such terms *canonical PR-effect values*, and do not distinguish any two such if they are identical apart from the ordering of the $d_j \cdot c_j$.

We say that an effect value of base type is in *weak canonical form* if (ignoring bracketing of `or`) it is an effect value of the form

$$E_1 \,\texttt{or}\, \ldots \,\texttt{or}\, E_n$$

where $n > 0$, the E_i are canonical PR-effect values, and no E_i occurs twice. (We could have simplified canonical forms further by applying the PR_i, obtaining a stronger canonical form. However, we did not do so as, in any case, we do not have an equational completeness result.)

Lemma 5.19. *Every program M of base type is provably equal to a weak canonical form* $\mathrm{CF}(M)$.

Proof. By Proposition 3.3, M can be proved equal to an effect value E. Using the associativity equation and the fact that `reward` and $+_p$ distribute over `or`, the effect value E can be proved equal to a term of the form $E_1 \,\texttt{or}\, \ldots \,\texttt{or}\, E_n$ where each E_i is a PR-effect term. □

Say that a theory Ax, valid in S_{T}, is *strongly purity complete for basic PR-effect values*, if for all PR-effect values $E : b$ we have:

$$\models_{\mathrm{S}_{\mathrm{T}}} E \downarrow_b \implies \exists c : b.\ \vdash_{\mathrm{Ax}} E = c : b$$

Lemma 5.20. Ax_i *is strongly purity complete for basic PR-effect values, for* $i = 1, 2, 3$.

Proof. For T_1 we have already noted that every PR-effect value is provably equal using Ax_1 to a term $E : b$ of the form $\sum_{j=1}^{m} p_j(d_j \cdot c_j)$ with no $d_j \cdot c_j$ repeated. For such a term $\models_{\mathrm{S}_{\mathrm{T}_1}} E \downarrow_b$ holds iff there is an $x \in [\![b]\!]$ such that $\mathcal{S}_{\mathrm{T}_1}[\![E]\!]\gamma = \eta_{\mathrm{T}_1}(b)$ for all $\gamma : [\![b]\!] \to \mathrm{R}$. Taking $\gamma = 0$, for example, we then see that $\sum_{j=1}^{m} p_j([\![d_j]\!] \cdot [\![c_j]\!]) = \eta_{\mathrm{T}_1}(b)$. As no $d_j \cdot c_j$ is repeated, neither is any $[\![d_j]\!] \cdot [\![c_j]\!]$. It follows that $m = 1$ and $d_1 = 0$. In that case the term is provably equal, using Ax_1, to c_1. The other two cases are similar: for T_2 we note that every PR-effect value is provably equal using Ax_2 to a term of the form $\sum_{j=1}^{m} p_i(d_j \cdot c_j)$ with no c_i repeated, and for T_3 we note that every PR-effect value is provably equal using Ax_3 to a term of the form $\sum_{j=1}^{m} p_i(d \cdot c_j)$ with no c_j repeated. □

In order to establish purity completeness we need a condition (C) on R. This is that for all $p \in (0,1)$ and $s < 0$ in R, there are $l, r \in \mathrm{R}$ such that $s + (r +_p l) > l$. Condition (C) evidently holds when there are no negative elements as in our example of the nonnegative reals $[0,\infty)$ with the addition monoid. It also holds for our other examples of the reals, $(-\infty,\infty)$, and the positive reals, $(0,\infty)$, the former with the addition monoid and the latter with the multiplication monoid. Two further examples satisfying the condition are the real intervals $(-\infty, 0]$ and $(0, 1]$, both with the usual ordering, the first with the sum monoid, and the second with the multiplication monoid. In all these examples we employ the usual convex combination, and the verification of Condition (C) is straightforward. We give a counterexample to the condition below.

There are natural conditions that imply Condition (C), and which, together, account for these examples. Consider the equation:

$$x +_{1/2} (y + z) = (x + y) +_{1/2} z \tag{5.10}$$

and say that condition (D) holds if, for all $p \in (0,1)$, there is an $l \in \mathrm{R}$ such that for all $s \in \mathrm{R}$ there is an $r \in \mathrm{R}$ such that $r +_p s > l$. Condition (C) is satisfied if Equation 5.10 holds or Condition (D) does. All our examples with the addition monoid satisfy the equation, and all our examples other than the nonpositive reals satisfy Condition (D).

Theorem 5.21 (General purity completeness)**.** *Suppose that* R *satisfies condition (C). Let* Ax *be a theory extending* Ax_1 *that is valid in* S_T*. If* Ax *is strongly purity complete for basic PR-effect values, then it is strongly purity complete at base types, i.e., for all programs* $M : b$ *we have:*

$$\models_{\mathrm{S}_\mathrm{T}} M \downarrow_b \implies \exists c : b.\ \vdash_{\mathrm{Ax}} M = c : b$$

Proof. We remark first that, in general, for any term $N : b$ and any PR-effect value $E : b$, if $\models_{\mathrm{S}_\mathrm{T}} N \downarrow_b$ and $\mathrm{S_T}[\![N]\!](\gamma) = \mathrm{S_T}[\![E]\!](\gamma)$ for some γ, then $\mathrm{S_T}[\![N]\!](\gamma) = \mathrm{S_T}[\![E]\!](\gamma)$ for any γ, and so, also, $\models_{\mathrm{S}_\mathrm{T}} E \downarrow_b$ and then $\vdash_{\mathrm{Ax}} E = c$, for some $c : b$ (this last using the strong purity completeness assumption).

It suffices to prove the claim for terms M in weak canonical form, i.e., of the form

$$E_1 \ \mathtt{or} \ \ldots \ \mathtt{or} \ E_n$$

where $n > 0$, and the E_i are canonical PR-effect values. We proceed by induction on n.

So suppose that $\models_{\mathrm{S}_\mathrm{T}} M \downarrow_b$. For some E_{i_0} we have $\mathrm{S_T}[\![M]\!](0) = \mathrm{S_T}[\![E_{i_0}]\!](0)$, and so, by the above remark, we see that $\mathrm{S_T}[\![N]\!] = \mathrm{S_T}[\![E_{i_0}]\!]$ and also that there is a $\overline{c} : b$ such that $\vdash_{\mathrm{Ax}} E_{i_0} = \overline{c}$.

In case $n = 1$ we have shown that $\vdash_{\mathrm{Ax}} M = c$ for some $c : b$, as required. Otherwise consider $E_{i_1} = \sum_{j=1}^{n} p_j(d_j \cdot c_j)$ for an $i_1 \neq i_0$. If every c_j is $\overline{c}$ then both E_{i_0} and E_{i_1} are in expectation PR-value form over $\overline{c}$, and so one of the equations PR_1–PR_4 can be used to reduce the size of M, and the induction hypothesis can be applied.

Otherwise, some c_j is not $\overline{c}$, and we show next that, for some γ we have

$$\mathbf{E}(\mathrm{S_T}[\![E_{i_1}]\!]\gamma \mid \gamma) \;>\; \mathbf{E}(\mathrm{S_T}[\![E_{i_0}]\!]\gamma \mid \gamma) \qquad (*)$$

There are two cases. In the first case no c_{j_1} is $\overline{c}$. Then choose $l < r \in \mathrm{R}$ and define $\gamma : [\![b]\!] \to \mathrm{R}$ by setting $\gamma(x) = r$ for $x \neq [\![\overline{c}]\!]$, and $\gamma([\![\overline{c}]\!]) = r_0 + l$, where r_0 is the least of the $[\![d_j]\!]$. Then we have:

$$\begin{array}{rcl}\mathbf{E}(\mathrm{S_T}[\![E_{i_1}]\!]\gamma \mid \gamma) & = & \sum_{j=1}^n p_j([\![d_j]\!] + \gamma(c_j))\\ & \geq & \sum_{j=1}^n p_j(r_0 + \gamma(c_j))\\ & = & \sum_{j=1}^n p_j(r_0 + r)\\ & = & r_0 + r\\ & > & r_0 + l\\ & = & \gamma(\overline{c})\\ & = & \mathbf{E}(\mathrm{S_T}[\![E_{i_0}]\!]\gamma \mid \gamma)\end{array}$$

In the second case $c_{j_0} = \overline{c}$ for some unique j_0. Setting $p = \sum_{j\neq j_0} p_j$, note that $p \in (0,1)$; then, setting $p'_j = p_j/p$ for $j \neq j_0$, note that $\sum_{j\neq j_0} p'_j = 1$. Taking r_0 to be the least of the $[\![d_j]\!]$ as before, there are l and r in R such that $r_0 + (r +_p l) > l$. For if $r_0 < 0$, condition (C) applies, and otherwise $r_0 \geq 0$ and we can choose any l, r with $l < r$. Define $\gamma : [\![b]\!] \to \mathrm{R}$ by setting $\gamma(x) = r$ for $x \neq [\![\overline{c}]\!]$, and $\gamma([\![\overline{c}]\!]) = l$. Then we have:

$$\begin{array}{rcl}\mathbf{E}(\mathrm{S_T}[\![E_{i_1}]\!]\gamma \mid \gamma) & = & \sum_{j=1}^n p_j([\![d_j]\!] + \gamma(c_j))\\ & \geq & \sum_{j=1}^n p_j(r_0 + \gamma(c_j))\\ & = & (\sum_{j\neq j_0} p'_j(r_0 + \gamma(c_j))) +_p (r_0 + \gamma(\overline{c}))\\ & = & (\sum_{j\neq j_0} p'_j(r_0 + r)) +_p (r_0 + l)\\ & = & (r_0 + r) +_p (r_0 + l)\\ & = & r_0 + (r +_p l)\\ & > & l\\ & = & \gamma(\overline{c})\\ & = & \mathbf{E}(\mathrm{S_T}[\![E_{i_0}]\!]\gamma \mid \gamma)\end{array}$$

This establishes $(*)$. So, for some E_{i_2}, with $i_2 \neq i_0$, $\mathrm{S_T}[\![M]\!](\gamma) = \mathrm{S_T}[\![E_{i_2}]\!](\gamma)$, and so $\mathrm{S_T}[\![M]\!] = \mathrm{S_T}[\![E_{i_2}]\!]$ and there is a $c_{i_2} : b$ such that $\vdash_{\mathrm{Ax}} E_{i_2} = c_{i_2}$. As Ax is valid in $\mathrm{S_T}$ we have $\mathrm{S_T}[\![\overline{c}]\!] = \mathrm{S_T}[\![c_{i_2}]\!]$ and so $\overline{c} = c_{i_2}$. (Monad units are always injective and so is $[\![\text{-}]\!] : \mathrm{Val}_b \to [\![b]\!]$.) We can therefore replace E_{i_0} and E_{i_2} by $\overline{c}$, apply one of PR_1–PR_4, to obtain a shorter canonical form, and then apply the induction hypothesis. This concludes the proof. □

So, using Lemma 5.20, we see that strong purity completeness at base types holds for T_i with respect to the Ax_i (assuming R satisfies condition (C))[1]. Regarding products of base types, strong purity completeness for general T at products of base types follows from Theorem 5.21 (under the same assumptions as those of the theorem), and so, then, for the T_i (assuming R satisfies condition (C)).

While Condition (C) is not attractive, it is necessary:

Theorem 5.22. *Suppose* R *does not satisfy condition (C). Then* Ax_1 *is not purity complete for* T_1*. That is, there is a term* M *such that* $\models_{\mathrm{S_T}} M \downarrow_b$ *holds but* $\vdash_{\mathrm{Ax}_1} M \downarrow_b$ *does not.*

[1] In [AP21] this was claimed without any assumption on R; however there was an error in the proof.

Proof. As the condition fails, we can choose $p \in (0,1)$ and $s < 0$ such that, for all l and r we have $s + (r +_p l) \leq l$. Take M to be the term $\mathtt{ff}\ \mathtt{or}\ c \cdot (\mathtt{tt} +_p \mathtt{ff})$ where $[\![c]\!] = s$. Then for all $\gamma : [\![\mathtt{Bool}]\!] \to \mathrm{R}$ we have $\mathbf{E}(\mathrm{S_T}[\![c \cdot (\mathtt{tt} +_p \mathtt{ff})]\!]\gamma \mid \gamma) \leq \mathbf{E}(\mathrm{S_T}[\![\mathtt{ff}]\!]\gamma \mid \gamma)$ and so $\models_{\mathrm{S_T}} M \downarrow_b$. However, switching to any R satisfying condition (C), we see that if $\vdash_{\mathrm{Ax_1}} M \downarrow_b$ then, by consistency, we would have $\models_{\mathrm{S_T}} M \downarrow_b$. But this is impossible as, using condition (C), we can find a $\gamma : [\![\mathtt{Bool}]\!] \to \mathrm{R}$ such that $\mathbf{E}(\mathrm{S_T}[\![c \cdot (\mathtt{tt} +_p \mathtt{ff})]\!]\gamma \mid \gamma) > \mathbf{E}(\mathrm{S_T}[\![\mathtt{ff}]\!]\gamma \mid \gamma)$ and so $\mathrm{S_T}[\![M]\!]\gamma = \mathrm{S_T}[\![c \cdot (\mathtt{tt} +_p \mathtt{ff})]\!]\gamma$ and this contradicts $\models_{\mathrm{S_T}} M \downarrow_b$ as $\mathrm{S_T}[\![c \cdot (\mathtt{tt} +_p \mathtt{ff})]\!]\gamma \neq \eta_{\mathrm{T_1}}(b)$ for any $b \in \mathtt{Bool}$. □

To conclude our discussion of purity we construct a counterexample to Condition (C). We make use of the free barycentric commutative algebra R_M over a commutative monoid $(M, +, 0)$. This is the set of finite probability distributions over M, with the usual convex combination operations, with *convolution* as the monoid operation, defined by:

$$(\sum_i p_i x_i) + (\sum_j q_j y_j) = \sum_{ij} p_i q_j (x_i + y_j)$$

and with 0 the Dirac distribution δ_0.

Consider the case where the monoid M is totally ordered, with the monoid operation preserving and reflecting the order. Every finite distribution over M can then be written uniquely in the form $\mu = \sum_{i=1}^n p_i x_i$ with $x_1 > \cdots > x_n$ (and no p_i zero). Set $\mathrm{m}(\mu) = x_n$, $\mathrm{w}(\mu) = p_n$, and, if $n > 1$, $\mathrm{p}(\mu) = \sum_{i=1}^{n-1} \frac{p_i}{1-p_n} x_i$. Note that $\mathrm{m}(x + \mu) = x + \mathrm{m}(\mu)$, for $x \in M$, and that $\mathrm{m}(\mu +_p \nu) = \min(\mathrm{m}(\mu), \mathrm{m}(\nu))$, for $p \in (0,1)$.

Let $\leq$ be the least relation on R_M such that:

$$\frac{\mathrm{m}(\mu) < \mathrm{m}(\nu)}{\mu \leq \nu} \qquad \frac{\mathrm{m}(\mu) = \mathrm{m}(\nu), \mathrm{w}(\mu) > \mathrm{w}(\nu)}{\mu \leq \nu} \qquad \frac{\mathrm{m}(\mu) = \mathrm{m}(\nu), \mathrm{w}(\mu) = \mathrm{w}(\nu) = 1}{\mu \leq \nu}$$

$$\frac{\mathrm{m}(\mu) = \mathrm{m}(\nu), \mathrm{w}(\mu) = \mathrm{w}(\nu) \neq 1, \mathrm{p}(\mu) \leq \mathrm{p}(\nu)}{\mu \leq \nu}$$

Intuitively, one decides whether $\mu \leq \nu$ or $\nu \leq \mu$ by comparing $\mathrm{m}(\mu)$ and $\mathrm{m}(\nu)$, and, if they are equal, comparing their corresponding probabilities, and then if they are equal, but not 1, proceeding recursively to the rest of μ and ν. It can be shown that $\leq$ is a total order, preserved and reflected by probabilistic choice and addition. Note that if $\mu \leq \nu$ then $\mathrm{m}(\mu) \leq \mathrm{m}(\nu)$.

Suppose now that M contains an element $s < 0$ (so M could, for example, be the nonpositive integers with the usual addition and order). Then we claim that R_M does not satisfy Condition (C). For, suppose there are l, r such that $s + (r +_p l) > l$. We have:

$$\begin{array}{rcl} \mathrm{m}(s + (r +_p l)) & = & s + \mathrm{m}(r +_p l) \\ & = & s + \min(\mathrm{m}(r), \mathrm{m}(l)) \\ & \leq & s + \mathrm{m}(l) \\ & < & \mathrm{m}(l) \end{array}$$

However, this contradicts $s + (r +_p l) > l$ as that implies that $\mathrm{m}(s + (r +_p l)) \geq \mathrm{m}(l)$.

6. Conclusion

This paper studies decision-making abstractions in the context of simple higher-order programming languages, focusing on their semantics, treating them operationally and denotationally. The denotational semantics are compositional. They are based on the selection monad, which has rich connections with logic and game theory. Unlike other programming-language research (e.g., [AJM00, HO00]), the treatment of games in this paper is extensional, focusing on choices but ignoring other aspects of computation, such as function calls and returns. Moreover, the games are one-player games. Going further, we have started to explore extensions of our languages with multiple players, where each choice and each reward is associated with one player. For example, writing A and E for the players, we can program a version of the classic prisoners's dilemma:

$$\begin{array}{l} \text{let}\,\mathtt{silent_A}, \mathtt{silent_E} : \mathtt{Bool}\,\text{be}\,(\mathtt{tt}\,\mathtt{or_A}\,\mathtt{ff}), (\mathtt{tt}\,\mathtt{or_E}\,\mathtt{ff})\,\text{in} \\ \mathtt{if}\ \mathtt{silent_A}\,\text{and}\,\mathtt{silent_E}\ \mathtt{then}\ -1 \cdot_{\mathtt{A}} -1 \cdot_{\mathtt{E}} * \\ \ \mathtt{else\ if}\ \mathtt{silent_A}\ \mathtt{then}\ -3 \cdot_{\mathtt{A}} * \\ \ \mathtt{else\ if}\ \mathtt{silent_E}\ \mathtt{then}\ -3 \cdot_{\mathtt{E}} * \\ \ \mathtt{else}\ -2 \cdot_{\mathtt{A}} -2 \cdot_{\mathtt{E}} * \end{array}$$

Here, $\mathtt{silent_A}$ and $\mathtt{silent_E}$ indicate whether the players remain silent, and the rewards, which are negative, correspond to years of prison. Semantically it would be natural to use the selection monad with $\mathbb{R}^2$ as the set of rewards, and with the writer monad as auxiliary monad. (One could envisage going further and treating probabilistic games via a combination of the writer monad and a monad for probability.) Many of our techniques carry over to languages with multiple players, which give rise to interesting semantic questions (e.g., should we favor some players over others? require Nash equilibria?) and may also be useful in practice.

Multi-objective optimization provides another area of interest. One could take R to be a product, with one component for each objective, and use the selection monad augmented with auxiliary monad the combination $\mathcal{P}_{\text{fin}}(\mathrm{R} \times -)$ of the finite powerset monad and a version of the writer monad enabling writing to different components. One would aim for a semantics returning Pareto optimal choices.

In describing Software 2.0, Karpathy suggested specifying some goal on the behavior of a desirable program, writing a "rough skeleton" of the code, and using the computational resources at our disposal to search for a program that works [Kar17]. While this vision may be attractive, realizing it requires developing not only search techniques but also the linguistic constructs to express goals and code skeletons. In the variant of this vision embodied in SmartChoices, the skeleton is actually a complete program, albeit in an extended language with decision-making abstractions. Thus, in the brave new world of Software 2.0 and its relatives, programming languages still have an important role to play, and their study should be part of their development. Our paper aims to contribute to one aspect of this project; much work remains.

In comparison with recent theoretical work on languages with differentiation (e.g,. [FST19, AP20, BCLG20, BMP20, CGM19, HSV20]), our languages are higher-level: they focus on how optimization or machine-learning may be made available to a programmer rather than on how they would be implemented. However, a convergence of these research lines is possible, and perhaps desirable. One thought is to extend our languages with differentiation primitives to construct selection functions that use gradient descent. These would be alternatives to argmax as discussed in the Introduction. Monadic reflection and reification, in the sense of Filinski [Fil94], could support the use of such alternatives, and more generally enhance programming flexibility. Similarly, it would be attractive to deepen the connections between our languages and probabilistic ones (e.g., [GMR$^+$12]). It may also be interesting to connect our semantics with particular techniques from the literature on MDPs and RL, and further to explore whether monadic ideas can contribute to implementations that include such techniques. Finally, at the type level, the monadic approach distinguishes "selected" values and "ordinary" ones; the "selected" values are reminiscent of the "uncertain" values of type Uncertain<`T`> [BMM14], and the distinction may be useful as in that setting.

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