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CARTESIAN CLOSED 2-CATEGORIES AND PERMUTATION EQUIVALENCE IN HIGHER-ORDER REWRITING

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Abstract. We propose a semantics for permutation equivalence in higher-order rewriting. This semantics takes place in cartesian closed 2-categories, and is proved sound and complete.

1. Introduction

Cartesian closed categories provide semantics for equational theories with variable binding [12, 4]. On the other hand, 2-categories with finite products provide semantics for term rewriting [3]. The present paper shows that *cartesian closed 2-categories* provide semantics for term rewriting with variable binding, as embodied by Brugginks's generalisation [1] of *permutation equivalence* [16, Chapter 8] to *higher-order rewriting* [11, 19, 15, 17].

We first define *cartesian closed 2-signatures*, which generalise higher-order rewrite systems, and organise them into a category Sig. We then construct an adjunction

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}, \arrow[l, bend left, "\mathcal{W}"] \end{tikzcd} \tag{1.1}$$

where $\mathsf{2CCCat}$ is the category of small cartesian closed 2-categories. From a given cartesian closed 2-signature S, the functor $\mathcal{H}$ constructs a cartesian closed 2-category, whose 2-cells are Bruggink's proof terms modulo permutation equivalence, which we prove is the free cartesian closed 2-category generated by S.

We review a number of examples and non-examples, and sketch an extension to deal with the latter.

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Related work. Our cartesian closed 2-signatures may be seen as a 2-dimensional refinement of *cartesian closed sketches* [18, 5, 10]. Bruggink's calculus of permutation equivalence is close in spirit to Hilken's 2-categorical semantics of the simply-typed λ-calculus [8], but technically different and generalised to arbitrary higher-order rewrite systems. Capriotti [2] proposes a semantics of so-called *flat* permutation equivalence in sesquicategories. More related work is discussed in Section 4.2.

2. Cartesian closed signatures and categories

We start by recalling the well-known adjunction [12, 4] between what we here call *(cartesian closed) 1-signatures* and cartesian closed categories.

For any set X, define *types* over X by the grammar:

$$A, B, \ldots \in \mathcal{L}_0(X) \quad ::= \quad x \mid 1 \mid A \times B \mid B^A,$$

with $x \in X$.

Proposition 2.1. *$\mathcal{L}_0$ defines a monad on* Set.

Let the set of *sequents* over a set X be $\mathcal{S}_0(X) = \mathcal{L}_0(X)^* \times \mathcal{L}_0(X)$, i.e., sequents are pairs of a list of types and a type. The assignment $X \mapsto \mathcal{S}_0(X)$ extends to an endofunctor on Set.

Definition 2.2. A *1-signature* consists of a set X_0 of *sorts*, and an $\mathcal{S}_0(X_0)$-indexed set X_1 of *operations*, or equivalently a map $X_1 \to \mathcal{S}_0(X_0)$.

A *morphism of 1-signatures* $(X_0, X_1) \to (Y_0, Y_1)$ is a pair (f_0, f_1) where $f_i : X_i \to Y_i$ such that

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ \mathcal{S}_0(X_0) \arrow[r, "\mathcal{S}_0(f_0)"'] & \mathcal{S}_0(Y_0) \end{tikzcd}$$

commutes. Morphisms compose in the obvious way, and we have:

Proposition 2.3. *Composition of morphisms is associative and unital, and hence 1-signatures and their morphisms form a category* Sig_1.

There is a well-known adjunction

$$\begin{tikzcd} \mathsf{Sig}_1 \arrow[r, bend left, "\mathcal{H}_1"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat} \arrow[l, bend left, "\mathcal{W}_1"] \end{tikzcd}$$

between 1-signatures and the category CCCat of small cartesian closed categories (with chosen structure) and (strict) cartesian closed functors, i.e., functors $F : \mathcal{C} \to \mathcal{D}$ preserving binary products, projections, and the terminal object on the nose, and such that, for all objects $A, B \in \mathcal{C}$, currying

$$F(B^A) \times F(A) = F(B^A \times A) \xrightarrow{F(ev_{A,B})} F(B)$$

yields an identity.

The functor $\mathcal{W}_1$ maps any cartesian closed category $\mathcal{C}$ to the signature with sorts $\mathcal{C}_0$, its set of objects, and with operations $A_1, \ldots, A_n \to A$ the set $\mathcal{C}([\![A_1 \times \ldots \times A_n]\!], [\![A]\!])$, where $[\![-]\!]$ denotes the function $\mathcal{L}_0(\mathcal{C}_0) \to \mathcal{C}_0$ defined by induction:

$$\begin{array}{rcll} [\![c]\!] & = & c & c \in \mathcal{C}_0 \\ [\![1]\!] & = & 1 & \\ [\![A \times B]\!] & = & [\![A]\!] \times [\![B]\!] & \\ [\![B^A]\!] & = & [\![B]\!]^{[\![A]\!]}. & \end{array} \tag{2.1}$$

Conversely, given a 1-signature X, consider the simply-typed λ-calculus with base types in X_0 and constants in X_1. I.e., for any $c \in X_1(G, A)$ and terms $\Gamma \vdash M_i{:}\, A_i$, for all $1 \leq i \leq n$, where $G = (A_1, \ldots, A_n)$, we have a term $\Gamma \vdash c(\!|M_1, \ldots, M_n|\!){:}\, A$, representing the application of the constant c to $M_1, \ldots, M_n$. We use special parentheses to avoid ambiguity with term application. Terms modulo $\beta\eta$ form a cartesian closed category $\mathcal{H}_1(X)$ with objects all types over X_0 and morphisms $A \to B$ all terms of type B with one free variable of type A.

A less often formulated observation, which is useful to us, is that the adjunction $\mathcal{H}_1 \dashv \mathcal{W}_1$ decomposes into two adjunctions

$$\begin{tikzcd} \mathsf{Sig}_1 \arrow[r, bend left, "\mathcal{K}_1"] \arrow[r, phantom, "\perp"] & \mathcal{L}_1\text{-Alg} \arrow[l, bend left, "\mathcal{U}_1"] \arrow[r, bend left, "\mathcal{F}_1"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat}, \arrow[l, bend left, "\mathcal{V}_1"] \end{tikzcd}$$

as follows.

Consider first the endofunctor $\mathcal{L}_1$ on Sig_1 defined on objects by mapping any 1-signature X to the 1-signature with

- as sorts the set X_0, and
- as operations $\Gamma \vdash A$ the λ-terms $\Gamma \vdash M{:}\, A$, with base types in X_0 and constants in X_1, as sketched above, modulo $\beta\eta$.

On morphisms of 1-signatures $X \xrightarrow{f} Y$, let $\mathcal{L}_1(f)$ substitute constants $c \in X_1$ with $f_1(c)$. We obtain

Proposition 2.4. *$\mathcal{L}_1$ is a monad on Sig_1, with unit and multiplication, say η and μ.*

Let now $\mathcal{L}_1$-Alg be the category of algebras for the monad $\mathcal{L}_1$ and $\mathcal{K}_1$ be the 'free algebra' functor $X \mapsto (\mathcal{L}_1(X), \mu_X)$.

The functor $\mathcal{V}_1$ maps any cartesian closed category $\mathcal{C}$ to the $\mathcal{L}_1$-algebra with base 1-signature $(\mathcal{C}_0, \mathcal{C}_1)$, defined as follows. First, $\mathcal{C}_0$ is the set of objects of $\mathcal{C}$. It has a canonical $\mathcal{L}_0$-algebra structure, say $h_0{:}\, \mathcal{L}_0(\mathcal{C}_0) \to \mathcal{C}_0$, obtained by interpreting type constructors in $\mathcal{C}$ as in (2.1). Extending this to contexts G by $h_0(G) = \prod_i h_0(G_i)$, let the operations in $\mathcal{C}_1(G, A)$ be the morphisms in $\mathcal{C}(h_0(G), h_0(A))$. Beware: the domain and codomain of such an operation are really G and A, not $h_0(G)$ and $h_0(A)$. Similarly, interpreting the λ-calculus in $\mathcal{C}$, the 1-signature $(\mathcal{C}_0, \mathcal{C}_1)$ has a canonical $\mathcal{L}_1$-algebra structure, say $h_1{:}\, \mathcal{L}_1(\mathcal{C}_0, \mathcal{C}_1) \to (\mathcal{C}_0, \mathcal{C}_1)$:

$$\begin{array}{rcl} h_1(G \vdash x_i{:}\, G_i) & = & \pi_i \\ h_1(G \vdash (){:}\, 1) & = & ! \\ h_1(G \vdash c(\!|M_1, \ldots, M_n|\!){:}\, A) & = & c \circ \langle h_1(M_1), \ldots, h_1(M_n) \rangle \\ h_1(G \vdash \lambda x{:}\, A.M{:}\, B^A) & = & \varphi(h_1(G, x{:}\, A \vdash M{:}\, B)) \\ h_1(G \vdash MN{:}\, B) & = & ev \circ \langle h_1(M), h_1(N) \rangle \\ h_1(G \vdash (M, N){:}\, A \times B) & = & \langle h_1(M), h_1(N) \rangle \\ h_1(G \vdash \pi M{:}\, A) & = & \pi \circ h_1(M) \\ h_1(G \vdash \pi' M{:}\, A) & = & \pi' \circ h_1(M), \end{array}$$

where ! is the unique morphism $h_0(G) \to 1$, φ is the bijection $\mathcal{C}(h_0(G,A), h_0(B)) \cong \mathcal{C}(h_0(G), h_0(B^A))$, and ev is the structure morphism $h_0(B^A \times A) \to h_0(B)$.

$\mathcal{L}_1$-algebras are much like cartesian closed categories whose objects are freely generated by their set of sorts. A perhaps useful analogy here is with multicategories $\mathcal{M}$, seen as being close to monoidal categories whose objects are freely generated by those of $\mathcal{M}$ by tensor and unit. Here, the functor $\mathcal{F}_1$ sends any $\mathcal{L}_1$-algebra (X, h) to the cartesian closed category with

- objects the types over X_0, i.e., $\mathcal{L}_0(X_0)$,
- morphisms $A \to B$ the set of operations in $X_1(A, B)$.

This canonically forms a cartesian closed category, with structure induced by the $\mathcal{L}_1$-algebra structure. We define it in more detail in dimension 2 in Section 7.2.

3. Cartesian closed 2-signatures

Given a 1-signature X, let $X_{||}$ denote the set of pairs of parallel operations, i.e., pairs of operations M, N over the same sequent. Otherwise said, $X_{||}$ is the pullback

$$\begin{tikzcd} X_{||} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ X_1 \arrow[r] & \mathcal{S}_0(X_0). \end{tikzcd}$$

Any morphism $f \colon X \to Y$ of 1-signatures yields a map $f_{||} \colon X_{||} \to Y_{||}$, via the dashed arrow (obtained by universal property of pullback) in

$$\begin{tikzcd} X_{||} \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X_1 \arrow[dr, "f_1"] \arrow[dd] & \\ & Y_{||} \arrow[rr] \arrow[dd] & & Y_1 \arrow[dd] \\ X_1 \arrow[rr] \arrow[dr, "f_1"'] & & \mathcal{S}_0(X_0) \arrow[dr] & \\ & Y_1 \arrow[rr, "\mathcal{S}_0(f_0)"] & & \mathcal{S}_0(Y_0). \end{tikzcd}$$

Definition 3.1. A *2-signature* consists of a 1-signature X, plus a set X_2 of *reduction rules* with a map $X_2 \to \mathcal{L}_1(X)_{||}$.

A *morphism of 2-signatures* $(X, X_2) \to (Y, Y_2)$ is a pair (f, f_2) where $f \colon X \to Y$ is a morphism of 1-signatures and $f_2 \colon X_2 \to Y_2$ makes the diagram

$$\begin{tikzcd} X_2 \arrow[r, "f_2"] \arrow[d] & Y_2 \arrow[d] \\ \mathcal{L}_1(X)_{||} \arrow[r, "\mathcal{L}_1(f_1)_{||}"'] & \mathcal{L}_1(Y)_{||} \end{tikzcd}$$

commute. We obtain:

Proposition 3.2. *Composition of morphisms is associative and unital, and hence 2-signatures and their morphisms form a category* Sig.

4. Examples

4.1. **Higher-order rewrite systems.** The prime example of a 2-signature is that for the pure λ-calculus: it has a sort t and operations

$$a\colon (t, t \vdash t) \qquad\qquad \ell\colon (t^t \vdash t),$$

with a reduction rule β over the pair $x{:}\,t^t, y{:}\,t \vdash a\llparenthesis \ell\llparenthesis x\rrparenthesis, y\rrparenthesis, x(y) : t$ in $\mathcal{L}_1(\{t\}, \{\ell, a\})_{||}$. Categorically, this will yield a 2-cell

$$\begin{tikzcd} & t \times t \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\beta"] & \\ t^t \times t \arrow[ur, "\ell\times t"] \arrow[rr, bend right, "ev"'] & {} & t. \end{tikzcd}$$

This is an example of a *higher-order rewrite system* in the sense of Nipkow [15]. Nipkow's definition is formally different, but his higher-order rewrite systems are in bijection with 2-signatures $h\colon X_2 \to \mathcal{L}_1(X)_{||}$ such that for all rules $r \in X_2$, letting $(\Gamma \vdash M, N\colon A) = h(r)$:

- M is not a variable,
- A is a sort,
- each variable occurring in Γ occurs free in M.

These restrictions help dealing with decidability problems on higher-order rewrite systems, whose extension to our setting we leave open.

Let us now anticipate over Adjunction (1.1) and our main results below and state our soundness and completeness theorem. Given a higher-order rewrite system X, i.e., a 2-signature satisfying the above conditions, let $\mathcal{R}(X)$ be the following locally-preordered 2-category. It has:

- objects are types in $\mathcal{L}_0(X_0)$;
- morphisms $A \to B$ are λ-terms in $\mathcal{L}_1(X)(A \vdash B)$, modulo $\beta\eta$;
- given two parallel morphisms M and N, there is one 2-cell $M \to N$ exactly when there is a sequence of reductions $M \to^* N$ in the usual sense [15].

Proposition 4.1. *$\mathcal{R}(X)$ is 2-cartesian closed.*

$\mathcal{R}(X)$ and $\mathcal{H}(X)$ have the same objects and morphisms. But because our inference rules for forming reductions are the same as deduction rules for proving the existence of a reduction in the usual sense, we may map any reduction $P\colon M \to N$ to the unique reduction $M \to N$ in $\mathcal{R}(X)$. Conversely, any standard reduction step has a proof, which provides a reduction P. We have proved:

Theorem 4.2 (Soundness and completeness)**.** *There exists an identity-on-objects, identity-on-morphisms, locally full cartesian closed 2-functor $\mathcal{H}(X) \xrightarrow{!} \mathcal{R}(X)$.*

4.2. **Theories with binding.** Understanding reduction rules as equations, it is easy to define the free cartesian closed category generated by a 2-signature. This yields the adjunction

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}'"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat} \arrow[l, bend left, "\mathcal{W}'"] \end{tikzcd} \tag{4.1}$$

recalled above.

This adjunction provides a categorical semantics for theories with binding, which is more general than other approaches by Fiore and Hur [7], Hirschowitz and Maggesi [9], and Zsidó [20].

If I understand correctly, the motivation for Fiore and Hur's subtle approach is the will to explain the λ-calculus by strictly less than itself. The present framework does not obey this specification, and instead tends to view the λ-calculus as a universal (parameterised) theory with binding.

We end this section by giving a formal construction of Adjunction (4.1). Cartesian closed categories form a full, reflective subcategory of $\mathsf{2CCCat}$, via the functor $\mathcal{J}\colon \mathsf{2CCCat} \to \mathsf{CCCat}$ mapping any cartesian closed 2-category $\mathcal{C}$ to the cartesian closed category with:

- objects those of $\mathcal{C}$,
- morphisms those of $\mathcal{C}$, modulo the congruence generated by $f \sim g$ iff there exists a 2-cell $f \to g$.

Here, $\mathcal{J}(\mathcal{C})$ is thought of as the free locally discrete cartesian closed 2-category. Adjunction (4.1) is obtained by composing the adjunctions

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{H}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat} \arrow[l, bend left, "\mathcal{W}"] \arrow[r, bend left, "\mathcal{J}"] \arrow[r, phantom, "\perp"] & \mathsf{CCCat}. \arrow[l, bend left] \end{tikzcd}$$

4.3. **Non-examples.** Non-examples are given by calculi whose reduction semantics is defined on terms modulo a so-called *structural congruence*, e.g., CCS [13], or the π-calculus [6, 14].

For example, consider the CCS term $(a \mid 0) \mid \overline{a}$. In CCS, it is *structurally* equivalent to $(a \mid \overline{a}) \mid 0$, which then reduces to $0 \mid 0$.

In order to account for this, we would have to consider a 2-signature with reduction rules for structural congruence, here $(M_1 \mid M_2) \mid M_3 \to M_1 \mid (M_2 \mid M_3)$ for associativity, and $M \mid N \to N \mid M$ for commutativity. But then, these reductions count as proper reductions, which departs from the desired computational behaviour. For example, the term $a \mid a$ has an infinite reduction sequence, using commutativity.

Anticipating the development in the next sections, a potential solution is to extend 2-signatures to *2-theories*. For any 2-signature X, let $X_{||}$ denote the set of pairs of reduction rules r, s with a common type $G \vdash M \to N \colon A$. A 2-theory is a 2-signature X, together with a set of equations between parallel reductions, i.e., a subset X_3 of $\mathcal{L}(X)_{||}$ (where $\mathcal{L}$ is defined in Section 5).

Another possibility would be to define 2-theories to constist of a pair of sets over $\mathcal{L}_1(X)_{||}$, one for structural equations, and the other for proper reduction rules.

The main adjunction announced above (1.1) extends to an adjunction between 2-theories and cartesian closed 2-categories. Using equations, we may specify that any reduction $M \to M$ using only structural rules be the identity on M, and consider the computational behaviour of a 2-category to consist of its non-invertible 2-cells, as proposed by Hilken [8]. A question is whether for a given calculus this can be done with finitely many equations.

5. A 2-lambda-calculus

We now begin the construction of Adjunction (1.1). We start in this section by defining a monad $\mathcal{L}$ on Sig, which we will use to factor Adjunction (1.1) as

$$\begin{tikzcd} \mathsf{Sig} \arrow[r, bend left, "\mathcal{K}"] \arrow[r, phantom, "\perp"] & \mathcal{L}\text{-Alg} \arrow[l, bend left, "\mathcal{U}"] \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}, \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

where:

- $\mathcal{L}$-Alg is the category of $\mathcal{L}$-algebras,
- $\mathcal{K}: \mathsf{Sig} \to \mathcal{L}$-Alg maps any X to the free $\mathcal{L}$-algebra $(\mathcal{L}^2 X \xrightarrow{\mu} \mathcal{L}X)$,
- $\mathcal{U}(\mathcal{L}X \xrightarrow{h} X) = X$,
- $\mathsf{2CCCat}$ is the category of cartesian closed 2-categories, which we define in Section 6.

The left-hand adjunction holds by $\mathcal{L}$ being a monad, thus we concentrate in Section 7 on establishing the right-hand one.

But for now, let us define the monad $\mathcal{L}$.

5.1. **Syntax.** Given a 2-signature $X = ((X_0, X_1), a: X_2 \to \mathcal{L}_1(X)_{||})$ (actually $\mathcal{L}_1(X)$ is $\mathcal{L}_1(X_0, X_1)$), we construct a new 2-signature $\mathcal{L}(X)$, whose reduction rules represent reduction sequences in the "higher-order rewrite system" defined by X, modulo permutation equivalence. The 2-signature $\mathcal{L}(X)$ has the same base 1-signature (X_0, X_1), and as reduction rules the terms of a 2λ-calculus (in the sense of Hilken [8]) modulo permutation equivalence, which we now define.

First, terms, called *reductions*, are defined by induction in Figure 1. The typing judgement has the shape $\Gamma \vdash P : M \to N : A$, where A is a type in $\mathcal{L}_0(X_0)$, Γ is a list of pairs of a variable and a type, with no variable appearing more than once, M and N are terms of type $\Gamma \vdash A$ modulo $\beta\eta$, and P is a reduction. In the sequel, we often forget the variables in such pairs $(\Gamma \vdash A)$, and identify them with sequents in $\mathcal{S}_0(X_0)$.

Remark 5.1. For any $\Gamma \vdash M : A$, we have a reduction $\Gamma \vdash M : M \to M : A$.

When clear from context, we abbreviate substitutions $[M_1/x_1, \ldots, M_n/x_n]$ of terms by $[M_1, \ldots, M_n]$. For a context G, G_i denotes its ith type. Also, for $(M, N) \in \mathcal{L}_1(X)_{||}$, we let $X(M, N)$ be the set of all reduction rules $r \in X_2$ such that $a(r) = (M, N)$. We write $X(\Gamma \vdash M, N : A)$ to indicate the common type of M and N. Similarly, $X(G \vdash A)$ denotes the set of operations in X_1 above $G \vdash A$.

5.2. **Substitution.** Next, we define substitution, which has "type"

$$\frac{\Gamma \vdash Q : N \to N' : \Delta \qquad \Delta \vdash P : M \to M' : A}{\Gamma \vdash P[Q] : M[N] \to M'[N'] : A,} \tag{5.1}$$

i.e., given a reduction P and a tuple of reductions Q, it produces a reduction of the indicated type, which we denote by $P[Q]$. Here, we denote by $\Gamma \vdash Q : N \to N' : \Delta$ a tuple of reductions $\Gamma \vdash Q_i : N_i \to N'_i : \Delta_i$, for $1 \leq i \leq |\Delta|$.

First, observe that we have a form of weakening: for any reduction $\Gamma \vdash P : M \to N : A$ and $x \notin \Gamma$, we also have $\Gamma, x : B \vdash P : M \to N : A$. We use this implicitly in the following.

The definition of substitution is a bit tricky:

$$\frac{\ldots \qquad \Gamma \vdash P_i : M_i \to N_i : G_i \qquad \ldots}{\Gamma \vdash r\langle\!\langle P_1, \ldots, P_n \rangle\!\rangle : M[M_1, \ldots, M_n] \to N[N_1, \ldots, N_n] : A} \; (r \in X(G \vdash M, N : A))$$

$$\frac{\Gamma \vdash P : M_1 \to M_2 : A \qquad \Gamma \vdash Q : M_2 \to M_3 : A}{\Gamma \vdash P \,;_{M_2} Q : M_1 \to M_3 : A}$$

$$\Gamma, x : A, \Delta \vdash x : x \to x : A \qquad\qquad \Gamma \vdash () : () \to () : 1$$

$$\frac{\Gamma \vdash P_1 : M_1 \to N_1 : G_1 \qquad \ldots \qquad \Gamma \vdash P_n : M_n \to N_n : G_n}{\Gamma \vdash c(\!| P_1, \ldots, P_n |\!) : c(\!| M_1, \ldots, M_n |\!) \to c(\!| N_1, \ldots, N_n |\!) : A} \; (c \in X_1(G \vdash A))$$

$$\frac{\Gamma, x : A \vdash P : M \to N : B}{\Gamma \vdash \lambda x : A.P : \lambda x : A.M \to \lambda x : A.N : B^A} \qquad \frac{\Gamma \vdash P : M \to M' : B^A \qquad \Gamma \vdash Q : N \to N' : A}{\Gamma \vdash PQ : MN \to M'N' : B}$$

$$\frac{\Gamma \vdash P : M \to M' : A \qquad \Gamma \vdash Q : N \to N' : B}{\Gamma \vdash (P, Q) : (M, N) \to (M', N') : A \times B}$$

$$\frac{\Gamma \vdash P : M \to N : A \times B}{\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \to \pi_{A,B} N : A} \qquad \frac{\Gamma \vdash P : M \to N : A \times B}{\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \to \pi'_{A,B} N : B}$$

Figure 1: Reductions

- first we define *left whiskering*, which has "type"

$$\frac{\Gamma \vdash Q : N \to N' : \Delta \qquad \Delta \vdash M : A}{\Gamma \vdash M[Q] : M[N] \to M'[N'] : A;}$$

- then we define *right whiskering*, which has "type"

$$\frac{\Gamma \vdash N : \Delta \qquad \Delta \vdash P : M \to M' : A}{\Gamma \vdash P[N] : M[N] \to M'[N] : A,}$$

  (where N denotes a tuple);
- then we define substitution by $P[Q] = (P[N] \,;_{M'[N]} M'[Q])$.

There is of course another legitimate definition, namely $M[Q] \,;_{M[N']} P[N']$. The two will be equated by permutation equivalence in the next section.

Left whiskering is defined inductively, with $\Delta = (x_1 : A_1, \ldots, x_n : A_n)$ and $Q = (Q_1, \ldots, Q_n)$, by:

$$\begin{array}{rcl}
()[Q] & = & () \\
x_i[Q] & = & Q_i \\
c(\!| M_1, \ldots, M_p |\!)[Q] & = & c(\!| M_1[Q], \ldots, M_p[Q] |\!) \\
(\lambda x : B.M)[Q] & = & \lambda x : B.(M[Q, x]) \quad (\text{for } x \notin \mathrm{dom}(\Delta)) \\
(MN)[Q] & = & (M[Q]N[Q]) \\
(M, N)[Q] & = & (M[Q], N[Q]) \\
(\pi_{A,B} M)[Q] & = & \pi_{A,B}(M[Q]) \\
(\pi'_{A,B} M)[Q] & = & \pi'_{A,B}(M[Q])
\end{array}$$

Right whiskering is defined inductively, with $\Delta = (x_1{:}\,A_1, \dots, x_n{:}\,A_n)$ and $N = (N_1, \dots, N_n)$, by:

$$\begin{array}{rcl}
(r\langle\!\langle P_1, \dots, P_p\rangle\!\rangle)[N] & = & r\langle\!\langle P_1[N], \dots, P_p[N]\rangle\!\rangle \\
(P_1 \,;_{M''}\, P_2)[N] & = & (P_1[N] \,;_{M''[N]}\, P_2[N]) \\
()[N] & = & () \\
x_i[N] & = & N_i \\
c(\!| P_1, \dots, P_p |\!)[N] & = & c(\!| P_1[N], \dots, P_p[N] |\!) \\
(\lambda x{:}\,B.P')[N] & = & \lambda x{:}\,B.(P'[N, x]) \quad (\text{for } x \notin \mathrm{dom}(\Delta)) \\
(P_1 P_2)[N] & = & (P_1[N] P_2[N]) \\
(P_1, P_2)[N] & = & (P_1[N], P_2[N]) \\
(\pi_{A,B} P')[N] & = & \pi_{A,B}(P'[N]) \\
(\pi'_{A,B} P')[N] & = & \pi'_{A,B}(P'[N]).
\end{array}$$

Definition 5.2. Let $P[Q] = (P[N] \,;_{M'[N]}\, M'[Q])$.

Proposition 5.3. *Given reductions P and Q as above, $P[Q]$ is a well-typed reduction $\Gamma \vdash P[Q] : M[N] \to M'[N'] {:}\, A$.*

5.3. **Permutation equivalence.** We now define *permutation equivalence* on reductions, by the equations in Figures 3, 4 and 5, in Appendix A. The *congruence* rules in Figure 3 are bureaucratic: they just say that permutation equivalence is a congruence. The *category* rules make reductions of a given type $\Gamma \vdash A$ into a category. In Figure 4, the *beta and eta* rules mirror the term-level beta and eta rules. Finally, the *lifting* rules lift composition of reductions towards toplevel.

So, $\mathcal{L}(X)$ has sorts X_0, operations X_1, and as reduction rules in $\mathcal{L}(X)(G \vdash M, N{:}\,A)$ all reductions $G \vdash P : M \to N{:}\,A$, modulo the equations.

This easily extends to:

Proposition 5.4. *$\mathcal{L}$ is a functor* $\mathsf{Sig} \to \mathsf{Sig}$.

Now, consider $\mathcal{L}\mathcal{L}(X)$. We define a mapping $\mu_X{:}\,\mathcal{L}\mathcal{L}(X) \to \mathcal{L}(X)$, by induction on reductions. The typing rule for reduction rules in $\mathcal{L}\mathcal{L}(X)$ specialises to:

$$\frac{(R \in \mathcal{L}(X)(G \vdash M, N{:}\,A)) \qquad \Gamma \vdash P_1 : M_1 \to N_1{:}\,G_1 \qquad \dots \qquad \Gamma \vdash P_n : M_n \to N_n{:}\,G_n}{\Gamma \vdash R\langle\!\langle P_1, \dots, P_n\rangle\!\rangle : M[M_1, \dots, M_n] \to N[N_1, \dots, N_n]{:}\,A}\,.$$

We set $\mu(R\langle\!\langle P_1, \dots, P_n\rangle\!\rangle) = R[\mu(P_1), \dots, \mu(P_n)]$. The other cases just propagate the substitution:

$$\begin{array}{rcl}
P \,;\, Q & \mapsto & \mu(P) \,;\, \mu(Q) \\
x & \mapsto & x \\
() & \mapsto & () \\
c(\!| P_1, \dots, P_n |\!) & \mapsto & c(\!| \mu(P_1), \dots, \mu(P_n) |\!) \\
\lambda x{:}\,A.P & \mapsto & \lambda x{:}\,A.\mu(P) \\
PQ & \mapsto & \mu(P)\mu(Q) \\
(P, Q) & \mapsto & (\mu(P), \mu(Q)) \\
\pi P & \mapsto & \pi(\mu(P)) \\
\pi' P & \mapsto & \pi'(\mu(P)).
\end{array}$$

Lemma 5.5. *This defines a natural transformation $\mu{:}\,\mathcal{L}^2 \to \mathcal{L}$, which makes the diagram*

$$\begin{tikzcd}
\mathcal{L}^3 \arrow[r, "\mathcal{L}\mu"] \arrow[d, "\mu\mathcal{L}"'] & \mathcal{L}^2 \arrow[d, "\mu"] \\
\mathcal{L}^2 \arrow[r, "\mu"'] & \mathcal{L}
\end{tikzcd}$$

commute.

Similarly, there is a natural transformation $\eta\colon id \to \mathcal{L}$, mapping each $r \in X(G \vdash M, N\colon A)$ to the reduction $G \vdash r\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle : M \to N\colon A$, and we have:

Lemma 5.6. *The diagram*

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\eta\mathcal{L}"] \arrow[dr, equal] & \mathcal{L}^2 \arrow[d, "\mu"] & \mathcal{L} \arrow[l, "\mathcal{L}\eta"'] \arrow[dl, equal] \\
& \mathcal{L} &
\end{tikzcd}$$

commutes.

Corollary 5.7. $(\mathcal{L}, \mu, \eta)$ *is a monad on* Sig*.*

A crucial result is:

Proposition 5.8. *For all* $\Gamma \vdash Q : N \to N'\colon \Delta$ *and* $\Delta \vdash P : M \to M'\colon A$*, we have:*

$$\Gamma \vdash P[Q] \equiv (M[Q] \,;_{M[N']}\, P[N']) : M[N] \to M'[N']\colon A.$$

Proof. We proceed by induction on P. Most cases are bureaucratic. Consider for instance $P = c\llparenthesis P_1, \ldots, P_p \rrparenthesis$. Then, by definition:

$$P[Q] = (c\llparenthesis P_1[N], \ldots, P_p[N] \rrparenthesis \,;_{c\llparenthesis M_1'[N], \ldots, M_p'[N] \rrparenthesis}\, c\llparenthesis M_1'[Q], \ldots, M_p'[Q] \rrparenthesis.$$

By the third lifting rule, this is permutation equivalent to

$$c\llparenthesis P_1[N] \,;_{M_1'[N]}\, M_1'[Q], \ldots, P_p[N] \,;_{M_p'[N]}\, M_p'[Q] \rrparenthesis.$$

By p applications of the induction hypothesis, we obtain

$$c\llparenthesis M_1[Q] \,;_{M_1[N']}\, P_1[N'], \ldots, M_p[Q] \,;_{M_p[N']}\, P_p[N'] \rrparenthesis,$$

which by lifting again yields the desired result:

$$c\llparenthesis M_1[Q], \ldots, M_p[Q] \rrparenthesis \,;_{c\llparenthesis M_1[N'], \ldots, M_p[N'] \rrparenthesis}\, c\llparenthesis P_1[N'], \ldots, P_p[N'] \rrparenthesis.$$

The case where something actually happens is $P = r\langle\!\langle P_1, \ldots, P_p \rangle\!\rangle$, with $r \in X(G \vdash M_0, M_0'\colon A)$ and each $\Delta \vdash P_i : M_i \to M_i'\colon G_i$. Then, the left-hand side is

$$r\langle\!\langle P_1[N], \ldots, P_p[N] \rangle\!\rangle \,;_{M_0'[M_1', \ldots, M_n'][N]}\, M_0'[M_1', \ldots, M_p'][Q].$$

By lifting, omitting indices of vertical compositions and implicitly using the *category* rules, we have

$$r\langle\!\langle P_1[N], \ldots, P_p[N] \rangle\!\rangle \equiv r\langle\!\langle M_1[N], \ldots, M_p[N] \rangle\!\rangle \,;\, M_0'[P_1[N], \ldots, P_p[N]].$$

Observing that $M_0'[M_1', \ldots, M_p'][Q] = M_0'[M_1'[Q], \ldots, M_p'[Q]]$, the whole is permutation equivalent to

$$\begin{array}{l}
r\langle\!\langle M_1[N], \ldots, M_p[N] \rangle\!\rangle; \\
M_0'[P_1[N], \ldots, P_p[N]]; \\
M_0'[M_1'[Q], \ldots, M_p'[Q]],
\end{array}$$

i.e., by lifting (inductively):

$$\begin{array}{l} r\langle\!\langle M_1[N], \ldots, M_p[N] \rangle\!\rangle; \\ M_0'[(P_1[N]\,;\, M_1'[Q]), \ldots, (P_p[N]\,;\, M_p'[Q])]. \end{array}$$

By induction hypothesis, this is permutation equivalent to

$$\begin{array}{l} r\langle\!\langle M_1[N], \ldots, M_p[N] \rangle\!\rangle; \\ M_0'[(M_1[Q]\,;\, P_1[N']), \ldots, (M_p[Q]\,;\, P_p[N'])], \end{array}$$

i.e., by lifting again to

$$\begin{array}{l} r\langle\!\langle M_1[N], \ldots, M_p[N] \rangle\!\rangle; \\ M_0'[M_1[Q], \ldots, M_p[Q]]; \\ M_0'[P_1[N'], \ldots, P_p[N']]. \end{array}$$

The second lifting rule then yields

$$\begin{array}{l} r\langle\!\langle M_1[Q], \ldots, M_p[Q] \rangle\!\rangle; \\ M_0'[P_1[N'], \ldots, P_p[N']], \end{array}$$

and hence

$$\begin{array}{l} M_0[M_1[Q], \ldots, M_p[Q]]; \\ r\langle\!\langle M_1[N'], \ldots, M_p[N'] \rangle\!\rangle; \\ M_0'[P_1[N'], \ldots, P_p[N']], \end{array}$$

so, by the second lifting rule again:

$$\begin{array}{l} M_0[M_1[Q], \ldots, M_p[Q]]; \\ r\langle\!\langle P_1[N'], \ldots, P_p[N'] \rangle\!\rangle, \end{array}$$

i.e., the right-hand side. □

6. Cartesian closed 2-categories

6.1. **Definition.** In a 2-category $\mathcal{C}$, a diagram $A \xleftarrow{p} C \xrightarrow{q} B$ is a *product diagram* iff for all object D, the induced functor

$$\mathcal{C}(D, C) \xrightarrow{\langle \mathcal{C}(D,p), \mathcal{C}(D,q) \rangle} \mathcal{C}(D, A) \times \mathcal{C}(D, B)$$

is an isomorphism of categories. Because this family of functors is 2-natural in D, the inverse functors will also be 2-natural.

Similarly, an object 1 of $\mathcal{C}$ is *terminal* iff for all D the unique functor

$$\mathcal{C}(D, 1) \xrightarrow{!} 1$$

is an isomorphism (where the right-hand 1 is the terminal category).

Definition 6.1. A *2-category with finite products*, or fp 2-category, is a 2-category $\mathcal{C}$, equipped with a terminal object and a 2-functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\times} \mathcal{C},$$

plus, for all A and B, a product diagram

$$A \xleftarrow{p} A \times B \xrightarrow{q} B.$$

In such an fp 2-category $\mathcal{C}$, given objects A and B, an *exponential* for them is a pair of an object B^A and a morphism $ev\colon A \times B^A \to B$, such that for all D, the functor

$$\begin{tikzcd} & \mathcal{C}(A,A)\times\mathcal{C}(D,B^A) \arrow[r, "\times"] & \mathcal{C}(A\times D, A\times B^A) \arrow[dr, "\mathcal{C}(A\times D,ev)"] & \\ \mathcal{C}(D,B^A) \arrow[ur, "(id_A!,id)"] & & & \mathcal{C}(A\times D,B) \end{tikzcd}$$

is an isomorphism. As above, because this family of functors is 2-natural in D, the inverse functors will also be 2-natural.

Definition 6.2. A *cartesian closed 2-category*, or cartesian closed 2-category, is an fp 2-category, equipped with a choice of exponentials for all pairs of objects. The category $\mathsf{2CCCat}$ has cartesian closed 2-categories as objects, and stricly structure-preserving functors between them as morphisms.

We observe in particular that this implies preservation of projections and evaluation morphisms.

7. Main adjunction

7.1. **Right adjoint.** Given a cartesian closed 2-category $\mathcal{C}$, define $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ as follows. First, let as in Section 2 $(\mathcal{C}_0, \mathcal{C}_1) = \mathcal{V}_1(\mathcal{C})$, and recall the canonical $\mathcal{L}_0$ and $\mathcal{L}_1$-algebra structures h_0 and h_1. Let then the reduction rules in $\mathcal{C}_2(G \vdash M, N{:}\, A)$ be the 2-cells in $\mathcal{C}(h_0(G), h_0(A))(h_1(M), h_1(N))$, abbreviated to $\mathcal{C}(G,A)(M,N)$ in the sequel.

This signature $\mathcal{VC}$ has a canonical $\mathcal{L}$-algebra structure $h_2{:}\, \mathcal{L}(\mathcal{VC}) \to \mathcal{VC}$, which we define by induction over reductions in Figure 2. In the case for λ, φ denotes the structure isomorphism $\mathcal{C}((\prod \Gamma) \times A, B) \cong \mathcal{C}(\prod \Gamma, B^A)$.

In order for the definition to make sense as a morphism $\mathcal{L}(\mathcal{VC}) \to \mathcal{VC}$, we have to check its compatibility with the equations. We have first:

Lemma 7.1. *For all $\Delta \vdash Q : N \to N'{:}\,\Gamma$ and $\Gamma \vdash P : M \to M'{:}\, A$ in $\mathcal{L}(\mathcal{VC})$,*

$$\begin{tikzcd} \Delta \arrow[r, bend left, "{M[N]}"] \arrow[r, bend right, "{M'[N']}"'] \arrow[r, phantom, "\Downarrow h_2(P[Q])"] & A \end{tikzcd} = \begin{tikzcd} \Delta \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow h_2(Q)"] & \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow h_2(P)"] & A. \end{tikzcd}$$

Proof. By induction on P and the axioms for cartesian closed 2-categories. □

Lemma 7.2. *Any two equated reductions are mapped to the same 2-cell in $\mathcal{C}$.*

Proof. We proceed by induction on the proof of the considered equation. The congruence and category rules of Figures 3 and 4 hold because, in $\mathcal{C}$, vertical composition is associative and unital, and equality is a congruence. The beta rule is less easy, so we spell it out.

The left-hand reduction is interpreted in $\mathcal{C}$ as

$$\begin{tikzcd} \prod \Gamma \arrow[r, bend left, "{\langle \varphi M, N\rangle}"] \arrow[r, bend right, "{\langle \varphi M', N'\rangle}"'] \arrow[r, phantom, "\Downarrow \langle \varphi P, Q\rangle"] & B^A \times A \arrow[r, "ev"] & B \end{tikzcd}$$

which is equal to

$$(G \vdash x_i : x_i \to x_i {:}\, G_i) \mapsto (id_{\pi_i} {:}\, \pi_i \to \pi_i {:} \textstyle\prod G \to G_i)$$

$$(G \vdash () : () \to () {:}\, 1) \mapsto (id_{!} {:}\, ! \to ! {:} \textstyle\prod G \to 1)$$

$$(\Gamma \vdash c(\!| P_1, \dots, P_n |\!) : c(\!| M_1, \dots, M_n |\!) \to c(\!| N_1, \dots, N_n |\!) {:}\, A) \mapsto$$

$$\begin{tikzcd}[column sep=large] \prod \Gamma \arrow[r, bend left, "\langle M_1,\dots,M_n\rangle"] \arrow[r, bend right, "\langle N_1,\dots,N_n\rangle"'] \arrow[r, phantom, "\Downarrow P"] & \prod G \arrow[r, "c"] & A \end{tikzcd} \quad (c \in \mathcal{C}_1(G,A), P = \langle P_1, \dots, P_n \rangle)$$

$$(\Gamma \vdash r\langle\!\langle P_1, \dots, P_n \rangle\!\rangle : M[M_1, \dots, M_n] \to N[N_1, \dots, N_n] {:}\, A) \mapsto$$

$$\begin{tikzcd}[column sep=large] \prod \Gamma \arrow[r, bend left, "\langle M_1,\dots,M_n\rangle"] \arrow[r, bend right, "\langle N_1,\dots,N_n\rangle"'] \arrow[r, phantom, "\Downarrow P"] & \prod G \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow r"] & A \end{tikzcd} \quad (P = \langle P_1, \dots, P_n \rangle)$$

$$(G \vdash P \,;_{M_2} Q : M_1 \to M_3 {:}\, A) \mapsto \begin{tikzcd}[column sep=huge] \prod G \arrow[r, bend left=50, "M_1"] \arrow[r] \arrow[r, bend right=50, "M_3"'] \arrow[r, phantom, shift left=3, "\Downarrow P"] \arrow[r, phantom, shift right=3, "\Downarrow Q"] & A \end{tikzcd}$$

$$(\Gamma \vdash \lambda x{:}A.P : \lambda x{:}A.M \to \lambda x{:}A.N {:}\, B^A) \mapsto \varphi(P {:}\, M \to N {:} (\textstyle\prod \Gamma) \times A \to B)$$

$$(\Gamma \vdash PQ : MN \to M'N' {:}\, B) \mapsto \begin{tikzcd}[column sep=huge] \prod \Gamma \arrow[r, bend left, "\langle M,N\rangle"] \arrow[r, bend right, "\langle M',N'\rangle"'] \arrow[r, phantom, "\Downarrow\langle P,Q\rangle"] & B^A \times A \arrow[r, "ev"] & B \end{tikzcd}$$

$$(\Gamma \vdash (P,Q) : (M,N) \to (M',N') {:}\, A \times B) \mapsto \begin{tikzcd}[column sep=huge] \prod \Gamma \arrow[r, bend left, "\langle M,N\rangle"] \arrow[r, bend right, "\langle M',N'\rangle"'] \arrow[r, phantom, "\Downarrow\langle P,Q\rangle"] & A \times B \end{tikzcd}$$

$$(\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \to \pi_{A,B} N {:}\, A) \mapsto \begin{tikzcd}[column sep=huge] \prod \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow P"] & A \times B \arrow[r, "\pi"] & A \end{tikzcd}$$

$$(\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \to \pi'_{A,B} N {:}\, B) \mapsto \begin{tikzcd}[column sep=huge] \prod \Gamma \arrow[r, bend left, "M"] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\Downarrow P"] & A \times B \arrow[r, "\pi'"] & B \end{tikzcd}$$

Figure 2: The $\mathcal{L}$-algebra structure on $\mathcal{V}(\mathcal{C})$

$$\begin{tikzcd} \prod\Gamma \arrow[r, bend left, "\langle id,N\rangle"] \arrow[r, bend right, "\langle id,N'\rangle"'] \arrow[r, phantom, "\Downarrow\langle id,Q\rangle"] & \prod\Gamma\times A \arrow[r, bend left, "\varphi M\times A"] \arrow[r, bend right, "\varphi M'\times A"'] \arrow[r, phantom, "\Downarrow\varphi P\times A"] & B^A\times A \arrow[r, "ev"] & B \end{tikzcd}$$

which is in turn equal (by cartesian closedness of $\mathcal{C}$) to:

$$\begin{tikzcd} \prod\Gamma \arrow[r, bend left, "\langle id,N\rangle"] \arrow[r, bend right, "\langle id,N'\rangle"'] \arrow[r, phantom, "\Downarrow\langle id,Q\rangle"] & \prod\Gamma\times A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow P"] & B \end{tikzcd}$$

and hence to the right-hand side of the equation by Lemma 7.1. The other beta and eta rules similarly hold by the properties of products, internal homs, and terminal object in $\mathcal{C}$.

The lifting rules hold by (particular cases of) the interchange law in $\mathcal{C}$ and functoriality of the structural isomorphisms

$$\mathcal{C}(A\times B,C)\cong\mathcal{C}(B,C^A) \qquad \text{and} \qquad \mathcal{C}(C,A\times B)\cong\mathcal{C}(C,A)\times\mathcal{C}(C,B),$$

which concludes the proof. □

This assignment extends to cartesian closed functors and we have:

Proposition 7.3. $\mathcal{V}$ *is a functor* $\mathsf{2CCCat}\to\mathsf{Sig}$.

7.2. **Left adjoint.** Given an $\mathcal{L}$-algebra $h{:}\,\mathcal{L}(X)\to X$, we now construct a cartesian closed 2-category $\mathcal{F}(X,h)$. It has:

- objects the types in $\mathcal{L}_0(X_0)$;
- 1-cells $A\to B$ the terms in $\mathcal{L}_1(X_0,X_1)(A,B)$;
- 2-cells $M\to N{:}\,A\to B$ the reduction rules in $X_2(M,N)$.

We then must define the cartesian closed 2-category structure, and we start with the 2-category structure. Composition of 1-cells $A\xrightarrow{M}B\xrightarrow{N}C$ is defined to be $A\xrightarrow{N[M]}C$. Identities are given by variables, as usual. Vertical composition of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left=50, "M_1"] \arrow[r, "M_2" description] \arrow[r, bend right=50, "M_3"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & B \end{tikzcd}$$

is given by $h(\eta(\alpha)\,;_{M_2}\eta(\beta))$.

Horizontal composition of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd} \tag{7.1}$$

is obtained as $h(\beta\langle\!\langle\eta(\alpha)\rangle\!\rangle)$.

The identity at $A\xrightarrow{M}B$ is $h(M)$.

To show that this yields a 2-category structure, the only non obvious point is the interchange law. We deal with it using the following series of results. First, consider the *left whiskering*

$$\begin{tikzcd} A \arrow[r, bend left, "M"] \arrow[r, bend right, "M'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "N"] & C \end{tikzcd}$$

of a 2-cell α by a 1-cell N, i.e., the composition $id_N \circ \alpha = h((h(N))\langle\!\langle \eta(\alpha) \rangle\!\rangle)$.

Lemma 7.4. *We have:* $h((h(N))\langle\!\langle \eta(\alpha) \rangle\!\rangle) = h(N[\eta(\alpha)])$.

Proof. Indeed, consider the term $N\langle\!\langle \eta(\eta(\alpha)) \rangle\!\rangle$ in $\mathcal{L}(\mathcal{L}(X))$. Its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h((h(N))\langle\!\langle \eta(\alpha) \rangle\!\rangle)$, i.e., $id_N \circ \alpha$, and $h(N[\eta(\alpha)])$. □

Similarly, consider the *right whiskering*

$$\begin{tikzcd} A \arrow[r, "M"] & B \arrow[r, bend left, "N"] \arrow[r, bend right, "N'"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

of a 2-cell γ by a 1-cell M, i.e., the composition $\gamma \circ id_N = h(\gamma\langle\!\langle \eta(h(M)) \rangle\!\rangle)$.

Lemma 7.5. *We have:* $h(\gamma\langle\!\langle \eta(h(M)) \rangle\!\rangle) = h(\gamma\langle\!\langle M \rangle\!\rangle)$.

Proof. Consider $(\eta\gamma)\langle\!\langle \eta M \rangle\!\rangle$ in $\mathcal{L}(\mathcal{L}(X))$. Its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h(\gamma\langle\!\langle \eta(h(M)) \rangle\!\rangle)$ and $h(\gamma\langle\!\langle M \rangle\!\rangle)$. □

Now, we prove that the two sensible ways of mimicking horizontal composition using whiskering coincide with actual horizontal composition:

Lemma 7.6. *For any cells as in* (7.1)*,*

$$(\beta \circ id_M) \,;\, (id_{N'} \circ \alpha) = \beta \circ \alpha = (id_N \circ \alpha) \,;\, (\beta \circ id_{M'}).$$

Proof. Consider first the reduction $\eta(\beta\langle\!\langle M \rangle\!\rangle) \,;\, \eta(N'[\eta(\alpha)])$ in $\mathcal{L}(\mathcal{L}(X))$. Taking $h \circ \mathcal{L}(h)$ and $h \circ \mu$ as above respectively yields

- $h(\eta(h(\beta\langle\!\langle M \rangle\!\rangle)) \,;\, \eta(h(N'[\eta(\alpha)])))$, and
- $h(\beta\langle\!\langle M \rangle\!\rangle \,;\, N'[\eta(\alpha)]) = h(\beta\langle\!\langle \eta(\alpha) \rangle\!\rangle)$,

hence the left-hand equality. Then consider $\eta(N[\eta(\alpha)]) \,;\, \eta(\beta(M'))$. Evaluating as before yields the right-hand equality. □

Furthermore, consider any configuration like:

$$\begin{tikzcd} A \arrow[r, bend left=40, "M"] \arrow[r, "M'" description] \arrow[r, bend right=40, "M''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \arrow[r, "N"] & C. \end{tikzcd}$$

Lemma 7.7. *We have* $(id_N \circ \alpha) \,;\, (id_N \circ \beta) = id_N \circ (\alpha \,;\, \beta)$.

Proof. Consider $\eta(N[\eta(\alpha)]) \,;\, \eta(N[\eta(\beta)])$. Evaluating yields equality of

- $h(\eta(h(N[\eta(\alpha)])) \,;\, \eta(h(N[\eta(\beta)])))$, i.e., the left-hand side, and
- $h(N[\eta(\alpha)] \,;\, N[\eta(\beta)])$, i.e., $h(N[\eta(\alpha) \,;\, \eta(\beta)])$ by lifting.

But now consider $N[\eta(\eta(\alpha)\,;\eta(\beta))]$. Evaluating yields equality of

- $h(N[\eta(\alpha)\,;\eta(\beta)])$, as above, and
- $h(N[\eta(h(\eta(\alpha)\,;\eta(\beta)))])$, i.e., $h(N[\eta(\alpha\,;\beta)])$ (where $\alpha\,;\beta$ denotes vertical composition in our candidate 2-category), i.e., the right-hand side. ☐

Finally, by a similar argument, we have:

Lemma 7.8. *For any*

$$\begin{tikzcd} A \arrow[r, "M"] & B \arrow[r, bend left=40, "N"{name=U}] \arrow[r, "N'"{name=Mid, description}] \arrow[r, bend right=40, "N''"'{name=D}] & C, \arrow[from=U, to=Mid, phantom, "\Downarrow\alpha"] \arrow[from=Mid, to=D, phantom, "\Downarrow\beta"] \end{tikzcd}$$

we have $(\alpha \circ id_M)\,;(\beta \circ id_M) = (\alpha\,;\beta)\circ id_M$.

Lemma 7.9. *The interchange law holds, i.e., for all reduction rules as in*

$$\begin{tikzcd} A \arrow[r, bend left=40, "M_1"{name=U1}] \arrow[r, "M_2"{name=M1, description}] \arrow[r, bend right=40, "M_3"'{name=D1}] & B \arrow[r, bend left=40, "N_1"{name=U2}] \arrow[r, "N_2"{name=M2, description}] \arrow[r, bend right=40, "N_3"'{name=D2}] & C, \arrow[from=U1, to=M1, phantom, "\Downarrow\alpha"] \arrow[from=M1, to=D1, phantom, "\Downarrow\beta"] \arrow[from=U2, to=M2, phantom, "\Downarrow\gamma"] \arrow[from=M2, to=D2, phantom, "\Downarrow\theta"] \end{tikzcd}$$

we have

$$(\gamma\,;\theta)\circ(\alpha\,;\beta) = (\gamma\circ\alpha)\,;(\theta\circ\beta).$$

Proof. By the previous results, we have

$$\begin{array}{l} (\gamma\,;\theta)\circ(\alpha\,;\beta) \\ = ((\gamma\,;\theta)\circ M_1)\,;(N_3\circ(\alpha\,;\beta)) \\ = (\gamma\circ M_1)\,;(\theta\circ M_1)\,;(N_3\circ\alpha)\,;(N_3\circ\beta) \\ = (\gamma\circ M_1)\,;(N_2\circ\alpha)\,;(\theta\circ M_2)\,;(N_3\circ\beta) \\ = (\gamma\circ\alpha)\,;(\theta\circ\beta). \end{array}$$

☐

Now, let us show cartesian closedness. We have a bijection of hom-sets $\mathcal{L}_1(X)(C \vdash A\times B) \cong \mathcal{L}_1(X)(C\vdash A)\times\mathcal{L}_1(X)(C\vdash B)$, given by

$$\begin{array}{rcl} \mathcal{L}_1(X)(C\vdash A\times B) & \to & \mathcal{L}_1(X)(C\vdash A)\times\mathcal{L}_1(X)(C\vdash B) \\ M & \mapsto & \pi M, \pi' M \end{array}$$

and

$$\begin{array}{rcl} \mathcal{L}_1(X)(C\vdash A)\times\mathcal{L}_1(X)(C\vdash B) & \to & \mathcal{L}_1(X)(C\vdash A\times B) \\ M, N & \mapsto & (M, N). \end{array}$$

These are mutually inverse thanks to the beta and eta rules for products in the simply-typed λ-calculus.

On 2-hom-sets, we have

$$\begin{array}{rcl} \mathcal{L}(X)(C\vdash M, N\colon A\times B) & \to & \mathcal{L}(X)(C\vdash \pi M, \pi N\colon A)\times\mathcal{L}(X)(C\vdash \pi' M, \pi' N\colon B) \\ P & \mapsto & \pi P, \pi' P \end{array}$$

and (omitting C)

$$\begin{array}{rcl} \mathcal{L}(X)(M_1, N_1\colon A)\times\mathcal{L}(X)(M_2, N_2\colon B) & \to & \mathcal{L}(X)((M_1, M_2), (N_1, N_2)\colon A\times B) \\ P_1, P_2 & \mapsto & (P_1, P_2), \end{array}$$

which are mutually inverse thanks to the beta and eta rules for products in Figure 4. We use these to define the desired isomorphism (u, v)

$$X_2(C \vdash M, N{:}\, A \times B) \cong X_2(C \vdash \pi M, \pi N{:}\, A) \times X_2(C \vdash \pi' M, \pi' N{:}\, B),$$

as in the diagrams

$$\begin{tikzcd} X_2(M,N) \arrow[rr, "u"] \arrow[d, "\eta"'] & & X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) \\ \mathcal{L}(X)(M,N) \arrow[r, "\cong"'] & \mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N) \arrow[ur, "h \times h"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) \arrow[r, "v"] \arrow[d, "\eta \times \eta"'] & X_2(M,N) \\ \mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N) \arrow[r, "\cong"'] & \mathcal{L}(X)(M,N). \arrow[u, "h"'] \end{tikzcd}$$

Starting from $r \in X_2(M, N)$, we obtain

$$v(u(r)) = h(\eta(h(\pi(\eta(r)))), \eta(h(\pi'(\eta(r))))).$$

But consider $(\eta(\pi\eta(r)), \eta(\pi'\eta(r)))$ in $\mathcal{L}(\mathcal{L}X)$; its images by $h \circ \mathcal{L}h$ and $h \circ \mu$ are respectively:

- $h(\eta(h(\pi(\eta r))), \eta(h(\pi'(\eta r))))$, and
- $h(\pi\eta(r), \pi'\eta(r))$, i.e., $h(\eta(r))$, i.e., r,

which must be equal because h is an $\mathcal{L}$-algebra, hence $v \circ u = id$.

Conversely, starting from $(r, s) \in X_2(M_1, M_2) \times X_2(N_1, N_2)$, we obtain the pair with components

$$h(\pi(\eta(h(\eta(r), \eta(s))))) \qquad \text{and} \qquad h(\pi'(\eta(h(\eta(r), \eta(s))))).$$

Considering $\pi(\eta(\eta(r), \eta(s))) \in \mathcal{L}(\mathcal{L}(X))$, its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ are respectively:

- $h(\pi(\eta(h(\eta(r), \eta(s)))))$, and
- $h(\pi(\eta(r), \eta(s))) = h(\eta(r)) = r$.

As above, they must be equal, and by symmetry the second component is s, and we have proved $u \circ v = id$. Similar reasoning for the terminal object and internal homs leads to:

Proposition 7.10. *This yields a cartesian closed 2-category structure on $\mathcal{C}$.*

This extends to morphisms of $\mathcal{L}$-algebras, so we have constructed a functor $\mathcal{F}{:}\, \mathcal{L}\text{-Alg} \to \mathsf{2CCCat}$.

7.3. **Adjunction.** Consider any $\mathcal{L}$-algebra (X, h). What does $(Y, k) = \mathcal{V}(\mathcal{F}(X, h))$ look like? Sorts in Y_0 are types in $\mathcal{L}_0(X_0)$. Operations $Y_1(G \vdash A)$ are terms in $\mathcal{L}_1(X_0, X_1)(\mu(\prod G) \vdash \mu(A))$, where μ denotes the monad multiplication for $\mathcal{L}_0$. Reduction rules in $Y_2(G \vdash M, N{:}\, B)$ are reduction rules in $X_2(\mu(\prod G) \vdash M', N'{:}\, \mu(B))$, where $M' = M[\pi_1 x/x_1, \ldots, \pi_n x/x_n]$ (and similarly for N').

Let $\eta^{\mathcal{L}}_X$ map:

- each sort $\iota \in X_0$ to the type $\iota \in \mathcal{L}_0(X_0)$ (this is the monad unit for $\mathcal{L}_0$),
- each operation $c \in X(G \vdash A)$ to the term $c(\!|\pi_1 x, \ldots, \pi_n x|\!)$, and

- each reduction rule $r \in X_2(G \vdash M, N{:}\,A)$ to the reduction rule
$$h(r\langle\!\langle \pi_1 x, \ldots, \pi_n x\rangle\!\rangle) \in X_2(\textstyle\prod G \vdash M', N'{:}\,A).$$
(Thanks to the fact that $\mu(\mathcal{L}_0(\eta)(A)) = A$.)

Theorem 7.11. *This $\eta^{\mathcal{L}}$ is a natural transformation which is the unit of an adjunction*

$$\begin{tikzcd} \mathcal{L}\text{-Alg} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathsf{2CCCat}. \arrow[l, bend left, "\mathcal{V}"] \end{tikzcd}$$

Proof. Consider any morphism $f{:}\,(X,h) \to \mathcal{V}(\mathcal{C})$, and let $\mathcal{X} = \mathcal{F}(X,h)$, $(Y,k) = \mathcal{V}(\mathcal{X})$, and $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, h_2{:}\,\mathcal{C}_2 \to \mathcal{C}_1)$. We now define a uniquely determined cartesian closed functor $f'{:}\,\mathcal{X} \to \mathcal{C}$ making the triangle

$$\begin{tikzcd} X \arrow[r, "\eta^{\mathcal{L}}_X"] \arrow[dr, "f"'] & \mathcal{V}(\mathcal{X}) \arrow[d, "\mathcal{V}(f')"] \\ & \mathcal{V}(\mathcal{C}) \end{tikzcd}$$

commute.

On objects, it is determined by induction: on sorts by f_0, and on type constructors by the requirement that f' be cartesian closed. On morphisms, it is similarly determined by f_1 and f' being cartesian closed. On 2-cells, define f' to be $f_2{:}\,X_2(A \vdash M, N{:}\,B) \to \mathcal{C}(f'(A), f'(B))(f'(M), f'(N))$, which is also the only possible choice from f.

This indeed makes the above triangle commute, because any $r \in X_2(G \vdash M, N{:}\,A)$ is first mapped to $h(r\langle\!\langle \pi_1 x, \ldots, \pi_n x\rangle\!\rangle) \in X_2(\prod G \vdash M', N'{:}\,A)$, and then to
$$h(r\langle\!\langle \pi_1 x, \ldots, \pi_n x\rangle\!\rangle)$$
in $\mathcal{C}$, which, because f is a morphism of $\mathcal{L}$-algebras, is equal to
$$h_2((f_2(r))\langle\!\langle \pi_1 x, \ldots, \pi_n x\rangle\!\rangle),$$
i.e., to $f_2(r)$.

It thus remains to show that f' is cartesian closed, which follows by f being a morphism of $\mathcal{L}$-algebras. For example, to show that binary pairings of reductions are preserved, consider $r \in X_2(C \vdash M_1, M_2{:}\,A)$ and $s \in X_2(C \vdash N_1, N_2{:}\,B)$. Their product in $\mathcal{F}(X)$ is obtained by considering the atomic reductions $x{:}\,C \vdash r\langle\!\langle x\rangle\!\rangle : M_1 \to M_2{:}\,A$ and $x{:}\,C \vdash s\langle\!\langle x\rangle\!\rangle : N_1 \to N_2{:}\,B$ and taking $h(r\langle\!\langle x\rangle\!\rangle, s\langle\!\langle x\rangle\!\rangle)$, which is mapped by f_2 to $f_2(h(r\langle\!\langle x\rangle\!\rangle, s\langle\!\langle x\rangle\!\rangle))$. But, because f is a morphism of $\mathcal{L}$-algebras, this is the same as $h_2((f_2(r))\langle\!\langle x\rangle\!\rangle, (f_2(s))\langle\!\langle x\rangle\!\rangle)$, which is by definition (i.e., Figure 2) the pairing $(f_2(r), f_2(s))$ in $\mathcal{C}$. $\square$

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Appendix A. Equations on reductions

Congruence

$$\frac{\Gamma \vdash P : M \to N{:}\, A}{\Gamma \vdash P \equiv P : M \to N{:}\, A} \qquad \frac{\Gamma \vdash P \equiv Q : M \to N{:}\, A}{\Gamma \vdash Q \equiv P : M \to N{:}\, A}$$

$$\frac{\Gamma \vdash P_1 \equiv P_2 : M \to N{:}\, A \qquad \Gamma \vdash P_2 \equiv P_3 : M \to N{:}\, A}{\Gamma \vdash P_1 \equiv P_3 : M \to N{:}\, A}$$

$$\frac{\Gamma \vdash P \equiv P' : M_1 \to M_2{:}\, A \qquad \Gamma \vdash Q \equiv Q' : M_2 \to M_3{:}\, A}{\Gamma \vdash (P \,;_{M_2} Q) \equiv (P' \,;_{M_2} Q') : M_1 \to M_3{:}\, A}$$

$$\frac{\begin{array}{c}(r \in X(G \vdash M, N{:}\, A)) \\ \Gamma \vdash P_1 \equiv Q_1 : M_1 \to N_1{:}\, G_1 \qquad \ldots \qquad \Gamma \vdash P_n \equiv Q_n : M_n \to N_n{:}\, G_n\end{array}}{\Gamma \vdash r\langle\!\langle P_1, \ldots, P_n \rangle\!\rangle \equiv r\langle\!\langle Q_1, \ldots, Q_n \rangle\!\rangle : M[M_1, \ldots, M_n] \to N[N_1, \ldots, N_n]{:}\, A}$$

$$\frac{\begin{array}{c}(c \in X(G \vdash A)) \\ \Gamma \vdash P_1 \equiv Q_1 : M_1 \to N_1{:}\, G_1 \qquad \ldots \qquad \Gamma \vdash P_n \equiv Q_n : M_n \to N_n{:}\, G_n\end{array}}{\Gamma \vdash c(\!| P_1, \ldots, P_n |\!) \equiv c(\!| Q_1, \ldots, Q_n |\!) : c(\!| M_1, \ldots, M_n |\!) \to c(\!| N_1, \ldots, N_n |\!){:}\, A}$$

$$\frac{\Gamma, x{:}\, A \vdash P \equiv Q : M \to N{:}\, B}{\Gamma \vdash (\lambda x{:}\, A.P) \equiv (\lambda x{:}\, A.Q) : \lambda x{:}\, A.M \to \lambda x{:}\, A.N{:}\, B^A}$$

$$\frac{\Gamma \vdash P \equiv P' : M \to M'{:}\, B^A \qquad \Gamma \vdash Q \equiv Q' : N \to N'{:}\, A}{\Gamma \vdash (PQ) \equiv (P'Q') : MN \to M'N'{:}\, B}$$

$$\frac{\Gamma \vdash P \equiv P' : M \to M'{:}\, A \qquad \Gamma \vdash Q \equiv Q' : N \to N'{:}\, B}{\Gamma \vdash (P, Q) \equiv (P', Q') : (M, N) \to (M', N'){:}\, A \times B}$$

$$\frac{\Gamma \vdash P \equiv Q : M \to N{:}\, A \times B}{\Gamma \vdash (\pi_{A,B} P) \equiv (\pi_{A,B} Q) : \pi_{A,B} M \to \pi_{A,B} N{:}\, A}$$

$$\frac{\Gamma \vdash P \equiv Q : M \to N{:}\, A \times B}{\Gamma \vdash (\pi'_{A,B} P) \equiv (\pi'_{A,B} Q) : \pi'_{A,B} M \to \pi'_{A,B} N{:}\, A}$$

Figure 3: Equations on reductions (Congruence)

Category

$$\frac{\Gamma \vdash P_1 : M_1 \to M_2 {:}\, A \qquad \Gamma \vdash P_2 : M_2 \to M_3 {:}\, A \qquad \Gamma \vdash P_3 : M_3 \to M_4 {:}\, A}{\Gamma \vdash (P_1 \,;_{M_2} (P_2 \,;_{M_3} P_3)) \equiv ((P_1 \,;_{M_2} P_2) \,;_{M_3} P_3) : M_1 \to M_4 {:}\, A}$$

$$\frac{\Gamma \vdash P : M \to N {:}\, A}{\Gamma \vdash (P \,;_N N) \equiv P : M \to N {:}\, A} \qquad \frac{\Gamma \vdash P : M \to N {:}\, A}{\Gamma \vdash (M \,;_M P) \equiv P : M \to N {:}\, A}$$

Beta and Eta

$$\frac{\Gamma, x{:}\, A \vdash P : M \to M' {:}\, B \qquad \Gamma \vdash Q : N \to N' {:}\, A}{\Gamma \vdash ((\lambda x{:}\, A.P)Q) \equiv P[Q/x] : (\lambda x{:}\, A.M)N \to M'[N'/x] {:}\, B}$$

$$\frac{\Gamma \vdash P : M \to N {:}\, B^A}{\Gamma \vdash P \equiv \lambda x{:}\, A.(Px) : M \to N {:}\, B^A}\ (x \notin \Gamma) \qquad \frac{\Gamma \vdash P : M_1 \to M_2 {:}\, A \qquad \Gamma \vdash Q : N_1 \to N_2 {:}\, B}{\Gamma \vdash \pi(P, Q) \equiv P : \pi(M_1, N_1) \to M_2 {:}\, A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2 {:}\, A \qquad \Gamma \vdash Q : N_1 \to N_2 {:}\, B}{\Gamma \vdash \pi'(P, Q) \equiv Q : \pi'(M_1, N_1) \to N_2 {:}\, A}$$

$$\frac{\Gamma \vdash P : (M_1, N_1) \to (M_2, N_2) {:}\, A \times B}{\Gamma \vdash P \equiv (\pi P, \pi' P) : (M_1, N_1) \to (M_2, N_2) {:}\, A \times B} \qquad \frac{\Gamma \vdash P : M \to N {:}\, 1}{\Gamma \vdash P \equiv () : M \to N {:}\, 1}$$

Figure 4: Equations on reductions (Category and Beta-Eta)

Lifting

$$\frac{(r \in X(\Gamma \vdash M_1, M_2{:}\, A)) \qquad \Delta \vdash P : N_1 \to N_2{:}\,\Gamma \qquad \Delta \vdash Q : N_2 \to N_3{:}\,\Gamma}{\Gamma \vdash r\langle\!\langle P \,;_{N_2} Q \rangle\!\rangle \equiv M_1[P] \,;_{M_1[N_2]} r\langle\!\langle Q \rangle\!\rangle : M_1[N_1] \to M_2[N_3]{:}\, A}$$

$$\frac{(r \in X(\Gamma \vdash M_1, M_2{:}\, A)) \qquad \Delta \vdash P : N_1 \to N_2{:}\,\Gamma \qquad \Delta \vdash Q : N_2 \to N_3{:}\,\Gamma}{\Gamma \vdash r\langle\!\langle P \,;_{N_2} Q \rangle\!\rangle \equiv r\langle\!\langle P \rangle\!\rangle \,;_{M_2[N_2]} M_2[Q] : M_1[N_1] \to M_2[N_3]{:}\, A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2{:}\, G \qquad \Gamma \vdash Q : M_2 \to M_3{:}\, G}{\Gamma \vdash (c(P \,;_{M_2} Q)) \equiv (c(P) \,;_{c(M_2)} c(Q)) : M_1 \to M_3{:}\, A}\ (c \in X(G \vdash A))$$

$$\frac{\Gamma, x{:}\, A \vdash P : M_1 \to M_2{:}\, B \qquad \Gamma, x{:}\, A \vdash Q : M_2 \to M_3{:}\, B}{\begin{array}{l}\Gamma \vdash (\lambda x{:}\, A.(P \,;_{M_2} Q)) \equiv ((\lambda x{:}\, A.P) \,;_{\lambda x{:}A.M_2} (\lambda x{:}\, A.Q)) \\ \hfill : \lambda x{:}\, A.M_1 \to \lambda x{:}\, A.M_3{:}\, B^A\end{array}}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : M_1 \to M_2{:}\, B^A \\ \Gamma \vdash P' : M_2 \to M_3{:}\, B^A \qquad \Gamma \vdash Q : N_1 \to N_2{:}\, A \qquad \Gamma \vdash Q' : N_2 \to N_3{:}\, A\end{array}}{\Gamma \vdash ((P \,;_{M_2} P')(Q \,;_{N_2} Q')) \equiv ((PQ) \,;_{M_2 N_2} (P'Q')) : M_1 N_1 \to M_3 N_3{:}\, B}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : M_1 \to M_2{:}\, A \\ \Gamma \vdash P' : M_2 \to M_3{:}\, A \qquad \Gamma \vdash Q : N_1 \to N_2{:}\, B \qquad \Gamma \vdash Q' : N_2 \to N_3{:}\, B\end{array}}{\begin{array}{l}\Gamma \vdash ((P \,;_{M_2} P'), (Q \,;_{N_2} Q')) \equiv ((P, Q) \,;_{(M_2, N_2)} (P', Q')) \\ \hfill : (M_1, N_1) \to (M_3, N_3){:}\, A \times B\end{array}}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2{:}\, A \times B \qquad \Gamma \vdash Q : M_2 \to M_3{:}\, A \times B}{\Gamma \vdash (\pi_{A,B}(P \,;_{M_2} Q)) \equiv (\pi_{A,B} P \,;_{\pi_{A,B} M_2} \pi_{A,B} Q) : M_1 \to M_3{:}\, A}$$

$$\frac{\Gamma \vdash P : M_1 \to M_2{:}\, A \times B \qquad \Gamma \vdash Q : M_2 \to M_3{:}\, A \times B}{\Gamma \vdash (\pi'_{A,B}(P \,;_{M_2} Q)) \equiv (\pi'_{A,B} P \,;_{\pi'_{A,B} M_2} \pi'_{A,B} Q) : M_1 \to M_3{:}\, B}$$

Figure 5: Equations on reductions (Lifting)