
CONSISTENCY AND COMPLETENESS OF REWRITING IN THE CALCULUS OF CONSTRUCTIONS*

DARIA WALUKIEWICZ-CHRZĄSZCZ AND JACEK CHRZĄSZCZ

Institute of Informatics, Warsaw University, ul. Banacha 2, 02-097 Warsaw, Poland
e-mail address: {daria,chrzaszcz}@mimuw.edu.pl

ABSTRACT. Adding rewriting to a proof assistant based on the Curry-Howard isomorphism, such as Coq, may greatly improve usability of the tool. Unfortunately adding an arbitrary set of rewrite rules may render the underlying formal system undecidable and inconsistent. While ways to ensure termination and confluence, and hence decidability of type-checking, have already been studied to some extent, logical consistency has got little attention so far.

In this paper we show that consistency is a consequence of canonicity, which in turn follows from the assumption that all functions defined by rewrite rules are complete. We provide a sound and terminating, but necessarily incomplete algorithm to verify this property. The algorithm accepts all definitions that follow dependent pattern matching schemes presented by Coquand and studied by McBride in his PhD thesis. It also accepts many definitions by rewriting including rules which depart from standard pattern matching.

1. INTRODUCTION

Equality is ubiquitous in mathematics. Yet it turns out that proof assistants based on the Curry-Howard isomorphism, such as Coq [11], are not very good at handling equality. While proving an equality is not a problem in itself, using already established equalities is quite problematic. Apart from equalities resulting from internal reductions (namely, beta and iota reductions), which can be used via the conversion rule of the calculus of inductive constructions without being recorded in the proof term, any other use of an equality requires giving all details about the context explicitly in the proof. As a result, proof terms may become extremely large, taking up memory and making type-checking time consuming: working with equations in Coq is not very convenient.

A straightforward idea for reducing the size of proof terms is to allow other equalities in the conversion, making their use transparent. This can be done by using user-defined *rewrite rules*. However, adding arbitrary rules may easily lead to logical inconsistency, making the

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proof environment useless. It is of course possible to put the responsibility on the user, but it is contrary to the current Coq policy to guarantee consistency of developments without axioms. Therefore it is desirable to retain this guarantee when rewriting is added to Coq. Since consistency is undecidable in the presence of rewriting in general, one has to find some decidable criteria satisfied only by rewriting systems which do not violate consistency.

The syntactical proof of consistency of the calculus of constructions, which is the basis of the formalism implemented in Coq, requires every term to have a normal form [2]. The same proof is also valid for the calculus of inductive constructions [24], which is even closer to the formalism implemented in Coq.

There exist several techniques to prove (strong) normalization of the calculus of constructions with rewriting [1, 7, 6, 21, 22], following numerous works about rewriting in the simply-typed lambda calculus. Practical criteria for ensuring other fundamental properties, like confluence, subject reduction and decidability of type-checking are addressed e.g. in [6].

Logical consistency is also studied in [6]. It is shown under the assumption that for every symbol f defined by rewriting, $f(t_1, \dots, t_n)$ is reducible if $t_1 \dots t_n$ are terms in normal form in the environment consisting of one type variable. Apart from a proof sketch that this is the case for the two rules defining the induction predicate for natural numbers and a remark that this property resembles the completeness of definitions, practical ways to satisfy the assumption of the consistency lemma are not discussed.

Techniques for checking completeness of definitions are known for almost 30 years for the first-order algebraic setting [14, 20, 15]. More recently, their adaptations to type theory appeared in [12, 16] and [18]. In this paper we show how the latter algorithm can be tailored to the calculus of constructions extended with rewriting. We study a system where the set of available function symbols and rewrite rules are not known from the beginning but may grow as the proof development advances, as it is the case with concrete implementations of modern proof assistants.

We show that logical consistency is an easy consequence of canonicity, which in turn can be proved from completeness of definitions by rewriting, provided that termination and confluence are proved first. Our completeness checking algorithm closes the list of necessary procedures needed to guarantee logical consistency of developments in a proof assistant based on the calculus of constructions with rewriting.

In fact, in this paper we work in a framework which is slightly more general than the calculus of constructions, namely that of pure type systems, of which the calculus of constructions is an instance. However, since termination and confluence are used both in our algorithm and in the proof of its correctness, our results are useful only if a termination and confluence criteria exist for a given pure type system extended with rewriting. Some work in this direction has been done, e.g., in [4].

2. REWRITING IN THE CALCULUS OF CONSTRUCTIONS

Let us briefly discuss how we imagine introducing rewriting in Coq and what problems we encounter on the way to a usable system.

From the user's perspective definitions by rewriting could be entered just as all other definitions:¹

¹The syntax of the definition by rewriting is inspired by the experimental "reécriture" branch of Coq developed by Blanqui. For the sake of clarity we omit certain details, like environments of rule variables and allow the infix $+$ in the definition.

```
Inductive nat : Set := 0 : nat | S : nat → nat.
```

```
Symbol + : nat → nat → nat
```

```
Rules
```

```
  0 + y → y
  x + 0 → x
  (S x) + y → S (x + y)
  x + (S y) → S (x + y)
  x + (y + z) → (x + y) + z.
```

```
Parameter n : nat.
```

The above fragment can be interpreted as an environment consisting of the inductive definition of natural numbers, symmetric definition by rewriting of addition and the declaration of a variable n of type nat . In this environment all rules for $+$ contribute to conversion. For instance both $\forall x: nat. x + 0 = x$ and $\forall x: nat. 0 + x = x$ can be proved by $\lambda x: nat. refl\ nat\ x$, where $refl$ is the only constructor of the Leibniz equality inductive predicate. Note that the definition of $+$ is terminating and confluent. The latter can be checked by an (automatic) examination of its critical pairs.

Rewrite rules can also be used to define higher-order and polymorphic functions, like the `map` function on polymorphic lists. In this example, the first two rules correspond to the usual definition of `map` by pattern matching and structural recursion and the third rule can be used to quickly get rid of the `map` function in case one knows that `f` is the identity function.

```
Symbol map : forall (A:Set), (A → A) → list A → list A
```

```
Rules
```

```
  map A f (nil A) → nil A
  map A f (cons A a l) → cons A (f a) (map A f l)
  map A (fun x ⇒ x) l → l
```

Even though we consider higher-order rewriting, we choose the simple matching modulo α -conversion. Higher-order matching is useful for example to encode logical languages by higher-order abstract syntax, but it is seldom used in Coq where modeling relies rather on inductive types. Instead of higher-order matching, one needs a possibility not to specify certain arguments in left-hand sides, and hence to work with rewrite rules built from terms that may be not typable. Consider, for example the type `tree` of trees with size, holding some Boolean values in the nodes, and the function `rotr` performing a right rotation in the root of the tree.

```
Inductive tree : nat → Set :=
```

```
  Leaf : tree 0
```

```
  | Node : forall n1:nat, tree n1 → bool → forall n2:nat, tree n2
    → tree (S(n1+n2)).
```

```
Symbol rotr : forall n:nat, (tree n) → (tree n)
```

```
Rules
```

```
  rotr 0 t → t
  rotr ? (Node 0 t1 a n2 t2) → Node 0 t1 a n2 t2
  rotr ?1 (Node ?2 (Node ?3 A b ?4 C) d ?5 E)
    → Node ?3 A b (S (?4 + ?5))(Node ?4 C d ?5 E)
```

The first argument of `rotr` is the size of the tree and the second is the tree itself. The first two rules cover the trees which cannot be rotated and the third one performs the rotation.

The `?` marks above should be treated as different variables. The information they hide is redundant for typable terms: if we take the third rule for example, the values of `?3`, `?4` and `?5` must correspond to the sizes of the trees `A`, `C` and `E` respectively, `?2` must be equal to `S(?3+?4)` and `?1` to `S(?2+?5)`. Note that by not writing these subterms we make the rule left-linear (and therefore easier to match) and avoid critical pairs with `+`, hereby helping the confluence proof.

This way of writing left-hand sides of rules was already used by Werner in [24] to define elimination rules for inductive types, making them orthogonal (the left-hand sides are of the form $I_{elim} P \vec{f} \vec{w} (c \vec{x})$, where P , \vec{f} , \vec{w} , \vec{x} are distinct variables and c is a constructor of I). In [6], Blanqui gives a precise account of these omissions using them to make more rewriting rules left-linear. Later, the authors of [8] show that these redundant subterms can be completely removed from terms (in a calculus without rewriting however). In [3], a new optimized convertibility test algorithm is presented for Coq, which ignores testing equality of these redundant arguments.

In our paper we do not explicitly specify which arguments should/could be replaced by `?` and do not restrict left-hand sides to be left-linear. Instead, we rely on an acceptance condition to suitably restrict the form of acceptable definitions by rewriting to guarantee the needed metatheoretical properties listed in the next section.

It is also interesting to note that when the first argument of `rotr` is `?1` then we may understand it as `S(?2+?5)` matched to terms modulo the convertibility relation and not just syntactically (i.e., modulo α -conversion).

3. PURE TYPE SYSTEMS WITH GENERATIVE DEFINITIONS

Even though most papers motivated by the development of Coq concentrate on the calculus of constructions, we present here a slightly more general formalization of a pure type system with inductive definitions and definitions by rewriting. The presentation, taken from [9, 10], is quite close to the way these elements could be implemented in Coq. The formalism is built upon a set of PTS sorts \mathcal{S} , a binary relation \mathcal{A} and a ternary relation \mathcal{R} over \mathcal{S} governing the typing rules (**Term/Ax**) and (**Term/Prod**) respectively (Figure 1). The syntactic class of pseudoterms is defined as follows:

$$t ::= v \mid s \mid t_1 t_2 \mid \lambda v:t_1.t_2 \mid (v:t_1)t_2$$

A pseudoterm can be a variable $v \in Var$, a sort $s \in \mathcal{S}$, an application, an abstraction or a product. We write $|t|$ to denote the size of the pseudoterm t , with $|v| = |s| = 1$. We use Greek letters γ, δ to denote substitutions which are finite partial maps from variables to pseudoterms. The postfix notation is used for the application of substitutions to pseudoterms.

Inductive definitions and definitions by rewriting are *generative*, i.e. they are stored in the environment and are used in terms only through names they “generate”. An environment is a sequence of declarations, each of them is a variable declaration $v : t$, an inductive definition $\text{Ind}(\Gamma^I := \Gamma^C)$, where Γ^I and Γ^C are environments providing names and types of (possibly mutually defined) inductive types and their constructors, or a definition by rewriting $\text{Rew}(\Gamma, R)$, where Γ is an environment providing names and types of (possibly mutually defined) function symbols and R is a set of rewrite rules defining them. Types of

inductive types, constructors and function symbols determine their arity: given $v : t$ in an inductive definition or a definition by rewriting, if t is of the form $(x_1 : t_1) \dots (x_n : t_n) \hat{t}$ where \hat{t} is not a product, then n is the arity of v .

A rewrite rule is a triple denoted by $\Delta \vdash l \longrightarrow r$, where l and r are pseudoterms and Δ is an environment, providing names and types of variables occurring in the left- and right-hand sides l and r .

Given an environment E , inductive types, constructors and function symbols declared in E are called constants (even though syntactically they are variables). We often write $h(e_1, \dots, e_n)$ to denote the application of a constant h to pseudoterms e_1, \dots, e_n , when n is the arity of h . General environments are denoted by E and environments containing only variable declarations are denoted by Γ, Δ, G, D . We assume that names of all declarations in environments are pairwise disjoint. A pair consisting of an environment E and a term e is called a sequent and denoted by $E \vdash e$. A sequent is well-typed if $E \vdash e : t$ for some t .

Definition 3.1. A *pure type system with generative definitions* is defined by the typing rules in Figure 1, where:

- POS is a positivity condition for inductive definitions (see assumptions below).
- ACC is an acceptance condition for definitions by rewriting (*idem*).
- The relation \approx used in the rule (**Term/Conv**) is the smallest congruence on well typed terms, generated by \longrightarrow which is the sum of beta and rewrite reductions, denoted by \longrightarrow_β and \longrightarrow_R respectively (for exact definition see [10], Section 2.8).
- The notation $\delta : \Gamma \rightarrow E$ means that δ is a *well-typed substitution*, i.e. $E \vdash v\delta : t\delta$ for all $v : t \in \Gamma$.

As in [22, 6], recursors and their reduction rules have no special status and they are supposed to be expressed by rewriting.

Assumptions. We assume that we are given a positivity condition POS for inductive definitions and an acceptance condition ACC for definitions by rewriting. Together with the right choice of the PTS they must imply the following properties:

- P1** subject reduction, i.e. $E \vdash e : t, E \vdash e \longrightarrow e'$ implies $E \vdash e' : t$
- P2** uniqueness of types, i.e. $E \vdash e : t, E \vdash e : t'$ implies $E \vdash t \approx t'$.
- P3** strong normalization, i.e. $E \vdash \text{ok}$ implies that reductions of all well-typed terms in E are finite
- P4** confluence, i.e. $E \vdash e : t, E \vdash e \longrightarrow^* e', E \vdash e \longrightarrow^* e''$ implies $E \vdash e' \longrightarrow^* \hat{e}$ and $E \vdash e'' \longrightarrow^* \hat{e}$ for some \hat{e} .

These properties are usually true in all well-behaved type theories. They are for example all proved for the calculus of algebraic constructions [6], an extension of the calculus of constructions with inductive types and rewriting, where POS is the strict positivity condition as defined in [17], and ACC is the General Schema.

From now on, we use the notation $t\downarrow$ for the unique normal form of t .

4. CONSISTENCY AND COMPLETENESS

Consistency of the calculus of constructions (resp. calculus of inductive constructions) can be shown by rejecting all cases of a hypothetical normalized proof e of $(x : *)x$ in a *closed* environment, i.e. empty environment (resp. an environment containing only inductive definitions and no axioms). Our goal is to extend the definition of closed environments to the

Let $\Gamma^I = I_1 : t_1^I \dots I_n : t_n^I$ and $\Gamma^C = c_1 : t_1^C \dots c_m : t_m^C$ $\frac{E \vdash t_j^I : s_j \quad t_j^I = \overrightarrow{(z : Z_j)} s'_j \quad \text{for } j = 1 \dots n}{E; \Gamma^I \vdash t_i^C : \hat{s}_i \quad t_i^C = \overrightarrow{(z : Z'_i)} I_{j_i} \vec{w} \quad \text{for } i = 1 \dots m} \text{ if } \text{POS}_E(\Gamma^I := \Gamma^C)$ $\frac{}{E \vdash \text{Ind}(\Gamma^I := \Gamma^C) : \text{correct}}$		
Let $\Gamma = f_1 : t_1 \dots f_n : t_n$ and $R = \{\Gamma_i : l_i \longrightarrow r_i\}_{i=1 \dots m}$, where $\Gamma_i = x_1^i : t_1^i; \dots; x_{n_i}^i : t_{n_i}^i$ $\frac{E \vdash t_k : s_k \quad \text{for } k = 1 \dots n}{E; \Gamma_i \vdash \text{ok} \quad FV(l_i, r_i) \subseteq \Gamma_i \quad \text{for } i = 1 \dots m} \text{ if } \text{ACC}_E(\Gamma, R)$ $\frac{}{E \vdash \text{Rew}(\Gamma, R) : \text{correct}}$		
$\frac{}{\epsilon \vdash \text{ok}}$	$\frac{E \vdash \text{ok} \quad E \vdash t : s}{E; v : t \vdash \text{ok}}$	
$\frac{E \vdash \text{ok} \quad E \vdash \text{Ind}(\Gamma^I := \Gamma^C) : \text{correct}}{E; \text{Ind}(\Gamma^I := \Gamma^C) \vdash \text{ok}}$	$\frac{E \vdash \text{ok} \quad E \vdash \text{Rew}(\Gamma, R) : \text{correct}}{E; \text{Rew}(\Gamma, R) \vdash \text{ok}}$	
$\frac{E_1; v : t; E_2 \vdash \text{ok}}{E_1; v : t; E_2 \vdash v : t}$		
$\frac{E \vdash \text{ok}}{E \vdash I_i : t_i^I}$	$\frac{E \vdash \text{ok}}{E \vdash c_i : t_i^C}$	where $\begin{cases} E = E_1; \text{Ind}(\Gamma^I := \Gamma^C); E_2 \\ \Gamma^I = I_1 : t_1^I \dots I_n : t_n^I \\ \Gamma^C = c_1 : t_1^C \dots c_m : t_m^C \end{cases}$
$\frac{E \vdash \text{ok}}{E \vdash f_i : t_i}$	$\frac{E \vdash \text{ok} \quad \delta : \Gamma_i \rightarrow E}{E \vdash l_i \delta \longrightarrow_R r_i \delta}$	where $\begin{cases} E = E_1; \text{Rew}(\Gamma, R); E_2 \\ \Gamma = f_1 : t_1 \dots f_n : t_n \\ R = \{\Gamma_i : l_i \longrightarrow r_i\}_{i=1 \dots m} \end{cases}$
(Term/Prod) $\frac{E \vdash t_1 : s_1 \quad E; v : t_1 \vdash t_2 : s_2}{E \vdash (v : t_1) t_2 : s_3}$ where $s_1, s_2, s_3 \in \mathcal{S}$	(Term/Abs) $\frac{E; v : t_1 \vdash e : t_2 \quad E \vdash (v : t_1) t_2 : s}{E \vdash \lambda v : t_1. e : (v : t_1) t_2}$	(Term/Ax) $\frac{E \vdash \text{ok}}{E \vdash s_1 : s_2}$ where $(s_1, s_2) \in \mathcal{A}$
(Term/App) $\frac{E \vdash e : (v : t_1) t_2 \quad E \vdash e' : t_1}{E \vdash e e' : t_2 \{v \mapsto e'\}}$	(Term/Conv) $\frac{E \vdash e : t \quad E \vdash t' : s \quad E \vdash t \approx t'}{E \vdash e : t'}$	

Figure 1: Definition correctness, environment correctness and lookup, PTS rules

calculus of constructions with rewriting, allowing it to include a certain class of definitions by rewriting.

Let us try to identify that class. If we reanalyze e in the new setting, the only new possible normal form of e is an application $f(\vec{e})$ of a function symbol f , coming from a rewrite definition $\text{Rew}(\Gamma, R)$, to some arguments in normal form. There is no obvious argument why such terms cannot be proofs of $(x : *)x$. On the other hand if we knew that such terms were always reducible, we could complete the consistency proof. Let us call $\text{COMP}(\Gamma, R)$ the condition on rewrite definitions we are looking for (i.e. $f(\vec{e})$ is always reducible), which can also be read as: the function symbols from Γ are completely defined by the set of rules R .

Note that the completeness of f has to be checked much earlier than it is used: we use it in a given closed environment $E = E_1; \text{Rew}(\Gamma, R); E_2$ but it has to be checked when f is added to the environment, i.e. in the environment E_1 . It implies that completeness checking has to account for environment extension and can be performed only with respect to arguments of such types, of which the set of normal forms could not change in the future. This is the case for arguments of inductive types.

The requirement that functions defined by rewriting are completely defined could very well be included in the condition ACC. On the other hand, the separation between ACC and COMP is motivated by the idea of working with abstract function symbols, equipped with some rewrite rules not defining them completely. For example if $+$ from Section 2 were declared using only the third rewrite rule, one could develop a theory of an associative function over natural numbers.

The intuition behind the definitions given below is the following. A rewrite definition $\text{Rew}(\Gamma, R)$ is *complete* (satisfies $\text{COMP}(\Gamma, R)$) if for all f in Γ , the goal $f(x_1, \dots, x_n)$ is *covered* by R . A goal is covered if all its instances are *immediately covered*, i.e. head-reducible by R . Following the discussion above we limit ourselves to *normalized canonical instances*, i.e. built from constructors wherever possible.

Definition 4.1 (Canonical form and canonical substitution). Given a judgment $E \vdash e : t$ we say that the term e is in *canonical* form if and only if:

- if $t \downarrow$ is an inductive type then $e = c(e_1, \dots, e_n)$ for some constructor c and terms e_1, \dots, e_n in canonical form
- otherwise e is arbitrary

Let Δ be a variable environment and E a correct environment. We call $\delta : \Delta \rightarrow E$ *canonical* if for every variable $x \in \Delta$, the term $x\delta$ is canonical.

From now on, let E be a global environment and let $\text{Rew}(\Gamma, R)$ be a rewrite definition such that $E \vdash \text{Rew}(\Gamma, R) : \text{correct}$. Let $f : (x_1 : t_1) \dots (x_n : t_n) t \in \Gamma$ be a function symbol of arity n .

Definition 4.2. A *goal* is a well-typed sequent $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$.

A *normalized canonical instance* of the goal $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$ is a well-typed sequent $E; \text{Rew}(\Gamma, R); E' \vdash f(e_1\delta \downarrow, \dots, e_n\delta \downarrow)$ for any canonical substitution $\delta : \Delta \rightarrow E; \text{Rew}(\Gamma, R); E'$.

A term e is *immediately covered* by R if there is a rule $G \vdash l \rightarrow r$ in R and a substitution γ such that $l\gamma = e$. By obvious extension we can also write that a goal or a normalized canonical instance is immediately covered by R .

A goal is *covered* by R if all its normalized canonical instances are immediately covered by R .

Note that, formally, a normalized canonical instance is not a goal. The difference is that the conversion corresponding to the environment of an instance contains reductions defined by R , while the one of a goal does not.

Definition 4.3 (Complete definition). A rewrite definition $\text{Rew}(\Gamma; R)$ is *complete* in the environment E , which is denoted by $\text{COMP}_E(\Gamma; R)$, if and only if for all function symbols $f : (x_1 : t_1) \dots (x_n : t_n) t \in \Gamma$ the goal $E; \Gamma; x_1 : t_1; \dots; x_n : t_n \vdash f(x_1, \dots, x_n)$ is covered by R .

Example 4.4. The terms $(S\ 0)$, $\lambda x:\text{nat}.x$ and $(\text{Node}\ 0\ \text{Leaf}\ \text{true}\ 0\ \text{Leaf})$ are canonical, while $(0 + 0)$ and $(\text{Node}\ nA\ A\ b\ 0\ \text{Leaf})$ are not. Given the definition of rotr from Section 2 consider the following terms:

$$\begin{aligned} t_1 &= \text{rotr}\ (S\ (nA + nC))\ (\text{Node}\ nA\ A\ b\ nC\ C) \\ t_2 &= \text{rotr}\ (S\ 0)\ (\text{Node}\ 0\ \text{Leaf}\ \text{true}\ 0\ \text{Leaf}) \end{aligned}$$

Both (with their respective environments) are goals for rotr , and t_2 (with a slightly different environment) is also a normalized canonical instance of t_1 . The goal t_1 is not immediately covered, but its instance t_2 is, as it is head-reducible by the second rule defining rotr . Since other instances of t_1 are also immediately covered, the goal is covered (see Example 5.20).

It follows that completeness of definitions by rewriting guarantees canonicity and logical consistency.

Definition 4.5. An environment E is *closed* if and only if it contains only inductive definitions and complete definitions by rewriting, i.e. for each partition of E into $E_1; \text{Rew}(\Gamma, R); E_2$ the condition $\text{COMP}_{E_1}(\Gamma, R)$ is satisfied.

Lemma 4.6 (Canonicity). *Let E be a closed environment. If $E \vdash e : t$ and e is in normal form then e is canonical.*

Proof. By induction on the size of e . If $t \downarrow$ is not an inductive type then any e is canonical. Otherwise, let us analyze the structure of e . It cannot be a product, an abstraction or a sort because $t \downarrow$ is an inductive type. Since E is closed, it is not a variable either. Hence e is of the form $e' e_1 \dots e_m$ (with m possibly equal 0), where e' is not an application. The term e' can be neither a product, nor a sort (they cannot be applied), nor a variable (E is closed). It is not an abstraction, since e is in normal form. The only possibility left is that e' is a constant h of arity $n \leq m$, and we get $e = h(e_1, \dots, e_n) e_{n+1} \dots e_m$.

Since $t \downarrow$ is an inductive type, h cannot be an inductive type. If it is a constructor then $n = m$ and by induction hypothesis e_1, \dots, e_n are in canonical form and so is $h(e_1, \dots, e_n)$. If h is a function symbol then $E = E_1; \text{Rew}(\Gamma, R); E_2$ for some E_1, E_2 and $h : (x_1 : t_1) \dots (x_n : t_n) \hat{t} \in \Gamma$ of arity $n \leq m$. Since E is closed, $\text{Rew}(\Gamma, R)$ is complete. Let us show that $E \vdash h(e_1, \dots, e_n)$ is a normalized canonical instance of $E_1; \Gamma; \Delta \vdash h(x_1, \dots, x_n)$, where $\Delta = x_1 : t_1; \dots; x_n : t_n$. By induction hypothesis, terms e_1, \dots, e_n are canonical and consequently $\delta : \Delta \rightarrow E$ defined by $\delta(x_i) = e_i$ is canonical. Moreover, $h(e_1, \dots, e_n) = h(x_1 \delta \downarrow, \dots, x_n \delta \downarrow)$ since e_1, \dots, e_n are in normal form. But every normalized canonical instance of a complete definition is reducible, which contradicts the assumption that $e = h(e_1, \dots, e_n) e_{n+1} \dots e_m$ is in normal form. \square

Theorem 4.7. *Every closed environment is consistent.*

Proof. Let E be a closed environment. Suppose that $E \vdash e : (x : \star)x$. Since $E \vdash \text{ok}$ and $E \vdash \text{Ind}(\text{False} : \star :=) : \text{correct}$ we have $E' \vdash \text{ok}$ where $E' = E; \text{Ind}(\text{False} : \star :=)$. Moreover E' is a closed environment.

Hence, we have $E' \vdash e\ \text{False} : \text{False}$. By Lemma 4.6, the normal form of $e\ \text{False}$ is canonical. Since False has no constructors, this is impossible. \square

5. CHECKING COMPLETENESS

The objective of this section is to provide an algorithm for checking completeness of definitions by rewriting. The algorithm presented in Subsection 5.2 checks that a goal is covered using successive *splitting* (Definition 5.3), i.e., replacement of variables of inductive types by constructor patterns. In order to know which constructor terms can replace a given variable, one has to compare types and hence an algorithm for unification modulo conversion is needed (Definition 5.2). Consider for example the first rule of the definition of `rotr`. It is clear that only `Leaf` can replace `t` in `rotr 0 t` because other trees have types that do not unify with `tree 0`.

Correctness of the completeness checking algorithm is proved in Lemma 5.19. It is done using an additional assumption on rewrite systems called *preservation of reducibility* which is discussed in Subsection 5.1.

Definition 5.1 (Unification problem). A quadruple $E, \Delta \vdash t \doteq s$, where E is an environment, Δ a variable environment and s, t are terms, is a *unification equation in E* . A *unification problem in E* is a finite set of unification equations. Without loss of generality we may assume that the variable environments Δ in all equations are the same.

A *unifier* or a *solution* of the unification problem U is a substitution $\gamma : \Delta \rightarrow E; E'$ such that $E; E' \vdash t\gamma \approx s\gamma$ for every $E, \Delta \vdash t \doteq s$ in U . We say that E' is the co-domain of γ , which is denoted by $Ran(\gamma)$.

A unifier γ is the *most general unifier* if $Ran(\gamma)$ is a variable environment Δ' and for every unifier $\delta : \Delta \rightarrow E; E''$ there exist a substitution $\delta' : \Delta' \rightarrow E; E''$, such that $E; E'' \vdash \delta \approx \gamma; \delta'$.

Definition 5.2 (Correct unification algorithm). A *unification algorithm* is a procedure which for every unification problem $U = \{E, \Delta \vdash t_i \doteq s_i\}$ returns a substitution γ , a bottom \perp , or a question mark $?$. The algorithm is *correct* if and only if: if it answers γ , it is the most general unifier $\gamma : \Delta \rightarrow E; \Delta'$ such that $\Delta' \subseteq \Delta$ and for all $x \in \Delta'$, $\gamma(x) = x$; if it answers \perp , U has no unifier.

Since unification modulo conversion is undecidable, every correct unification algorithm must return $?$ in some cases, which may be seen as too difficult for the algorithm. An example of such a partial unification algorithm is constructor unification, that is first-order unification with constructors and type constructors as rigid symbols, answering $?$ whenever one compares a non-trivial pair of terms involving non-rigid symbols.

From now on we assume the existence of a correct (partial) unification algorithm Alg .

Definition 5.3 (Splitting). Let $E; \Gamma; \Delta \vdash f(\vec{e})$ be a goal. A variable x is a *splitting variable* if $x : t \in \Delta$ and $t \downarrow = I\vec{u}$ for some inductive type $I \in E$.

A *splitting operation* considers all constructors c of the inductive type I and for each of them constructs the following unification problem U_c :

$$E; \Gamma, \Delta; \Delta_c \vdash x \doteq c(z_1, \dots, z_k) \quad E; \Gamma, \Delta; \Delta_c \vdash I\vec{u} \doteq I\vec{w}$$

where $c : (z_1 : Z_1) \dots (z_k : Z_k). I\vec{w}$ and $\Delta_c = z_1 : Z_1, \dots, z_k : Z_k$.

If for all constructors c , $Alg(U_c) \neq ?$, the splitting is *successful*. In that case, let $Sp(x) = \{\sigma_c \mid \sigma_c = Alg(U_c) \wedge Alg(U_c) \neq \perp\}$. The result of splitting is the set of goals $\{E; \Gamma; Ran(\sigma_c) \vdash f(\vec{e})\sigma_c\}_{\sigma_c \in Sp(x)}$.

If $Alg(U_c) = ?$ for some c , the splitting *fails*.

Example 5.4. If one splits the goal $\text{rotr } n \ t$ along n , one gets two goals: $\text{rotr } 0 \ t$ and $\text{rotr } (S \ m) \ t$. The first one is immediately covered by the first rule for rotr and if we split the second one along t , the **Leaf** case is impossible, because $\text{tree } 0$ does not unify with $\text{tree } (S \ m)$ and the **Node** case gives $\text{rotr } (S \ (nA + nC))$ (**Node** $nA \ A \ b \ nC \ C$).

The following lemma states the correctness of splitting, i.e. that splitting does not decrease the set of normalized canonical instances. Note that the lemma would also hold if we had a unification algorithm returning an arbitrary set of most general solutions, but in order for the coverage checking algorithm to terminate the set of goals resulting from splitting must be finite.

Lemma 5.5. *Let $E; \Gamma; \Delta \vdash f(\vec{e})$ be a coverage goal and let $\{E; \Gamma; \text{Ran}(\sigma_c) \vdash f(\vec{e})\sigma_c\}_{\sigma_c \in \text{Sp}(x)}$ be the result of successful splitting along $x : I\vec{u} \in \Delta$. Then every normalized canonical instance of $E; \Gamma; \Delta \vdash f(\vec{e})$ is a normalized canonical instance of $E; \Gamma; \text{Ran}(\sigma_c) \vdash f(\vec{e})\sigma_c$ for some $\sigma_c \in \text{Sp}(x)$.*

Proof. Let $E; \text{Rew}(\Gamma, R); E' \vdash f(e_1\delta\downarrow, \dots, e_n\delta\downarrow)$ be a normalized canonical instance according to a substitution $\delta : \Delta \rightarrow E; \text{Rew}(\Gamma, R); E'$. Since δ is canonical, $x\delta$ is a constructor term $c(s_1, \dots, s_k)$ for some constructor $c : (z_1 : Z_1) \dots (z_k : Z_k). I\vec{w}$ of I . Let us show that $E; \text{Rew}(\Gamma, R); E' \vdash f(e_1\delta\downarrow, \dots, e_n\delta\downarrow)$ is a normalized canonical instance of $E; \Gamma; \text{Ran}(\sigma_c) \vdash f(e_1\sigma_c, \dots, e_n\sigma_c)$. Let $\Delta_c = z_1 : Z_1, \dots, z_k : Z_k$.

First note that $\delta \cup [\vec{s}/\vec{z}] : \Delta; \Delta_c \rightarrow E; \text{Rew}(\Gamma, R); E'$ is a solution of the unification problem $E; \Gamma, \Delta; \Delta_c \vdash x \doteq c(z_1, \dots, z_k)$ and $E; \Gamma, \Delta; \Delta_c \vdash I\vec{u} \doteq I\vec{w}$, from the definition of splitting. Indeed, $x\delta = c(s_1, \dots, s_k) = c(z_1, \dots, z_k)[\vec{s}/\vec{z}]$ and $E; \text{Rew}(\Gamma, R); E' \vdash (I\vec{u})\delta \approx (I\vec{w})[\vec{s}/\vec{z}]$ since they are both types of $c(s_1, \dots, s_k)$ in $E; \text{Rew}(\Gamma, R); E'$.

By definition of σ_c , which is the most general unifier computed by a correct unification algorithm, $E; \Gamma; \text{Ran}(\sigma_c) \vdash \delta \cup [\vec{s}/\vec{z}] \approx \sigma_c; \delta'$ for some $\delta' : \text{Ran}(\sigma_c) \rightarrow E; \text{Rew}(\Gamma, R); E'$, where $\text{Ran}(\sigma_c) \subseteq \Delta; \Delta_c$. Consequently, $E; \Gamma; \text{Ran}(\sigma_c) \vdash \delta'(z_m) \approx s_m$ for $z_m \in \Delta_c$ and $E; \Gamma; \text{Ran}(\sigma_c) \vdash \delta'(y) \approx \delta(y)$ for $y \in \Delta$. Since \vec{s} are canonical terms $\delta'\downarrow$ is a canonical substitution.

Let us look closely at $E; \text{Rew}(\Gamma, R); E' \vdash f((e_1\sigma_c)(\delta'\downarrow)\downarrow, \dots, (e_n\sigma_c)(\delta'\downarrow)\downarrow)$ which is a normalized $\delta'\downarrow$ -instance of $E; \Gamma; \text{Ran}(\sigma_c) \vdash f(e_1\sigma_c, \dots, e_n\sigma_c)$. Since $E; \Gamma; \text{Ran}(\sigma_c) \vdash \delta \cup [\vec{s}/\vec{z}] \approx \sigma_c; \delta'$, we have $(e_m\sigma_c)(\delta'\downarrow)\downarrow = (e_m\sigma_c\delta')\downarrow = (e_m\delta)\downarrow$ for every m . Consequently, $E; \text{Rew}(\Gamma, R); E' \vdash f(e_1\delta\downarrow, \dots, e_n\delta\downarrow)$ is a normalized canonical instance of $E; \Gamma; \text{Ran}(\sigma_c) \vdash f(e_1\sigma_c, \dots, e_n\sigma_c)$. \square

5.1. Preservation of Reducibility. Although one would expect that an immediately covered goal is also covered, it is not always true, even for confluent systems. It turns out that we need a property of critical pairs that is stronger than just joinability. Let us suppose that $\text{or} : \text{bool} \rightarrow \text{bool} \rightarrow \text{bool}$ is defined by four rules by cases over **true** and **false** and that $\text{if} : \text{bool} \rightarrow \text{bool} \rightarrow \text{bool} \rightarrow \text{bool}$ is defined by two rules by cases on the first argument.

Inductive I : $\text{bool} \rightarrow \text{Set} := \text{C} : \text{forall } b:\text{bool}, I \ (\text{or } b \ b)$.

Symbol f: $\text{forall } b:\text{bool}, I \ b \rightarrow \text{bool}$

Rules

$f \ (\text{or } b \ b) \ (\text{C } b) \longrightarrow \text{if } b \ (f \ \text{true} \ (\text{C } \text{true})) \ (f \ \text{false} \ (\text{C } \text{false}))$

In the example presented above all expressions used in types and rules are in normal form, all critical pairs are joinable, the system is terminating, and splitting of $\mathbf{f} \ \mathbf{b} \ \mathbf{i}$ along \mathbf{i} results in the only reducible goal $\mathbf{f} \ (\mathbf{or} \ \mathbf{b} \ \mathbf{b}) \ (\mathbf{C} \ \mathbf{b})$. In spite of that \mathbf{f} is not completely defined, as $\mathbf{f} \ \mathbf{true} \ (\mathbf{C} \ \mathbf{true})$ is a normalized canonical instance of $\mathbf{f} \ (\mathbf{or} \ \mathbf{b} \ \mathbf{b}) \ (\mathbf{C} \ \mathbf{b})$ and it is not reducible. In order to know that an immediately covered goal is always covered we need one more condition on rewrite rules, called preservation of reducibility.

Definition 5.6. Definition by rewriting $\text{Rew}(\Gamma, R)$ *preserves reducibility* in an environment E if for every critical pair $\langle f(\vec{u}), r\delta \rangle$ of a rule $G_1 \vdash f(\vec{e}) \longrightarrow r$ in R with a rule $G_2 \vdash g \longrightarrow d$ coming from R or from some other rewrite definition in E , the term $f(\vec{u}\downarrow)$ is head-reducible by R .

Note that by using $?$ variables in rewrite rules one can get rid of (some) critical pairs and hence make a definition by rewriting satisfy this property. In the example above one could write $\mathbf{f} \ ? \ (\mathbf{C} \ \mathbf{b})$ as the left-hand side. This would also make the system non-terminating, and show that f is not really well-defined.

Of course all orthogonal rewrite systems, in particular inductive elimination schemes, as defined in [24], preserve reducibility.

Lemma 5.7. *Let $E \vdash e : t$ and $e = f(e_1, \dots, e_n)$, where f of arity n comes from $\text{Rew}(\Gamma, R)$ which preserves reducibility. If e is head-reducible by R then $f(e_1\downarrow, \dots, e_n\downarrow)$ is also head-reducible by R .*

Proof. By induction on \longrightarrow . If e_1, \dots, e_n are in normal forms then the conclusion is obvious. Otherwise, let $G_1 \vdash f(\vec{l}) \longrightarrow r$ be a rule from R and γ a substitution such that $f(\vec{e}) = f(\vec{l})\gamma$ and let us make one reduction step $e_i \longrightarrow e'_i$, using the rule $G_2 \vdash g \longrightarrow d$.

There are two possibilities: the reduction in e_i happens either in substitution γ , i.e. in the term $\gamma(x)$, where x is a free variable of $f(\vec{l})$, or it happens on a position p that belongs to $f(\vec{l})$. In the former case, let us do identical reduction in all other instances of x . Obviously, we get a term $f(e'_1, \dots, e'_n)$ that is smaller than e in \longrightarrow and is still an instance of $f(\vec{l})$. Hence by induction hypothesis we get the desired conclusion.

Otherwise, $f(\vec{l})$ and g superpose at some nonvariable position and we have $f(\vec{l})|_p\gamma = g\xi$ for some position p and substitution ξ . Since we may suppose that free variables of $f(\vec{l})$ and g are different, we get $f(\vec{l})|_p(\gamma \cup \xi) = g(\gamma \cup \xi)$. Let δ be the most general unifier of $f(\vec{l})|_p$ and g and let $\langle f(\vec{u}), r\delta \rangle$ be the corresponding critical pair. Since δ is the most general unifier, there exists σ such that $(\gamma \cup \xi) = \delta; \sigma$ and $f(\vec{e}) = f(\vec{l})\gamma = f(\vec{l})(\gamma \cup \xi) = f(\vec{l})\delta\sigma$ with $f(\vec{l})\delta\sigma \rightarrow_R f(\vec{u})\sigma = f(e_1, \dots, e'_i, \dots, e_n)$. By preservation of reducibility $f(\vec{u}\downarrow)$ is head-reducible by R . Hence $f(\vec{u}\downarrow)\sigma$ is also head-reducible by R . Like above we can apply induction hypothesis and deduce that $f(\vec{e}\downarrow)$ is head-reducible by R . \square

Lemma 5.8. *Let $\text{Rew}(\Gamma, R)$ preserve reducibility in an environment E , let $f \in \Gamma$ and let $E; \Gamma; \Delta \vdash f(\vec{e})$ be a goal. If it is immediately covered then it is covered.*

Proof. Let $E; \Gamma; \Delta \vdash f(\vec{e})$ be a goal immediately covered by R and $\delta : \Delta \rightarrow E; \text{Rew}(\Gamma, R); E'$ be a canonical substitution. Obviously, $E; \text{Rew}(\Gamma, R); E' \vdash f(\vec{e}\delta)$ is immediately covered by R . Hence, by Lemma 5.7 $E; \text{Rew}(\Gamma, R); E' \vdash f(\vec{e}\delta\downarrow)$ is also immediately covered by R , i.e. $E; \Gamma; \Delta \vdash f(\vec{e})$ is covered. \square

$\frac{t_1 <_{p.1} t'_1 \Rightarrow S_1 \quad t_2 <_{p.2} t'_2 \Rightarrow S_2}{(x:t_1)t_2 <_p (x:t'_1)t'_2 \Rightarrow S_1 \cup S_2}$	$\frac{t_1 <_{p.1} t'_1 \Rightarrow S_1 \quad t_2 <_{p.2} t'_2 \Rightarrow S_2}{\lambda x:t_1.t_2 <_p \lambda x:t'_1.t'_2 \Rightarrow S_1 \cup S_2}$
$\frac{t_1 <_{p.1} t'_1 \Rightarrow S_1 \quad t_2 <_{p.2} t'_2 \Rightarrow S_2}{t_1 t_2 <_p t'_1 t'_2 \Rightarrow S_1 \cup S_2}$	
$\frac{t_1 <_{p.1} t'_1 \Rightarrow S_1 \quad \dots \quad t_n <_{p.n} t'_n \Rightarrow S_n}{h(t_1, \dots, t_n) <_p h(t'_1, \dots, t'_n) \Rightarrow S_1 \cup \dots \cup S_n} \quad h \text{ is a constant}$	
$\frac{t_1, t_2 \notin \text{Var} \quad \text{head}(t_1) \neq \text{head}(t_2)}{t_1 <_p t_2 \Rightarrow \{\perp\}}$	$\frac{t_1 \in \Delta_1 \quad t_2 \notin (\text{Var} \setminus \Delta_2)}{t_1 <_p t_2 \Rightarrow \{p\}}$
$\frac{(t_1 \in (\text{Var} \setminus \Delta_1) \wedge t_1 = t_2) \vee (t_1 \notin \text{Var} \wedge t_2 \in \Delta_2)}{t_1 <_p t_2 \Rightarrow \emptyset}$	$\frac{(t_1 \in (\text{Var} \setminus \Delta_1) \wedge t_1 \neq t_2) \vee (t_2 \in (\text{Var} \setminus \Delta_2) \wedge t_1 \neq t_2)}{t_1 <_p t_2 \Rightarrow \{\perp\}}$

Figure 2: Splitting matching rules, parametrized by Δ_1, Δ_2

5.2. Coverage Checking Algorithm. In this section we present an algorithm checking whether a set of goals is covered by the given set of rewrite rules. The algorithm is correct only for definitions that preserve reducibility. The algorithm, in a loop, picks a goal, checks whether it is immediately covered, and if not, splits the goal replacing it by the subgoals resulting from splitting. In order to ensure termination, splitting is limited to *safe splitting variables*. Intuitively, a splitting variable is safe if it lies within the contour of the left-hand side of some rule when we superpose the tree representation of the left-hand side with the tree representation of the goal. The number of nodes that have to be added to the goal in order to fill the tree of the left-hand side is called a distance, and a sum of distances over all rules is called a measure. Since the measures of goals resulting from splitting are smaller than the measure of the original goal, the coverage checking algorithm terminates.

This subsection is organized as follows. We start by defining the splitting matching algorithm which is used to define safe splitting variables. Next, we provide definitions and lemmas needed to prove termination of the coverage checking algorithm and then we give the algorithm itself and the proof of its correctness. We conclude this subsection with some positive and negative examples leading to an extension of the algorithm allowing us to accept definitions by case analysis even if the unification algorithm is not strong enough.

Let us start with the splitting matching algorithm which finds variables in t_1 that lie within the contour of t_2 .

Definition 5.9 (Splitting matching). The *splitting matching algorithm* is defined in Figure 2. Given two sequents $\Delta_1 \vdash t_1$ and $\Delta_2 \vdash t_2$, it returns the unique set S , such that $t_1 <_{\Delta} t_2 \Rightarrow S$ is derivable. The set S is a subset of $\{\perp\} \cup \{p \in \text{Pos}(t_1) \mid t_1|_p \in \Delta_1\}$.

Definition 5.10 (Safe splitting variable). Let $\Delta_1 \vdash t_1$ and $\Delta_2 \vdash t_2$ be sequents such that t_2 is a left-hand side of a rule from R and let S be a set such that $t_1 <_{\Delta} t_2 \Rightarrow S$ and $\perp \notin S$. A variable $x \in \Delta_1$ is a *safe splitting variable for $\Delta_1 \vdash t_1$ along $\Delta_2 \vdash t_2$* if it is a splitting variable and there exists $p \in S$ such that $t_1|_p = x$ and either $t_2|_p$ is a variable declared in Δ_2 or $t_2|_p = c(\vec{e})$ for some constructor c and some terms \vec{e} .

The set of safe splitting variables for the sequent $\Delta_1 \vdash t_1$ along $\Delta_2 \vdash t_2$ is denoted by $SV(\Delta_1 \vdash t_1, \Delta_2 \vdash t_2)$ or $SV(t_1, t_2)$ for short. $SV(t, R)$ is the set of safe splitting variables for t along left-hand sides of rules from R .

Example 5.11. In the goal `rotr (S (S nC)) (Node 0 Leaf b (S nC) C)` there are two safe splitting variables `b` and `C` along the left-hand sides of the rules defining `rotr`.

Definition 5.12 (Distance). Let $\Delta_1 \vdash t_1$ and $\Delta_2 \vdash t_2$ be sequents and S be a set such that $t_1 <_{\Lambda} t_2 \Rightarrow S$. If $\perp \notin S$ then the *distance* of $\Delta_1 \vdash t_1$ from $\Delta_2 \vdash t_2$, denoted by $dist(\Delta_1 \vdash t_1, \Delta_2 \vdash t_2)$ or $dist(t_1, t_2)$, equals $\sum_{p \in S} |t_2|_p$; otherwise it is equal to 0.

The following two lemmas state that the distance of a term decreases when we apply a substitution, and it decreases strictly if it is a substitution resulting from splitting.

Lemma 5.13 (Distance of a substituted sequent). *Let $\Delta_1 \vdash t_1$ and $\Delta_2 \vdash t_2$ be sequents and let S be a set such that $t_1 <_{\Lambda} t_2 \Rightarrow S$. Then for every substitution $\gamma : \Delta_1 \rightarrow \Delta'$ we have $dist(\Delta' \vdash t_1\gamma, \Delta_2 \vdash t_2) \leq dist(\Delta_1 \vdash t_1, \Delta_2 \vdash t_2)$.*

Moreover, if $\perp \in S$ then $dist(\Delta' \vdash t_1\gamma, \Delta_2 \vdash t_2) = dist(\Delta_1 \vdash t_1, \Delta_2 \vdash t_2) = 0$.

Proof. Let S_{γ} be a set such that $t_1\gamma <_{\Lambda} t_2 \Rightarrow S_{\gamma}$ and let us denote $dist(\Delta_1 \vdash t_1, \Delta_2 \vdash t_2)$ by d and $dist(\Delta' \vdash t_1\gamma, \Delta_2 \vdash t_2)$ by d_{γ} .

If $\perp \in S$ then $d = 0$. Note that $\perp \in S$ if and only if there is a position p such that subterms occurring at p in t_1 and t_2 either have different head symbols, or $t_2|_p$ (resp. $t_1|_p$) is a bound variable in t_2 (resp. t_1) and $t_1|_p \neq t_2|_p$. Of course, if we compare $t_1\gamma|_p$ and $t_2|_p$ then either they still have different head-symbols or $t_2|_p$ (resp. $t_1|_p$) is a bound variable and $t_1\gamma|_p \neq t_2|_p$. Hence $d_{\gamma} = 0$.

If $\perp \notin S$ then $d = \sum_{p \in S} |t_2|_p \geq 0$. If $\perp \in S_{\gamma}$ then obviously $0 = d_{\gamma} \leq d$. Otherwise, let us take $p \in S$ and the set $Q_p = \{q \in S_{\gamma} \mid p \preceq q\}$, where \preceq is the prefix ordering. Since all positions from Q_p are independent (as $t_1\gamma|_q \in Var$ for every $q \in S_{\gamma}$) we have $\sum_{q \in Q_p} |t_2|_q \leq |t_2|_p$ and the equality holds only if $Q_p = \{p\}$. Let us show that $\forall q \in S_{\gamma} \exists p \in S \ p \preceq q$. Indeed, assuming that $\perp \notin S_{\gamma}$, $q \in S_{\gamma}$ either because $q \in S$ and $(t_1|_q)\gamma \in \Delta'$ or because there is a position $p \in S$ such that $q = p \cdot q'$ for some q' and $(t_1|_p\gamma)|_{q'} \in \Delta'$. Of course, since positions in S are independent, the sets Q_p are disjoint for different p .

Hence $S_{\gamma} = \bigcup_{p \in S} Q_p$ and $d_{\gamma} = \sum_{q \in S_{\gamma}} |t_2|_q = \sum_{p \in S} \sum_{q \in Q_p} |t_2|_q \leq \sum_{p \in S} |t_2|_p = d$. \square

Lemma 5.14 (Distance after splitting strictly decreases). *Let $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$ be a goal, $t = f(e_1, \dots, e_n)$, let $G \vdash l \rightarrow r$ be one of the rewrite rules for f in R and let S be a set such that $t <_{\Lambda} l \Rightarrow S$ and $\perp \notin S$. If $x : I\vec{u} \in SV(t, l)$ is a safe splitting variable and splitting t along x is successful then $dist(Ran(\sigma_c) \vdash t\sigma_c, G \vdash l) < dist(\Delta \vdash t, G \vdash l)$ for every $\sigma_c \in Sp(x)$.*

Proof. Let $\sigma_c \in Sp(x)$ and let S_c be a set such that $t\sigma_c <_{\Lambda} l \Rightarrow S_c$. By Lemma 5.13 we have $dist(t\sigma_c, l) \leq dist(t, l)$. Let us analyze the proof of that lemma and show that in case of a substitution resulting from splitting there is a strict inequality between $dist(t\sigma_c, l)$ and $dist(t, l)$. In the proof it was noticed that for every $p \in S$, $\sum_{q \in Q_p} |l|_q \leq |l|_p$, where $Q_p = \{q \in S_c \mid p \preceq q\}$ and that $\sum_{q \in Q_p} |l|_q = |l|_p$ only if $Q_p = \{p\}$. Consequently, if we show that there exists a position p such that $p \notin Q_p$, we immediately get $dist(t\sigma_c, l) < dist(t, l)$.

Since $x : I\vec{u}$ is a safe splitting variable for t along l , there exists a position $p \in S$ such that $t|_p = x$ and $l|_p \in G$ or $l|_p = c'(\vec{a})$ for some constructor c' . Since σ_c results from successful splitting, $x\sigma_c = c(\vec{b})$ for some \vec{b} . Now, there are three cases. If $l|_p \in G$ then

$t\sigma_c <_{\Lambda} l \Rightarrow \emptyset$, $Q_p = \emptyset$ (and hence $p \notin Q_p$). Otherwise, if $c' \neq c$ then we fall into an easy case when $\perp \in S_c$ and $dist(t\sigma_c, l) = 0 < |c'(\vec{a})| \leq dist(t, l)$. Finally, if $c' = c$, the computation of S_c passes through the step $c(\vec{b}) <_p c(\vec{a}) \Rightarrow _$. This means that all positions in Q_p come from \vec{b} and that they are longer than p or that $Q_p = \emptyset$. Thus $p \notin Q_p$ and $dist(t\sigma_c, l) < dist(t, l)$. \square

Definition 5.15 (Measure of a goal). Let $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$ be a goal and let $R_f = \{G_i \vdash l_i \longrightarrow r_i\}_{i=1..m}$ be the set of rules for f . The *measure of $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$* equals $\sum_{i=1..m} dist(\Delta \vdash f(e_1, \dots, e_n), G_i \vdash l_i)$.

It follows directly from Lemmas 5.13 and 5.14 that the measure of a goal strictly decreases after applying a substitution resulting from splitting.

Lemma 5.16 (Measure after splitting strictly decreases). *Let $E; \Gamma; \Delta \vdash f(e_1, \dots, e_n)$ be a goal, $t = f(e_1, \dots, e_n)$, $R_f = \{G_i \vdash l_i \longrightarrow r_i\}_{i=1..m}$ be the set of rules for f and S be a set such that $t <_{\Lambda} l_j \Rightarrow S$ for some $j \in \{1, \dots, m\}$ and $\perp \notin S$. If $x : I\vec{u} \in SV(t, R)$ is a safe splitting variable and $\{\phi_1, \dots, \phi_n\}$ is the result of successful splitting of t along x then the measure of every ϕ_i is strictly smaller than the measure of t .*

Proof. For every $\sigma_c \in Sp(x)$, we have to show that $\sum_{i=1..m} dist(t\sigma_c, l_i) < \sum_{i=1..m} dist(t, l_i)$. This follows from $dist(t\sigma_c, l_j) < dist(t, l_j)$, which is the consequence of Lemma 5.14 and $dist(t\sigma_c, l_i) \leq dist(t, l_i)$ for all $i = 1 \dots m$, $i \neq j$, which follows from Lemma 5.13. \square

Definition 5.17 (Coverage checking algorithm). Let W be a set of pairs consisting of a goal and a set of safe variables of that goal along left-hand sides of rules from R and let CE be a set of goals. The coverage checking algorithm works as follows:

Initialize

$$W = \{(E; \Gamma; x_1 : t_1; \dots; x_n : t_n \vdash f(x_1, \dots, x_n), SV(f(x_1, \dots, x_n), R))\}$$

$$CE = \emptyset$$

Repeat

- (1) choose a pair (ϕ, X) from W ,
- (2) if ϕ is immediately covered by one of the rules from R then

$$W := W \setminus \{(\phi, X)\}$$
- (3) otherwise
 - (a) if $X = \emptyset$ then $W := W \setminus \{(\phi, X)\}$, $CE := CE \cup \{\phi\}$
 - (b) otherwise choose $x \in X$; split ϕ along x
 - (i) if splitting is successful and returns $\{\phi_1, \dots, \phi_n\}$ then

$$W := W \setminus \{(\phi, X)\} \cup \{(\phi_i, SV(\phi_i, R))\}_{i=1..n},$$
 - (ii) otherwise $W := W \setminus \{(\phi, X)\} \cup \{(\phi, X \setminus \{x\})\}$

until $W = \emptyset$

Lemma 5.18. *The cover checking algorithm terminates.*

Proof. Let us consider the following measure $M(W)$: the multiset of lexicographically ordered pairs consisting of the measure of ϕ and the size of X , for all $(\phi, X) \in W$. We will show that every loop of the algorithm strictly decreases $M(W)$. Consider $(\phi, X) \in W$. If ϕ is immediately covered then obviously the measure of $W \setminus \{(\phi, X)\}$ is strictly smaller than the measure of W . Otherwise, we split ϕ along some $x \in X$. If splitting fails then (ϕ, X) is replaced by $(\phi, X \setminus \{x\})$ and the size of the second component strictly decreases. If splitting is successful and returns $\{\phi_1, \dots, \phi_n\}$ then (ϕ, X) is replaced by $\{(\phi_i, SV(\phi_i, R))\}_{i=1..n}$. By

Lemma 5.16 the measures of goals from $\{\phi_1, \dots, \phi_n\}$ are strictly smaller than the measure of ϕ and consequently $M(W)$ strictly decreases. \square

Lemma 5.19. *If $\text{Rew}(\Gamma, R)$ preserves reducibility and the algorithm stops with $CE = \emptyset$ then the initial goal is covered.*

Proof. Let us consider a successful run of the algorithm, performing a finite number of times the body of the Repeat loop and resulting in $CE = \emptyset$. By induction on n , the number of Repeat steps until the end of the algorithm, we prove that the goals appearing in W are covered.

The base case, for $n = 0$, is trivial since W_0 is empty.

Now suppose that n steps before the end of the algorithm all goals in W_n are covered and let us check that this was true $n + 1$ steps before the end, i.e. one step of the algorithm earlier.

In case 2, W_{n+1} contains all goals from W_n and one goal ϕ which is immediately covered by a rule in R . By preservation of reducibility (Lemma 5.8) every normalized canonical instance of ϕ is also immediately covered and consequently all goals of W_{n+1} are covered.

Case 3(a) is impossible since it makes the set CE non-empty.

In case 3(b)i, W_{n+1} contains some of the goals from W_n and one goal ϕ whose subgoals resulting from successful splitting are already in W_n . By Lemma 5.5 the set of normalized canonical instances of these subgoals contains the set of normalized canonical instances of ϕ . Hence W_{n+1} is covered.

In case 3(b)ii the set of goals in W_{n+1} and W_n are equal.

Hence the initial goal in W is also covered. \square

Example 5.20. The beginning of a possible run of the algorithm for the function `rotr` is presented already in Example 5.4. Both splitting operations are performed on safe variables, as required. We are left with the goal `rotr (S (nA + nC)) (Node nA A b nC C)`. Splitting along `A` results in:

```
rotr (S (0 + nC)) (Node 0 Leaf b nC C)
rotr (S((S(nX+nZ))+nC)) (Node (S(nX+xZ)) (Node nX X y nZ Z) b nC C)
```

immediately covered by the second and the third rule respectively.

Since we started with the initial goal `rotr n t` and since the definition of `rotr` preserves reducibility, it is complete.

When the coverage checking algorithm stops with $CE \neq \emptyset$, we cannot deduce that R is complete. The set CE contains potential counterexamples. They can be true counterexamples, false counterexamples, or goals for which splitting failed along all safe variables, due to incompleteness of the unification algorithm. In some cases further splitting of a false counterexample may result in reducible goals or in the elimination of the goal as uninhabited, but it may also loop. Some solutions preventing looping (finitary splitting) can be found in [18].

Unfortunately splitting failure due to incompleteness of the unification may happen while checking coverage of a definition by case analysis over complex dependent inductive types (for example trees of size 2), even if rules for all constructors are given. Therefore, it is advisable to add a second phase to our algorithm, which would treat undefined output of unification as success. Using this second phase of the algorithm, one can accept all definitions by case analysis that can be written in Coq.

Example 5.21. Let g be a function defined by case analysis and let g' be its version defined by rewriting (without the impossible `Leaf` case):

```

Definition g (t : tree (S (S 0))) : bool := match t with
  Leaf => false
| Node _ Leaf _ _ => true
| Node _ (Node _ _ _ _) _ => false
end.

```

Symbol $g' : \text{tree } (S (S 0)) \rightarrow \text{bool}$

Rules

```

g' (Node ?1 Leaf b ?2 t') → true
g' (Node ?1 (Node ?2 t b ?3 t') b' ?4 t'') → false

```

Our algorithm starts with the goal $g' \ x$, splits it along x , easily detects that `Leaf` case is not possible but is stuck on `Node n t b m t'`, because this requires deciding that $S \ (n+m)$ is unifiable with $S \ (S \ 0)$, which may be too hard for a unification algorithm². In that case the initial goal $g' \ x$ becomes a potential counterexample.

Accepting all definitions by case analysis. The second phase of our algorithm would start only for the goals with safe splitting variables, i.e. where regular splitting failed because the unification was too weak. In this phase, the splitting would become lax by treating ? unification result as successful and returning simple substitutions $\sigma_c = \{x \mapsto c(\vec{z})\}$ for such cases (see Definition 5.3). As a result the goals would not be well-typed sequents anymore, which has to be taken into account by the unification algorithm. On the other hand typability is not required for splitting matching and the rest of the algorithm which would work just like described in Definition 5.17. Both arguments of termination and correctness of the algorithm would hold.

Going back to our example. Redoing the lax splitting on x in the goal $g' \ x$, one gets again that `Leaf` is impossible, but `Node` is now accepted and leads to an (untyped) goal $g' \ (\text{Node } n \ t \ b \ m \ t')$. Splitting on t is now successful for both constructors and both resulting goals get reduced.

6. MORE EXAMPLES

6.1. Heterogeneous equality. Consider the inductive predicate `JMeq` of heterogeneous equality with its non-standard elimination rule:

Inductive `JMeq (A:Set)(a:A): forall B:Set, B → Set := JMrefl: JMeq A a A a.`

Symbol `JMelim : forall (A:Set)(a:A)(P: forall b:A, JMeq A a A b → Set), P a (JMrefl A a) → forall (b: A) (e: JMeq A a A b), (P b e)`

Rules

```

JMelim A a P h a (JMrefl A a) → h

```

²Note that a better unification algorithm could find the two most general solutions $n=0, m=(S \ 0)$ and $n=(S \ 0), m=0$. Then splitting would result in two goals immediately covered by rules for g' .

One splitting of $\text{JMelim } A \ a \ P \ h \ b \ c$ over c results in $\text{JMelim } A \ a \ P \ h \ a$ ($\text{JMrefl } A \ a$) which is equal to the left-hand side of the rule. Hence this rule completely defines JMelim .

6.2. Uniqueness of Identity Proofs and Streicher's axiom K. Consider the type eq and the definition of function UIP , proving that identity proofs are unique:

Inductive $\text{eq } (A:\text{Set})(a:A): A \rightarrow \text{Set} \quad := \quad \text{refl}: \text{eq } A \ a \ a$.

Symbol $\text{UIP} : \text{forall } (A:\text{Set})(a \ b:A)(p \ q: \text{eq } A \ a \ b), (\text{eq } (\text{eq } A \ a \ b) \ p \ q)$

Rules

$\text{UIP } A \ a \ a \ (\text{refl } A \ a) \ (\text{refl } A \ a) \longrightarrow \text{refl } (\text{eq } A \ a \ a) \ (\text{refl } A \ a)$.

The function UIP is completely defined since two subsequent splittings of $\text{UIP } A \ a \ b \ p \ q$, along p and along q , result in $\text{UIP } A \ a \ a \ (\text{refl } A \ a) \ (\text{refl } A \ a)$ which is exactly the left-hand side of the only rule for UIP .

The rule for Streicher's axiom K can also easily be proved complete:

Symbol $\text{K} : \text{forall } (A:\text{Set}) \ (a:A) \ (P:\text{eq } A \ a \ a \rightarrow \text{Set}),$

$P \ (\text{refl } A \ a) \rightarrow \text{forall } p: \text{eq } A \ a \ a, P \ p$

Rules

$\text{K } A \ a \ P \ h \ (\text{refl } A \ a) \longrightarrow h$

Note that both rules for UIP and K can also be written in a left-linear form:

$\text{UIP } A \ a \ ?1 \ (\text{refl } ?2 \ ?3) \ (\text{refl } ?4 \ ?5) \longrightarrow \text{refl } (\text{eq } A \ a \ a) \ (\text{refl } A \ a)$

$\text{K } A \ a \ P \ h \ (\text{refl } ?1 \ ?2) \longrightarrow h$

6.3. Non pattern matching rules. These are two examples of complete definitions which do not follow the pattern matching schemes as defined in [12] and [16].

Symbol $\text{or}' : \text{bool} \rightarrow \text{bool} \rightarrow \text{bool}$

Rules

$\text{or}' \ x \ x \longrightarrow x$

$\text{or}' \ \text{true} \ y \longrightarrow \text{true}$

$\text{or}' \ x \ \text{true} \longrightarrow \text{true}$

Symbol $\text{lt}, \text{diff} : \text{nat} \rightarrow \text{nat} \rightarrow \text{bool}$

Rules

$\text{lt} \ 0 \ y \longrightarrow \text{diff} \ 0 \ y$

$\text{lt} \ x \ 0 \longrightarrow \text{false}$

$\text{lt} \ (S \ x) \ (S \ y) \longrightarrow \text{lt} \ x \ y$

$\text{diff} \ x \ x \longrightarrow \text{false}$

$\text{diff} \ 0 \ (S \ y) \longrightarrow \text{true}$

$\text{diff} \ (S \ x) \ 0 \longrightarrow \text{true}$

$\text{diff} \ (S \ x) \ (S \ y) \longrightarrow \text{diff} \ x \ y$

7. CONCLUSIONS AND RELATED WORK

In this paper we study consistency of the calculus of constructions with rewriting. More precisely, we propose a formal system extending an arbitrary PTS with inductive definitions and definitions by rewriting. Assuming that suitable positivity and acceptance conditions guarantee termination and confluence, we formalize the notion of a complete definition by rewriting. We show that in every environment consisting only of inductive definitions and complete definitions by rewriting there is no proof of $(x : *)x$. Moreover, we present a sound and terminating algorithm for checking completeness of definitions. It is necessarily incomplete, since in presence of dependent types emptiness of types trivially reduces to completeness and the former is undecidable.

Our coverage checking algorithm resembles the one proposed by Coquand in [12] for Martin-Löf type theory and used by McBride for his OLEG calculus [16]. In these works the procedure consisting in successive case-splittings is used to interactively built pattern matching equations, or to check that a given set of equations can be built this way. Unlike in our paper, Coquand and McBride do not have to worry whether all instances of a reducible subgoal are reducible. Indeed, in [12] pattern matching equations are meant to be applied to terms modulo conversion, and in [16] equations (or rather the order of splittings in the successful run of the coverage checking procedure) serve as a guideline to construct an OLEG term verifying the equations. Equations themselves are never used for reduction and the constructed term reduces according to existing rules.

In our paper rewrite rules are matched against terms modulo α -conversion. Rewriting has to be confluent, strongly normalizing and has to preserve reducibility. Under these assumptions we can prove completeness for all examples from [12] and for the class of pattern matching equations considered in [16]. In particular we can deal with elimination rules for inductive types and with Streicher's axiom K. Moreover, we can accept definitions which depart from standard pattern matching, like `rotr` and `+`.

The formal presentation of our algorithm is directly inspired by the work of Pfenning and Schürmann [18]. A motivation for that paper was to verify that a logic program in the Twelf prover covers all possible cases. In LF, the base calculus of Twelf, there is no polymorphism, no rewriting and conversion is modulo $\beta\eta$ -conversion. The authors use higher-order matching modulo $\beta\eta$ -conversion, which is decidable for patterns a la Miller and strict patterns. Moreover, since all types and function symbols are known in advance, the coverage is checked with respect to all available function symbols. In our paper, conversion contains rewriting and it cannot be used for matching; instead we use matching modulo α . This simplifies the algorithm searching for safe splitting variables, but on the other hand it does not fit well with instantiation and normalization. To overcome this problem we introduce the notions of normalized canonical instance and preservation of reducibility which were not present in previously mentioned papers. Finally, since the sets of function symbols and rewrite rules grow as the environment extends, coverage is checked with respect to constructors only.

Even though the worst-case complexity of the coverage checking is clearly exponential, for practical examples the algorithm should be quite efficient. It is very similar in spirit to the algorithms checking exhaustiveness of definitions by pattern matching in functional programming languages and these are known to work effectively in practice.

An important issue which is not addressed in this paper is to know how much we extend conversion. Of course it depends on the choice of conditions ACC and POS and on

the unification algorithm used for coverage checking. In particular, some of the definitions by pattern matching can be encoded by recursors [13], so if ACC is strict, we may have no extension at all. In general there seems to be at least two kinds of extensions. The first are non-standard elimination rules for inductive types, but the work of McBride shows that the axiom K is sufficient to encode all other definitions by pattern matching considered by Coquand. The second are additional rules which extend a definition by pattern matching (like associativity for $+$). It is known that for first-order rewriting, these rules are inductive consequences of the pattern matching ones, i.e. all their canonical instances are satisfied as equations (see e.g. Theorem 7.6.5 in [19]). Unfortunately, this is no longer true for higher-order rules over inductive types with functional arguments. Nevertheless it seems that such rules are inductive consequences of the pattern matching rules if the corresponding equality is extensional.

Finally, our completeness condition COMP verifies closure properties defined in [9, 10]. Hence, it is adequate for a smooth integration of rewriting with the module system present in Coq since its version 7.4.

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