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THE OMEGA RULE IS $\mathbf{\Pi^1_1}$-COMPLETE IN THE $\lambda\beta$-CALCULUS

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Abstract. In a functional calculus, the so called ω-rule states that if two terms P and Q applied to any closed term N return the same value (i.e. $PN = QN$), then they are equal (i.e. $P = Q$ holds). As it is well known, in the $\lambda\beta$-calculus the ω-rule does not hold, even when the η-rule (weak extensionality) is added to the calculus. A long-standing problem of H. Barendregt (1975) concerns the determination of the logical power of the ω-rule when added to the $\lambda\beta$-calculus. In this paper we solve the problem, by showing that the resulting theory is $\mathbf{\Pi^1_1}$-Complete.

Introduction

In a functional calculus, the so called ω-rule states that if two terms P and Q applied to any closed term N return the same value (i.e. $PN = QN$), then they are equal (i.e. $P = Q$ holds). As it is well known, in the $\lambda\beta$-calculus the ω-rule does not hold, even when the η-rule (weak extensionality) is added to the calculus.

It is therefore natural to investigate the logical status of the ω-rule in λ-theories.

We have first considered constructive forms of such rule in [7], obtaining r.e. λ-theories which are closed under the ω-rule. This gives the counterintuitive result that closure under the ω-rule does not necessarily give rise to non constructive λ-theories, thus solving a problem of A. Cantini (see [3]).

Then we have considered the ω-rule with respect to the highly non constructive λ-theory $\mathcal{H}$. The theory $\mathcal{H}$ is obtained extending β-conversion by identifying all closed unsolvables. $\mathcal{H}\omega$ is the closure of this theory under the ω-rule (and β-conversion). A long-standing conjecture of H. Barendregt ([1], Conjecture 17.4.15) stated that the provable equations of $\mathcal{H}\omega$ form a $\mathbf{\Pi^1_1}$-Complete set. In [8], we solved in the affirmative the problem.

Of course the most important problem is to determine the logical power of ω-rule when added to the *pure* $\lambda\beta$-calculus.

As in [1], we call $\lambda\omega$ the theory that results from adding the ω-rule to the *pure* $\lambda\beta$-calculus. In [6], we showed that the $\lambda\omega$ is *not* recursively enumerable, by giving a many-one

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reduction of the set of true $\mathbf{\Pi^0_2}$ sentences to the set of closed equalities provable in $\lambda\omega$, thus solving a problem originated with H. Barendregt and re-raised in [4].

The problem of the logical upper bound to $\lambda\omega$ remained open. That this bound is $\mathbf{\Pi^1_1}$ has been conjectured again by H. Barendregt in the well known Open Problems List, which ends the 1975 Conference on "λ-Calculus and Computer Science Theory", edited by C. Böhm [2]. Here we solve in the affirmative this conjecture. The celebrated *Plotkin terms* (introduced in [10]) furnish the main technical tool.

0.1. **Remarks on the Structure of the Proof.** The present paper is a revised and improved version of [9]. It is self-contained, with the exception of some specific points where we use results and methods from [6]. Such points will be precisely indicated in Section 3 and in Section 4. The authors are working to a comprehensive formalism to give a unified presentation of all the results. At present, however, this could not have been done without great complications.

To help the reader, we now describe in an informal way the general idea of the proof.

As already for the result in [6], the proof relies on suitable modifications of the mentioned Plotkin terms. Roughly speaking, Plotkin's construction gives rise, in the usual $\lambda\beta\eta$-calculus, to pairs of closed terms P_0 and P_1 such that for every closed term M, P_0M and P_1M are $\beta\eta$-convertible. On the other hand, P_0 and P_1 are *not* themselves $\beta\eta$-convertible (see [1], 17.3.26).

When we add the ω-rule to the $\lambda\beta$-calculus, such terms - suitably modified - become a way to express various forms of *universal quantification.* Intuitively, P_0 and P_1 are equal if and only if *for all* M belonging to some given set of terms, P_0M and P_1M are equal.

There are two points that must be stressed.

- First, different quantifiers require different specific constructions of suitable Plotkin terms.
- Second, to properly use equality between P_0 and P_1 as a test for quantification, one must exclude that $P_0M = P_1M$ holds for some M not belonging to the set of interest.

Focusing on the second problem, the technical tool that we have used - both in [6] and in the present paper - is to cast proofs in the $\lambda\beta$-calculus with the ω-rule, in some kind of "normal form". (Observe that, in presence of the ω-rule, proofs become infinitary objects.) In particular as "normal form" for proofs, we have used in [6] the notion of *cascaded proof.* Here we use the notion of *canonical proof* introduced in Section 2. In both cases, the intuitive idea is to extensively use the ω-rule to limit the use of β-reductions. This makes the behavior of the (various) Plotkin terms more controllable, which, in turn, makes the mentioned problem solvable. It turns out that one cannot use a unique "normal form", or at least we were not able to do this. In particular, observe that we need terms for two kinds of quantifier:

- Arithmetical quantification over recursive enumerable sets of terms.
- One second order universal quantification to express $\mathbf{\Pi^1_1}$-complete problems.

Different kinds of terms are used to express, *via* their equality, the two kinds of quantification. So in the present paper we concentrate on *canonical proofs.* This kind of "normal form" is suitable to cope with terms whose equality is used to express a $\mathbf{\Pi^1_1}$-complete problem. It is not suitable, however, to properly control the behavior of terms related to first-order quantification. For such terms, we rely on the methods used in [6] for the analysis of *cascaded proofs.* The general scheme of the proof is as follows:

- In Section 2, we introduce the notion of *canonical proof* and prove that every provable equality has a canonical proof.
- In Section 3, we introduce suitable Plotkin terms to express quantification over Church numerals.
- In Section 4, we introduce suitable Plotkin terms to express second order quantification on sequences of numbers to reduce the $\mathbf{\Pi_1^1}$-complete problem of well-foundedness of recursive trees to equality of terms in the $\lambda\beta$-calculus with the ω-rule.

1. The ω-rule

Notation will be standard and we refer to [1], for terminology and results on λ-calculus. In particular:

- $\equiv$ denotes syntactical identity;
- $\longrightarrow_\beta$, $\longrightarrow_\eta$ and $\longrightarrow_{\beta\eta}$ denote β-, η- and, respectively, $\beta\eta$-reduction and $\longrightarrow^*_\beta$, $\longrightarrow^*_\eta$ and $\longrightarrow^*_{\beta\eta}$ their respective reflexive and transitive closures;
- $=_\beta$ and $=_{\beta\eta}$ denote β- and, respectively, $\beta\eta$-conversion;
- combinators (i.e. closed λ-terms) such e.g. $\mathbf{I}$ have the usual meaning;
- $\underline{k}$ denotes the kth Church numeral.

λ-terms are denoted by capital letters: in particular we adopt the convention that F, G, H, J, M, N, P, $Q, \ldots$ are *closed* terms and U, V, X, Y, W, Z are possibly *open* terms.

For a λ-term the notion of *having order* 0 has the usual meaning ([1] 17.3.2). We shall also call *zero-term* a term of order 0. As usual, we say that a term has *positive order* if it is not of order zero. We shall refer to a β-reduction performed not within the scope of a λ as *a weak β-reduction.* In the sequel, we shall need the following notions. We define the notions of *trace* and *extended trace* (shortly *etrace*) as follows. Given the reduction $F \longrightarrow^*_{\beta\eta} G$ and the closed subterm M of F the *traces* of M in the terms of the reduction are simply the copies of M until each is either deleted by a contraction of a redex with a dummy λ or altered by a reduction internal to M or by a reduction with M at the head (when M begins with λ). The notion of *etrace* is the same except that we allow internal reductions, so that a copy of M altered by an internal reduction continues to be an etrace.

By $\lambda\beta$ we denote the theory of β-convertibility (see [1]). The theory $\lambda\omega$ is obtained by adding the so called *ω-rule* to $\lambda\beta$, see [1] 4.1.10.

We formulate $\lambda\omega$ slightly differently. In particular, we want a formulation of the theory such that only equalities between *closed* terms can be proven. Moreover it will be convenient to use $\beta\eta$-conversion. The so called η-rule (that is $(\lambda x.Mx) =_\eta M$) obviously holds in $\lambda\omega$. Nevertheless it will be useful to have this rule at disposal to put proofs in some specified forms.

Definition 1.1. *Equality in $\lambda\omega$* (denoted by $=_\omega$) is defined by the following rules:

- $\beta\eta$-conversion:
$$\text{if } M =_{\beta\eta} N \text{ then } M =_\omega N$$
- the rule of substituting equals for equals in the form:
$$\text{if } M =_\omega N \text{ then } PM =_\omega PN$$
- transitivity and symmetry of equality,

• the ω-rule itself:
$$\frac{\forall M,\ M\ closed,\ PM =_\omega QM}{P\ =_\omega\ Q}$$

We leave to the reader to check that the formulation above is equivalent to the standard one (see Chapter 4 of [1]).

As usual proofs in $\lambda\omega$ can be thought of as (possibly infinite) well-founded trees. In particular the tree of a proof either ends with an instance of the ω-rule or has an end piece consisting of a finite tree of equality inferences all of whose leaves are either $\beta\eta$ conversions or direct conclusions of the ω-rule. It is easy to see that each such endpiece can be put in the form:

$$F =_{\beta\eta} G_1M_1 =_\omega G_1N_1 =_{\beta\eta} G_2M_2 =_\omega G_2N_2 =_{\beta\eta} \ldots G_tM_t =_\omega G_tN_t =_{\beta\eta} H$$

where $M_i =_\omega N_i$, for $1 \le i \le t$ are direct conclusions of the ω-rule. See [8], Section 5, for more details. While the context is slightly different, the argument is *verbatim* the same. This is a particular case of a general result due to the second author of the present paper, see [12]. Moreover, by the Church-Rosser Theorem this configuration of inferences can be put in the form

$$\begin{gathered} F \longrightarrow^*_{\beta\eta} J_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1M_1 =_\omega G_1N_1 \longrightarrow^*_{\beta\eta} J_2 \ {}^*_{\beta\eta}\!\longleftarrow \\ {}^*_{\beta\eta}\!\longleftarrow G_2M_2 =_\omega G_2N_2 =_{\beta\eta}\longrightarrow^*_{\beta\eta} \ldots \\ \ldots \ {}^*_{\beta\eta}\!\longleftarrow G_tM_t =_\omega G_tN_t \longrightarrow^*_{\beta\eta} J_{t+1} \ {}^*_{\beta\eta}\!\longleftarrow H \end{gathered} \tag{1.1}$$

where $M_i =_\omega N_i$, for $1 \le i \le t$, are as above. We shall call the sequence (1.1) *the standard form* for the endpiece of a proof.

Since proofs are infinite trees (denoted by symbols $\mathcal{T}$, $\mathcal{T}'$ etc.), they can be assigned countable ordinals. We shall need a few facts about countable ordinals, that we briefly mention in the following. For the basic notions on countable ordinals, see e.g. [11].

(a) Cantor Normal Form to the Base Omega (ω)
Every countable ordinal α can be written uniquely in the form $\omega^{\alpha_1} * n_1 + \cdots + \omega^{\alpha_k} * n_k$ where $n_1, \ldots, n_k$ are positive integers and $\alpha_1 > \cdots > \alpha_k$ are ordinals.

(b) Hessenberg Sum
Write $\alpha = \omega^{\alpha_1} * n_1 + \cdots + \omega^{\alpha_k} * n_k$ and $\gamma = \omega^{\alpha_1} * m_1 + \cdots + \omega^{\alpha_k} * m_k$ where some of the n_i and m_j may be 0. Then the *Hessenberg Sum* is defined as follows: $\alpha \oplus \gamma =_{def} \omega^{\alpha_1} * (n_1 + m_1) + \cdots + \omega^{\alpha_k} * (n_k + m_k)$.

Hessenberg sum is strictly increasing on both arguments. That is, for α, γ different from 0, we have: $\alpha, \gamma < \alpha \oplus \gamma$.

(c) Hessenberg Product
We only need this for product with an integer. We put: $\alpha \odot n =_{def} \alpha \oplus \cdots \oplus\ \alpha$ n-times.

Coming back to proofs, observe first that we can assume that if a proof has an endpiece, then this endpiece is in standard form (see above). The ordinal that we want to assign to a proof $\mathcal{T}$ (considered as a tree) is the transfinite ordinal $\mathrm{ord}(\mathcal{T})$, *the order of* $\mathcal{T}$, defined recursively by

Definition 1.2.

- If $\mathcal{T}$ ends in an endpiece computation of the form (1.1) with no instances of the ω-rule ($t = 0$), that is consisting of a unique $\beta\eta$-conversion, then $\mathrm{ord}(\mathcal{T}) =_{def} 1$;
- If $\mathcal{T}$ ends in an instance of the ω-rule whose premises have trees resp. $\mathcal{T}_1, \dots \mathcal{T}_i, \dots$ then $\mathrm{ord}(\mathcal{T}) =_{def} \omega^\theta$, with $\theta = Sup\{\mathrm{ord}(\mathcal{T}_1) \oplus \cdots \oplus \mathrm{ord}(\mathcal{T}_i) : i = 1, 2, \dots\}$;
- If $\mathcal{T}$ ends in an endpiece computation of the form (1.1), with $t > 0$ instances of the ω-rule, and the t premises $M_1 =_\omega N_1, \dots, M_t =_\omega N_t$ have resp. trees $\mathcal{T}_1, \dots, \mathcal{T}_t$ then $\mathrm{ord}(\mathcal{T}) =_{def} 1 \oplus \mathrm{ord}(\mathcal{T}_1) \oplus \mathrm{ord}(\mathcal{T}_2) \cdots \oplus \mathrm{ord}(\mathcal{T}_t)$.

Here $\oplus$ is the Hessenberg sum of ordinals defined above.

We shall need also the following notion.

Definition 1.3. If $\mathcal{T}$ ends in an endpiece computation of the form (1.1), with $t > 0$ instances of the ω-rule, and the t premises $M_1 =_\omega N_1, \dots, M_t =_\omega N_t$ have resp. trees $\mathcal{T}_1, \dots, \mathcal{T}_t$ then $rank(\mathcal{T})$, the *rank* of $\mathcal{T}$, is the maximum of $\mathrm{ord}(\mathcal{T}_1)$, $\mathrm{ord}(\mathcal{T}_2)$, $\dots$, $\mathrm{ord}(\mathcal{T}_t)$.

We need the following propositions.

Proposition 1.4. *If $\mathcal{T}$ ends in an endpiece computation of the form (1.1), with $t > 0$, and the equations $M_1 =_\omega N_1, \dots, M_t =_\omega N_t$, have resp. trees $\mathcal{T}_1, \dots, \mathcal{T}_t$ then $\mathrm{ord}(\mathcal{T}) > \mathrm{ord}(\mathcal{T}_i)$, for each $i = 1, \dots, t$.*

Proof. $\mathrm{ord}(\mathcal{T}_i) > 0$ and $\oplus$ is strictly increasing on its arguments. □

Proposition 1.5. *Assume that $\mathcal{T}$ ends in an instance of the ω-rule whose premises have, respectively, trees $\mathcal{T}_1, \dots, \mathcal{T}_t, \dots$ Then for any integers $t, n_1, \dots, n_t$,*

$$\mathrm{ord}(\mathcal{T}) > \mathrm{ord}(\mathcal{T}_1) \odot n_1 \oplus \cdots \oplus \mathrm{ord}(\mathcal{T}_t) \odot n_t \ .$$

Proof. Let $\mathrm{ord}(\mathcal{T}_i) = \alpha_i$, for $1 \le i \le t$ and put all $\alpha_1, \dots, \alpha_t$ into Cantor normal form:

$$\alpha_1 = \omega^{\beta_1} * n_{11} + \cdots + \omega^{\beta_k} * n_{1k} \quad \dots \quad \alpha_t = \omega^{\beta_1} * n_{t1} + \cdots + \omega^{\beta_k} * n_{tk} \ .$$

Let $n = \max\{n_r, n_{ij}\} + 1$, with $j, r = 1 \dots t$ and $i = 1 \dots k$. Then

$$\begin{aligned} \mathrm{ord}(\alpha_1) \odot n_1 \oplus \cdots \oplus \mathrm{ord}(\alpha_t) \odot n_t &< \mathrm{ord}(\alpha_1) \odot n \oplus \cdots \oplus \mathrm{ord}(\alpha_t) \odot n \\ &= (\alpha_1 \oplus \cdots \oplus \alpha_t) \odot n \\ &\le \omega^{\beta_1} * n * k * t * n \ . \end{aligned}$$

Now let $\theta = Sup\{\mathrm{ord}(\mathcal{T}_1) \oplus \cdots \oplus \mathrm{ord}(\mathcal{T}_i) : i = 1, 2, \dots\}$. We have $\omega^{\beta_1} < \theta \le \omega^\theta = \mathrm{ord}(\mathcal{T})$. But $\mathrm{ord}(\mathcal{T})$ is a countable ordinal of the form ω^γ and is thus closed under addition. Hence $\omega^{\beta_1} * n * k * t * n < \mathrm{ord}(\mathcal{T})$. This ends the proof. □

Remark 1.6. Since for proofs $\mathcal{T}$ we shall mainly use $\mathrm{ord}(\mathcal{T})$, we sometimes refer to $\mathrm{ord}(\mathcal{T})$ simply as *the ordinal* of the proof $\mathcal{T}$.

2. Canonical Proofs

We want to show that proofs in $\lambda\omega$ can be set in a suitable form.

Definition 2.1. We say that M has *the same form as* N iff

- in case of $N \equiv \lambda y_1 \dots y_n.YL_1 \dots L_m$, where Y begins with λ, we have
  - $M \equiv \lambda y_1 \dots y_n.ZP_1 \dots P_m$, where Z begins with λ,

  - $\lambda y_1 \ldots y_n.Y =_\omega \lambda y_1 \ldots y_n.Z$,
  - and for every i with $1 \leq i \leq m$,

$$\lambda y_1 \ldots y_n.L_i =_\omega \lambda y_1 \ldots y_n.P_i \ ,$$

    where possibly $n = 0$;
- in case of $N \equiv \lambda y_1 \ldots y_n.y_j L_1 \ldots L_m$, we have
  - $M \equiv \lambda y_1 \ldots y_n.y_j P_1 \ldots P_m$,
  - and for every i, with $1 \leq i \leq m$,

$$\lambda y_1 \ldots y_n.L_i =_\omega \lambda y_1 \ldots y_n.P_i \ .$$

Recall that a set $\mathcal{X}$ of closed terms, is *cofinal* for $\beta\eta$-reductions, if every closed term M has a $\beta\eta$-reduct in $\mathcal{X}$.

Definition 2.2. We say that a set $\mathcal{X}$ of closed terms is *supercofinal* if it is cofinal and contains all the terms that do not reduce to a zero-term.

Remark 2.3. In the previous Definition, observe that, due to the cofinality of $\mathcal{X}$, if a term reduces to a zero-term then it reduces to a zero-term which is in $\mathcal{X}$.

In the following, let $\mathcal{X}$ be a specified supercofinal set.

Definition 2.4. An endpiece in standard form

$$\begin{aligned} F \longrightarrow^*_{\beta\eta} H_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1 M_1 =_\omega G_1 N_1 \longrightarrow^*_{\beta\eta} H_2 \ {}^*_{\beta\eta}\!\longleftarrow \\ {}^*_{\beta\eta}\!\longleftarrow G_2 M_2 =_\omega G_2 N_2 =_{\beta\eta}\longrightarrow^*_{\beta\eta} \cdots \\ \ldots \ {}^*_{\beta\eta}\!\longleftarrow G_t M_t =_\omega G_t N_t \longrightarrow^*_{\beta\eta} H_{t+1} \ {}^*_{\beta\eta}\!\longleftarrow F' \end{aligned} \tag{2.1}$$

is called *an $\mathcal{X}$-canonical endpiece* (or, when $\mathcal{X}$ is clear from the context, simply *a canonical endpiece*) iff

(1) for every i, $i = 1, \ldots, t+1$, the confluence terms H_i belong to $\mathcal{X}$;
(2) for every i, $i = 1, \ldots, t+1$, there exist terms Y, $L_1, \ldots, L_m$ $Z_1 \ldots Z_n$ (possibly different for different i) such that G_i has the form

$$G_i \equiv \lambda x.\lambda y_1 \ldots y_n.((\lambda y.Y)L_1 \ldots L_m)Z_1 \ldots Z_n \ ,$$

and such that the following holds:
   (a) (Conditions on the Left Facing Arrows)
   for every i, $i = 1, \ldots, t$, the sequence of left reductions

$$H_i \ {}^*_{\beta\eta}\!\longleftarrow G_i M_i$$

   has the following structure:
   - a one step β-reduction of the form

$$[M_i/x](\lambda y_1 \ldots y_n.((\lambda y.Y)L_1 \ldots L_m)Z_1 \ldots Z_n) \ {}_\beta\!\longleftarrow G_i M_i$$

   - followed by a sequence of non-head β-reductions,
   - followed by a sequence of η-reductions.
   (b) (Condition on the Right Facing Arrows)
   for every i, $i = 1, \ldots, t$, the sequence of right reductions

$$G_i N_i \longrightarrow^*_{\beta\eta} H_{i+1}$$

has the following structure

$$\begin{aligned}
G_iN_i &\longrightarrow^*_{\beta\eta} [N_i/x](\lambda y_1 \dots y_n.((\lambda y.Y)L_1 \dots L_m)Z_1 \dots Z_n) \\
&\longrightarrow^*_{\beta\eta} \lambda y_1 \dots y_n.[N_i/x](\lambda y.Y)[N_i/x]L_1 \dots [N_i/x]L_m)y_1 \dots y_n) \\
&\longrightarrow^*_{\eta} [N_i/x](\lambda y.Y)[N_i/x]L_1 \dots [N_i/x]L_m \\
&\longrightarrow^*_{\beta\eta} J_0J_1 \dots J_m \longrightarrow^*_{\beta\eta} H_{i+1}
\end{aligned}$$

where

$$J_0 \equiv \begin{cases} \text{the } \beta\eta\text{-normal form of } [N_i/x]\lambda y.Y & \text{if exists;} \\ [N_i/x]\lambda y.Y & \text{otherwise.} \end{cases}$$

and for $k = 1, \dots, m$

$$J_k \equiv \begin{cases} \text{the } \beta\eta\text{-normal form of } [N_i/x]L_k & \text{if exists;} \\ [N_i/x]L_k & \text{otherwise.} \end{cases}$$

In the following definition, recall that an endpiece can be considered as a finite tree of equality inferences.

Definition 2.5. Given the supercofinal set $\mathcal{X}$, the notion of $\mathcal{X}$*-canonical proof* is defined inductively as follows.

- A $\beta\eta$-conversion is $\mathcal{X}$*-canonical* if the confluence term belongs to $\mathcal{X}$.
- An instance of the ω-rule is $\mathcal{X}$*-canonical* if the proofs of the premisses of the instances are $\mathcal{X}$-canonical.
- Otherwise a proof is *canonical* if its endpiece is an $\mathcal{X}$-canonical endpiece and all the proofs of the leaves which are direct conclusions of the ω-rule are $\mathcal{X}$-canonical.

Proposition 2.6. *For every supercofinal set $\mathcal{X}$, every provable equality $M =_\omega N$ has an $\mathcal{X}$-canonical proof.*

Proof. Let $\mathcal{X}$ be fixed. We prove this proposition by induction on the ordinal ord$(\mathcal{T})$ of a proof $\mathcal{T}$ of $M =_\omega N$. For the basis case just suppose that $M =_{\beta\eta} N$ and use the Church-Rosser theorem.

For the induction step we distinguish two cases.

First Case. $M =_\omega N$ is the direct conclusion of the ω-rule. This follows directly from the induction hypothesis.

Second Case 2. $\mathcal{T}$ has an endpiece of the form

$$\begin{aligned}
M \longrightarrow^*_{\beta\eta} H_1 \; {}^*_{\beta\eta}\!\!\longleftarrow G_1M_1 =_\omega G_1N_1 \longrightarrow^*_{\beta\eta} H_2 \; {}^*_{\beta\eta}\!\!\longleftarrow \\
{}^*_{\beta\eta}\!\!\longleftarrow G_2M_2 =_\omega G_2N_2 =_{\beta\eta}\longrightarrow^*_{\beta\eta} \dots \\
\dots \; {}^*_{\beta\eta}\!\!\longleftarrow G_tM_t =_\omega G_tN_t \longrightarrow^*_{\beta\eta} H_{t+1} \; {}^*_{\beta\eta}\!\!\longleftarrow N
\end{aligned} \tag{2.2}$$

where, for each $i = 1 \dots t$, $M_i =_\omega N_i$ is the conclusion of an instance of the ω-rule.

Observe that, without changing the ordinal of the proof, we can assume that every H_i, with $1 \le i \le t+1$, is in $\mathcal{X}$.

Consider the first component of the endpiece (2.2)

$$M \longrightarrow^*_{\beta\eta} H_1 \; {}^*_{\beta\eta}\!\!\longleftarrow G_1M_1 =_\omega G_1N_1$$

Let σ be a standard $\beta\eta$-reduction $G_1M_1 \longrightarrow^*_{\beta\eta} H_1$, with all the η-reductions postponed. We have now different subcases.

First Subcase. No etrace of M_1 appears in functional position in a head redex neither in the head part of σ, nor in H_1 itself (that is H_1 has not a head redex of the form $(\lambda x.U)V$, with $\lambda x.U$ an etrace of M_1).

In this case, the same head reductions can be performed (up to a substitution of M_1 by N_1) in the G_1N_1 side. Thus simply replacing G_1, we may freely assume that this head part is missing at all and thus σ is composed only of non-head β-reductions followed by η-reductions. Moreover, by our hypothesis, we can also assume that G_1 has not the form: $\lambda xy_1 \dots y_p.\ xY_1 \cdots Y_q$.

Moreover we can also assume that G_1 begins with a λ. For otherwise, assume that in the head part of σ, a λ never appears at the beginning of the reducts of G_1. Therefore all the reduction σ is internal to G_1 and M_1, and this implies that H_1 has the form $G'_1M'_1$, where G_1 $\beta\eta$-reduces to G'_1 and M_1 $\beta\eta$-reduces to M'_1, respectively. Thus, replacing G_1 with $\lambda x.G'_1x$, we obtain a term of the required form.

On the G_1N_1 side, the *Conditions on the Right Facing Arrows* may require a reduction of G_1N_1 to a suitable term H^+.

By the Church-Rosser Theorem and the cofinality of $\mathcal{X}$, let $\overline{H}$ be a term in $\mathcal{X}$, which is a common reduct of H^+ and H_2. Now, there exists a proof $\mathcal{T}'$ of $\overline{H} =_\omega N$, with $\mathrm{ord}(\mathcal{T}') < \mathrm{ord}(\mathcal{T})$ (where N is the final term of the endpiece (2.2)). Thus by induction hypothesis there exists a canonical proof $\mathcal{T}_1$ of $\overline{H} =_\omega N$. Now, the required canonical proof is obtained by concatenating the component

$$M \longrightarrow^*_{\beta\eta} H_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1M_1 =_\omega G_1N_1 \longrightarrow^*_{\beta\eta} \overline{H}\ ;$$

with $\mathcal{T}_1$.

That this concatenation results in a canonical proof can be easily checked in case $\mathcal{T}_1$ ends in an instance of the ω-rule as well as in case $\mathcal{T}_1$ ends in an endpiece.

Second Subcase. Assume that:

- an etrace of M_1 appears in functional position in a head redex of the head part of σ, or in H_1 itself;
- a λ appears at the beginning of some term in the head part of σ.

Thus we have $G_1M_1 \longrightarrow^*_{\beta\eta} \lambda u.U \longrightarrow^*_{\beta\eta} H_1$, for some U. For any closed term R, consider the reduction:

$$G_1M_1R \longrightarrow^*_{\beta\eta} (\lambda u.U)R \longrightarrow_{\beta\eta} [R/u]U \longrightarrow^*_{\beta\eta} H'\ .$$

Here H' is $[R/u]H_1$. This can be done for every λ appearing in the head part of σ. Thus for each choice of closed $R_1 \dots R_n$ we have a standard $\beta\eta$-reduction σ' of $G_1M_1R_1 \cdots R_n$ to a term H'', which is H_1 with each abstracted variable u_j substituted by the corresponding closed term R_j (unless this variable has been eliminated by η-reduction: in this case the resulting term is applied to R_j).

Now, being $\mathcal{X}$ supercofinal, either H'' is in $\mathcal{X}$ or H'' $\beta\eta$-reduces to a zero-term H^0 in $\mathcal{X}$, by a reduction σ''. In this reduction some new λ may appear at the beginning of the term (since we have also η-reductions), and we treat this λ as before, by applying all the terms in the reduction some other R. Thus we extend the sequence $R_1 \dots R_n$ to a new sequence $R_1 \dots R_n, R'_1 \dots R'_m$. Since H^0 is a zero-term all the external λ appearing in σ'' are eventually eliminated by η-reductions. Therefore, starting from $H''R'_1 \cdots R'_m$ and applying the reductions in σ'', we obtain the term $H^0R'_1 \cdots R'_m$. Now $H^0R'_1 \cdots R'_m$ is a zero-term, so that if it is not in $\mathcal{X}$, the reduction to a suitable term in $\mathcal{X}$ adds no new λs at the beginning of the term. So, without loss of generality we can assume that H'' is in $\mathcal{X}$, and that σ' is a standard $\beta\eta$-reduction of $G_1M_1R_1\ldots R_n$ to H'', such that no term in the head part of σ' begins with λ.

Now in the head reduction part of σ', we come to a term V with a head redex of the form: $(\lambda u.W)U$, where $M_1 \longrightarrow^*_{\beta\eta} \lambda u.W$. Let $V \equiv (\lambda u.W)UU_1\cdots U_v$, we write V in the form $(\lambda u.W)[V_1/x_1,\ldots,V_{r_1}/x_{r_1}]\overrightarrow{X}_1$, showing all the etraces $V_1,\ldots,V_{r_1}$ of M_1 in V. Then

$$M_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 =_\omega N_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 \qquad (*)$$

has a proof with ordinal (much) less than $\mathrm{ord}(\mathcal{T})$. Now, consider the component

$$MR_1\ldots R_n \longrightarrow^*_{\beta\eta} H'' \;{}^*_{\beta\eta}\!\longleftarrow M_1[M_1/x_1,\ldots,M_1/x_{r_1}]\overrightarrow{X}_1 =_\omega M_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1$$

The reduction $M_1[M_1/x_1,\ldots,M_1/x_{r_1}]\overrightarrow{X}_1 \longrightarrow^*_{\beta\eta} H''$ has a head part shorter than σ'. Thus, iterating the previous transformation for each occurrence M_1 in functional position in the head reduction part of σ', we arrive to a final sequence of terms $\overrightarrow{X}_s$ such that $M_1[M_1/x_1,\ldots,M_1/x_{r_s}]\overrightarrow{X}_s$ is the last such occurrence of M_1. Therefore, for what concerns the component

$$MR_1\ldots R_n \longrightarrow^*_{\beta\eta} H'' \;{}^*_{\beta\eta}\!\longleftarrow M_1[M_1/x_1,\ldots,M_1/x_{r_s}]\overrightarrow{X}_s =_\omega=_\omega M_1[N_1/x_1,\ldots,N_1/x_{r_s}]\overrightarrow{X}_s$$

we can argue as in the First Subcase above.

On the right hand side, observe that the iteration of the previous argument gives rise to a chain of equalities (where for simplicity, we do not consider reduction internal to M_1; this does not affect the argument)

$$\begin{aligned}
&N_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 =_\omega NR_1\ldots R_n\\
&N_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 =_\omega M_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1\\
&M_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 =_\omega M_1[M_1/x_1,\ldots,M_1/x_{r_1}]\overrightarrow{X}_1\\
&M_1[M_1/x_1,\ldots,M_1/x_{r_1}]\overrightarrow{X}_1 \longrightarrow^*_{\beta\eta} M_1[M_1/x_1,\ldots,M_1/x_{r_2}]\overrightarrow{X}_2\\
&M_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_1 \longrightarrow^*_{\beta\eta} N_1[N_1/x_1,\ldots,N_1/x_{r_2}]\overrightarrow{X}_2\\
&M_1[M_1/x_1,\ldots,M_1/x_{r_2}]\overrightarrow{X}_2 =_\omega M_1[N_1/x_1,\ldots,N_1/x_{r_2}]\overrightarrow{X}_2\\
&M_1[N_1/x_1,\ldots,N_1/x_{r_2}]\overrightarrow{X}_2 =_\omega N_1[N_1/x_1,\ldots,N_1/x_{r_2}]\overrightarrow{X}_2\\
&\ldots\\
&M_1[N_1/x_1,\ldots,N_1/x_{r_s}]\overrightarrow{X}_s =_\omega N_1[N_1/x_1,\ldots,N_1/x_{r_1}]\overrightarrow{X}_s
\end{aligned}$$

From this chain, by Proposition 1.5 of Section 1, one obtains a proof of

$$M_1[N_1/x_1,\ldots,N_1/x_{r_s}]\overrightarrow{X}_s =_\omega NR_1\ldots R_n\ ,$$

with an ordinal less than $\mathrm{ord}(\mathcal{T})$. We can also substitute $M_1[N_1/x_1,\ldots,N_1/x_{r_s}]\overrightarrow{X}_s$ with a suitable reduct $\overline{H}$, meeting both the *Conditions on the Right Facing Arrows* w.r.t. $M_1[N_1/x_1,\ldots,N_1/x_{r_s}]\overrightarrow{X}_s$ and the cofinality condition w.r.t. $\mathcal{X}$. Still, $\overline{H} =_\omega NR_1\ldots R_n$ has a proof with ordinal less than $\mathrm{ord}(\mathcal{T})$. Thus by induction hypothesis there exists a canonical proof $\mathcal{T}_1$ of $\overline{H} =_\omega NR_1\ldots R_n$.

Now, we can concatenate the component

$$MR_1 \dots R_n \longrightarrow^*_{\beta\eta} H'' \ {}^*_{\beta\eta}\!\longleftarrow (\lambda x.M_1[x/x_1, \dots, x/x_{r_s}]\overrightarrow{X}_s)M_1 =_\omega$$
$$=_\omega (\lambda x.M_1[x/x_1, \dots, x/x_{r_s}]\overrightarrow{X}_s)N_1 \longrightarrow^*_{\beta\eta} \overline{H} \ ;$$

with $\mathcal{T}_1$. That this concatenation results in a canonical proof can be easily checked in case $\mathcal{T}_1$ ends in an instance of the ω-rule as well as in case $\mathcal{T}_1$ ends in an endpiece.

Thus we have proved the following:

for every $R_1 \dots R_n$, there exists a canonical proof of $MR_1 \dots R_n =_\omega NR_1 \dots R_n$.

Now, n applications of the ω-rule give the required canonical proof of $M =_\omega N$.

Third Subcase.

- an etrace of M_1 appears in functional position in a head redex of the head part of σ, or in H_1 itself;
- no λ appears at the beginning of some term in the head part of σ.

This case can be treated as the previous one, with the difference that the resulting canonical proof ends in a canonical endpiece, rather than in an instance of the ω-rule. □

We shall need the following result on $\mathcal{X}$-canonical proofs.

Proposition 2.7. *Let $\mathcal{T}$ be an $\mathcal{X}$-canonical proof of $M =_\omega N$ ending in an endpiece. Then for every sequence of terms $P_1, \dots, P_m$, there exist terms $R_1, \dots, R_n$ such that the equality $MP_1 \cdots P_m R_1 \cdots R_n =_\omega NP_1 \cdots P_m R_1 \cdots R_n$ has an $\mathcal{X}$-canonical proof $\mathcal{T}_1$, also ending in an endpiece, with $rank(\mathcal{T}_1) = rank(\mathcal{T})$.*

Proof. Assume that $\mathcal{T}$ has an endpiece of the form:

$$\begin{aligned} M \longrightarrow^*_{\beta\eta} H_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1M_1 =_\omega G_1N_1 \longrightarrow^*_{\beta\eta} H_2 \ {}^*_{\beta\eta}\!\longleftarrow \\ {}^*_{\beta\eta}\!\longleftarrow G_2M_2 =_\omega G_2N_2 =_{\beta\eta}\longrightarrow^*_{\beta\eta} \dots \\ \dots \ {}^*_{\beta\eta}\!\longleftarrow G_tM_t =_\omega G_tN_t \longrightarrow^*_{\beta\eta} H_{t+1} \ {}^*_{\beta\eta}\!\longleftarrow N \end{aligned} \tag{2.3}$$

where, for each $i = 1 \dots t$, $M_i =_\omega N_i$ is the conclusion of an instance of the ω-rule.

We argue by induction on t. Assume $t = 1$. Consider the first (and unique) component of the endpiece (2.3)

$$M \longrightarrow^*_{\beta\eta} H_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1M_1 =_\omega G_1N_1$$

Let $P_1, \dots, P_m$ be given. We have two cases.

First Case. $H_1P_1 \cdots P_m$ is in $\mathcal{X}$. In this case, the component can directly be transformed into a component of the right form, using the equality $(\lambda x.(G'_1[x]P_1 \cdots P_m))M_1 =_\omega ((\lambda x.G'_1[x]P_1 \cdots P_m))N_1$, where the applicative context $G'_1[\,]$ is $G_1\ [\,]$, that is G_1 applied to the hole $[\,]$.

Second Case. $H_1P_1 \cdots P_m$ is not in $\mathcal{X}$. In this case, $H_1P_1 \cdots P_m$ reduces to a suitable zero-term H' in $\mathcal{X}$. To obtain a component of the right form, we have to transform $(\lambda x.(G'_1[x]P_1 \cdots P_m))M_1$ as in the proof of the previous proposition. This can be done - as shown in the second subcase of such proof - at the cost (in the worst case) of applying $(\lambda x.(G'_1[x]P_1 \cdots P_m))M_1$ to a sequence $R_1, \dots, R_n$ of terms and introducing some additional leaves each one of ordinal not greater than the one of $M_1 =_\omega N_1$.

Hence the result follows for $t = 1$. Now assume $t > 1$. Let $P_1, \dots, P_m$ be given. By induction hypothesis, for some $R_1, \dots, R_n$ there is a proof with an endpiece of rank less or equal to $rank(\mathcal{T})$ of $G_1N_1P_1\cdots P_mR_1\cdots R_n =_\omega NP_1\cdots P_mR_1\cdots R_n$. Now consider the first component of the endpiece (2.3)

$$M \longrightarrow^*_{\beta\eta} H_1 \ {}^*_{\beta\eta}\!\longleftarrow G_1M_1 =_\omega G_1N_1$$

Again we have two cases.

First Case. $H_1P_1\cdots P_mR_1\cdots R_n$ is in $\mathcal{X}$. In this case, the component can directly be transformed into a component of the right form, using the equality

$$(\lambda x.(G'_1[x]P_1\cdots P_mR_1\cdots R_n))M_1 =_\omega (\lambda x.(G'_1[x]P_1\cdots P_mR_1\cdots R_n))N_1$$

for a suitable applicative context $G'_1[\,]$.

Second Case. $H_1P_1\cdots P_mR_1\cdots R_n$ is not in $\mathcal{X}$. In this case, $H_1P_1\cdots P_mR_1\cdots R_n$ reduces to a suitable zero-term H' in $\mathcal{X}$. To obtain a component of the right form, we have to transform $(\lambda x.G'_1[x]P_1\cdots P_m)M_1$ as in the proof of the previous proposition. This can be done - as shown in the second subcase of such proof - at the cost (in the worst case) of applying $(\lambda x.(G'_1[x]P_1\cdots P_mR_1\cdots R_n))M_1$ to a sequence $R'_1,\ldots,R'_k$ of terms and introducing some additional leaves each one of ordinal not greater than the one of $M_1 =_\omega N_1$.

Now again by induction hypothesis there exist $R''_1,\ldots,R''_s$ such that there is a proof with an endpiece of rank less or equal to $rank(\mathcal{T})$ of

$$G_1N_1P_1\cdots P_mR_1\cdots R_nR'_1\cdots R'_kR''_1\cdots R''_s =_\omega NP_1\cdots P_mR_1\cdots R_nR'_1\cdots R'_kR''_1\cdots R''_s\ .$$

Observe now that since $H'R'_1\cdots R'_k$ is a zero-term, we can obtain a term in $\mathcal{X}$ which is a reduct of $H'R'_1\cdots R'_kR''_1\cdots R''_s$ without introducing new λs but (possibly) only other leaves each one of ordinal not greater than the one of $M_1 =_\omega N_1$.

So in both cases, the result follows. □

3. Plotkin Terms

Recall that H, M, N, P, Q always denote closed terms. Let $\lceil M\rceil$ denote the Church numeral corresponding to the Gödel number of the term M. We can of course require that any term occurs infinitely many times (up to ω-equality) in the enumeration. By Kleene's enumerator construction ([1] 8.1.6) there exists a combinator $\mathbf{J}$ such that $\mathbf{J}\lceil M\rceil$ β-converts to M, for every M.
The combinator $\mathbf{J}$ can be used to enumerate various r.e. sets of closed terms. In particular, let $\mathcal{X}$ be a r.e. set of terms, and let $T_\mathcal{X}$ be a term representing the r.e. function that enumerates $\mathcal{X}$. Set $\mathbf{E} \equiv \lambda x.\mathbf{J}(T_\mathcal{X}x)$. It is well known that we can assume that $\mathbf{E}$ is in $\beta\eta$-normal form. We call $\mathbf{E}$ a *generator of* $\mathcal{X}$. As usual we shorten $\mathbf{E}\underline{n}$ with $\mathbf{E}_n$. We also suppress the dependency of $\mathbf{E}$ from $\mathbf{J}$ and $\mathcal{X}$, when it is clear from the context.
Now, by the methods of proof used in [6], which make use of modified forms of the celebrated *Plotkin terms* ([1] 17.3.26), one can prove the following:

Lemma 3.1. *Given a r.e. set of terms $\mathcal{X}$ and a generator $\mathbf{E}$ of $\mathcal{X}$, there exists a term H such that for every M the following holds*

$$H\mathbf{E}_0 =_\omega HM \quad \textit{iff for some } k, \quad M =_\omega \mathbf{E}_k\ . \qquad \square$$

Remark 3.2. The Lemma's proof is identical to the proof of Proposition 5 of [6], and consists of two parts:

(1) to show that if $M =_\omega \mathbf{E}_k$, for some k, then $H\mathbf{E}_0 =_\omega HM$; this is done by the standard argument based on the structure of Plotkin terms;
(2) to show that if $M \neq_\omega \mathbf{E}_k$, for every k, then $H\mathbf{E}_0 \neq_\omega HM$; this difficult point requires a detailed analysis of proofs in $\lambda\omega$, as formulated in [6]; this analysis is done in [6] and is based on casting such proofs in a suitable normal form, called in [6] *cascaded proofs*.

The proof of the following result has the same structure. We define suitable Plotkin terms, which makes the "if part" easy to check, and we rely on the analysis based on *cascaded proofs* for the "only if part". As the external structure of the involved Plotkin terms is the same (zero-terms obtained by applying suitable $\beta\eta$-normal forms to other $\beta\eta$-normal forms), the proof strictly follows the pattern of the proof of Proposition 5 of [6] and is omitted.

Proposition 3.3. *There exist two terms* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *such that for every* M

$$\mathbf{H_1}M =_\omega \mathbf{H_2} \quad \textit{iff} \;\; \textit{for all } k, \quad M \neq_\omega \underline{k}\ .$$

Proof. In [6], we constructed Plotkin terms P and Q such that for every n

$$P\underline{n} =_\omega Q\underline{n} \quad \text{iff} \quad \begin{array}{l} n \text{ is the Gödel number of a closed term} \\ \text{which does not } \beta\eta\text{-convert to a Church numeral.} \end{array}$$

Now, let the Plotkin terms F and G be such that

$$F_nG_nMM_1M_2 \longrightarrow_\beta F_n(F_{n+1}G_{n+1}M\langle \underline{n}, M, P\underline{n}\rangle\langle \underline{n}, \mathbf{J}\underline{n}, Q\underline{n}\rangle)\mathbf{\Omega\Omega\Omega} \tag{3.1}$$

where $\mathbf{\Omega} \equiv (\lambda x.xx)(\lambda x.xx)$ and, as usual, the notation $\langle X_1, X_2, X_3\rangle$ stands for Church triple, i.e. $\langle X_1, X_2, X_3\rangle \equiv \lambda z.zX_1X_2X_3$, and

$$G_n \longrightarrow_\beta F_{n+1}G_{n+1}(\mathbf{J}\underline{n})\langle \underline{n}, \mathbf{J}\underline{n}, Q\underline{n}\rangle\langle \underline{n}, \mathbf{J}\underline{n}, Q\underline{n}\rangle \tag{3.2}$$

Claim. We claim that for every M

$$F_0G_0MMM =_\omega F_0G_0\mathbf{\Omega\Omega\Omega} \quad \text{iff} \;\; \text{for all } k, \quad M \neq_\omega \underline{k}\ .$$

To prove the claim, we first show that for every k

$$F_0G_0\mathbf{\Omega\Omega\Omega} =_\omega F_0(F_1(\ldots(F_kG_k\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega}. \tag{3.3}$$

Indeed, let k be fixed. Then there exists a $k' \geq 1$, such that $\mathbf{J}(\underline{k+k'}) =_\omega \mathbf{\Omega}$. Then, since $\mathbf{\Omega} \neq_\omega \underline{n}$ for every n, we have that

$$\langle \underline{k+k'}, \mathbf{\Omega}, P(\underline{k+k'})\rangle =_\omega \langle \underline{k+k'}, \mathbf{J}(\underline{k+k'}), Q(\underline{k+k'})\rangle$$

and (repeatedly applying (3.1)) both terms of equation (3.3) are $\lambda\omega$-equal to

$$\begin{aligned} &F_0(F_1(\ldots(F_k(F_{k+1}(\ldots(F_{k+k'}(F_{k+k'+1}G_{k+k'+1}\mathbf{\Omega}\langle \underline{k+k'}, \mathbf{\Omega}, P(\underline{k+k'})\rangle \\ &\quad \langle \underline{k+k'}, \mathbf{J}(\underline{k+k'}), Q(\underline{k+k'})\rangle)\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega} =_\omega \quad \text{(by (3.2))} \\ &=_\omega F_0(F_1(\ldots(F_k(F_{k+1}(\ldots(F_{k+k'}G_{k+k'}\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega}\ . \end{aligned}$$

Assume that M is such that for all n, $M \neq_\omega \underline{n}$. Let k such that $M =_\omega \mathbf{J}\underline{k}$. By hypothesis, $\langle \underline{k}, M, P\underline{k}\rangle =_\omega \langle \underline{k}, \mathbf{J}\underline{k}, Q\underline{k}\rangle$. It follows that

$$F_0G_0MMM =_\omega F_0(F_1(\ldots(F_kG_k\mathbf{\Omega\Omega\Omega})\ldots)\mathbf{\Omega\Omega\Omega})\mathbf{\Omega\Omega\Omega} =_\omega F_0G_0\mathbf{\Omega\Omega\Omega}\ .$$

Now assume that M is such that for some n, $M =_\omega \underline{n}$. It follows that, for every k, $\langle \underline{k}, M, P\underline{k}\rangle \neq_\omega \langle \underline{k}, \mathbf{J}\underline{k}, Q\underline{k}\rangle$ and, roughly speaking, in the term F_0G_0MMM, M can never be eliminated by G. A formal proof argues by contradiction on a *cascaded proof* of

$$F_0G_0MMM =_\omega F_0G_0\mathbf{\Omega\Omega\Omega}\ .$$

This ends the proof of the claim.

Now define

$$\mathbf{H_1} \equiv \lambda x.F_0G_0xxx \quad \text{and} \quad \mathbf{H_2} \equiv F_0G_0\mathbf{\Omega\Omega\Omega}\ . \qquad \square$$

We shall make extensive use of terms $\mathbf{H_1}$ and $\mathbf{H_2}$ in the following Section.

4. Barendregt Construction

In the present Section, we shall make use of the Proposition 4 of [6], that we restate here for the sake of the reader.

If $M =_\omega N$ and M has a $\beta\eta$-normal form then N has the same normal form. Therefore two $\beta\eta$-normal forms equalized in $\lambda\beta\omega$ are identical.

We make the following definitions, which will hold in all the present and the next Section:

Definition 4.1.
(1) $\mathbf{\Theta} \equiv (\lambda ab.b(aab))(\lambda ab.b(aab))$ (Turing's fixed point).
(2) $\mathbf{W} \equiv \lambda xy.xyy$
(3) $\mathbf{L} \equiv (\lambda xyz.\lambda abc.xy(z(yc))bac)$
(4) $F \equiv \mathbf{\Theta L H_1} \equiv (\lambda ab.b(aab))(\lambda ab.b(aab))(\lambda xyz.\lambda abc.xy(z(yc))bac)\mathbf{H_1}$
(5) $G \equiv \mathbf{\Theta W H_2} \equiv (\lambda ab.b(aab))(\lambda ab.b(aab))(\lambda xy.xyy)\mathbf{H_2}$

Observe:

(i)
$$\begin{aligned} G &\longrightarrow_{\beta\eta} (\lambda b.b(\mathbf{\Theta}b))\mathbf{W H_2} \\ &\longrightarrow_{\beta\eta} \mathbf{W}(\mathbf{\Theta W})\mathbf{H_2} \\ &\longrightarrow_{\beta\eta} \mathbf{\Theta W H_2 H_2} \equiv G\mathbf{H_2} \end{aligned}$$

(ii)
$$\begin{aligned} FZABC &\longrightarrow_{\beta\eta} (\lambda b.b(\mathbf{\Theta}b))\mathbf{L H_1}ZABC \\ &\longrightarrow_{\beta\eta} \mathbf{L}(\mathbf{\Theta L})\mathbf{H_1}ZABC \\ &\longrightarrow_{\beta\eta} (\lambda yz.\lambda abc.\mathbf{\Theta L}y(z(yc))bac)\mathbf{H_1}ZABC \\ &\longrightarrow_{\beta\eta} (\lambda z.\lambda abc.\mathbf{\Theta L H_1}(z(\mathbf{H_1}c))bac)ZABC \\ &\longrightarrow_{\beta\eta} (\lambda abc.\mathbf{\Theta L H_1}(Z(\mathbf{H_1}c))bac)ABC \\ &\longrightarrow_{\beta\eta} (\lambda bc.\mathbf{\Theta L H_1}(Z(\mathbf{H_1}c))bAc)BC \\ &\longrightarrow_{\beta\eta} (\lambda c.\mathbf{\Theta L H_1}(Z(\mathbf{H_1}c))BAc)C \\ &\longrightarrow_{\beta\eta} \mathbf{\Theta L H_1}(Z(\mathbf{H_1}C))BAC \equiv FZ^*BAC \end{aligned}$$

where we have shortened $Z(\mathbf{H_1}C)$ to Z^*.

Let gk be the cofinal Gross-Knuth strategy defined in [1] 13.2.7. By writing $gk(M)$, we mean the term obtained by starting with the term M and applying (once) the gk strategy.

Then the reduction sequences

$$\begin{aligned} G \longrightarrow^*_{\beta\eta} G(gk(\mathbf{H_2})) \longrightarrow^*_{\beta\eta} G\mathbf{H_2}(gk(\mathbf{H_2})) \longrightarrow^*_{\beta\eta} \\ \longrightarrow^*_{\beta\eta} G(gk(\mathbf{H_2}))(gk(gk(\mathbf{H_2}))) \longrightarrow^*_{\beta\eta} \cdots \end{aligned} \tag{4.1}$$

$$\begin{aligned} FZABC &\longrightarrow^*_{\beta\eta} F(gk(Z^*))(gk(B))(gk(A))(gk(C)) \longrightarrow^*_{\beta\eta} \\ &\longrightarrow^*_{\beta\eta} F(gk((gk(Z^*))^*))(gk(gk(A)))(gk(gk(B)))(gk(gk(C))) \longrightarrow^*_{\beta\eta} \cdots \end{aligned} \tag{4.2}$$

(where again the notation X^* is a shortening of $X(\mathbf{H_1}C)$) are cofinal for $\beta\eta$-reductions starting with G and, respectively, with $FZABC$.

Let P be the initial term or an intermediate term of the reduction sequence of the form (4.1), we indicate by $\overline{gk}(P)$ the first term in the sequence, which has the form displayed in (4.1), obtained from P by the reductions in (4.1).

Similarly, if P is the initial term or an intermediate term of a reduction sequence of the form (4.2), starting from $FM_1M_2M_3M_4$, for some M_1, M_2, M_3, M_4, we indicate by $\overline{gk}(P)$ the first term in the sequence, which has the form displayed in (4.2), obtained from P by the reductions in (4.2).

Now, we choose the cofinal set $\mathcal{X}$ as follows: for every closed term M,

- if M $\beta\eta$-reduces, by the leftmost outermost reduction strategy, to a term of the form $PN_1 \cdots N_k$, where P is a term of the sequence (4.1) then $\overline{gk}(P)gk(N_1)\cdots gk(N_k)$ is the reduct of M in $\mathcal{X}$;
- if M $\beta\eta$-reduces, by the leftmost outermost reduction strategy, to a term the form $PN_1 \cdots N_k$, where P is a term of a sequence (4.2), starting from $FM_1M_2M_3M_4$, for some M_1, M_2, M_3, M_4, then
$$\overline{gk}(FM_1M_2M_3M_4)gk(N_1)\cdots gk(N_k)$$
is the reduct of M in $\mathcal{X}$;
- M is in $\mathcal{X}$, otherwise.

Observe that we use the leftmost outermost reduction strategy, since it is cofinal (see [1], 13.1.3). The following Lemma is immediate.

Lemma 4.2. *$\mathcal{X}$ is supercofinal.* □

Lemma 4.3. *If $GM_1 \ldots M_m =_\omega GN_1 \ldots N_m$ then, for each k, $1 \le k \le m$, $M_k =_\omega N_k$.*

Proof. By induction on the ordinal of a canonical proof $\mathcal{T}$ of $GM_1 \ldots M_m =_\omega GN_1 \ldots N_m$.

Basis: $\mathrm{ord}(\mathcal{T})$ is 1. This case is clear since G is of order 0.

Induction step:

Case 1. $\mathcal{T}$ ends in an application of the ω-rule. Apply the induction hypothesis to the subproof of $GM_1 \ldots M_m\mathbf{I} =_\omega GN_1 \ldots N_m\mathbf{I}$.

Case 2. $\mathcal{T}$ has the endpiece

$$\begin{gathered} GM_1 \ldots M_m \longrightarrow^*_{\beta\eta} R_1 \ {}^*_{\beta\eta}\!\longleftarrow L_1P_1 =_\omega L_1Q_1 \longrightarrow^*_{\beta\eta} R_2 \ {}^*_{\beta\eta}\!\longleftarrow L_2P_2 =_\omega L_2Q_2 \\ \longrightarrow^*_{\beta\eta} \ldots \longrightarrow^*_{\beta\eta} R_{t+1} \ {}^*_{\beta\eta}\!\longleftarrow GN_1 \ldots N_m \ . \end{gathered}$$

Since $\mathcal{T}$ is canonical, every term R_i, with $1 \le i \le t+1$, has the form

$$\mathbf{\Theta W} H^*_{i,1} \ldots H^*_{i,n_i} M^*_{i,1} \ldots M^*_{i,m}$$

where $H^*_{i,j} =_\omega \mathbf{H_2}$, for $j = 1, \ldots, n_i$, and $M^*_{i,k} =_\omega M_k$ for $k = 1, \ldots, m$. Since G is of order 0, we must also have $M^*_{t+1,k} =_\omega N_k$ for $k = 1, \ldots, m$. This completes the proof. □

By an inspection of the proof of the previous Lemma, the following stronger result can be obtained.

Lemma 4.4. *Assume that $GM_1 \dots M_m =_\omega GN_1 \dots N_m$ has a canonical proof $\mathcal{T}$. Then for each k, $1 \leq k \leq m$, there is a canonical proof $\mathcal{T}_k$ of $M_k =_\omega N_k$, with the ordinal of $\mathcal{T}_k$ not greater than the ordinal of $\mathcal{T}$.* □

For the proof of the following Lemma, we need Proposition 4 of [6], stated above.

Lemma 4.5. *Suppose that:*

- $FL_1P_1Q_1\underline{n}M_1 \dots M_m =_\omega FL_2P_2Q_2\underline{n}N_1 \dots N_m$;
- $L_1 =_\omega G(\mathbf{H_1}\underline{n}) \dots (\mathbf{H_1}\underline{n})$, *$k$ times;*
- $L_2 =_\omega G(\mathbf{H_1}\underline{n}) \dots (\mathbf{H_1}\underline{n})$, *$l$ times.*

Then either $P_1 =_\omega P_2, Q_1 =_\omega Q_2$ and $k = l$ mod 2 or $P_1 =_\omega Q_2$, $Q_1 =_\omega P_2$ and $k = l + 1$ mod2, where, possibly, $k = 0$ or $l = 0$.

Proof. By induction on the ordinal of a canonical proof of

$$FL_1P_1Q_1\underline{n}M_1 \dots M_m =_\omega FL_2P_2Q_2\underline{n}N_1 \dots N_m$$

Basis: The ordinal is 1 and we have a $\beta\eta$-conversion. Use a standard argument, taking into account that by Proposition 3.3 the copies of $\mathbf{H_2}$ are distinct, w.r.t. ω-equality, from the copies of $\mathbf{H_1}\underline{n}$. Therefore the β-reduction of G cannot affect the count of the copies of $\mathbf{H_1}\underline{n}$.

Induction step:
Case 1. The proof ends in an application of the ω-rule. Just apply the induction hypothesis to any of the premises.
Case 2. The proof has a canonical endpiece beginning with a component

$$FL_1P_1Q_1\underline{n}M_1 \dots M_m \longrightarrow^*_{\beta\eta} H \;{}^*_{\beta\eta}\!\longleftarrow LQ =_\omega LR \longrightarrow^*_{\beta\eta} H^+ \ .$$

Now H has the same form as $FL_1P_1Q_1\underline{n}M_1 \dots M_m$ by the choice of the cofinal set. W.l.o.g. we can assume that the reduction from $FL_1P_1Q_1\underline{n}M_1 \dots M_m$ to H is a standard β-reduction followed by a sequence of η-reductions. The 8 term head reduction cycle of F with 4 arguments must be completed an integral number of times to result in a term which η-reduces to one with F at the head. Suppose that this cycle is completed s times. Let $r = k + s$.

On the other hand, since the endpiece is canonical L, after a sequence (possibly empty) of η-reductions, reduces to a term of the form

$$\lambda z.X_0X_2X_3X_4X_5X_6X_7X_8Y_1 \dots Y_m$$

where $X_0 \equiv \lambda x.X_1$. Indeed, the form of the external structure of H must be

$$X_0'X_2'X_3'X_4'X_5'X_6'X_7'X_8'Y_1' \dots Y_m' \ ,$$

since this is the form of any term, in the cofinal sequence, starting from $FL_1P_1Q_1\underline{n}M_1 \dots M_m$. Therefore L must have the form

$$\lambda z.X_0X_2X_3X_4X_5X_6X_7X_8Y_1 \dots Y_m \ ,$$

since we have to obtain H by internal reductions and Q is not substituted for a variable in functional position in a head redex. It follows, using for some items Proposition 4 of [6], that:

- $[Q/z]X_0 \longrightarrow^*_{\beta\eta} \lambda ab.b(aab)$ (since $\lambda ab.b(aab)$ is in $\beta\eta$-normal form);
- $[Q/z]X_2 \longrightarrow^*_{\beta\eta} \lambda ab.b(aab)$;
- $[Q/z]X_3 \longrightarrow^*_{\beta\eta} \mathbf{L}$ (since $\mathbf{L}$ is in $\beta\eta$-normal form);
- $[Q/z]X_4 =_\omega \mathbf{H_1}$
- $[Q/z]X_5 =_\omega G\mathbf{H_2}\ldots\mathbf{H_2}(\mathbf{H_1}\underline{n})\ldots(\mathbf{H_1}\underline{n})$,
  with t occurrences of $\mathbf{H_2}$, due to the possible β-reduction of G, and r occurrences of $\mathbf{H_1}\underline{n}$, since we have started with k copies of $\mathbf{H_1}\underline{n}$, and each reduction cycle of F adds a copy.
- $[Q/z]X_6 =_\omega P_1$ if $s \equiv 0$ mod 2 or
  $[Q/z]X_6 =_\omega Q_1$ if $s \equiv 1$ mod 2,
  this item, and the following one, results from the fact the each reduction cycle of F interchanges P_1 and Q_1;
- $[Q/z]X_7 =_\omega Q_1$ if $s \equiv 0$ mod 2 or
  $[Q/z]X_7 =_\omega P_1$ if $s \equiv 1$ mod 2;
- $[Q/z]X_8 \longrightarrow^*_{\beta\eta} \underline{n}$;
- $[Q/z]Y_i =_\omega M_i$, for every $1 \le i \le m$.

From the fact that $P =_\omega R$, and using again Proposition 4 of [6], we have:

- $[R/z]X_0 \longrightarrow^*_{\beta\eta} \lambda ab.b(aab)$;
- $[R/z]X_2 \longrightarrow^*_{\beta\eta} \lambda ab.b(aab)$;
- $[R/z]X_3 \longrightarrow^*_{\beta\eta} \mathbf{L}$;
- $[R/z]X_4 =_\omega \mathbf{H_1}$;
- $[R/z]X_5 =_\omega G\mathbf{H_2}\ldots\mathbf{H_2}(\mathbf{H_1}\underline{n})\ldots(\mathbf{H_1}\underline{n})$ with t occurrences of $\mathbf{H_2}$ and r occurrences of $\mathbf{H_1}\underline{n}$;
- $[R/z]X_6 =_\omega P_1$ if $s \equiv 0$ mod 2;
- $[R/z]X_6 =_\omega Q_1$ if $s \equiv 1$ mod 2;
- $[R/z]X_7 =_\omega Q_1$ if $s \equiv 0$ mod 2;
- $[R/z]X_7 =_\omega P_1$ if $s \equiv 1$ mod 2;
- $[R/z]X_8 \longrightarrow^*_{\beta\eta} \underline{n}$;
- $[R/z]Y_i =_\omega M_i$, for every $1 \le i \le m$.

Observe moreover that H^+, because of its construction, has the same form as H (up to some η-reductions). Say $H^+ \equiv FL_1^+P_1^+Q_1^+\underline{n}M_1^+\ldots M_m^+$. Moreover, we can freely assume that L_1^+ is $G(\mathbf{H_1}\underline{n})\ldots(\mathbf{H_1}\underline{n})$ with no occurrence of $\mathbf{H_2}$ and r occurrences of $\mathbf{H_1}\underline{n}$. This amounts to start with a different term and then perform t β-reductions of G. By Proposition 3.3 the copies of $\mathbf{H_2}$ are distinct, w.r.t. ω-equality, from the copies of $\mathbf{H_1}\underline{n}$. Therefore the β-reduction of G cannot affect the count of the copies of $\mathbf{H_1}\underline{n}$.

The part of the proof beginning with H^+ is a canonical proof of the fact that $H^+ =_\omega FL_2P_2Q_2\underline{n}N_1\ldots N_m$, because the cofinality restriction met for LR also works for H^+ . Thus the induction hypothesis applies to this proof.

Now the idea is that r and l have "to be in accordance" by induction hypothesis. On the other hand k differs from r only for s cycles of F, and therefore they behave in the right way. So the required property is obtained by transitivity. Formally:

Subcase 2.1. $P_1^+ =_\omega P_2$, $Q_1^+ =_\omega Q_2$ and $r \equiv l$ mod 2.
In case s is even we have $P_1 =_\omega P_2$ and $Q_1 =_\omega Q_2$ and $k \equiv l$ mod 2. In case s is odd we have k and l with opposite parity and $Q_1 =_\omega P_1^+ =_\omega P_2$, $P_1 =_\omega Q_2^+ =_\omega Q_2$.

Subcase 2.2. $P_1^+ =_\omega Q_2$, $Q_1^+ =_\omega P_2$ and $r \equiv l+1$ mod 2.
In case s is even we have $P_1 =_\omega Q_2$ and $Q_1 =_\omega P_2$ and $k \equiv l+1$ mod 2. In case s is odd we

have k and l with the same parity and $P_1 =_\omega Q_1^+ =_\omega P_2$, $Q_1 =_\omega P_2^+ =_\omega Q_2$. This completes the proof. □

Also in this case, by an inspection of the proof, the following stronger result can be obtained.

Lemma 4.6. *Suppose that*

- $FL_1P_1Q_1\underline{n}M_1 \dots M_m =_\omega FL_2P_2Q_2\underline{n}N_1 \dots N_m$ *has a canonical proof* $\mathcal{T}$*;*
- $L_1 =_\omega G(\mathbf{H_1}\underline{n}) \dots (\mathbf{H_1}\underline{n})$*, k times, has a canonical proof* $\mathcal{T}_1$*;*
- $L_2 =_\omega G(\mathbf{H_1}\underline{n}) \dots (\mathbf{H_1}\underline{n})$*, l times, has a canonical proof* $\mathcal{T}_2$*.*

Then:

- *either* $k = l \bmod 2$ *and* $P_1 =_\omega P_2, Q_1 =_\omega Q_2$ *have canonical proofs* $\mathcal{T}_3$ *and, respectively,* $\mathcal{T}_4$*,*
- *or* $k = l + 1 \bmod 2$ *and* $P_1 =_\omega Q_2$*,* $Q_1 =_\omega P_2$ *have canonical proofs* $\mathcal{T}_3$ *and, respectively,* $\mathcal{T}_4$*.*

Here, possibly, $k = 0$ *or* $l = 0$*, and* $\max\{\mathrm{ord}(\mathcal{T}_3), \mathrm{ord}(\mathcal{T}_4)\} \leq \max\{\mathrm{ord}(\mathcal{T}), \mathrm{ord}(\mathcal{T}_1), \mathrm{ord}(\mathcal{T}_2)\}$. □

4.1. **Well Founded Trees.** We assume that we have encoded sequences of numbers as numbers, with 0 encoding the empty sequence. $\langle n \rangle$ is the sequence consisting of n alone (singleton) and $*$ is the concatenation function. For simplicity, we shall use these notations ambiguously for the corresponding λ-terms. We require only that the term $y * \langle z \rangle$ is in $\beta\eta$-normal form with $z\mathbf{II}$ at its head (this construction can be obtained "making normal" a term representing $*$, see [13]).

Our proof of $\mathbf{\Pi_1^1}$-completeness of $\lambda\omega$ is inspired by the argument in Section 17.4 of [1] (however, we will substantially modify Barendregt's construction). The starting point is the following well known theorem (see [5] Ch.16 Th.20):

Theorem 4.7. *The set of (indices of) well-founded recursive trees is* $\mathbf{\Pi_1^1}$*-complete.* □

The idea is now to reduce the well-foundedness of a recursive tree to the equality of two suitable terms in $\lambda\omega$.

Suppose that we have a primitive recursive tree $\mathbf{t}$ with a representing term $\mathbf{T}$ such that

$$\mathbf{T}\underline{n} \longrightarrow^*_{\beta\eta} \begin{cases} \mathbf{I} & \text{if } n \text{ is the number of a sequence in } \mathbf{t}; \\ \mathbf{K}^* & \text{otherwise.} \end{cases}$$

Define

$$\begin{aligned} A \equiv_{def} & \mathbf{\Theta}(\lambda x.\lambda a.a(\lambda y.\mathbf{T}y(\lambda z.FG(x\mathbf{K}(y * \langle z \rangle)) \\ & (x\mathbf{K}^*(y * \langle z \rangle))z))(\lambda y.\mathbf{T}y(\lambda z.FG(x\mathbf{K}^*(y * \langle z \rangle)) \\ & (x\mathbf{K}(y * \langle z \rangle))z)))\mathbf{K} \\ B \equiv_{def} & \mathbf{\Theta}(\lambda x.\lambda a.a(\lambda y.\mathbf{T}y(\lambda z.FG(x\mathbf{K}(y * \langle z \rangle)) \\ & (x\mathbf{K}^*(y * \langle z \rangle))z))(\lambda y.\mathbf{T}y(\lambda z.FG(x\mathbf{K}^*(y * \langle z \rangle)) \\ & (x\mathbf{K}(y * \langle z \rangle))z)))\mathbf{K}^* \end{aligned}$$

Clearly:

$$\begin{aligned} A &\longrightarrow^*_{\beta\eta} \lambda y.\mathbf{T}y(\lambda z.FG(A(y * \langle z \rangle))(B(y * \langle z \rangle))z) \\ B &\longrightarrow^*_{\beta\eta} \lambda y.\mathbf{T}y(\lambda z.FG(B(y * \langle z \rangle))(A(y * \langle z \rangle))z) \end{aligned}$$

Now we state a corollary to Lemma 4.6.

Corollary 4.8. *If* $FG(A\underline{n})(B\underline{n})\underline{n}M_1 \ldots M_m =_\omega FG(B\underline{n})(A\underline{n})\underline{n}N_1 \ldots N_m$ *has a canonical proof* $\mathcal{T}$ *then* $A\underline{n} =_\omega B\underline{n}$ *has a canonical proof* $\mathcal{T}_1$*, with* $\mathrm{ord}(\mathcal{T}_1) \leq \mathrm{ord}(\mathcal{T})$*.* □

Lemma 4.9. *If the subtree* $\mathbf{t}(n)$ *of the tree* $\mathbf{t}$ *rooted at* n *is well-founded then* $A\underline{n} =_\omega B\underline{n}$*.*

Proof. By induction on the ordinal of the subtree $\mathbf{t}(n)$, which is defined in the natural way. Note that if n is not the number of a sequence in the tree then $\mathbf{T}\underline{n} \longrightarrow^*_{\beta\eta} \mathbf{K}^*$ so $A\underline{n} \longrightarrow^*_{\beta\eta} \mathbf{I} \;{}^*_{\beta\eta}\!\longleftarrow B\underline{n}$.

Basis. The ordinal is 0 so the tree $\mathbf{t}(n)$ contains only the empty sequence. Suppose that 0 is the number of the empty sequence. Then

$$A\underline{0} \longrightarrow^*_{\beta\eta} \lambda z.FG(A(\underline{0} * \langle z\rangle))(B(\underline{0} * \langle z\rangle))z$$

and

$$B\underline{0} \longrightarrow^*_{\beta\eta} \lambda z.FG(B(\underline{0} * \langle z\rangle))(A(\underline{0} * \langle z\rangle))z$$

and if N $\beta\eta$-converts to a Church numeral then

$$A\underline{0}N \longrightarrow^*_{\beta\eta} FG\mathbf{II}N \;{}^*_{\beta\eta}\!\longleftarrow B\underline{0}N$$

and if N does not $\beta\eta$-convert to a Church numeral then

$$\begin{aligned} A\underline{0}N &\longrightarrow^*_{\beta\eta} FG(A(\underline{0} * \langle N\rangle))(B(\underline{0} * \langle N\rangle))N \longrightarrow^*_{\beta\eta} \\ &\longrightarrow^*_{\beta\eta} F(G(\mathbf{H_1}N))(B(\underline{0} * \langle N\rangle))(A(\underline{0} * \langle N\rangle))N =_\omega \\ &=_\omega F(G\mathbf{H_2})(B(\underline{0} * \langle N\rangle))(A(\underline{0} * \langle N\rangle))N \;{}^*_{\beta\eta}\!\longleftarrow B\underline{0}N \;. \end{aligned}$$

So, by the ω-rule, $A\underline{0} =_\omega B\underline{0}$.

Induction Step. The ordinal of the subtree rooted at n is larger than 0. We have

$$A\underline{n} \longrightarrow^*_{\beta\eta} \lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z$$
$$B\underline{n} \longrightarrow^*_{\beta\eta} \lambda z.FG(B(\underline{n} * \langle z\rangle))(A(\underline{n} * \langle z\rangle))z$$

Now, if N $\beta\eta$-converts to a Church numeral, then

$$\begin{aligned} A\underline{n}N &\longrightarrow^*_{\beta\eta} FG(A(\underline{n} * \langle N\rangle))(B(\underline{n} * \langle N\rangle))N =_\omega \quad \text{(by induction hypothesis)} \\ &=_\omega FG(B(\underline{n} * \langle N\rangle))(A(\underline{n} * \langle N\rangle))N \;{}^*_{\beta\eta}\!\longleftarrow B\underline{n}N \end{aligned}$$

and if N does not $\beta\eta$-convert to a Church numeral, then

$$\begin{aligned} A\underline{n}N &\longrightarrow^*_{\beta\eta} FG(A(\underline{n} * \langle N\rangle))(B(\underline{n} * \langle N\rangle))N \longrightarrow^*_{\beta\eta} \\ &\longrightarrow^*_{\beta\eta} F(G(\mathbf{H_1}N))(B(\underline{n} * \langle N\rangle))(A(\underline{n} * \langle N\rangle))N =_\omega \\ &=_\omega F(G\mathbf{H_2})(B(\underline{n} * \langle N\rangle))(A(\underline{n} * \langle N\rangle))N \;{}^*_{\beta\eta}\!\longleftarrow B\underline{n}N \;. \end{aligned}$$

So by the ω-rule $A\underline{n} =_\omega B\underline{n}$. This completes the proof. □

Lemma 4.10. *If* $A\underline{n} =_\omega B\underline{n}$ *then the subtree* $\mathbf{t}(n)$ *rooted at* n *is well-founded or* n *is not in the tree* $\mathbf{t}$*.*

Proof. Consider all canonical proofs of smallest ordinal of $A\underline{n} =_\omega B\underline{n}$ for n in the tree $\mathbf{t}$, and assume that the subtree $\mathbf{t}(n)$ rooted at n is not well-founded. Let $\mathcal{T}$ be such a proof.

Case 1. $\mathcal{T}$ is a $\beta\eta$-conversion. It is easily seen that this is impossible. Indeed, assume that $A\underline{n} =_{\beta\eta} B\underline{n}$; by the Church-Rosser Theorem a common $\beta\eta$-reduct must exist.

On the other hand, since n is in $\mathbf{t}$, we have

$$A\underline{n} \longrightarrow^*_{\beta\eta} \lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z$$

and

$$B\underline{n} \longrightarrow^*_{\beta\eta} \lambda z.FG(B(\underline{n} * \langle z\rangle))(A(\underline{n} * \langle z\rangle))z\ .$$

However that $\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z$ and $\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z$ have a common reduct is impossible, considering that $A(\underline{n} * \langle z\rangle)$ and $B(\underline{n} * \langle z\rangle)$ are not $\beta\eta$-convertible and at each reduction step of F they are interchanged and a new term $\mathbf{H_1}$ is generated. So the reducts never have the right "parity" to be identical (see also Lemma 4.5).

Case 2. $\mathcal{T}$ ends in the ω-rule. Then for each m, $A\underline{nm} =_\omega B\underline{nm}$ has a canonical proof of smaller ordinal. Now

$$\begin{aligned} A\underline{nm} &\longrightarrow^*_{\beta\eta} \lambda y.\mathbf{T}y(\lambda z.FG(A(y * \langle z\rangle))(B(y * \langle z\rangle))z)\underline{nm} \\ &\longrightarrow_{\beta\eta} \mathbf{T}\underline{n}(\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z)\underline{m} \\ &\longrightarrow^*_{\beta\eta} (\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z)\underline{m} \\ &\longrightarrow_{\beta\eta} FG(A(\underline{n} * \langle \underline{m}\rangle))(B(\underline{n} * \langle \underline{m}\rangle))\underline{m} \end{aligned}$$

and reducing in a similar way $B\underline{nm}$, we see that

$$FG(A(\underline{n} * \langle \underline{m}\rangle))(B(\underline{n} * \langle \underline{m}\rangle))\underline{m} =_\omega FG(B(\underline{n} * \langle \underline{m}\rangle))(A(\underline{n} * \langle \underline{m}\rangle))\underline{m}$$

has a proof of the same (smaller) ordinal. Thus, by Corollary 4.8,

$$A(\underline{n} * \langle \underline{m}\rangle) =_\omega B(\underline{n} * \langle \underline{m}\rangle)$$

has a proof with the same or smaller ordinal.

Thus by induction hypothesis, the extension of $n*\ \langle m\rangle$ in the tree is well-founded. So, every extension of n in the tree is well-founded. Thus the subtree rooted at n is well-founded. This contradicts the choice of n.

Case 3. $\mathcal{T}$ has an endpiece. Now, by Proposition 2.7, for each m there exist term $R_1, \ldots, R_k$ such that we have a canonical proof, with an endpiece of the same rank as $\mathcal{T}$, of

$$\begin{aligned} A\underline{nm}R_1\cdots R_k &\longrightarrow^*_{\beta\eta} (\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z)\underline{m}R_1\cdots R_k \\ &\longrightarrow^*_{\beta\eta} \ldots \ {}^*_{\beta\eta}\!\longleftarrow B\underline{nm}R_1\cdots R_k\ . \end{aligned}$$

Now consider that this endpiece is $\mathcal{X}$-canonical.
So, to equalize

$$FG(A(\underline{n} * \langle \underline{m}\rangle))(B(\underline{n} * \langle \underline{m}\rangle))\underline{m}R_1\cdots R_k$$

with

$$FG(B(\underline{n} * \langle \underline{m}\rangle))(A(\underline{n} * \langle \underline{m}\rangle))\underline{m}R_1\cdots R_k\ ,$$

it is necessary that some of instances of the ω-rule, occurring in the endpiece, supplies a proof of $A(\underline{n} * \langle \underline{m}\rangle) =_\omega B(\underline{n} * \langle \underline{m}\rangle)$.

To see this consider the particular case when there is only one leaf which is a direct conclusion of the ω-rule.

$$\begin{aligned} A\underline{nm}R_1\cdots R_k &\longrightarrow^*_{\beta\eta} (\lambda z.FG(A(\underline{n} * \langle z\rangle))(B(\underline{n} * \langle z\rangle))z)\underline{m}R_1\cdots R_k \longrightarrow^*_{\beta\eta} \\ &\longrightarrow^*_{\beta\eta} H\ {}^*_{\beta\eta}\!\longleftarrow LQ =_\omega LR \longrightarrow^*_{\beta\eta} H^+\ {}^*_{\beta\eta}\!\longleftarrow \\ &{}^*_{\beta\eta}\!\longleftarrow\ (\lambda z.FG(B(\underline{n} * \langle z\rangle))(A(\underline{n} * \langle z\rangle))z)\underline{m}R_1\cdots R_k\ {}^*_{\beta\eta}\!\longleftarrow \\ &{}^*_{\beta\eta}\!\longleftarrow B\underline{nm}R_1\cdots R_k\ . \end{aligned}$$

Since the endpiece is $\mathcal{X}$-canonical, it follows that LQ has the form of $FG(A(\underline{n}*\langle m\rangle))(B(\underline{n}*\langle m\rangle))\underline{m}R_1\cdots R_k$ and LR has the form of $FG(B(\underline{n} * \langle m\rangle))(A(\underline{n} * \langle m\rangle))\underline{m}R_1\cdots R_k$.

Now let N be any term. By the definition of canonical proof, there is a $\mathcal{X}$-canonical proof, with ordinal less than $\mathcal{T}$, of $LQN =_\omega LRN$ and therefore a proof with ordinal less than $\mathcal{T}$, of

$$FG(A(\underline{n} * \langle m\rangle))(B(\underline{n} * \langle m\rangle))\underline{m}R_1 \cdots R_k N =_\omega FG(B(\underline{n} * \langle m\rangle))(A(\underline{n} * \langle m\rangle))\underline{m}R_1 \cdots R_k N \ .$$

Again by Lemma 4.5, Lemma 4.3 and Proposition 3.3, $A(\underline{n}* < \underline{m} >) =_\omega B(\underline{n}* < \underline{m} >)$ has a proof with the same or smaller ordinal as

$$FG(A(\underline{n} * \langle m\rangle))(B(\underline{n} * \langle m\rangle))\underline{m}R_1 \cdots R_k N =_\omega FG(B(\underline{n} * \langle m\rangle))(A(\underline{n} * \langle m\rangle))\underline{m}R_1 \cdots R_k N \ .$$

Thus, by induction hypothesis, the extension of $n * \ \langle m\rangle$ in the tree is well-founded. Thus every extension of n in the tree is well-founded and again we contradict the choice of n.

The case with multiple leaves can be treated by induction on the number of leaves, in the endpiece, that are direct conclusions of the ω-rule.

Considering such leaves from left to right, and using the fact that the endpiece is $\mathcal{X}$-canonical

- when the left hand side and the right hand side of the leaf have both the form:
  $$FG(A(\underline{n} * \langle \underline{m}\rangle))(B(\underline{n} * \langle \underline{m}\rangle))\underline{m}R_1 \cdots R_k$$
  then we move to the next leaf;
- at some leaf, we must have that the left hand side has the form
  $$FG(A(\underline{n} * \langle \underline{m}\rangle))(B(\underline{n} * \langle \underline{m}\rangle))\underline{m}R_1 \cdots R_k$$
  and the right hand side of the leaf has the form
  $$FG(B(\underline{n} * \langle m\rangle))(A(\underline{n} * \langle m\rangle))\underline{m}R_1 \cdots R_k \ ,$$
  this case is treated as the one above.

This completes the proof. □

We have thus proved:

Proposition 4.11. *$A\underline{n} =_\omega B\underline{n}$ iff the subtree $\mathbf{t}(n)$ rooted at n is well-founded or n is not in the tree $\mathbf{t}$.* □

Proposition 4.12. (Main Theorem) *The set $\{(M,N)|M =_\omega N\}$ is $\mathbf{\Pi^1_1}$-complete.*

Proof. It easy to see that equality in $\lambda\omega$ is $\mathbf{\Pi^1_1}$. On the other hand, given any recursive tree $\mathbf{t}$ construct the terms A and B (observe that the construction is effective and uniform on (the term $\mathbf{T}$ representing) $\mathbf{t}$). Then use Proposition 4.11 to determine (*via* equality in $\lambda\omega$) if $\mathbf{t} = \mathbf{t}(0)$ is well-founded. □

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