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A COINDUCTIVE REFORMULATION OF MILNER'S PROOF SYSTEM FOR REGULAR EXPRESSIONS MODULO BISIMILARITY

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Abstract. Milner defined an operational semantics for regular expressions as finite-state processes. In order to axiomatize bisimilarity of regular expressions under this process semantics, he adapted Salomaa's complete proof system for equality of regular expressions under the language semantics. Apart from most equational axioms, Milner's system Mil inherits from Salomaa's system a non-algebraic rule for solving fixed-point equations. Recognizing distinctive properties of the process semantics that render Salomaa's proof strategy inapplicable, Milner posed completeness of the system Mil as an open question.

As a proof-theoretic approach to this problem we characterize the derivational power that the fixed-point rule adds to the purely equational part Mil^{-} of Mil. We do so by means of a coinductive rule that permits cyclic derivations that consist of a finite process graph (maybe with empty steps) that satisfies the layered loop existence and elimination property LLEE, and two of its Mil^{-}-provable solutions. By adding this rule instead of the fixed-point rule in Mil, we define the coinductive reformulation cMil as an extension of Mil^{-}. For showing that cMil and Mil are theorem equivalent we develop effective proof transformations from Mil to cMil, and vice versa. Since it is located half-way in between bisimulations and proofs in Milner's system Mil, cMil may become a beachhead for a completeness proof of Mil.

This article extends our contribution to the CALCO 2021 proceedings. Here we refine the proof transformations by framing them as eliminations of derivable and admissible rules, and we link coinductive proofs to a coalgebraic formulation of solutions of process graphs.

1. Introduction

Milner introduced in [Mil84] a process semantics $[\![\cdot]\!]_P$ for regular expressions e as finite-state process graphs $[\![e]\!]_P$. Informally the process interpretation is defined as follows, for regular expressions built from the constants 0 and 1 by using the regular operators $+$, $\cdot$, and $(\cdot)^*$:

- 0 stands for deadlock, 1 for successful termination, letters a for atomic actions,
- the operators $+$ and $\cdot$ are interpreted as choice and concatenation of processes, respectively,
- (unary) Kleene star $(\cdot)^*$ denotes iteration with the option to terminate successfully before each execution of the iteration body (then even infinitely many iterations are possible).

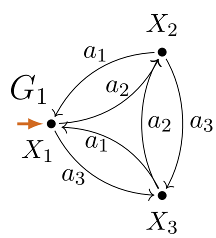
Key words and phrases: regular expressions, process theory, bisimilarity, coinduction, interpretational proof theory, proof transformations, derivable and admissible inference rules.

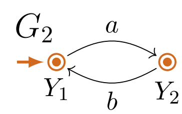
* This is a special-journal version of the paper "A Coinductive Version of Milner's Proof System for Regular Expressions Modulo Bisimilarity" [Gra21b] presented at CALCO 2021.

Milner called regular expressions 'star expressions' when they are interpreted as processes. He formulated this semantics after developing a complete equational proof system for equality of 'a class of regular behaviors'. By that he understood the bisimilarity equivalence classes of finite-state processes that are represented by μ-terms. Then he defined the process semantics by interpreting regular expressions as μ-term representations of finite-state processes. In doing so, he defined 'star behaviors', the bisimilarity equivalence classes of the interpretations of star expressions as a subclass of 'regular behaviors'. As an afterthought to the complete proof system for regular behaviors, he was interested in an axiomatization of equality of 'star behavior' directly on star expressions (instead of on μ-term representations). For this purpose he appropriately adapted Salomaa's complete proof system [Sal66] for language equivalence on regular expressions to a system Mil that is sound for equality of denoted star behaviors. But Milner noticed that completeness of Mil cannot be shown in analogy to Salomaa's completeness proof. He formulated completeness of Mil as an open problem, because he realized a significant difficulty due to a peculiarity by which the process semantics contrasts starkly with the language semantics of regular expressions.

The process semantics of regular expressions is incomplete in the following sense. While for every finite-state automaton M there is a regular expression e whose language interpretation $[\![e]\!]_L$ coincides with the language accepted by M (formally $L(M) = [\![e]\!]_L$), it is not the case that every finite-state process is the process interpretation of some star expression, not even modulo bisimilarity. Giving a counterexample that demonstrates this, Milner proved in [Mil84] that the process graph G_1 below with linear recursive equational specification $\mathcal{S}(G_1)$ does not define a star behavior, and hence is not bisimilar to the process interpretation of a star expression. He conjectured that the same is true for the process graph G_2 below with specification $\mathcal{S}(G_2)$. That was confirmed later by Bosscher [Bos97].

$$\mathcal{S}(G_1) = \begin{cases} X_1 = a_2 \cdot X_2 + a_3 \cdot X_3 \\ X_2 = a_1 \cdot X_1 + a_3 \cdot X_3 \\ X_3 = a_1 \cdot X_1 + a_2 \cdot X_2 \end{cases}$$





$$\mathcal{S}(G_2) = \begin{cases} Y_1 = 1 + a \cdot Y_2 \\ Y_2 = 1 + b \cdot Y_1 \end{cases}$$

(Here and later we highlight the start vertex of a process graph by a brown arrow →, and emphasize a vertex v with immediate termination in brown as ◉ including a boldface ring.) It follows that the systems of $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ of guarded equations with star expressions cannot be solved by star expressions modulo bisimilarity. Due to soundness of Mil, also the specifications $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ are unsolvable by star expressions when equality is interpreted as provability in Mil. However, if all actions in the process graphs G_1 and G_2 are replaced by a single action a, obtaining graphs $G_1^{(a)}$ and $G_2^{(a)}$, then the arising specifications $\mathcal{S}(G_1^{(a)})$ and $\mathcal{S}(G_2^{(a)})$ are solvable, modulo bisimilarity, and also with respect to provability in Mil. Indeed it is easy to verify that solutions are obtained by letting $X_1 := X_2 := X_3 := a^* \cdot 0$ in $\mathcal{S}(G_1^{(a)})$, and by letting $Y_1 := Y_2 := a^*$ in $\mathcal{S}(G_2^{(a)})$.

The extraction procedure of solutions of specifications in Salomaa's proof completeness is able to solve every linear system of recursion equations, independently of the actions occurring. It follows that an analogous procedure is not possible for solving systems of linear recursion equations in the process semantics. The extraction procedure for linear specifications with respect to the language semantics is possible because both laws for distributing $\cdot$ over $+$ are available, and indeed are part of Salomaa's proof system. But Mil

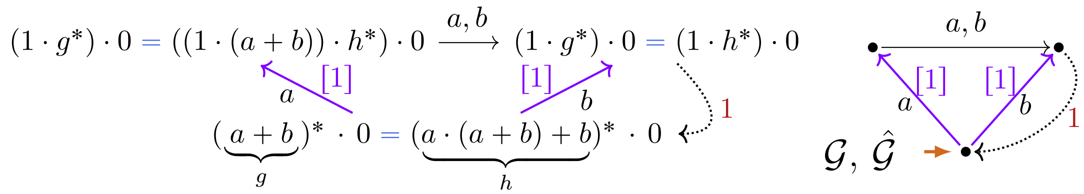


Figure 1: A LLEE-witnessed coinductive proof of $(a+b)^* \cdot 0 = (a \cdot (a+b) + b)^* \cdot 0$ with underlying 1-chart $\mathcal{G}$ and LLEE-witness $\hat{\mathcal{G}}$ (with colored, indexed loop-entry transitions). The proof uses the abbreviations $g := a + b$ and $h := a \cdot (a+b) + b$.

does not contain the *left*-distributivity law $x \cdot (y+z) = x \cdot y + x \cdot z$, because it famously is not sound under bisimilarity. In the presence of only *right*-distributivity $(x+y) \cdot z = x \cdot z + y \cdot z$ in Mil the extraction procedure from Salomaa's proof does not work, because failure of *left*-distributivity oftentimes prevents expressions to be rewritten in such a way that the fixed-point rule RSP* in Mil can be applied successfully. But if RSP* is replaced in Mil by a general unique-solvability rule scheme for guarded systems of equations (see Definition 2.11), then a complete system arises (noted in [Gra06]). Therefore completeness of Mil hinges on whether the fixed-point rule RSP* enables to prove equal any two star-expression solutions of a given guarded system of equations, on the basis of the purely equational part Mil^- of Mil.

As a stepping stone for tackling this difficult question, we characterize the derivational power that the fixed-point rule RSP* adds to the subsystem Mil^- of Mil. We do so by means of 'coinductive proofs' whose shapes have the 'loop existence and elimination property' LEE from [GF20a]. This property stems from the interpretation of (1-free) star expressions, which is defined by induction on syntax trees, creating a hierarchy of 'loop subgraphs'. Crucially for our purpose, linear guarded systems of equations that correspond to finite process graphs with LEE *are* uniquely solvable modulo provability in Mil^-. The reason is that process graphs with LEE, which need not be in the image of the process semantics, are amenable to applying *right*-distributivity and the rule RSP* for an extraction procedure like in Salomaa's proof (see Section 6). These process graphs can be expressed modulo bisimilarity by some star expression, which can be used to show that any two solutions modulo Mil^- of a specification of LEE-shape are Mil-provably equal. This is a crucial step in the completeness proof by Fokkink and myself in [GF20a] for the tailored restriction BBP of Milner's system Mil to '1-free' star expressions.

Thus motivated, we define a 'LLEE-witnessed coinductive proof' as a process graph $\mathcal{G}$ with 'layered' LEE (LLEE) whose vertices are labeled by equations between star expressions. The left- and the right-hand sides of the equations in the vertices of $\mathcal{G}$ have to form a solution vector of a specification corresponding to the process graph $\mathcal{G}$. That specification, however, needs to be satisfied only up to provability in Mil^- from sound assumptions. Such coinductive derivations are typically circular, like the one depicted in Figure 1 of the semantically valid equation $(a+b)^* \cdot 0 = (a \cdot (a+b) + b)^* \cdot 0$. That example is intended to give a first impression of the concepts involved, despite of the fact that some details can only be appreciated later, when this example will be revisited in Example 5.3. We describe these concepts below.

The process graph $\mathcal{G}$ in Figure 1, which is given together with a labeling $\hat{\mathcal{G}}$ that is a 'LLEE-witness' of $\mathcal{G}$. The colored transitions with marking labels $[n]$, for $n \in \mathbb{N}^+$, indicate the LLEE-structure of $\mathcal{G}$, see Section 4. The graph $\mathcal{G}$ underlies the coinductive proof on the left (see Example 5.3 for a justification). $\mathcal{G}$ is a '1-chart' that is, a process graph with

$$\frac{\overbrace{(a+b)^*}^{e_0^*} \cdot 0 = \overbrace{(a\cdot(a+b)+b)}^{f} \cdot (\overbrace{(a+b)^*}^{e_0^*} \cdot 0) + 0}{(a+b)^* \cdot 0 = (a\cdot(a+b)+b)^* \cdot 0}\;\iota,\ \text{RSP}^* \quad \frac{e_0^* \cdot 0 = f \cdot (e_0^* \cdot 0) + 0}{e_0^* \cdot 0 = f^* \cdot 0}$$

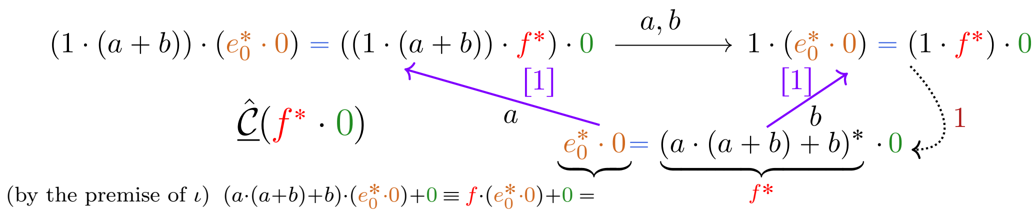


Figure 2: Mimicking an instance ι of the fixed-point rule RSP* (above) in Milner's system $\mathsf{Mil} = \mathsf{Mil}^- + \text{RSP}^*$ by a coinductive proof (below) over $\mathsf{Mil}^- + \{\text{premise of } \iota\}$ with LLEE-witness $\hat{\underline{\mathcal{C}}}(f^* \cdot 0)$.

1-transitions that represent empty steps. Here and later we depict 1-transitions as dotted arrows. For 1-charts, '1-bisimulation' is the adequate concept of bisimulation (Definition 2.4).

We showed in [Gra21c, Gra20] that the process (chart) interpretation $\mathcal{C}(e)$ of a star expression e is the image of a 1-chart $\underline{\mathcal{C}}(e)$ with LLEE under a functional 1-bisimulation. In this example, $\mathcal{G} = \underline{\mathcal{C}}(h^* \cdot 0)$ maps by a functional 1-bisimulation to interpretations of both expressions in the conclusion. The correctness conditions for such coinductive proofs are formed by the requirement that the left-, and respectively, the right-hand sides of formal equations form 'Mil^--provable solutions' of the underlying process graph: an expression at a vertex v can be reconstructed, provably in Mil^-, from the transitions to, and the expressions at, immediate successor vertices of v. Crucially we establish in Section 6, by a generalization of arguments in [GF20a, GF20b] using RSP*, that every LLEE-witnessed coinductive proof over Mil^- can be transformed into a derivation in Mil with the same conclusion.

This raises the question of whether the fixed-point rule RSP* of Mil adds any derivational power to Mil^- that goes beyond those of LLEE-witnessed coinductive proofs over Mil^-, and if so, how far precisely. In Section 7 we show that every instance of the fixed-point rule RSP* can be mimicked by a LLEE-witnessed coinductive proof over Mil^- in which also the premise of the rule may be used. It follows that the derivational power that RSP* adds to Mil^- within Mil consists of iterating such LLEE-witnessed coinductive proofs along finite (meta-)prooftrees. The example in Figure 2 is intended to give a first idea of the construction that we will use (in the proof of Lemma 7.2) to mimic instances of RSP*. Here this construction results in a coinductive proof that only differs slightly from the one with the same underlying LLEE-1-chart we saw earlier. We will revisit this example in Example 7.6.

Based on the two transformations from coinductive proofs to derivations in Mil, and of applications of the fixed-point rule to coinductive proofs, we reformulate Milner's system Mil as a theorem-equivalent proof system cMil. For this, we replace the fixed-point rule RSP* in Mil with a rule that permits to infer an equation $e = f$ from a finite set Γ of equations if there is a LLEE-witnessed coinductive proof over Mil^- plus the equations in Γ that has conclusion $e = f$. We also define a theorem-equivalent system CLC ('combining LLEE-witnessed coinductive provability') with the equational coinductive proof rule alone. In the formalization of these systems we depart from the the exposition in [Gra21a, Gra21b]. There, we used a hybrid concept of formulas that included entire coinductive proofs, which then could be used as specific rule premises. Here, the proof systems are purely equational, and coinductive proofs occur only as side-conditions of rules that formalize coinductive provability.

Additionally, we formulate proof systems $\overline{\mathsf{cMil}}$ and CC that arise from cMil and CLC by dropping 'LLEE-witnessed' as a requirement for coinductive proofs. These systems are (obviously) complete for bisimilarity of process interpretations, because they can mimic the unique solvability rule scheme for guarded systems of specifications mentioned before.

Inspiration for cyclic proofs from related and previous work. Apart from their origin from a question in process theory, the results described here were inspired by coinductively motivated proof systems with derivations of cyclic form, and by our previous work on their proof-theoretic links to traditional equational proof systems. This is a brief account of those direct influences, without drawing wider connections to work on cyclic proofs.

About proofs by cyclic arguments that express bisimulations we learned from Rutten and Jacobs [JR97, Rut98], and about formalized cyclic derivations via the coinductively motivated proof systems by Brandt and Henglein [BH98]: $\mathsf{BH}^{=}$ for unwinding-equivalence, and $\mathsf{BH}^{\leqslant}$ for the subtyping relation between recursive types in μ-term notation. Derivations in $\mathsf{BH}^{=}$ roughly represent bisimulations up to transitivity and symmetry. Via a connection of $\mathsf{BH}^{=}$ to a tableaux-like system $\mathsf{AK}^{=}$ by Ariola and Klop [AK96] with cyclic deductions we later recognized that a tableaux system HS with loop-detecting deductions of cyclic form similar to $\mathsf{AK}^{=}$ had already been used earlier by Hüttel and Stirling in [HS91, HS98] to show that bisimilarity of normed context-free processes is decidable. In [Gra05b] we developed a simple coinductively motivated proof system $\mathsf{G}^{=_L}$ for language equivalence $=_L$ of regular expressions. That system was later refined substantially (using more flexible rules, similar to $\mathsf{BH}^{=}$), and generalized (similar to as $\mathsf{BH}^{\leqslant}$ generalized $\mathsf{BH}^{=}$) to one for language containment of regular expressions by Henglein and Nielsen in [HN11].

However, all of these proof systems use derivations in the form of proof-trees. Thus they permit cyclic derivations only of 'palm-tree' form (ordered trees with backlinks, [Tar72]). In contrast, we will permit cyclic derivations to have the form of general transition graphs. In this manner we 'free' proof-graphs from the requirement to only exhibit 'vertical sharing' [Blo01], and move close to informal reasoning as used in coalgebra like in [JR97, Rut98].

The transformations that we construct in Section 6 and Section 7 have been inspired by the proof-theoretic interpretations that we developed in [Gra05a] between the proof system $\mathsf{AC}^{=}$ for unwinding-equality of recursive types by Amadio and Cardelli [AC93] (a Hilbert-style proof system with a fixed-point rule analogous to RSP*) and the system $\mathsf{BH}^{=}$ by Brandt and Henglein. The transformation from cMil to Mil in Section 6 is also similar to one we described in [Gra05b] that transforms derivations in $\mathsf{G}^{=_L}$ into derivations in Salomaa's system $\mathsf{F_1}$ [Sal66] for $=_L$ (where $\mathsf{F_1}$ contains a fixed-point rule just like RSP*).

Relation with the conference article. This article provides significantly more details and explanations than the article [Gra21b] in the proceedings of CALCO 2021. Furthermore it contains the following additions of content:

- ▻ Detailed proofs for the proof transformations from cMil to Mil (in Section 6), and from Mil to cMil (in Section 7).
- ▻ Proof-theoretic explanation of the transformations as the elimination of rules that are derivable or admissible (based on Definition 2.9, Lemma 2.10 in Section 2).

- – For the proof transformation from the coinductive reformulation cMil to Milner's system Mil we show that circular coinductive proofs over the purely equational part Mil^{-} of Mil are admissible in Mil (see Lemma 6.10).
- – For the proof transformation from Milner's system Mil to its coinductive reformulation cMil we show that the fixed-point rule RSP* of Mil is derivable in cMil (see Lemma 7.4).

▻ A statement that illustrates that the transformation from cMil to Mil can provide inroads for a completeness proof of Milner's system Mil (see Corollary 6.14).

▻ A diagram that gives an overview of all developed proof transformations (see Figure 18).

▻ An example that provides a sanity-check on the proof transformation from cMil to Mil. It demonstrates that this transformation cannot work for mimicking instances of the fixed-point rule without guardedness side-condition (see Non-Example 7.7).

▻ An illustration of the difference between LEE-witnesses (witnesses of the loop existence and elimination condition LEE), and witnesses of 'layered LEE' (LLEE-witnesses) by an example that is based on different runs of the loop elimination procedure (see Figure 5).

▻ A section in which we informally link the concept of 'provable solution' (Definition 2.16) that is the basis for our concept of 'coinductive proof' to a coalgebraic formulation of this concept by Schmid, Rot, and Silva in [SRS21] (see Section 3).

Relation with the completeness proof of Milner's system in [Gra22a]**.** The completeness proof of Milner's system Mil summarized in [Gra22a] with report [Gra22b] was finished and written only after the article [Gra21b] for CALCO 2021. Indeed, we found the results in Section 6 of [Gra21b] and here in Section 7 (that instances of the fixed-point rule RSP* can be mimicked by LLEE-witnessed coinductive proofs) in an effort to prepare for that completeness proof. In particular, we wanted to be able to argue for the expedience of the use of LLEE-1-charts (see Definition 4.9) despite of the fact that reasoning with LLEE-1-charts towards a completeness proof of Mil encounters a crucial obstacle[1]. Without any argumentation that links derivations in Milner's system closely to LLEE-1-charts, it could be conceivable that this obstacle does not have any wider significance. Namely, it could be entirely specific to the use of LLEE-1-charts, while a completeness proof might possibly be based on quite different concepts. The situation changed, however, after we realized that instances of the fixed-point rule can always be modeled (see Lemma 7.4) by cyclic proofs of the shape of guarded LLEE-1-charts (see Definition 5.1), and proofs in Milner's system can be transformed (see Theorem 7.8) into meta-prooftrees of such cyclic proofs (derivations in the system CLC, see Definition 5.8). On the basis of these results we could argue that in principle every completeness proof of Milner's system Mil can be routed through (see Section 8) arguments in which LLEE-1-charts appear front and central.

The completeness proof of Mil in [Gra22a, Gra22b] uses additional observations and concepts (above all, a 'crystallization procedure' of LLEE-1-charts for minimization under 1-bisimilarity), and is not formulated in terms of the cyclic proof systems that we introduce here. However, the results of Section 6, the transformation of LLEE-witnessed coinductive proofs into derivations in Mil (see Proposition 6.8) are of central importance for formulating the completeness proof in [Gra22a, Gra22b]. Indeed, they prove the lemmas **(E)** (extraction of provable solutions from guarded LLEE-1-charts) and **(SE)** (provable solution equality in guarded LLEE-1-charts) of the completeness proof as listed in Section 5 of [Gra22a, Gra22b].

[1] Namely the fact that LLEE-1-charts are not closed under '1-bisimulation collapse', an observation that is central for the crystallization procedure sketched in [Gra22a].

Conversely, we here use another one of the lemmas in Section 5 of [Gra22a, Gra22b], the lemma **(T)** (transfer of provable solutions conversely along functional 1-bisimulations), for illustrating the results of Section 6: we prove two (specific) completeness properties of LLEE-witnessed coinductive proofs in relation to Milner's system (see Corollary 6.14).

The completeness proof for Milner's system Mil with respect to process semantics equality of star expressions implies that the coinductive versions cMil and CLC of Mil that we introduce here are complete (in the same sense) as well. This is because our main result (see Theorem 7.8) states that cMil and CLC have the same derivational power as Mil.

Contents

Overview. We start in Section 2 with introducing basic definitions concerning the process semantics of regular expressions, and concepts that we will need. We define star expressions, finite process graphs with 1-transitions, (1-)bisimulations and 1-bisimilarity, and the process semantics of star expressions. Then we introduce equational-logic based, and equation-based proof systems, with Milner's system Mil and two variants as first examples. Also, we define when inference rules are derivable or admissible in such a proof system, and establish easy interconnections. Finally we define the concept of solution for 1-charts with respect to an equational proof system. In Section 3 we link to an insightful coalgebraic characterization of provable solutions of 1-charts that is due to Schmid, Rot, and Silva in [SRS21]. We reformulate it in our terminology, but do not prove it in detail, as our development does not depend on it. In Section 4 we explain concepts and definitions concerning the (layered) loop existence and elimination property (L)LEE from [GF20b, GF20a], and recall the '1-chart interpretation' of star expressions from [Gra20, Gra21c], which guarantees LLEE.

In Section 5 we introduce 'coinductive proofs' over equational proof systems. We formulate proof systems CC and CLC with appropriate rule schemes that permit to use and combine coinductive proofs, and respectively, LLEE-witnessed coinductive proofs. Then we introduce the coinductive reformulation cMil of Mil as an extension of the equational part Mil^{-} of Mil. We also establish basic proof-theoretic connections between these new systems.

In Section 6 we show that coinductive proofs over proof systems with derivational power not greater than Milner's system Mil can be transformed into derivations in Mil. We use this fact to obtain a proof transformation from cMil to Mil.

In Section 7 we demonstrate that every instance ι of the fixed-point rule RSP^* of Mil can be mimicked by a coinductive proof of the conclusion of ι where (correctness conditions of) that proof may use the equational part Mil^- of Mil plus the premise equation of ι. We apply this central observation for defining a proof transformation from Mil to cMil. With this transformation and the one constructed in Section 6 we prove that the proof systems CLC and cMil are theorem-equivalent with Milner's system Mil.

In the final section, Section 8, we recapitulate our motivation for introducing coinductive circular proofs, and summarize our results. We argue that the coinductive proof systems CLC and cMil can be viewed as being located roughly half-way in between derivations in Mil and bisimulations between process interpretations of star expressions. We conclude with initial ideas about a proof strategy for a completeness proof of CLC and cMil, which would yield a completeness of Mil.

2. Process semantics for star expressions, and Milner's proof system

Here we fix terminology concerning star expressions, 1-charts, 1-bisimulations; we exhibit Milner's system (and a few variants), and recall the chart interpretation of star expressions.

Definition 2.1 (star expressions)**.** Let A be a set of *actions*. The set $\mathit{StExp}(A)$ of *star expressions over actions in* A are strings that are defined by the following grammar:

$$e, e_1, e_2 ::= 0 \mid 1 \mid a \mid (e_1 + e_2) \mid (e_1 \cdot e_2) \mid e^* \qquad (\text{where } a \in A)$$

We will drop outermost brackets, and those that are expendable according to the precedence of star * over composition $\cdot$ and choice $+$, and of composition $\cdot$ over choice $+$. We use e, f, g, h, possibly indexed and/or decorated, as identifiers (for reasoning on the meta-level like with 'syntactical variables' [Sho67]) for star expressions. We write $\equiv$ for syntactic equality between star expressions denoted by such identifiers, and values of star expression functions, in a given context, but we permit $=$ in formal equations between star expressions. We denote by $\mathit{Eq}(A)$ the set of formal equations $e = f$ between two star expressions $e, f \in \mathit{StExp}(A)$.

We define sum expressions $\sum_{i=1}^{n} e_i$ inductively as 0 if $n = 0$, as e_1 if $n = 1$, and as $(\sum_{i=1}^{n-1} e_i) + e_n$ if $n > 0$, for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The *(syntactic) star height* $|e|_*$ of a star expression $e \in \mathit{StExp}(A)$ is the maximal nesting depth of stars in e, defined inductively by: $|0|_* := |1|_* := |a|_* := 0$, $|e_1 + e_2|_* := |e_1 \cdot e_2|_* := \max\{|e_1|_*, |e_2|_*\}$, and $|e^*|_* := 1 + |e|_*$.

Definition 2.2 (1-charts, and charts)**.** A 1*-chart* is a 6-tuple $\langle V, A, 1, v_s, \to, \downarrow\rangle$ where V is a <u>finite</u> set of *vertices*, A is a set of *(proper) action labels*, $1 \notin A$ is the specified *empty step label*, $v_s \in V$ is the *start vertex* (hence $V \neq \varnothing$), $\to \subseteq V \times \underline{A} \times V$ is the *labeled transition relation*, where $\underline{A} := A \cup \{1\}$ is the set of action labels including 1, and $\downarrow \subseteq V$ is a set of *vertices with immediate termination*. In such a 1-chart, we call a transition in $\to \cap (V \times A \times V)$ (labeled by a *proper action* in A) a *proper transition*, and a transition in $\to \cap (V \times \{1\} \times V)$ (labeled by the *empty-step symbol* 1) a 1*-transition*. Reserving non-underlined action labels like $a, b, \ldots$ for proper actions, we use underlined action label symbols like $\underline{a}$ for actions labels in the set $\underline{A}$; in doing so we highlight also in firebrick transition labels that may involve 1.

We say that a 1-chart is *weakly guarded* if it does not contain cycles of 1-transitions.

By a *chart* we mean a 1-chart $\underline{\mathcal{C}}$ that is 1*-transition free* in the sense that all of its transitions are proper. We will use the symbols $\underline{\mathcal{C}}$ and $\mathcal{C}$ (also with subscripts) as identifiers for 1-charts, and charts, respectively. We use the notations $V(\underline{\mathcal{C}})$, and $V(\mathcal{C})$ for quick reference to the set of vertices of a 1-chart $\underline{\mathcal{C}}$, and of a chart $\mathcal{C}$.

Below we define the process semantics of star (regular) expressions as (1-free) charts, and hence as finite, rooted labeled transition systems, which will be compared with (1-)bisimilarity. The charts that will be obtained in this way correspond to non-deterministic finite-state automata that are defined by iterating partial derivatives [Ant96] (1996) of Antimirov (who did not aim at a process semantics). Indeed, Antimirov's result that every regular expression only has finitely many iterated partial derivatives (Corollary 3.5 in [Ant96]) guarantees finiteness of chart interpretations as defined below. We will use the notation $\mathcal{C}(e)$ with as meaning 'the chart induced by (the process interpretation of) the star expression e'.

Definition 2.3. The *chart interpretation of* a star expression $e \in \mathit{StExp}(A)$ is the 1-transition free chart $\mathcal{C}(e) = \langle V(e), A, 1, e, \rightarrow \cap (V(e) \times A \times V(e)), {\downarrow} \cap V(e)\rangle$, where $V(e)$ consists of all star expressions that are reachable from e via the labeled transition relation $\rightarrow \subseteq \mathit{StExp}(A) \times A \times \mathit{StExp}(A)$ that is defined, together with the immediate-termination relation ${\downarrow} \subseteq \mathit{StExp}(A)$, via derivability in the transition system specification (TSS) $\mathcal{T}(A)$, for $a \in A$, $e, e_1, e_2, e', e'_1, e'_2 \in \mathit{StExp}(A)$:

$$\frac{}{1{\downarrow}} \qquad \frac{e_i{\downarrow}}{(e_1 + e_2){\downarrow}} \qquad \frac{e_1{\downarrow} \qquad e_2{\downarrow}}{(e_1 \cdot e_2){\downarrow}} \qquad \frac{}{(e^*){\downarrow}}$$

$$\frac{}{a \xrightarrow{a} 1} \qquad \frac{e_i \xrightarrow{a} e'_i}{e_1 + e_2 \xrightarrow{a} e'_i} \qquad \frac{e_1 \xrightarrow{a} e'_1}{e_1 \cdot e_2 \xrightarrow{a} e'_1 \cdot e_2} \qquad \frac{e_1{\downarrow} \qquad e_2 \xrightarrow{a} e'_2}{e_1 \cdot e_2 \xrightarrow{a} e'_2} \qquad \frac{e \xrightarrow{a} e'}{e^* \xrightarrow{a} e' \cdot e^*}$$

If $e \xrightarrow{a} e'$ is derivable in $\mathcal{T}(A)$, for $e, e' \in \mathit{StExp}(A)$, $a \in A$, then we say that e' is a *derivative* of e. If $e{\downarrow}$ is derivable in $\mathcal{T}(A)$, then we say that e *permits immediate termination.*

In Section 4 we define a refinement of this interpretation from [Gra21c] into a 1-chart interpretation. In both versions, (1-)charts obtained will be compared with respect to 1-bisimilarity that relates the behavior of 'induced transitions' of 1-charts. By an *induced a-transition* $v \xrightarrow{(a)} w$, for a proper action $a \in A$, in a 1-chart $\underline{\mathcal{C}}$ we mean a path $v \xrightarrow{1} \cdots \xrightarrow{1} \cdot \xrightarrow{a} w$ in $\underline{\mathcal{C}}$ that consists of a finite number of 1-transitions that ends with a proper a-transition. By *induced termination* $v{\downarrow}^{(1)}$, for $v \in V$ we mean that there is a path $v \xrightarrow{1} \cdots \xrightarrow{1} \tilde{v}$ with $\tilde{v}{\downarrow}$ in $\underline{\mathcal{C}}$.

Definition 2.4 ((1-)bisimulation)**.** Let $\underline{\mathcal{C}}_i = \langle V_i, A, 1, v_{\mathrm{s},i}, \rightarrow_i, {\downarrow_i}\rangle$ be 1-charts, for $i \in \{1, 2\}$.

By a 1-*bisimulation between* $\underline{\mathcal{C}}_1$ *and* $\underline{\mathcal{C}}_2$ we mean a binary relation $B \subseteq V_1 \times V_2$ such that $\langle v_{\mathrm{s},1}, v_{\mathrm{s},2}\rangle \in B$ holds (that is, B relates the start vertices of $\mathcal{C}_1$ and $\mathcal{C}_2$), and for every $\langle v_1, v_2\rangle \in B$ the following three conditions hold:

(forth): $\forall v'_1 \in V_1 \forall a \in A\big(\, v_1 \xrightarrow{(a)}_1 v'_1 \implies \exists v'_2 \in V_2\big(\, v_2 \xrightarrow{(a)}_2 v'_2 \wedge \langle v'_1, v'_2\rangle \in B\,\big)\big)$,

(back): $\forall v'_2 \in V_2 \forall a \in A\big(\, \exists v'_1 \in V_1\big(\, v_1 \xrightarrow{(a)}_1 v'_1 \wedge \langle v'_1, v'_2\rangle \in B\,\big) \impliedby v_2 \xrightarrow{(a)}_2 v'_2\,\big)$,

(termination): $v_1{\downarrow}_1^{(1)} \iff v_2{\downarrow}_2^{(1)}$.

We write $\underline{\mathcal{C}}_1 \leftrightarrow_B \underline{\mathcal{C}}_2$ if B is a 1-bisimulation between $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$. We denote by $\underline{\mathcal{C}}_1 \leftrightarrow \underline{\mathcal{C}}_2$ and say that $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$ are 1-*bisimilar*, if there is a 1-bisimulation between $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$.

By a *functional* 1-bisimulation *from* $\underline{\mathcal{C}}_1$ *to* $\underline{\mathcal{C}}_2$ we mean a 1-bisimulation B between $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$ that is defined by a function $\phi : V_1 \rightarrow V_2$ as its graph, that is, by $B = \mathit{graph}(\phi) = \{\langle v, \phi(v)\rangle \mid v \in V\}$; in this case we write $\underline{\mathcal{C}}_1 \rightrightarrows_\phi \underline{\mathcal{C}}_2$. We write $\underline{\mathcal{C}}_1 \rightrightarrows \underline{\mathcal{C}}_2$ if there is a functional 1-bisimulation from $\underline{\mathcal{C}}_1$ to $\underline{\mathcal{C}}_2$.

We note that for 1-transition-free 1-charts the bisimulation conditions specialize to their usual form: the induced transitions $\xrightarrow{(\cdot)}$ in (forth) and (back) specialize to proper transitions $\xrightarrow{\cdot}$, and induced termination ${\downarrow}^{(1)}$ in (termination) specializes to immediate termination ${\downarrow}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be charts (1-transition-free 1-charts). We write $\mathcal{C}_1 \leftrightarrow_B \mathcal{C}_2$, and say that B

is a *bisimulation between* $\mathcal{C}_1$ and $\mathcal{C}_2$ if B is a 1-bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. We write $\mathcal{C}_1 \leftrightarroweq \mathcal{C}_2$, and say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *bisimilar* if there is a bisimulation between $\mathcal{C}_1$ and $\mathcal{C}_2$. We write $\mathcal{C}_1 \rightrightarrows \mathcal{C}_2$ if there is 1-bisimulation from chart $\mathcal{C}_1$ to chart $\mathcal{C}_2$).

Let $\underline{\mathcal{C}}$ be a 1-chart, and $B \subseteq V(\underline{\mathcal{C}}) \times V(\underline{\mathcal{C}})$. We say that B is a 1-bisimulation *on* $\underline{\mathcal{C}}$ if B is a 1-bisimulation between $\underline{\mathcal{C}}$ and $\underline{\mathcal{C}}$. Let $\mathcal{C}$ be a chart, and $B \subseteq V(\mathcal{C}) \times V(\mathcal{C})$. We say that B is a bisimulation *on* $\mathcal{C}$ if B is a bisimulation between $\mathcal{C}$ and $\mathcal{C}$.

We now define 'process semantics equality' of two star expressions as bisimilarity of their chart interpretations. We do not introduce the process semantics of star expressions as 'star behaviors' (bisimilarity equivalence classes of their chart interpretations) as Milner in [Mil84], but only the relation that two star expressions denote the same star behavior.

Definition 2.5 (process semantics equality)**.** We define *process semantics equality* as the binary relation $=_{[\![\cdot]\!]_P} \subseteq \mathit{StExp}(A) \times \mathit{StExp}(A)$ by stipulating it, for all $e, f \in \mathit{StExp}(A)$, as bisimilarity of the (1-free) chart interpretations of e and f :

$$e =_{[\![\cdot]\!]_P} f \quad :\Longleftrightarrow \quad \mathcal{C}(e) \leftrightarroweq \mathcal{C}(f)\,.$$

Definition 2.6 (proof system $\mathcal{EL}$, $\mathit{Eq}(\cdot)$-based/$\mathcal{EL}$-based proof systems)**.** Let A be a set.

By an *$\mathit{Eq}(A)$-based proof system* we will mean a Hilbert-style proof system whose *formulas* are the equations in $\mathit{Eq}(A)$ between star expressions over A. For an $\mathit{Eq}(A)$-based proof system $\mathcal{S}$ and a set $\Gamma \subseteq \mathit{Eq}(A)$ we denote by $\mathcal{S}+\Gamma$ the $\mathit{Eq}(A)$-based proof system whose *rules* are those of $\mathcal{S}$, and whose *axioms* are those of $\mathcal{S}$ plus the equations in Γ.

The *basic proof system $\mathcal{EL}(A)$ of equational logic* for star expressions over A is an $\mathit{Eq}(A)$-based proof system that has the following *rules*:

$$\frac{}{e = e}\,\mathrm{Refl} \qquad \frac{e = f}{f = e}\,\mathrm{Symm} \qquad \frac{e = f \qquad f = g}{e = g}\,\mathrm{Trans} \qquad \frac{e = f}{C[e] = C[f]}\,\mathrm{Cxt}$$

that is, the rules Refl (for reflexivity), and the rules Symm (for symmetry), Trans (for transitivity), and Cxt (for filling a context), where $C[]$ is a 1-hole star expression context.

By an *$\mathcal{EL}(A)$-based system* we mean an $\mathit{Eq}(A)$-based proof system whose rules include the rules of the basic system $\mathcal{EL}(A)$ of equational logic (additionally, it may specify an arbitrary set of axioms). We will use the letter $\mathcal{S}$ as identifier for $\mathcal{EL}$-based proof systems.

Definition 2.7. Let $\mathcal{S}$ be an $\mathit{Eq}(A)$-based proof system. Let $e, f \in \mathit{StExp}(A)$. We say that $e = f$ is *derivable in* $\mathcal{S}$, which we denote here by $e_1 =_{\mathcal{S}} e_2$ (instead of the more commonly used notation $\vdash_{\mathcal{S}} e_1 = e_2$), if there is a derivation without assumptions in $\mathcal{S}$ that has conclusion $e = f$. If $e = f$ is derivable in $\mathcal{S}$, we also say that $e = f$ is a *theorem* of $\mathcal{S}$.

Definition 2.8 (sub-system, theorem equivalence/subsumption of $\mathit{Eq}(\cdot)$-based proof systems)**.** Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $\mathit{Eq}(A)$-based proof systems.

We say that $\mathcal{S}_1$ is a *sub-system* of $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \sqsubseteq \mathcal{S}_2$, if every axiom of $\mathcal{S}_1$ is an axiom of $\mathcal{S}_2$, and every rule of $\mathcal{S}_1$ is also a rule of $\mathcal{S}_2$. We say that $\mathcal{S}_1$ *is theorem-subsumed by* $\mathcal{S}_2$, denoted by $\mathcal{S}_1 \lesssim \mathcal{S}_2$, if every theorem of $\mathcal{S}_1$ is also a theorem of $\mathcal{S}_2$, that is, if $e =_{\mathcal{S}_1} f$ implies $e =_{\mathcal{S}_2} f$, for all $e, f \in \mathit{StExp}(A)$. We say that $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are theorem-equivalent*, denoted by $\mathcal{S}_1 \sim \mathcal{S}_2$, if $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same theorems (that is, if $\mathcal{S}_1 \lesssim \mathcal{S}_2$, and $\mathcal{S}_2 \lesssim \mathcal{S}_1$).

For the definitions of the concept of 'derivable', 'correct', and 'admissible' rule in Definition 2.9 below for an $\mathit{Eq}(A)$-based proof system we introduce an informal concept of derivation rule that will suffice for our purpose. For abstract formulations of rules, and for the concepts of derivability, correctness, and admissibility of rules we refer:

(i) to [Gra05a], where these concepts have been gathered and formally treated for Hilbert-style proof systems (as well as for natural-deduction style proof systems),
(ii) and to [Gra04], where for 'abstract Hilbert systems', systematic connections between these concepts of rules have been studied, also with respect to how rules can be eliminated from derivations.

Let $\mathcal{S}$ be an $Eq(A)$-based proof system for star expressions over A. Let $n \in \mathbb{N}$. By a(n) *(n-premise) rule R for $\mathcal{S}$* we mean an inference scheme all of whose instances are of the form:

$$\frac{e_1 = f_1 \quad \ldots \quad e_n = f_n}{e = f}\, R$$

with star expressions $e, f, e_1, \ldots, e_n, f_1, \ldots, f_n \in StExp(A)$. For such a rule R for $\mathcal{S}$ we denote by $\mathcal{S}+R$ the $Eq(A)$-based proof system that extends $\mathcal{S}$ by adding R as an additional rule.

Definition 2.9 (derivable, correct, and admissible rules)**.** Let $\mathcal{S}$ be an $Eq(A)$-based proof system. Let R be a rule for $\mathcal{S}$.

We say that R is *derivable in $\mathcal{S}$* if every instance ι of R can be mimicked by a derivation $\mathcal{D}_\iota$ in $\mathcal{S}$ by which we mean that the set of assumptions of $\mathcal{D}_\iota$ is contained in the set of premises of ι, and the conclusion of $\mathcal{D}_\iota$ is the conclusion of ι.

We say that R is *correct for $\mathcal{S}$* if instances of R can be eliminated from derivations in $\mathcal{S}+R$ in the following limited sense: for every derivation $\mathcal{D}$ in $\mathcal{S}+R$ without assumptions that terminates with an instance of R but all of whose immediate subderivations are derivations in $\mathcal{S}$ there is a derivation $\mathcal{D}'$ in $\mathcal{S}$ without assumptions, and with the same conclusion as $\mathcal{D}$.

We say that R is *admissible in $\mathcal{S}$* if $\mathcal{S}+R \sim \mathcal{S}$ holds, that is, the addition of R to $\mathcal{S}$ does not extend the derivable formulas (the theorems) of $\mathcal{S}$.

The definition of 'R is admissible in $\mathcal{S}$' is easily understood to be equivalent with the statement that instances of R can be eliminated from derivations in $\mathcal{S}+R$ without assumptions in the unlimited sense: for every derivation $\mathcal{D}$ in $\mathcal{S}+R$ without assumptions there is a derivation $\mathcal{D}'$ in $\mathcal{S}$ without assumptions, and with the same conclusion as $\mathcal{D}$. Therefore rule admissibility implies rule correctness. This justifies the implication "$\Rightarrow$" in item (i) of the lemma below that gathers basic relationships between the three properties of rules with respect to a proof system as defined above.

Lemma 2.10. *Let R be a rule for an $\mathcal{EL}$-based proof system $\mathcal{S}$ for star expressions over A. Then the following statements link derivability, correctness, and admissibility of R in/for $\mathcal{S}$:*
(i) *R is admissible in $\mathcal{S}$ if and only if R is correct for $\mathcal{S}$.*
(ii) *If R is derivable in $\mathcal{S}$, then R is also correct for $\mathcal{S}$, and due to (i) also admissible in $\mathcal{S}$. However, rule admissibility and correctness does not imply derivability in general.*

Proof. Concerning statement (i) of the lemma we have already argued for the direction "$\Rightarrow$" just above. The direction "$\Leftarrow$" can be established by showing that, if R is correct for $\mathcal{S}$, then every given derivation $\mathcal{D}$ in $\mathcal{S}+R$ can be transformed into a derivation $\mathcal{D}'$ in $\mathcal{S}$ with the same conclusion by eliminating instances of R in top-down direction, using derivation replacements as guaranteed by the defining statement of 'R is correct for $\mathcal{S}$'.

For showing the main part of statement (ii), we consider an n-premise rule R for $\mathcal{S}$ that is a derivable rule of $\mathcal{S}$. In order to show that R is correct for $\mathcal{S}$, we have to show that every derivation $\mathcal{D}$ in $\mathcal{S}+R$ that terminates with an instance ι of R but has immediate subderivations in $\mathcal{S}$ can be transformed into a derivation $\mathcal{D}'$ in $\mathcal{S}$ with the same conclusion. Let $\mathcal{D}$ be such a derivation in $\mathcal{S}+R$ with instance ι of R at the bottom, as illustrated on the right below. Since R is derivable in $\mathcal{S}$ there is a derivation $\mathcal{D}_\iota$ in $\mathcal{S}$ that derives the conclusion of ι from its n premises. Then $\mathcal{D}$ can be transformed according to the following step:

$$\mathcal{D}\left\{\quad \iota\,\frac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n\\ e_1 = f_1 & \ldots & e_n = f_n\end{matrix}}{e = f}\,R\right. \qquad\Longrightarrow\qquad \left.\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n\\ [e_1 = f_1] & \ldots & [e_n = f_n]\\ & \mathcal{D}_\iota & \\ & e = f & \end{matrix}\quad\right\}\mathcal{D}' \tag{2.1}$$

(where $[e_1 = f_1], \ldots [e_n = f_n]$ denote the assumption classes of $e_1 = f_n, \ldots, e_1 = f_n$ in leafs at the top of the prooftree $\mathcal{D}_\iota$). The result of this step is a derivation $\mathcal{D}'$ in $\mathcal{S}$ with the same conclusion as $\mathcal{D}$. Since $\mathcal{D}$ was chosen arbitrary in this statement but with a bottommost instance of R and immediate subderivations in $\mathcal{S}$, we have shown the desired transformation statement, which guarantees that R is correct for $\mathcal{S}$.

If a rule R is correct and admissible in an $\mathcal{EL}$-based proof system $\mathcal{S}$, then R does not need to be derivable. This is because correctness of R in $\mathcal{S}$ cannot be used to mimic such instances of R that do not have theorems of $\mathcal{S}$ as conclusion by derivations in $\mathcal{S}$. As a trivial example we take $\mathcal{S} = \mathcal{EL}(A)$. In $\mathcal{S}$ only reflexivity axioms are theorems. A 1-premise rule that leaves its premise unchanged is clearly admissible in $\mathcal{S}$, but not derivable, because instances with formulas $e = f$ where $e \not\equiv f$ cannot be mimicked by derivations in $\mathcal{S}$. ☐

Now we introduce Milner's proof system Mil, and two of its variants Mil' and $\overline{\mathsf{Mil}}'$. Afterwards we gather basic connections between these systems.

Definition 2.11 (Milner's system Mil, variants and subsystems)**.** Let A be a set of actions.

By the proof system $\mathsf{Mil}^-(A)$ we mean the $\mathcal{EL}(A)$-based proof system for star expressions over A with the following *axiom schemes*:

$$\begin{array}{rcrc}
(\text{assoc}(+)) & (e+f)+g = e+(f+g) & (\text{id}_\text{l}(\cdot)) & 1\cdot e = e\\
(\text{neutr}(+)) & e+0 = e & (\text{id}_\text{r}(\cdot)) & e\cdot 1 = e\\
(\text{comm}(+)) & e+f = f+e & (\text{deadlock}) & 0\cdot e = 0\\
(\text{idempot}(+)) & e+e = e & (\text{rec}(^*)) & e^* = 1+e\cdot e^*\\
(\text{assoc}(\cdot)) & (e\cdot f)\cdot g = e\cdot(f\cdot g) & (\text{trm-body}(^*)) & e^* = (1+e)^*\\
(\text{r-distr}(+,\cdot)) & (e+f)\cdot g = e\cdot g+f\cdot g & &
\end{array}$$

where $e, f, g \in \mathit{StExp}(A)$, and with the *rules* of the system $\mathcal{EL}(A)$ of equational logic.

The *recursive specification principle for star iteration* RSP*, the *unique solvability principle for star iteration* USP_1, and the *general unique solvability principle* USP are the schematically defined rules with side-conditions of the following forms:

$$\frac{e = f\cdot e+g}{e = f^*\cdot g}\,\text{RSP}^*\ \ (\text{if } f{\not\downarrow}) \qquad \frac{e_1 = f\cdot e_1+g \qquad e_2 = f\cdot e_2+g}{e_1 = e_2}\,\text{USP}_1\ \ (\text{if } f{\not\downarrow})$$

$$\frac{\Big\{ e_{i,1} = \Big(\sum_{j=1}^{n_i} f_{i,j}\cdot e_{j,1}\Big)+g_i \Big) \quad e_{i,2} = \Big(\sum_{j=1}^{n_i} f_{i,j}\cdot e_{j,2}\Big)+g_i \Big) \Big\}_{i=1,\ldots,n}}{e_{1,1} = e_{1,2}}\,\text{USP}\ \ \begin{pmatrix}\text{if } f_{i,j}{\not\downarrow}\\ \text{for all } i,j\end{pmatrix}$$

Milner's proof system $\mathsf{Mil}(A)$ is the extension of $\mathsf{Mil}^-(A)$ by adding the rule RSP*. Its variant systems $\mathsf{Mil}'(A)$, and $\overline{\mathsf{Mil}}'(A)$, arise from $\mathsf{Mil}^-(A)$ by adding (instead of RSP*) the rule USP_1, and respectively, the rule USP. $\mathsf{ACI}(A)$ is the system with the axioms for associativity, commutativity, and idempotency for +. We will keep the action set A implicit in the notation.

Lemma 2.12. *Milner's system* Mil *and its variants* Mil' *and* $\overline{\mathsf{Mil}}'$ *are related as follows:*

(i) $\mathsf{Mil}' \lesssim \overline{\mathsf{Mil}}'$,
(ii) $\mathsf{Mil} \sim \mathsf{Mil}'$.

Proof. Statement (i) of the lemma is due to the fact that instances the rule USP_1 are also instances of USP, and therefore $\mathsf{Mil}' = \mathsf{Mil}^{-}{+}\mathrm{USP}_1 \lesssim \mathsf{Mil}^{-}{+}\mathrm{USP} = \overline{\mathsf{Mil}}'$ follows.

For establishing statement (ii) we show that USP_1 is a derivable rule in Mil, and that RSP^* is a derivable rule in Mil'. Then, by Lemma 2.10, (ii), USP_1 is an admissible rule of Mil, thus $\mathsf{Mil}{+}\mathrm{USP}_1 \sim \mathsf{Mil}$, and RSP^* is an admissible rule of Mil', hence $\mathsf{Mil}'{+}\mathrm{USP}_1 \sim \mathsf{Mil}'$. With this we can argue as follows:

$$\mathsf{Mil} \sim \mathsf{Mil}{+}\mathrm{USP}_1 = (\mathsf{Mil}^{-}{+}\mathrm{RSP}^*){+}\mathrm{USP}_1 = (\mathsf{Mil}^{-}{+}\mathrm{USP}_1){+}\mathrm{RSP}^* = \mathsf{Mil}'{+}\mathrm{RSP}^* \sim \mathsf{Mil}'$$

Then we obtain $\mathsf{Mil} \sim \mathsf{Mil}'$ by transitivity of theorem equivalence $\sim$.

For showing that RSP^* is derivable in Mil', we consider an instance of RSP^* as in Definition 2.11, for fixed star expressions e, f with $f{\not\downarrow}$, and g. From its premise $e = f \cdot e + g$ we have to show that the conclusion $e = f^* \cdot g$ of the RSP^* instance can be derived by inferences in $\mathsf{Mil}' = \mathsf{Mil}^{-}{+}\mathrm{USP}_1$. By stepwise use of axioms of Mil^{-} we obtain:

$$\begin{aligned} f^* \cdot g &=_{\mathsf{Mil}^-} (1 + f \cdot f^*) \cdot g =_{\mathsf{Mil}^-} 1 \cdot g + (f \cdot f^*) \cdot g \\ &=_{\mathsf{Mil}^-} 1 \cdot g + f \cdot (f^* \cdot g) =_{\mathsf{Mil}^-} g + f \cdot (f^* \cdot g) =_{\mathsf{Mil}^-} f \cdot (f^* \cdot g) + g \end{aligned}$$

Hence there is a derivation of $f^* \cdot g = f \cdot (f^* \cdot g) + g$ in Mil'. This derivation can be extended, due to $f{\not\downarrow}$, by an instance of USP_1 that is applied to $e = f \cdot e + g$ and $f^* \cdot g = f \cdot (f^* \cdot g) + g$. We obtain a derivation of $e = f^* \cdot g$ in Mil' from the assumption $e = f \cdot e + g$.

For showing that USP_1 is derivable in Mil, we consider an instance of USP_1 as in Definition 2.11, with premises $e_1 = f \cdot e_1 + g$, and $e_2 = f \cdot e_2 + g$, for fixed star expressions e_1, e_2, f with $f{\not\downarrow}$, and g. By two instances of RSP^* we get $e_1 = f^* \cdot g$, and $e_2 = f^* \cdot g$. By applying Symm below $e_2 = f^* \cdot g$, we obtain $f^* \cdot g = e_2$. Then by applying Trans to $e_1 = f^* \cdot g$ and $f^* \cdot g = e_2$ we obtain the conclusion $e_1 = e_2$ of the USP_1 instance. □

Now we define soundness and completeness of equation-based proof systems for star expressions with respect to equivalence relations on star expressions. Then we formulate soundness of Milner's system from [Mil84], and recall Milner's completeness question.

Definition 2.13. Let $\mathcal{S}$ be an $Eq(A)$-based proof system and let $\simeq$ be an equivalence relation on $StExp(A)$. We say that $\mathcal{S}$ *is sound for* $\simeq$ if, for all $e, f \in StExp(A)$, $e =_{\mathcal{S}} f$ implies $e \simeq f$. We say that $\mathcal{S}$ *is complete for* $\simeq$ if, for all $e, f \in StExp(A)$, $e \simeq f$ implies $e =_{\mathcal{S}} f$.

Proposition 2.14 [Mil84]**.** Mil *is sound for process semantics equality* $=_{[\![\cdot]\!]_P}$ *on regular expressions. That is, for all* $e, f \in StExp(A)$ *it holds:* $(\, e =_{\mathsf{Mil}} f \implies e =_{[\![\cdot]\!]_P} f \,)$, *and hence* $(\, e =_{\mathsf{Mil}} f \implies e =_{[\![\cdot]\!]_P} f \,)$.

Question 2.15 [Mil84]**.** Is Mil complete for bisimilarity of process interpretations? That is, does for all $e, f \in StExp(A)$ the implication $(\, e =_{\mathsf{Mil}} f \impliedby \mathcal{C}(e) \leftrightarrow \mathcal{C}(f) \,)$ hold?

Finally we define the crucial concept of provable solution of a 1-chart with respect to an $\mathcal{EL}$-based proof system. Intuitively, a 'provable solution' of a 1-chart $\underline{\mathcal{C}}$ is a provable solution of some recursive specification $\mathcal{S}(\underline{\mathcal{C}})$ that is associated with $\underline{\mathcal{C}}$ in a natural way (see for example the two examples on page 2). Since associating specifications $\mathcal{S}(\underline{\mathcal{C}})$ to 1-charts $\underline{\mathcal{C}}$ presupposes the use of some list representation for the set $T_{\underline{\mathcal{C}}}(v)$ of transitions from vertex v, for every vertex v of $\underline{\mathcal{C}}$, any such association map cannot be unique. The definition of provable solutions of 1-charts below uses such list representations implicitly, and assumes that associativity, commutativity, and reflexivity axioms are present in the underlying proof system. In this way the concept of provable solution permits us to avoid defining associated specifications for 1-charts in some canonical (but still necessarily arbitrary) way. In the next section we explain an alternative characterization of provable solutions.

Definition 2.16 (provable solutions)**.** Let $\mathcal{S}$ be an $\mathcal{EL}$-based proof system for star expressions over A that extends ACI. Let $\underline{\mathcal{C}} = \langle V, A, \underline{1}, v_{\mathrm{s}}, \rightarrow, \downarrow\rangle$ be a 1-chart.

By a *star expression function on* $\underline{\mathcal{C}}$ we mean a function $s : V \rightarrow \mathit{StExp}(A)$ on the vertices of $\underline{\mathcal{C}}$. Let $v \in V$. We say that such a star expression function s on $\underline{\mathcal{C}}$ is an *$\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$ at v* if it holds that:

$$s(v) \;=_{\mathcal{S}}\; \tau_{\underline{\mathcal{C}}}(v) + \sum_{i=1}^{n} \underline{a_i} \cdot s(v_i)\,, \tag{2.2}$$

given the (possibly redundant) list representation $T_{\underline{\mathcal{C}}}(v) = \left\{ v \xrightarrow{\underline{a_i}} v_i \;\middle|\; i \in \{1, \ldots, n\} \right\}$, of transitions from v in $\underline{\mathcal{C}}$ and where $\tau_{\underline{\mathcal{C}}}(v)$ is the *termination constant* $\tau_{\underline{\mathcal{C}}}(v)$ *of* $\underline{\mathcal{C}}$ *at* v defined as 0 if $v\not\downarrow$, and as 1 if $v\downarrow$. This definition does not depend on the specifically chosen list representation of $T_{\underline{\mathcal{C}}}(v)$, because $\mathcal{S}$ extends ACI, and therefore it contains the associativity, commutativity, and idempotency axioms for $+$.

By an *$\mathcal{S}$-provable solution of* $\underline{\mathcal{C}}$ (with *principal value* $s(v_{\mathrm{s}})$ at the start vertex v_{s}) we mean a star expression function s on $\underline{\mathcal{C}}$ that is an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$ at every vertex of $\underline{\mathcal{C}}$.

3. Characterization of provable solutions of 1-charts

This section is an intermezzo in which we link to an elegant coalgebraic formulation of the concept of provable solution by Schmid, Rot, and Silva in [SRS21]. Their observation is a crucial first part of a detailed and beautiful coalgebraic analysis of the completeness proof in [GF20b, GF20a] by Fokkink and myself for a tailored restriction of Milner's system Mil to '1-free star expressions'. Here we reformulate their characterization of provable solution by means of the terminology that we are using here, and explain the connection, but do not prove the statements in detail. This is because we will only use this characterization later as an additional motivation for our concept of 'coinductive proof', but not for developing the proof transformations to and from the coinductive reformulation cMil of Milner's system Mil.

Schmid, Rot, and Silva construe the operational process semantics that a transition system like that in Definition 2.3 induces on the set *StExp* of all star expressions as a coalgebra (also denoted by) *StExp*. Charts can also be represented as (finite) coalgebras due to their structure as transition graphs. On this basis, they obtain the following characterization of provable solutions for proof systems $\mathcal{S}$ like Mil and Mil^{-}.

Lemma 3.1 ($\sim$ Lemma 2.2 in [SRS21])**.** *For every chart $\mathcal{C}$, for every star-expression function $s : V(\mathcal{C}) \rightarrow \mathit{StExp}$, and for $\mathcal{S} \in \{\mathsf{Mil}, \mathsf{Mil}^{-}\}$, the following two statements are equivalent:*

(i) *s is an $\mathcal{S}$-provable solution of a chart $\mathcal{C}$.*

(ii) *$\mathcal{C} \xrightarrow{[s]_{=_{\mathcal{S}}}} \mathit{StExp}/_{=_{\mathcal{S}}}$ is a coalgebra homomorphism, where* $[s]_{=_{\mathcal{S}}} \colon V(\mathcal{C}) \rightarrow \mathit{StExp}/_{=_{\mathcal{S}}}$, $v \mapsto [s(v)]_{=_{\mathcal{S}}}$.

By analyzing the proof of this statement on the basis of terminology we use here, we find the following. Schmid, Rot, and Silva noticed that statements (a) and (b) below hold, and used them in conjunction with (c) (which is analogous to Proposition 2.9 in [GF20b, GF20a]) to obtain the characterization for charts above. We extend it to 1-charts here in (d):

(a) Provability in a system $\mathcal{S}$ like Milner's defines a bisimulation relation on the the set of star expressions when that is endowed with the process semantics. (See Lemma 3.3).
(b) Due to (a) a factor chart $\mathcal{C}(e)/_{=_\mathcal{S}}$ can be defined such that $\mathcal{C}(e) \rightrightarrows \mathcal{C}(e)/_{=_\mathcal{S}}$ holds, that is, there is a functional bisimulation from $\mathcal{C}(e)$ to $\mathcal{C}(e)/_{=_\mathcal{S}}$. (See Lemma 3.5).
(c) Every star expression e is the principal value of a Mil-provable solution of the chart interpretation $\mathcal{C}(e)$ of e. (See Lemma 3.7). Equivalently, every star expression e can be Mil-provably reconstructed from the transitions to its derivatives in the process semantics. (See Lemma 3.6.)
(d) A star expression function s with principal value e is a Mil-provable solution of a 1-chart $\underline{\mathcal{C}}$ if and only if the relativization $[s]_{=_\mathsf{Mil}}$ of s to $=_\mathsf{Mil}$-equivalence classes defines a 1-bisimulation from $\underline{\mathcal{C}}$ to $\mathcal{C}(e)/_{=_\mathcal{S}}$. (See Proposition 3.8.)

We formulate these statements more precisely here below. We start with a general definition of factor charts. But the property (b) of factor charts of the chart interpretation with respect to provability will then only be shown in Lemma 3.4 below, after the formulation of the property (a) in Lemma 3.3.

Definition 3.2 (factor chart)**.** Let $\mathcal{C} = \langle V, A, 1, v_\mathrm{s}, \rightarrow, \downarrow\rangle$ be a chart (that is, a 1-transition free 1-chart). Let $\simeq$ be an equivalence relation on V. Then we define the *factor chart* $\mathcal{C}/_\simeq$ of $\mathcal{C}$ with respect to $\simeq$ by:

$$\mathcal{C}/_\simeq = \langle V/_\simeq, A, 1, [v_\mathrm{s}]_\simeq, \rightarrow/_\simeq, \downarrow/_\simeq\rangle$$

where: $V/_\simeq := \{[v]_\simeq \mid v \in V\}$, and for all $v, v_1, v_2 \in V$ and $a \in A$:

$$[v_1]_\simeq \xrightarrow{a}/_\simeq [v_2]_\simeq \;:\Longleftrightarrow\; \text{there are } \tilde{v}_1, \tilde{v}_2 \in V \text{ such that } v_1 \simeq \tilde{v}_1 \xrightarrow{a} \tilde{v}_2 \simeq v_2\,,$$
$$([v]_\simeq)\downarrow/_\simeq \;:\Longleftrightarrow\; v\downarrow.$$

By the projection function from $\mathcal{C}$ to $\mathcal{C}/_\simeq$ we mean the function $\pi_\simeq : V \to V/_\simeq$, $v \mapsto [v]_\simeq$.

Lemma 3.3. *Let $\mathcal{S} \in \{\mathsf{Mil}^-(A), \mathsf{Mil}(A)\}$. Then provability $=_\mathcal{S}$ with respect to $\mathcal{S}$ is a bisimulation on the chart interpretation $\mathcal{C}(e)$ of e, for every $e \in \mathit{StExp}(A)$.*

Proof (Idea). By verifying the bisimulation conditions (forth), (back), (termination) for conclusions of derivations in $\mathcal{S}$, proceeding by induction on the depth of derivations in $\mathcal{S}$. In the base case, this is settled for the axioms of Mil^-. In the induction step, it is settled for the conclusions of the reflexivity, symmetry, and transitivity rules of $\mathcal{EL}$, and of the fixed-point rule RSP* in Mil. (The arguments are similar to the proof of Theorem 2.1 in [SRS21].) $\square$

Lemma 3.4. *Let $\mathcal{C} = \langle V, A, 1, v_\mathrm{s}, \rightarrow, \downarrow\rangle$ be a (1-free) chart. Let $\simeq$ be an equivalence relation on V that is a bisimulation on $\mathcal{C}$.*

Then $\mathcal{C} \rightrightarrows_{\pi_\simeq} \mathcal{C}/_\simeq$ holds, that is, $\pi_\simeq$ defines a functional bisimulation from $\mathcal{C}$ to $\mathcal{C}/_\simeq$.

Proof (hint). The (forth) and (termination) conditions for the graph $\mathit{graph}(\pi_\simeq)$ of $\pi_\simeq$ to be a bisimulation are easy to verify. For demonstrating also the (back) condition for $\mathit{graph}(\pi_\simeq)$ to be a bisimulation it is crucial to use the assumption that $\simeq$ is a bisimulation on $\mathcal{C}$. $\square$

Lemma 3.5. *$\mathcal{C}(e) \rightrightarrows_{\pi_{=_\mathcal{S}}} \mathcal{C}(e)/_{=_\mathcal{S}}$ for every $e \in \mathit{StExp}(A)$, and $\mathcal{S} \in \{\mathsf{Mil}^-, \mathsf{Mil}\}$.*

Proof. Let $\mathcal{S} \in \{\mathsf{Mil}^-, \mathsf{Mil}\}$. Due to Lemma 3.3, $=_\mathcal{S}$ is a bisimulation on $\mathcal{C}(e)$. Then we obtain $\mathcal{C}(e) \rightrightarrows_{\pi_{=_\mathcal{S}}} \mathcal{C}(e)/_{=_\mathcal{S}}$ by applying Lemma 3.4. $\square$

The following lemma states that every star expression e can be reconstructed, provably in Mil^{-}, from the transitions that it facilitates in the process semantics, and the targets of these transitions. Statements like this are frequently viewed as being analogous to the fundamental theorem of calculus, which states that every differentiable function can be reconstructed from its derivative function via integration.

Lemma 3.6. *$e =_{\mathsf{Mil}^{-}} \tau_{\mathcal{C}(e)}(e) + \sum_{i=1}^{n} a_i \cdot e'_i$ holds, given a list representation $T_{\mathcal{C}(e)}(w) = \{e \xrightarrow{a_i} e'_i \mid i \in \{1,\ldots,n\}\}$ of the transitions from e in the chart interpretation $\mathcal{C}(e)$ of e.*

Proof (hint). The proof proceeds by induction on the structure of the star expression e. All axioms of Mil^{-} (and hence all axioms of Mil) are necessary in the arguments. An analogous statement that can be viewed as the restriction of the statement of Lemma 3.6 for '1-free star expressions' was proved as Lemma A.2 in [GF20b] , and as Theorem 2.2 in [SRS21]. Here we will prove an analogous statement, Lemma 4.20, in the next section. □

Lemma 3.7. *For every star expression $e \in \mathit{StExp}(A)$ with chart interpretation $\mathcal{C}(e) = \langle V(e), A, 1, e, \to, \downarrow\rangle$ the identical star-expression function $\mathrm{id}_{V(e)} : V(e) \to \mathit{StExp}(A)$, $e \mapsto e$ is a Mil^{-}-provable solution of $\mathcal{C}(e)$ with principal value e.*

Proof. The correctness conditions for the star-expression function $\mathrm{id}_{V(e)}$ to be a Mil^{-}-provable solution of the chart interpretation $\mathcal{C}(e)$ of e are guaranteed by Lemma 3.6. □

On the basis of these preparations we now reformulate the characterization of provable solutions of charts by Schmid, Rot, and Silva as a characterization of provable solutions of 1-charts via functional 1-bisimulations to factor charts of appropriate chart interpretations.

Proposition 3.8. *Let $\underline{\mathcal{C}} = \langle V, A, v_{\mathrm{s}}, 1, \to, \downarrow\rangle$ be a 1-chart. Let $\mathcal{S} \in \{\mathsf{Mil}(A), \mathsf{Mil}^{-}(A)\}$. Then for all star expression functions $s : V \to \mathit{StExp}(A)$ it holds:*

$$s \text{ is } \mathcal{S}\text{-provable solution of } \underline{\mathcal{C}} \quad\Longleftrightarrow\quad \underline{\mathcal{C}} \Rightarrow_{[s]_{=_{\mathcal{S}}}} \mathcal{C}(s(v_{\mathrm{s}}))/_{=_{\mathcal{S}}} \,. \tag{3.1}$$

where $[s]_{=_{\mathcal{S}}} := \pi_{=_{\mathcal{S}}} \circ s$ with the projection $\pi_{=_{\mathcal{S}}} : \mathit{StExp}(A) \to \mathit{StExp}(A)/_{=_{\mathcal{S}}}$, $e \mapsto [e]_{=_{\mathcal{S}}}$.

Proof (sketch). A technical part of the proof consists in showing that for every star-expression function $s : V \to \mathit{StExp}(A)$ on a 1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \to, \downarrow\rangle$ the following two statements are equivalent:

(i) s is an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$,

(ii) s is an $\mathcal{S}$-provable solution of the 'induced chart' $\underline{\mathcal{C}}_{(\cdot)} = \langle V, A, 1, \xrightarrow{(\cdot)}, \downarrow^{(1)}\rangle$ of $\underline{\mathcal{C}}$ that results by using induced transitions as transitions, and induced termination as termination, that is, with:

$$\xrightarrow{(\cdot)} \;:=\; \{\langle v, a, v'\rangle \mid a \in A,\, v, v' \in V,\, v \xrightarrow{(a)} v'\}\,, \qquad \downarrow^{(1)} \;:=\; \{v \mid v \in V,\, v\downarrow^{(1)}\}\,.$$

For the implication '⇒' in (3.1) we assume that s is a $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$. By the auxiliary statement above, s is then also a $\mathcal{S}$-provable solution of the induced chart $\underline{\mathcal{C}}_{(\cdot)}$ of $\underline{\mathcal{C}}$. Then it is not difficult to verify the 1-bisimulation conditions (forth), (back), and (termination) for $[s]_{=_{\mathcal{S}}}$ to define a 1-bisimulation from $\underline{\mathcal{C}}$ to $\mathcal{C}(s(v_{\mathrm{s}}))/_{=_{\mathcal{S}}}$.

For the converse implication '⇐' in (3.1), we assume $s : V \to \mathit{StExp}(A)$ as a star expression function with $\underline{\mathcal{C}} \Rightarrow_{[s]_{=_{\mathcal{S}}}} \mathcal{C}(s(v_{\mathrm{s}}))/_{=_{\mathcal{S}}}$. Here Lemma 3.6, the possibility to $\mathcal{S}$-provably reconstruct a star expression e from the transitions to its derivatives, can be employed in order to recognize s as a $\mathcal{S}$-provable solution of the induced chart $\underline{\mathcal{C}}_{(\cdot)}$ of $\underline{\mathcal{C}}$. Then by applying the auxiliary statement, we obtain that s is also a $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$. □

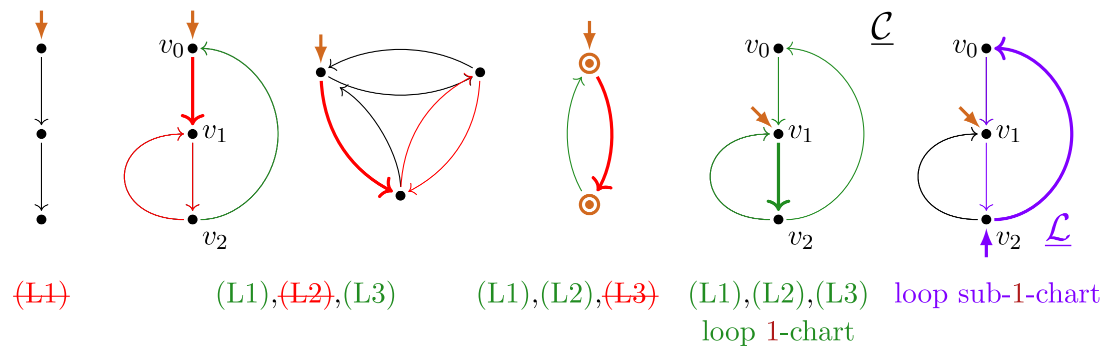


Figure 3: Four 1-charts (action labels ignored) that violate at least one loop 1-chart condition (L1), (L2), or (L3), and a loop 1-chart $\underline{\mathcal{C}}$ with one of its loop sub-1-charts $\underline{\mathcal{L}}$.

4. Layered loop existence and elimination, and LLEE-witnesses

In this subsection we recall definitions from [GF20a, Gra21c] of the loop existence and elimination condition LEE, its 'layered' version LLEE, and of chart labelings that witness these conditions. Specifically we will use the adaptation of these concepts to 1-charts that has been introduced in [Gra21c], because the use of 1-charts with 1-transitions will be crucial for the concept of 'LLEE-witnessed coinductive proof' in Section 5. For this purpose we also recall the '1-chart interpretation' of star expressions as introduced in [Gra21c] for which the property LLEE is guaranteed in contrast to the chart interpretation from Definition 2.3. We will keep formalities to a minimum as these are necessary for our purpose here, and have to refer to [GF20a, Gra21c] and the appertaining reports [GF20b, Gra20] for more details.

We start with the definitions of loop 1-charts, and of loop sub(-1)charts, and examples for these concepts.

Definition 4.1 (loop 1-chart)**.** A 1-chart $\underline{\mathcal{L}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ is called a *loop 1-chart* if it satisfies three conditions:

(L1) There is an infinite path from the start vertex v_s.

(L2) Every infinite path from v_s returns to v_s after a positive number of transitions.

(L3) Immediate termination is only permitted at the start vertex, that is, $\downarrow \subseteq \{v_s\}$.

Transitions from v_s are called *loop-entry transitions*, all other transitions *loop-body transitions*.

Example 4.2. In Figure 3 we have gathered, on the left, four examples of 1-charts (with action labels ignored) that are *not* loop 1-charts: each of them violates one of the conditions (L1), (L2), or (L3). The paths in red indicate violations of (L2), and (L3), respectively, where the thicker arrows from the start vertex indicate transitions that would need to be (but are not) loop-entry transitions. However, the 1-chart $\underline{\mathcal{C}}$ in Figure 3 is indeed a loop 1-chart.

Definition 4.3 (loop sub-1-chart of 1-chart)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart.

A *loop sub-1-chart of* a 1-chart $\underline{\mathcal{C}}$ is a loop 1-chart $\underline{\mathcal{L}}$ that is a sub-1-chart of $\underline{\mathcal{C}}$ with some vertex $v \in V$ of $\underline{\mathcal{C}}$ as start vertex such that $\underline{\mathcal{L}}$ is formed, for a nonempty set U of transitions of $\underline{\mathcal{C}}$ from v, by all vertices and transitions on paths that start with a transition in U and continue onward until v is reached again; in this case the transitions in U are the loop-entry transitions of $\underline{\mathcal{L}}$, and we say that the transitions in U *induce* $\underline{\mathcal{L}}$.

Example 4.4. In the 1-chart $\underline{\mathcal{C}}$ in Figure 3 we have illustrated (in the right copy of $\underline{\mathcal{C}}$) a loop sub-1-chart $\underline{\mathcal{L}}$ of $\underline{\mathcal{C}}$ with start vertex v_2 that is induced by the set $U := \{\langle v_2, a, v_0\rangle\}$ that consists of the single loop-entry transition from v_2 to v_0, assuming that its action label is a. Then $\underline{\mathcal{L}}$ consists of all colored transitions. – We note that also the generated sub-1-chart $\underline{\mathcal{C}}{\downarrow}^{v_2}_{*}$ of $\underline{\mathcal{C}}$ that is rooted at v_2 is a loop sub-1-chart of $\underline{\mathcal{C}}$, because it is a loop 1-chart, and it that is generated by the set of both of the two transitions from v_2.

Definition 4.5 ((single-/multi-step) loop elimination)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart. Suppose that $\underline{\mathcal{L}}$, $\underline{\mathcal{L}}_1$, ..., $\underline{\mathcal{L}}_n$ are loop sub-1-charts with sets $U, U_1, \ldots, U_n$ of loop-entry transitions from their start vertices $v, v_1, \ldots, v_n \in V$, respectively, for $n \in \mathbb{N}\backslash\{0\}$.

The 1-chart $\underline{\mathcal{C}}'$ that results by the *elimination of (the loop sub-1-chart)* $\underline{\mathcal{L}}$ *from* $\underline{\mathcal{C}}$ arises by removing all loop-entry transitions in U of $\underline{\mathcal{L}}$ from $\underline{\mathcal{C}}$, and then also removing all vertices and transitions that become unreachable; in this case we write $\underline{\mathcal{C}} \Rightarrow_{\text{elim}} \underline{\mathcal{C}}'$, and also say that $\underline{\mathcal{C}}'$ results by a *single-step loop elimination* from $\underline{\mathcal{C}}$.

Suppose that the loop sub-1-charts $\underline{\mathcal{L}}_1$, ..., $\underline{\mathcal{L}}_n$ satisfy the following two conditions:

(ms-1) their sets $U_1, \ldots, U_n$ of loop-entry transitions are disjoint (that is, $U_i \cap U_j \neq \varnothing$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$),

(ms-2) no start vertex of a loop sub-1-chart $\underline{\mathcal{L}}_i$ is in the body of another one $\underline{\mathcal{L}}_j$, for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Then we say that a 1-chart $\underline{\mathcal{C}}'$ results by the *multi-step loop elimination of* $\underline{\mathcal{L}}_1$, ..., $\underline{\mathcal{L}}_n$ *from* $\underline{\mathcal{C}}$ if $\underline{\mathcal{C}}'$ arises from $\underline{\mathcal{C}}$ by removing all loop-entry transitions in $U_1, \ldots, U_n$ of $\underline{\mathcal{L}}_1$, ..., $\underline{\mathcal{L}}_n$ from $\underline{\mathcal{C}}$, and then also removing all vertices and transitions that become unreachable; in this case we write $\underline{\mathcal{C}} \Rightarrow\!\!\!\!\ominus_{\text{elim}} \underline{\mathcal{C}}'$, and say that $\underline{\mathcal{C}}'$ results by a *multi-step loop elimination* from $\underline{\mathcal{C}}$.

Lemma 4.6. $\Rightarrow_{\text{elim}} \subseteq \Rightarrow\!\!\!\!\ominus_{\text{elim}} \subseteq \Rightarrow^{+}_{\text{elim}}$, *and consequently* $\Rightarrow^{*}_{\text{elim}} = \Rightarrow\!\!\!\!\ominus^{*}_{\text{elim}}$.

Proof. First, $\Rightarrow_{\text{elim}} \subseteq \Rightarrow\!\!\!\!\ominus_{\text{elim}}$ holds because every single-step loop elimination is also a multi-step loop elimination. Crucially, $\Rightarrow\!\!\!\!\ominus_{\text{elim}} \subseteq \Rightarrow^{+}_{\text{elim}}$ holds, because every multi-step loop elimination of loop sub-1-charts $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_n$ in a 1-chart $\underline{\mathcal{C}}$ with loop-entry transitions $U_1, \ldots, U_n$ can be implemented as a sequence of single-step loop eliminations of $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_n$ irrespective of the chosen order: hereby (ms-1) guarantees that every loop-entry transition belongs uniquely to one of $\underline{\mathcal{L}}_1, \ldots, \underline{\mathcal{L}}_n$ and thus is removed in precisely one step; and (ms-2) ensures that, after the elimination of a loop sub-1-chart $\underline{\mathcal{L}}_i$, another one $\underline{\mathcal{L}}_j$ with $j \neq i$ that has not yet been eliminated is still a loop sub-1-chart. Finally these statements imply that the many-step versions of single-step and multi-step loop elimination coincide. ☐

Definition 4.7 (LEE and LLEE)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart.

We say that $\underline{\mathcal{C}}$ has the *loop existence and elimination property* (LEE) if repeated loop elimination started on $\underline{\mathcal{C}}$ leads to a 1-chart without an infinite path, that is, if there is multi-step loop elimination reduction sequence $\underline{\mathcal{C}} \Rightarrow\!\!\!\!\ominus^{*}_{\text{elim}} \underline{\mathcal{C}}'$ (or by Lemma 4.6 equivalently, a single-step loop elimination reduction sequence $\underline{\mathcal{C}} \Rightarrow^{*}_{\text{elim}} \underline{\mathcal{C}}'$) that leads to a 1-chart $\underline{\mathcal{C}}'$ without an infinite path.

If, in a successful elimination sequence from a 1-chart $\underline{\mathcal{C}}$, loop-entry transitions are never removed that depart from a vertex in the body of a previously eliminated loop sub-1-chart, then we say that $\underline{\mathcal{C}}$ satisfies *layered* LEE (LLEE), and that $\underline{\mathcal{C}}$ is a LLEE-1-*chart*.

Example 4.8. In Figure 4 we have illustrated a successful run of the loop elimination procedure for the 1-chart $\underline{\mathcal{C}}$ there. The loop-entry transitions of loop sub-1-charts that are eliminated in the next step, respectively, are marked in bold. We have neglected action

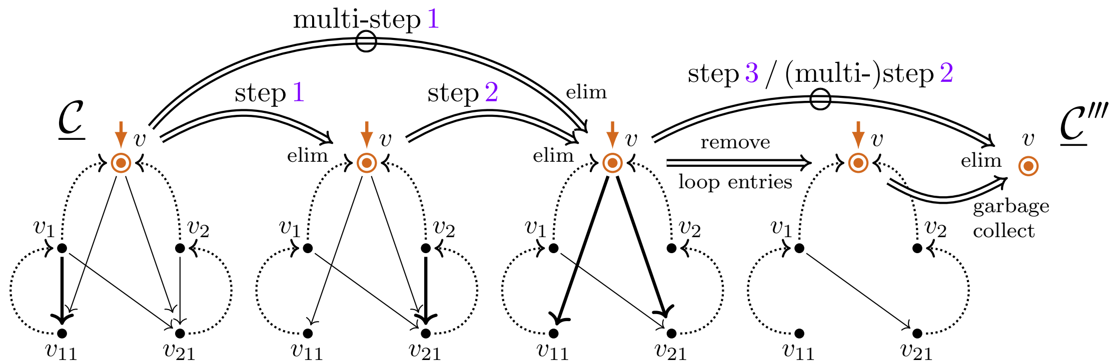


Figure 4: Example of a loop elimination process that witnesses LEE/LLEE for the 1-chart $\underline{\mathcal{C}}$. Three single-step loop eliminations from $\underline{\mathcal{C}}$ reach the same result $\underline{\mathcal{C}}'''$ as two multi-step loop eliminations (where the second multi-step is also a single step).

labels there, except for indicating 1-transitions by dotted arrows. Since the graph $\underline{\mathcal{C}}'''$ that is reached by three loop-subgraph elimination steps $\underline{\mathcal{C}} \Rightarrow^{+}_{\text{elim}} \underline{\mathcal{C}}'''$ from the 1-chart $\underline{\mathcal{C}}$ does not have an infinite path, and since no loop-entry transitions have been removed from a previously eliminated loop sub-1-chart, we conclude that $\underline{\mathcal{C}}$ satisfies LEE and LLEE.

In Figure 5 we illustrate two runs of the loop elimination procedure from a 1-chart $\underline{\mathcal{E}}$: The one from $\underline{\mathcal{E}}$ to the left only witnesses LEE but not LLEE, since in the second elimination step a loop-entry transition (drawn red) is removed from the body of the loop sub-1-chart that is eliminated in the first step (drawn in green). The one from $\underline{\mathcal{E}}$ to the right witnesses LLEE, because transitions are only removed sequentially at the same vertex, and hence no loop-entry transition is removed from the body of a loop 1-chart that was eliminated before.

The two process graphs G_1 and G_2 on page 2, which are not expressible by star expressions modulo bisimilarity, do not satisfy LLEE nor LEE: neither of them has a loop subchart (as argued in Example 4.2), yet both of them facilitate infinite paths.

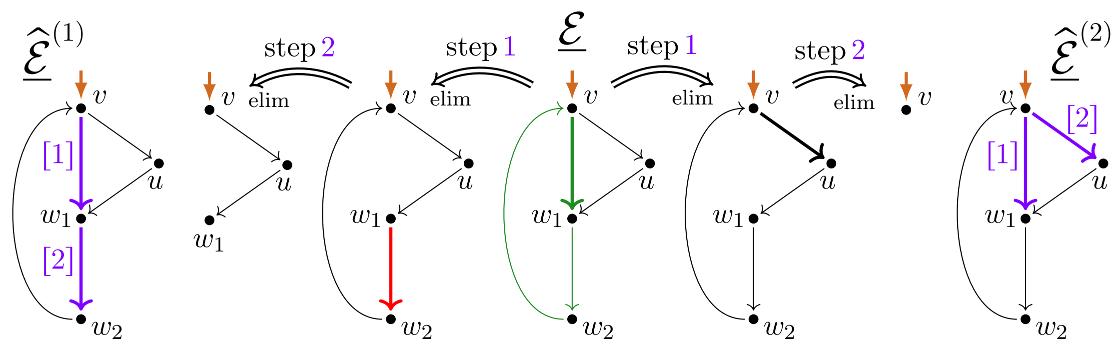


Figure 5: Two runs of the loop elimination procedure on the 1-chart $\underline{\mathcal{E}}$ in the middle: the one to the left witnesses LEE (but not LLEE, due to the removal of the red loop-entry transition from the body of the green loop subchart removed earlier), its recording is the (not layered) LEE-witness $\underline{\widehat{\mathcal{E}}}^{(1)}$ of $\underline{\mathcal{E}}$; the one to the right witnesses layered LEE (LLEE), its recording is the LLEE-witness (layered LEE-witn.) $\underline{\widehat{\mathcal{E}}}^{(2)}$ of $\underline{\mathcal{E}}$.

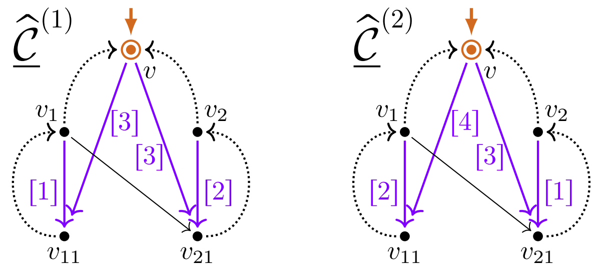


Figure 6: Three LLEE-witnesses of the 1-chart $\underline{\mathcal{C}}$ in Figure 4, of which $\widehat{\underline{\mathcal{C}}}^{(1)}$ and $\widehat{\underline{\mathcal{C}}}^{(3)}$ are the recordings of the successful single-step and multi-step runs of the loop elimination procedure in Figure 4, respectively.

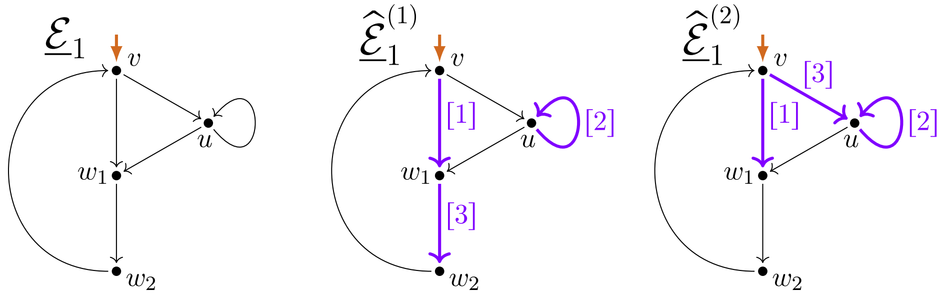


Figure 7: A LEE-witness that is not layered (in the middle), and a LLEE-witness (right) for a variation $\underline{\mathcal{E}}_1$ of the LLEE-1-chart $\underline{\mathcal{E}}$ in Figure 5.

Definition 4.9 (LLEE-witness)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \rightarrow, \downarrow\rangle$ be a 1-chart.

By an *entry/body-labeling* of $\underline{\mathcal{C}}$ we mean a 1-chart $\hat{\underline{\mathcal{C}}} = \langle V, A \times \mathbb{N}, 1, v_s, \hat{\rightarrow}, \downarrow\rangle$ with actions in $A \times \mathbb{N}$ that results from $\underline{\mathcal{C}}$ by attaching to every transition of $\underline{\mathcal{C}}$ an additional *marking label* in $\mathbb{N}$ (the transitions in $\hat{\rightarrow}$ are marking-labeled versions of the transitions in $\rightarrow$).

A LLEE-*witness* $\hat{\underline{\mathcal{C}}}$ *of* a 1-chart $\underline{\mathcal{C}}$ is an entry/body-labeling of $\underline{\mathcal{C}}$ that is the recording of a LLEE-guaranteeing, successful run $\underline{\mathcal{C}} \Longrightarrow^*_{\text{elim}} \underline{\mathcal{C}}'$ of the multi-step loop elimination procedure on $\underline{\mathcal{C}}$ that results by attaching to a transition τ of $\underline{\mathcal{C}}$ the marking label n for $n \in \mathbb{N}^+$ (in pictures indicated as $[n]$, in steps as $\rightarrow_{[n]}$) forming a *loop-entry transition* if τ is eliminated in the n-th multi-step, and by attaching marking label 0 to all other transitions of $\underline{\mathcal{C}}$ (in pictures neglected, in steps indicated as $\rightarrow_{\text{bo}}$) forming a *body transition*.

We say that a LLEE-witness $\hat{\underline{\mathcal{C}}}$ of a 1-chart $\underline{\mathcal{C}}$ *is guarded* if the action labels of the loop-entry transitions of $\hat{\underline{\mathcal{C}}}$ are proper (different from 1). We say that a LLEE-1-chart $\underline{\mathcal{C}}$ is *guarded* if $\underline{\mathcal{C}}$ has a guarded LLEE-witness.

The definition above of guardedness for LLEE-witnesses is justified in view of the fact that loop-entry transitions divide infinite paths in LLEE-witnesses into finite segments that consist only of body transitions with perhaps a leading loop-entry transition. This is a consequence of the fact that LLEE-witnesses do not permit infinite paths of body transitions (see Lemma 4.13, (ii)). Therefore guarded LLEE-witnesses, in which loop-entry transitions must be proper, do not permit infinite paths of 1-transitions. It also follows that the underlying (LLEE-)1-chart of a guarded LLEE-witness is weakly guarded.

Example 4.10. The entry/body-labelings $\hat{\underline{\mathcal{C}}}_1$ and $\hat{\underline{\mathcal{C}}}_3$ in Figure 6 of the 1-chart $\underline{\mathcal{C}}$ from Figure 4 are LLEE-witnesses that arise from the successful runs of the loop elimination procedure in Example 4.9: $\hat{\underline{\mathcal{C}}}_1$ is the recording on $\underline{\mathcal{C}}$ of the three single-step loop eliminations (viewed as trivial multi-steps in order to apply the clause for a LLEE-witness in Definition 4.9) that lead to $\underline{\mathcal{C}}'''$, and $\hat{\underline{\mathcal{C}}}_3$ is the recording on $\underline{\mathcal{C}}$ of the two multi-step loop eliminations from $\underline{\mathcal{C}}$ to $\hat{\underline{\mathcal{C}}}'''$. The entry/body-labeling $\hat{\underline{\mathcal{C}}}_2$ in Figure 6 is another LLEE-witness of $\underline{\mathcal{C}}$ that records the successful process of four elimination steps of four loop sub-1-charts each of which is induced by only a single loop-entry transition. The 1-chart $\underline{\mathcal{C}}$ in Figure 4 has a property that does not hold in general: $\underline{\mathcal{C}}$ only admits layered LEE-witnesses.

Indeed, this does not hold for the 1-chart $\underline{\mathcal{E}}$ in Figure 5: the entry/body-labeling $\widehat{\underline{\mathcal{E}}}^{(1)}$ is not a layered LLEE-witness, because it arises from a run of the loop elimination process in which in the second step a loop-entry transition is eliminated from the body of a loop sub-1-chart that was eliminated in the first step. But the entry/body-labeling $\widehat{\underline{\mathcal{E}}}^{(2)}$ there is a layered LLEE-witness of $\underline{\mathcal{E}}$. The situation is analogous for the two entry/body-labelings $\widehat{\underline{\mathcal{E}}}_1^{(1)}$ and $\widehat{\underline{\mathcal{E}}}_1^{(2)}$ of the slightly more involved LLEE-1-chart $\underline{\mathcal{E}}_1$ in Figure 7, where $\widehat{\underline{\mathcal{E}}}_1^{(2)}$ is a layered LLEE-witness of $\underline{\mathcal{E}}_1$, but $\widehat{\underline{\mathcal{E}}}_1^{(1)}$ is a LEE-witness of $\underline{\mathcal{E}}$ that is not layered.

Remark 4.11 (from LEE-witnesses to LLEE-witnesses)**.** It can be shown that every LEE-witness that is not layered can be transformed into a layered LEE-witness (LLEE-witness) of the same underlying 1-chart. Indeed, the step from the (not layered) LEE-witness $\widehat{\underline{\mathcal{E}}}_1^{(1)}$ to the LLEE-witness $\widehat{\underline{\mathcal{E}}}_1^{(2)}$ in the example in Figure 7, which transfers the loop-entry transition marking label [3] from the transition from w_1 to w_2 over to the transition from v to u, hints at the proof of this statement. However, we do not need this result, because we will be able to use the guaranteed existence of LLEE-witnesses (see Theorem 4.18) for the 1-chart interpretation below (see Definition 4.15).

For the proofs in Section 6 we will need the 'descends-in-loop-to' relation $\curvearrowright$ as defined below, and the fact that it constitutes a 'descent' in a LLEE-witness. The latter is expressed by the subsequent lemma together with termination of the body-step relation $\to_{\mathrm{bo}}$. Both of these properties can be established by arguing with the successful runs of the loop sub-1-chart elimination procedure that underlies a LLEE-witness.

Definition 4.12. Let $\hat{\underline{\mathcal{C}}}$ be a LLEE-witness of a 1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \to, \downarrow \rangle$.

Let $v, w \in V$. We denote by $v \curvearrowright w$, and by $w \curvearrowleft v$, and say that v *descends in a loop to* w, if w is in the body of the loop sub-1-chart at v, which means that there is a path $v \to_{[n]} v' \to_{\mathrm{bo}}^{*} w$ from v via a loop-entry transition and subsequent body transitions <u>without</u> encountering v again.

Lemma 4.13. *The relations $\curvearrowright$ and $\to_{\mathrm{bo}}$ defined by a* LLEE-*witness* $\hat{\underline{\mathcal{C}}}$ *of a* 1-*chart* $\underline{\mathcal{C}}$ *satisfy:*

(i) $\curvearrowleft^{+}$ *is a well-founded, strict partial order on* V.
(ii) $\leftarrow_{\mathrm{bo}}^{+}$ *is a well-founded strict partial order on* V.

Proof. Well-foundedness and irreflexivity of each of $\curvearrowleft^{+}$ and $\leftarrow_{\mathrm{bo}}^{+}$ follows from termination of $\curvearrowright$ and $\to_{\mathrm{bo}}$, respectively. These termination properties can be established in the same way as for LLEE-charts without 1-transitions, for which they follow immediately from Lemma 5.2 in [GF20a, GF20b]. Since $\curvearrowleft^{+}$ and $\leftarrow_{\mathrm{bo}}^{+}$ are transitive by definition, it follows that both relations are well-founded strict partial orders. ☐

While chart interpretations of '1-free' star expressions always satisfy LEE, see [GF20b]), we observed in [Gra21c] that this is not true for the chart interpretations of star expressions in general. As a remedy for this failure of LEE for chart interpretations, we introduced '1-chart interpretations' of star expressions [Gra21c]. For such 1-chart interpretations we showed that LEE is guaranteed, and that they refine chart interpretations in the sense that there always is a functional 1-bisimulation from the 1-chart interpretation of a star expression to its chart interpretation (see Theorem 4.18 below). For the definition of 1-chart interpretations we extended the syntax of star expressions to obtain 'stacked star expressions', see the definition below. The intuition behind the use of the 'stack product' symbol $\circledast$ is to keep track of when a transition has descended into the body of an iteration expression such that the iteration can be interpreted as a loop sub-1-chart or a tower of nested (and possibly partially overlapping) loop sub-1-charts. This feature of $\circledast$ makes a transition system specification possible (see Definition 4.15) which introduces 1-transitions only as 'backlinks' that lead from the body (the internal vertices) of some loop sub-1-chart $\underline{\mathcal{L}}$ of a 1-chart interpretation back to the start vertex of $\underline{\mathcal{L}}$.

Definition 4.14 (stacked star expressions)**.** Let A be a set whose members we call *actions*. The set $\mathit{StExp}^{(\circledast)}(A)$ of *stacked star expressions over (actions in) A* is defined by the grammar:

$$E \;\; ::= \;\; e \;\mid\; E \cdot e \;\mid\; E \circledast e^{*} \qquad (\text{where } e \in \mathit{StExp}(A))\,.$$

Note that the set $\mathit{StExp}(A)$ of star expressions would arise if the clause $E \circledast e^{*}$ were dropped.

The *star height* $|E|_{*}$ of stacked star expressions E is defined by adding the two clauses $|E \cdot e|_{*} := \max\{|E|_{*}, |e|_{*}\}$, and $|E \circledast e^{*}|_{*} := \max\{|E|_{*}, |e^{*}|_{*}\}$ to the definition of the star height of star expressions.

The *projection function* $\pi : \mathit{StExp}^{(\circledast)}(A) \to \mathit{StExp}(A)$ is defined by interpreting $\circledast$ as $\cdot$ by the clauses: $\pi(E \cdot e) := \pi(E) \cdot e$, $\pi(E \circledast e^{*}) := \pi(E) \cdot e^{*}$, and $\pi(e) := e$, for all stacked star expressions $E \in \mathit{StExp}^{(\circledast)}(A)$, and star expressions $e \in \mathit{StExp}(A)$.

In line with [Gra21c] we introduce the 1-chart interpretation of a star expression e with notation $\underline{\mathcal{C}}(e)$ as 'the 1-chart induced by (the process interpretation of) e'. For understanding the TSS in its definition below it is key to note that, by the rules for iterations, the stacked product operation $\circledast$ helps to record that an expression has descended from the iteration expression on the right-hand side of $\circledast$. This feature is used by the rule for stacked product to introduce 1-transitions only as backlinks to expressions from which they have descended. The rules for iteration expressions define loop-entry transitions and body transitions, respectively, dependent on whether e is 'strongly normed' (symbolically denoted by $\mathit{nd}^{+}(e)$) in the sense of facilitating a process trace to termination, and hence dependent on whether an iteration induces a loop sub-1-chart (outside of inner loop sub-1-charts).

Definition 4.15 (1-chart interpretation of star expressions)**.** By the *1-chart interpretation* $\underline{\mathcal{C}}(e)$ of a star expression e we mean the 1-chart that arises together with the entry/body-labeling $\hat{\underline{\mathcal{C}}}(e)$ as the e-rooted labeled transition system with 1-transitions (1-LTS) generated by $\{e\}$ according to the following TSS on the set $\mathit{StExp}^{(\circledast)}$ of stacked star expressions, where $l \in \{\mathrm{bo}\} \cup \{[n] \mid n \in \mathbb{N}^{+}\}$ are marking labels:

$$\frac{}{a \xrightarrow{a}_{\mathrm{bo}} 1} \qquad \frac{e_i \xrightarrow{a}_{l} E_i' \quad (i \in \{1,2\})}{e_1 + e_2 \xrightarrow{a}_{\mathrm{bo}} E_i'} \qquad \frac{e \xrightarrow{a}_{l} E' \quad (\text{if } \mathit{nd}^{+}(e))}{e^{*} \xrightarrow{a}_{[|e^{*}|_{*}]} E' \circledast e^{*}} \qquad \frac{e \xrightarrow{a}_{l} E' \quad (\text{if } \neg\mathit{nd}^{+}(e))}{e^{*} \xrightarrow{a}_{\mathrm{bo}} E' \circledast e^{*}}$$

$$\frac{E_1 \xrightarrow{a}_{l} E_1'}{E_1 \cdot e_2 \xrightarrow{a}_{l} E_1' \cdot e_2} \qquad \frac{E_1 \xrightarrow{a}_{l} E_1'}{E_1 \circledast e_2^{*} \xrightarrow{a}_{l} E_1' \circledast e_2^{*}} \qquad \frac{e_1{\downarrow} \qquad e_2 \xrightarrow{a}_{l} E_2'}{e_1 \cdot e_2 \xrightarrow{a}_{\mathrm{bo}} E_2'} \qquad \frac{e_1{\downarrow}}{e_1 \circledast e_2^{*} \xrightarrow{1}_{\mathrm{bo}} e_2^{*}}$$

The condition $\mathit{nd}^{+}(e)$ means a strengthening of normedness, namely, that e permits a positive length path to an expression f with $f{\downarrow}$; it is definable by induction. Immediate termination for expressions of $\underline{\mathcal{C}}(e)$ is defined by the same rules as in Definition 2.3 (for star expressions only, preventing immediate termination for expressions with stacked product $\circledast$). We note that finiteness of $\underline{\mathcal{C}}(e)$ as a 1-chart is guaranteed by Theorem 4.18, (ii), below.

We also extend the 1-chart interpretation of star expressions in the obvious way to all stacked star expressions $E \in \mathit{StExp}^{(*)}$: by $\underline{\mathcal{C}}(E)$ we mean the E-rooted sub-1-LTS generated by $\{E\}$ in the 1-LTS generated by the TSS above.

Definition 4.16. For every stacked star expression $E \in \mathit{StExp}^{(*)}(A)$, we define the set $\underline{A\partial}(E)$ of *action (partial) 1-derivatives of* E, and the set $\underline{\partial}(E)$ of *(partial) 1-derivatives of* E by:

$$\underline{A\partial}(E) := \left\{\langle \underline{a}, E'\rangle \mid E \xrightarrow{\underline{a}} E'\right\} \subseteq \underline{A} \times \mathit{StExp}^{(*)}(A)\,,$$

$$\underline{\partial}(E) := \left\{E' \mid E \xrightarrow{\underline{a}} E' \text{ for some } \underline{a} \in \underline{A}\right\} \subseteq \mathit{StExp}^{(*)}(A)\,,$$

where the transitions are defined by the TSS in Definition 4.15.

Lemma 4.17. *The action 1-derivatives $\underline{A\partial}(E)$ of a stacked star expression E over actions in A satisfy the following recursive equations, for all $a \in A$, $e, e_1, e_2 \in \mathit{StExp}(A)$, and stacked star expressions E_1 over actions in A:*

$$\underline{A\partial}(0) := \underline{A\partial}(1) := \varnothing\,,$$

$$\underline{A\partial}(a) := \{\langle a, 1\rangle\}\,,$$

$$\underline{A\partial}(e_1 + e_2) := \underline{A\partial}(e_1) \cup \underline{A\partial}(e_2)\,,$$

$$\underline{A\partial}(E_1 \cdot e_2) := \begin{cases} \left\{\langle \underline{a}, E_1' \cdot e_2\rangle \mid \langle \underline{a}, E_1'\rangle \in \underline{A\partial}(E_1)\right\} & \text{if } E_1 {\downarrow\!\!\!\!/}\,, \\ \left\{\langle \underline{a}, E_1' \cdot e_2\rangle \mid \langle \underline{a}, E_1'\rangle \in \underline{A\partial}(E_1)\right\} \cup \underline{A\partial}(E_2) & \text{if } E_1{\downarrow}\,, \end{cases}$$

$$\underline{A\partial}(E_1 \circledast e_2^*) := \begin{cases} \left\{\langle \underline{a}, E_1' \cdot e_2^*\rangle \mid \langle \underline{a}, E_1'\rangle \in \underline{A\partial}(E_1)\right\} & \text{if } E_1 {\downarrow\!\!\!\!/}\,, \\ \left\{\langle \underline{a}, E_1' \cdot e_2^*\rangle \mid \langle \underline{a}, E_1'\rangle \in \underline{A\partial}(E_1)\right\} \cup \{\langle 1, e_2^*\rangle\} & \text{if } E_1{\downarrow}\,, \end{cases}$$

$$\underline{A\partial}(e^*) := \left\{\langle a, E' \circledast e^*\rangle \mid \langle a, E'\rangle \in \underline{A\partial}(e)\right\}.$$

Proof. By case-wise inspection of the definition of the TSS in Def. 4.15. □

Theorem 4.18 [Gra20, Gra21c]**.** *For every $e \in \mathit{StExp}(A)$, the following statements hold for the concepts as introduced in Definition 4.15:*

(i) *The entry/body-labeling $\hat{\underline{\mathcal{C}}}(e)$ of $\underline{\mathcal{C}}(e)$ is a guarded* LLEE*-witness of $\underline{\mathcal{C}}(e)$.*

(ii) *The projection function π defines a 1-bisimulation from the 1-chart interpretation $\underline{\mathcal{C}}(e)$ of e to the chart interpretation $\mathcal{C}(e)$ of e, that is symbolically, $\underline{\mathcal{C}}(e) \rightrightarrows_{\pi} \mathcal{C}(e)$, and hence also $\underline{\mathcal{C}}(e) \rightrightarrows \mathcal{C}(e)$. Since the set of stacked star expressions that form the pre-image of a star expression under the projection function is always finite, it follows that 1-chart interpretations of star expressions are always finite as well.*

For the proof of Lemma 4.21 below we will need the second of the two subsequent lemmas, Lemma 4.20. Its proof uses the first lemma, which crucially states that every star expression e with immediate termination can Mil^{-}-provably be written as a star expression $1 + f$ where f does not permit immediate termination.

Lemma 4.19. *If $e{\downarrow}$ for a star expression $e \in \mathit{StExp}(A)$, then there is a star expression $f \in \mathit{StExp}(A)$ with $f{\downarrow\!\!\!\!/}$, $e =_{\mathsf{Mil}^{-}} 1 + f$, $|f|_* = |e|_*$, and $((\mathrm{id} \times \pi) \circ \underline{A\partial})(f) = ((\mathrm{id} \times \pi) \circ \underline{A\partial})(e)$.*

Proof. By a proof by induction on structure of e, in which all axioms of Mil^{-} are used. □

Lemma 4.20. *$\pi(E) =_{\mathsf{Mil}^-} \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{i=1}^{n} \underline{a_i} \cdot \pi(E'_i)$, given a list representation $T_{\underline{\mathcal{C}}(E)}(w) = \{E \xrightarrow{\underline{a_i}} E'_i \mid i \in \{1,\ldots,n\}\}$ of the transitions from E in $\underline{\mathcal{C}}(E)$.*

Proof. We establish the lemma by induction on the star height $|E|_*$ of E with a subinduction on the syntactical structure of E. All cases of stacked star expressions can be dealt with in a quite straightforward manner, except for the case of star expressions with an outermost iteration. There, an appeal to Lemma 4.19 is crucial. We treat this case in detail below.

Suppose that $E \equiv e^*$ for some star expression $e \in \mathit{StExp}(A)$ (without occurrences of stacked product $\circledast$). For showing the representation of $\pi(E)$ as stated by the lemma, we assume that the transitions from e in $\underline{\mathcal{C}}(e)$ as defined in Definition 4.15 are as follows:

$$T_{\underline{\mathcal{C}}(e)}(e) = \{e \xrightarrow{a_i} E'_i \mid i = 1,\ldots n\}\,, \tag{4.1}$$

for some stacked star expression $E'_1,\ldots,E'_n$, which are 1-derivatives of e. Note that according to the TSS in Definition 4.15 only proper transitions (those with proper action labels in A) can depart from the star expression e (which does not contain stacked products $\circledast$). Then it follows, again from the TSS in Definition 4.15 that:

$$T_{\underline{\mathcal{C}}(E)}(E) = T_{\underline{\mathcal{C}}(e^*)}(e^*) = \{e^* \xrightarrow{a_i} E'_i \circledast e^* \mid i = 1,\ldots n\}\,. \tag{4.2}$$

We assume now that $e{\downarrow}$ holds. (We will see that if $e{\not\downarrow}$ holds, the argumentation below becomes easier). Then by Lemma 4.19 there is a star expression $f \in \mathit{StExp}(A)$ with $f{\not\downarrow}$, and such that $1 + f =_{\mathsf{Mil}^-} e$, $|f|_* = |e|_*$, and $((\mathrm{id} \times \pi) \circ \underline{A\partial})(f) = ((\mathrm{id} \times \pi) \circ \underline{A\partial})(e)$ hold. From the latter it follows with (4.1):

$$T_{\underline{\mathcal{C}}(f)}(f) = \{f \xrightarrow{a_i} F'_i \mid i = 1,\ldots n\}\,, \tag{4.3}$$

$$\pi(F'_i) = \pi(E'_i) \qquad \text{(for all } i = 1,\ldots n)\,, \tag{4.4}$$

for some stacked star expression $F'_1,\ldots,F'_n$, which are 1-derivatives of f. Note again that only proper transitions can depart from the star expression f according to Definition 4.15. On the basis of these assumptions we can now argue as follows:

$$\begin{aligned}
\pi(E) &\equiv \pi(e^*) && \text{(due to } E \equiv e^* \text{ in this case)}\\
&\equiv e^* && \text{(since } e^* \text{ does not contain } \circledast)\\
&=_{\mathsf{Mil}^-} (1+f)^* && \text{(by the choice of } f \text{ with } 1+f =_{\mathsf{Mil}^-} e)\\
&=_{\mathsf{Mil}^-} f^* && \text{(by axiom (trm-body(}^*)))\\
&=_{\mathsf{Mil}^-} 1 + f \cdot f^* && \text{(by axiom (rec(}^*)))\\
&=_{\mathsf{Mil}^-} 1 + f \cdot (1+f)^* && \text{(by axiom (trm-body(}^*)))\\
&=_{\mathsf{Mil}^-} 1 + f \cdot e^* && \text{(by the choice of } f \text{ with } 1+f =_{\mathsf{Mil}^-} e)\\
&=_{\mathsf{Mil}^-} 1 + \Big(\tau_{\underline{\mathcal{C}}(f)}(f) + \sum_{i=1}^{n} a_i \cdot \pi(F'_i)\Big) \cdot e^* && \begin{array}{l}\text{(by the induction hypothesis, due to}\\ \ |f|_* = |e|_* < |e|_* + 1 = |e^*|_* = |E|_*,\\ \ \text{and in view of (4.3))}\end{array}\\
&\equiv 1 + \Big(0 + \sum_{i=1}^{n} a_i \cdot \pi(E'_i)\Big) \cdot e^* && \begin{array}{l}\text{(by } \tau_{\underline{\mathcal{C}}(f)}(f) \equiv 0 \text{ due to } f{\not\downarrow}\\ \text{and by using (4.4))}\end{array}\\
&=_{\mathsf{Mil}^-} \tau_{\underline{\mathcal{C}}(e^*)}(e^*) + \sum_{i=1}^{n} a_i \cdot (\pi(E'_i) \cdot e^*) && \begin{array}{l}\text{(by axioms (neutr(+)), (r-distr(+},\cdot)),\\ \text{(assoc(}\cdot)), \tau_{\underline{\mathcal{C}}(e^*)}(e^*) \equiv 1 \text{ due to } (e^*){\downarrow}\end{array}
\end{aligned}$$

$$\begin{tikzcd}
 & \underline{\mathcal{C}} \ \text{LLEE, guarded} \arrow[dl, Rightarrow, "{[L_1]_{=_{\mathsf{Mil}}}}"'] \arrow[dr, Rightarrow, "{[L_2]_{=_{\mathsf{Mil}}}}"] & \\
\mathcal{C}(e_1)/_{=_{\mathsf{Mil}}} & & \mathcal{C}(e_2)/_{=_{\mathsf{Mil}}}
\end{tikzcd}$$

Figure 8: A coinductive proof $\langle \underline{\mathcal{C}}, L\rangle$ over Mil of $e_1 = e_2$ gives rise to functional 1-bisimulations to factor charts of $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ with respect to provability in Mil.

$$\equiv \qquad \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{i=1}^{n} a_i \cdot \pi(E_i' * e^*) \qquad \text{(by the definition of the projection } \pi\text{)}.$$

In view of (4.2), this chain of Mil^{-}-provable equalities verifies the statement in the lemma in this case $E = e^*$ with $e{\downarrow}$. If $e{\not\downarrow}$ holds, then the detour via f is not necessary, and the argument is much simpler. The statement of the lemma holds true then as well. □

Lemma 4.21. *For every star expression $e \in StExp(A)$ with 1-chart interpretation $\underline{\mathcal{C}}(e) = \langle V(e), A, 1, e, \rightarrow, \downarrow\rangle$ the star-expression function $s : V(e) \rightarrow StExp(A)$, $E \mapsto \pi(E)$ is a Mil^{-}-provable solution of $\underline{\mathcal{C}}(e)$ with principal value e.*

Proof. The statement of the lemma is an immediate consequence of Lemma 4.20. □

5. Coinductive version of Milner's proof system

In this section we motivate and define 'coinductive proofs', introduce coinductive versions of Milner's system Mil, and establish first interconnections between these proof systems.

As the central concept we now introduce 'coinductive proofs' over $\mathcal{EL}$-based proof systems $\mathcal{S}$. We have seen examples for such circular deductions earlier in Figure 1 and Figure 2. We define a coinductive proof over $\mathcal{S}$ as a weakly guarded 1-chart $\underline{\mathcal{C}}$ whose vertices are labeled by equations between star expressions such that the left-, and the right-hand sides of the equations form $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$. The conclusion of such a proof is the equation that labels the start vertex of $\underline{\mathcal{C}}$. If $\mathcal{S}$ is theorem-subsumed by Mil (formally, $\mathcal{S} \precsim \mathsf{Mil}$ holds), then a coinductive proof with 1-chart $\underline{\mathcal{C}}$, conclusion $e_1 = e_2$, and left- and right-hand side labeling functions L_1 and L_2 can be viewed, due to Proposition 3.8, as a pair of 1-bisimulations defined by $[L_1]_{=_{\mathsf{Mil}}}$ and by $[L_2]_{=_{\mathsf{Mil}}}$ from $\underline{\mathcal{C}}$ to $\mathcal{C}(e_1)/_{=_{\mathsf{Mil}}}$, and to $\mathcal{C}(e_2)/_{=_{\mathsf{Mil}}}$ (see Figure 8). In this case we can show that the conclusion $e_1 = e_2$ of the coinductive proof is semantically sound (see Proposition 5.6). Indeed a stronger statement holds, and its proof will form the central part of Section 6: if $\mathcal{S} \precsim \mathsf{Mil}$ holds, and the underlying chart of the coinductive proof of $e_1 = e_2$ over $\mathcal{S}$ is a LLEE-1-chart, then that proof can be transformed into a proof of $e_1 = e_2$ in Milner's system Mil (see Proposition 6.8 in the next section).

In order to guarantee that coinductive proofs over a proof system $\mathcal{S}$ can only derive semantically valid equations, it is necessary to demand that $\mathcal{S}$ is sound for process-semantics equality $=_{[\![\cdot]\!]_P}$ (for example if $\mathcal{S} \precsim \mathsf{Mil}$, using Proposition 2.14). This notwithstanding, we do not include this requirement in the definition below, but add it later to statements when it is needed. The reason is that in Section 7 we want to be able to show (see Lemma 7.3) that even instances of the fixed-point rule RSP^* with premises that are not semantically valid can be mimicked by coinductive proofs over appropriate proof systems that are unsound.

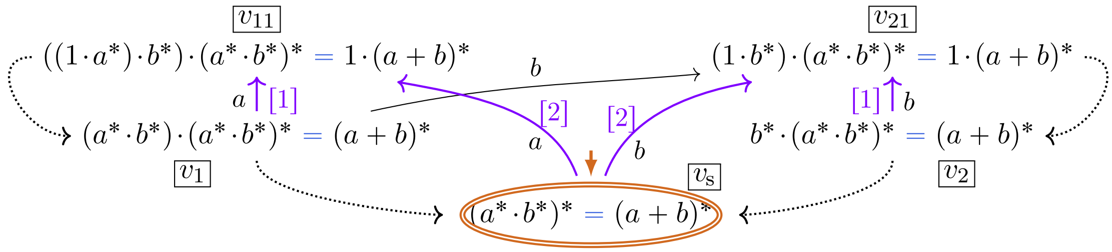


Figure 9: LLEE-witnessed coinductive proof of $(a^* \cdot b^*)^* = (a+b)^*$ over Mil^- (see Ex. 5.2).

Definition 5.1 ((LLEE-witnessed) coinductive proofs)**.** Let A be a set of actions. Let $\mathcal{S}$ be an $\mathcal{EL}(A)$-based proof system with $\mathsf{ACI} \sqsubseteq \mathcal{S}$. Let $e_1, e_2 \in \mathit{StExp}(A)$ be star expressions.

By a *coinductive proof over $\mathcal{S}$ of* $e_1 = e_2$ we mean a pair $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$ that consists of a weakly guarded 1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_\mathrm{s}, \rightarrow, \downarrow \rangle$, and a labeling function $L : V \rightarrow \mathit{Eq}(A)$ of vertices of $\underline{\mathcal{C}}$ by formal equations over A such that for the functions $L_1, L_2 : V \rightarrow \mathit{StExp}(A)$ that denote the star expressions $L_1(v)$, and $L_2(v)$, on the left-, and on the right-hand side of the equation $L(v)$, respectively, the following conditions hold:

(cp1) L_1 and L_2 are $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$,

(cp2) $e_1 \equiv L_1(v_\mathrm{s})$ and $e_2 \equiv L_2(v_\mathrm{s})$ (e_1 and e_2 are principal values of L_1 and L_2, respectively).

We denote by $e_1 \overset{\text{coind}}{=\!=\!=}_{\mathcal{S}} e_2$ that there is a coinductive proof over $\mathcal{S}$ of $e_1 = e_2$.

By a LLEE-*witnessed coinductive proof* over $\mathcal{S}$ we mean a coinductive proof $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$ where $\underline{\mathcal{C}}$ is a guarded LLEE-1-chart. We denote by $e_1 \overset{\text{LLEE}}{=\!=\!=}_{\mathcal{S}} e_2$ that there is a LLEE-witnessed coinductive proof over $\mathcal{S}$ of $e_1 = e_2$.

While the restriction to guardedness of the LLEE-1-chart underlying LLEE-witnessed coinductive proofs could be relaxed to weak guardedness, we have required guardedness in this definition in order to (somewhat) reduce technicalities in the proofs in Section 6.

We provide two examples of LLEE-witnessed coinductive proofs. First we develop a new one, and then we revisit and justify the example in Figure 1 from the introduction.

Example 5.2. In Figure 9 we have illustrated a LLEE-witnessed coinductive proof over Mil^- of the statement $(a^* \cdot b^*)^* \overset{\text{LLEE}}{=\!=\!=}_{\mathsf{Mil}^-} (a+b)^*$. Formally this proof is of the form $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$ where $\underline{\mathcal{C}} = \underline{\mathcal{C}}((a^* \cdot b^*)^*)$ has the guarded LLEE-witness $\hat{\underline{\mathcal{C}}}((a^* \cdot b^*)^*)$ (see Theorem 4.18) as indicated in Figure 9 where framed boxes contain vertex names.

In this illustration we have drawn the 1-chart $\underline{\mathcal{C}}$ that carries the equations with its start vertex below in order to adhere to the prooftree intuition for the represented derivation, namely with the conclusion at the bottom. We will do so repeatedly also below. Solution correctness for the left-hand sides L_1 of the equations L on $\underline{\mathcal{C}}$ in Figure 9 follow from Lemma 4.21, because $\underline{\mathcal{C}} = \underline{\mathcal{C}}((a^* \cdot b^*)^*)$ where $(a^* \cdot b^*)^*$ is the left-hand side of the conclusion. This notwithstanding, below we verify the correctness conditions in $\underline{\mathcal{C}}$ for the left-hand side L_1 and the right-hand side L_2 of the equation labeling function L for the (most involved) case of vertex v_1 as follows (we neglect some associative brackets, and combine some applications of axioms in Mil^-):

$$\begin{aligned}
L_1(v_1) \equiv{} & (a^* \cdot b^*) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & ((1 + a \cdot a^*) \cdot (1 + b \cdot b^*)) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & (1 \cdot 1 + a \cdot a^* \cdot 1 + 1 \cdot b \cdot b^* + a \cdot a^* \cdot b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & (1 + a \cdot a^* + a \cdot a^* \cdot b \cdot b^* + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & (1 + a \cdot a^* \cdot (1 + b \cdot b^*) + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & (1 + a \cdot a^* \cdot b^* + b \cdot b^*) \cdot (a^* \cdot b^*)^* \\
={}_{\mathsf{Mil}^-} & 1 \cdot (a^* \cdot b^*)^* + a \cdot (((1 \cdot a^*) \cdot b^*) \cdot (a^* \cdot b^*)^*) + b \cdot ((1 \cdot b^*) \cdot (a^* \cdot b^*)^*) \\
\equiv{} & 1 \cdot L_1(v_{\mathrm{s}}) + a \cdot L_1(v_{11}) + b \cdot L_1(v_{21})
\end{aligned}$$

$$\begin{aligned}
L_2(v_1) \equiv{} & (a + b)^* \\
={}_{\mathsf{Mil}^-} & (a + b)^* + (a + b)^* ={}_{\mathsf{Mil}^-} 1 + (a + b) \cdot (a + b)^* + 1 + (a + b) \cdot (a + b)^* \\
={}_{\mathsf{Mil}^-} & 1 + 1 + (a + b) \cdot (a + b)^* + a \cdot (a + b)^* + b \cdot (a + b)^* \\
={}_{\mathsf{Mil}^-} & 1 + (a + b) \cdot (a + b)^* + a \cdot (1 \cdot (a + b)^*) + b \cdot (1 \cdot (a + b)^*) \\
={}_{\mathsf{Mil}^-} & 1 \cdot (a + b)^* + a \cdot (1 \cdot (a + b)^*) + b \cdot (1 \cdot (a + b)^*) \\
\equiv{} & 1 \cdot L_2(v_{\mathrm{s}}) + a \cdot L_2(v_{11}) + b \cdot L_2(v_{21})
\end{aligned}$$

Note that the form of these two correctness conditions at v_1 arise from the outgoing transitions from v_1 in $\underline{\mathcal{C}}$ in Figure 9: the 1-transition from v_1 to v_{s}, the a-transition from v_1 to v_{11}, and the b-transition from v_1 to v_{21}.

The solution conditions for $L = \langle L_1, L_2 \rangle$ at the vertices v and v_2 can be verified analogously. At v_{11} and at v_{21} the solution conditions follow by using the axiom $\mathrm{id}_{\mathrm{l}}(\cdot)$ of Mil^-.

Example 5.3. For the statement $g^* \cdot 0 \equiv (a+b)^* \cdot 0 \overset{\text{LLEE}}{=\!=\!=}_{\mathsf{Mil}^-} (a \cdot (a+b)+b)^* \cdot 0 \equiv h^* \cdot 0$, we illustrated in Figure 2 the coinductive proof $\mathcal{CP} = \langle \underline{\mathcal{C}}(h^* \cdot 0), L \rangle$ over Mil^- with underlying guarded LLEE-witness $\hat{\underline{\mathcal{C}}}(h^* \cdot 0)$, where $\underline{\mathcal{C}}(h^* \cdot 0)$ and $\hat{\underline{\mathcal{C}}}(h^* \cdot 0)$ is the 1-chart interpretation as defined according to Definition 4.15, and the equation-labeling function L on $\underline{\mathcal{C}}(h^* \cdot 0)$ is defined as in the figure.

The correctness conditions at the start vertex (at the bottom) can be verified as follows:

$$\begin{aligned}
g^* \cdot 0 \equiv{} & (a + b)^* \cdot 0 ={}_{\mathsf{Mil}^-} (1 + (a + b) \cdot (a + b)^*) \cdot 0 ={}_{\mathsf{Mil}^-} 1 \cdot 0 + ((a + b) \cdot g^*) \cdot 0 \\
={}_{\mathsf{Mil}^-} & 0 + (a \cdot g^* + b \cdot g^*) \cdot 0 ={}_{\mathsf{Mil}^-} (a \cdot g^* + b \cdot g^*) \cdot 0 \\
={}_{\mathsf{Mil}^-} & (a \cdot g^*) \cdot 0 + (b \cdot g^*) \cdot 0 ={}_{\mathsf{Mil}^-} a \cdot (g^* \cdot 0) + b \cdot (g^* \cdot 0) \\
={}_{\mathsf{Mil}^-} & a \cdot ((1 \cdot g^*) \cdot 0) + b \cdot ((1 \cdot g^*) \cdot 0)\,,
\end{aligned}$$

$$\begin{aligned}
h^* \cdot 0 \equiv{} & (a \cdot (a + b) + b)^* \cdot 0 ={}_{\mathsf{Mil}^-} (1 + (a \cdot (a + b) + b) \cdot (a \cdot (a + b) + b)^*) \cdot 0 \\
={}_{\mathsf{Mil}^-} & 1 \cdot 0 + ((a \cdot (a + b) + b) \cdot h^*) \cdot 0 ={}_{\mathsf{Mil}^-} 0 + ((a \cdot (a + b)) \cdot h^* + b \cdot h^*) \cdot 0 \\
={}_{\mathsf{Mil}^-} & (a \cdot ((a + b) \cdot h^*) + b \cdot h^*) \cdot 0 ={}_{\mathsf{Mil}^-} ((a \cdot ((a + b) \cdot h^*)) \cdot 0 + (b \cdot h^*) \cdot 0 \\
={}_{\mathsf{Mil}^-} & a \cdot ((a + b) \cdot h^*) \cdot 0) + b \cdot (h^* \cdot 0) \\
={}_{\mathsf{Mil}^-} & a \cdot ((1 \cdot (a + b)) \cdot h^*) \cdot 0) + b \cdot ((1 \cdot h^*) \cdot 0)\,.
\end{aligned}$$

From the provable equality for $g^* \cdot 0$ the correctness condition for $(1 \cdot g^*) \cdot 0$ at the left upper vertex of $\underline{\mathcal{C}}(h^* \cdot 0)$ can be obtained by additional uses of the axiom $(\mathrm{id}_{\mathrm{l}}(\cdot))$. The correctness

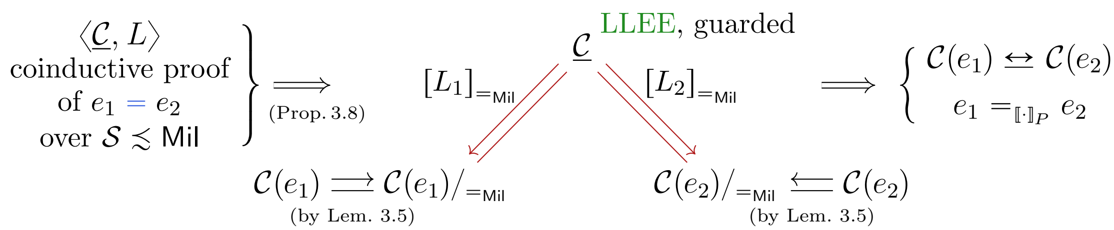


Figure 10: Coinductive proofs over $\mathcal{S} \lesssim \mathsf{Mil}$ guarantee equality of process semantics.

condition for $((1 \cdot (a+b)) \cdot h^*) \cdot 0$ at the left upper vertex of $\underline{\mathcal{C}}(h^* \cdot 0)$ can be verified as follows:

$$\begin{aligned}((1 \cdot (a+b)) \cdot h^*) \cdot 0 &=_{\mathsf{Mil}^-} ((a+b) \cdot h^*) \cdot 0 =_{\mathsf{Mil}^-} (a \cdot h^* + b \cdot h^*) \cdot 0 \\ &=_{\mathsf{Mil}^-} (a \cdot h^*) \cdot 0 + (b \cdot h^*) \cdot 0 =_{\mathsf{Mil}^-} a \cdot (h^* \cdot 0) + b \cdot (h^* \cdot 0) \\ &=_{\mathsf{Mil}^-} a \cdot ((1 \cdot h^*) \cdot 0) + b \cdot ((1 \cdot h^*) \cdot 0)\,.\end{aligned}$$

Finally, the correctness conditions at the right upper vertex of $\underline{\mathcal{C}}(h^* \cdot 0)$ can be obtained by applications of the axiom $(\mathrm{id}_{\mathrm{l}}(\cdot))$ only.

As a direct consequence of Definition 5.1, the following lemma states that LLEE-witnessed coinductive provability of an equation implies its coinductive provability. The subsequent lemma states easy observations about the composition of coinductive provability over a proof system $\mathcal{S}$ with provability $=_{\mathcal{S}}$ in $\mathcal{S}$.

Lemma 5.4. *$e_1 \overset{\text{LLEE}}{=\!=\!=}_{\mathcal{S}} e_2$ implies $e_1 \overset{\text{coind}}{=\!=\!=}_{\mathcal{S}} e_2$, for all $e_1, e_2 \in \mathit{StExp}(A)$, where $\mathcal{S}$ is an $\mathcal{EL}$-based proof system over $\mathit{StExp}(A)$ with $\mathsf{ACI} \sqsubseteq \mathcal{S}$ that is sound with respect to $=_{[\![\cdot]\!]_P}$.*

Lemma 5.5. *Let $R \in \{\overset{\text{coind}}{=\!=\!=}_{\mathcal{S}}, \overset{\text{LLEE}}{=\!=\!=}_{\mathcal{S}}\}$ for some $\mathcal{EL}$-based proof system $\mathcal{S}$ with $\mathsf{ACI} \sqsubseteq \mathcal{S}$. Then R is reflexive, symmetric, and satisfies $=_{\mathcal{S}} \circ R \subseteq R$, $R \circ =_{\mathcal{S}} \subseteq R$, and $=_{\mathcal{S}} \subseteq R$.*

The proposition below, and the subsequent remark are evidence for the fact, mentioned at the start of this section, that coinductive proofs over proof systems that are sound with respect to $=_{[\![\cdot]\!]_P}$ derive semantically valid conclusions themselves. Rather than formulating their statements for all semantically sound proof systems, we restrict our attention to systems that are theorem-subsumed by Mil.

Proposition 5.6. *Let $\mathcal{S}$ be an $\mathcal{EL}$-based proof system over $\mathit{StExp}(A)$ with $\mathsf{ACI} \sqsubseteq \mathcal{S} \lesssim \mathsf{Mil}$. Then for all $e_1, e_2 \in \mathit{StExp}(A)$ it holds:*

$$e_1 \overset{\text{coind}}{=\!=\!=}_{\mathcal{S}} e_2 \implies \mathcal{C}(e_1) \leftrightarrow \mathcal{C}(e_2)\,, \tag{5.1}$$

That is, if there is a coinductive proof over $\mathcal{S}$ of $e_1 = e_2$, then the chart interpretations of e_1 and e_2 are bisimilar.

Proof. We have illustrated the proof of this proposition in Figure 10: In every coinductive proof $\langle \underline{\mathcal{C}}, L\rangle$ over $\mathcal{S}$ with $\mathcal{S} \lesssim \mathsf{Mil}$ of an equation $e_1 = e_2$, the star-expression functions L_1 and L_2 are Mil-provable solutions of $\underline{\mathcal{C}}$. Then by using Proposition 3.8 and Lemma 3.5 we get the link of functional (1-)bisimulations between $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ as drawn in that picture. Since (functional) bisimulations compose with (functional) 1-bisimulations to 1-bisimulations, and 1-bisimulations between charts are bisimulations, we obtain that $\mathcal{C}(e_1) \leftrightarrow \mathcal{C}(e_2)$ holds, and consequently, that $e_1 =_{[\![\cdot]\!]_P} e_2$ holds. □

Remark 5.7. For every coinductive proof $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$, whether $\mathcal{CP}$ is LLEE-witnessed or not, over an $\mathcal{EL}$-based proof system $\mathcal{S}$ with $\mathsf{ACI} \sqsubseteq \mathcal{S} \lesssim \mathsf{Mil}$ the finite relation defined by:

$$B := \left\{ \left\langle \tau_{\underline{\mathcal{C}}}(v) + \sum_{i=1}^{n} \underline{a_i} \cdot L_1(v_i),\, \tau_{\underline{\mathcal{C}}}(v) + \sum_{i=1}^{n} \underline{a_i} \cdot L_2(v_i) \right\rangle \,\middle|\, \begin{array}{l} T_{\underline{\mathcal{C}}}(v) = \{ v \xrightarrow{\underline{a_i}} v_i \mid i \in \{1, \ldots, n\} \}, \\ v \in V(\underline{\mathcal{C}}), \text{ the set of vertices of } \underline{\mathcal{C}} \end{array} \right\}$$

is a 1-bisimulation up to $=_{\mathcal{S}}$ with respect to the labeled transition system on all star expressions that is defined by the TSS in Definition 2.3. This can be shown by using that the left-hand sides $L_1(v)$, and respectively the right-hand sides $L_2(v)$, of the equations $L(v)$ in $\mathcal{CP}$, for $v \in V(\underline{\mathcal{C}})$, form $\mathcal{S}$-provable solutions of the 1-chart $\underline{\mathcal{C}}$ that underlies $\mathcal{CP}$.

We now define two proof systems CLC and CC for combining LLEE-witnessed coinductive provability. Each of these systems consists of a single rule scheme, a more specific one for CLC, and a more liberal one for CC. Instances of rules of these two schemes formalize LLEE-witnessed coinductive provability in CLC, and respectively coinductive provability in CC, of equations between star expressions from assumed equations. Different from the exposition in [Gra21a, Gra21b], where we permitted entire coinductive proofs as formulas and as premises of rules, we here keep the proof systems $Eq(A)$-based by externalizing coinductive proofs from the rules by 'hiding' them in side-conditions.[2] The more restricted proof system CLC will form the core of our coinductive reformulation of Milner's system.

Definition 5.8 (proof systems CLC, CC for combining (LLEE-witn.) coinductive provability)**.** Let A be a set of actions. We define $Eq(A)$-based proof systems $\mathsf{CLC}(A)$ and $\mathsf{CC}(A)$.

The *proof system* $\mathsf{CLC}(A)$ *for combining* LLEE-*witnessed coinductive provability (over extensions of* $\mathsf{Mil}^-(A)$*) of equations between star expressions over* A is an $Eq(A)$-based proof system without *axioms*, but with the *rules* of the scheme:

$$\frac{g_1 = h_1 \quad \ldots \quad g_n = h_n}{e = f}\,\mathrm{LCoProof}_n \quad (\text{if (5.3) holds}) \tag{5.2}$$

$$e \overset{\text{LLEE}}{=\!=}_{\mathsf{Mil}^- + \Gamma} f \text{ for } \Gamma = \{g_1 = h_1, \ldots, g_n = h_n\} \text{ with } n \in \mathbb{N} \text{ (including } n = 0). \tag{5.3}$$

The *proof system* $\mathsf{CC}(A)$ *for combining coinductive provability (over extensions of* $\mathsf{Mil}^-(A)$*)* is an $Eq(A)$-based proof system without *axioms*, but with the *rules* of the scheme:

$$\frac{g_1 = h_1 \quad \ldots \quad g_n = h_n}{e = f}\,\mathrm{CoProof}_n \quad (\text{if (5.4) holds})$$

$$e \overset{\text{coind}}{=\!=}_{\mathsf{Mil}^- + \Gamma} f \text{ for } \Gamma = \{g_1 = h_1, \ldots, g_n = h_n\} \text{ with } n \in \mathbb{N} \text{ (including } n = 0). \tag{5.4}$$

We will keep the set A implicit, and write CLC and CC for $\mathsf{CLC}(A)$ and $\mathsf{CC}(A)$, respectively.

Note that the systems CLC and CC do not contain the rules of $\mathcal{EL}$ nor any axioms. Instead, derivations in these systems have to start with 0-premise instances of $\mathrm{LCoProof}_0$ or $\mathrm{CoProof}_0$. Due to Lemma 5.4 every instance of the rule $\mathrm{LCoProof}_n$ of CLC for some $n \in \mathbb{N}$ is also an instance of the rule $\mathrm{CoProof}_n$ of CC. It follows that derivability of an equation $e = f$ in CLC implies derivability of $e = f$ in CC, that is, $\mathsf{CLC} \lesssim \mathsf{CC}$ holds, see Lemma 5.11, (i), below.

Based on CLC, we now define the system that we call the coinductive variant cMil of Milner's proof system Mil. For this, we replace the fixed-point rule in Mil by the rule scheme

[2] Keeping the systems equation-based (by avoiding coinductive proofs as formulas as in [Gra21a, Gra21b]) permits us to then compare the coinductive proof systems CLC, CC, and later cMil, and $\overline{\mathsf{cMil}}$ via rule derivability and admissibility to Milner's system Mil and its variants Mil', and $\overline{\mathsf{Mil}}'$.

of CLC, or equivalently, by adding this rule scheme to the purely equational part Mil^{-} of Mil. By adding the rule scheme of CC to Mil^{-}, we also define an extension $\overline{\mathsf{cMil}}$ of cMil.

Definition 5.9 (proof systems cMil, cMil_1, and $\overline{\mathsf{cMil}}$)**.** Let A be a set of actions.

The proof system $\mathsf{cMil}(A)$, the *coinductive variant of* $\mathsf{Mil}(A)$, is an $\mathcal{EL}$-based proof system whose *axioms* are those of $\mathsf{Mil}^{-}(A)$, and whose *rules* are those of $\mathcal{EL}(A)$, plus the *rule scheme* $\{\mathrm{LCoProof}_n\}_{n\in\mathbb{N}}$ from $\mathsf{CLC}(A)$. By $\mathsf{cMil}_1(A)$ we mean *the simple coinductive variant of* $\mathsf{Mil}(A)$, an $\mathcal{EL}$-based proof system that arises by only adding the rule $\mathrm{LCoProof}_1$ of $\mathsf{CLC}(A)$ to the rules and axioms of $\mathsf{Mil}^{-}(A)$.

By $\overline{\mathsf{cMil}}(A)$ we mean the variant of $\mathsf{cMil}(A)$ in which the more general *rule scheme* $\{\mathrm{CoProof}_n\}_{n\in\mathbb{N}}$ from $\mathsf{CC}(A)$ is used (instead of $\{\mathrm{LCoProof}_n\}_{n\in\mathbb{N}}$ from $\mathsf{CLC}(A)$).

We again permit to write cMil, cMil_1, $\overline{\mathsf{cMil}}$ for $\mathsf{cMil}(A)$, $\mathsf{cMil}_1(A)$, and $\overline{\mathsf{cMil}}(A)$, respectively.

We now prove a lemma (Lemma 5.11 below) that gathers elementary theorem equivalence and theorem subsumption statements between the coinductive variants of Milner's system defined above. For its proof we argue with subsystem relationships as gathered by Lemma 5.10 below, and we explain basic proof transformations between these systems.

Lemma 5.10. *The following subsystem relationships hold between the coinductive proof systems defined above:* (i) $\mathsf{CLC} \sqsubseteq \mathsf{cMil}$, (ii) $\mathsf{cMil}_1 \sqsubseteq \mathsf{cMil}$, (iii) $\mathsf{CC} \sqsubseteq \overline{\mathsf{cMil}}$.

Lemma 5.11. *The following theorem subsumption and theorem equivalence statements hold:*

(i) $\mathsf{CLC} \lesssim \mathsf{CC}$,
(ii) $\mathsf{cMil}_1 \lesssim \mathsf{cMil} \lesssim \overline{\mathsf{cMil}}$,
(iii) $\mathsf{CLC} \sim \mathsf{cMil}$,
(iv) $\mathsf{CC} \sim \overline{\mathsf{cMil}}$.

Proof. We have argued for statement (i) above, below Definition 5.8: every instance of the rule $\mathrm{LCoProof}_n$ of CLC, for $n \in \mathbb{N}$, is also an instance of the rule $\mathrm{CoProof}_n$ of CC. This also implies the part $\mathsf{cMil} \lesssim \overline{\mathsf{cMil}}$ of statement (ii), because it shows in every derivation $\mathcal{D}$ of an equation $e = f$ in cMil every instance of $\mathrm{LCoProof}_n$, for $n \in \mathbb{N}$, can be replaced by an instance of $\mathrm{CoProof}_n$ with as result a derivation $\mathcal{D}'$ of $e = f$ in $\overline{\mathsf{cMil}}$. The part $\mathsf{cMil}_1 \lesssim \mathsf{cMil}$ of statement (ii) follows from the fact that cMil_1 is a subsystem of cMil by Lemma 5.10, (ii).

For statement (iii), the theorem-subsumption part $\mathsf{CLC} \lesssim \mathsf{cMil}$ follows from $\mathsf{CLC} \sqsubseteq \mathsf{cMil}$, see Lemma 5.10, (i). For showing the converse implication, $\mathsf{CLC} \gtrsim \mathsf{cMil}$, we will demonstrate the proof-transformation statement (5.5) below by first showing its special case (5.6):

$$\left.\begin{array}{l}\text{Every derivation } \mathcal{D} \text{ in } \mathsf{cMil} \text{ can be transformed into a derivation } \mathcal{D}' \text{ in } \mathsf{CLC}\\ \text{with the same conclusion.}\end{array}\right\} \tag{5.5}$$

$$\left.\begin{array}{l}\text{The transformation statement (5.5) holds for every derivation } \mathcal{D} \text{ in } \mathsf{cMil} \text{ with}\\ \text{an instance of } \mathrm{LCoProof}_n, \text{ for some } n \in \mathbb{N}, \text{ at the bottom.}\end{array}\right\} \tag{5.6}$$

The idea for both of these proof transformation statements is to 'hide' derivation parts that consist of axioms and rules in Mil^{-} into the correctness statements of coinductive proofs that appear as side-conditions in instances of the coinductive rule in CLC. More precisely, the idea is to replace derivation parts $\mathcal{D}_0$ in Mil^{-} of a derivation $\mathcal{D}$ in cMil, where $\mathcal{D}_0$ consists of the inference in Mil^{-} of an equation $g = h$ from a set Γ of m assumption equations, by an instance of the coinductive rule $\mathrm{LCoProof}_m$ that has the m assumptions of $\mathcal{D}_0$ in Γ as its premises. Then whenever $g = h$ is needed to justify a correctness conditions it can be reconstructed from the premises, provably in Mil^{-}, on the basis of the derivation $\mathcal{D}_0$ in Mil^{-}.

We start by showing (5.6), and proceed for this purpose by induction on the depth $|\mathcal{D}|$ of $\mathcal{D}$. Suppose that $\mathcal{D}$ is a derivation in cMil with an occurrence of an instance ι of $\mathrm{LCoProof}_n$ at the bottom. To perform the induction step for $\mathcal{D}$, we need to transform $\mathcal{D}$ into a derivation $\mathcal{D}'$ in CLC with the same conclusion. We may assume that the immediate subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of $\mathcal{D}$ (just above the instance ι) contain axioms and/or rules of Mil^{-}, because otherwise $\mathcal{D}$ is already a derivation in CLC. To keep the illustration of the transformation step simple, we assume that only the i-th subderivation $\mathcal{D}_i$ contains axioms and/or rules of Mil; the general case will become clear through this example. We assume that $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\mathcal{D}_1}{g_1 = h_1} \quad \ldots \quad \dfrac{\dfrac{\mathcal{D}_{i1}}{(g_{i1} = h_{i1})} \;\cdots\; \dfrac{\mathcal{D}_{im}}{(g_{im} = h_{im})}}{\mathcal{D}_{i0} \atop g_i = h_i} \quad \ldots \quad \dfrac{\mathcal{D}_n}{g_n = h_n}}{e = f}\ \mathrm{LCoProof}_n \tag{5.7}$$

$$\text{with side-condition } \; e \overset{\mathrm{LLEE}}{=\!=\!=}_{\mathsf{Mil}^{-}+\Gamma} f \;\text{ for } \Gamma = \{g_1 = h_1, \ldots g_n = h_n\}\,, \tag{5.8}$$

where $\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}, \mathcal{D}_{i+1}, \ldots \mathcal{D}_n$ are already derivations in CLC (with bottommost instances of $\mathrm{LCoProof}_{n_i}$ that are suggested by dashed lines), but $\mathcal{D}_i$ contains axioms and/or rule instances of Mil^{-}, and can be construed with a bottom part $\mathcal{D}_{i0}$ in Mil^{-} below m conclusions $g_{i1} = h_{i1}, \ldots, g_{im} = h_{im}$ of instances of coinductive rules from $\{\mathrm{LCoProof}_j\}_{j\in\mathbb{N}}$.

Then we apply the induction hypothesis to the subderivations $\mathcal{D}_{i1}, \ldots, \mathcal{D}_{im}$ of $\mathcal{D}_0$, which is possible due to $|\mathcal{D}_{i1}|, \ldots, |\mathcal{D}_{im}| < |\mathcal{D}_i| < |\mathcal{D}|$, to obtain derivations $\mathcal{D}'_{i1}, \ldots, \mathcal{D}'_{im}$ in CLC with the same conclusions $g_{i1} = h_{i1}, \ldots, g_{im} = h_{im}$, respectively. Then we transform $\mathcal{D}$ by replacing the instance of $\mathrm{LCoProof}_n$ at the bottom by an instance of $\mathrm{LCoProof}_{n+m-1}$, keeping the first $i-1$ and the last $n+1-i$ premises and their subderivations, but replacing the i-th premise and its immediate subderivation $\mathcal{D}_i$ by m additional premises with immediate subderivations $\mathcal{D}'_{i1}, \ldots, \mathcal{D}'_{im}$, thereby obtaining:

$$\dfrac{\dfrac{\mathcal{D}_1}{g_1 = h_1} \quad \ldots \quad \dfrac{\mathcal{D}'_{i1}}{g_{i1} = h_{i1}} \quad \ldots \quad \dfrac{\mathcal{D}'_{im}}{g_{im} = h_{im}} \quad \ldots \quad \dfrac{\mathcal{D}_n}{g_n = h_n}}{e = f}\ \mathrm{LCoProof}_{n+m-1} \tag{5.9}$$

However, we need to show the side-condition for the displayed instance of $\mathrm{LCoProof}_{n+m-1}$:

$$\begin{aligned} e \overset{\mathrm{LLEE}}{=\!=\!=}_{\mathsf{Mil}^{-}+\Gamma'} f \;\text{ with } \Gamma' = \big\{g_1 = h_1, \ldots g_{i-1} = h_{i-1}, g_{i+1} = h_{i+1}, \ldots, g_n = h_n\big\} \cup \Delta\,, \\ \text{where: } \Delta := \big\{g_{i1} = h_{i1}, \ldots g_{im} = h_{im}\big\} \\ \text{the set of conclusions of } \mathcal{D}'_{i1}, \ldots, \mathcal{D}'_{im}. \end{aligned} \tag{5.10}$$

Now due to (5.8) there is a LLEE-witnessed coinductive proof $\mathcal{LCP}$ of $e = f$ over $\mathsf{Mil}^{-}+\Gamma$. But now $\mathcal{LCP}$ is also a LLEE-witnessed coinductive proof of $e = f$ over $\mathsf{Mil}^{-}+\Gamma'$, and thus over a different set of premises, which we recognize as follows. The equations in Δ, which have been added to Γ in order to get Γ' after removing $g_i = h_i$, are derivable in $\mathsf{Mil}^{-}+\Gamma$ by means for the derivation $\mathcal{D}_0$. Therefore the correctness conditions for $\mathcal{LCP}$ as a LLEE-witnessed coinductive proof over $\mathsf{Mil}^{-}+\Gamma$ imply the correctness conditions for $\mathcal{LCP}$ as a LLEE-witnessed coinductive proof over $\mathsf{Mil}^{-}+\Gamma'$. This shows (5.10). Therefore the resulting derivation $\mathcal{D}'$ in (5.9) is a derivation in CLC with the same conclusion as $\mathcal{D}$. (This transformation step can obviously be generalized to the situation in which not just $\mathcal{D}_i$, but also others among the immediate subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of $\mathcal{D}$ contain axioms and/or rules of Mil^{-}.) In this way we have performed the induction step.

Finally we establish (5.5) in full generality. For this we consider the remaining situation in which the derivation $\mathcal{D}$ in cMil does not terminate with an instance of a coinductive rule. Then $\mathcal{D}$ can be construed with a part derivation $\mathcal{D}_0$ in Mil^{-} above its conclusion, and below $m \in \mathbb{N}$ subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_m$, each of which terminates with a coinductive rule (see below on the left). Note that $m = 0$ is possible if $\mathcal{D}$ is a derivation in Mil^{-}. By applying (5.6) to $\mathcal{D}_1, \ldots, \mathcal{D}_m$ we obtain derivations $\mathcal{D}'_1, \ldots, \mathcal{D}'_m$ in CLC with the same conclusions, respectively. By combining these derivations in CLC with an instance of LCoProof_m we can perform the following transformation step in order to obtain a derivation $\mathcal{D}'$ in CLC:

$$\begin{array}{c} \mathcal{D}_1 \qquad\qquad \mathcal{D}_m \\ (e_1 = f_1) \;\ldots\; (e_m = f_m) \\ \mathcal{D}_0 \\ e = f \end{array} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{D}'_1 \\ e_1 = f_1\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}'_m \\ e_m = f_m\end{array}}{e = f}\,\text{LCoProof}_m \tag{5.11}$$

Here we need to establish the following side-condition for the instance of LCoProof_m at the bottom of $\mathcal{D}'$:

$$e \overset{\text{LLEE}}{=\!=\!=}_{\mathsf{Mil}^{-}+\Gamma} f \quad \text{where } \Gamma := \{e_1 = f_1, \ldots, e_m = f_m\} \,. \tag{5.12}$$

We can establish this coinductive-provability statement by recognizing that $\langle \underline{\mathcal{C}}(e), L\rangle$ with $\underline{\mathcal{C}}(e)$ the 1-chart interpretation of e, and with equation labeling function:

$$L : V(\underline{\mathcal{C}}(e)) \to \mathit{StExp}(A)\,,\;\; G \mapsto \begin{cases} G = G & \text{if } G \not\equiv e \\ e = f & \text{if } G \equiv e \end{cases}$$

is a LLEE-witnessed coinductive proof over $\mathsf{Mil}^{-}+\Gamma$ of $e = f$. To verify this statement, we use that $\underline{\mathcal{C}}(e)$ is a guarded LLEE-1-chart by Theorem 4.18, (i), and we have to check the correctness conditions for L_1 and L_2 with $L = \langle L_1, L_2\rangle$ to be $\mathsf{Mil}^{-}+\Gamma$-provable solutions of $\underline{\mathcal{C}}(e)$. We first note that $e = f$ is provable in $\mathsf{Mil}^{-}+\Gamma$ (i.e. $e =_{\mathsf{Mil}^{-}+\Gamma} f$) since $\mathcal{D}_0$ is a derivation of $e = f$ in Mil^{-} from the assumptions in Γ. Then we argue as follows:

- The correctness conditions for L_1 to be a $(\mathsf{Mil}^{-}+\Gamma)$-provable solution of $\underline{\mathcal{C}}(e)$ follow from the fact that L_1 is a Mil^{-}-provable solution of $\underline{\mathcal{C}}(e)$ due to Lemma 4.21.
- L_2 differs from L_1 only in the value for the start vertex e, where $L_2(e) \equiv f$, but $L_1(e) \equiv e$. From this it follows, in view of $e =_{\mathsf{Mil}^{-}+\Gamma} f$, that the correctness conditions for L_1 imply the correctness conditions for L_2, because they differ only up to expressions that are provably equal in $\mathsf{Mil}^{-}+\Gamma$ and also need to hold up to $(\mathsf{Mil}^{-}+\Gamma)$-provability.

This argument shows (5.12), which justifies the side-condition of the instance of LCoProof_m at the bottom of the derivation $\mathcal{D}'$ on the right in (5.11). Therefore we have indeed obtained from $\mathcal{D}$ a derivation in CLC with the same conclusion as $\mathcal{D}$. In this way we have completed the proof of $\mathsf{cMil} \lesssim \mathsf{CLC}$, the remaining part of (iii).

Statement (iv) can be shown entirely analogously as statement (iii). □

Remark 5.12 (completeness of CC, $\overline{\mathsf{cMil}}$, $\overline{\mathsf{Mil}}'$)**.** The proof systems CC and $\overline{\mathsf{cMil}}$, as well as the variant $\overline{\mathsf{Mil}}'$ of Milner's system with the general unique solvability principle USP are complete for bisimilarity of star expressions. This can be established along Salomaa's completeness proof for his inference system for language equality of regular expressions [Sal66], by an argument that we can outline as follows. Given star expressions e and f with $\mathcal{C}(e) \leftrightarrow \mathcal{C}(f)$, e and f can be shown to be principal values of Mil^{-}-provable solutions of $\mathcal{C}(e)$ and $\mathcal{C}(f)$, respectively (by a lemma for the chart interpretation similar to Lemma 4.21).

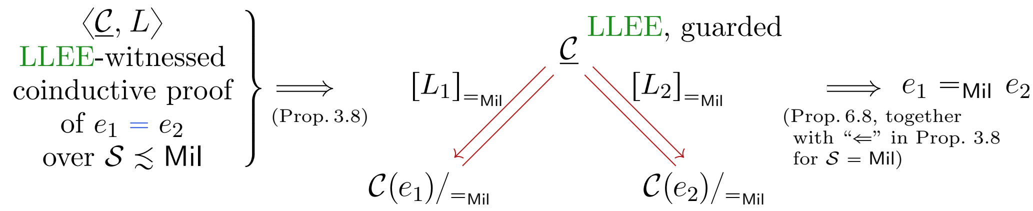


Figure 11: Statement underlying the proof transformation from cMil to Mil.

These solutions can be transferred to the (1-free) product chart $\mathcal{C}$ of $\mathcal{C}(e)$ and $\mathcal{C}(f)$, with e and f as principal values of Mil^{-}-provable solutions L_1 and L_2 of $\mathcal{C}$, respectively. From this we obtain a (not necessarily LLEE-witnessed) coinductive proof $\langle \mathcal{C}, L \rangle$ of $e = f$ over Mil^{-}. It follows that $e = f$ is provable in CC, and in $\overline{\mathsf{cMil}}$. Now since the correctness conditions for the Mil^{-}-provable solutions L_1 and L_2 of $\mathcal{C}$ at each of the vertices of $\mathcal{C}$ together form a guarded system of linear equations to which the rule USP can be applied (as $\mathcal{C}$ is 1-free), we obtain that $e = f$ is also provable in $\overline{\mathsf{Mil}}'$.

6. From LLEE-witnessed coinductive proofs to Milner's system

In this Section we develop a proof-theoretic interpretation of the coinductive variant system cMil of Mil in Milner's original system Mil. Since cMil and Mil differ only by the coinductive rule scheme $\{\mathrm{LCoProof}_n\}_{n \in \mathbb{N}}$ (which is part of cMil, but not of Mil), and by the fixed-point rule RSP* (which is part of Mil, but not of cMil), the crucial step for this proof transformation is to show that instances of $\mathrm{LCoProof}_n$, for $n \in \mathbb{N}$, can be mimicked in Mil if their premise equations are derivable in Mil. We will do so by showing that the rules $\mathrm{LCoProof}_n$, for $n \in \mathbb{N}$, are correct for Mil. This implies that these rules are admissible in Mil (by Lemma 2.10), and also, that they can be eliminated from derivations in $\mathsf{Mil}+\{\mathrm{LCoProof}_n\}_{n \in \mathbb{N}}$, which is an extension of cMil. In this way we obtain the proof-theoretic interpretation of cMil in Mil.

For proving correctness of $\mathrm{LCoProof}_n$ for Mil, where $n \in \mathbb{N}$, we will show that every LLEE-witnessed coinductive proof $\langle \underline{\mathcal{C}}, L \rangle$ over a proof system $\mathcal{S}$ that is theorem-subsumed by Mil of an equation $e_1 = e_2$ can also be established by a proof of $e_1 = e_2$ in Milner's system Mil. Informally, this statement is illustrated in Figure 11 by informally employing the characterization in Proposition 3.8 of the provable solutions in such LLEE-witnessed coinductive proofs. We establish the indicated informal second step in this section, where it will be guaranteed by Proposition 6.8. In particular, our proof of this step will use the following two statements:

- **(SE)** *(solution extraction)* from a LLEE-witness $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$ a Mil-provable solution $s_{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$ can be extracted (Lemma 6.4), and
- **(SU)** *(solution uniqueness)* every Mil-provable solution of the LLEE-1-chart $\underline{\mathcal{C}}$ (such as L_1 and L_2) is Mil-provably equal to the solution $s_{\underline{\mathcal{C}}}$ extracted from $\hat{\underline{\mathcal{C}}}$ (Lemma 6.7).

By these statements we will obtain for every LLEE-witnessed coinductive proof $\langle \underline{\mathcal{C}}, L \rangle$ of $e_1 = e_2$, assuming that v_s is the star vertex of $\underline{\mathcal{C}}$ and hence $L_1(v_s) \equiv e_1$ as well as $L_2(v_s) \equiv e_2$ hold, that $e_1 \equiv L_1(v_s) =_{\mathsf{Mil}} s_{\hat{\underline{\mathcal{C}}}}(v_s) =_{\mathsf{Mil}} L_2(v_s) \equiv e_2$ holds, and therefore $e_1 =_{\mathsf{Mil}} e_2$.

The proofs of the statements **(SE)** and **(SU)** below are adaptations to LLEE-1-charts of proofs of analogous statements for LLEE-charts in Section 5 of [GF20a, GF20b]. We have to, at places substantially, refine the extraction technique of star expressions from process graphs with LLEE that was first described in [GF20a, GF20b]. However, we will use the simplification to only reason about guarded LLEE-witnesses, in which loop-entry transition are proper transitions. We can do so because the 1-chart interpretation $\underline{\mathcal{C}}(e)$ of a star expression e is guaranteed to have a guarded LLEE-witness $\hat{\underline{\mathcal{C}}}(e)$ by Theorem 4.18.

For developing and proving the extraction statement **(SE)** we use that the hierarchical loop structure of a 1-chart $\underline{\mathcal{C}}$ with LLEE-witness $\hat{\underline{\mathcal{C}}}$ facilitates the extraction of a Mil^{-}-provable solution of $\underline{\mathcal{C}}$ (see Lemma 6.4). The reason is as follows. The process behavior at every vertex w in $\underline{\mathcal{C}}$ can be split into an iteration part that is induced via the loop-entry transitions from w in $\hat{\underline{\mathcal{C}}}$ (which induce loop sub-1-charts with inner loop sub-1-charts whose behavior can be synthesized recursively), and an exit part that is induced via the body transitions from w in $\hat{\underline{\mathcal{C}}}$. This intuition leads us to the definition below. We define the 'extraction function' $s_{\hat{\underline{\mathcal{C}}}}$ of $\hat{\underline{\mathcal{C}}}$ from a 'relative extraction function' $t_{\hat{\underline{\mathcal{C}}}}$ of $\hat{\underline{\mathcal{C}}}$, whose values $t_{\hat{\underline{\mathcal{C}}}}(w, v)$ capture the behavior at w in a loop sub-1-chart at v until v is reached.

Definition 6.1 ((relative) extraction function)**.** Let $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \rightarrow, \downarrow \rangle$ be a (guarded) LLEE-1-chart with guarded LLEE-witness $\hat{\underline{\mathcal{C}}}$.

The *extraction function* $s_{\hat{\underline{\mathcal{C}}}} : V \rightarrow \mathit{StExp}(A)$ *of* $\hat{\underline{\mathcal{C}}}$ is defined from the *relative extraction function* $t_{\hat{\underline{\mathcal{C}}}} : \{\langle w, v \rangle \mid w, v \in V(\underline{\mathcal{C}}),\ w \curvearrowleft^{=} v\} \rightarrow \mathit{StExp}(A)$ *of* $\hat{\underline{\mathcal{C}}}$ for $w, v \in V$:

$$t_{\hat{\underline{\mathcal{C}}}}(w, v) := \begin{cases} 1 & \text{if } w = v, \\ \Big(\sum_{i=1}^{n} a_i \cdot t_{\hat{\underline{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\hat{\underline{\mathcal{C}}}}(u_i, v)\Big) & \text{if } w \curvearrowleft v, \end{cases}$$

$$s_{\hat{\underline{\mathcal{C}}}}(w) := \Big(\sum_{i=1}^{n} a_i \cdot t_{\hat{\underline{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \Big(\tau_{\underline{\mathcal{C}}}(w) + \sum_{i=1}^{m} \underline{b_i} \cdot s_{\hat{\underline{\mathcal{C}}}}(u_i)\Big),$$

provided: $T_{\hat{\underline{\mathcal{C}}}}(w) = \big\{ w \xrightarrow{a_i}_{[l_i]} w_i \bigm| l_i \in \mathbb{N}^{+},\ i \in \{1, \ldots, n\} \big\} \cup \big\{ w \xrightarrow{\underline{b_i}}_{[\mathrm{bo}]} u_i \bigm| i \in \{1, \ldots, m\} \big\}$,
induction for $t_{\hat{\underline{\mathcal{C}}}}$ on: $\langle w_1, v_1 \rangle <_{\mathrm{lex}} \langle w_2, v_2 \rangle \ :\Longleftrightarrow\ v_1 \curvearrowleft^{+} v_2 \vee (v_1 = v_2 \wedge w_1 \leftarrow^{+}_{\mathrm{bo}} w_2)$,
induction for $s_{\hat{\underline{\mathcal{C}}}}$ on the strict partial order $\leftarrow^{+}_{\mathrm{bo}}$ (see Lemma 4.13),

where $<_{\mathrm{lex}}$ is a well-founded strict partial order due to Lemma 4.13. The choice of the list representations of action-target sets of $\hat{\underline{\mathcal{C}}}$ changes the definitions of these functions only up to provability in ACI.

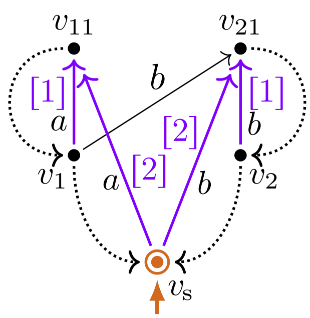
We exemplify the extraction process defined above by a concrete example.

Example 6.2. We consider the 1-chart $\underline{\mathcal{C}}$, and the LLEE-witness $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$, in the LLEE-witnessed coinductive proof $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$ of $(a^{*} \cdot b^{*})^{*} = (a + b)^{*}$ in Example 5.2. We detail in Figure 12 the process of computing the principal value $s_{\hat{\underline{\mathcal{C}}}}(v_{\mathrm{s}})$ of the extraction function $s_{\hat{\underline{\mathcal{C}}}}$ of $\hat{\underline{\mathcal{C}}}$. The statement of Lemma 6.4 below will guarantee that $s_{\hat{\underline{\mathcal{C}}}}$ is a Mil^{-}-provable solution of $\underline{\mathcal{C}}$.

In order to show that the extraction function of a guarded LLEE-witness of a 1-chart $\underline{\mathcal{C}}$ defines a Mil^{-}-provable solution of $\underline{\mathcal{C}}$, see Lemma 6.4 and its proof later, we first have to establish a Mil^{-}-provable connection between the relative extraction function and the extraction function of a guarded LLEE-witness. For this we prove the following lemma.

Lemma 6.3. *Let* $\underline{\mathcal{C}}$ *be a (guarded)* LLEE*-1-chart with guarded* LLEE*-witness* $\hat{\underline{\mathcal{C}}}$*. Then* $s_{\hat{\underline{\mathcal{C}}}}(w) =_{\mathsf{Mil}^{-}} t_{\hat{\underline{\mathcal{C}}}}(w, v) \cdot s_{\hat{\underline{\mathcal{C}}}}(v)$ *holds for all vertices* $w, v \in V(\underline{\mathcal{C}})$ *such that* $w \curvearrowleft^{=} v$*.*

Proof. Let $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \rightarrow, \downarrow \rangle$ be a LLEE-1-chart with guarded LLEE-witness $\hat{\underline{\mathcal{C}}}$.



$\underline{\mathcal{C}}, \hat{\underline{\mathcal{C}}}$

$$
\begin{aligned}
t_{\hat{\underline{\mathcal{C}}}}(v_{21}, v_2) &:= \; 0^* \cdot (1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_2, v_2)) \equiv 0^* \cdot (1 \cdot 1) =_{\mathsf{Mil}^-} 1 \\
t_{\hat{\underline{\mathcal{C}}}}(v_2, v_{\mathrm{s}}) &:= \; (b \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{21}, v_2))^* \cdot (1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{\mathrm{s}}, v_{\mathrm{s}})) =_{\mathsf{Mil}^-} b^* \\
t_{\hat{\underline{\mathcal{C}}}}(v_{21}, v_{\mathrm{s}}) &:= \; 0^* \cdot (1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_2, v_{\mathrm{s}})) =_{\mathsf{Mil}^-} 1 \cdot b^* =_{\mathsf{Mil}^-} b^* \\
t_{\hat{\underline{\mathcal{C}}}}(v_{11}, v_1) &:= \; 0^* \cdot (1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_1, v_1)) \equiv 0^* \cdot (1 \cdot 1) =_{\mathsf{Mil}^-} 1 \\
t_{\hat{\underline{\mathcal{C}}}}(v_1, v_{\mathrm{s}}) &:= \; (a \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{11}, v_{\mathrm{s}}))^* \cdot (b \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{21}, v_{\mathrm{s}}) + 1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{\mathrm{s}}, v_{\mathrm{s}})) \\
&=_{\mathsf{Mil}^-} a^* \cdot (b \cdot b^* + 1) =_{\mathsf{Mil}^-} a^* \cdot b^* \\
t_{\hat{\underline{\mathcal{C}}}}(v_{11}, v_{\mathrm{s}}) &:= \; 0^* \cdot (1 \cdot t_{\hat{\underline{\mathcal{C}}}}(v_1, v_{\mathrm{s}})) =_{\mathsf{Mil}^-} a^* \cdot b^* \\
s_{\hat{\underline{\mathcal{C}}}}(v_{\mathrm{s}}) &:= \; (a \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{11}, v_{\mathrm{s}}) + b \cdot t_{\hat{\underline{\mathcal{C}}}}(v_{21}, v_{\mathrm{s}}))^* \cdot 1 \\
&=_{\mathsf{Mil}^-} (a \cdot (a^* \cdot b^*) + b \cdot b^*)^*
\end{aligned}
$$

Figure 12: Extraction of the principal value of a Mil^--provable solution $s_{\hat{\underline{\mathcal{C}}}}$ from the LLEE-witness $\hat{\mathcal{C}}$ in the coinductive proof in Example 5.2, with Mil^--provable simplifications.

We have to show that $s(w) =_{\mathsf{Mil}^-} t_{\hat{\underline{\mathcal{C}}}}(w, v) \cdot s(v)$ holds for all $w, v \in V$ with $w \curvearrowleft^{=} v$. We first notice that this statement holds obviously for $w = v$, due $t_{\hat{\underline{\mathcal{C}}}}(w, v) \equiv t_{\hat{\underline{\mathcal{C}}}}(v, v) \equiv 1$, and the presence of the axiom $(\mathrm{id}_{\mathrm{l}}(\cdot))$ in Mil^-. Therefore it suffices to show, by also using this fact, that $s(w) =_{\mathsf{Mil}^-} t_{\hat{\underline{\mathcal{C}}}}(w, v) \cdot s(v)$ holds for all $w, v \in V$ with $w \curvearrowleft v$. We will show this by using the same induction as for the definition of the relative extraction function $t_{\hat{\underline{\mathcal{C}}}}$ in Definition 6.1, that is, by complete induction on the (converse) lexicographic partial order $<_{\mathrm{lex}}$ of $\curvearrowleft^{+}$ and $\leftarrow_{\mathrm{bo}}^{+}$ on $V \times V$ defined by: $\langle w_1, v_1 \rangle <_{\mathrm{lex}} \langle w_2, v_2 \rangle :\Longleftrightarrow v_1 \curvearrowleft^{+} v_2 \vee (v_1 = v_2 \wedge w_1 \leftarrow_{\mathrm{bo}}^{+} w_2)$, which is well-founded by Lemma 4.13. For our argument we assume to have given, underlying the definition of the relative extraction function $t_{\hat{\underline{\mathcal{C}}}}$ and the extraction function $s_{\hat{\underline{\mathcal{C}}}}$, list representations $T_{\hat{\underline{\mathcal{C}}}}(w)$ of the transitions from w in $\hat{\underline{\mathcal{C}}}$ as in Definition 6.1, for all $w \in V$.

In order to carry out the induction step, we let $w, v \in V$ be arbitrary, but such that $w \curvearrowleft v$. On the basis of the form of $T_{\hat{\underline{\mathcal{C}}}}(w)$ as in Definition 6.1 we argue as follows, starting with a step in which we use the definition of $s_{\hat{\underline{\mathcal{C}}}}$, and followed by a second step in which we use that $\tau_{\underline{\mathcal{C}}}(w) \equiv 0$ holds, because w cannot have immediate termination as due to $w \curvearrowleft v$ it is in the body of the loop at v (cf. condition (L3) for loop 1-charts in Section 4):

$$
\begin{aligned}
s_{\hat{\underline{\mathcal{C}}}}(w) &\equiv \Big(\sum_{i=1}^{n} a_i \cdot t_{\hat{\underline{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\hat{\underline{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\equiv \Big(\sum_{i=1}^{n} a_i \cdot t_{\hat{\underline{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\hat{\underline{\mathcal{C}}}}(u_i)\Big) + 0\Big) \\
&=_{\mathsf{Mil}^-} \Big(\sum_{i=1}^{n} a_i \cdot t_{\hat{\underline{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\hat{\underline{\mathcal{C}}}}(u_i)\Big) \\
&\qquad \text{(by axiom (neutr(+)))}
\end{aligned}
$$

$$
\begin{aligned}
&=_{\mathsf{Mil}^-} \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \sum_{i=1}^{m} \underline{b}_i \cdot (t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \cdot s_{\underline{\hat{\mathcal{C}}}}(v)) \\
&\qquad \big(\text{if } u_i = v, \text{ then } s_{\underline{\hat{\mathcal{C}}}}(u_i) =_{\mathsf{Mil}^-} t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \cdot s_{\underline{\mathcal{C}}}(v)) \text{ due to } t_{\underline{\hat{\mathcal{C}}}}(v, v) = 1; \\
&\qquad\ \text{if } u_i \neq v, \text{ we can apply the induction hypothesis to } s_{\underline{\hat{\mathcal{C}}}}(u_i), \\
&\qquad\quad \text{as } w \rightarrow_{\mathrm{bo}} u_i \text{ (see } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Def. 6.1) and } u_i \neq v \\
&\qquad\quad \text{imply } u_i \leftharpoonup v, \text{ and } u_i \leftarrow_{\mathrm{bo}} w \text{ entails } \langle u_i, v\rangle <_{\mathrm{lex}} \langle w, v\rangle\big) \\
&=_{\mathsf{Mil}^-} \Big(\Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \Big(\sum_{i=1}^{m} \underline{b}_i \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v)\Big)\Big) \cdot s_{\underline{\hat{\mathcal{C}}}}(v) \\
&\qquad \text{(by axioms (r-distr}(+,\cdot)\text{), and (assoc}(\cdot)\text{))} \\
&\equiv \quad t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s_{\underline{\hat{\mathcal{C}}}}(v) \\
&\qquad \text{(by } w \leftharpoonup v, \text{ and the definition of } t_{\underline{\hat{\mathcal{C}}}}(w, v) \text{ in Def. 6.1)}
\end{aligned}
$$

We note that this reasoning also applies for the special cases $n = 0$, and with a slight change also for $m = 0$, where $\sum_{i=1}^{m} \underline{b}_i \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i) \equiv \sum_{i=1}^{m} \underline{b}_i \cdot (t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \cdot s_{\underline{\hat{\mathcal{C}}}}(v)) \equiv \sum_{i=1}^{m} \underline{b}_i \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \equiv 0$, and then an axiom (deadlock) has to be used. In this way we have shown, due to $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^-$, the desired Mil^--provable equality $s(w) =_{\mathsf{Mil}^-} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v)$ for the vertices v and w that we picked with the property $w \leftharpoonup v$.

Since $w, v \in V$ with $w \leftharpoonup v$ were arbitrary above, we have successfully carried out the proof by induction that $s_{\underline{\hat{\mathcal{C}}}}(w) =_{\mathsf{Mil}^-} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s_{\underline{\mathcal{C}}}(v)$ holds for all $w, v \in V$ with $w \leftharpoonup v$. As we have argued that the statement also holds for $w = v$, we have proved the lemma. □

Lemma 6.4 (extracted function is provable solution)**.** *Let $\underline{\mathcal{C}}$ be a* LLEE-1*-chart with guarded* LLEE-*witness $\underline{\hat{\mathcal{C}}}$. Then the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ is a* Mil^-*-provable solution of $\underline{\mathcal{C}}$.*

Proof. Let $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \rightarrow, \downarrow\rangle$ be a (guarded) LLEE-1-chart with guarded LLEE-witness $\underline{\hat{\mathcal{C}}}$.

We show that the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ is a Mil^--provable solution of $\underline{\mathcal{C}}$ by verifying the Mil^--provable correctness conditions for $s_{\underline{\hat{\mathcal{C}}}}$ at every vertex $w \in V$. For the argument we assume to have given, underlying the definition of the relative extraction function $t_{\underline{\hat{\mathcal{C}}}}$ and the extraction function $s_{\underline{\hat{\mathcal{C}}}}$, list representations $T_{\underline{\hat{\mathcal{C}}}}(w)$ of the transitions from w in $\underline{\hat{\mathcal{C}}}$ as written in Definition 6.1, for all vertices $w \in V$.

We let $w \in V$ be arbitrary. Starting from the definition of $s_{\underline{\hat{\mathcal{C}}}}$ in Definition 6.1 on the basis of the form of $T_{\underline{\hat{\mathcal{C}}}}(w)$, we argue by the following steps:

$$
\begin{aligned}
s_{\underline{\hat{\mathcal{C}}}}(w) \;&\equiv\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^* \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b}_i \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&=_{\mathsf{Mil}^-} \Big(1 + \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^*\Big) \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b}_i \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\qquad \text{(by axiom (rec}(^*)\text{))}
\end{aligned}
$$

$$
\begin{aligned}
=_{\mathsf{Mil}^-} \;& \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
& + \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot \Big(\Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big)\Big) \\
& \quad \text{(by axioms (r-distr}(+,\cdot)\text{), (id}_{\mathrm{l}}(\cdot)\text{), and (assoc}(\cdot)\text{))} \\
\equiv \;& \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) + \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot s_{\underline{\hat{\mathcal{C}}}}(w) \\
& \quad \text{(by definition of } s_{\underline{\hat{\mathcal{C}}}}(w) \text{ in Def. 6.1)} \\
=_{\mathsf{Mil}^-} \;& \Big(\sum_{i=1}^{n} a_i \cdot \big(t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s_{\underline{\hat{\mathcal{C}}}}(w)\big)\Big) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
& \quad \text{(by axioms (comm}(+)\text{), (r-distr}(+,\cdot)\text{), and (assoc}(\cdot)\text{))} \\
=_{\mathsf{Mil}^-} \;& \Big(\sum_{i=1}^{n} a_i \cdot s_{\underline{\hat{\mathcal{C}}}}(w_i)\Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w) \\
& \quad \text{(by Lemma 6.3, due to } w_i \curvearrowleft^{=} w \text{, which follows from } w \to_{[l_i]} w_i \\
& \quad\; \text{(see } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Def. 6.1), and by axioms (assoc}(+)\text{)).} \\
=_{\mathsf{ACI}} \;& \tau_{\underline{\mathcal{C}}}(w) + \Big(\sum_{i=1}^{n} a_i \cdot s_{\underline{\hat{\mathcal{C}}}}(w_i)\Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) .
\end{aligned}
$$

Since $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^-$, this chain of equalities yields a Mil^--provable equality that establishes, in view of $T_{\underline{\hat{\mathcal{C}}}}(w)$ as in Definition 6.1, the correctness condition for $s_{\underline{\hat{\mathcal{C}}}}$ to be a Mil^--provable solution at the vertex w that we picked.

Since $w \in V$ was arbitrary, we have established that the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ is a Mil^--provable solution of $\underline{\mathcal{C}}$. ☐

For showing the solution uniqueness statement **(SU)** we can also use the hierarchical loop structure of a 1-chart $\underline{\mathcal{C}}$ with LLEE-witness $\underline{\hat{\mathcal{C}}}$ for carrying out proofs by induction. We repurpose the two-step approach of the proof of Lemma 6.4 that used the Mil-provable relationship between the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ and the relative extraction function $t_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ in Lemma 6.3. In doing so we first establish, for every Mil-provable solution s of $\underline{\mathcal{C}}$, a connection with the relative extraction function $t_{\underline{\mathcal{C}}}$ of $\underline{\hat{\mathcal{C}}}$, see Lemma 6.5 below. The proof of this lemma proceeds by an induction that starts at innermost loop sub-1-charts of the given LLEE-witness, and then progresses to outer loop sub-1-charts. Different from Lemma 6.3, it will be crucial here to employ the fixed-point rule RSP* of Mil in the proof.

Lemma 6.5. *Let $\underline{\mathcal{C}}$ be a* LLEE*-1-chart with guarded* LLEE*-witness $\underline{\hat{\mathcal{C}}}$. Furthermore, let $\mathcal{S}$ be an $\mathcal{EL}(A)$-based proof system such that $\mathsf{ACI} \sqsubseteq \mathcal{S} \precsim \mathsf{Mil}$.*

Let $s : V(\underline{\mathcal{C}}) \to \mathit{StExp}(A)$ be an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$. Then $s(w) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v)$ holds for all vertices $w, v \in V(\underline{\mathcal{C}})$ with $w \curvearrowleft^{=} v$.

Proof. Let $\underline{\hat{\mathcal{C}}}$ be a guarded LLEE-witness of a (guarded) LLEE-1-chart $\underline{\mathcal{C}} = \langle V, A, 1, v_{\mathrm{s}}, \to, \downarrow \rangle$. Let $s : V \to \mathit{StExp}(A)$ be an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$.

We have to show that $s(w) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v)$ holds for all $w, v \in V$ with $w \curvearrowleft^{=} v$. We first notice that this statement holds obviously for $w = v$, due $t_{\underline{\hat{\mathcal{C}}}}(w, v) \equiv t_{\underline{\hat{\mathcal{C}}}}(v, v) \equiv 1$.

Therefore it suffices to show, by also using this fact, that $s(w) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w,v) \cdot s(v)$ holds for all $w, v \in V$ with $w \curvearrowleft v$. We will show this by using the same induction as for the definition of the relative extraction function in Definition 6.1, that is, by complete induction on the (converse) lexicographic partial order $<_{\text{lex}}$ of $\curvearrowleft^{+}$ and $\leftarrow_{\text{bo}}^{+}$ on $V \times V$ defined by: $\langle w_1, v_1 \rangle <_{\text{lex}} \langle w_2, v_2 \rangle \;:\Longleftrightarrow\; v_1 \curvearrowleft^{+} v_2 \vee (\, v_1 = v_2 \wedge w_1 \leftarrow_{\text{bo}}^{+} w_2 \,)$, which is well-founded due to Lemma 4.13. For our argument we suppose to have given, underlying the definition of the relative extraction function $t_{\underline{\hat{\mathcal{C}}}}$ and the extraction function $s_{\underline{\hat{\mathcal{C}}}}$, list representations $T_{\underline{\hat{\mathcal{C}}}}(w)$ of the transitions from w in $\underline{\hat{\mathcal{C}}}$ as written in Definition 6.1, for all $w \in V$.

In order to carry out the induction step, we let $w, v \in V$ be arbitrary such that $w \curvearrowleft v$. On the basis of the form of $T_{\underline{\hat{\mathcal{C}}}}(w)$ as in Definition 6.1 we argue as follows, starting with a step in which we use that s is an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$, and followed by a second step in which we use that $\tau_{\underline{\mathcal{C}}}(w) \equiv 0$ holds, because w cannot have immediate termination as due to $w \curvearrowleft v$ it is in the body of the loop at v (see condition (L3) for loop 1-charts in Section 4):

$$
\begin{aligned}
s(w) \;&=_{\mathcal{S}}\;\; \tau_{\underline{\mathcal{C}}}(w) + \Big(\Big(\sum_{i=1}^{n} a_i \cdot s(w_i) \Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i) \Big) \Big) \\
&=_{\mathsf{Mil}^{-}}\;\; \Big(\sum_{i=1}^{n} a_i \cdot s(w_i) \Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i) \Big) \\
&=_{\mathsf{Mil}}\;\; \Big(\sum_{i=1}^{n} a_i \cdot \big(t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s(w) \big) \Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i) \Big) \\
&\qquad \big(\text{if } w_i = w, \text{ then } s(w_i) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s(w)) \text{ due to } t_{\underline{\hat{\mathcal{C}}}}(w,w) = 1; \\
&\qquad\;\; \text{if } w_i \neq w, \text{ we can apply the induction hypothesis to } s(w_i), \\
&\qquad\quad\;\; \text{because then } w \rightarrow_{[l_i]} w_i \text{ (see } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Def. 6.1) implies } w_i \curvearrowleft w, \\
&\qquad\quad\;\; \text{and due to } w \curvearrowleft v \text{ we get } \langle w_i, w \rangle <_{\text{lex}} \langle w, v \rangle \big) \\
&=_{\mathsf{Mil}}\;\; \Big(\sum_{i=1}^{n} a_i \cdot \big(t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s(w) \big) \Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot \big(t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \cdot s(v) \big) \Big) \\
&\qquad \big(\text{if } u_i = v, \text{ then } s(u_i) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \cdot s(v)) \text{ due to } t_{\underline{\hat{\mathcal{C}}}}(v,v) = 1; \\
&\qquad\;\; \text{if } u_i \neq v, \text{ we can apply the induction hypothesis to } s(u_i), \\
&\qquad\quad\;\; \text{as } w \rightarrow_{\text{bo}} u_i \text{ (see } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Def. 6.1) and } u_i \neq v \\
&\qquad\quad\;\; \text{imply } u_i \curvearrowleft v, \text{ and } u_i \leftarrow_{\text{bo}} w \text{ entails } \langle u_i, v \rangle <_{\text{lex}} \langle w, v \rangle \big) \\
&=_{\mathsf{Mil}^{-}}\;\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \Big) \cdot s(w) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \Big) \cdot s(v) \\
&\qquad (\text{by axioms } (\text{assoc}(\cdot)), \text{ and } (\text{r-distr}(+,\cdot))).
\end{aligned}
$$

We note that these equalities also hold for the special cases in which $n = 0$ or/and $m = 0$, where in the case $m = 0$ an axiom (deadlock) needs to be used in the last step. Since $\mathsf{ACI} \sqsubseteq \mathcal{S} \lesssim \mathsf{Mil}$, and $\mathsf{Mil}^{-} \sqsubseteq \mathsf{Mil}$, we have obtained the following provable equality:

$$
s(w) \;=_{\mathsf{Mil}}\;\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \Big) \cdot s(w) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v) \Big) \cdot s(v)\,,
$$

Since $\left(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\right)\!\not\downarrow$ holds, we can apply RSP* in order to obtain, and reason further:

$$
\begin{aligned}
s(w) \;&=_{\mathsf{Mil}}\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \bigg(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v)\Big) \cdot s(v)\bigg) \\
&=_{\mathsf{Mil}^-}\; \bigg(\Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \Big(\sum_{i=1}^{m} \underline{b_i} \cdot t_{\underline{\hat{\mathcal{C}}}}(u_i, v)\Big)\bigg) \cdot s(v) \\
&\qquad \text{(by axiom (assoc}(\cdot)))\\
&\equiv\; t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v) \\
&\qquad \text{(by } w \curvearrowleft v\text{, and the definition of } t_{\underline{\hat{\mathcal{C}}}}(w, v) \text{ in Def. 6.1)}
\end{aligned}
$$

In this way we have shown, due to $\mathsf{Mil}^- \sqsubseteq \mathsf{Mil}$, the desired Mil-provable equality $s(w) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v)$ for the vertices v and w that we picked with the property $w \curvearrowleft v$.

Since $w, v \in V$ with $w \curvearrowleft v$ were arbitrary for this argument, we have successfully carried out the proof by induction $s(w) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w, v) \cdot s(v)$ holds for all $w, v \in V$ with $w \curvearrowleft v$. As we have argued that the statement also holds for $w = v$, we have proved the lemma. □

Definition 6.6. For an $Eq(A)$-based proof system $\mathcal{S}$ we say that two star expression functions $s_1, s_2 : V \to StExp(A)$ are *$\mathcal{S}$-provably equal* if $s_1(v) =_{\mathcal{S}} s_2(v)$ holds for all $v \in V$.

Now we use the relationship of arbitrary Mil-provable solutions of a guarded LLEE-witness $\underline{\hat{\mathcal{C}}}$ with the relative extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$ as stated by Lemma 4.13 in order to demonstrate the solution uniqueness statement **(SU)**. The proof can be viewed as proceeding on the maximal length of body transition paths from vertices v, where for descents from v via a loop-entry transition into an inner loop the statement of Lemma 6.5 is used. Again the use of the fixed-point rule RSP* of Mil^- is crucial, because any two Mil^--provable solutions of a guarded LLEE-1-chart cannot be expected to be Mil^--provably equal in general.[3]

Lemma 6.7 (provable equality of solutions of LLEE-1-charts)**.** *Let $\underline{\mathcal{C}}$ be a guarded* LLEE-*1-chart, and let $\mathcal{S}$ be an $\mathcal{EL}$-based proof system over $StExp(A)$ such that* $\mathsf{ACI} \sqsubseteq \mathcal{S} \precsim \mathsf{Mil}$.

Then any two $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$ are Mil*-provably equal.*

Proof. Let $\underline{\mathcal{C}} = \langle V, A, 1, v_s, \to, \downarrow\rangle$ be a LLEE-1-chart with guarded LLEE-witness $\underline{\hat{\mathcal{C}}}$, and let $\mathcal{S}$ an $\mathcal{EL}$-based proof system as assumed in the lemma. In order to show that any two $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$ are Mil-provably equal, it suffices to show that every $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$ is Mil-provably equal to the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of $\underline{\hat{\mathcal{C}}}$.

For demonstrating this, let $s : V \to StExp(A)$ be an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$. We have to show that $s(w) =_{\mathsf{Mil}} s_{\underline{\hat{\mathcal{C}}}}(w)$ holds for all $w \in V$. We proceed by complete induction on the well-founded relation $\leftarrow^{+}_{\mathrm{bo}}$ (see Lemma 4.13, (ii)), which does not require us to treat base cases separately. For our argument we assume to have given, underlying the definition of the relative extraction function $t_{\underline{\hat{\mathcal{C}}}}$ and the extraction function $s_{\underline{\mathcal{C}}}$, list representations $T_{\underline{\hat{\mathcal{C}}}}(w)$ of the transitions from w in $\underline{\hat{\mathcal{C}}}$ as written in Definition 6.1, for all $w \in V$.

Let $w \in V$ be arbitrary. On the basis of $T_{\underline{\hat{\mathcal{C}}}}(w)$ as in Definition 6.1 we argue as follows, starting with a step in which we use that s is an $\mathcal{S}$-provable solution of $\underline{\mathcal{C}}$ in view of the

[3]As a simple example, the use of RSP* is necessary for proving equal in Mil the two Mil^--provable solutions of the guarded LLEE-1-chart $\underline{\mathcal{C}}((a \cdot a)^* \cdot 0)$ with the principal values $a^* \cdot 0$ and $(a \cdot a)^* \cdot 0$, respectively.

assumed form of $T_{\underline{\hat{\mathcal{C}}}}(w)$:

$$\begin{aligned}
s(w) \;&=_{\mathcal{S}}\; \tau_{\underline{\mathcal{C}}}(w) + \Big(\Big(\sum_{i=1}^{n} a_i \cdot s(w_i)\Big) + \Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i)\Big)\Big) \\
&=_{\mathsf{ACI}}\; \Big(\sum_{i=1}^{n} a_i \cdot s(w_i)\Big) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big)\Big) \\
&=_{\mathsf{Mil}}\; \Big(\sum_{i=1}^{n} a_i \cdot (t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s(w))\Big) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\qquad (w \rightarrow_{[l_i]} w_i \text{ (due to } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Definition 6.1) implies } w_i \curvearrowleft^{=} w, \\
&\qquad\ \text{from which Lemma 6.5 yields } s(w_i) =_{\mathsf{Mil}} t_{\underline{\hat{\mathcal{C}}}}(w_i, w) \cdot s(w)\ ) \\
&=_{\mathsf{Mil}^-}\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\qquad \text{(by axioms (assoc}(\cdot)) \text{ and (r-distr}(+,\cdot))) \\
&=_{\mathsf{Mil}}\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\qquad \text{(due to } w \rightarrow_{\mathrm{bo}} u_i \text{ (see } T_{\underline{\hat{\mathcal{C}}}}(w) \text{ as in Definition 6.1), and hence } u_i \leftarrow_{\mathrm{bo}} w, \\
&\qquad\ s(u_i) =_{\mathsf{Mil}} s_{\underline{\hat{\mathcal{C}}}}(u_i) \text{ follows from the induction hypothesis).}
\end{aligned}$$

Since $\mathsf{ACI} \sqsubseteq \mathcal{S} \lesssim \mathsf{Mil}$, and $\mathsf{Mil}^- \sqsubseteq \mathsf{Mil}$, we have obtained the following provable equality:

$$s(w) =_{\mathsf{Mil}} \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big) \cdot s(w) + \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big)$$

Now since $\big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\big){\downarrow}$ holds, we can apply the rule RSP* to this in order to obtain:

$$\begin{aligned}
s(w) \;&=_{\mathsf{Mil}}\; \Big(\sum_{i=1}^{n} a_i \cdot t_{\underline{\hat{\mathcal{C}}}}(w_i, w)\Big)^{*} \cdot \Big(\Big(\sum_{i=1}^{m} \underline{b_i} \cdot s_{\underline{\hat{\mathcal{C}}}}(u_i)\Big) + \tau_{\underline{\mathcal{C}}}(w)\Big) \\
&\equiv\; s_{\underline{\hat{\mathcal{C}}}}(w) \qquad \text{(by the definition of } s_{\underline{\hat{\mathcal{C}}}} \text{ in Definition 6.1)}
\end{aligned}$$

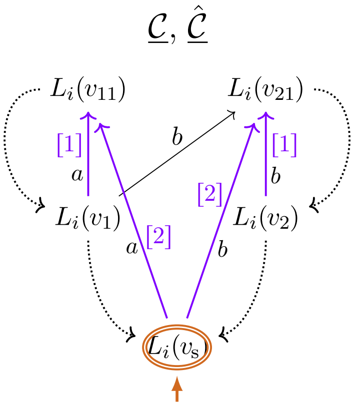
Thus we have verified the proof obligation $s(w) =_{\mathsf{Mil}} s_{\underline{\hat{\mathcal{C}}}}(w)$ for the induction step, for the vertex w as picked.

By having performed the induction step, we have successfully carried out the proof by complete induction on $\leftarrow_{\mathrm{bo}}$ that $s(w) =_{\mathsf{Mil}} s_{\underline{\hat{\mathcal{C}}}}(w)$ holds for all $w \in V$, and for an arbitrary $\mathcal{S}$-provable solution s of $\underline{\mathcal{C}}$. This implies the statement of the lemma, that any two $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$ are Mil-provably equal. □

Proposition 6.8. *For every $\mathcal{EL}$-based proof system $\mathcal{S}$ over $StExp(A)$ with $\mathsf{ACI} \sqsubseteq \mathcal{S} \lesssim \mathsf{Mil}$, provability by* LLEE*-witnessed coinductive proofs over $\mathcal{S}$ implies derivability in* Mil*:*

$$\big(\ e_1 \overset{\mathrm{LLEE}}{=\!=}_{\mathcal{S}} e_2 \quad\Longrightarrow\quad e_1 =_{\mathsf{Mil}} e_2\ \big) \qquad \textit{for all } e_1, e_2 \in StExp(A). \tag{6.1}$$

Proof. For showing (6.1), let $e, f \in StExp(A)$ be such that $e \overset{\mathrm{LLEE}}{=\!=}_{\mathcal{S}} f$. Then there is a LLEE-witnessed coinductive proof $\mathcal{LCP} = \langle \underline{\mathcal{C}}, L \rangle$ of $e_1 = e_2$ over $\mathcal{S}$, which consists of a guarded LLEE-1-chart $\underline{\mathcal{C}}$ and $\mathcal{S}$-provable solutions $L_1, L_2 : V(\underline{\mathcal{C}}) \to StExp(A)$ of $\underline{\mathcal{C}}$ with $e_1 \equiv L_1(v_{\mathrm{s}})$ and $e_2 \equiv L_2(v_{\mathrm{s}})$. By applying Lemma 6.7 to $\mathcal{LCP}$ we find that L_1 and L_2 are Mil-provably equal. This entails $e_1 \equiv L_1(v_{\mathrm{s}}) =_{\mathsf{Mil}} L_2(v_{\mathrm{s}}) \equiv e_2$, and thus $e_1 =_{\mathsf{Mil}} e_2$. □



$$
\begin{aligned}
L_i(v_{21}) &=^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 \cdot L_i(v_2) =_{\mathsf{Mil}^-} L_i(v_2) \\
&\qquad (=^{(\mathrm{sol})}_{\mathsf{Mil}^-} \text{ means use of `is } \mathsf{Mil}^-\text{-provable solution'}) \\
L_i(v_2) &=^{(\mathrm{sol})}_{\mathsf{Mil}^-} b \cdot L_i(v_{21}) + 1 \cdot L_i(v_\mathrm{s}) =_{\mathsf{Mil}^-} b \cdot L_i(v_2) + L_i(v_\mathrm{s}) \\
&\Downarrow \text{ applying RSP}^* \\
L_i(v_2) &=_{\mathsf{Mil}} b^* \cdot L_i(v_\mathrm{s}) \\
L_i(v_{11}) &=^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 \cdot L_i(v_1) =_{\mathsf{Mil}^-} L_i(v_1) \\
L_i(v_1) &=_{\mathsf{Mil}^-} a \cdot L_i(v_{11}) + b \cdot L_i(v_{21}) + 1 \cdot L_i(v_\mathrm{s}) \\
&=_{\mathsf{Mil}} a \cdot L_i(v_1) + (b \cdot b^* + 1) \cdot L_i(v_\mathrm{s}) \\
&=_{\mathsf{Mil}^-} a \cdot L_i(v_1) + b^* \cdot L_i(v_\mathrm{s}) \\
&\Downarrow \text{ applying RSP}^* \\
L_i(v_1) &=_{\mathsf{Mil}} a^* \cdot (b^* \cdot L_i(v_\mathrm{s})) =_{\mathsf{Mil}^-} (a^* \cdot b^*) \cdot L_i(v_\mathrm{s})
\end{aligned}
$$

$$
\begin{aligned}
L_i(v_\mathrm{s}) &=^{(\mathrm{sol})}_{\mathsf{Mil}^-} 1 + a \cdot L_i(v_{11}) + b \cdot L_i(v_{21}) =_{\mathsf{Mil}^-} 1 + a \cdot L_i(v_1) + b \cdot L_i(v_2) \\
&=_{\mathsf{Mil}} (a \cdot (a^* \cdot b^*) + b \cdot b^*) \cdot L_i(v_\mathrm{s}) + 1 \\
&\Downarrow \text{ applying RSP}^* \\
L_i(v_\mathrm{s}) &=_{\mathsf{Mil}} (a \cdot (a^* \cdot b^*) + b \cdot b^*)^* \cdot 1 =_{\mathsf{Mil}^-} (a \cdot (a^* \cdot b^*) + b \cdot b^*)^* =_{\mathsf{Mil}^-} s_{\hat{\underline{\mathcal{C}}}}(v_\mathrm{s})
\end{aligned}
$$

Figure 13: Showing for the coinductive proof $\langle \underline{\mathcal{C}}, L \rangle$ in Example 5.2 that the principal value $L_i(v_\mathrm{s})$ of the Mil^--provable solution L_i for $i \in \{1, 2\}$ is Mil-provably equal to the principal value $s_{\hat{\underline{\mathcal{C}}}}(v_\mathrm{s})$ of the solution $s_{\hat{\underline{\mathcal{C}}}}$ extracted the underlying LLEE-witness $\hat{\underline{\mathcal{C}}}$.

Example 6.9. We consider again the LLEE-witnessed coinductive proof $\mathcal{CP} = \langle \underline{\mathcal{C}}, L \rangle$ of $(a^* \cdot b^*)^* = (a + b)^*$ in Example 5.2. In Figure 13 we exhibit the extraction process of derivations in Mil of $L_1(v_\mathrm{s}) = s_{\underline{\mathcal{C}}}(v)$ and $L_2(v_\mathrm{s}) = s_{\underline{\mathcal{C}}}(v)$ from the guarded LLEE-witness $\hat{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}$. These two derivations in Mil can be combined by using $\mathcal{EL}$ rules in order to obtain a derivation in Mil of $(a^* \cdot b^*)^* \equiv L_1(v_\mathrm{s}) = L_2(v_\mathrm{s}) \equiv (a + b)^*$.

Lemma 6.10. *The rules* $\mathrm{LCoProof}_n$ *are correct for* Mil*, for all* $n \in \mathbb{N}$*. This statement also holds effectively: Every derivation* $\mathcal{D}$ *in* $\mathsf{Mil}+\mathrm{LCoProof}_n$ *that consists of a bottommost instance of* $\mathrm{LCoProof}_n$*, where* $n \in \mathbb{N}$*, whose immediate subderivations are derivations in* Mil *can be transformed effectively into a derivation* $\mathcal{D}'$ *in* Mil *that has the same conclusion as* $\mathcal{D}$*.*

Proof. We let $n \in \mathbb{N}$. In order to show correctness of the rule $\mathrm{LCoProof}_n$ for Mil, we consider a derivation $\mathcal{D}$ in $\mathsf{Mil}+\mathrm{LCoProof}_n$ that has immediate subderivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in Mil, and that terminates with an instance ι of $\mathrm{LCoProof}_n$, where $\Gamma := \{g_1 = h_1, \ldots, g_1 = h_1\}$:

$$
\iota \; \frac{\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ g_1 = h_1 & \ldots & g_n = h_n \end{array}}{e = f} \; \mathrm{LCoProof}_n
$$

and $e \overset{\text{LLEE}}{=\!=}_{\mathsf{Mil}^-+\Gamma} f$ holds as side-condition on the instance ι of $\mathrm{LCoProof}_n$. Then there is a LLEE-witnessed coinductive proof $\mathcal{LCP} = \langle \underline{\mathcal{C}}, L \rangle$ of $e = f$ over $\mathsf{Mil}^-+\Gamma$. We have to show that there is a derivation $\mathcal{D}'$ in Mil with the same conclusion $e = f$.

Since $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are derivations in Mil, their conclusions in Γ are derivable in Mil. This implies $\mathsf{Mil}+\Gamma \lesssim \mathsf{Mil}$. It follows that $\mathcal{LCP}$ is also a LLEE-witnessed coinductive proof of

$e = f$ over Mil, that is, it holds:

$$e \overset{\text{LLEE}}{=\!=}_{\mathsf{Mil}} f\,.$$

From this we obtain $e =_{\mathsf{Mil}} f$ by applying Proposition 6.8, which guarantees a derivation $\mathcal{D}'$ in Mil with conclusion $e = f$ as desired. The proof of Proposition 6.8 furthermore guarantees that such a derivation $\mathcal{D}'$ in Mil can be constructed effectively from the coinductive proof of $e = f$ over $\mathsf{Mil}^{-}{+}\Gamma$ (and hence over Mil) and the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in Mil. □

Theorem 6.11. $\mathsf{cMil} \lesssim \mathsf{Mil}$. *Moreover, every derivation in* cMil *with conclusion* $e = f$ *can be transformed effectively into a derivation in* Mil *that has the same conclusion.*

Proof. Due to Lemma 6.10, every rule LCoProof_n, for $n \in \mathbb{N}$, is correct for Mil. Then by using Lemma 2.10, (i), we find that each of these rules are admissible in Mil. This means that $\mathsf{Mil}{+}\text{LCoProof}_n \sim \mathsf{Mil}$ holds for all $n \in \mathbb{N}$, which implies, with an argument by induction on the prooftree size of derivations in $\mathsf{Mil}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}$, that $\mathsf{Mil}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}} \sim \mathsf{Mil}$ holds as well. With this statement we can now argue as follows:

$$\begin{aligned}
\mathsf{cMil} &= \mathsf{Mil}^{-}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}} && \text{(by Definition 5.9)}\\
&\sqsubseteq (\mathsf{Mil}^{-}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}){+}\text{RSP}^{*} && \text{(by extension via adding the rule RSP}^{*}\text{)}\\
&= (\mathsf{Mil}^{-}{+}\text{RSP}^{*}){+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}} && \text{(by construing the same system differently)}\\
&= \mathsf{Mil}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}} && \text{(by Definition 2.11)}\\
&\sim \mathsf{Mil} && \text{(as argued above)}\,.
\end{aligned}$$

From this we obtain $\mathsf{cMil} \lesssim \mathsf{Mil}$ in view of $(\sqsubseteq \cdot \sim) \subseteq (\lesssim \cdot \sim) \subseteq \lesssim$.

For demonstrating the effective transformation statement of the theorem we use the transformation from the proof of the implication "$\Leftarrow$" in Lemma 2.10, (i), which states that correct rules are also admissible. We have to show that every derivation $\mathcal{D}$ in cMil can be transformed effectively into a derivation $\mathcal{D}'$ in Mil with the same conclusion. In order to establish this statement by induction we prove it for all derivations $\mathcal{D}$ in the extension $\mathsf{cMil}{+}\text{RSP}^{*} = \mathsf{Mil}^{-}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}{+}\text{RSP}^{*} = \mathsf{Mil}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}$ of cMil.

We proceed by induction on the number of instances of rules LCoProof_n, for $n \in \mathbb{N}$, in $\mathcal{D}$. Let $\mathcal{D}$ be a derivation $\mathsf{Mil}^{-}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}{+}\text{RSP}^{*}$. If $\mathcal{D}$ does not contain an instance of LCoProof_n with $n \in \mathbb{N}$, then $\mathcal{D}$ is already a derivation in $\mathsf{Mil} = \mathsf{Mil}^{-}{+}\text{RSP}^{*}$, and no further transformation is necessary. Otherwise $\mathcal{D}$ contains at least one instance of LCoProof_n with $n \in \mathbb{N}$. We pick an instance ι in $\mathcal{D}$ of a rule LCoProof_{n_0} with $n_0 \in \mathbb{N}$ that is topmost among the instances of the coinductive rule in $\mathcal{D}$, that is, none of the immediate subderivations of the instance ι in $\mathcal{D}$ contains any instance of a rule LCoProof_n for $n \in \mathbb{N}$. Let $\mathcal{D}_0$ be the subderivation of $\mathcal{D}$ that ends in ι. Since ι is topmost, all of the immediate subderivations of ι and $\mathcal{D}_0$ in $\mathcal{D}$ are derivations in $\mathsf{Mil} = \mathsf{Mil}^{-}{+}\text{RSP}^{*}$, and $\mathcal{D}_0$ is a derivation in $\mathsf{Mil}^{-}{+}\text{LCoProof}_{n_0}{+}\text{RSP}^{*} = \mathsf{Mil}{+}\text{LCoProof}_{n_0}$. Therefore we can apply the effective part of Lemma 6.10 to the subderivation $\mathcal{D}_0$. We obtain a derivation $\mathcal{D}_0'$ in Mil with the same conclusion as ι and $\mathcal{D}_0$. Then by replacing $\mathcal{D}_0$ in $\mathcal{D}$ with $\mathcal{D}_0'$ we obtain a derivation $\tilde{\mathcal{D}}$ in $\mathsf{Mil}^{-}{+}\{\text{LCoProof}_n\}_{n\in\mathbb{N}}{+}\text{RSP}^{*}$ that has the same conclusion as $\mathcal{D}$, but that has one instance of a coinductive rule less than $\mathcal{D}$. Now we can apply the induction hypothesis to $\tilde{\mathcal{D}}$ in order to effectively transform it in a derivation $\mathcal{D}'$ in Mil that has the same conclusion as $\mathcal{D}$. In this way we have performed the induction step. □

We conclude this section with an illustrative application of the results obtained here that provides in-roads for a completeness proof for Milner's system Mil. Specifically we

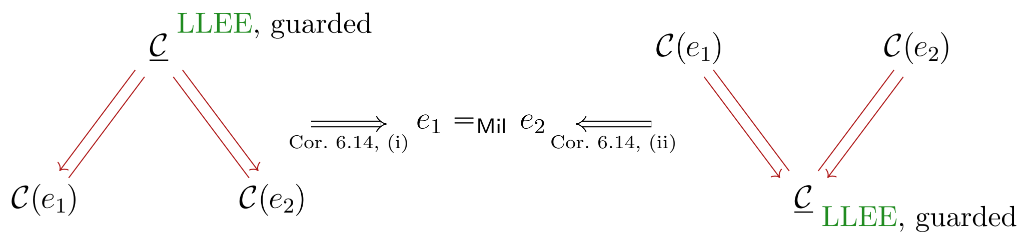


Figure 14: Illustration of the statements (for arbitrary given $e_1, e_2 \in \mathit{StExp}$) of Corollary 6.14: Milner's system Mil is complete for 1-bisimilarity of chart interpretations via guarded LLEE-1-charts as joint expansion or as joint minimization.

apply Proposition 6.8, the transformation of LLEE-witnessed coinductive proofs over Mil into derivations in Mil. We show (see Corollary 6.14 below) that Milner's system is complete for bisimilarity of chart interpretations of star expressions when bisimilarity is witnessed by joint expansion or joint minimization to a guarded LLEE-1-chart via functional 1-bisimulations (see Definition 6.12). For showing this statement we must, however, use here without proof a technical result from [Gra22a]: Mil-provable solutions of 1-charts can be transferred backwards over functional 1-bisimulations (see Lemma 6.13). This statement is a generalization to 1-charts of Proposition 5.1 in [GF20a], which states that $\mathcal{S}$-provable solutions of charts, for $\mathsf{ACI} \sqsupseteq \mathcal{S}$, can be transferred backwards over functional bisimulations.

Definition 6.12. Let $\underline{\mathcal{C}}$, $\underline{\mathcal{C}}_1$, and $\underline{\mathcal{C}}_2$ be 1-charts. We say that $\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$ are 1-bisimilar *via* $\underline{\mathcal{C}}$ *as (their) joint expansion* ($\underline{\mathcal{C}}_1$ and $\underline{\mathcal{C}}_2$ are 1-bisimilar *via* $\underline{\mathcal{C}}$ *as (their) joint minimization*) if $\underline{\mathcal{C}}_1 \Leftarrow \underline{\mathcal{C}} \Rightarrow \underline{\mathcal{C}}_2$ holds (respectively, if $\underline{\mathcal{C}}_1 \Rightarrow \underline{\mathcal{C}} \Leftarrow \underline{\mathcal{C}}_2$ holds).

Lemma 6.13 ($\sim$ Lemma 3.8, (i), in [Gra22a])**.** Mil-*Provable solvability with principal value e is preserved under converse functional 1-bisimilarity, on weakly guarded 1-charts, for all star expressions $e \in \mathit{StExp}$.*

Corollary 6.14. *The following two statements hold (see Figure 14 for their illustrations):* (i) Mil *is complete for 1-bisimilarity of chart interpretations of two star expressions via a guarded* LLEE-1*-chart as joint expansion.* (ii) Mil *is complete for 1-bisimilarity of chart interpretations of two star expressions via a guarded* LLEE-1*-chart as joint minimization.*

Proof. For showing statement (i), let $e_1, e_2 \in \mathit{StExp}(A)$ be such that there is a guarded LLEE-1-chart $\underline{\mathcal{C}}$ with $\mathcal{C}(e_1) \Leftarrow \underline{\mathcal{C}} \Rightarrow \mathcal{C}(e_2)$. We have to show $e_1 =_{\mathsf{Mil}} e_2$.

Due to Lemma 3.7, there are Mil^{-}-provable (and thus Mil-provable) solutions of $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ with principal values e_1, and e_2, respectively. Then we obtain by applying Lemma 6.13, in view of the converse functional 1-bisimulations from $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ to $\underline{\mathcal{C}}$, that there are two Mil-provable solutions L_1 and L_2 of $\underline{\mathcal{C}}$ with the principal values e_1 and e_2, respectively. These two Mil-provable solutions of the guarded LLEE-1-chart $\underline{\mathcal{C}}$ can be combined to obtain a LLEE-witnessed coinductive proof $\langle \underline{\mathcal{C}}, L \rangle$ of $e_1 = e_2$ over Mil. Thus we obtain:

$$e_1 \overset{\text{LLEE}}{=\!=\!=}_{\mathsf{Mil}} e_2 \,.$$

From this we arrive at $e_1 =_{\mathsf{Mil}} e_2$ by Proposition 6.8.

For showing statement (ii), let $e_1, e_2 \in \mathit{StExp}(A)$ be such that there is a guarded LLEE-1-chart $\underline{\mathcal{C}}$ with $\mathcal{C}(e_1) \Rightarrow \underline{\mathcal{C}} \Leftarrow \mathcal{C}(e_2)$. We have to show $e_1 =_{\mathsf{Mil}} e_2$.

We first note that due to Lemma 6.3 the extraction function $s_{\underline{\hat{\mathcal{C}}}}$ of a (guarded) LLEE-witness $\underline{\hat{\mathcal{C}}}$ of the guarded LLEE-1-chart $\underline{\mathcal{C}}$ yields a Mil^{-}-provable (and hence also a Mil-provable) solution s of $\underline{\mathcal{C}}$ whose principal value we denote by e. Next we apply Theorem 4.18, (ii), to extend the assumed functional 1-bisimulations to $\underline{\mathcal{C}}$ above the chart interpretations $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ to start from the 1-chart interpretations $\underline{\mathcal{C}}(e_1)$ and $\underline{\mathcal{C}}(e_2)$: we obtain $\underline{\mathcal{C}}(e_1) \rightarrow \mathcal{C}(e_1) \rightarrow \underline{\mathcal{C}} \leftarrow \mathcal{C}(e_2) \leftarrow \underline{\mathcal{C}}(e_2)$. By Theorem 4.18, (i), we find that $\underline{\mathcal{C}}(e_1)$ and $\underline{\mathcal{C}}(e_2)$ are guarded LLEE-1-charts. By transitivity of $\rightarrow$ we obtain $\underline{\mathcal{C}}(e_1) \rightarrow \underline{\mathcal{C}} \leftarrow \underline{\mathcal{C}}(e_2)$. Now we can apply Lemma 6.13 to obtain, from the Mil-provable solution s of $\underline{\mathcal{C}}$, a Mil-provable solution $L_{e_1,2}$ of $\underline{\mathcal{C}}(e_1)$ with principal value e, and $L_{e_2,2}$ of $\underline{\mathcal{C}}(e_2)$ also with principal value e. By Lemma 4.21 we furthermore obtain Mil^{-}-provable (and hence Mil-provable) solutions $L_{e_1,1}$ of $\underline{\mathcal{C}}(e_1)$ with principal value e_1, and $L_{e_2,1}$ of $\underline{\mathcal{C}}(e_2)$ with principal value e_2. The two Mil-provable solutions $L_{e_1,1}$ and $L_{e_1,2}$ of the guarded LLEE-1-chart $\underline{\mathcal{C}}(e_1)$ can be combined to a LLEE-witnessed coinductive proof $\langle \underline{\mathcal{C}}(e_1), L_{e_1} \rangle$ of $e_1 = e$ over Mil. Analogously, the two Mil-provable solutions $L_{e_2,1}$ and $L_{e_2,2}$ of the guarded LLEE-1-chart $\underline{\mathcal{C}}(e_2)$ can be combined to a LLEE-witnessed coinductive proof $\langle \underline{\mathcal{C}}(e_2), L_{e_2} \rangle$ of $e_2 = e$ over Mil. Together we obtain:

$$e_1 \overset{\text{LLEE}}{=\!=}_{\mathsf{Mil}} e \quad \text{and} \quad e_2 \overset{\text{LLEE}}{=\!=}_{\mathsf{Mil}} e\,.$$

Now by an appeal to Proposition 6.8 we obtain $e_1 =_{\mathsf{Mil}} e$ and $e_2 =_{\mathsf{Mil}} e$. Finally, by applying of symmetry and transitivity rules in Mil we obtain $e_1 =_{\mathsf{Mil}} e_2$. $\square$

Remark 6.15. While both of the statements (i) and (ii) of Corollary 6.14 were important pieces of the puzzle for constructing the completeness proof for Mil as sketched in [Gra22a], neither of them yields such a proof directly. This is because of the following two facts: First, two bisimilar chart interpretations do not always have a guarded LLEE-1-chart as their joint expansion along $\leftarrow$. This can be demonstrated with the counterexample of the charts in Example 4.1 of [GF20a]. As a consequence, (i) is sometimes not applicable. Second, two bisimilar chart interpretations do not in general have a guarded LLEE-1-chart as their joint minimization along $\rightarrow$. This is an easy consequence of the counterexample that is described in [Gra22a, Sect. 6]. Therefore direct application of (ii) is also excluded in some situations.

Yet more sophisticated applications of (i) and (ii) can still turn out to be expedient. Indeed, a strengthening of (ii) has helped us settle a subcase, and a refinement of (ii) has led to completeness proof for Mil as sketched in [Gra22a]. In [GF20a], Fokkink and I have employed the stronger and more specific version of the joint minimization idea in Corollary 6.14, (ii), for a completeness proof of an adaptation BBP by Bergstra, Bethke, and Ponse of Mil to '1-free star expressions' (without 1, and with binary iteration instead of unary iteration). Concretely, we used that bisimilar chart interpretations of 1-free star expressions have LLEE-charts as their bisimulation collapses (and thus have guarded LLEE-1-charts as joint minimizations). The completeness proof for Mil as summarized in [Gra22a] has been built around a more involved argument that employs 'crystallized' LLEE-1-chart approximations of the joint bisimulation collapse of bisimilar chart interpretations.

7. From Milner's system to LLEE-witnessed coinductive proofs

In this section we develop a proof-theoretic interpretation of Mil in the subsystem cMil_1 of cMil, and hence also a proof-theoretic interpretation of Mil in cMil. Since Mil and cMil_1 differ only by the fixed-point rule RSP^* (which is part of Mil, but not of cMil) and the

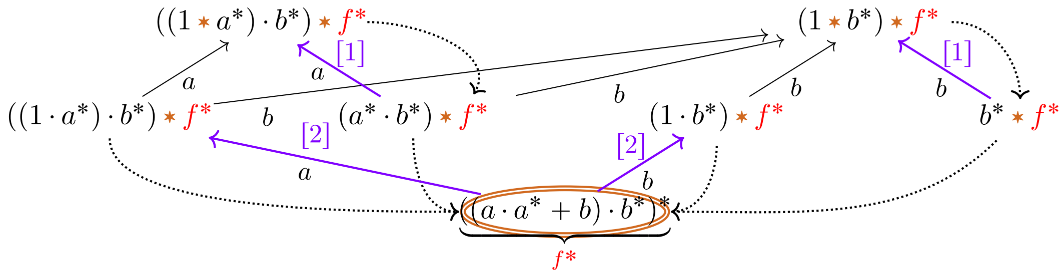


Figure 15: The 1-chart interpretation $\underline{\mathcal{C}}(f^*)$ for f^* as in (7.1) in Example 7.1.

rule LCoProof_1 (which is part of cMil_1, but not of Mil), the crucial step for this proof transformation is to show that instances of RSP^* can be mimicked in cMil_1.

We will do so by showing that instances of RSP^* are derivable in cMil_1, and in particular, can be mimicked by instances of LCoProof_1. More precisely, we will show that every instance ι of the fixed-point rule RSP^* of Mil can be mimicked by an instance of LCoProof_1 that has the same premise and conclusion, and that uses as its side-condition a LLEE-witnessed coinductive proof over Mil^- in which the premise of ι may be used. Still more explicitly, we show that every RSP^*-instance with premise $e = f \cdot e + g$ such that $f{\downarrow}$ and with conclusion $e = f^* \cdot g$ gives rise to a coinductive proof of $e = f^* \cdot g$ over $\mathsf{Mil}^- + \{e = f \cdot e + g\}$ with underlying 1-chart $\underline{\mathcal{C}}(f^* \cdot g)$ and guarded LLEE-witness $\hat{\underline{\mathcal{C}}}(f^* \cdot g)$.

We first illustrate this mimicking step by a concrete example (see Example 7.1), in order to motivate and convey the idea of this proof transformation. It will be built on three auxiliary statements (see after Example 7.1) two of which we have shown already in Section 4. Subsequently we prove the remaining crucial auxiliary statement (Lemma 7.2), and then establish the transformation by showing that RSP^* is a derivable rule in cMil_1 (Lemma 7.4, using Lemma 7.3). Finally we use derivability of RSP^* in cMil_1 in order to obtain the proof transformation from Mil to cMil_1 (see Theorem 7.5).

Example 7.1. We consider an instance of RSP^* that corresponds, up to an application of $\text{r-distr}(+,\cdot)$, to the instance of RSP^* at the bottom in Figure 13:

$$\frac{\overbrace{(a+b)^*}^{e} = \overbrace{((a \cdot a^* + b) \cdot b^*)}^{f} \cdot \overbrace{(a+b)^*}^{e} + \overbrace{\;1\;}^{g}}{\underbrace{(a+b)^*}_{e} = \underbrace{((a \cdot a^* + b) \cdot b^*)^*}_{f^*} \cdot \underbrace{\;1\;}_{g}} \text{RSP}^* \quad (\text{where } f{\not\downarrow}) \tag{7.1}$$

We want to mimic this instance of RSP^* by an instance of LCoProof_1 that uses a LLEE-witnessed coinductive proof of $e = f^* \cdot g$ over Mil^- plus the premise of the RSP^* instance (7.1). We first obtain the 1-chart interpretation $\underline{\mathcal{C}}(f^*)$ of f^* according to Definition 4.15, see Figure 15, together with its LLEE-witness $\hat{\underline{\mathcal{C}}}(f^*)$ that is guaranteed by Theorem 4.18.

Due to Lemma 4.21 the iterated partial 1-derivatives as depicted define a Mil^--provable solution of $\underline{\mathcal{C}}(f^*)$ when stacked products $\ast$ are replaced by products $\cdot$. From this LLEE-witness that carries a Mil-provable solution we now obtain a LLEE-witnessed coinductive proof of $f \cdot e + g = f^* \cdot g$ under the assumption of $e = f \cdot e + g$, as follows. By replacing parts $(\ldots) \ast f^*$ by $\pi((\ldots)) \cdot e$ in the Mil-provable solution of $\underline{\mathcal{C}}(f^*)$, and respectively, by replacing $(\ldots) \ast f^*$ by $(\pi((\ldots)) \cdot f^*) \cdot g$ we obtain the left- and the right-hand sides of the

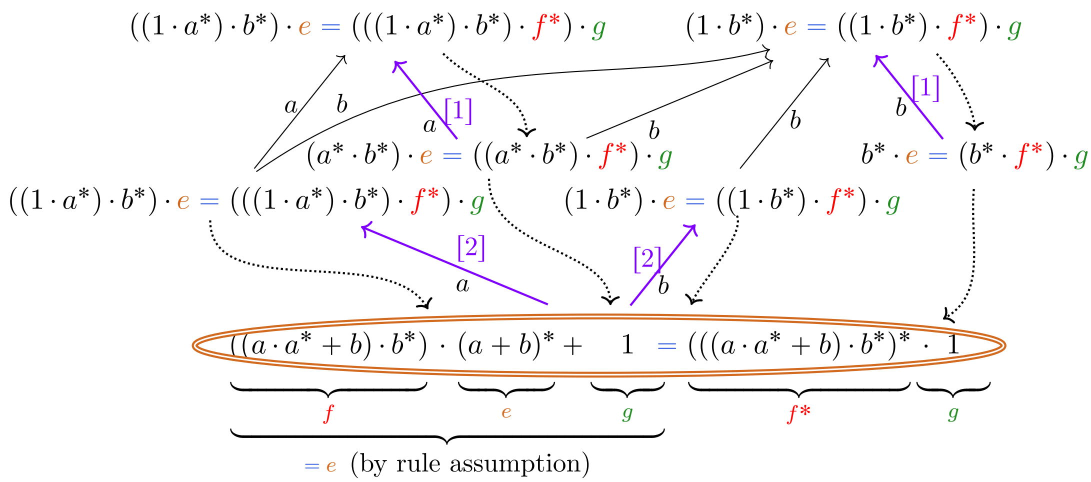


Figure 16: LLEE-witnessed coinductive proof of $f \cdot e + g = f^* \cdot g$ over $\mathsf{Mil}^{-}+\{e = f \cdot e + g\}$.

formal equations in the cyclic derivation in Figure 16. That derivation is a LLEE-witnessed coinductive proof $\mathcal{LCP}$ of $f \cdot e + g = f^* \cdot g$ over $\mathsf{Mil}^{-}+\{e = f \cdot e + g\}$: The right-hand sides form a Mil-provable solution of $\mathcal{C}(f^* \cdot g)$ due to Lemma 4.21 (note that $\mathcal{C}(f^* \cdot g)$ is isomorphic to $\mathcal{C}(f^*)$ due to $g \equiv 1$). The left-hand sides also form a solution of $\mathcal{C}(f^* \cdot g)$ (see Lemma 7.2 below), noting that for the 1-transitions back to the conclusion the assumption $e = f \cdot e + g$ must be used in addition to Mil^{-}. By using this assumption again, the result $\mathcal{LCP}'$ of replacing $f \cdot e + g$ in the conclusion of $\mathcal{LCP}$ by e is also a LLEE-witnessed coinductive proof over $\mathsf{Mil}^{-}+\{e = f \cdot e + g\}$. Consequently:

$$\frac{e = f \cdot e + g \qquad \mathcal{LCP}_{\mathsf{Mil}^{-}+\{e = f \cdot e + g\}}(e = f^* \cdot g)}{e = f^* \cdot g} \,\mathrm{LCoProof}_1 \tag{7.2}$$

is a rule instance of cMil and CLC by which we have mimicked the RSP^* instance in (7.1).

By examining the steps that we used in the example above, we find that three main auxiliary statements were used for the construction of a LLEE-witnessed coinductive proof that mimics an instance of the fixed-point rule RSP^* by an instance of $\mathrm{LCoProof}_1$. In relation to an instance of RSP^* of the generic form as in Definition 2.11, these are the statements that, for all star expressions e, f, and g, it holds:

(a) The 1-chart interpretation $\underline{\mathcal{C}}(e)$ of a star expressions e is a guarded LLEE-1-chart.
(b) e is the principal value of a Mil^{-}-provable solution of the 1-chart interpretation $\underline{\mathcal{C}}(e)$ of e.
(c) e is the principal value of a $(\mathsf{Mil}^{-}+\{e = f \cdot e + g\})$-provable solution of the 1-chart interpretation $\underline{\mathcal{C}}(f^* \cdot g)$ of $f^* \cdot g$.

While (a) is guaranteed by Theorem 4.18, and (b) by Lemma 4.21, we are now going to justify the central statement (c) by proving the following lemma.

Lemma 7.2. *Let $e, f, g \in StExp(A)$ with $f{\downarrow}$, and let $\Gamma := \{e = f \cdot e + g\}$. Then e is the principal value of a $(\mathsf{Mil}^{-}+\Gamma)$-provable solution of the 1-chart interpretation $\underline{\mathcal{C}}(f^* \cdot g)$ of $f^* \cdot g$.*

Proof. First, it can be verified that the vertices of $\underline{\mathcal{C}}(f^* \cdot g)$ are of either of three forms:

$$V(\underline{\mathcal{C}}(f^* \cdot g)) = \{f^* \cdot g\} \cup \big\{(F \star f^*) \cdot g \;\big|\; F \in \underline{\partial}^{+}(f)\big\} \cup \big\{G \mid G \in \underline{\partial}^{+}(g)\big\}\,, \tag{7.3}$$

where $\underline{\partial}^{+}(f)$ means the set of iterated 1-derivatives of f according to the TSS in Def. 4.15.
This facilitates to define a function $s : V(\underline{\mathcal{C}}(f^* \cdot g)) \to \mathit{StExp}(A)$ on $\underline{\mathcal{C}}(f^* \cdot g)$ by:

$$\begin{aligned} s(f^* \cdot g) &:= e\,, \\ s((F \ast f^*) \cdot g) &:= \pi(F) \cdot e\,, && (\text{for } F \in \underline{\partial}^{+}(f)), \\ s(G) &:= \pi(G) && (\text{for } G \in \underline{\partial}^{+}(g)), \end{aligned}$$

We will show that s is a $(\mathsf{Mil}^{-}{+}\Gamma)$-provable solution of $\underline{\mathcal{C}}(f^* \cdot g)$. Instead of verifying the correctness conditions for s for list representations of transitions, we will argue more loosely with sums over action 1-derivatives sets $\underline{A\partial}(H)$ of stacked star expressions H where such sums are only well-defined up to ACI. Due to $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^{-}$ such an argumentation is possible. Specifically we will demonstrate, for all $E \in V(\underline{\mathcal{C}}(f^* \cdot g))$, that s is a $(\mathsf{Mil}^{-}{+}\Gamma)$-provable solution at E, that is, that it holds:

$$s(E) =_{\mathsf{Mil}^{-}+\Gamma} \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{\langle \underline{a}, E' \rangle \in \underline{A\partial}(E)} \underline{a} \cdot s(E')\,, \tag{7.4}$$

where by the sum on the right-hand side we mean an arbitrary representative of the ACI equivalence class of star expressions that is described by the sum expression of this form.

For showing (7.4), we distinguish the three cases of vertices $E \in V(\underline{\mathcal{C}}(f^* \cdot g))$ according to (7.3), that is, $E \equiv f^* \cdot g$, $E \equiv (F \ast f^*) \cdot g$ for some $F \in \underline{\partial}^{+}(f)$, and $E \equiv G$ for some $G \in \underline{\partial}^{+}(g)$. We will see that the assumption Γ will only be needed for the treatment of the first case.

In the first case, we consider $E \equiv f^* \cdot g$. We find by Lemma 4.17 (or by inspecting the TSS in Definition 4.15), and in view of (7.3):

$$\underline{A\partial}(f^* \cdot g) = \{\langle \underline{a}, (F \ast f^*) \cdot g \rangle \mid \langle \underline{a}, F \rangle \in \underline{A\partial}(f)\} \cup \underline{A\partial}(g)\,, \tag{7.5}$$

$$\begin{aligned} \underline{\partial}(f^* \cdot g) &= \{(F \ast f^*) \cdot g \mid F \in \underline{\partial}(f)\} \cup \underline{\partial}(g) \\ &\subseteq \big\{(F \ast f^*) \cdot g \mid F \in \underline{\partial}^{+}(f)\big\} \cup \underline{\partial}^{+}(g) \subseteq V(\underline{\mathcal{C}}(f^* \cdot g))\,. \end{aligned} \tag{7.6}$$

Then (7.6) guarantees that s is defined for all partial 1-derivatives of $E \equiv f^* \cdot g$. With this knowledge we can argue as follows:

$$\begin{aligned} s(E) &\equiv s(f^* \cdot g) && (\text{by } E \equiv f^* \cdot g) \\ &\equiv e && (\text{by the definition of } s) \\ &=_{\mathsf{Mil}^{-}+\Gamma} f \cdot e + g && (\text{since } \Gamma = \{e = f \cdot e + g\}) \\ &=_{\mathsf{Mil}^{-}} \Big(\tau_{\underline{\mathcal{C}}(f)}(f) + \sum_{\langle \underline{a}, F \rangle \in \underline{A\partial}(f)} \underline{a} \cdot \pi(F)\Big) \cdot e + \Big(\tau_{\underline{\mathcal{C}}(g)}(g) + \sum_{\langle \underline{a}, G \rangle \in \underline{A\partial}(g)} \underline{a} \cdot \pi(G)\Big) \\ & \qquad (\text{by using Lemma 4.20}) \\ &=_{\mathsf{Mil}^{-}} \Big(\tau_{\underline{\mathcal{C}}(f)}(f) \cdot e + \sum_{\langle \underline{a}, F \rangle \in \underline{A\partial}(f)} \underline{a} \cdot (\pi(F) \cdot e)\Big) + \Big(\tau_{\underline{\mathcal{C}}(g)}(g) + \sum_{\langle \underline{a}, G \rangle \in \underline{A\partial}(g)} \underline{a} \cdot \pi(G)\Big) \\ & \qquad (\text{by using } (\text{r-distr}(+,\cdot)) \text{ and } (\text{assoc}(\cdot))) \\ &=_{\mathsf{Mil}^{-}} \Big(\sum_{\langle \underline{a}, F \rangle \in \underline{A\partial}(f)} \underline{a} \cdot (\pi(F) \cdot e)\Big) + \Big(\tau_{\underline{\mathcal{C}}(f^* \cdot g)}(f^* \cdot g) + \sum_{\langle \underline{a}, G \rangle \in \underline{A\partial}(g)} \underline{a} \cdot \pi(G)\Big) \\ & \qquad (\text{since } \tau_{\underline{\mathcal{C}}(f)}(f) \equiv 0 \text{ due to } f{\downarrow}\!\!\!/, \text{ by using axiom (deadlock)}, \\ & \qquad \text{and } \tau_{\underline{\mathcal{C}}(f^* \cdot g)}(f^* \cdot g) \equiv \tau_{\underline{\mathcal{C}}(g)}(g)) \end{aligned}$$

$$
\begin{aligned}
&=_{\mathsf{ACI}} \quad \tau_{\underline{\mathcal{C}}(f^*\cdot g)}(f^*\cdot g) + \Big(\sum_{\langle \underline{a}, F\rangle \in \underline{A\partial}(f)} \underline{a}\cdot(s((F * f^*)\cdot g))\Big) + \sum_{\langle \underline{a}, G\rangle \in \underline{A\partial}(g)} \underline{a}\cdot s(G) \\
&\qquad\qquad \text{(by definition of } s\text{, axioms (comm(+)))} \\
&=_{\mathsf{ACI}} \quad \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{\langle \underline{a}, E'\rangle \in \underline{A\partial}(f^*\cdot g) = \underline{A\partial}(E)} \underline{a}\cdot s(E') \\
&\qquad\qquad \text{(by } E \equiv f^*\cdot g \text{ and (7.5)).}
\end{aligned}
$$

Due to $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^- \sqsubseteq \mathsf{Mil}^-{+}\Gamma$ this chain of equalities is provable in $\mathsf{Mil}^-{+}\Gamma$, which verifies (7.4) for E as considered here, or in other words, s is a $(\mathsf{Mil}^-{+}\Gamma)$-provable solution of at E.

In the second case we consider $E \equiv (F * f^*)\cdot g \in V(\underline{\mathcal{C}}(f^*\cdot g))$. Then $F \in \underline{\partial}^+(f)$, and $\tau_{\underline{\mathcal{C}}(E)}(E) \equiv \tau_{\underline{\mathcal{C}}((F * f^*)\cdot g)}((F * f^*)\cdot g) \equiv 0$ holds, because expressions with stacked product occurring do not have immediate termination by Definition 4.15. We distinguish the subcases $F{\downarrow}$ and $F{\not\downarrow}$.

For the first subcase we assume $F{\not\downarrow}$. Then $\tau_{\underline{\mathcal{C}}(F)}(F) \equiv 0$ holds, and we find by Lemma 4.17 (or by inspecting the TSS in Definition 4.15), by $F \in \underline{\partial}^+(f)$, and from (7.3):

$$
\underline{A\partial}((F * f^*)\cdot g) = \big\{\langle \underline{a}, (F' * f^*)\cdot g\rangle \mid \langle \underline{a}, F'\rangle \in \underline{A\partial}(F)\big\}\ , \tag{7.7}
$$

$$
\begin{aligned}
\underline{\partial}((F * f^*)\cdot g) &= \big\{(F' * f^*)\cdot g \mid F' \in \underline{\partial}(F)\big\} \\
&\subseteq \big\{(F' * f^*)\cdot g \mid F' \in \underline{\partial}^+(f)\big\} \subseteq V(\underline{\mathcal{C}}(f^*\cdot g))\,.
\end{aligned} \tag{7.8}
$$

Due to (7.8), s is defined for all partial 1-derivatives of $E \equiv (F * f^*)\cdot g$. We argue as follows:

$$
\begin{aligned}
s(E) &\equiv s((F * f^*)\cdot g) && \text{(by } E \equiv (F * f^*)\cdot g\text{)} \\
&\equiv \pi(F)\cdot e && \text{(by the definition of } s\text{)} \\
&=_{\mathsf{Mil}^-} \Big(\tau_{\underline{\mathcal{C}}(F)}(F) + \sum_{\langle \underline{a}, F'\rangle \in \underline{A\partial}(F)} \underline{a}\cdot\pi(F')\Big)\cdot e && \text{(by using Lemma 4.20)} \\
&=_{\mathsf{Mil}^-} 0\cdot e + \sum_{\langle \underline{a}, F'\rangle \in \underline{A\partial}(F)} \underline{a}\cdot(\pi(F')\cdot e) && \begin{array}{l}\text{(by } \tau_{\underline{\mathcal{C}}(F)}(F) \equiv 0\text{, due to } F{\not\downarrow}\text{, and} \\ \text{axioms (r-distr}(+,\cdot)\text{), (assoc}(\cdot)\text{))}\end{array} \\
&=_{\mathsf{Mil}^-} 0 + \sum_{\langle \underline{a}, F'\rangle \in \underline{A\partial}(F)} \underline{a}\cdot s((F' * f^*)\cdot g) && \text{(by ax. (deadlock) and def. of } s\text{)} \\
&=_{\mathsf{ACI}} \tau_{\underline{\mathcal{C}}((F * f^*)\cdot g)}((F * f^*)\cdot g) + \sum_{\langle \underline{a}, E'\rangle \in \underline{A\partial}((F * f^*)\cdot g)} \underline{a}\cdot s(E') && \text{(due to (7.7), and } \tau_{\underline{\mathcal{C}}(E)}(E) \equiv 0\text{)} \\
&\equiv \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{\langle \underline{a}, E'\rangle \in \underline{A\partial}(E)} \underline{a}\cdot s(E') && \text{(by } E \equiv (F * f^*)\cdot g\text{).}
\end{aligned}
$$

For the second subcase we assume $F{\downarrow}$. Then $F \in \mathit{StExp}(A)$ (that is, F does not contain a stacked product symbol), and $\tau_{\underline{\mathcal{C}}(F)}(F) \equiv 1$ holds. Furthermore, we find, again by inspecting the TSS in Definition 4.15, by $F \in \underline{\partial}^+(f)$, and in view of (7.3):

$$
\underline{A\partial}((F * f^*)\cdot g) = \{\langle \underline{1}, f^*\cdot g\rangle\} \cup \big\{\langle \underline{a}, (F' * f^*)\cdot g\rangle \mid \langle \underline{a}, F'\rangle \in \underline{A\partial}(F)\big\}\ . \tag{7.9}
$$

$$
\begin{aligned}
\underline{\partial}((F * f^*)\cdot g) &= \{f^*\cdot g\} \cup \big\{(F' * f^*)\cdot g \mid F' \in \underline{\partial}(F)\big\} \\
&\subseteq \{f^*\cdot g\} \cup \big\{(F' * f^*)\cdot g \mid F' \in \underline{\partial}^+(f)\big\} \subseteq V(\underline{\mathcal{C}}(f^*\cdot g))\,.
\end{aligned} \tag{7.10}
$$

Due to (7.10), s is defined for all partial 1-derivatives of $E \equiv (F * f^*) \cdot g$ also in this subcase. Then we can argue as follows:

$$\begin{aligned}
s(E) &\equiv s((F * f^*) \cdot e) && \text{(by } E \equiv (F * f^*) \cdot g\text{)}\\
&\equiv \pi(F) \cdot e && \text{(by the definition of } s\text{)}\\
&=_{\mathsf{Mil}^-} \Big(\tau_{\underline{\mathcal{C}}(F)}(F) + \sum_{\langle \underline{a}, F' \rangle \in \underline{A\partial}(F)} \underline{a} \cdot \pi(F')\Big) \cdot e && \text{(by using Lemma 4.20)}\\
&=_{\mathsf{Mil}^-} 1 \cdot e + \sum_{\langle \underline{a}, F' \rangle \in \underline{A\partial}(f)} \underline{a} \cdot (\pi(F') \cdot e) && \text{(by } \tau_{\underline{\mathcal{C}}(F)}(F) \equiv 1\text{, and axioms (r-distr}(+,\cdot)\text{), (assoc}(\cdot)\text{))}\\
&\equiv 1 \cdot s(f^* \cdot g) + \sum_{\langle \underline{a}, F' \rangle \in \underline{A\partial}(f)} \underline{a} \cdot s((F' * f^*) \cdot g) && \text{(by the definition of } s\text{)}\\
&=_{\mathsf{ACI}} 0 + \sum_{\langle \underline{a}, E' \rangle \in \underline{A\partial}((F * f^*) \cdot g)} \underline{a} \cdot s(E') && \text{(by (7.9), using axioms (comm}(+)\text{), and (assoc}(+)\text{))}\\
&\equiv \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{\langle \underline{a}, E' \rangle \in \underline{A\partial}(E)} \underline{a} \cdot s(E') && \text{(by } E \equiv (F * f^*) \cdot g\text{, and } \tau_{\underline{\mathcal{C}}(E)}(E) \equiv 0\text{).}
\end{aligned}$$

Due to $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^- \sqsubseteq \mathsf{Mil}^-{+}\Gamma$ the chains of equalities in both subcases are provable in $\mathsf{Mil}^-{+}\Gamma$, and therefore we have now verified (7.4) also in the (entire) second case, that is, that s is a $(\mathsf{Mil}^-{+}\Gamma)$-provable solution of $\underline{\mathcal{C}}(f^* \cdot g)$ at E as in this case.

In the final case, $E \equiv G$ with $G \in \underline{\partial}^+(g)$. Since then 1-derivatives of G are in $\underline{\partial}^+(g)$ as well, and hence by (7.3) also in $V(\underline{\mathcal{C}}(f^* \cdot g))$, it follows that s is defined for all 1-derivatives of G and E. With this knowledge we can argue as follows:

$$\begin{aligned}
s(E) &\equiv s(G) && \text{(by } E \equiv G\text{)}\\
&\equiv \pi(G) && \text{(by the definition of } s\text{)}\\
&=_{\mathsf{Mil}^-} \tau_{\underline{\mathcal{C}}(G)}(G) + \sum_{\langle \underline{a}, G' \rangle \in \underline{A\partial}(G)} \underline{a} \cdot \pi(G') && \text{(by using Lemma 4.20)}\\
&=_{\mathsf{ACI}} \tau_{\underline{\mathcal{C}}(G)}(G) + \sum_{\langle \underline{a}, G' \rangle \in \underline{A\partial}(G)} \underline{a} \cdot s(G') && \text{(by the definition of } s\text{)}\\
&=_{\mathsf{ACI}} \tau_{\underline{\mathcal{C}}(E)}(E) + \sum_{\langle \underline{a}, E' \rangle \in \underline{A\partial}(E)} \underline{a} \cdot s(E') && \text{(by } E \equiv G\text{).}
\end{aligned}$$

Due to $\mathsf{ACI} \sqsubseteq \mathsf{Mil}^- \sqsubseteq \mathsf{Mil}^-{+}\Gamma$ this chain of equalities verifies (7.4) also in this case.

By having established (7.4) for the, according to (7.3), three possible forms of stacked star expressions that are vertices of $\underline{\mathcal{C}}(f^* \cdot g)$, we have shown that s is indeed a $(\mathsf{Mil}^-{+}\Gamma)$-provable solution of $\underline{\mathcal{C}}(f^* \cdot g)$. $\square$

After having proved statement (c), we can combine the statements (a), (b), and (c) as above in order construct LLEE-witnessed coinductive proofs with which instances of RSP* can be mimicked by instances of LCoProof_1. This leads us to Lemma 7.3 below, and, as it can show derivability of RSP* in cMil_1, to Lemma 7.4.

Lemma 7.3. *Let $e, f, g \in StExp(A)$ with $f{\downarrow}$, and let $\Gamma := \{e = f \cdot e + g\}$. Then it holds that $e \overset{\text{LLEE}}{=\!=\!=}_{(\mathsf{Mil}^-+\Gamma)} f^* \cdot g$.*

Proof. First, there is a $\mathsf{Mil}^{-}+\Gamma$-provable solution s_1 of $\underline{\mathcal{C}}(f^* \cdot g)$ with $s_1(f^* \cdot g) \equiv e$, due to Lemma 7.2. Second, there is a Mil^{-}-provable solution s_2 of $\underline{\mathcal{C}}(f^* \cdot g)$ with $s_2(f^* \cdot g) \equiv f^* \cdot g$, due to Lemma 4.21. Then $\langle \underline{\mathcal{C}}(f^* \cdot g), L \rangle$ with $L(v) := s_1(v) = s_2(v)$ for all $v \in V(\underline{\mathcal{C}}(f^* \cdot g))$ is a LLEE-witnessed coinductive proof of $e = f^* \cdot g$ over $\mathsf{Mil}^{-}+\Gamma$, because $\underline{\mathcal{C}}(f^* \cdot g)$ has the guarded LLEE-witness $\hat{\underline{\mathcal{C}}}(f^* \cdot g)$ by Theorem 4.18. $\square$

We note that the equation in the set Γ in the assumption of Lemma 7.3 does not need to be sound semantically. Therefore it was crucial for the formulation of this lemma that we did not require proof systems $\mathcal{S}$ to be sound with respect to $=_{[\![\cdot]\!]_P}$ for the definition of coinductive proofs in Definition 5.1. Indeed we have done so there in order to be able to formulate this lemma, which states that also instances of the fixed-point rule RSP^* with premises that are not semantically sound can be mimicked by appropriate coinductive proofs.

Lemma 7.4. RSP^* *is a derivable rule in* cMil.

Proof. Every instance ι of RSP^* can be replaced by a mimicking derivation $\mathcal{D}_\iota$ in cMil_1 according to the following step, where $f{\downarrow}$ holds as the side-condition of the instance of RSP^*:

$$\iota\,\frac{e = f \cdot e + g}{e = f^* \cdot g}\,\mathrm{RSP}^* \qquad \Longrightarrow \qquad \frac{e = f \cdot e + g}{e = f^* \cdot g}\,\mathrm{LCoProof}_1 \tag{7.11}$$

Here the side-condition $e \overset{\text{LLEE}}{=\!=\!=}_{(\mathsf{Mil}^{-}+\{e=f\cdot e+g\})} f^* \cdot g$ of the instance of $\mathrm{LCoProof}_1$ in the derivation $\mathcal{D}_\iota$ on the right is guaranteed by Lemma 7.3. $\square$

We now can show the main result of this section, the proof transformation from Mil to cMil_1. We obtain this transformation by using derivability of RSP^* in cMil_1 as stated by this lemma, and by combining basic proof-theoretic transformations that eliminate derivable, and hence correct and admissible, rules from derivations as described in the proof of Lemma 2.10.

Theorem 7.5. $\mathsf{Mil} \lesssim \mathsf{cMil}_1$. *What is more, every derivation in* Mil *with conclusion* $e = f$ *can be transformed effectively into a derivation with conclusion* $e = f$ *in* cMil_1.

Proof. Due to Lemma 7.4, RSP^* is a derivable rule in cMil_1. Then by Lemma 2.10, (ii), RSP^* is also an admissible rule in cMil_1, and hence $\mathsf{cMil}_1+\mathrm{RSP}^* \sim \mathsf{cMil}_1$ holds. With that we can now argue as follows:

$$\begin{aligned}\mathsf{Mil} = \mathsf{Mil}^{-}+\mathrm{RSP}^* &\sqsubseteq (\mathsf{Mil}^{-}+\mathrm{RSP}^*)+\mathrm{LCoProof}_1 \\ &= (\mathsf{Mil}^{-}+\mathrm{LCoProof}_1)+\mathrm{RSP}^* = \mathsf{cMil}_1+\mathrm{RSP}^* \sim \mathsf{cMil}_1\,.\end{aligned}$$

From this we can infer $\mathsf{Mil} \lesssim \mathsf{cMil}_1$, because $\sqsubseteq$ implies $\lesssim$, and $\lesssim \cdot \sim\; \subseteq\; \sim$. That every derivation $\mathcal{D}$ in Mil can be transformed effectively into a derivation $\mathcal{D}'$ in cMil with the same conclusion follows from derivability of RSP^* in cMil: then in $\mathcal{D}$ every instance of RSP^* can be replaced by a corresponding instance of $\mathrm{LCoProof}_1$ as described in (7.11) of the proof of Lemma 7.4 with as result a derivation $\mathcal{D}'$ in cMil_1 with the same conclusion as $\mathcal{D}$. This argument instantiates the implication from rule derivability to rule admissibility, and the transformations explained in the proof of Lemma 2.10, (i) and (ii), specifically (2.1). $\square$

Example 7.6. In Figure 2 we provided a first illustration for translating an instance of the fixed-point rule into a coinductive proof in Figure 2 on page 4. Specifically, we mimicked the instance ι (see below) of the fixed-point rule RSP^* in Milner's system $\mathsf{Mil} = \mathsf{Mil}^{-}+\mathrm{RSP}^*$ by a coinductive proof over $\mathsf{Mil}^{-}+\{\text{premise of } \iota\}$ with LLEE-witness $\hat{\underline{\mathcal{C}}}(f^* \cdot 0)$.

The correctness conditions that have to be satisfied for the right-hand sides in order to recognize this prooftree as a LLEE-witnessed coinductive proof $\mathsf{Mil}^{-}+\{\text{premise of } \iota\}$ are the same as those that we have verified for the right-hand sides of the coinductive proof over Mil^{-} with the same LLEE-witness in Example 5.3. Note that the premise of ι is not used for the correctness conditions of the right-hand sides. The correctness condition for the left-hand side $e_0^* \cdot 0$ at the bottom vertex of $\underline{\mathcal{C}}(f^* \cdot 0)$ can be verified as follows, now making use of the premise of the considered instance ι of RSP*:

$$\begin{aligned}
e_0^* \cdot 0 &=_{\{\text{premise of } \iota\}} f \cdot (e_0^* \cdot 0) + 0 \\
&=_{\mathsf{Mil}^-} (a \cdot (a+b) + b) \cdot (e_0^* \cdot 0) \\
&=_{\mathsf{Mil}^-} (a \cdot (a+b)) \cdot (e_0^* \cdot 0) + b \cdot (e_0^* \cdot 0) \\
&=_{\mathsf{Mil}^-} a \cdot ((a+b) \cdot (e_0^* \cdot 0)) + b \cdot (1 \cdot (e_0^* \cdot 0)) \\
&=_{\mathsf{Mil}^-} a \cdot ((1 \cdot (a+b)) \cdot (e_0^* \cdot 0)) + b \cdot (1 \cdot (e_0^* \cdot 0))
\end{aligned}$$

Together this yields the provable equation:

$$e_0^* \cdot 0 =_{\mathsf{Mil}^- + \{\text{premise of } \iota\}} a \cdot ((1 \cdot (a+b)) \cdot (e_0^* \cdot 0)) + b \cdot (1 \cdot (e_0^* \cdot 0)),$$

which demonstrates the correctness condition for the left-hand side $e_0^* \cdot 0$ at the bottom vertex of $\underline{\mathcal{C}}(f^* \cdot 0)$. The correctness condition for the left-hand side $a \cdot ((1 \cdot (a+b))$ at the top left vertex of $\underline{\mathcal{C}}(f^* \cdot 0)$ can be verified without using the premise of ι as follows:

$$\begin{aligned}
((1 \cdot (a+b)) \cdot (e_0^* \cdot 0) &=_{\mathsf{Mil}^-} ((a+b) \cdot (e_0^* \cdot 0) =_{\mathsf{Mil}^-} a \cdot (e_0^* \cdot 0) + b \cdot (e_0^* \cdot 0) \\
&=_{\mathsf{Mil}^-} a \cdot (1 \cdot (e_0^* \cdot 0)) + b \cdot (1 \cdot (e_0^* \cdot 0)).
\end{aligned}$$

Finally, the correctness condition of the left-hand side $1 \cdot (e_0^* \cdot 0)$ at the right upper vertex of $\underline{\mathcal{C}}(f^* \cdot 0)$ can be obtained by an application of the axiom $(\mathrm{id}_{\mathrm{l}}(\cdot))$ only.

We close this section by giving an example that provides an additional sanity check for the proof transformation from Mil to cMil that we developed above. The example below shows that the construction of a LLEE-witnessed coinductive proof fails for an inference that is <u>not</u> an instance of the fixed-point rule RSP* because the side-condition is violated.

Non-Example 7.7. We consider the following (not semantically valid) inference according to a (semantically unsound) extension $\cancel{\text{RSP}}^*$ of the fixed-point rule RSP* that does not require the guardedness side-condition $f\not\downarrow$ (see Definition 2.11):

$$\frac{\overbrace{(a+c)^*}^{e} = \overbrace{(a+1)}^{f} \cdot \overbrace{(a+c)^*}^{e} + \overbrace{1}^{g}}{(a+c)^* = \underbrace{(a+1)^*}_{f^*} \cdot \underbrace{1}_{g}} \; \cancel{\text{RSP}}^* \quad (\text{but not RSP}^* \text{ since } f \equiv (a+1)\downarrow)$$

The premise is semantically valid by Proposition 2.14 because it is provable in Mil^-:

$$\begin{aligned}
(a+c)^* &=_{\mathsf{Mil}^-} 1 + (a+c) \cdot (a+c)^* =_{\mathsf{Mil}^-} 1 + a \cdot (a+c)^* + c \cdot (a+c)^* \\
&=_{\mathsf{Mil}^-} 1 + a \cdot (a+c)^* + 1 + a \cdot (a+c)^* + c \cdot (a+c)^* \\
&=_{\mathsf{Mil}^-} 1 + a \cdot (a+c)^* + (a+c)^* =_{\mathsf{Mil}^-} 1 + a \cdot (a+c)^* + 1 \cdot (a+c)^* \\
&=_{\mathsf{Mil}^-} 1 + (a+1) \cdot (a+c)^* =_{\mathsf{Mil}^-} (a+1) \cdot (a+c)^* + 1.
\end{aligned}$$

But the conclusion of the inference is obviously not valid semantically, because its left-hand side can iterate c-transitions, while its right-hand side does not permit c-transitions.

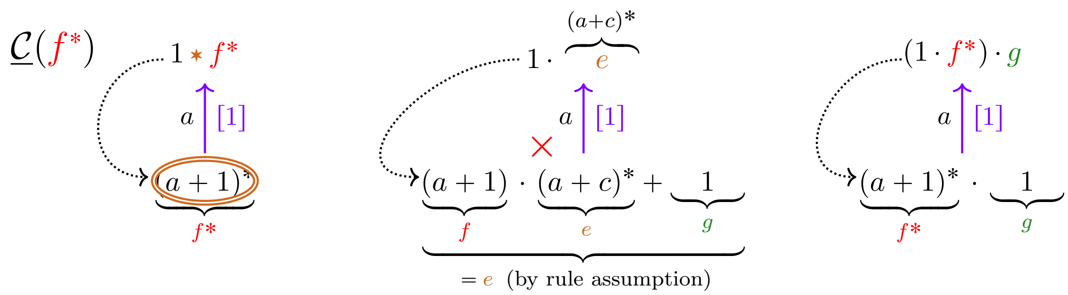


Figure 17: Failure of the construction a Mil^--provable solution for a coinductive proof that would mimic an instance of a fixed-point rule like RSP^* without side-condition.

Now by mechanically performing the same construction of a coinductive proof as we illustrated it in Example 7.1 and in Example 7.6, we obtain the LLEE-1-chart $\underline{\mathcal{C}}(f^*)$ and the star expression assignments to it as in Figure 17. There we recognize that, while $f^* \cdot g$ is the principal value of a Mil^--provable solution of $\underline{\mathcal{C}}(f^* \cdot g)$ (see on the right), we have not obtained a $\mathsf{Mil}+\{e = f \cdot e + g\}$-provable solution of $\underline{\mathcal{C}}(f^* \cdot g)$ with principal value $f \cdot e + g$ (see in the middle). This is because the correctness condition is violated at the bottom vertex, because $(a+1) \cdot (a+c)^* + 1 =_{\mathsf{Mil}^-} 1 + a \cdot (1 \cdot (a+c)^*)$ does not hold: otherwise it would have to be semantically valid by Proposition 2.14, but it is not, because only the left-hand side permits an initial c-transition.

Therefore the construction does not give rise to a LLEE-witnessed coinductive proof of the (not semantically valid) formal equation $(a+c)^* = (a+1) \cdot (a+c)^* + 1$.

Based on the proof transformations that we have developed in this section and earlier in Section 6, we now obtain our main result. It justifies that we called the proof system cMil a reformulation of Milner's system Mil.

Theorem 7.8. $\mathsf{Mil} \sim \mathsf{cMil}_1 \sim \mathsf{cMil} \sim \mathsf{CLC}$, *i.e. these proof systems are theorem-equivalent.*

Proof. By combining the statements of Theorem 7.5, Lemma 5.11, and Theorem 6.11 we obtain the theorem-subsumption statements $\mathsf{Mil} \lesssim \mathsf{cMil}_1 \lesssim \mathsf{cMil}\ (\sim \mathsf{CLC}) \lesssim \mathsf{Mil}$, which together justify the theorem-equivalence of Mil with each of cMil_1, cMil, and CLC. ☐

8. Summary and Conclusion

We set out on a proof-theoretic investigation of the problem of whether Milner's system Mil is complete for process-semantics equality $=_{[\![\cdot]\!]_P}$ on regular expressions (which Milner calls 'star expressions' for disambiguation when interpreted according to the process semantics). Specifically we aimed at characterizing the derivational power that the fixed-point rule RSP^* in Mil adds to its purely equational part Mil^-.

In order to define a substitute for the rule RSP^* we based ours on two results that we have obtained earlier (the first in joint work with Fokkink):

(S/U) Linear specifications of the shape[4] of transition graphs that satisfy the layered loop existence and elimination property LLEE are:

[4]The two examples on page 2 illustrate the correspondence between a recursive specifications and its associated transition graph. But note that these graphs do not satisfy LLEE (see Example 4.8).

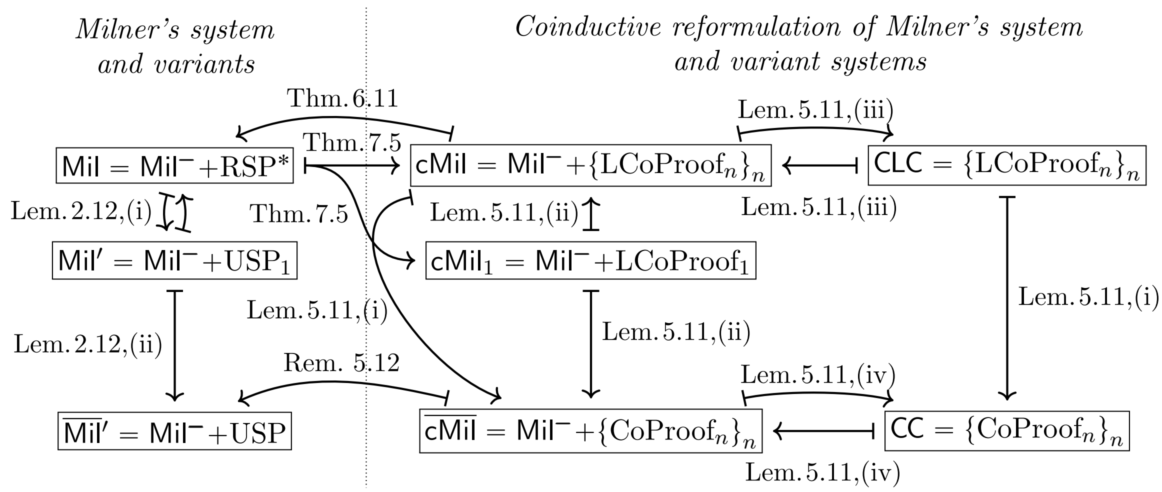


Figure 18: Web of proof transformations between Milner's system Mil and variants, and the coinductive reformulation cMil of Mil with kernel system CLC and variants.

(S) solvable by star expressions modulo provability in Mil^- (and modulo $\underline{\leftrightarrow}$).
(U) uniquely solvable by star expressions modulo provability in Mil (and mod. $\underline{\leftrightarrow}$).
These statements were crucial steps in the completeness proof [GF20b, GF20a] by Fokkink and myself for a tailored restriction BBP of Mil to '1-free' star expressions.

(IV)$_1$ While the chart interpretation $\mathcal{C}(e)$ of a star expression e does not always satisfy LLEE, there is a variant 1-chart interpretation $\underline{\mathcal{C}}(e)$ (typically with 1-transitions) that is (1-)bisimilar to e, and satisfies LLEE, for all star expressions e. (See Definition 4.15, and [Gra20, Gra21c]). Hereby 1-transitions are interpreted as empty-step processes.

Now in order to obtain results for 1-charts that are analogous to **(S)** and **(U)**, we have generalized these statements to guarded 1-charts in Section 6. In doing so, we obtained:

(S/U)$_1$ Guarded linear specifications of the shape of 1-charts that satisfy the loop existence and elimination property LLEE are:
(S)$_1$ solvable by star expressions modulo provability in Mil^- (see **(SE)**),
(U)$_1$ uniquely solvable by star expressions modulo provability in Mil (see **(SU)**).
These statements correspond to the statements **(SE)** and **(SU)** in Section 6 which we have proved as Lemma 6.4, and Lemma 6.7, respectively.

These statements motivated us to define 'coinductive proofs' as pairs of solutions for guarded LLEE-1-charts, because every pair of Mil^--provable, or Mil-provable solutions can be proved equal in Mil. On the basis of this idea we developed the following concepts and results (we emphasize earlier introduced terminology again here below):

① LLEE-*witnessed coinductive proof over an equational proof system* $\mathcal{S}$ (Definition 5.1): We defined a coinductive proof as a weakly guarded LLEE-1-chart $\underline{\mathcal{C}}$ whose vertices are labeled by equations between the values of two $\mathcal{S}$-provable solutions of $\underline{\mathcal{C}}$.

② *Coinductive reformulation* cMil of Milner's system Mil (Definition 5.9): We defined cMil as the result of replacing the fixed-point rule RSP* in Mil with rules of the scheme $\{\mathrm{LCoProof}_i\}_{i\in\mathbb{N}}$ that formalize provability by LLEE-witnessed coinductive proofs.

$$
\begin{tikzcd}[column sep=huge]
e_1 =_{\mathsf{Mil}} e_2 \arrow[r, bend right, "\text{Thm.}\,7.5"'] & e_1 =_{\mathsf{cMil}} e_2 \arrow[l, bend right, "\text{Thm.}\,6.11"'] \arrow[r, bend right, "\text{Lem.}\,5.11,\text{(iii)}"'] & e_1 =_{\mathsf{CLC}} e_2 \arrow[l, bend right, "\text{Lem.}\,5.11,\text{(iii)}"'] \arrow[r, bend right, "\text{due to Prop.}\,2.14"'] & \begin{array}{l} \mathcal{C}(e_1) \leftrightarrow \mathcal{C}(e_2) \\ e_1 =_{[\![\cdot]\!]_P} e_2 \end{array} \arrow[l, bend right, dashed, "\textbf{?}"']
\end{tikzcd}
$$

Figure 19: The coinductive proof system CLC (next to cMil) as potential beachhead for a completeness proof of Mil. We label with a question mark the transformation that is not defined here (it can be extracted from the completeness proof of Mil).

- ▻ As the 'kernel' of cMil we defined the system CLC (Definition 5.8) for combining LLEE-witnessed coinductive proofs with only the rules of the scheme $\{\text{LCoProof}_i\}_{i\in\mathbb{N}}$.

③ *Proof transformations* between cMil and Mil that show their theorem-equivalence.

- ▻ We showed that the rules of $\{\text{LCoProof}_i\}_{i\in\mathbb{N}}$ are correct and admissible for Mil, and that this implies that every derivation in cMil can be transformed into one in Mil with the same conclusion by eliminating occurrences of these rules. (See Section 6.)
- ▻ We showed that the fixed-point rule RSP* of Mil is derivable in cMil, because every instance ι can be mimicked by an instance of LCoProof_1 that uses the premise of ι, and, as a side-condition, a LLEE-witnessed coinductive proof over Mil^{-}. As a consequence we showed that every derivation in Mil can be transformed into one in cMil with the same conclusion by eliminating occurrences of RSP*. (See Section 7.)

④ *Coinductive reformulation* $\overline{\mathsf{cMil}}$ of the variant $\overline{\mathsf{Mil}}'$ (with the powerful rule USP, see Definition 2.11) of Mil: We formulated systems $\overline{\mathsf{cMil}}$ (Definition 5.9) and $\overline{\mathsf{CC}}$ (Definition 5.8) based on coinductive proofs without LLEE-witnesses. The systems $\overline{\mathsf{cMil}}$ and $\overline{\mathsf{CC}}$ can be recognized as complete variants of cMil and CLC. We have, however, only argued that in Remark 5.12 (on the basis of earlier work [Gra06]), but not proved it in detail here.

⑤ *Proof transformations* between cMil and CLC, and $\overline{\mathsf{cMil}}$ and $\overline{\mathsf{CC}}$ that show their theorem-equivalence. Here the idea of the non-trivial transformations from cMil to CLC was to 'hide' all derivation parts that consist of axioms or rules of the purely equational part Mil^{-} of Mil into the correctness conditions of solutions in LLEE-witnessed coinductive proofs, which occur as side-conditions of instances of rules of $\{\text{LCoProof}_i\}_{i\in\mathbb{N}}$ in CLC. The transformation from $\overline{\mathsf{cMil}}$ to $\overline{\mathsf{CC}}$ operates analogously. (See Lemma 5.11, (iii), and (iv).)

In Figure 18 we have illustrated the web of proof transformations between, on the one hand, Milner's systems and its variants as defined in Definition 2.11, and, on the other hand, the coinductive reformulations cMil and cMil_1 of Mil, and $\overline{\mathsf{cMil}}$ of $\overline{\mathsf{Mil}}'$ in Definition 5.9, as well as the coinductive kernel systems CLC of cMil, and $\overline{\mathsf{CC}}$ of $\overline{\mathsf{cMil}}$ in Definition 5.8.

The coinductive reformulation cMil of Mil, and its coinductive kernel CLC, can be looked upon as being situated roughly half-way in between Mil and bisimulations between chart interpretations of star expressions. We illustrate this in Figure 19. This picture arises from the highest level in Figure 18, when instantiated for specified conclusions $e_1 =_{\mathsf{Mil}} e_2$, $e_1 =_{\mathsf{cMil}} e_2$, and $e_1 =_{\mathsf{CLC}} e_2$, and by extending it further to $\mathcal{C}(e_1) \leftrightarrow \mathcal{C}(e_2)$, and hence to $e_1 =_{[\![\cdot]\!]_P} e_2$. For the last step we use soundness of CLC, which follows from soundness of Mil with respect to $=_{[\![\cdot]\!]_P}$, see Proposition 2.14, in view of the proof transformations that link CLC via cMil to Mil. Now we note that derivations of CLC represent proof-trees of coinductive proofs each of which defines a bisimulation up to provability by Proposition 5.6, and Remark 5.7. Therefore we can argue that prooftrees in CLC, and by proof-theoretic association also prooftrees in cMil, are situated roughly half-way in between prooftrees in Mil and bisimulations between chart interpretation of star expressions. We think that Figure 19 provides a reasonable suggestion of the proof-theoretic closeness of these systems.

The proof-theoretic connections of CLC with cMil and Mil guarantee that completeness of Mil (with respect to $=_{[\![\cdot]\!]_P}$) is equivalent to completeness of cMil, and also to completeness of CLC. Stronger still, the proof transformations between CLC, cMil, and Mil guarantee that every completeness proof of Mil can be 'routed through' CLC (and also through cMil). Such a 'rerouting' through CLC of a completeness proof of Mil does, however, not need to be equally natural as a 'direct' completeness proof of Mil. But since CLC is intuitively much closer to $=_{[\![\cdot]\!]_P}$ than Mil (as suggested by Figure 19), much hope was warranted to obtain a completeness proof for Mil by finding a completeness proof for CLC first, or to at least to use concepts that we have introduced here. (The latter hope turned out to be justified.)

Indeed, since the proof systems cMil and CLC are tied to process graphs via the circular deductions they permit and to bisimulations up to provability, and since cMil and CLC are theorem-equivalent with Mil, they were conceived as (and still can be) natural beachheads for a completeness proof of Milner's system. In Figure 19 we have indicated the step to CLC that is missing *here* for a completeness proof of Mil with a question mark: a completeness proof of CLC, and that is, an argument that, for every bisimulation between $\mathcal{C}(e_1)$ and $\mathcal{C}(e_2)$ where e_1 and e_2 are given star expressions, yields a prooftree in CLC with conclusion $e_1 = e_2$.

Due to their feature of permitting derivations that can be construed as combinations of 1-bisimulations (up to provability), we were confident that the proof system cMil and its coinductive kernel CLC substantially increase the space for graph-based approaches to finding a completeness proof of Mil. A concrete indication for this expectation was the following. By closely analyzing the completeness proof in [GF20b, GF20a] for the tailored restriction BBP of Mil to '1-free' star expressions we find:

> Valid equations between 1-free star expressions admit derivations in CLC of depth less than or equal to 2. (8.1)

This fact suggests the following research question:

> Can derivations in CLC (derivations in cMil) always be simplified to some kind of normal form that is of bounded depth (respectively, of bounded nesting depth of LLEE-witnessed coinductive proofs)? (8.2)

Despite of the fact that this question admits a trivial answer in view of the completeness proof of Mil in [Gra22a], see (C2) and (C3) below, it can be desirable to also find an answer that is based on a proof-theoretic analysis, and that leads to a workable concept of 'normal form' for derivations in CLC or in cMil. Intuitions for finding such a concept may be found in developing simplification steps of 1-charts with LLEE under 1-bisimilarity, as those are used in the completeness proofs for BBP with respect to '1-free' star expressions in [GF20b, GF20a], and for Mil with respect to all star expressions in [Gra22a, Gra22b]. We close by listing consequences of the latter (one is direct, the other two are a bit technical).

Consequences of the completeness proof of Mil in [Gra22a]. Below we list the most important consequences that the completeness proof of Mil as summarized in [Gra22a] has for the line of investigation on coinductive versions of Mil as reported here:

(C1) The coinductive versions cMil and CLC of Milner's system Mil are (just like Mil) complete with respect to process semantics equality $=_{[\![\cdot]\!]_P}$ of star expressions. (This follows directly from Theorem 5.1 in [Gra22a] in view of Theorem 7.8 here.)

(C2) Valid equations between star expressions admit derivations in CLC of depth less than or equal to 2. (This generalization of (8.1) can be shown by a close analysis of the structure of the completeness proof of Mil in Section 5 of [Gra22a]. The proof is similar to that of Corollary 6.14, (ii), and also similar to the argumentation for the analogous statement concerning '1-free' star expressions and the system BBP as reported above.)

(C3) As a consequence of (C2) the research question (8.2) admits the trivial answer "yes", albeit one that avoids a close proof-theoretic analysis. (Yet we still think that an answer that is based on a fine-grained proof-theoretic analysis would be more desirable).

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