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Structure Theorem and Strict Alternation Hierarchy for $\mathbf{FO^2}$ on Words *

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Abstract. It is well-known that every first-order property on words is expressible using at most three variables. The subclass of properties expressible with only two variables is also quite interesting and well-studied. We prove precise structure theorems that characterize the exact expressive power of first-order logic with two variables on words. Our results apply to both the case with and without a successor relation.

For both languages, our structure theorems show exactly what is expressible using a given quantifier depth, n, and using m blocks of alternating quantifiers, for any $m \leq n$. Using these characterizations, we prove, among other results, that there is a strict hierarchy of alternating quantifiers for both languages. The question whether there was such a hierarchy had been completely open. As another consequence of our structural results, we show that satisfiability for first-order logic with two variables without successor, which is NEXP-complete in general, becomes NP-complete once we only consider alphabets of a bounded size.

1. Introduction

It is well-known that every first-order property on words is expressible using at most three variables [7, 8]. The subclass of properties expressible with only two variables is also quite interesting and well-studied (Fact 1.1).

In this paper we prove precise structure theorems that characterize the exact expressive power of first-order logic with two variables on words. Our results apply to $\mathrm{FO}^2[<]$ and $\mathrm{FO}^2[<, \mathrm{Suc}]$, the latter of which includes the binary successor relation in addition to the linear ordering on string positions.

For both languages, our structure theorems show exactly what is expressible using a given quantifier depth, n, and using m blocks of alternating quantifiers, for any $m \leq n$. Using these characterizations, we prove that there is a strict hierarchy of alternating quantifiers for both languages. The question whether there was such a hierarchy had been completely open since it was asked in [3, 4]. As another consequence of our structural results,

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we show that satisfiability for $\mathrm{FO}^2[<]$, which is NEXP-complete in general [4], becomes NP-complete once we only consider alphabets of a bounded size.

Our motivation for studying FO^2 on words comes from the desire to understand the trade-off between formula size and number of variables. This is of great interest because, as is well-known, this is equivalent to the trade-off between parallel time and number of processors [6]. Adler and Immerman [1] introduced a game that can be used to determine the minimum size of first-order formulas with a given number of variables needed to express a given property. These games, which are closely related to the communication complexity games of Karchmer and Wigderson [9], were used to prove two optimal size bounds for temporal logics [1]. Later Grohe and Schweikardt used similar methods to study the size versus variable trade-off for first-order logic on unary words [5]. They proved that all first-order expressible properties of unary words are already expressible with two variables and that the variable-size trade-off between two versus three variables is polynomial whereas the trade-off between three versus four variables is exponential. They left open the trade-off between k and $k+1$ variables for $k \geq 4$. While we do not directly address that question here, our classification of FO^2 on words is a step towards the general understanding of the expressive power of FO needed for progress on such trade-offs.

Our characterization of $\mathrm{FO}^2[<]$ and $\mathrm{FO}^2[<, \mathrm{Suc}]$ on words is based on the very natural notion of n-ranker (Definition 3.2). Informally, a ranker is the position of a certain combination of letters in a word. For example, $\triangleright_{\mathtt{a}}$ and $\triangleleft_{\mathtt{b}}$ are 1-rankers where $\triangleright_{\mathtt{a}}(w)$ is the position of the first $\mathtt{a}$ in w (from the left) and $\triangleleft_{\mathtt{b}}(w)$ is the position of the first $\mathtt{b}$ in w from the right. Similarly, the 2-ranker $r_2 = \triangleright_{\mathtt{a}}\triangleright_{\mathtt{c}}$ denotes the position of the first $\mathtt{c}$ to the right of the first $\mathtt{a}$, and the 3-ranker, $r_3 = \triangleright_{\mathtt{a}} \triangleright_{\mathtt{c}} \triangleleft_{\mathtt{b}}$ denotes the position of the first $\mathtt{b}$ to the left of r_2. If there is no such letter then the ranker is undefined. For example, $r_3(\mathtt{cababcba}) = 5$ and $r_3(\mathtt{acbbca})$ is undefined.

Our first structure theorem (Theorem 3.8) says that the properties expressible in $\mathrm{FO}^2_n[<]$, i.e. first-order logic with two variables and quantifier depth n, are exactly boolean combinations of statements of the form, "r is defined", and "r is to the left (right) of r'" for k-rankers, r, and k'-rankers, r', with $k \leq n$ and $k' < n$. A non-quantitative version of this theorem was previously known [13].[1] Furthermore, a quantitative version in terms of iterated block products of the variety of semi-lattices is presented in [16], based on work by Straubing and Thérien [14].

Surprisingly, Theorem 3.8 can be generalized in almost exactly the same form to characterize $\mathrm{FO}^2_{m,n}[<]$ where there are at most m blocks of alternating quantifiers, $m \leq n$. This second structure theorem (Theorem 4.5) uses the notion of (m, n)-ranker where there are m blocks of $\triangleright$'s or $\triangleleft$'s, that is, changing direction in rankers corresponds exactly to alternation of quantifiers. Using Theorem 4.5 we prove that there is a strict alternation hierarchy for $\mathrm{FO}^2_n[<]$ (Theorem 4.11) but that exactly at most $|\Sigma| + 1$ alternations are useful, where $|\Sigma|$ is the size of the alphabet (Theorem 4.7).

The language $\mathrm{FO}^2[<, \mathrm{Suc}]$ is more expressive than $\mathrm{FO}^2[<]$ because it allows us to talk about consecutive strings of symbols[2]. For $\mathrm{FO}^2[<, \mathrm{Suc}]$, a straightforward generalization of n-ranker to n-successor-ranker allows us to prove exact analogs of Theorems 3.8 and 4.5. We use the latter to prove that there is also a strict alternation hierarchy for $\mathrm{FO}^2_n[<, \mathrm{Suc}]$

[1] See item 7 in Fact 1.1: a "turtle language" is a language of the form "r is defined", for some ranker, r.

[2] With three variables we can express $\mathrm{Suc}(x, y)$ using the ordering: $x < y \wedge \forall z(z \leq x \vee y \leq z)$.

(Theorem 5.6). Since in the presence of successor we can encode an arbitrary alphabet in binary, no analog of Theorem 4.7 holds for $FO^2[<, Suc]$.

The expressive power of first-order logic with three or more variables on words has been well-studied. The languages expressible are of course the star-free regular languages [10]. The dot-depth hierarchy is the natural hierarchy of these languages. This hierarchy is strict [2] and identical to the first-order quantifier alternation hierarchy [18, 19].

Many beautiful results on FO^2 on words were also already known. The main significant outstanding question was whether there was an alternation hierarchy. The following is a summary of the main previously known characterizations of $FO^2[<]$ on words. For a detailed treatment of all these characterizations, we refer the reader to [15].

Fact 1.1. [3, 4, 11, 12, 17, 13] Let $R \subseteq \Sigma^\star$. The following conditions are equivalent:
(1) $R \in FO^2[<]$
(2) R is expressible in unary temporal logic
(3) $R \in \Sigma_2 \cap \Pi_2[<]$
(4) R is an unambiguous regular language
(5) The syntactic semi-group of R is a member of **DA**
(6) R is recognizable by a partially-ordered 2-way automaton
(7) R is a boolean combination of "turtle languages"

The proofs of our structure theorems are self-contained applications of Ehrenfeucht-Fraïssé games. All of the above characterizations follow from these results. Furthermore, we have now exactly connected quantifier and alternation depth to the picture, thus adding tight bounds and further insight to the above results.

For example, one can best understand item 4 above – that $FO^2[<]$ on words corresponds to the unambiguous regular languages – via Theorem 3.12 which states that any $FO^2_n[<]$ formula with one free variable that is always true of at most one position in any string, necessarily denotes an n-ranker.

In the conclusion of [13], the authors define the subclasses of rankers with one and two blocks of alternation. They write that, "...turtle languages might turn out to be a helpful tool for further studies in algebraic language theory." We feel that the present paper fully justifies that prediction. Turtle languages — aka rankers — do provide an exceptionally clear and precise understanding of the expressive power of FO^2 on words, with and without successor.

In summary, our structure theorems provide a complete classification of the expressive power of FO^2 on words in terms of both quantifier depth and alternation. They also tighten several previous characterizations and lead to the alternation hierarchy results.

We begin the remainder of this paper with a brief review of logical background including Ehrenfeucht-Fraïssé games, our main tool. In Sect. 3 we formally define rankers and present our structure theorem for $FO^2_n[<]$. The structure theorem for $FO^2_{m,n}[<]$ is covered in Sect. 4, including our alternation hierarchy result that follows from it. Sect. 5 extends our structure theorems and the alternation hierarchy result to $FO^2[<, Suc]$. Finally, we discuss applications of our structural results to satisfiability for $FO^2[<]$ in Sect. 6.

2. Background and Definitions

We recall some notation concerning strings, first-order logic, and Ehrenfeucht-Fraïssé games. See [6] for more details, including the proof of Facts 2.1 and 2.2.

Σ will always denote a finite alphabet and ε the empty string. For a word $w \in \Sigma^\ell$ and $i \in [1,\ell]$, let w_i be the i-th letter of w; and for $[i,j]$ a subinterval of $[1,\ell]$, let $w_{[i,j]}$ be the substring $w_i \dots w_j$. Slightly abusing notation, we identify a word $w \in \Sigma^\ell$ with the logical structure $w = (\{1,\dots,\ell\}; Q^w_{\mathtt{a}}, \mathtt{a} \in \Sigma; x^w; y^w)$. Here $Q_{\mathtt{a}}, \mathtt{a} \in \Sigma$ are all unary relation symbols, and x and y are the only two variables. If not specified otherwise, we have $x^w = y^w = 1$ by default, and for all $\mathtt{a} \in \Sigma$, $Q^w_{\mathtt{a}} = \{1 \le i \le \ell \mid w_i = \mathtt{a}\}$. Furthermore, we write (w,i,j) for the word structure w with the two variables set to i and j, respectively, and (w,i) for the word structure w with $x^w = i$. Thus $w = (w,1,1)$, and $(w,i) \models Q_{\mathtt{a}}(x)$ iff $w_i = \mathtt{a}$.

We use FO[$<$] to denote first-order logic with a binary linear order predicate $<$, and FO $=$ FO[$<$, Suc] for first-order logic with an additional binary successor predicate. FO^2_n refers to the restriction of first-order logic to use at most two distinct variables, and quantifier depth n. $\mathrm{FO}^2_{m,n}$ is the further restriction to formulas such that any path in their parse tree has at most m blocks of alternating quantifiers, and $\mathrm{FO}^2\text{–ALT}[m] = \bigcup_{n \ge m} \mathrm{FO}^2_{m,n}$. We write $u \equiv^2_n v$ to mean that u and v agree on all formulas from FO^2_n, and $u \equiv^2_{m,n} v$ if they agree on $\mathrm{FO}^2_{m,n}$.

We assume that the reader is familiar with our main tool: the Ehrenfeucht-Fraïssé game. In each of the n moves of the game $\mathrm{FO}^2_n(u,v)$, Samson places one of the two pebble pairs, x or y on a position in one of the two words and Delilah then answers by placing that pebble's mate on a position of the other word. Samson wins if after any move, the map from the chosen points in u to those in v, i.e., $x^u \mapsto x^v$, $y^u \mapsto y^v$ is not an isomorphism of the induced substructures; and Delilah wins otherwise. The fundamental theorem of Ehrenfeucht-Fraïssé games is the following:

Fact 2.1. Let $u, v \in \Sigma^\star$, $n \in \mathbb{N}$. Delilah has a winning strategy for the game $\mathrm{FO}^2_n(u,v)$ iff $u \equiv^2_n v$.

Thus, Ehrenfeucht-Fraïssé games are a perfect tool for determining what is expressible in first-order logic with a given quantifier-depth and number of variables. The game $\mathrm{FO}^2_{m,n}(u,v)$ is the restriction of the game $\mathrm{FO}^2_n(u,v)$ in which Samson may change which word he plays on at most $m-1$ times.

Fact 2.2. Let $u, v \in \Sigma^\star$ and let $m, n \in \mathbb{N}$ with $m \le n$. Delilah has a winning strategy for the game $\mathrm{FO}^2_{m,n}(u,v)$ iff $u \equiv^2_{m,n} v$.

We end this section with a simple lemma that will be useful whenever we want to prove that there is a formula expressing a property of strings. With this lemma, it suffices to show that for any pair of strings, one with the property in question and one without, there is a formula that distinguishes between these two particular strings.

Lemma 2.3. *Let $P \subseteq \Sigma^\star$ and let L be a logic closed under boolean operations with only finitely many inequivalent formulas. If for every $u \in P$ and every $v \in \overline{P}$ there is a formula $\varphi_{u,v} \in L$ such that $u \models \varphi_{u,v}$ and $v \not\models \varphi_{u,v}$, then there is a formula $\varphi \in L$ such that for all $w \in \Sigma^\star$, $w \models \varphi \iff w \in P$.*

Proof. Let $\Gamma = \{\psi_{u,v} \mid u \in P, v \in \overline{P}\}$, and let Γ' be a maximal subset of Γ containing only inequivalent formulas. Since L contains only finitely many inequivalent formulas, Γ' is finite. For every $u \in P$, we define the finite sets of formulas $\Gamma'_u = \{\psi \in \Gamma' \mid u \models \psi\}$. Since all these sets are subsets of the finite set Γ', there can only be finitely many of them. Thus there is a finite set $P' \subseteq P$ such that $\{\Gamma'_u \mid u \in P\} = \{\Gamma'_u \mid u \in P'\}$. Now we set

$$\varphi = \bigvee_{u \in P'} \bigwedge_{\psi \in \Gamma'_u} \psi$$

We have $\varphi \in L$ and for every $w \in \Sigma^\star$, $w \in P \iff w \models \varphi$ as required. □

It is well-known [6] that for any $m, n \in \mathbb{N}$, the logics FO^2_n and $\mathrm{FO}^2_{m,n}$, both with and without the successor predicate, have only finitely many inequivalent formulas. Thus the above lemma applies to these logics.

3. Structure Theorem for $\mathrm{FO}^2[<]$

We define boundary positions that point to the first or last occurrences of a letter in a word, and define an n-ranker as a sequence of n boundary positions. In terms of [13], boundary positions are turtle instructions and n-rankers are turtle programs of length n. The following three lemmas show that basic properties about the definedness and position of these rankers can be expressed in $\mathrm{FO}^2[<]$, and we use these results to prove our structure theorem.

Definition 3.1. A *boundary position* denotes the first or last occurrence of a letter in a given word. Boundary positions are of the form $d_{\mathtt{a}}$ where $d \in \{\triangleright, \triangleleft\}$ and $\mathtt{a} \in \Sigma$. The interpretation of a boundary position $d_{\mathtt{a}}$ on a word $w = w_1 \dots w_{|w|} \in \Sigma^\star$ is defined as follows.

$$d_{\mathtt{a}}(w) = \begin{cases} \min\{i \in [1, |w|] \mid w_i = \mathtt{a}\} & \text{if } d = \triangleright \\ \max\{i \in [1, |w|] \mid w_i = \mathtt{a}\} & \text{if } d = \triangleleft \end{cases}$$

Here we set $\min\{\}$ and $\max\{\}$ to be undefined, thus $d_{\mathtt{a}}(w)$ is undefined if a does not occur in w. A boundary position can also be specified with respect to a position $q \in [1, |w|]$.

$$d_{\mathtt{a}}(w, q) = \begin{cases} \min\{i \in [q+1, |w|] \mid w_i = \mathtt{a}\} & \text{if } d = \triangleright \\ \max\{i \in [1, q-1] \mid w_i = \mathtt{a}\} & \text{if } d = \triangleleft \end{cases}$$

Definition 3.2. Let n be a positive integer. An *n-ranker* r is a sequence of n boundary positions. The interpretation of an n-ranker $r = (p_1, \dots, p_n)$ on a word w is defined as follows.

$$r(w) := \begin{cases} p_1(w) & \text{if } r = (p_1) \\ \text{undefined} & \text{if } (p_1, \dots, p_{n-1})(w) \text{ is undefined} \\ p_n(w, (p_1, \dots, p_{n-1})(w)) & \text{otherwise} \end{cases}$$

Instead of writing n-rankers as a formal sequence $(p_1, \dots, p_n)$, we often use the simpler notation $p_1 \dots p_n$. We denote the set of all n-rankers by R_n, and the set of all n-rankers that are defined over a word w by $R_n(w)$. Furthermore, we set $R^\star_n := \bigcup_{i \in [1,n]} R_i$ and $R^\star_n(w) := \bigcup_{i \in [1,n]} R_i(w)$.

Definition 3.3. Let r be an n-ranker. As defined above, we have $r = (p_1, \dots, p_n)$ for boundary positions p_i. The *k-prefix ranker* of r for $k \in [1, n]$ is $r_k := (p_1, \dots, p_k)$.

Definition 3.4. Let $i, j \in \mathbb{N}$. The *order type* of i and j is defined as

$$\mathrm{ord}(i,j) = \begin{cases} < & \text{if } i < j \\ = & \text{if } i = j \\ > & \text{if } i > j \end{cases}$$

Lemma 3.5 (distinguishing points on opposite sides of a ranker)**.** *Let n be a positive integer, let $u, v \in \Sigma^\star$ and let $r \in R_n(u) \cap R_n(v)$. Samson wins the game $FO^2_n(u,v)$ where initially $ord(x^u, r(u)) \neq ord(x^v, r(v))$.*

Proof. We only look at the case where $x^u \geq r(u)$ and $x^v < r(v)$ since all other cases are symmetric to this one. For $n = 1$ Samson has a winning strategy: If r is the first occurrence of a letter, then Samson places y on $r(u)$ and Delilah cannot reply. If r marks the last occurrence of a letter in the whole word, then Samson places y on $r(v)$. Again, Delilah cannot reply with any position and thus loses.

For $n > 1$, we look at the prefix ranker r_{n-1} of r. One of the following two cases applies.

(1) $r_{n-1}(u) < r(u)$, as shown in Fig. 1. Samson places pebble y on $r(u)$, and Delilah has to reply with a position that is to the left of x^v. She cannot choose a position in the interval $(r_{n-1}(v), r(v))$, because this section does not contain the letter $u_{r(u)}$. Thus she has to choose a position left of or equal to $r_{n-1}(v)$. By induction Samson wins the remaining game.

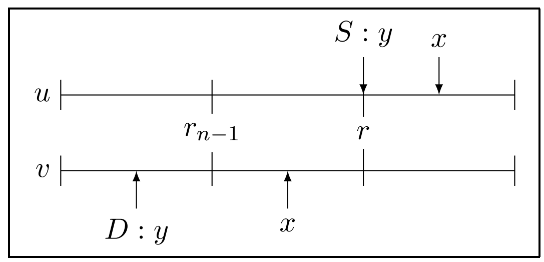

Figure 1: The case $r_{n-1}(u) < r(u)$

(2) $r(u) < r_{n-1}(u)$, as shown in Fig. 2. Samson places y on $r(v)$, and Delilah has to reply with a position to the right of x^u and thus to the right of $r(u)$. She cannot choose any position in $(r(u), r_{n-1}(u))$, because this interval does not contain the letter $v_{r(v)}$, thus Delilah has to choose a position to the right of or equal to $r_{n-1}(u)$. By induction Samson wins the remaining game. ☐

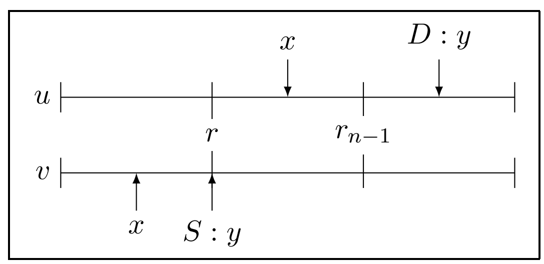

Figure 2: The case $r(u) < r_{n-1}(u)$

Lemma 3.6 (expressing the definedness of a ranker)**.** *Let n be a positive integer, and let $r \in R_n$. There is a formula $\varphi_r \in FO^2_n[<]$ such that for all $w \in \Sigma^\star$, $w \models \varphi_r \iff r \in R_n(w)$.*

Proof. Using Lemma 2.3 it suffices to consider arbitrary $u, v \in \Sigma^\star$ with $r \in R_n(u)$ and $r \notin R_n(v)$, and using Fact 2.1, it suffices to show that Samson wins the game $\mathrm{FO}^2_n(u,v)$. If r_1, the shortest prefix ranker of r, is not defined over v, the letter referred to by r_1 occurs in u but does not occur in v. Thus Samson easily wins in one move.

Otherwise we let $r_i = (p_1, \ldots, p_i)$ be the shortest prefix ranker of r that is undefined over v. Thus r_{i-1} is defined over both words. Without loss of generality we assume that $p_i = \lhd_{\mathtt{a}}$. This situation is illustrated in Fig. 3. Notice that v does not contain any $\mathtt{a}$'s to the left of $r_{i-1}(v)$, otherwise r_i would be defined over v. Samson places x in u on $r_i(u)$, and Delilah has to reply with a position right of or equal to $r_{i-1}(v)$. Now Lemma 3.5 applies and Samson wins in $i-1$ more moves. $\square$

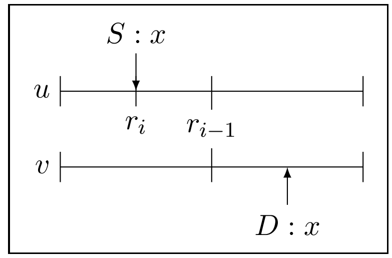

Figure 3: $r_i(v)$ is undefined

Lemma 3.7 (position of a ranker)**.** *Let n be a positive integer and let $r \in R_n$. There is a formula $\psi_r \in FO^2_n[<]$ such that for all $w \in \Sigma^\star$ and for all $i \in [1, |w|]$, $(w, i) \models \psi_r \iff i = r(w)$.*

Proof. As in the proof of Lemma 3.6, it suffices to show that for arbitrary $u, v \in \Sigma^\star$, Samson wins the game $\mathrm{FO}^2_n(u, v)$ where initially $x^u = r(u)$ and $x^v \neq r(v)$. If $r(v)$ is defined over v, then we can apply Lemma 3.5 immediately to get the desired strategy for Samson. Otherwise we use the strategy from Lemma 3.6. $\square$

Theorem 3.8 (structure of $\mathrm{FO}^2_n[<]$)**.** *Let u and v be finite words, and let $n \in \mathbb{N}$. The following two conditions are equivalent.*

(i) (a) *$R_n(u) = R_n(v)$, and,*
(b) *for all $r \in R^\star_n(u)$ and $r' \in R^\star_{n-1}(u)$, $ord(r(u), r'(u)) = ord(r(v), r'(v))$*

(ii) *$u \equiv^2_n v$*

Notice that condition (i)(a) is equivalent to $R^\star_n(u) = R^\star_n(v)$. Instead of proving Theorem 3.8 directly, we prove the following more general version on words with two interpreted variables.

Theorem 3.9. *Let u and v be finite words, let $i_1, i_2 \in [1, |u|]$, let $j_1, j_2 \in [1, |v|]$, and let $n \in \mathbb{N}$. The following two conditions are equivalent.*

(i) (a) *$R_n(u) = R_n(v)$, and,*
(b) *for all $r \in R^\star_n(u)$ and $r' \in R^\star_{n-1}(u)$, $ord(r(u), r'(u)) = ord(r(v), r'(v))$, and,*
(c) *$(u, i_1, i_2) \equiv^2_0 (v, j_1, j_2)$, and,*
(d) *for all $r \in R^\star_n(u)$, $ord(i_1, r(u)) = ord(j_1, r(v))$ and $ord(i_2, r(u)) = ord(j_2, r(v))$*

(ii) *$(u, i_1, i_2) \equiv^2_n (v, j_1, j_2)$*

Proof. For $n = 0$, (i)(a), (i)(b) and (i)(d) are vacuous, and (i)(c) is equivalent to (ii). For $n \geq 1$, we prove the two implications individually using induction on n.

We first show "$\neg$(i) $\Rightarrow$ $\neg$(ii)". Assuming that (i) holds for $n \in \mathbb{N}$ but fails for $n+1$, we show that $(u, i_1, i_2) \not\equiv^2_{n+1} (v, j_1, j_2)$ by giving a winning strategy for Samson in the FO^2_{n+1} game on the two structures. If (i)(c) does not hold, then Samson wins immediately. If (i)(d) does not hold for $n+1$, then Samson wins by Lemma 3.5. If (i)(a) or (i)(b) do not hold for $n+1$, then one of the following three cases applies.

(1) There is an $(n+1)$-ranker that is defined over one word but not over the other.
(2) There are two n-rankers that do not agree on their ordering in u and v.
(3) There is an $(n+1)$-ranker that does not appear in the same order on both structures with respect to a k-ranker where $k \leq n$.

We first look at case (2) where there are two rankers $r, r' \in R^\star_n(u)$ that disagree on their ordering in u and v. Without loss of generality we assume that $r(u) \leq r'(u)$ and

$r(v) > r'(v)$, and present a winning strategy for Samson in the FO^2_{n+1} game. In the first move he places x on $r(u)$ in u. Delilah has to reply with $r(v)$ in v, otherwise she would lose the remaining n-move game as shown in Lemma 3.5. Let r'_{n-1} be the $(n-1)$-prefix-ranker of r'. We look at two different cases depending on the ordering of r'_{n-1} and r'.

For $r'_{n-1}(u) < r'(u)$, the situation is illustrated in Fig. 4. In his second move, Samson places y on $r'(v)$. Delilah has to reply with a position to the left of x^u, but she cannot choose any position from the interval $(r'_{n-1}(u), r'(u))$ because it does not contain the letter v_{y^v}. So she has to reply with a position left of or equal to $r'_{n-1}(u)$, and Samson wins the remaining FO^2_{n-1} game as shown in Lemma 3.5.

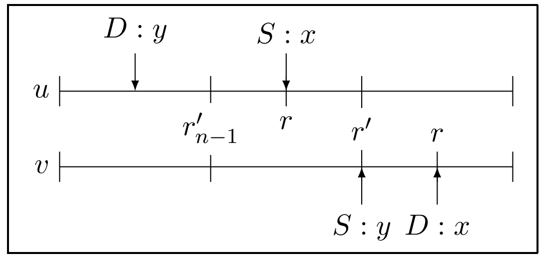

Figure 4: Two n-rankers appear in different order and r' ends with $\triangleright$.

For $r'_{n-1}(u) > r'(u)$, the situation is illustrated in Fig. 5. In his second move, Samson places pebble y on $r'(u)$, and Delilah has to reply with a position to the right of x^v, but she cannot choose anything from the interval $(r'(v), r'_{n-1}(v))$ because this section does not contain the letter u_{y^u}. Thus she has to reply with a position right of or equal to $r'_{n-1}(v)$, and Samson wins the remaining FO^2_{n-1} game as shown in Lemma 3.5.

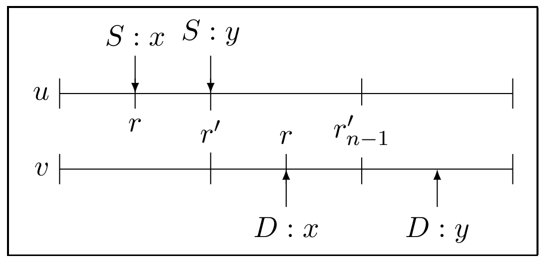

Figure 5: Two n-rankers appear in different order and r' ends with $\triangleleft$.

Now we look at cases (1) and (3), assuming that case (2) does not apply. We know that condition (i) from the statement of the theorem fails, but still all n-rankers agree on their ordering. In both case (1) and case (3), there are two consecutive n-rankers $r, r' \in R_n(u)$ with $r(u) < r'(u)$ and a letter $\mathtt{a} \in \Sigma$ such that without loss of generality $\mathtt{a}$ occurs in the segment $u_{((r(u),r'(u))}$ but not in the segment $v_{(r(v),r'(v))}$. We describe a winning strategy for Samson in the game $\mathrm{FO}^2_{n+1}(u,v)$. He places x on an $\mathtt{a}$ in the segment $(r(u), r'(u))$ of u, as shown in Fig. 6. Delilah cannot reply with anything in the interval $(r(v), r'(v))$. If she replies with a position left of or equal to $r(v)$, then x is on different sides of the n-ranker r in the two words. Thus Lemma 3.5 applies and Samson wins the remaining n-move game. If Delilah replies with a position right of or equal to $r'(v)$, then we can apply Lemma 3.5 to r' and get a winning strategy for the remaining game as well. This concludes the proof of "$\neg$(i) $\Rightarrow$ $\neg$(ii)".

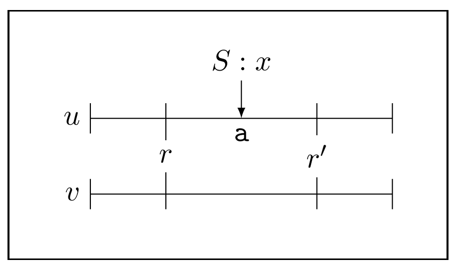

Figure 6: A letter $\mathtt{a}$ occurs between n-rankers r, r' in u but not in v

To show "(i) $\Rightarrow$ (ii)", we assume (i) for $n+1$, and present a winning strategy for Delilah in the FO^2_{n+1} game on the two structures. In his first move Samson picks up one of the two pebbles, and places it on a new position. Without loss of generality we assume that Samson picks up x and places it on u in his first move. If $x^u = r(u)$ for any ranker $r \in R^{\star}_{n+1}(u)$, then Delilah replies with $x^v = r(v)$. This establishes (i)(c) and (i)(d) for n, and thus Delilah has a winning strategy for the remaining FO^2_n game by induction.

If Samson does not place x^u on any ranker from $R^\star_{n+1}(u)$, then we look at the closest rankers from $R^\star_n(u)$ to the left and right of x^u, denoted by λ and ρ, respectively. Let $\mathtt{a} := u_{x^u}$ and define the $(n+1)$-ranker $s = (\lambda, \triangleright_\mathtt{a})$. On u we have $\lambda(u) < s(u) < \rho(u)$. Because of (i)(a) s is defined on v as well, and because of (i)(b), we have $\lambda(v) < s(v) < \rho(v)$. If y^u is not contained in the interval $(\lambda(u), \rho(u))$, then Delilah places x on $s(v)$, which establishes (i)(c) and (i)(d) for n. Thus by induction Delilah has a winning strategy for the remaining FO^2_n game.

If both pebbles x^u and y^u occur in the interval $(\lambda(u), \rho(u))$, then we need to be more careful. Without loss of generality we assume $y^u < x^u$ as illustrated in Fig. 7. Thus Delilah has to place x in the interval $(y^v, \rho(v))$ and at a position with letter $\mathtt{a} := u_{x^u}$. We define the $n+1$-ranker $s = (\rho, \triangleleft_\mathtt{a})$. From (i)(d) we know that s appears on the same side of y in both structures, thus we have $y^v < s(v) < \rho(v)$. Delilah places her pebble x on $s(v)$, and thus establishes (i)(c) and (i)(d) for n. By induction, Delilah has a winning strategy for the remaining FO^2_n game. □

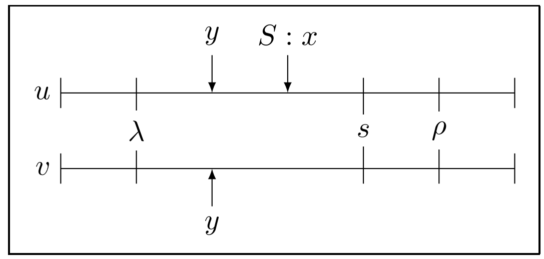

Figure 7: x and y are in the same section

A fundamental property of an n-ranker is that it uniquely describes a position in a given word. Now we show that the converse holds as well: Any position in a word that can be uniquely described with an $\mathrm{FO}^2[<]$ formula can also be described by a ranker (Lemma 3.11). Furthermore, any $\mathrm{FO}^2[<]$ formula that describes a unique position in any given word is equivalent to a boolean combination of rankers (Theorem 3.12).

Definition 3.10 (unique position formula)**.** A formula $\varphi \in \mathrm{FO}^2[<]$ with x as a free variable is a *unique position formula* if for all $w \in \Sigma^\star$ there is at most one $i \in [1, |w|]$ such that $(w, i) \models \varphi$.

Lemma 3.11. *Let n be a positive integer and let $\varphi \in FO^2_n[<]$ be a unique position formula. Let $u \in \Sigma^\star$ and let $i \in [1, |u|]$ such that $(u, i) \models \varphi$. Then $i = r(u)$ for some ranker $r \in R^\star_n$.*

Proof. Suppose for the sake of a contradiction that there is no ranker $r \in R^\star_n$ such that $(u, i) \models \varphi_r$. Because the first and last positions in u are described by 1-rankers, we know that $i \notin \{1, |u|\}$. We construct a new word v by doubling the symbol at position i in u, $v = u_1 \ldots u_{i-1} u_i u_i u_{i+1} \ldots u_{|u|}$. By assumption, there is no n-ranker that describes position i in u. A brief argument by contradiction shows that there are also no n-rankers that describe positions i or $i+1$ in v: Assuming that such a ranker exists, let r be the shortest such ranker. Thus none of the prefix rankers of r point to either positions i or $i+1$ in v. This means that all prefix rankers of r are interpreted in exactly the same way on both u and v, and irrespective of whether $r(v)$ points to i or $i+1$, we have have $r(u) = i$, a contradiction. Hence all n-rankers are insensitive to the doubling of u_i, and the two words u and v agree on the definedness of all n-rankers and on their ordering. By Theorem 3.9, we thus have $(u, i) \equiv^2_n (v, i) \equiv^2_n (v, i+1)$, which contradicts the fact that φ is a unique position formula. □

Theorem 3.12. *Let n be a positive integer and let $\varphi \in FO^2_n[<]$ be a unique position formula. There is a $k \in \mathbb{N}$, and there are mutually exclusive formulas $\alpha_i \in FO^2_n[<]$ and* *rankers $r_i \in R^\star_n$ such that*

$$\varphi \equiv \bigvee_{i\in[1,k]} (\alpha_i \wedge \varphi_{r_i})$$

where $\varphi_{r_i} \in FO^2_n[<]$ is the formula from Lemma 3.7 that uniquely describes the ranker r_i.

Proof. Let $\mathcal{T}$ be the set of all $\mathrm{FO}^2_n[<]$ types of words over Σ with one interpreted variable. Because there are only finitely many inequivalent formulas in $\mathrm{FO}^2_n[<]$, $\mathcal{T}$ is finite. Let $\mathcal{T}' \subseteq \mathcal{T}$ be the set of all types that satisfy φ. We set $\mathcal{T}' = \{T_1, \dots, T_k\}$ and let $\alpha_i \in \mathrm{FO}^2_n[<]$ be a description of type T_i. Thus $\varphi \equiv \bigvee_{i\in[1,k]} \alpha_i$.

Now suppose that $(u,j) \models \varphi$. Thus $(u,j) \models \alpha_i$ for some i. By Lemma 3.11 $(u,j) \models \varphi_{r_i}$ for some $r_i \in R^\star_n$. Thus $\alpha_i \to \varphi_{r_i}$ since $\varphi_{r_i} \in \mathrm{FO}^2_n$ and α_i is a complete FO^2_n formula. Thus $\alpha_i \equiv \alpha_i \wedge \varphi_{r_i}$ so φ is in the desired form. ☐

4. Alternation hierarchy for $\mathrm{FO}^2[<]$

We define alternation rankers and prove our structure theorem (Theorem 4.5) for $\mathrm{FO}^2_{m,n}[<]$. Surprisingly the number of alternating blocks of $\triangleleft$ and $\triangleright$ in the rankers corresponds exactly to the number of alternating quantifier blocks. The main ideas from our proof of Theorem 3.8 still apply here, but keeping track of the number of alternations does add complications.

Definition 4.1 (m-alternation n-ranker)**.** Let $m, n \in \mathbb{N}$ with $m \le n$. An *m-alternation n-ranker*, or (m,n)-ranker, is an n-ranker with exactly m blocks of boundary positions that alternate between $\triangleright$ and $\triangleleft$.

We use the following notation for alternation rankers.

$$\begin{aligned}
R_{m,n}(w) &:= \{r \mid r \text{ is an } m\text{-alternation } n\text{-ranker and defined over the word } w\} \\
R_{m\triangleright,n}(w) &:= \{r \in R_{m,n}(w) \mid r \text{ ends with } \triangleright\} \\
R^\star_{m,n}(w) &:= \bigcup_{i\in[1,m], j\in[1,n]} R_{i,j}(w) \\
R^\star_{m\triangleright,n}(w) &:= R^\star_{m-1,n}(w) \cup \bigcup_{i\in[1,n]} R_{m\triangleright,i}(w)
\end{aligned}$$

Lemma 4.2. *Let m and n be positive integers with $m \le n$, let $u, v \in \Sigma^\star$, and let $r \in R_{m,n}(u) \cap R_{m,n}(v)$. Samson wins the game $FO^2_{m,n}(u,v)$ where initially $ord(r(u), x^u) \neq ord(r(v), x^v)$.*

Furthermore, Samson can start the game with a move on u if r ends with $\triangleright$, $r(u) \le x^u$ and $r(v) \ge x^v$, or if r ends with $\triangleleft$, $r(u) \ge x^u$ and $r(v) \le x^v$. He can start the game with a move on v if r ends with $\triangleright$, $r(u) \ge x^u$ and $r(v) \le x^v$, or if r ends with $\triangleleft$, $r(u) \le x^u$ and $r(v) \ge x^v$.

Proof. If $m = n = 1$, then we can immediately apply the base case from the proof of Lemma 3.5. Samson wins in one move, placing his pebble on u or v as specified.

For the remaining cases, we assume without loss of generality that r ends with $\triangleright$ and that $x^u \ge r(u)$ and $x^v \le r(v)$. Let r_{n-1} be the $(n-1)$-prefix ranker of r. This situation is illustrated in Fig. 1 of Lemma 3.5. Samson places y on $r(u)$, and creates a situation where $y^u > r_{n-1}(u)$ and $y^v \le r_{n-1}(v)$. If r_{n-1} ends with $\triangleleft$, then by induction Samson wins the remaining $\mathrm{FO}^2_{m-1,n-1}$ game and thus he has a winning strategy for the $\mathrm{FO}^2_{m,n}$ game. If r_{n-1} ends with $\triangleright$, then by induction Samson wins the remaining $\mathrm{FO}^2_{m,n-1}$ game starting with a move on u, and thus he has a winning strategy for the $\mathrm{FO}^2_{m,n}$ game. □

Lemma 4.3. *Let m and n be positive integers with $m \leq n$ and let $r \in R_{m,n}$. There is a $\varphi_r \in FO^2_{m,n}[<]$ such that for all $w \in \Sigma^\star$, $w \models \varphi_r \iff r \in R_{m,n}(w)$.*

Proof. Using Lemma 2.3 it suffices to consider arbitrary $u, v \in \Sigma^\star$ with $r \in R_{m,n}(u)$ and $r \notin R_{m,n}(v)$, and using Fact 2.1, it suffices to show that Samson wins the game $\mathrm{FO}^2_{m,n}(u,v)$. Let $r_i = (p_1, \ldots, p_i)$ be the shortest prefix ranker of r that is undefined over v, and we assume without loss of generality that this ranker ends with the boundary position $p_i = \triangleleft_{\mathtt{a}}$ for some $\mathtt{a} \in \Sigma$. This situation is illustrated in Fig. 3 for Lemma 3.7. In his first move Samson places x on $r_i(u)$ and thus forces a situation where $x^u < r_{i-1}(u)$ and $x^v \geq r_{i-1}(v)$. If r_{i-1} ends with $\triangleleft$, then according to Lemma 4.2, Samson wins the remaining $\mathrm{FO}^2_{m,n-1}$ game starting with a move on u. Otherwise r_{i-1} ends with $\triangleright$, and thus by Lemma 4.2 Samson wins the remaining $\mathrm{FO}^2_{m-1,n-1}$ game starting with a move on v. □

Lemma 4.4. *Let m and n be positive integers with $m \leq n$ and let $r \in R_{m,n}$. There is a formula $\psi_r \in FO^2_{m,n}[<]$ such that for all $w \in \Sigma^\star$ and for all $i \in [1, |w|]$, $(w, i) \models \psi_r \iff i = r(w)$.*

Proof. As in the proof of Lemma 4.3, it suffices to show that Samson wins the game $\mathrm{FO}^2_{m,n}(u,v)$ where initially $x^u = r(u)$ and $x^v \neq r(v)$. Depending on whether r is defined over v, we use the strategies from Lemma 4.2 or Lemma 4.3. □

Theorem 4.5 (structure of $\mathrm{FO}^2_{m,n}[<]$)**.** *Let u and v be finite words, and let $m, n \in \mathbb{N}$ with $m \leq n$. The following two conditions are equivalent.*

(i) (a) *$R_{m,n}(u) = R_{m,n}(v)$, and,*
 (b) *for all $r \in R^\star_{m,n}(u)$ and for all $r' \in R^\star_{m-1,n-1}(u)$, we have $ord(r(u), r'(u)) = ord(r(v), r'(v))$, and,*
 (c) *for all $r \in R^\star_{m,n}(u)$ and $r' \in R^\star_{m,n-1}(u)$ such that r and r' end with different directions, $ord(r(u), r'(u)) = ord(r(v), r'(v))$*

(ii) $u \equiv^2_{m,n} v$

Just as before with Theorem 3.8, instead of proving Theorem 4.5 directly, we prove a more general version that applies to words with two interpreted variables. The statement of the general version is asymmetric with respect to the roles of the two structures u and v. This is necessary because of the correspondence between quantifier alternations (i.e. alternations between u and v in the game) and alternations of directions in the rankers. This asymmetry already affected the statement of Lemma 4.2, where Samson's winning strategy starts with a move on the specified structure. In fact, as the proof of the following theorem shows, he does not have a winning strategy that starts with a move on the other structure. We remark that conditions (i)(a) through (i)(e) of the general theorem are completely symmetric with respect to the roles of u and v, and only conditions (i)(f) and (ii) are asymmetric. Theorem 4.5 follows directly from the general theorem, since here $i_1 = i_2 = j_1 = j_2 = 1$, thus conditions (i)(e) and (i)(f) or trivially true, and the equivalence holds with the roles of u and v reversed as well.

Theorem 4.6. *Let u and v be finite words, let $i_1, i_2 \in [1, |u|]$, let $j_1, j_2 \in [1, |v|]$, and let $m, n \in \mathbb{N}$ with $m \leq n$. The following two conditions are equivalent.*

(i) (a) *$R_{m,n}(u) = R_{m,n}(v)$, and,*

(b) *for all $r \in R^\star_{m,n}(u)$ and for all $r' \in R^\star_{m-1,n-1}(u)$, we have $ord(r(u), r'(u)) = ord(r(v), r'(v))$, and,*

(c) *for all $r \in R^\star_{m,n}(u)$ and $r' \in R^\star_{m,n-1}(u)$ such that r and r' end with different directions, $ord(r(u), r'(u)) = ord(r(v), r'(v))$*

(d) *$(u, i_1, i_2) \equiv^2_0 (v, j_1, j_2)$, and,*

(e) *for all $r \in R^\star_{m-1,n}(u)$, $ord(r(u), i_1) = ord(r(v), j_1)$ and $ord(r(u), i_2) = ord(r(v), j_2)$, and,*

(f) *for all $r \in R^\star_{m,n}(u)$, and $(i, j) \in \{(i_1, j_1), (i_2, j_2)\}$,*

(f_1) *if r ends on $\triangleright$ and $r(u) = i$, then $r(v) \leq j$*

(f_2) *if r ends on $\triangleright$ and $r(u) < i$, then $r(v) < j$*

(f_3) *if r ends on $\triangleleft$ and $r(u) = i$, then $r(v) \geq j$*

(f_4) *if r ends on $\triangleleft$ and $r(u) > i$, then $r(v) > j$*

(ii) *Delilah wins the game $FO^2_{m,n}[<]((u, i_1, i_2), (v, j_1, j_2))$ if Samson starts with a move on (u, i_1, i_2).*

Proof. As in the proof of Theorem 3.8, we use induction on n. For $n = 0$, condition (i)(d) just by itself is equivalent to (ii), and all other conditions of (i) are vacuous. For $n \geq 1$, we we first show "¬ (i) ⇒ ¬ (ii)".

Suppose that (i) holds for (m, n), but fails for $(m, n + 1)$. If (i)(d) does not hold then Samson wins immediately. If (i)(e) does not hold for $(m, n+1)$, then by Lemma 4.2, Samson wins the $(m, n + 1)$-game on (u, v), starting with a move on either u or v. If Samson can start with a move on u, we have established that (ii) is false. Otherwise, we reverse the roles of u and v, and observe that condition (i)(e) still remains the same. Thus, even if Samson needs to start with a move on v, he still has a winning strategy, and (ii) does not hold for $(m, n + 1)$. If (i)(f) does not hold for $(m, n + 1)$, then again by using Lemma 4.2, Samson wins the $(m, n + 1)$-game on (u, v) starting with a move on u.

If one of (i)(a), (i)(b) or (i)(c) fail, then we show that Samson has a winning strategy for the game $\mathrm{FO}^2_{m,n+1}(u, v)$. We observe that it does not matter what structure Samson chooses for his first move, since all of (i)(a), (i)(b) and (i)(c) are completely symmetric with respect to the roles of u and v. Thus if Samson's winning strategy starts with a move on v, we can reverse the roles of u and v and get a winning strategy starting with move on u. One of the following cases applies.

(1) There is a ranker $r \in R_{m,n+1}$ that is defined over one structure but not over the other.

This first case applies if (a) fails for $(m, n + 1)$. If condition (2) fails for $(m, n + 1)$, then there are two n-rankers for which it fails, or an $(n + 1)$-ranker and an n-ranker. This leads to the following two cases.

(2) There are two rankers $r \in R_{m,n}(u)$ and $r' \in R_{m-1,n}(u)$ that disagree on their order, i.e. $\mathrm{ord}(r(u), r'(u)) \neq \mathrm{ord}(r(v), r'(v))$.

(3) There are two rankers $r \in R_{m,n+1}(u)$ and $r' \in R_{m-1,n}(u)$ that disagree on their order.

In a similar fashion, we obtain the remaining two cases if condition (3) fails for $(m, n + 1)$.

(4) There are rankers $r, r' \in R_{m,n}(u)$ that end on different directions and disagree on their order.

(5) There are rankers $r \in R_{m,n+1}(u)$ and $r' \in R_{m,n}(u)$ that end on different directions and disagree on their order.

We look at the cases (2) and (4) first, then deal with case (1) assuming that cases (2) and (4) do not apply, and finally look at cases (3) and (5).

For case (2), we assume that $r(u) \leq r'(u)$, as illustrated in Fig. 8. The situation for $r(u) \geq r'(u)$ is completely symmetric. Depending on the last boundary position of r, one of the following two subcases applies.

- r ends with $\triangleright$. Samson places x on $r(u)$ in his first move. If Delilah replies with a position to the left of $r'(v)$ or equal to $r'(v)$, then $x^v < r(v)$. Thus we can apply Lemma 4.2 to get a winning strategy for Samson in the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on u. If Delilah replies with a position to the right of $r'(v)$, Samson has a winning strategy for the remaining $\mathrm{FO}^2_{m-1,n}$ game. Thus we have a winning strategy for Samson in the $\mathrm{FO}^2_{m,n+1}$ game.

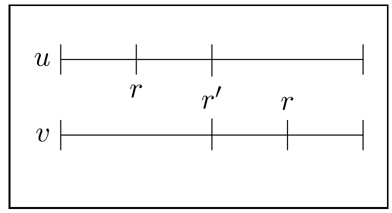

Figure 8: r and r' appear in different order

- r ends with $\triangleleft$. This is similar to the previous case, but now Samson places x on $r(v)$ in his first move. If Delilah replies with a position to the right of $r'(u)$, or equal to $r'(u)$, then as above we get a winning strategy for Samson in the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on v. Otherwise we get a winning strategy for Samson with only $m-1$ alternations for the remaining game. Thus again he has a winning strategy for the $\mathrm{FO}^2_{m,n+1}$ game.

For case (4), Samson's winning strategy is very similar to the previous case. If $r(u) \leq r'(u)$ and r ends with $\triangleright$, then Samson places x on $r(u)$ in his first move. If Delilah replies with a position to the right of $r(u)$, then Samson's winning strategy is as above. Otherwise x is on different sides of r' and Samson has a winning strategy for the remaining $\mathrm{FO}^2_{m,n}$ game that starts with a move on u. All together, he has a winning strategy for the $\mathrm{FO}^2_{m,n+1}$ game. The remaining three cases (ordering of $r(u)$ and $r'(u)$ and ending direction of r) work in the same way.

Similar to what we did in the proof of Theorem 3.8, we can reduce the remaining cases to an easier situation where a certain segment contains a certain letter in one structure, but not in the other structure, and then apply Lemma 4.2 to obtain a winning strategy for Samson.

To deal with case (1), we assume that the previous two cases, (2) and (4), do not apply. Without loss of generality, say that the $(m, n+1)$-ranker r is defined over u but not over v. Let $\mathtt{a} := u_{r(u)}$ be the letter in u at position $r(u)$. We define the following sets of rankers.

$$R_\ell := \{s \in R^\star_{m\triangleright,n}(u) \mid s(u) < r(u)\}$$
$$R_r := \{s \in R^\star_{m\triangleleft,n}(u) \mid s(u) > r(u)\}$$

Notice that all rankers from R_ℓ appear to the left of all rankers from R_r in u. From the inductive hypothesis, and from the fact that both cases (2) and (4) do not apply, it follows that over v, all rankers from R_ℓ appear to the left of all rankers from R_r as well. However, the rankers from R_ℓ and R_r by themselves do not necessarily appear in the same order in both structures. We look at the ordering of these rankers in v, and let λ be the rightmost ranker from R_ℓ and ρ be the leftmost ranker from R_r. By construction, we have

$\lambda(u) < r(u) < \rho(u)$, so the segment (λ, ρ) in u contains the letter $\mathtt{a}$. Let r_n be the n-prefix-ranker of r, and observe that r_n is defined on both structures and that r_n is contained in either R_ℓ or R_r. Because r is not defined on v, the letter $\mathtt{a}$ does not occur in v either to the right of r_n if $r_n \in R_\ell$, or to the left of r_n if $r_n \in R_r$. Thus the segment (λ, ρ) does not contain the letter $\mathtt{a}$ in v.

Now we know that $\mathtt{a}$ occurs in the segment (λ, ρ) in u but not in v, and thus we have established the situation illustrated in Fig. 9. Samson places his first pebble on an $\mathtt{a}$ within this section of u, and Delilah has to reply with a position outside of this section. No matter what side of the segment she chooses, with Lemma 4.2 Samson has a winning strategy for the remaining game and thus wins the $\mathrm{FO}^2_{m,n+1}$ game.

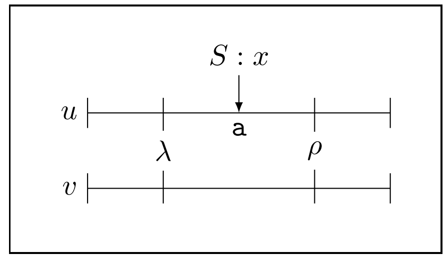

Figure 9: A letter occurs between rankers r, r' in u but not in v

In cases (3) and (5), we again assume that cases (2) and (4) do not apply, and we look at the same sets of rankers, R_ℓ and R_r, and at r_n, the n-prefix-ranker of r. We assume that $r(u) \leq r'(u)$ and that r ends with $\triangleright$, all three other cases are completely symmetric. Notice that r_n is an $(m-1,n)$-ranker, or an (m,n)-ranker that ends with $\triangleright$. Thus both structures agree on the ordering of r_n and r'. The relative positions of all these rankers are illustrated in Fig. 10. As above, let λ be the rightmost ranker from R_ℓ and let ρ be the leftmost ranker from R_r, with respect to the ordering of these rankers on v. Again we know that $\lambda(u) < r(u) < \rho(u)$ and therefore the segment (λ, ρ) of u contains an $\mathtt{a}$. Notice that $r_n \in R_\ell$ and $r' \in R_r$, thus $r_n(v) \leq \lambda(v) < \rho(v) \leq r'(v)$. Thus the segment (λ, ρ) does not contain the letter $\mathtt{a}$ in v, providing Samson with a winning strategy as argued above.

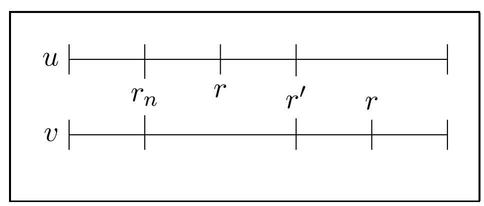

Figure 10: Ranker positions, case (4)

To prove "(i) $\Rightarrow$ (ii)", we assume that the theorem holds for n, and that (i) holds for $(m, n+1)$, and we present a winning strategy for Delilah in the game $\mathrm{FO}^2_{m,n+1}(u,v)$ where Samson starts with a move on u.

If Samson places x on a ranker $r \in R^\star_{m-1,n}(u)$, then Delilah replies by placing x on the same ranker on v. Since (i)(b) holds for $(m, n+1)$, this establishes (i)(e) and (i)(f) for (m, n). It also establishes (i)(e) and (i)(f) for $(m-1, n)$ with reversed roles of u and v. Thus we can apply the inductive hypothesis to get a winning strategy for Delilah in the remaining game.

If $x^u = y^u$ after Samson's first move, then Delilah replies with $x^v = y^v$. We use the inductive hypothesis to argue that Delilah wins the remaining n-move game, no matter what structure Samson chooses for his next move. If he chooses to play on u, then the remaining game is an (m,n)-game. Since in the first move Delilah set $x^v = y^v$, we have (i)(e) and (i)(f) for (m, n), and thus the inductive hypothesis applies and Delilah wins the remaining game. On the other hand, if Samson chooses to play on v for the next move, the remaining game is an $(m-1, n)$-game, since he started with a move on u. Because Delilah set $x^v = y^v$ in the first move, (i)(e) for $(m, n+1)$ implies both (i)(e) and (i)(f) for $(m-1, n)$ with reversed roles of u and v. Thus we can again use the inductive hypothesis to get a winning strategy for Delilah in the remaining game.

Otherwise we assume that $x^u < y^u$ after Samson's first move, the case for $x^u > y^u$ is completely symmetric. We look at the following two sets of rankers.

$$R_\ell := \{r \in R^\star_{m\triangleright,n}(u) \mid r(u) < x^u\}$$
$$R_r := \{r \in R^\star_{m\triangleleft,n}(u) \mid r(u) > x^u\}$$

On u, all rankers from R_ℓ occur to the left of all rankers from R_r. Since (i)(c) holds for $(m, n+1)$, this is also true for the positions of these rankers on v. Let $\mathtt{a}$ be the letter Samson places his pebble on. To establish both (i)(e) and (i)(f) for (m, n), Delilah needs to find an $\mathtt{a}$ in v that is to the right of all rankers from R_ℓ and to the left of all rankers from R_r. We define

$$R^0_\ell = \{r \in R^\star_{m\triangleright,n}(u) - R^\star_{m-1,n}(u) \mid r(u) = x^u\}$$
$$R^0_r = \{r \in R^\star_{m\triangleleft,n}(u) - R^\star_{m-1,n}(u) \mid r(u) = x^u\}$$
$$R'_\ell := \{r\triangleright_{\mathtt{a}} \mid r \in R_\ell\} \cup R^0_\ell$$

and have Delilah place her pebble x^v on the rightmost ranker from R'_ℓ on v. This position of course is labeled with an $\mathtt{a}$. Since on u all rankers from R'_ℓ occur to the left of or at x^u, all of them occur strictly to the left of y^u. Since all rankers in R'_ℓ are from $R^\star_{m-1,n+1}(u)$ or $R^\star_{m\triangleright,n+1}(u)$, we can apply (i)(e) and (i)(f_2), and we see that all of these rankers also appear to the left of y^v. Therefore we have $x^v < y^v$, which makes sure that Delilah does not lose in this move, and also establishes (i)(d).

To complete the inductive step, we need to argue that Delilah's move also establishes (i)(e) and (i)(f), both for (m, n), and for $(m-1, n)$ with reversed roles of u and v. Then, using the inductive hypothesis, Delilah has a winning strategy for the remaining game, no matter what side Samson chooses for his next move.

We observe that all rankers from R'_ℓ appear to the right of the rankers from R_r. This is true by definition on u, and holds for v because (i)(b) and (i)(c) hold for $(m, n+1)$. Since Delilah placed x^v on a ranker from R'_ℓ, we have (i)(e), (i)(f_2) and (i)(f_4) for (m, n) for all all rankers from R_r. And since Delilah placed x^v on the rightmost of the rankers from R'_ℓ, we know that all rankers from R_ℓ appear to the left of x^v, just as they do on u. Thus we have (i)(e), (i)(f_2) and (i)(f_4) for the rankers from R_ℓ as well, and therefore for all rankers mentioned in those conditions.

All rankers from $R^\star_{m\triangleright,n}$ that appear at x^u are in R^0_ℓ, since we already dealt with the case where x^u does appear at a ranker from $R^\star_{m-1,n}$. Since Delilah chose x^v as the rightmost ranker from R'_ℓ, all of these rankers appear to the left of or at x^v, and we have established (i)(f_1) for (m, n). For condition (i)(f_3), we need to argue about R^0_r. From (i)(b) and (i)(c) for $(m, n+1)$, we know that all rankers from R^0_r appear to the right of or at the same position as the rankers from R'_ℓ on v, just as they do on u. Thus (i)(f_3) holds as well.

Now that we have established (i) for (m, n), we use the inductive hypothesis to get a winning strategy for Delilah for the remaining game if Samson's next move is on u. For the case where his next move is on v, we only need to establish (i) for $(m-1, n)$, but with reversed roles of u and v. Reversing the roles of the two structures only affects condition (i)(f), and (i)(f) for $(m-1, n)$ follows immediately from (i)(e) for (m, n). Thus Delilah also wins the remaining game if Samson's next move is on v. □

Using Theorem 4.5, we show that for any fixed alphabet Σ, at most $|\Sigma| + 1$ alternations are useful. Intuitively, each boundary position in a ranker says that a certain letter does not occur in some part of a word. Alternations are only useful if they visit one of these previous parts again. Once we visited one part of a word $|\Sigma|$ times, this part cannot contain any more letters and thus is empty.

Theorem 4.7. *Let Σ be a finite alphabet, let $u, v \in \Sigma^\star$ and $n \in \mathbb{N}$. If $u \equiv^2_{|\Sigma|+1,n} v$, then $u \equiv^2_n v$.*

Proof. Suppose for the sake of a contradiction that $u \equiv^2_{|\Sigma|+1,n} v$ and $u \not\equiv^2_n v$. Thus, using Theorem 4.5, u and v agree on the definedness of all $(|\Sigma|+1, n)$-rankers, and on their order with respect to all $(|\Sigma|, n-1)$-rankers and some $(|\Sigma|+1, n-1)$-rankers. But since $u \not\equiv^2_n v$, u and v need to disagree on the properties of some other ranker. Let $r = (p_1, \ldots, p_t)$ with $t \in \mathbb{N}$ be the shortest such ranker. We know that r has more than $|\Sigma|$ blocks of alternating directions, say r is an m-alternation ranker for some $m > |\Sigma|$. Let $1 \leq k_1, \ldots, k_m \leq t$ be the indices of the boundary positions at the end of each block, i.e. where p_{k_i}, $1 \leq i < m$ points to a different direction than p_{k_i+1}. For the last of those indices we have $k_m = t$.

We look at the prefix rankers of r up to the end of each alternating block, $r_{k_i} := (p_1, \ldots, p_{k_i})$, and the intervals defined by these prefix rankers. We set $I_0(u) := [1, |u|]$, $r_0(u) = 0$ if p_1 points to the right, and $r_0(u) = |u| + 1$ if p_1 points to the left. For all $i \in [1, m]$ let,

$$I_i(u) := \begin{cases} [r_{k_i-1}(u) + 1, r_{k_i}(u) - 1] & \text{if } p_{k_i} \text{ points to the right} \\ [r_{k_i}(u) + 1, r_{k_i-1}(u) - 1] & \text{if } p_{k_i} \text{ points to the left} \end{cases}$$

Notice that by definition the letter mentioned in p_{k_i} does not occur in the interval I_i.

Suppose that for all $i \in [1, m]$ we have $r_{k_i}(u) \in I_{i-1}(u)$. Then the letter mentioned in p_{k_i} has to occur in the interval $I_{i-1}(u)$ of u, but the interval $I_{|\Sigma|}(u)$ of u cannot contain any of the $|\Sigma|$ distinct letters. Therefore $r_{k_{|\Sigma|+1}} \notin I_{|\Sigma|}$ and we have a contradiction.

Otherwise there is an $i \in [1, m]$ such that $r_{k_i}(u) \notin I_{i-1}(u)$. We will construct a ranker r' that is shorter than r, does not have more alternations than r and occurs at exactly the same position as r in both u and v. The main idea for this construction is that if $r_{k_i}(u) \notin I_{i-1}(u)$, then it is not useful to enter this interval at all. By our assumption, u and v disagree on some property of the ranker r, and thus on some property of the shorter ranker r'. This contradicts our assumption that r was the shortest such ranker.

Now we show how to construct a shorter ranker r' that occurs at the same position as r. We assume without loss of generality that p_{k_i} points to the left. In this case we have $r_{k_i}(u) \notin I_{i-1}(u) = [r_{k_{i-1}-1}(u) + 1, r_{k_{i-1}}(u) - 1]$. We look at the relative positions of the rankers $r_{k_{i-1}+1}, \ldots, r_{k_i}$ with respect to the ranker $r_{k_{i-1}-1}$. We know that $r_{k_i}(u) \leq r_{k_{i-1}-1}(u)$, and we are interested in the right-most of the rankers $r_{k_{i-1}+1}, \ldots, r_{k_i}$ that is still outside of the interval $I_{i-1}(u)$. Let r_j be this ranker. Thus we have

$$r_{k_i}(u) < \ldots < r_j(u) \leq r_{k_{i-1}-1}(u) < r_{j-1}(u) < \ldots < r_{k_{i-1}+1}(u) < r_{k_{i-1}}(u)$$

We know that $u \equiv^2_{|\Sigma|+1,n} v$, thus by Theorem 4.5, these rankers occur in exactly the same order in v. Now we set $s := (r_{k_{i-1}-1}, p_j, \ldots, p_{k_i})$. Because u and v agree on the ordering of the relevant rankers, we have $s(u) = r_{k_i}(u)$ and $s(v) = r_{k_i}(v)$. Therefore we have reduced the size of a prefix of r without increasing the number of alternations, and thus have a shorter ranker r' that occurs at the same position as r in both structures. □

In order to prove that the alternation hierarchy for FO^2 is strict, we define example languages that can be separated by a formula of a given alternation depth m, but that cannot be separated by any formula of lower alternation depth. As Theorem 4.7 shows, we need to increase the size of the alphabet with increasing alternation depth. We inductively define the example words $u_{m,n}$ and $v_{m,n}$ and the example languages K_m and L_m over finite alphabets $\Sigma_m = \{\mathtt{a}_0, \ldots, \mathtt{a}_{m-1}\}$. Here i, m and n are positive integers.

$$
\begin{aligned}
u_{1,n} &:= \mathtt{a}_0 & v_{1,n} &:= \varepsilon \\
u_{2,n} &:= \mathtt{a}_0(\mathtt{a}_1\mathtt{a}_0)^{2n} & v_{2,n} &:= (\mathtt{a}_1\mathtt{a}_0)^{2n} \\
u_{2i+1,n} &:= (\mathtt{a}_0 \ldots \mathtt{a}_{2i})^n \; u_{2i,n} & v_{2i+1,n} &:= (\mathtt{a}_0 \ldots \mathtt{a}_{2i})^n \; v_{2i,n} \\
u_{2i+2,n} &:= u_{2i+1,n} \; (\mathtt{a}_{2i+1} \ldots \mathtt{a}_0)^n & v_{2i+2,n} &:= v_{2i+1,n} \; (\mathtt{a}_{2i+1} \ldots \mathtt{a}_0)^n
\end{aligned}
$$

Notice that $u_{m,n}$ and $v_{m,n}$ are almost identical – if we delete only one $\mathtt{a}_0$ from $u_{m,n}$, we get $v_{m,n}$. Finally, we set $K_m := \bigcup_{n\geq 1}\{u_{m,n}\}$ and $L_m := \bigcup_{n\geq 1}\{v_{m,n}\}$.

Definition 4.8. A formula φ *separates* two languages $K, L \subseteq \Sigma^\star$ if for all $w \in K$ we have $w \models \varphi$ and for all $w \in L$ we have $w \not\models \varphi$ or vice versa.

Lemma 4.9. *For all $m \in \mathbb{N}$, there is a formula $\varphi_m \in FO^2[<]$-$ALT[m]$ that separates K_m and L_m.*

Proof. For $m = 1$, we can easily separate $K_1 = \{\mathtt{a}_0\}$ and $L_1 = \{\varepsilon\}$ with the formula $\exists x(x = x)$. For all larger m, we show that the two languages K_m and L_m differ on the ordering of two $(m-1)$-alternation rankers. Then by Theorem 4.5 there is an $FO^2_{m,m}[<]$ formula that separates K_m and L_m. We inductively define the rankers

$$
\begin{aligned}
r_2 &:= \triangleright_{\mathtt{a}_0} & s_2 &:= \triangleright_{\mathtt{a}_1} \\
r_{2i+1} &:= \triangleleft_{\mathtt{a}_{2i}} r_{2i} & s_{2i+1} &:= \triangleleft_{\mathtt{a}_{2i}} s_{2i} \\
r_{2i+2} &:= \triangleright_{\mathtt{a}_{2i+1}} r_{2i+1} & s_{2i+2} &:= \triangleright_{\mathtt{a}_{2i+1}} s_{2i+1}
\end{aligned}
$$

For $m = 2$, it is easy to see that $r_2(u_{2,n}) < s_2(u_{2,n})$, but $r_2(v_{2,n}) > s_2(v_{2,n})$. For $m > 2$, these rankers disagree on their order as well. To prove this, we prove the following two equalities.

$$r_{2i+2}(u_{2i+2,n}) = r_{2i+1}(u_{2i+1,n}) = (2i+1)n + r_{2i}(u_{2i,n})$$

To prove this, we first use the definitions above and write

$$r_{2i+2}(u_{2i+2,n}) = (\triangleright_{\mathtt{a}_{2i+1}} r_{2i+1})(u_{2i+1,n} \; (\mathtt{a}_{2i+1} \ldots \mathtt{a}_0)^n)$$

The letter $\mathtt{a}_{2i+1}$ does not occur in the word $u_{2i+1,n}$, and thus $\triangleright_{\mathtt{a}_{2i+1}}(u_{2i+2,n})$ points to the first position in $u_{2i+2,n}$ right after the copy of $u_{2i+1,n}$. We observe that r_{2i+1} starts with $\triangleleft$, and that r_{2i+1} is defined on $u_{2i+1,n}$. Thus the evaluation of the remainder of r_{2i+2} on $u_{2i+2,n}$ never leaves the copy of $u_{2i+1,n}$, and we have

$$r_{2i+2}(u_{2i+2,n}) = r_{2i+1}(u_{2i+1,n})$$

For the second part of the equality, we have

$$r_{2i+1}(u_{2i+1,n}) = (\triangleleft_{\mathtt{a}_{2i}} r_{2i})((\mathtt{a}_0 \ldots \mathtt{a}_{2i})^n \; u_{2i,n})$$

As above, the letter $\mathtt{a}_{2i}$ does not occur in the word $u_{2i,n}$, and thus $\triangleleft_{\mathtt{a}_{2i}}(u_{2i+1,n})$ points to the position in $u_{2i+1,n}$ right before the copy of $u_{2i,n}$. The ranker r_{2i} starts with $\triangleright$, and r_{2i} is defined on $u_{2i,n}$. Thus, just as above, the evaluation of the remainder of r_{2i+1} on $u_{2i+1,n}$ never leaves the copy of $u_{2i,n}$, and we have

$$r_{2i+1}(u_{2i+1,n}) = (2i+1)n + r_{2i}(u_{2i,n})$$

Exactly the same holds for the other rankers $(s_2, \ldots)$ and words $(v_{2,n}, \ldots)$. We have

$$\begin{aligned} r_{2i+2}(u_{2i+2,n}) = r_{2i+1}(u_{2i+1,n}) &= (2i+1)n + r_{2i}(u_{2i,n}) \\ s_{2i+2}(u_{2i+2,n}) = s_{2i+1}(u_{2i+1,n}) &= (2i+1)n + s_{2i}(u_{2i,n}) \\ r_{2i+2}(v_{2i+2,n}) = r_{2i+1}(v_{2i+1,n}) &= (2i+1)n + r_{2i}(v_{2i,n}) \\ s_{2i+2}(v_{2i+2,n}) = s_{2i+1}(v_{2i+1,n}) &= (2i+1)n + s_{2i}(v_{2i,n}) \end{aligned}$$

Now an easy inductive argument, based on the two equalities we just proved, shows that the rankers disagree on their order. Therefore condition (i)(b) of Theorem 4.5 fails for any pair of words, and there is a formula in $\mathrm{FO}^2_{m,m}[<]$ that separates K_m and L_m. ☐

Lemma 4.10. *For $m \in \mathbb{N}$, $m \geq 1$, and all $n \in \mathbb{N}$, we have $u_{m,n} \equiv^2_{m-1,n} v_{m,n}$.*

Proof. Because we do not have constants, there are no quantifier-free sentences. Thus $\mathrm{FO}^2_{0,n}[<]$ does not contain any formulas and the statement holds trivially for $m = 1$.

For $m \geq 2$ and any $n \geq m$, we claim that exactly the same $(m-1,n)$-rankers are defined over $u_{m,n}$ and $v_{m,n}$, and that all $(m-1,n)$-rankers appear in the same order with respect to all $(m-2,n-1)$-rankers and all $(m-1,n-1)$-rankers that end on a different direction. Once we established this claim, the lemma follows immediately from Theorem 4.5. We already observed that $u_{m,n}$ and $v_{m,n}$ are almost identical. The only difference between the two words is that $u_{m,n}$ contains the letter a_0 in the middle whereas $v_{m,n}$ does not. Thus we only have to consider rankers that are affected by this middle a_0.

We claim that any ranker that points to the middle a_0 of $u_{m,n}$ requires at least $m-1$ alternations. Furthermore, we claim that any such ranker needs to start with $\triangleright$ for even m and with $\triangleleft$ for odd m. We prove this by induction on m.

For $m = 2$ we have $u_{2,n} = \mathsf{a}_0(\mathsf{a}_1\mathsf{a}_0)^n$. Any n-ranker that starts with $\triangleleft$ cannot reach the first a_0, thus we need a ranker that starts with $\triangleright$.

For odd $m > 2$ we have $u_{m,n} = (\mathsf{a}_0 \ldots \mathsf{a}_{m-1})^n u_{m-1,n}$. Any n-ranker that starts with $\triangleright$ cannot leave the first block of $n \cdot m$ symbols of this word and thus not reach the middle a_0. Therefore we need to start with $\triangleleft$, and in fact use $\triangleleft_{\mathsf{a}_{m-1}}$ at some point, because we would not be able to leave the last section of $u_{m-1,n}$ otherwise. But with $\triangleleft_{\mathsf{a}_{m-1}}$ we move past all of $u_{m-1,n}$, and we need one alternation to turn around again. By induction, we need at least $m-2$ alternations within $u_{m-1,n}$, and thus $m-1$ alternations total.

The argument for even m is completely symmetric. Thus we showed that we need at least $m-1$ alternation blocks to point to the middle a_0. Furthermore, we showed that if we have exactly $m-1$ alternation blocks, then the last of these blocks uses $\triangleright$. Therefore we only need to consider $(m-1)$-alternation rankers that end on $\triangleright$ and pass through the middle a_0. It is easy to see that all of these rankers agree on their ordering with respect to all other $(m-2)$-alternation rankers, and with respect to all $(m-1)$-alternation rankers that end on $\triangleleft$.

To summarize, we showed that $u_{m,n}$ and $v_{m,n}$ satisfy condition (i) from Theorem 4.5 for $m-1$ alternations. Thus the two words agree on all formulas from $\mathrm{FO}^2_{m-1,n}[<]$. ☐

Theorem 4.11 (alternation hierarchy for $\mathrm{FO}^2[<]$)**.** *For any positive integer m, there is a $\varphi_m \in \mathrm{FO}^2[<]\text{–ALT}[m]$ and there are two languages K_m, L_m such that φ_m separates K_m and L_m, but no $\psi \in \mathrm{FO}^2[<]\text{–ALT}[m-1]$ separates K_m and L_m.*

Proof. The theorem immediately follows from Lemma 4.9 and Lemma 4.10. ☐

Theorem 4.11 resolves an open question from [3, 4].

5. Structure Theorem and Alternation Hierarchy for FO$^2[<, \mathrm{Suc}]$

We extend our definitions of boundary positions and rankers from Sect. 3 to include the substrings of a given length that occur immediately before and after the position of the ranker.

Definition 5.1. A (k, ℓ)-*neighborhood boundary position* denotes the first or last occurrence of a substring in a word. More precisely, a (k, ℓ)-neighborhood boundary position is of the form $d_{(s,\mathtt{a},t)}$ with $d \in \{\triangleright, \triangleleft\}$, $s \in \Sigma^k$, $\mathtt{a} \in \Sigma$ and $t \in \Sigma^\ell$. The interpretation of a (k, ℓ)-neighborhood boundary position $p = d_{(s,\mathtt{a},t)}$ on a word $w = w_1 \dots w_{|w|}$ is defined as follows.

$$p(w) = \begin{cases} \min\{i \in [k+1, |w| - \ell] \mid w_{i-k} \dots w_{i+\ell} = s\,\mathtt{a}\,t\} & \text{if } d = \triangleright \\ \max\{i \in [k+1, |w| - \ell] \mid w_{i-k} \dots w_{i+\ell} = s\,\mathtt{a}\,t\} & \text{if } d = \triangleleft \end{cases}$$

Notice that $p(w)$ is undefined if the sequence $s\mathtt{a}t$ does not occur in w. A (k, ℓ)-neighborhood boundary position can also be specified with respect to a position $q \in [1, |w|]$.

$$p(w, q) = \begin{cases} \min\{i \in [\max\{q+1, k+1\}, |w| - \ell] \mid w_{i-k} \dots w_{i+\ell} = s\,\mathtt{a}\,t\} & \text{if } d = \triangleright \\ \max\{i \in [k+1, \min\{q-1, |w| - \ell\}] \mid w_{i-k} \dots w_{i+\ell} = s\,\mathtt{a}\,t\} & \text{if } d = \triangleleft \end{cases}$$

Observe that $(0, 0)$-neighborhood boundary positions are identical to the boundary positions from Definition 3.1. As before in the case without successor, we build rankers out of these boundary positions. The size of the boundary position neighborhoods grows linearly from the first boundary position to the last one, reflecting the remaining quantifier depth for successor moves at those positions.

Definition 5.2. An n-*successor-ranker* r is a sequence of n neighborhood boundary positions, $r = (p_1, \dots, p_n)$, where p_i is a (k_i, ℓ_i)-neighborhood boundary position and $k_i, \ell_i \in [0, i-1]$. The interpretation of an n-successor-ranker r on a word w is defined as follows.

$$r(w) := \begin{cases} p_1(w) & \text{if } r = (p_1) \\ \text{undefined} & \text{if } (p_1, \dots, p_{n-1})(w) \text{ is undefined} \\ p_n(w, (p_1, \dots, p_{n-1})(w)) & \text{otherwise} \end{cases}$$

We denote the set of all n-successor-rankers that are defined over a word w by $SR_n(w)$, and set $SR_n^\star(w) := \bigcup_{i \in [1,n]} SR_i(w)$.

Because we now have the additional atomic relation Suc, we need to extend our definition of order type as well.

Definition 5.3. Let $i, j \in \mathbb{N}$. The *successor order type* of i and j is defined as

$$\mathrm{ord_S}(i, j) = \begin{cases} \ll & \text{if } i < j - 1 \\ -1 & \text{if } i = j - 1 \\ = & \text{if } i = j \\ +1 & \text{if } i = j + 1 \\ \gg & \text{if } i > j + 1 \end{cases}$$

With this new definition of n-successor-rankers, our proofs for Lemmas 3.5, 3.6, 3.7 and Theorem 3.8 go through with only minor modifications. Instead of working through all the details again, we simply point out the differences.

First we notice that 1-successor-rankers are simply 1-rankers, so the base case of all inductions remains unchanged. In the proofs of Lemmas 3.5, 3.6 and 3.7, and in the proof of "(ii) $\Rightarrow$ (i)" from Theorem 3.8, we argued that Delilah cannot reply with a position in a given section because it does not contain a certain ranker and therefore it does not contain the symbol used to define this ranker. Now we need to know more – we need to show that Delilah cannot reply with a certain letter in a given section that is surrounded by a specified neighborhood, given that this section does not contain the corresponding successor-ranker. Whenever Samson's winning strategy depends on the fact that an n-successor-ranker does not occur in a given section, he has $n-1$ additional moves left. So if Delilah does not reply with a position with the same letter and the same neighborhood, Samson can point out a difference in the neighborhood with at most $(n-1)$ additional moves.

For the other direction of Theorem 3.8, we need to make sure that Delilah can reply with a position that is contained in the correct interval, has the same symbol and is surrounded by the same neighborhood. Where we previously defined the n-ranker $s := (\lambda, \triangleright_{\mathtt{a}})$ or $s := (\rho, \triangleleft_{\mathtt{a}})$, we now include the $(n-1)$-neighborhood of the respective positions chosen by Samson. Thus we make sure that Samson cannot point out a difference in the two words, and Delilah still has a winning strategy. Thus we have the following three theorems for $\mathrm{FO}^2[<, \mathrm{Suc}]$.

Theorem 5.4 (structure of $\mathrm{FO}^2_n[<, \mathrm{Suc}]$)**.** *Let u and v be finite words, and let $n \in \mathbb{N}$. The following two conditions are equivalent.*

(i) (a) $SR_n(u) = SR_n(v)$, *and,*
 (b) *for all $r \in SR^\star_n(u)$ and for all $r' \in SR^\star_{n-1}(u)$,*
 $ord_S(r(u), r'(u)) = ord_S(r(v), r'(v))$
(ii) $u \equiv^2_n v$

Theorem 5.5 (structure of $\mathrm{FO}^2_{m,n}[<, \mathrm{Suc}]$)**.** *Let u and v be finite words, and let $m, n \in \mathbb{N}$ with $m \le n$. The following two conditions are equivalent.*

(i) (a) $SR_{m,n}(u) = SR_{m,n}(v)$, *and,*
 (b) *for all $r \in SR^\star_{m,n}(u)$ and for all $r' \in SR^\star_{m-1,n-1}(u)$,*
 $ord_S(r(u), r'(u)) = ord_S(r(v), r'(v))$, *and,*
 (c) *for all $r \in SR^\star_{m,n}(u)$ and $r' \in SR^\star_{m,n-1}(u)$ such that r and r' end with different directions,* $ord_S(r(u), r'(u)) = ord_S(r(v), r'(v))$
(ii) $u \equiv^2_{m,n} v$

Theorem 5.6 (alternation hierarchy for $\mathrm{FO}^2[<, \mathrm{Suc}]$)**.** *Let m be a positive integer. There is a $\varphi_m \in FO^2[<, \mathrm{Suc}]$–$ALT[m]$ and there are two languages $K_m, L_m \subseteq \Sigma^\star$ such that φ_m separates K_m and L_m, but there is no $\psi \in FO^2[<, \mathrm{Suc}]$–$ALT[m-1]$ that separates K_m and L_m.*

Proof. We use the same ideas as before in Theorem 4.11. We define example languages that now include an extra letter $\mathtt{b}$ to ensure that the successor predicate is of no use. As before, we inductively construct the words $u_{m,n}$ and $v_{m,n}$ and use them to define the languages K_m and L_m.

$$
\begin{aligned}
u_{1,n} &:= \mathtt{b}^{2n}\mathtt{a}_0\mathtt{b}^{2n} & v_{1,n} &:= \mathtt{b}^{2n} \\
u_{2,n} &:= u_{1,n}\ (\mathtt{a}_1\mathtt{b}^{2n}\mathtt{a}_0\mathtt{b}^{2n})^{2n} & v_{2,n} &:= v_{1,n}\ (\mathtt{a}_1\mathtt{b}^{2n}\mathtt{a}_0\mathtt{b}^{2n})^{2n} \\
u_{2i+1,n} &:= (\mathtt{b}^{2n}\mathtt{a}_0\mathtt{b}^{2n}\ldots\mathtt{b}^{2n}\mathtt{a}_{2i})^n\ u_{2i,n} & v_{2i+1,n} &:= (\mathtt{b}^{2n}\mathtt{a}_0\mathtt{b}^{2n}\ldots\mathtt{b}^{2n}\mathtt{a}_{2i})^n\ v_{2i,n} \\
u_{2i+2,n} &:= u_{2i+1,n}\ (\mathtt{a}_{2i+1}\mathtt{b}^{2n}\ldots\mathtt{a}_0\mathtt{b}^{2n})^n & v_{2i+2,n} &:= v_{2i+1,n}\ (\mathtt{a}_{2i+1}\mathtt{b}^{2n}\ldots\mathtt{a}_0\mathtt{b}^{2n})^n
\end{aligned}
$$

Finally we set $K_m := \bigcup_{n\geq 1}\{u_{m,n}\}$ and $L_m := \bigcup_{n\geq 1}\{v_{m,n}\}$. Notice that the $\mathtt{b}$s are not necessary to distinguish between the two languages K_m and L_m, and thus the proof of Lemma 4.9 goes through unchanged and we have a formula $\varphi_m \in \mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}[m]$ that separates K_m and L_m. To see that no $\mathrm{FO}^2[<,\mathrm{Suc}]\text{–ALT}[m-1]$ formula can separate K_m and L_m, we observe that any $(n-1)$-neighborhood in the words $u_{m,n}$ and $v_{m,n}$ contains all $\mathtt{b}$s except for at most one letter $\mathtt{a}_i$ for some $i \in [0, m-1]$. Thus the proof of Lemma 4.10 goes through here as well. $\square$

6. Small Models and Satisfiability for FO$^2[<]$

The complexity of satisfiability for $\mathrm{FO}^2[<]$ was investigated in [4]. There it is shown that any satisfiable $\mathrm{FO}^2_n[<]$ formula has a model of size at most exponential in n. It follows that satisfiability for $\mathrm{FO}^2[<]$ is in NEXP, and a reduction from TILING shows that satisfiability for $\mathrm{FO}^2[<]$ is NEXP-complete. Using our characterization of $\mathrm{FO}^2[<]$, Wilke observed that satisfiability becomes NP-complete if we look at binary alphabets only [21]. We generalize this observation and show that satisfiability for $\mathrm{FO}^2[<]$ is NP-complete for any fixed alphabet size. In contrast to this, satisfiability for $\mathrm{FO}^2[<,\mathrm{Suc}]$ is NEXP-complete even for binary alphabets [4], since in the presence of a successor predicate we can encode an arbitrary alphabet in binary. Before we state and prove the two theorems of this section, we prove a simple technical lemma first.

Lemma 6.1. *Let $u, v, v', w \in \Sigma^\star$. If $v \equiv^2_n v'$, then $uvw \equiv^2_n uv'w$.*

Proof. We argue that Delilah has a winning strategy for the game $\mathrm{FO}^2_n(uvw, uv'w)$: If Samson places a pebble in u or w, Delilah replies with the identical position in u or w in the other structure. If Samson places a pebble in v or v', Delilah replies according to her winning strategy in the game $\mathrm{FO}^2_n(v, v')$. All of these moves obviously preserve the ordering of the pebbles, and thus Delilah wins. $\square$

Theorem 6.2 (Small Model Property for Bounded Alphabets)**.** *Let $n \in \mathbb{N}$ and let $\varphi \in FO^2_n[<]$ be a formula over a k-letter alphabet. If φ is satisfiable, then φ has a model of size $O(n^k)$.*

Proof. Let w be an arbitrary model of φ. We use induction on k to show how to construct a new model of size $O(n^k)$ that satisfies φ. For $k = 1$, i.e. a single letter alphabet, we observe that an n-ranker can only point to a position within the first or last n letters of w. We let w' be a copy of w with all letters after the first n letters and before the last n letters removed. The words w and w' agree on the existence and ordering of all n-rankers, thus we can apply Theorem 3.8 and it follows that $w' \models \varphi$.

For the inductive case, we partition w into segments, where each segment is a maximal sequence of the same letter. For example, the word $\mathtt{aaabb}$ has two segments, $\mathtt{aaa}$ and $\mathtt{bb}$. First, we let w' be a copy of w where we cut down all segments that are longer than $2n$ to exactly $2n$ letters. Since no n-ranker can point to a position within any segment after the first n letters and before the last n letters of that segment, we have $w' \models \varphi$.

Now we partition the word w' such that $w' = u_1 s_1 u_2 \dots u_r s_r u_{r+1}$, where $r \in \mathbb{N}$ and for every $1 \leq i \leq r$, u_i is a string of maximal length that uses exactly k different letters, s_i is a segment, and u_{r+1} is a string over at most a k-letter alphabet. We observe that this partition is unique: If a is the last of the $(k+1)$ letters in our alphabet to appear in w', starting from the left, then s_1 is the left-most segment of a's, and u_1 is everything up to that segment. Now s_2 is the left-most segment after s_1 of the letter that appears last after s_1, and so on. We can point to a position in segment s_n with an n-ranker, but no n-ranker that starts with $\triangleright$ can point to a position to the right of s_n. Similarly, we partition w', now starting from the right, such that $w' = v_{q+1} t_q v_q \dots v_2 t_1 v_1$, where $q \in \mathbb{N}$ and for every $1 \leq i \leq q$, v_i is a string of maximal length that uses exactly k different letters, t_i is a segment, and v_{q+1} is a string over at most a k-letter alphabet. Again, this partition is unique and any n-ranker that starts with $\triangleleft$ cannot point to a position to the left of t_n. We also notice that both partitions have the same number of segments, i.e. $r = q$, since any substring $u_i s_i$ from the first partition contains all letters of the alphabet and thus has to contain at least one segment t_j from the second partition, and vice versa.

If both partitions use more than $2n$ segments, then the segment s_n of the first partition occurs to the left of the segment t_n of the second partition. In this case we construct the word $w'' = u_1 s_1 u_2 \dots u_n s_n t_n v_n \dots v_2 t_1 v_1$. w'' agrees with w' on all n-rankers, and thus $w'' \models \varphi$. Every one of the strings $u_1, \dots, u_n$ and $v_1, \dots v_n$ uses at most k different letters, therefore we can apply the inductive hypothesis and replace each of these strings with an equivalent string of length $O(n^k)$, as explained in Lemma 6.1. Thus we have constructed a word of length $O(n^{k+1})$ that satisfies φ.

If the partitions have at most $2n$ segments, then we combine the two partitions such that $w' = w_1 x_1 \dots x_p w_{p+1}$, where $p \leq 4n$, and for every $1 \leq i \leq p$, x_p is one of the original segments $s_1, \dots, s_r$ and $t_1, \dots, t_q$. As above, we use the inductive hypothesis to replace all strings x_i with equivalent strings of length $O(n^k)$, and thus construct a new string of length $O(n^{k+1})$ that satisfies φ. □

Theorem 6.3. *Satisfiability for* $FO^2[<]$ *where the size of the alphabet is bounded by some fixed* $k \geq 2$ *is* NP*-complete.*

Proof. Membership in NP follows immediately from Theorem 6.2 – we nondeterministically guess a model of size $O(n^k)$ where n is the quantifier depth of the given formula, and verify that it is a model of the formula. Now we give a reduction from SAT. Let α be a boolean formula in conjunctive normal form over the variables $X_1, \dots, X_n$. We construct a $\mathrm{FO}^2[<]$ formula $\varphi = \varphi_n \wedge \alpha[\xi_i/X_i]$, where φ_n says that every model has size exactly n, and where we replace every occurrence of X_i in α with a formula ξ_i of length $O(n)$ which says that the i-th letter is a 1. The total length of φ is $O(|\alpha| \cdot n)$, and φ is satisfiable iff α is satisfiable. □

7. Conclusion

We proved precise structure theorems for FO^2, with and without the successor predicate, that completely characterize the expressive power of the respective logics, including exact bounds on the quantifier depth and on the alternation depth. Using our structure theorems, we showed that the quantifier alternation hierarchy for FO^2 is strict, settling an open question from [3, 4]. Both our structure theorems and the alternation hierarchy results add further insight to and simplify previous characterizations of FO^2. We hope that the insights gained in our study of FO^2 on words will be useful in future investigations of the trade-off between formula size and number of variables.

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References

[1] Adler, M., and Immerman, N. An n! lower bound on formula size. *ACM Transactions on Computational Logic 4*, 3 (2003), 296–314.
[2] Brzozowski, J., and Knast, R. The dot-depth hierarchy of star-free languages is infinite. *Journal of Computer and System Science 16* (1978), 37–55.
[3] Etessami, K., Vardi, M. Y., and Wilke, T. First-order logic with two variables and unary temporal logic. In *IEEE Symposium on Logic in Computer Science* (1997).
[4] Etessami, K., Vardi, M. Y., and Wilke, T. First-order logic with two variables and unary temporal logic. *Information and Computation 179*, 2 (2002), 279–295.
[5] Grohe, M., and Schweikardt, N. The succinctness of first-order logic on linear orders. *Logical Methods in Computer Science 1*, 1:6 (2005), 1–25.
[6] Immerman, N. *Descriptive Complexity*. Springer, 1999.
[7] Immerman, N., and Kozen, D. Definability with bounded number of bound variables. *Information and Computation 83*, 2 (1989), 121–139.
[8] Kamp, J. A. *Tense logic and the theory of linear order*. PhD thesis, University of California, Los Angeles, 1968.
[9] Karchmer, M., and Wigderson, A. Monotone circuits for connectivity require super-logarithmic depth. *SIAM Journal of Discrete Mathematics 3*, 2 (1990), 255–265.
[10] McNaughton, R., and Papert, S. A. *Counter-free automata*. MIT Press, Cambridge, MA, 1971.
[11] Pin, J.-E., and Weil, P. Polynomial closure and unambiguous product. *Theory of Computing Systems 30* (1997), 1–39.
[12] Schützenberger, M. P. Sur le produit de concatenation non ambigu. *Semigroup Forum 13* (1976), 47–75.
[13] Schwentick, T., Thérien, D., and Vollmer, H. Partially-ordered two-way automata: a new characterization of DA. In *Developments in Language Theory* (2001).
[14] Straubing, H., and Thérien, D. Weakly iterated block products. In *Latin American Theoretical Informatics Conference* (2002).
[15] Tesson, P., and Thérien, D. Diamonds are forever: the variety DA. In *Semigroups, Algorithms, Automata and Languages* (2001).
[16] Tesson, P., and Thérien, D. Algebra meets logic: the case of regular languages. *Logical Methods in Computer Science 3*, 1:4 (2007).
[17] Thérien, D., and Wilke, T. Over words, two variables are as powerful as one quantifier alternation. In *ACM Symposium on Theory of Computing* (1998).
[18] Thomas, W. Classifying regular events in symbolic logic. *Journal of Computer and System Science 25* (1982), 360–376.
[19] Thomas, W. An application of the Ehrenfeucht-Fraïssé game in formal language theory. *Mémoires de la S.M.F. 16* (1984), 11–21.
[20] Weis, P., and Immerman, N. Structure theorem and strict alternation hiearchy for FO^2 on words. In *Computer Science Logic* (2007).
[21] Wilke, T. Personal communication, 2007.