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DOMAIN REPRESENTATIONS INDUCED BY DYADIC SUBBASES

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Abstract. We study domain representations induced by dyadic subbases and show that a proper dyadic subbase of a second-countable regular space X induces an embedding of X in the set of minimal limit elements of a subdomain D of $\{0,1,\bot\}^\omega$. In particular, if X is compact, then X is a retract of the set of limit elements of D.

1. Introduction

From a computational point of view, it is natural to consider a subbase of a second-countable T_0 space X as a collection of primitive properties of X through which one can identify each point of X. In this way, by fixing a numbering of the subbase, one can represent each point of X as a subset of $\mathbb{N}$ and construct a domain representation of X in the domain P_ω of subsets of $\mathbb{N}$ [3, 17]. Note that P_ω is isomorphic to the domain of infinite sequences of the Sierpinski space $\{1,\bot\}$.

On the other hand, each regular open set A (i.e., an open set which is equal to the interior of its closure) of a topological space X divides X into three parts: A, the exterior of A, and their common boundary. Therefore, one can consider a pair of regular open subsets which are exteriors of each other as a pair of primitive properties and use a subbase which is composed of such pairs of open sets in representing the space. Such a subbase is called a dyadic subbase and a dyadic subbase of a space X induces a domain representation of X in the domain $\mathbb{T}^\omega$ of infinite sequences of $\mathbb{T}=\{0,1,\bot\}$. In [20] and [12], the authors introduced to a dyadic subbase the properness property which expresses a kind of orthogonality between the components and studied domain representations of Hausdorff spaces induced by proper dyadic subbases. In this representation, the domain is fixed to $\mathbb{T}^\omega$ and an embedding φ_S of a Hausdorff space X in $\mathbb{T}^\omega$ is derived from a proper dyadic subbase S of X.

In this paper, we derive from a dyadic subbase S a domain (i.e., an ω-algebraic pointed dcpo) D_S and a bounded complete domain $\widehat{D}_S$ which are subdomains of $\mathbb{T}^\omega$ containing $\varphi_S(X)$ as subspaces. The domain D_S has the following properties. (1) If X is a strongly nonadhesive Hausdorff space (Definition 5.4), then the set $L(D_S)$ of limit (i.e., non-compact)

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elements of D_S has minimal elements. (2) If X is regular, then φ_S is an embedding of X in the set of minimal elements of $L(D_S)$. (3) If X is compact, then there is a retraction ρ_S from $L(D_S)$ to X. That is, every infinite strictly increasing sequence in $K(D_S)$ represents a point of X through ρ_S and $(D_S, L(D_S), \rho_S)$ is the kind of domain representations studied in [19]. The domain $\widehat{D}_S$ also has the properties (1) to (3) and, in addition, it is bounded complete.

We study properties of representations for second-countable Hausdorff spaces and investigate which property holds under each of the above-mentioned conditions. Therefore, a space in this paper means a second-countable Hausdorff space unless otherwise noted. We are mainly interested in the case where X is a regular space because the corresponding domain representations have good properties as we mentioned above. In addition, it is proved in [11] that every second-countable regular space has a proper dyadic subbase and in [12] that every dense-in-itself second-countable regular space has an independent subbase, which is a proper dyadic subbase with an additional property.

We review proper dyadic subbases and their properties in the next section, and we study TTE-representations and domain representations in $\mathbb{T}^\omega$ derived from (proper) dyadic subbases in Section 3. We introduce the domains D_S and $\widehat{D}_S$ in Section 4, and present the strongly nonadhesiveness condition in Section 5. Then we study domain representations in these domains for the case X is regular in Section 6. Finally, in Section 7, we study the small inductive dimension of the T_0-spaces $L(D_S)$ and $L(\widehat{D}_S)$ based on a result in [19].

Preliminaries and Notations:

Bottomed Sequences: Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{2}$ be the set $\{0, 1\}$. Let $\mathbb{T}$ be the set $\{0, 1, \bot\}$ where $\bot$ is called the bottom character which means undefinedness. The set of infinite sequences of a set Σ is denoted by Σ^ω. Each element of $\mathbb{T}^\omega$ is called a *bottomed sequence* and each copy of 0 and 1 which appears in a bottomed sequence p is called a *digit* of p. A *finite bottomed sequence* is a bottomed sequence with a finite number of digits, and the set of all finite bottomed sequences is denoted by $\mathbb{T}^*$. We sometimes omit $\bot^\omega$ at the end of a finite bottomed sequence and identify a finite bottomed sequence with a finite sequence of $\mathbb{T}$. The set of finite sequences of $\mathbb{2}$ is denoted by $\mathbb{2}^*$.

We define the partial order relation $\sqsubseteq$ on $\mathbb{T}$ by $\bot \sqsubseteq 0$ and $\bot \sqsubseteq 1$, and its product order on $\mathbb{T}^\omega$ is denoted by the same symbol $\sqsubseteq$, i.e., for every $p, q \in \mathbb{T}^\omega$, $p \sqsubseteq q$ if $p(n) \sqsubseteq q(n)$ for each $n \in \mathbb{N}$. Then $\mathbb{2}^\omega$ is the set of maximal elements of $\mathbb{T}^\omega$. We consider the T_0-topology $\{\varnothing, \{0\}, \{1\}, \{0, 1\}, \mathbb{T}\}$ on $\mathbb{T}$, and its product topology on $\mathbb{T}^\omega$. We write $\mathrm{dom}(p) = \{k : p(k) \neq \bot\}$ for $p \in \mathbb{T}^\omega$. For a finite bottomed sequence $e \in \mathbb{T}^*$, the length $|e|$ of e is the maximal number n such that $e(n-1) \neq \bot$. We denote by $p|_n$ the finite bottomed sequence with $\mathrm{dom}(p|_n) = \mathrm{dom}(p) \cap \{0, 1, \ldots, n-1\}$ such that $p|_n \sqsubseteq p$. That is, $p|_n(k) = p(k)$ if $k < n$ and $p|_n(k) = \bot$ if $k \geqslant n$. Note that the notation $p|_n$ is used with a different meaning in [20].

The letters a and b will be used for elements of $\mathbb{2}$, c for elements of $\mathbb{T}$, i, j, k, l, m, n for elements of $\mathbb{N}$, p and q for bottomed sequences, and d and e for finite bottomed sequences. We write $c^\omega = (c, c, \cdots) \in \mathbb{T}^\omega$ for $c \in \mathbb{T}$. We denote by $p[n := a]$ the bottomed sequence q such that $q(n) = a$ and $q(i) = p(i)$ for $i \neq n$.

Topology: Throughout this paper, X denotes a second-countable Hausdorff space unless otherwise noted. Therefore, if X is regular, then X is separable metrizable by Urysohn's metrization theorem. Recall that a subset U of X is *regular open* if U is the interior of its closure.

A *filter* $\mathfrak{F}$ on the space X is a family of subsets of X with the following properties.
(1) $\emptyset \notin \mathfrak{F}$.
(2) If $A \in \mathfrak{F}$ and $A \subseteq A' \subseteq X$, then $A' \in \mathfrak{F}$.
(3) If $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$.
Let $\mathfrak{V}(x)$ denote the family of neighbourhoods of $x \in X$. For a filter $\mathfrak{F}$ on X and a point $x \in X$, if we have $\mathfrak{V}(x) \subseteq \mathfrak{F}$ then we say that $\mathfrak{F}$ *converges to* x.

A family $\mathfrak{B}$ of subsets of X is called a *filter base* if it satisfies $\emptyset \notin \mathfrak{B}$, $\mathfrak{B} \neq \emptyset$, and that for all $A, B \in \mathfrak{B}$ there exists $C \in \mathfrak{B}$ such that $C \subseteq A \cap B$. A filter generated by a filter base $\mathfrak{B}$ is defined as the minimum filter containing $\mathfrak{B}$. We say that a filter base *converges to* $x \in X$ if it generates a filter which converges to x.

We denote by $\mathrm{cl}_Y A$, $\mathrm{bd}_Y A$, and $\mathrm{ext}_Y A$ the closure, boundary, and exterior of a set A in a space Y, respectively, and we omit the subscript if the space is obvious.

Domain Theory: Let $(P, \sqsubseteq)$ be a partially ordered set (poset). We say that two elements p and p' of a poset P are *compatible* if $p \sqsubseteq q$ and $p' \sqsubseteq q$ for some $q \in P$, and write $p \uparrow p'$ if p and p' are compatible. For $p \in P$ and $A \subseteq P$, we define $\uparrow p = \{q : q \sqsupseteq p\}$, $\downarrow p = \{q : q \sqsubseteq p\}$, $\uparrow A = \cup\{\uparrow q : q \in A\}$, and $\downarrow A = \cup\{\downarrow q : q \in A\}$. Therefore, we have $\downarrow\uparrow p = \{q : q \uparrow p\}$. We say that A is *downwards-closed* if $A = \downarrow A$, and *upwards-closed* if $A = \uparrow A$.

A subset A of a poset P is called *directed* if it is nonempty and each pair of elements of A has an upper bound in A. A *directed complete partial order (dcpo)* is a partially ordered set in which every directed subset A has a *least upper bound* (*lub*) $\sqcup A$. A dcpo is *pointed* if it has a least element.

Let $(D, \sqsubseteq)$ be a dcpo. A *compact element* of D is an element $d \in D$ such that for every directed subset A, if $d \sqsubseteq \sqcup A$ then $d \in \downarrow A$. An element of D is called a *limit element* if it is not compact. We write $K(D)$ for the set of compact elements of D, and $L(D)$ for the set of limit elements of D.

For $x \in D$, we define $K_x = K(D) \cap \downarrow x$. A dcpo D is *algebraic* if K_x is directed and $\sqcup K_x = x$ for each $x \in D$, and it is *ω-algebraic* if D is algebraic and $K(D)$ is countable. In this paper, a *domain* means an ω-algebraic pointed dcpo. The *Scott topology* of a domain D is the topology generated by $\{\uparrow d : d \in K(D)\}$. In this paper, we consider a domain D as a topological space with the Scott topology. A poset is *bounded complete* if every subset which has an upper bound also has a least upper bound. $\mathbb{T}^\omega$ is a bounded complete domain such that $K(\mathbb{T}^\omega) = \mathbb{T}^*$.

An *ideal* of a poset P is a directed downwards-closed subset. The set of ideals of P ordered by set inclusion is denoted by $\mathrm{Idl}(P)$. The poset $\mathrm{Idl}(P)$ becomes a domain called the *ideal completion* of P if P is countable. We have an order isomorphism $K(\mathrm{Idl}(P)) \cong P$ for each countable poset P with a least element. On the other hand, for a domain D, we have $\mathrm{Idl}(K(D)) \cong D$. Therefore, $K(D)$, the set of compact elements of D, determines the structure of D. We say that an ideal of $K(D)$ is *principal* if its least upper bound is in $K(D)$. An infinite strictly increasing sequence $d_0 \sqsubsetneq d_1 \sqsubsetneq d_2 \sqsubsetneq \cdots$ in $K(D)$ determines a non-principal ideal $\{e \in K(D) : e \sqsubseteq d_i \text{ for some } i\}$ of $K(D)$ and thus determines a point of $L(D)$.

A poset P is a *conditional upper semilattice with least element (cusl)* if it has a least element and every pair of compatible elements has a least upper bound. If P is a cusl, then $\mathrm{Idl}(P)$ is a bounded complete domain. For background material on domains, see [8, 1, 15].

Representation: We write $f :\subseteq A \to B$ if f is a partial function from A to B. For a finite or a countably infinite alphabet Σ, a surjective partial function from Σ^ω to X is called a *(TTE-)representation* of X. We say that a continuous function $\gamma :\subseteq \Sigma^\omega \to X$ is *reducible* (resp. *continuously reducible*) to $\delta :\subseteq \Sigma^\omega \to X$ if there exists a computable function (resp. continuous function) $\phi :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\gamma(p) = \delta(\phi(p))$ for every $p \in \mathrm{dom}(\gamma)$. Two representations $\delta, \delta' :\subseteq \Sigma^\omega \to X$ are *equivalent* (resp. *continuously equivalent*) if they are reducible (reps. continuously reducible) to each other. A representation $\delta :\subseteq \Sigma^\omega \to X$ is called *admissible* if δ is continuous and every continuous function $\gamma :\subseteq \Sigma^\omega \to X$ is continuously reducible to δ.

Let X be a T_0-space and $B = \{B_n : n \in \mathbb{N}\}$ be a subbase of X indexed by $\mathbb{N}$. Consider the representation $\delta_B :\subseteq \mathbb{N}^\omega \to X$ such that $\delta_B(p) = x$ if and only if $\{p(k) : k \in \mathbb{N}\} = \{n \in \mathbb{N} : x \in B_n\}$. δ_B is called a *standard representation* of X with respect to B. Any representation which is continuously equivalent to a standard representation is admissible [23, 14, 22].

Domain representation: Let D be a domain, D^R a subspace of D, and μ a quotient map from D^R onto X. The triple (D, D^R, μ) is called a *domain representation* of X. Note that we do not require D to be bounded-complete or each element of D^R to be total (i.e., condense) in this paper. See [2, 3] for the notion of totality. A domain representation is called a *retract domain representation* if μ is a retraction, and a *homeomorphic domain representation* if μ is a homeomorphism.

A domain representation (D, D^R, μ) of X is *upwards-closed* if D^R is upwards-closed and $\mu(d) = \mu(e)$ for every $d \sqsupseteq e \in D^R$. A domain representation (D, D^R, μ) is called *admissible* if for every pair (E, E^R) of a domain E and a dense subset $E^R \subseteq E$ and for every continuous function $\nu : E^R \to X$, there is a continuous function $\phi : E \to D$ such that $\nu(x) = \mu(\phi(x))$ holds for all $x \in E^R$. A domain representation $E = (E, E^R, \nu)$ *reduces continuously* to a domain representation $D = (D, D^R, \mu)$ if there is a continuous function $\phi : E \to D$ such that $\phi(E^R) \subseteq D^R$ and $\nu(x) = \mu(\phi(x))$ for all $x \in E^R$. For more about (admissible) domain representations, see [3, 9, 16, 17].

2. Proper dyadic subbases

Recall that a space means a Hausdorff space unless otherwise noted.

Definition 2.1. A *dyadic subbase* S of a space X is a family $\{S_{n,a} : n \in \mathbb{N}, a \in \mathbb{2}\}$ of regular open sets indexed by $\mathbb{N} \times \mathbb{2}$ such that (1) $S_{n,1}$ is the exterior of $S_{n,0}$ for each $n \in \mathbb{N}$ and (2) it forms a subbase of X.

Note that we allow duplications in $S_{n,a}$ and therefore, for example, a one point set $X = \{x\}$ has a dyadic subbase $S_{n,0} = X, S_{n,1} = \varnothing$ $(n = 0, 1, \ldots)$. Note also that this definition is applicable also to non-Hausdorff spaces, though we only consider the case X is Hausdorff in this paper. We denote by $S_{n,\perp}$ the set $X \setminus (S_{n,0} \cup S_{n,1})$. Since $S_{n,0}$ is regular open, we get $\mathrm{bd}\, S_{n,0} = \mathrm{bd}\, S_{n,1} = S_{n,\perp}$. Note that $S_{n,\perp}$ is defined differently in [20].

A topological space is called *semiregular* if the family of regular open sets forms a base of X. It is immediate that a regular space is semiregular. From the definition, a space with a dyadic subbase is a second-countable semiregular space. On the other hand, it is shown in [20] that every second-countable semiregular space has a dyadic subbase.

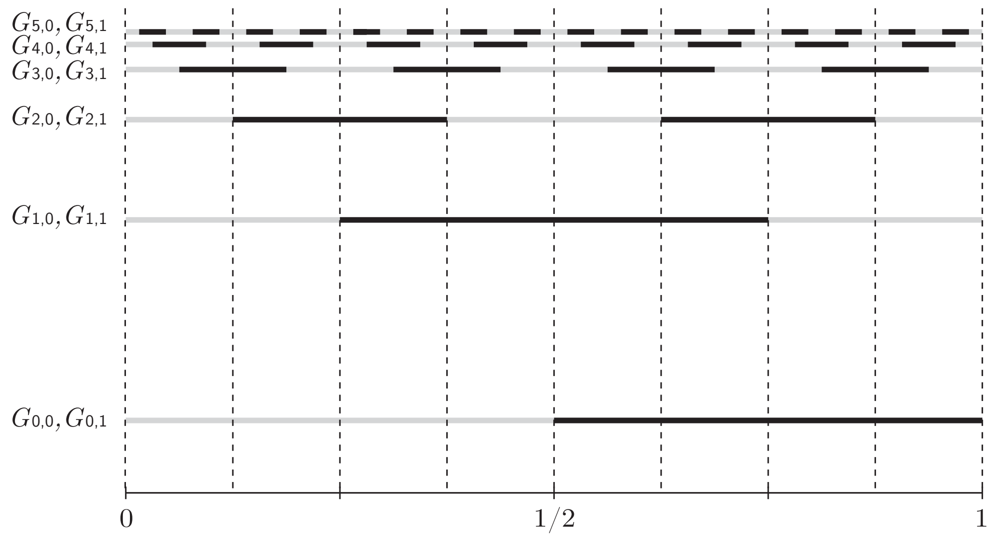

Figure 1: Gray subbase of the unit interval $\mathbb{I}$.

From a dyadic subbase S, we obtain a topological embedding $\varphi_S : X \to \mathbb{T}^\omega$ as follows.

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}), \\ 1 & (x \in S_{n,1}), \\ \bot & (x \in S_{n,\bot}). \end{cases}$$

We denote by $\tilde{x}$ the sequence $\varphi_S(x) \in \mathbb{T}^\omega$ and denote by $\widetilde{X}$ the set $\varphi_S(X) \subseteq \mathbb{T}^\omega$ if there is no ambiguity of S.

In the sequence $\tilde{x}$, if $\tilde{x}(n) = a$ for $n \in \mathbb{N}$ and $a \in \mathbb{2}$, then this fact holds for some neighbourhood A of x because $S_{n,a}$ is open. On the other hand, if $\tilde{x}(n) = \bot$, then every neighbourhood A of x contains points y_0 and y_1 with $\tilde{y_0}(n) = 0$ and $\tilde{y_1}(n) = 1$. Therefore, if $\tilde{x}(n) = \bot$, then every neighbourhood A of x does not exclude both of the possibilities $\tilde{x}(n) = 0$ and $\tilde{x}(n) = 1$.

Example 2.2 (Gray subbase)**.** Let $\mathbb{I} = [0, 1]$ be the unit interval and let $X_0 = [0, 1/2)$ and $X_1 = (1/2, 1]$ be subsets of $\mathbb{I}$. The tent function is the function $t : \mathbb{I} \to \mathbb{I}$ defined as

$$t(x) = \begin{cases} 2x & (x \in \operatorname{cl} X_0), \\ 2(1 - x) & (x \in \operatorname{cl} X_1). \end{cases}$$

We define the dyadic subbase G as

$$G_{n,a} = \{x : t^n(x) \in X_a\}$$

for $n \in \mathbb{N}$ and $a \in \mathbb{2}$. The map φ_G is an embedding of the unit interval in $\mathbb{T}^\omega$ [7, 18]. If x is a dyadic rational number other than 0 or 1, then $\varphi_G(x)$ has the form $e\bot 10^\omega$ for $e \in \mathbb{2}^*$, and it is in $\mathbb{2}^\omega$ otherwise. Figure 1 shows the Gray subbase, with the gray lines representing $G_{n,0}$ and the black lines representing $G_{n,1}$.

For a dyadic subbase S and $p \in \mathbb{T}^\omega$, let

$$S(p) = \bigcap_{k \in \mathrm{dom}(p)} S_{k,p(k)}, \tag{2.1}$$

$$\bar{S}(p) = \bigcap_{k \in \mathrm{dom}(p)} \mathrm{cl}\, S_{k,p(k)} = \bigcap_{k \in \mathrm{dom}(p)} (S_{k,p(k)} \cup S_{k,\perp}) \tag{2.2}$$

denote the corresponding subsets of X. Note that, for $x \in X$ and $p \in \mathbb{T}^\omega$,

$$x \in S(p) \Leftrightarrow \tilde{x}(k) = p(k) \text{ for } k \in \mathrm{dom}(p) \Leftrightarrow p \sqsubseteq \tilde{x}, \tag{2.3}$$

$$x \in \bar{S}(p) \Leftrightarrow \tilde{x}(k) \sqsubseteq p(k) \text{ for } k \in \mathrm{dom}(p) \Leftrightarrow p \uparrow \tilde{x}. \tag{2.4}$$

For $e \in \mathbb{T}^* = K(\mathbb{T}^\omega)$, $S(e)$ is an element of the base generated by the subbase S. We denote by $\mathfrak{B}_S$ the base $\{S(e) : e \in \mathbb{T}^*\} \backslash \{\emptyset\}$. On the other hand, $L(\mathbb{T}^\omega)$ is the space in which X is represented as the following proposition shows.

Proposition 2.3. *Suppose that S is a dyadic subbase of a space X.*
(1) *$S(\tilde{x}) = \{x\}$ for all $x \in X$.*
(2) *$\widetilde{X} \subseteq L(\mathbb{T}^\omega)$.*

Proof.
(1) Let x, y be distinct elements in X. Since X is T_1, there exists $e \in \mathbb{T}^*$ such that $x \in S(e)$ and $y \notin S(e)$. From (2.3), we have $e \sqsubseteq \tilde{x}$ and $e \not\sqsubseteq \tilde{y}$. So we get $\tilde{x} \not\sqsubseteq \tilde{y}$, therefore, $y \notin S(\tilde{x})$.
(2) Suppose that $\mathrm{dom}(\tilde{x})$ is finite. Then $S(\tilde{x})$ is an open set and thus $\{x\}$ is a clopen set which contradicts the fact that x is on the boundary of $S_{n,a}$ for $n \notin \mathrm{dom}(\tilde{x})$. □

Definition 2.4. We say that a dyadic subbase S is *proper* if $\mathrm{cl}\, S(e) = \bar{S}(e)$ for every $e \in \mathbb{T}^*$.

If S is a proper dyadic subbase, then $\bar{S}(e)$ is the closure of the base element $S(e)$. Therefore, by (2.4), the sequence $\tilde{x}$ codes not only base elements to which x belongs but also base elements to whose closure x belongs.

Proposition 2.5. *Suppose that S is a proper dyadic subbase of a space X.*
(1) *If $x \in X$ and $p \sqsupseteq \tilde{x}$, then the family $\{S(e) : e \in K_p\}$ is a filter base which converges to $x \in X$.*
(2) *If $x \neq y \in X$, then $x \in S_{n,a}$ and $y \in S_{n,1-a}$ for some $n \in \mathbb{N}$ and $a \in \mathbb{2}$. That is, x and y are separated by some subbase element.*
(3) *If $x \in X$ and $p \sqsupseteq \tilde{x}$, then $\bar{S}(p) = \{x\}$.*
(4) *If $p \in \mathbb{2}^\omega$, then $\bar{S}(p)$ is either a one-point set $\{x\}$ for some $x \in X$ or the empty set.*

Proof.
(1) Since we have $\tilde{x} \uparrow e$ for every $e \in K_p$, we obtain $\mathrm{cl}\, S(e) = \bar{S}(e) \neq \emptyset$. Therefore, we get $\emptyset \notin \{S(e) : e \in K_p\}$.
(2) Since X is Hausdorff, there is $e \in \mathbb{T}^*$ such that $x \in S(e)$ and $y \notin \mathrm{cl}\, S(e) = \bar{S}(e)$. That is, $e \sqsubseteq \tilde{x}$ and $e \not\uparrow \tilde{y}$ by (2.3) and (2.4). Therefore, we get $\tilde{x} \not\uparrow \tilde{y}$.
(3) From (2), we have $\bar{S}(\tilde{x}) = \{x\}$. We get $\bar{S}(p) \subseteq \bar{S}(\tilde{x}) = \{x\}$ from $p \sqsupseteq \tilde{x}$, and $\bar{S}(p) \ni x$ from $p \uparrow \tilde{x}$.
(4) Let $p \in \mathbb{2}^\omega$. If $p \sqsupseteq \tilde{x}$ for some $x \in X$, then $\bar{S}(p)$ is a one-point set $\{x\}$ by (3). If $p \not\sqsupseteq \tilde{x}$ for all $x \in X$, then $p \not\uparrow \tilde{x}$, because p is maximal. Therefore, $\bar{S}(p)$ is empty. □

[20] contains an example of a non-proper dyadic subbase for which Proposition 2.5 (1) to (4) do not hold.

Finally, we define a property of a dyadic subbase which is stronger than properness.

Definition 2.6. An *independent subbase* is a dyadic subbase such that $S(e)$ is not empty for every $e \in \mathbb{T}^*$.

Proposition 2.7 ([12])**.** *An independent subbase is proper.*

The Gray subbase in Example 2.2 is an independent subbase and we show many independent subbases as examples of proper dyadic subbases. When S is an independent subbase, we have $S(d) \supseteq S(e)$ if and only if $d \sqsubseteq e$. In particular, $S(d) \neq S(e)$ if $d \neq e$. Therefore, for an independent subbase S, the poset $(\mathfrak{B}_S, \supseteq)$ ordered by reverse inclusion is isomorphic to $\mathbb{T}^*$.

3. Representations and domain representations derived from dyadic subbases

We study some representations and domain representations of a space X derived from a (proper) dyadic subbase of X.

We introduce two representations. The first one is immediately derived from a dyadic subbase. If S is a dyadic subbase of X, then the inverse φ_S^{-1} of the embedding φ_S is a representation of X with the alphabet $\Gamma = \{0, 1, \perp\}$ where $\perp$ is considered as an ordinary character of Γ. Each point is represented uniquely with this representation and it is easy to show that $\varphi_S^{-1} :\subseteq \Gamma^\omega \to X$ is an admissible representation if and only if $S_{n,\perp} = \varnothing$ for every n.

The second one is derived from a proper dyadic subbase. If S is a proper dyadic subbase of X, from Proposition 2.5 (3), we have a map ρ_S from $\uparrow\widetilde{X} \subset \mathbb{T}^\omega$ to X such that $\rho_S(p)$ is the unique element in $\bar{S}(p)$. In particular, from Proposition 2.5 (4), ρ_S restricted to the maximal elements $\mathbb{2}^\omega$ is a partial surjective map from $\mathbb{2}^\omega$ to X, that is, it is a representation of X which we denote by ρ'_S.

Example 3.1. For the Gray subbase G of $\mathbb{I}$, ρ'_G is a total function from $\mathbb{2}^\omega$ to $\mathbb{I}$ which is called the Gray expansion of $\mathbb{I}$ [18]. ρ'_G is equivalent to the binary expansion through simple conversion functions.

As this example suggests, we consider that ρ'_S is a generalization of the binary expansion representation to a proper dyadic subbase S. We study its continuity in Proposition 3.4. It is not admissible in general as the following proposition shows.

Proposition 3.2. *Suppose that S is a proper dyadic subbase of a space X. ρ'_S is admissible if and only if $S_{n,\perp} = \varnothing$ for every n.*

Proof. Only if part: suppose that ρ'_S is admissible and $x \in S_{n,\perp}$. Theorem 12 of [4] says that every admissible representation has a continuously equivalent open restriction. Suppose that $\delta :\subseteq \mathbb{2}^\omega \to X$ is such an open restriction of ρ'_S and $x = \delta(p)$. Let $a = p(n)$. Since δ is an open map, $\delta(\{q \in \mathbb{2}^\omega : q(n) = a\})$ is an open neighbourhood of x, and since δ is a restriction of ρ'_S, $\delta(\{q \in \mathbb{2}^\omega : q(n) = a\}) \subseteq \rho'_S(\{q \in \mathbb{2}^\omega : q(n) = a\}) = S_{n,a} \cup S_{n,\perp}$. Therefore, $S_{n,a} \cup S_{n,\perp}$ is a neighbourhood of x, which contradicts with $x \in S_{n,\perp}$.

If part: since the base $\mathfrak{B}_S$ is composed of closed and open sets, X is regular and therefore ρ'_S is continuous by Proposition 3.4 below. Since $S_{n,\perp}$ is empty, $x \in S_{n,0}$ or $x \in S_{n,1}$ holds for every $x \in X$. Therefore, one can construct a reduction from the standard representation of X with respect to an enumeration of the subbase S to ρ'_S. □

Next, we study domain representations. We start with a general construction of a domain representation from a base of a space. Suppose that $\mathfrak{B}$ is a base of a space X such that $\emptyset \notin \mathfrak{B}$, $X \in \mathfrak{B}$, and $\mathfrak{B}$ is closed under finite non-empty intersection. For the domain $D_{\mathfrak{B}}$ obtained as the ideal completion of the poset $(\mathfrak{B}, \supseteq)$ with the reverse inclusion and for the map $\iota(x) = \{U \in \mathfrak{B} : x \in U\}$ from X to $D_{\mathfrak{B}}$, ι is a homeomorphic embedding of X in $D_{\mathfrak{B}}$. Therefore, $(D_{\mathfrak{B}}, \iota(X), \iota^{-1})$ is a homeomorphic domain representation which is known to be admissible [3, 9, 17].

We introduce two domain representations derived from (proper) dyadic subbases. The first one is $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$, which is defined for a space X with a dyadic subbase S. Since φ_S is an embedding, it is a homeomorphic domain representation. In particular, if S is an independent subbase, then the poset $\mathfrak{B}_S$ is isomorphic to the poset $\mathbb{T}^*$. Therefore, the domain $D_{\mathfrak{B}_S}$ is isomorphic to $\mathbb{T}^\omega$ and thus the domain representations $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$ and $(D_{\mathfrak{B}_S}, \iota(X), \iota^{-1})$ coincide. However, if S is a dyadic subbase which is not independent, then the poset $\mathbb{T}^*$ provides only a "notation" of the base $\mathfrak{B}_S$, and a set $S(d)$ may be the same as $S(e)$ for $d \neq e \in \mathbb{T}^*$. We show that $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$ is an admissible domain representation even for this case.

Proposition 3.3. *If S is a dyadic subbase of a space X, then $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$ is an admissible domain representation.*

Proof. Suppose that E^R is a subset of a domain E and μ is a continuous map from E^R to X. Define a function $\phi : K(E) \to \mathbb{T}^\omega$ as $\phi(e)(n) = a$ if and only if $\mu(\uparrow e \cap E^R) \subseteq S_{n,a}$. Since ϕ is monotonic, it has a continuous extension to E, which is a continuous function from E to $\mathbb{T}^\omega$. It is also denoted by ϕ. We show that the function ϕ satisfies $\varphi_S^{-1}(\phi(p)) = \mu(p)$ for $p \in E^R$. We have $\phi(p)(n) = \sqcup_{e \in K_p} \phi(e)(n)$. Therefore, for $a \in \mathbb{2}$, $\phi(p)(n) = a$ if and only if $(\exists e \in K_p)(\phi(e)(n) = a)$, if and only if $(\exists e \in K_p)(\mu(\uparrow e \cap E^R) \subseteq S_{n,a})$, if and only if $\mu(p) \in S_{n,a}$. Therefore, $\phi(p) = \varphi_S(\mu(p))$. □

The other domain representation is $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$, which is defined for a regular space X with a proper dyadic subbase S. Suppose that S is a proper dyadic subbase of a space X. From Proposition 2.5, ρ_S is a map from $\uparrow\widetilde{X}$ to X. We have $\varphi_S(\rho_S(p)) \sqsubseteq p$ and $\rho_S(\varphi_S(x)) = x$. Therefore, $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$ is an upwards-closed retract domain representation if and only if ρ_S is a quotient map. Blanck showed in Theorem 5.10 of [3] that if a topological space has an upwards-closed retract Scott domain representation, then it is a regular Hausdorff space. Therefore, $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$ is a domain representation only if X is regular. We show this fact as a corollary to the following equivalence.

Proposition 3.4. *Let S be a proper dyadic subbase of a space X. The followings are equivalent.*

(1) *X is regular.*
(2) *$\rho_S : \uparrow\widetilde{X} \to X$ is continuous.*
(3) *$\rho'_S :\subseteq \mathbb{2}^\omega \to X$ is continuous.*

Proof.

$(1 \Rightarrow 2)$: Let $p \in \uparrow\widetilde{X}$ and $x = \rho_S(p)$. Since $\{S(\tilde{x}|_n) : n \in \mathbb{N}\}$ is a neighbourhood base of x in X and $\{\uparrow p|_m \cap \uparrow\widetilde{X} : m \in \mathbb{N}\}$ is a neighbourhood base of p in $\uparrow\widetilde{X}$, it suffices to show that for every n, there is m such that $\rho_S(\uparrow p|_m \cap \uparrow \widetilde{X}) \subseteq S(\tilde{x}|_n)$. Since X is regular, there is $m > n$ such that $x \in S(\tilde{x}|_m) \subseteq \operatorname{cl} S(\tilde{x}|_m) \subseteq S(\tilde{x}|_n)$. Then, for all $q \in \uparrow \widetilde{X}$ such that $q \sqsupseteq p|_m$, we have $\varphi_S(\rho_S(q)) \uparrow \tilde{x}|_m$ because $\varphi_S(\rho_S(q)) \sqsubseteq q \sqsupseteq p|_m \sqsupseteq \tilde{x}|_m$. Thus, $\rho_S(q) \in \bar{S}(\tilde{x}|_m)$. Therefore, $\rho_S(q) \in \bar{S}(\tilde{x}|_m) = \operatorname{cl} S(\tilde{x}|_m) \subseteq S(\tilde{x}|_n)$.

(2 ⇒ 3): Immediate.

(3 ⇒ 1): Suppose that $x \in X$ and $n \in \mathbb{N}$. For each $p \in \uparrow\tilde{x} \cap \mathbb{2}^\omega$, since ρ'_S is continuous on p, there exists $e_p \in K_p$ such that $\rho'_S(\uparrow e_p \cap \mathbb{2}^\omega) \subseteq S(\tilde{x}|_n)$. It means that $\bar{S}(e_p) \subseteq S(\tilde{x}|_n)$. Here, we can assume that $\tilde{x}|_{|e_p|} \sqsubseteq e_p$ by replacing e_p with $e_p \sqcup \tilde{x}|_{|e_p|}$. Note that $\uparrow e_p \cap \mathbb{2}^\omega$ for $p \in \uparrow\tilde{x} \cap \mathbb{2}^\omega$ is an open cover of $\uparrow\tilde{x} \cap \mathbb{2}^\omega$ and $\uparrow\tilde{x} \cap \mathbb{2}^\omega$ is compact because it is homeomorphic to $\mathbb{2}^j$ for some $j \leqslant \omega$. Therefore, for some finite subset $\{p_0, \dots, p_{h-1}\}$ of $\uparrow\tilde{x} \cap \mathbb{2}^\omega$, we have $\cup_{i<h}\uparrow e_{p_i} \supseteq \uparrow\tilde{x} \cap \mathbb{2}^\omega$. Let m be the maximal length of e_{p_i} for $i < h$ and let $l = m - |\operatorname{dom}(x|_m)|$. Let $d_0, \dots, d_{2^l-1}$ be sequences of length m obtained by filling the first l bottoms of $\tilde{x}|_m$ with 0 and 1. We have $(\uparrow d_0 \cup \dots \cup \uparrow d_{2^l-1}) \cap \mathbb{2}^\omega = \uparrow\tilde{x}|_m \cap \mathbb{2}^\omega$. Therefore, $\cup_{i<2^l-1}\bar{S}(d_i) = \bar{S}(\tilde{x}|_m)$. On the other hand, for each $i < 2^l$, there is $j < h$ such that $d_i \sqsupseteq e_{p_j}$. Therefore, $\bar{S}(d_i) \subseteq \bar{S}(e_{p_j}) \subseteq S(\tilde{x}|_n)$. Thus, we have $\bar{S}(\tilde{x}|_m) \subseteq S(\tilde{x}|_n)$. Since $\bar{S}(\tilde{x}|_m) = \operatorname{cl} S(\tilde{x}|_m)$, it means that X is a regular space. □

Corollary 3.5. *Suppose that S is a proper dyadic subbase of a space X. The triple $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$ is a domain representation if and only if X is regular. In this case, it is an admissible retract domain representation.*

Proof. Suppose that X is regular. From Proposition 3.4, ρ_S is a retraction with right inverse φ_S. Therefore, ρ_S is a quotient map. Since $\widetilde{X} \subseteq \uparrow\widetilde{X}$ and φ_S^{-1} is a restriction of ρ_S to $\widetilde{X}$, the identity map on $\mathbb{T}^\omega$ is a reduction map from the admissible domain representation $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$ to $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$. □

4. Domains D_S and $\widehat{D}_S$

In the previous section, we studied domain representations in the domain $\mathbb{T}^\omega$. In the following sections, we study domain representations in subdomains D_S and $\widehat{D}_S$ of $\mathbb{T}^\omega$. Before that, we consider the domain E_S which is defined as the closure of $\widetilde{X}$ in $\mathbb{T}^\omega$. It is easy to show that the triple $(E_S, \widetilde{X}, \varphi_S^{-1})$ is a dense domain representation of X and, if in addition S is proper and X is regular, then $(E_S, \uparrow\widetilde{X}, \rho_S)$ is a dense admissible retract domain representation of X. In these domain representations, we have $S(e) \neq \varnothing$ for every $e \in K(E_S)$ and the family $\{S(e) : e \in K_p\}$ forms a filter base for every $p \in L(E_S)$. In this sense, one can say that E_S does not contain superfluous elements. However, E_S is identical to $\mathbb{T}^\omega$ if S is an independent subbase and the domain E_S does not have information about X in this case. We consider further restrictions of $\mathbb{T}^\omega$ and define the domains D_S and $\widehat{D}_S$ as follows.

Definition 4.1. Let S be a dyadic subbase of a space X.

(1) We define the poset $K_S \subseteq \mathbb{T}^*$ as

$$K_S = \{p|_m : p \in \widetilde{X}, m \in \mathbb{N}\}$$

and define $D_S = \operatorname{Idl}(K_S)$.

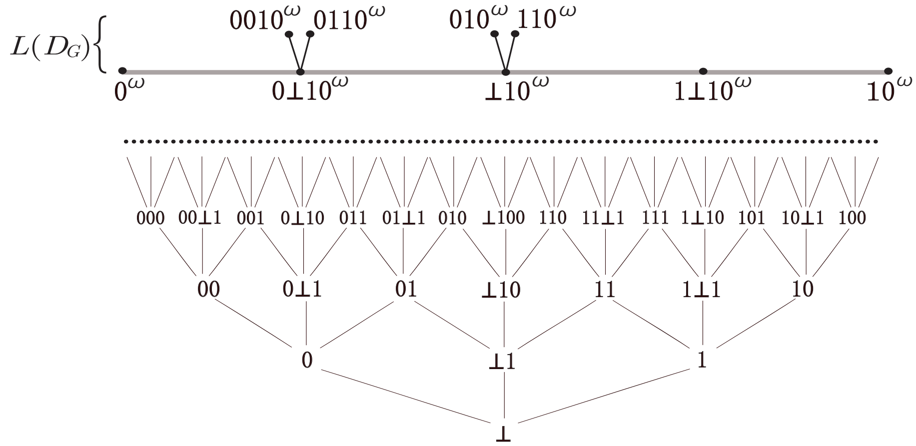

Figure 2: The domain D_G.

(2) We define the poset $\widehat{K}_S \subseteq \mathbb{T}^*$ as

$$\widehat{K}_S = \{p|_m : p \in \uparrow\widetilde{X}, m \in \mathbb{N}\}$$

and define $\widehat{D}_S = \mathrm{Idl}(\widehat{K}_S)$.

For the Gray-subbase G of $\mathbb{I}$, we have $\bot 1 \in K_G$ because $\varphi_G(1/2) = \bot 10^\omega$, but $\bot 0 \notin K_G$ and $\bot\bot 1 \notin K_G$. Figure 2 shows the structure of $D_G = \widehat{D}_G$.

We have $K_S \subseteq \widehat{K}_S \subseteq \mathbb{T}^*$ and $D_S \subseteq \widehat{D}_S \subseteq \mathbb{T}^\omega$ for a dyadic subbase S. We also have $\tilde{X} \subseteq D_S$ and $\uparrow\tilde{X} \subseteq \widehat{D}_S$.

Proposition 4.2.
(1) *If S is a dyadic subbase of a space X, then $\widetilde{X}$ is dense in D_S.*
(2) *If S is a proper dyadic subbase of a space X, then $\widetilde{X}$ is dense in $\widehat{D}_S$.*

Proof.
(1) $\widetilde{X}$ is dense in D_S because $S(e)$ is not empty for every $e \in K_S$.
(2) By Proposition 2.5(1), $S(e)$ is not empty for every $e \in \widehat{K}_S$. □

The domains D_S and $\widehat{D}_S$ are not equal in general as the following example shows.

Example 4.3. Let W be the space obtained by glueing four copies of $\mathbb{I}$ at one of the endpoints. That is, $W = \mathbb{2} \times \mathbb{2} \times \mathbb{I}/\!\sim$ for $\sim$ the equivalence relation identifying $(a, b, 0)$ for $a, b \in \mathbb{2}$. Let z be the identified point. That is, $z = [(a, b, 0)]$ for $a, b \in \mathbb{2}$. Let R be the dyadic subbase defined as

$$R_{0,c} = \{c\} \times \mathbb{2} \times (0, 1]/\sim,$$
$$R_{1,c} = \mathbb{2} \times \{c\} \times (0, 1]/\sim,$$
$$R_{n+2,c} = \mathbb{2} \times \mathbb{2} \times G_{n,c}/\sim,$$

for $n \in \mathbb{N}$ and $c \in \mathbb{2}$. We have $\tilde{z} = \bot\bot 0^\omega \in L(D_R)$ and $ab0^\omega \in L(D_R)$ for $a, b \in \mathbb{2}$. However, $a\bot 0^\omega \notin L(D_R)$ for $a \in \mathbb{2}$ and $\bot b0^\omega \notin L(D_R)$ for $b \in \mathbb{2}$. On the other hand, we have $a\bot 0^\omega \in L(\widehat{D}_R)$ for $a \in \mathbb{2}$ and $\bot b0^\omega \in L(\widehat{D}_R)$ for $b \in \mathbb{2}$.

Proposition 4.4. *If S is a dyadic subbase of a space X, then $\widehat{K}_S$ is a cusl and therefore $\widehat{D}_S$ is a bounded complete domain.*

Proof. Let $d = p|_m, e = q|_n \in \widehat{K}_S$ for $p, q \in \uparrow\widetilde{X}$ and $m \leqslant n \in \mathbb{N}$. Suppose that $d \uparrow e$ in $\widehat{K}_S$ and let $f = d \sqcup e$ be their least upper bound in $\mathbb{T}^*$. Then, since d and $q|_n$ are compatible and $|d| \leqslant n$, $d \uparrow q$ in $\mathbb{T}^\omega$ and $r = d \sqcup q$ satisfies $r|_n = f$. Since $r \sqsupseteq q \sqsupseteq \tilde{x}$ for some $x \in X$, we have $f \in \widehat{K}_S$. □

Proposition 4.5.

- *If S is a dyadic subbase of a space X, then the domain representation $(\widehat{D}_S, \widetilde{X}, \varphi_S^{-1})$ is admissible.*
- *If S is a proper dyadic subbase of a regular space X, then the domain representation $(\widehat{D}_S, \uparrow\widetilde{X}, \rho_S)$ is admissible.*

Proof.

(1) We show that there is a reduction from the admissible domain representation $(\mathbb{T}^\omega, \widetilde{X}, \varphi_S^{-1})$ to $(\widehat{D}_S, \widetilde{X}, \varphi_S^{-1})$. Since $\widehat{D}_S$ is bounded complete, we define $\phi : \mathbb{T}^\omega \to \widehat{D}_S$ as $\phi(p) = \sqcup\{e \in \widehat{K}_S : e \sqsubseteq p\}$. It preserves $\widetilde{X}$ because $\{S(e) : e \in \widehat{K}_S, e \sqsubseteq \tilde{x}\}$ contains $S(\tilde{x}|_n)$ for every n.

(2) The map ϕ preserves $\uparrow\widetilde{X}$ and it is a reduction also from $(\mathbb{T}^\omega, \uparrow\widetilde{X}, \rho_S)$ to $(\widehat{D}_S, \uparrow\widetilde{X}, \rho_S)$. □

As Example 4.6 shows, D_S is not bounded complete in general. It is left open whether the results corresponding to Proposition 4.5 hold for D_S.

Example 4.6. Let Y be the space obtained by glueing $1/4$ and $3/4$ in $\mathbb{I}$. That is, Y is the quotient space of $\mathbb{I}$ with the equivalence relation generated by $1/4 \sim 3/4$. Let T be the independent subbase of Y such that $T_{0,0} = (G_{0,0}\backslash\{1/4\})/\sim$, $T_{0,1} = (G_{0,1}\backslash\{3/4\})/\sim$, and $T_{n,a} = G_{n,a}/\sim$ for $n > 0$. We have $\varphi_T(z) = \bot\bot 10^\omega$ for $z = [1/4] = [3/4]$ and $\varphi_T([x]) = \varphi_G(x)$ for $x \notin \{1/4, 3/4\}$. Therefore, K_S contains $\bot\bot 1$ and $\bot 1 (= \varphi_T([1/2])|_2)$, which are bounded above by $011 = \varphi_T([1/3])|_3$ and $111 = \varphi_T([2/3])|_3$. However, $\bot 11$, which is the least upper bound of $\bot\bot 1$ and $\bot 1$ in $\mathbb{T}^\omega$, does not belong to K_T. Therefore, K_T is not a cusl and D_T is not a bounded complete domain. Note that the poset $\widehat{K}_T$ contains $\bot 11$ because $\bot 110^\omega \sqsupseteq \varphi_T(z)$.

In Example 4.6, $\uparrow\varphi_T(z)$ in D_T is the set $\{\bot\bot p, 00p, 01p, 10p, 11p, 1\bot p, 0\bot p\}$ for $p = 10^\omega$. Therefore, it is different from $\uparrow\varphi_T(z)$ in $\mathbb{T}^\omega$ which contains also $\bot 0p$ and $\bot 1p$. As Example 4.6 and 4.3 show, $L(D_S) \subsetneq L(\widehat{D}_S)$ in general. However, for a proper dyadic subbase S, D_S and $\widehat{D}_S$ coincide on the top elements as Proposition 4.8 shows.

Lemma 4.7. *Let S be a dyadic subbase of a space X.*

(1) *For $p \in D_S$ and $n \in \mathbb{N}$, we have $p|_n \in K_S$.*

(2) *For $p \in \widehat{D}_S$ and $n \in \mathbb{N}$, we have $p|_n \in \widehat{K}_S$.*

Proof.

(1) Suppose that p is the least upper bound of an ideal $\{\tilde{x}_i|_{m_i} : i \in I\}$. Then, $p|_n$ is the least upper bound of the ideal $\{\tilde{x}_i|_{m_i}|_n : i \in I\} = \{\tilde{x}_i|_{\min\{m_i,n\}} : i \in I\}$, whose length is no more than n.

(2) It is proved similarly to (1). □

Proposition 4.8. *For a proper dyadic subbase S of a space X, $D_S \cap \mathbb{2}^\omega = \widehat{D}_S \cap \mathbb{2}^\omega$.*

Proof. Let $p \in \widehat{D}_S \cap \mathbb{2}^\omega$ and $n \in \mathbb{N}$. By Lemma 4.7, we have $p|_n \in \widehat{K}_S$. Therefore, $S(p|_n) \neq \emptyset$ by Proposition 4.2(2). For every $y \in S(p|_n)$, we have $p|_n = \tilde{y}|_n$ since $p \in \mathbb{2}^\omega$. Thus, $p|_n \in K_S$ for every $n \in \mathbb{N}$ and we have $p \in D_S$. □

5. Domains with minimal-limit sets

We study structures of D_S and $\widehat{D}_S$ and present a condition on X which ensures the existence of minimal elements of $L(D_S)$ and $L(\widehat{D}_S)$.

Definition 5.1. Let P be a poset.

(1) $x \in P$ is a *minimal element* if $y \sqsubseteq x$ implies $y = x$ for all $y \in P$. We write $\min(P)$ for the set of all minimal elements of P.
(2) We say that P has *enough minimal elements* if, for all $y \in P$, there exists $x \in \min(P)$ such that $x \sqsubseteq y$.
(3) For a domain D, if $L(D)$ has enough minimal elements, we call $\min(L(D))$ the *minimal-limit set* of D.

The poset $L(\mathbb{T}^\omega)$ does not have enough minimal elements.

Definition 5.2.

(1) Let $(P, \sqsubseteq)$ be a pointed poset with the least element $\bot_P$. The *level* of $x \in P$, if it exists, is the maximal length n of a chain $\bot_P = y_0 \sqsubsetneq y_1 \sqsubsetneq \ldots \sqsubsetneq y_n = x$, and it is denoted by $\mathrm{level}(x)$.
(2) A poset P is *stratified* if it is pointed and every element of P has a level.
(3) We say that y is an *immediate successor* of x if $x \sqsubsetneq y$ and there is no element z such that $x \sqsubsetneq z \sqsubsetneq y$. We write $\mathrm{succ}(x)$ for the set of immediate successors of x.
(4) We say that a stratified poset P is *finite-branching* if $\mathrm{succ}(x)$ is finite for every $x \in P$.

In [19], the following proposition is proved with a slightly stronger definition of finite-branchingness that contains the condition $\mathrm{level}(y) = \mathrm{level}(x) + 1$ for $y \in \mathrm{succ}(x)$. However, one can check that this condition is not used in the proof and it holds with our definition of finite-branchingness.

Proposition 5.3 (Proposition 4.13 of [19])**.** *If D is a domain such that $K(D)$ is finite-branching, then $L(D)$ has enough minimal elements and $\min(L(D))$ is compact.* □

Definition 5.4.

(1) We say that a space X is *adhesive* if X has at least two points and closures of any two non-empty open sets have non-empty intersection.
(2) We say that X is *nonadhesive* if it is not adhesive.
(3) We say that X is *strongly nonadhesive* if every open subspace is nonadhesive.

Nonadhesiveness (and even strongly nonadhesiveness) is a weak condition that many of the Hausdorff spaces satisfy. A space is called Urysohn (or completely Hausdorff or $T_{2\frac{1}{2}}$ in some literature) if any two distinct points can be separated by closed neighbourhoods. A regular space is always Urysohn.

Proposition 5.5. *Every Urysohn space is strongly nonadhesive.* □

Note that there is an adhesive Hausdorff space as the following example shows.

Example 5.6. Let P be the set of dyadic irrational numbers in $\mathbb{I} = [0, 1]$ and $\mathbb{N}^+$ be the set of positive integers. We define our space $A = P \cup \mathbb{N}^+$. A neighbourhood base of $x \in P$ is $U \cap P$ for U a Euclidean neighbourhood of $x \in \mathbb{I}$. A neighbourhood base of $n \in \mathbb{N}^+$ is the union of $\{n\}$ and $U \cap P$ for U a Euclidean neighbourhood of $\{k/2^n : k \text{ is an odd number}, 0 < k < 2^n\}$. One can easily verify that A is Hausdorff. The closure of $U \cap P$ is $(U \cap P) \cup \{n \in \mathbb{N}^+ : k/2^n \in U \text{ for some odd number } k\}$ and it contains $\{n \in \mathbb{N}^+ : n > m\}$ for some m. Therefore A is adhesive. The space A has the following independent subbase S^A.

$$S^A_{n,a} = (G_{n,a} \cap P) \cup \{n \in \mathbb{N}^+ : k/2^n \in G_{n,a} \text{ for all odd number } k < 2^n \}.$$

We have $\varphi_{S^A}(x) = \varphi_G(x)$ for $x \in P$, and $\varphi_{S^A}(n) = \perp^n 10^\omega$ for $n \in \mathbb{N}^+$.

As Propositions 5.7 and 5.8 show, adhesiveness of X and finite-branchingness of D_S are closely related. Recall again that a (non)adhesive space means a (non)adhesive Hausdorff space.

Proposition 5.7. *Suppose that X is an adhesive space and S is a proper dyadic subbase of X such that $S_{n,a} \neq \emptyset$ for every $n \in \mathbb{N}$ and $a \in \mathbb{2}$. Then, $\mathrm{succ}(\perp^\omega)$ in K_S is infinite. Therefore, K_S is not finite-branching.*

Proof. All the elements of $\mathrm{succ}(\perp^\omega)$ have the form $\perp^k a$ for $k \in \mathbb{N}$ and $a \in \mathbb{2}$. Suppose that $\mathrm{succ}(\perp^\omega)$ is finite and let $\perp^{n-1}a$ be an element with the maximal length. For $b = 1-a$, take $x \in S_{n,b}$ and $p \in \uparrow\tilde{x} \cap \mathbb{2}^\omega$. For $d = p|_n$, $d \in \widehat{K}_S$ holds and therefore $S(d) \neq \emptyset$ by Proposition 4.2. Let $e \in \mathbb{2}^n$ be the bitwise complement of d. Since $\perp^{n-1}a \in K_S$, $e \in \widehat{K}_S$ and therefore $S(e) \neq \emptyset$ by Proposition 4.2. Therefore, closures of $S(d)$ and $S(e)$ intersect. Since S is proper, $\mathrm{cl}\, S(d) = \bar{S}(d)$ and $\mathrm{cl}\, S(e) = \bar{S}(e)$. Therefore, there exists $y \in \bar{S}(d) \cap \bar{S}(e)$. Since $\tilde{y}|_n = \perp^\omega$, the smallest index of digits in $\tilde{y}$ is greater than n, and we have contradiction. □

For the independent subbase S^A of A in Example 5.6, $\mathrm{succ}(\perp^\omega) = \{\perp^k 1 : k \in \mathbb{N}\}$.

Proposition 5.8. *Suppose that X is a nonadhesive space and S is a proper dyadic subbase of X. Then, $\mathrm{succ}(\perp^\omega)$ in K_S is finite.*

Proof. Since S is nonadhesive, for some $p, q \in \mathbb{T}^*$, $S(p) \neq \emptyset$, $S(q) \neq \emptyset$, and $\mathrm{cl}\, S(p) \cap \mathrm{cl}\, S(q) = \emptyset$ hold. Since S is proper, $\bar{S}(p) \cap \bar{S}(q) = \emptyset$. Let $n = \max\{|p|, |q|\}$. If $\tilde{x}|_n = \perp^\omega$ for some $x \in X$, then $x \in \bar{S}(p)$ and $x \in \bar{S}(q)$ and we have contradiction. Thus, in K_S, $\mathrm{succ}(\perp^\omega) \subseteq \{\perp^k a \perp^\omega : k < n, a \in \mathbb{2}\}$. □

Lemma 5.9. *Suppose that S is a proper dyadic subbase of a space X and $e \in \mathbb{T}^*$. Let ν be an enumeration of $\mathbb{N} \setminus \mathrm{dom}(e)$. Then,*

$$T_{n,a} = S_{\nu(n),a} \cap S(e) \quad (n \in \mathbb{N}, a \in \mathbb{2})$$

is a proper dyadic subbase of $S(e)$. □

Proof. Let A be the regular open set $S(e)$. First, note that if P is a regular open subset of X, then $A \cap P$ is a regular open subset of A and $\mathrm{ext}_A(A \cap P) = A \cap \mathrm{ext}_X P$. Therefore, T

is a dyadic subbase. Note also that $\mathrm{cl}_A(A \cap P) = A \cap \mathrm{cl}_X P$. Therefore, for $d \in \mathbb{T}^*$, we have

$$\begin{aligned}\mathrm{cl}_A \bigcap_{k \in \mathrm{dom}(d)} T_{k,d(k)} &= \mathrm{cl}_A(A \cap \bigcap_{k \in \mathrm{dom}(d)} S_{\nu(k),d(k)}) \\ &= A \cap \mathrm{cl}_X \bigcap_{k \in \mathrm{dom}(d)} S_{\nu(k),d(k)} \\ &= A \cap \bigcap_{k \in \mathrm{dom}(d)} \mathrm{cl}_X S_{\nu(k),d(k)} \\ &= \bigcap_{k \in \mathrm{dom}(d)} (A \cap \mathrm{cl}_X S_{\nu(k),d(k)}) \\ &= \bigcap_{k \in \mathrm{dom}(d)} \mathrm{cl}_A(A \cap S_{\nu(k),d(k)}) \\ &= \bigcap_{k \in \mathrm{dom}(d)} \mathrm{cl}_A T_{k,d(k)}.\end{aligned}$$

Therefore, T is proper. □

Proposition 5.10. *Suppose that X is a strongly nonadhesive space and S is a proper dyadic subbase of X.*
(1) *The poset K_S is finite-branching.*
(2) *The poset $\widehat{K}_S$ is finite-branching.*

Proof.
(1) Let $e \in \mathbb{T}^*$. By applying Proposition 5.8 to the proper dyadic subbase T on $S(e)$ in Lemma 5.9, $\mathrm{succ}(\perp)$ is finite in the poset K_T. Since K_T is identical to $\uparrow e$ in K_S, $\mathrm{succ}(e)$ is finite in K_S.
(2) In this proof, $\mathrm{succ}(d)$ for $d \in K_S$ means $\mathrm{succ}(d)$ in K_S. Let $e \in \widehat{K}_S$ and let k be the maximal length of elements in $\cup_{d \in \downarrow e \cap K_S} \mathrm{succ}(d)$, which exists by (1). Suppose that, for some $n \geqslant k$ and $a \in \mathbb{2}$, $e[n := a] \in \widehat{K}_S$. Then, for some $x \in X$ and $p \sqsupseteq \tilde{x}$, $p|_{n+1} = e[n := a]$. Therefore, $\tilde{x}|_n \sqsubseteq e$. For $d_0 = \tilde{x}|_n$, let $m > n$ be the least integer such that $\tilde{x}|_m \sqsupsetneq d_0$. The set $\mathrm{succ}(d_0)$ contains $\tilde{x}|_m$ and we have contradiction. □

Theorem 5.11. *Suppose that X is a strongly nonadhesive space and S is a proper dyadic subbase of X.*
(1) *$L(D_S)$ has enough minimal elements and $\min(L(D_S))$ is compact.*
(2) *$L(\widehat{D}_S)$ has enough minimal elements and $\min(L(\widehat{D}_S))$ is compact.*

Proof. From Proposition 5.3 and 5.10. □

Note that, as Proposition 5.5 shows, Theorem 5.11 is applicable to all the Urysohn spaces, in particular, to regular spaces. Note also that the premise of Theorem 5.11 is not a necessary condition for $L(D_S)$ to have enough minimal elements. For example, for the space A and the dyadic subbase S^A in Example 5.6, the domain D_{S^A} has enough minimal elements and $\varphi_{S^A}(A) \subseteq \min(L(D_{S^A}))$.

It is shown in [21] that there is a Hausdorff space X and an independent subbase S of X such that D_S is equal to $\mathbb{T}^\omega$ and therefore $L(D_S)$ does not have enough minimal elements.

6. Domain representations in minimal-limit sets

Now, we show that X is embedded in $\min(L(D_S))$ and $\min(L(\widehat{D}_S))$ for the case S is a proper dyadic subbase of a regular space X. We start with new notations and a small lemma.

Definition 6.1. For a dyadic subbase S of a space X, $p \in \mathbb{T}^\omega$, and $n \in \mathbb{N}$, we define $S^n_{\mathrm{ex}}(p) \subseteq X$ and $\bar{S}^n_{\mathrm{ex}}(p) \subseteq X$ as follows.

$$S^n_{\mathrm{ex}}(p) = \bigcap_{k<n} S_{k,p(k)} = S(p|_n) \cap \bigcap_{\substack{k < n, \\ k \notin \mathrm{dom}(p)}} S_{k,\perp},$$

$$\bar{S}^n_{\mathrm{ex}}(p) = \bigcap_{k<n} \mathrm{cl}\, S_{k,p(k)} = \bar{S}(p|_n) \cap \bigcap_{\substack{k < n, \\ k \notin \mathrm{dom}(p)}} S_{k,\perp}.$$

Lemma 6.2. *Let $e \in \mathbb{T}^*$ and $n = |e|$.*
(1) *$S^n_{\mathrm{ex}}(e) \neq \varnothing$ if and only if $e \in K_S$.*
(2) *$\bar{S}^n_{\mathrm{ex}}(e) \neq \varnothing$ if and only if $e \in \widehat{K}_S$.*

Proof.
(1) $x \in S^n_{\mathrm{ex}}(e)$ if and only if $e = \tilde{x}|_n$.
(2) $x \in \bar{S}^n_{\mathrm{ex}}(e)$ if and only if $\tilde{x}|_n \sqsubseteq e$ if and only if there exists $q \sqsupseteq \tilde{x}$ such that $e = q|_n$. □

Theorem 6.3. *Suppose that S is a proper dyadic subbase of a regular space X and $D \in \{D_S, \widehat{D}_S\}$. If $p \in L(D)$ and p is compatible with $\tilde{x}$ in $\mathbb{T}^\omega$, then $p \sqsupseteq \tilde{x}$. In particular, $\widetilde{X} \subseteq \min(L(D))$.*

Proof. Suppose that $p \in L(\mathbb{T}^\omega)$ satisfies $p \uparrow \tilde{x}$ and $p \not\sqsupseteq \tilde{x}$. There is an index $n \in \mathbb{N}$ such that $\tilde{x}(n) \neq \perp$ and $p(n) = \perp$. We assume that $\tilde{x}(n) = 0$. That is, $x \in S_{n,0}$. Since X is regular and S is proper, $x \in S(e) \subseteq \mathrm{cl}\, S(e) = \bar{S}(e) \subseteq S_{n,0}$ for some $e \in K(D_S)$. We can assume that $e = \tilde{x}|_m$ for some $m > n$ such that $p(m-1) \neq \perp$.

We have

$$\bar{S}(e) = \bigcap_{k \in \mathrm{dom}(e)} (S_{k,e(k)} \cup S_{k,\perp}),$$

$$\bar{S}^m_{\mathrm{ex}}(p) = \bigcap_{\substack{k < m, \\ k \in \mathrm{dom}(p)}} (S_{k,p(k)} \cup S_{k,\perp}) \cap \bigcap_{\substack{k < m, \\ k \notin \mathrm{dom}(p)}} S_{k,\perp}.$$

Therefore, since $p \uparrow e$, we have $S_{n,0} \supseteq \bar{S}(e) \supseteq \bar{S}^m_{\mathrm{ex}}(p) \supseteq S^m_{\mathrm{ex}}(p)$. On the other hand, since $p(n) = \perp$, we have $S_{n,0} \cap \bar{S}^m_{\mathrm{ex}}(p) = \varnothing$. Therefore, we can conclude that both $\bar{S}^m_{\mathrm{ex}}(p) = \bar{S}^m_{\mathrm{ex}}(p|_m)$ and $S^m_{\mathrm{ex}}(p) = S^m_{\mathrm{ex}}(p|_m)$ are empty. Thus, by Lemma 6.2, we have $p|_m \notin K_S$ and $p|_m \notin \widehat{K}_S$. Then, from Lemma 4.7, we have $p \notin D_S$ and $p \notin \widehat{D}_S$. □

Theorem 6.4. *Suppose that S is a proper dyadic subbase of a compact Hausdorff space X and $D \in \{D_S, \widehat{D}_S\}$. We have $\widetilde{X} = \min(L(D))$ and X is a retract of $L(D)$.*

Proof. Since a compact Hausdorff space is regular, we have $\widetilde{X} \subseteq \min(L(D))$ by Theorem 6.3. Assume that there exists $p \in \min(L(D)) \backslash \widetilde{X}$. For every $x \in X$, $\tilde{x}$ and p are not compatible in $\mathbb{T}^\omega$ by Theorem 6.3. Therefore, $\tilde{x}(k)$ and $p(k)$ are different digits for some k. Thus, we have an open covering $X = \bigcup_{k \in \mathrm{dom}(p)} S_{k,1-p(k)}$. Since $\bar{S}(p|_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there is no finite subcovering. Therefore, X is not compact. □

We state properties of domain representations as a corollary. Here, $\uparrow\widetilde{X}$ in $(D_S, \uparrow\widetilde{X}, \rho_S)$ is the upwards-closure of $\widetilde{X}$ in D_S which may be different from the upwards-closure of $\widetilde{X}$ in $\mathbb{T}^\omega$.

Corollary 6.5. *Suppose that S is a proper dyadic subbase of a regular space X and $D \in \{D_S, \widehat{D}_S\}$.*

(1) *In the dense domain representation $(D, \tilde{X}, \varphi^{-1})$, we have $\tilde{X} \subseteq \min(L(D))$. In particular, if X is compact, then $\tilde{X} = \min(L(D))$.*
(2) *In the retract domain representation $(D, \uparrow\widetilde{X}, \rho_S)$, $\uparrow\widetilde{X}$ is downwards-closed in $L(D)$. In particular, if X is compact, then $\uparrow\widetilde{X} = L(D)$.*

Proof. By Theorem 6.3 and 6.4. □

As Corollary 6.5 shows, if X is compact, then $(D_S, L(D_S), \rho_S)$ and $(\widehat{D}_S, L(\widehat{D}_S), \rho_S)$ are representations of X as minimal-limit sets of domains studied in [19]. In both of the domains, all the strictly increasing sequences in the set of compact elements denote points of X via ρ_S.

As we have seen, if S is a proper dyadic subbase of a regular space X, then $\min(L(D_S))$ is a compact space in which X is embedded densely. Therefore, $\min(L(D_S))$ is a kind of compactification of X. However, it is not a Hausdorff compactification, in general, as Example 6.6 shows.

Example 6.6. Let $Z = \mathbb{I} \times \mathbb{I}$ be a unit square. An independent subbase H of Z is defined as

$$H_{2k,a} = G_{k,a} \times \mathbb{I}, \quad H_{2k+1,a} = \mathbb{I} \times G_{k,a}, \qquad \text{for } k \in \mathbb{N},\ a \in \mathbb{2}.$$

We have $\varphi_H((1/2, 1/2)) = \bot\bot110^\omega$. We set $A = \{00p, 01p, 10p, 11p\}$ and $B = \{0\bot p, 1\bot p, \bot 0p, \bot 1p\}$ where $p = 110^\omega \in \mathbb{T}^\omega$. Note that $\uparrow\bot\bot p = \{\bot\bot p\} \cup A \cup B$.

Let $Z^0 = \mathbb{I} \times \mathbb{I} \backslash \{(1/2, 1/2)\}$ be a subspace of Z. The independent subbase of Z^0 which is obtained by restricting each element of H to Z^0 is denoted by H^0. We have $L(D_{H^0}) = L(D_H) \backslash \{\bot\bot p\}$ and we get

$$\min(L(D_{H^0})) = (\min(L(D_H)) \backslash \{\bot\bot p\}) \cup B.$$

Since the set B contains a pair of compatible bottomed sequences, $\min(L(D_{H^0}))$ is not Hausdorff.

Example 6.7. Let $Z^1 = Z \backslash \{(1/2, 1/3)\}$ be a subspace of Z and H^1 be a dyadic subbase of Z^1 defined similarly to Example 6.6. We have

$$\widetilde{Z}^1 = \min(L(D_H)) \backslash \{q\}$$

where $q = \bot 011(01)^\omega$. However, since we have

$$\left| y - \frac{1}{3} \right| < \frac{1}{3 \cdot 2^n} \Rightarrow \varphi_{H^1}((1/2, y))|_{2n} = q|_{2n}$$

for all $n \in \mathbb{N}$, we get $L(D_{H^1}) = L(D_H)$. Therefore $\min(L(D_{H^1}))$ is a Hausdorff compactification of Z^1.

Example 6.8. We set $Z^2 = Z^1 \cup \{x_0, x_1\}$ with $\tilde{x}_a = q[0 := a]$ for q in Example 6.7 and $a \in \mathbb{2}$, and let H^2 be the corresponding dyadic subbase. The space Z^2 is a non-regular Hausdorff space and we have $\min(L(D_{H^2})) = \min(L(D_H))$. Since we have $\tilde{x}_a \notin \min(L(D_{H^2}))$, we get $\tilde{X} \nsubseteq \min(L(D_{H^2}))$.

7. Height of $L(D_S)$ and the dimension of X

We finally study relations between the degree of a proper dyadic subbase S and structures of $L(D_S)$ and $L(\widehat{D}_S)$.

Definition 7.1. For a dyadic subbase S of a space X and $x \in X$, we define $\deg_S(x) = |\{n \in \mathbb{N} : x \in S_{n,\perp}\}|$ and $\deg S = \sup\{\deg_S(x) : x \in X\}$.

If $\deg S = m$, then $\varphi_S(x)$ contains at most m copies of $\perp$ for $x \in X$. It is proved in [11] that every separable metrizable space X with $\dim X = m$ has a proper dyadic subbase S with $\deg S = m$. Here, $\dim X$ is the covering dimension of X. It is known that $\dim X$ is equal to the small inductive dimension $\operatorname{ind} X$ of X for a separable metrizable space X. See, for example, [5] and [6] for dimension theory.

For a domain D, we consider the small inductive dimension $\operatorname{ind} L(D)$ of the topological space $L(D)$ with the subspace topology of the Scott topology of D. In Theorem 6.11 of [19], it is proved that if D is a domain with property M, then $\operatorname{ind} L(D) = \operatorname{height} L(D)$ holds. Here, $\operatorname{height} P$ is the maximal length of a chain $a_0 \sqsubsetneq a_1 \sqsubsetneq \ldots \sqsubsetneq a_n$ in a poset P. Property M is defined as follows.

Definition 7.2. (1) We say that a poset P is *mub-complete* if for every finite subset $A \subseteq P$, the set of upper bounds of A has enough minimal elements. That is, if p is an upper bound of A, then there exists a minimal upper bound q of A such that $q \sqsubseteq p$.
(2) We say that a domain D has *property M* if $K(D)$ is mub-complete and each finite subset $A \subseteq K(D)$ has a finite set of minimal upper bounds.

Property M is equivalent to Lawson-compactness for ω-algebraic dcpo by the 2/3 SFP Theorem [13]. Domains with property M are studied in [10].

Proposition 7.3. *Suppose that S is a proper dyadic subbase of a regular space X.*
(1) *The domains D_S and $\widehat{D}_S$ have property M.*
(2) $\operatorname{ind} L(\widehat{D}_S) = \operatorname{height} L(\widehat{D}_S) \geqslant \operatorname{ind} L(D_S) = \operatorname{height} L(D_S) \geqslant \dim X$.
(3) *If X is compact, then* $\operatorname{ind} L(\widehat{D}_S) = \deg S$.

Proof.
(1) Since bounded completeness implies property M, $\widehat{D}_S$ has property M. For D_S, suppose that a finite subset $A \subseteq K_S$ has an upper bound. Let d be the least upper bound of A in $\mathbb{T}^*$. Then, $e \in K_S$ is an upper bound of A in K_S if and only if $e \sqsupseteq d$. If $d \in K_S$, then it is the only minimal upper bound of A. Suppose that $d \notin K_S$, $e \sqsupseteq d$, and $e \in K_S$. Then, for $n = |d|$, $e|_n \sqsupseteq d$ and $e|_n \in K_S$ by Lemma 4.7(1). Therefore, if e is a minimal upper bound of A, then $e = e|_n$ and the length of e is no more than n. Therefore, the set of minimal upper bounds of A is finite.

(2) The equation $\operatorname{ind} L(D) = \operatorname{height} L(D)$ for $D \in \{D_S, \widehat{D}_S\}$ is derived from (1) and Theorem 6.11 of [19]. We have $\operatorname{ind} L(D_S) \geqslant \operatorname{ind} X$ because X is embedded in $L(D_S)$ and $\operatorname{ind} X = \dim X$ for a separable metrizable space X.

(3) Since $\uparrow\tilde{x}$ in $\widehat{D}_S$ and $\uparrow\tilde{x}$ in $\mathbb{T}^\omega$ are the same set for $x \in X$, the maximum number of bottoms in $\tilde{x}$ for $x \in X$ is equal to the height of $L(\widehat{D}_S)$, which is equal to the small inductive dimension of $L(\widehat{D}_S)$ by (2). □

Note that $\operatorname{ind} L(D_S)$ may not be equal to $\deg S$ even for an independent subbase of a compact space X as Example 4.3 shows. In this example, the height of $L(D_R)$ is one, whereas that of $L(\widehat{D}_R)$ is two.

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