Logical Methods in Computer Science
Vol. 9(2:05)2013, pp. 1–30
www.lmcs-online.org

Submitted May. 29, 2011
Published May. 28, 2013

TOTAL REPRESENTATIONS

VICTOR SELIVANOV

A.P. Ershov Institute of Informatics Systems, Siberian Branch of Russian Academy of Sciences, Lavrentyev prosp. 6, Novosibirsk, 630090, Russia
e-mail address: vseliv@iis.nsk.su

Abstract. Almost all representations considered in computable analysis are partial. We provide arguments in favor of total representations (by elements of the Baire space). Total representations make the well known analogy between numberings and representations closer, unify some terminology, simplify some technical details, suggest interesting open questions and new invariants of topological spaces relevant to computable analysis.

1. Introduction

A *numbering* of a set S is a surjection from ω onto S. Numberings are used to transfer the computability theory on ω to many countable structures. A *representation* of a set S is a partial surjection from the Baire space $\mathcal{N} = \omega^\omega$ onto S. Representations are used to transfer the computability theory on $\mathcal{N}$ to many structures of continuum cardinality. Numbering theory was mostly systematized by the Novosibirsk group of researchers in computability theory [Er73a, Er75, Er77] while representation theory was mostly systematized by the Hagen group of researchers in computable analysis (CA) [Wei00].

Although the analogy of representation theory to numbering theory is well known, there is a striking difference between them: numberings are in most cases total functions while representations considered so far are almost always partial functions (among rare exceptions are Section 1.1.6 of [Ba00] and [Bre13]). Note that total numberings have a better theory than partial numberings and are sufficient for many important topics like computable model theory (although partial numberings also have some advantages, in particular the corresponding category is known to be cartesian closed while the category of total numberings is not). One might expect that total representations are also useful in some situations.

2012 ACM CCS: [**Mathematics of computing**]: Continuous mathematics—Topology—Point-set topology; Continuous mathematics—Continuous functions; [**Theory of computation**]: Models of computation—Computability—Turing machines.

2010 Mathematics Subject Classification: 03D78, 03D45, 03D55, 03D30.

Key words and phrases: Representation, total representation, space, Baire space, hierarchy, reducibility, continuous function.

Supported by DFG-RFBR (Grant 436 RUS 113/1002/01, 09-01-91334), by RFBR Grant 13-01-00015, and by a Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme.

DOI:10.2168/LMCS-9(2:05)2013

In this paper we argue in favor of total representations by elements of $\mathcal{N}$ which we call just total representations (TRs). Of course, similar to the numbering theory, some properties of partial representations are better than those of total representations. In particular, the category of sequential admissibly represented spaces is known to be cartesian closed [Sch02] while the corresponding category of sequential admissibly totally represented spaces is not (see Section 8). On the other hand, the class of all sequential admissibly represented spaces is in a sense too large (it coincides with the class of sequential T_0-spaces which are quotients of countably based spaces, and only few of such spaces are really interesting for CA).

From recent results of M. de Brecht [Bre13] it follows that the countably based admissibly totally representable spaces coincide with the so called quasi-Polish spaces (see Section 8 for details). This class of quasi-Polish spaces is a good solution to the problem from [Se08] of finding a natural class of spaces that includes the Polish spaces and the ω-continuous domains and has a reasonable descriptive set theory (DST). Note that while DST for countably based spaces is well understood, this is not the case for the non-countably based spaces. Since some non-countably based spaces are quite important for Computation Theory and CA, the development of some DST for them seems very desirable. We hope that admissible totally represented spaces could be of use in such a developments (see Section 8 for additional remarks).

As is well known, representation theory, in contrast to numbering theory, has a strong topological flavor. In fact, many notions of CA have two versions: computable and topological, the second being in a sense a "limit case" of the first. The topological version turns out to be fundamental for understanding many phenomena related to the computable version; this is explained by the simple but important fact that computable functions between topological spaces are sequentially continuous [Sch02]. In this paper we are mainly concerned with the topological versions, only from time to time making comments on the computable versions. We have to warn the reader that TRs are useful mostly for understanding the topological aspects of CA.

Apart from CA, TRs are useful also for other fields, in particular for the hierarchy theory. In Sections 5 and 6 we show that all levels of the most popular hierarchies in arbitrary countably-based spaces have principal TRs (a notion analogous to the corresponding notion from numbering theory) which turn out to be acceptable and precomplete. This extends or improves some earlier facts from DST [Ke94, Mos80, Bra05]. Note that hierarchies are also of primary interest to CA because they provide natural tools to measure the topological complexity (known also as the discontinuity degrees [Her93, Her96]) of non-computable problems in CA.

We also show in Section 7 that analogs of the main attractive properties of admissible representations hold for the so called principal continuous TRs (defined again by analogy to some notions from numbering theory). Section 8 relates principal continuous TRs with admissible (partial) representations which are very popular in CA. In Section 9 we develop topological analogs of some results on computable numberings [Er77, Er06], this again demonstrates that topological analogs of results from computability theory are often easier. We include also some previous remarks on TRs from [Se92, Se04, Se07a].

We also observe that such important notions of DST as the continuous and Borel reducibilities of equivalence relations on the Baire space are analogous to similar notions considered in the context of numbering theory. Moreover, we show in Section 11 that most of the natural substructures of the structure of continuous degrees of equivalence relations have undecidable first-order theories. This is done via establishing a close relation of this reducibility to a version of one of the Weihrauch's reducibilities [Her93, KSZ10].

Thus, the results of this paper show that sticking to TRs leads to some natural classes of spaces, makes the analogy between numberings and representations closer, unifies terminology, simplifies some technical details, suggests interesting open questions and new invariants of topological spaces relevant to CA. In contrast, total numberings seem to be less important than partial numberings in the study of effective theory of countable topological spaces [Sp01]. Note that, though we do present quite a few apparently new results in the technical sections of this paper, we also give a reasonable space to discussing the analogy with numbering theory and to citing known facts which confirm our claim that TRs deserve a special attention.

In Section 2 we mention the spaces relevant to this paper and discuss a technical notion of a family of pointclasses. In Section 3 we discuss classical hierarchies of DST in arbitrary spaces. In Section 4 several reducibility relations on TRs are introduced and discussed. In Sections 5 and 6 we introduce the important notion of a principal TR and show that natural TRs of levels of the standard hierarchies in the countably based spaces are principal and precomplete. Section 7 shows that the principal continuous TRs of spaces hold the main attractive properties of the admissible representations. In Section 8 we discuss admissible TRs (putting emphasis on the spaces of open sets of countably based spaces) which turns out to be an important subclass of admissible partial representations. Section 9 investigates semilattices of TRs of open sets in countably based spaces. Section 10 presents the category of TRs which is useful in some contexts. In Section 11 we discuss some reducibility notions for equivalence relations on the Baire space.

2. Spaces and Pointclasses

Here we discuss spaces considered in the sequel and a technical notion of a family of pointclasses that is useful in hierarchy theory.

We freely use the standard set-theoretic notation like $|X|$ for the cardinality of X, $X \times Y$ for the cartesian product, $pr_X(A) = \{x \mid \exists y \in Y(x,y) \in A\}$ for the projection of $A \subseteq X \times Y$ to X, Y^X for the set of functions $f : X \to Y$, $P(X)$ for the set of all subsets of X. For $A \subseteq X$, $\overline{A}$ denotes the complement $X \setminus A$ of A in X. For $\mathcal{A} \subseteq P(X)$, $BC(\mathcal{A})$ denotes the Boolean closure of $\mathcal{A}$, i.e. the set of finite Boolean combinations of sets in $\mathcal{A}$.

We assume the reader to be familiar with basic notions of topology. The set of open subsets of a space X is sometimes denoted $O(X)$. We often abbreviate "topological space" to "space". A space X is *Polish* if it is countably based and metrizable with a metric d such that (X,d) is a complete metric space. A space X is *quasi-Polish* [Bre13] if it is countably based and quasi-metrizable with a quasi-metric d such that (X,d) is a complete quasi-metric space. A *quasi-metric* on X is a function from $X \times X$ to the nonnegative reals such that $d(x,y) = d(y,x) = 0$ iff $x = y$, and $d(x,y) \leq d(x,z) + d(z,y)$. Since for the quasi-metric spaces different notions of completeness and of a Cauchy sequence are considered, the definition of quasi-Polish spaces should be made more precise (see [Bre13] for additional details). We skip these details because we will in fact use other characterizations of these spaces given in the sequel.

Let ω be the space of non-negative integers with the discrete topology. Of course, the spaces $\omega \times \omega = \omega^2$, and $\omega \sqcup \omega$ are homeomorphic to ω, the first homeomorphism is realized by the Cantor pairing function $\langle x, y \rangle$.

Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi : \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. By $\sigma\xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\cdots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x[n] = x(0)\cdots x(n-1)$ denote the initial segment of x of length n. Notations in the style of regular expressions like 0^ω, 0^*1 or 0^m1^n have the obvious standard meaning.

Define the topology on $\mathcal{N}$ by taking arbitrary unions of sets of the form $\sigma \cdot \mathcal{N}$, where $\sigma \in \omega^*$, as the open sets. The space $\mathcal{N}$ with this topology known as the *Baire space* is of primary importance for DST and CA. The importance stems from the fact that many countable objects are coded straightforwardly by elements of $\mathcal{N}$, and it has very specific topological properties. In particular, it is zero-dimensional and the spaces $\mathcal{N}^2$, $\mathcal{N}^\omega$, $\omega \times \mathcal{N} = \mathcal{N} \sqcup \mathcal{N} \sqcup \cdots$ are homeomorphic to $\mathcal{N}$. The well known homeomorphisms are given by the formulas $\langle x, y\rangle(2n) = x(n)$ and $\langle x, y\rangle(2n+1) = y(n)$, $\langle x_0, x_1, \ldots\rangle(\langle m, n\rangle) = x_m(n)$, $(n, x) \mapsto n \cdot x$, respectively.

For any finite alphabet A with at least two symbols, let A^ω be the set of ω-words over A. This set may be topologized similar to the Baire space. The resulting space is known as *Cantor space* (more often the last name is applied to the space $\mathcal{C} = 2^\omega$ of infinite binary sequences). Although in representation theory the Baire and Cantor spaces are equivalent as the sets of names, in the study of total representations Baire space is more suitable. The reason is that Cantor space is compact, hence all its continuous images are also compact.

We also need the space $P\omega$ of subsets of ω with the Scott topology on the complete lattice $(P(\omega); \subseteq)$. The basic open sets of this topology are of the form $\{A \subseteq \omega \mid F \subseteq A\}$ where F runs through the finite subsets of ω.

We conclude this section by recalling (in a slightly generalized form) a technical notion from DST. A *pointclass* in X is a subset of $P(X)$. A *family of pointclasses* is a family $\Gamma = \{\Gamma(X)\}$ indexed by arbitrary spaces such that $\Gamma(X) \subseteq P(X)$ for any space X, and $f^{-1}(A) \in \Gamma(X)$ for any $A \in \Gamma(Y)$ and any continuous function $f : X \to Y$. In particular, any pointclass $\Gamma(X)$ in such a family is downward closed under the Wadge reducibility in X. Recall that $B \subseteq X$ is *Wadge reducible* to $A \subseteq X$ (in symbols $B \leq_W A$) if $B = f^{-1}(A)$ for some continuous function f on X. A basic example of a family of pointclasses is $O = \{O(X)\}$ where $O(X)$ is the set of open sets in X. There are also two trivial examples of families E, F where $E(X) = \{\emptyset\}$ and $F(X) = \{X\}$ for any space X.

We define some operations on families of pointclasses which are relevant to hierarchy theory. First, we can use the usual set-theoretic operations pointwise. E.g., the union $\bigcup_i \Gamma_i$ of families $\Gamma_0, \Gamma_1, \ldots$ is defined by $(\bigcup_i \Gamma_i)(X) = \bigcup_i \Gamma_i(X)$.

Secondly, a large class of such operations is induced by the set-theoretic operations of L.V. Kantorovich and E.M. Livenson which are now better known under the name "ω-Boolean operations". Relate to any $\mathcal{A} \subseteq P(\omega)$ the operation $\Gamma \mapsto \Gamma_\mathcal{A}$ on families of pointclasses as follows: $\Gamma_\mathcal{A}(X) = \{\mathcal{A}(C_0, C_1, \ldots) \mid C_0, C_1, \ldots \in \Gamma(X)\}$ where

$$\mathcal{A}(C_0, C_1, \ldots) = \bigcup_{A \in \mathcal{A}} ((\bigcap_{n \in A} C_n) \cap (\bigcap_{n \in \overline{A}} \overline{C}_n)).$$

The operation $\Gamma \mapsto \Gamma_\mathcal{A}$ includes many useful concrete operations including the operation $\Gamma \mapsto \Gamma_\sigma$ where $\Gamma_\sigma(X)$ is the set of all countable unions of sets in $\Gamma(X)$, the operation $\Gamma \mapsto \Gamma_c$ where $\Gamma_c(X)$ is the set of all complements of sets in $\Gamma(X)$, and the operation $\Gamma \mapsto \Gamma_d$ where

$\Gamma_d(X)$ is the set of all differences of sets in $\Gamma(X)$. E.g., the first operation is obtained from the general scheme if $\mathcal{A}$ is the set of all non-empty subsets of $P(\omega)$.

Finally, we will need the operation $\Gamma \mapsto \Gamma_p$ defined by $\Gamma_p(X) = \{pr_X(A) \mid A \in \Gamma(\mathcal{N} \times X)\}$.

We will see that some properties of families are preserved by these operations. In this section we state this for the following simple property [Ke94]. A family of pointclasses Γ is *reasonable* if for any numbering $\nu : \omega \to \Gamma(X)$ its universal set $\{(n,x) \mid x \in \nu(n)\}$ is in $\Gamma(\omega \times X)$. Note that the converse implication ($\{(n,x) \mid x \in \nu(n)\} \in \Gamma(\omega \times X)$ implies that $\nu(n) \in \Gamma(X)$ for all $n < \omega$) holds for any family because $x \mapsto (n,x)$ is a continuous function from X to $\omega \times X$. One easily checks that the families E, F, O are reasonable.

The next result is straightforward, so the proof is omitted.

Lemma 2.1.

(1) *If Γ is a reasonable family of pointclasses then Γ_σ is reasonable.*
(2) *Let Γ be reasonable and $\mathcal{A} \subseteq P(\omega)$. Then $\Gamma_{\mathcal{A}}$ is reasonable.*
(3) *If Γ is reasonable then so is also Γ_p.*

3. Hierarchies

Here we briefly discuss some hierarchies of subsets in arbitrary spaces which are often of use in DST and CA. First we recall definition of Borel hierarchy in arbitrary spaces from [Se04a] (some particular cases were considered in [Sco76, T79, Se82a, Se84]). Let ω_1 be the first non-countable ordinal.

Definition 3.1. Define the sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ of pointclasses in arbitrary space X by induction on α as follows: $\mathbf{\Sigma}^0_0(X) = \{\emptyset\}$, $\mathbf{\Sigma}^0_1(X)$ is the class of open sets in X, $\mathbf{\Sigma}^0_2(X)$ is the class of countable unions of finite Boolean combinations of open sets, and $\mathbf{\Sigma}^0_\alpha(X)$ for $\alpha > 2$ is the class of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$, where $\mathbf{\Pi}^0_\beta(X) = \{A \mid \overline{A} \in \mathbf{\Sigma}^0_\beta(X)\}$.

The sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ is called *Borel hierarchy* in X. The pointclasses $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$ are the *non-selfdual levels* and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$ are the *self-dual levels* of the hierarchy (as is usual in DST, we apply the last terms also to levels of other hierarchies below). The pointclass $\mathbf{B}(X)$ of *Borel sets* in X is the union of all levels of the Borel hierarchy. Let us state the inclusions of levels which are well known for Polish spaces.

Proposition 3.2. *For any space X and for all α, β with $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_\beta(X)$.*

Remark 3.3. Definition 3.1 applies to arbitrary topological space, and Proposition 3.2 holds true in the full generality. Note that Definition 3.1 differs from the classical definition for Polish spaces [Ke94] only for the level 2, and that for the case of Polish spaces our definition of Borel hierarchy is equivalent to the classical one. The classical definition applied, say, to ω-continuous domains does not in general have the properties one expects from a hierarchy. E.g., Proposition 3.2 is true for our definition but is in general false for the classical one.

Note that, in notation of the previous section, we have $\mathbf{\Sigma}^0_0 = E$, $\mathbf{\Sigma}^0_1 = O$, $\mathbf{\Sigma}^0_2 = ((\mathbf{\Sigma}^0_1)_d)_\sigma\}$ (because $\mathbf{\Sigma}^0_2(X)$ obviously coincides with the set of countable unions of differences of open sets in X), $\mathbf{\Sigma}^0_{\alpha+1} = ((\mathbf{\Sigma}^0_\alpha)_c)_\sigma$ for any countable $\alpha \geq 2$, and $\mathbf{\Sigma}^0_\lambda = (\bigcup_{\alpha<\lambda} \mathbf{\Sigma}^0_\alpha)_\sigma$ for any limit countable ordinal λ. Thus, by Lemma 2.1 any fixed non-self-dual level of Borel hierarchy is a reasonable family of pointclasses.

For any non-zero ordinal $\theta < \omega_1$, let $\{\mathbf{\Sigma}^{-1,\theta}_\alpha\}_{\alpha<\omega_1}$ be the Hausdorff difference hierarchy over $\mathbf{\Sigma}^0_\theta$. We recall the definition. An ordinal α is *even* (resp. *odd*) if $\alpha = \lambda + n$ where λ is either zero or a limit ordinal and $n < \omega$, and the number n is even (resp., odd). For an ordinal α, let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$, otherwise. For any ordinal α, define the operation D_α sending sequences of sets $\{A_\beta\}_{\beta<\alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup\{A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma \mid \beta < \alpha,\, r(\beta) \neq r(\alpha)\}.$$

For any ordinal $\alpha < \omega_1$ and any pointclass $\mathcal{E}$ in X, let $D_\alpha(\mathcal{E})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{E}$ for all $\beta < \alpha$. Finally, let $\mathbf{\Sigma}^{-1,\theta}_\alpha(X) = D_\alpha(\mathbf{\Sigma}^0_\theta(X))$ for any space X and for all $\alpha, \theta < \omega$, $\theta > 0$.

It is easy to see that for any $\alpha < \omega_1$ there is $\mathcal{D}_\alpha \subseteq P(\omega)$ such that $\mathcal{D}_\alpha(\mathbf{\Sigma}^0_\theta(X)) = \mathbf{\Sigma}^{-1,\theta}_\alpha(X)$ for all non-zero $\theta < \omega_1$ and all X. Thus, by Lemma 2.1 any fixed non-self-dual level of the difference hierarchy is a reasonable family of pointclasses. It is well known and easy to check that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^{-1,\theta}_\alpha(X) \subseteq \mathbf{\Delta}^0_{\theta+1}(X)$ for all $0 < \theta < \omega_1$ and X.

Let $\{\mathbf{\Sigma}^1_n(X)\}_{1\leq n<\omega}$ be the Luzin's projective hierarchy in X. Using the corresponding operation on families of pointclasses from the previous section we have $\mathbf{\Sigma}^1_1(X) = (\mathbf{\Pi}^0_2(X))_p$ and $\mathbf{\Sigma}^1_{n+1}(X) = (\mathbf{\Pi}^1_n(X))_p$ for any $n \geq 1$. The reason why the definition of the first level is distinct from the classical definition $\mathbf{\Sigma}^1_1(X) = (\mathbf{\Pi}^0_1(X))_p$ for Polish spaces is again the difference of our definition of $\mathbf{\Sigma}^0_2$ from the classical one. Again, by Lemma 2.1 any fixed non-self-dual level of the projective hierarchy is a reasonable family of pointclasses. It is well known and easy to check that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Delta}^1_1(X)$. It is easy to see that, similar to a known fact for Polish spaces, $\mathbf{\Sigma}^1_1 = \mathbf{B}_p$.

For a further reference, we summarize some of the above remarks.

Lemma 3.4.

(1) *In notation of the previous section,* $\mathbf{\Sigma}^0_0 = E$, $\mathbf{\Sigma}^0_1 = O$, $\mathbf{\Sigma}^0_2 = ((\mathbf{\Sigma}^0_1)_d)_\sigma$, $\mathbf{\Sigma}^0_{\alpha+1} = ((\mathbf{\Sigma}^0_\alpha)_c)_\sigma$ *for any countable* $\alpha \geq 2$, *and* $\mathbf{\Sigma}^0_\lambda = (\bigcup_{\alpha<\lambda} \mathbf{\Sigma}^0_\alpha)_\sigma$ *for any limit countable ordinal* λ.

(2) *For any* $\alpha < \omega_1$ *there is* $\mathcal{D}_\alpha \subseteq P(\omega)$ *such that* $\mathcal{D}_\alpha(\mathbf{\Sigma}^0_\theta) = \mathbf{\Sigma}^{-1,\theta}_\alpha$ *for all non-zero* $\theta < \omega_1$.

(3) *We have* $\mathbf{\Sigma}^1_1 = (\mathbf{\Pi}^0_2)_p$ *and* $\mathbf{\Sigma}^1_{n+1} = (\mathbf{\Pi}^1_n)_p$ *for any* $n \geq 1$.

(4) *Any non-self-dual level of any of the three hierarchies is a reasonable family of pointclasses.*

For levels of the difference and projective hierarchies we have the natural inclusions similar to those in Proposition 3.2. Note that for Polish spaces the class $\mathbf{\Sigma}^1_1$ of analytic sets has several nice equivalent characterizations (in particular, as the class of continuous images of Polish spaces or as the class of sets obtained from the closed sets by applying the Suslin A-operation). In Lemma 56 of [Bre13] it was observed that the characterization in terms of continuous images extends to the quasi-Polish spaces. In Section 8 we will see that this characterization fails in general for non-countably based admissibly totally represented spaces.

Next we establish important structural properties of $\mathbf{\Sigma}$-levels of the Borel hierarchy which are well known for Polish spaces [Ke94]. This result demonstrates that our extension of the classical definition to arbitrary spaces is natural.

Let Γ be a family of pointclasses. A pointclass $\Gamma(X)$ has the *ω-reduction property* if for each countable sequence $A_0, A_1, \ldots$ in $\Gamma(X)$ there is a countable sequence $D_0, D_1, \ldots$ in $\Gamma(X)$ such that $D_i \subseteq A_i$, $D_i \cap D_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i<\omega} D_i = \bigcup_{i<\omega} A_i$. A pointclass

$\Gamma(X)$ has the *ω-uniformization property* if for any $A \in \Gamma(\omega \times X)$ there is $D \in \Gamma(\omega \times X)$ such that $D \subseteq A$, $pr_X(D) = pr_X(A)$, and for any $x \in X$ there is at most one $n \in \omega$ with $(n, x) \in D$; we say that such set D *uniformizes* A. Just as in [Ke94] one can check that if Γ is reasonable then $\Gamma(X)$ has the ω-uniformization property iff it has the ω-reduction property.

Theorem 3.5. *For any space X and any $2 \leq \alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X)$ has the ω-reduction and ω-uniformization properties. If X is zero-dimensional, the same holds for the class $\mathbf{\Sigma}^0_1(X)$ of open sets.*

Proof. By item 4 of Lemma 3.4 and remarks before the formulation, it suffices to establish the ω-uniformization property. We consider only the second level. For $\alpha > 2$ and $\alpha = 1$ the proof is almost the same.

Let $A \in \mathbf{\Sigma}^0_2(\omega \times X)$, then $A = \bigcup_n (B_n \setminus C_n)$ for some $B_n, C_n \in \mathbf{\Sigma}^0_1(\omega \times X)$. Then $x \in pr_X A$ iff $\exists n(x \in A(n))$ iff $\exists n, m(x \in B_m(n) \setminus C_m(n))$ where $A(n) = \{x \mid (n, x) \in A\}$. Let

$$E_{m,n} = \{x \mid x \in B_m(n) \setminus C_m(n) \wedge \neg\exists\langle m_1, n_1\rangle < \langle m, n\rangle(x \in B_{m_1}(n_1) \setminus C_{m_1}(n_1))\}.$$

Then $E_{m,n} \cap E_{m_1,n_1} = \emptyset$ for any distinct $\langle m, n\rangle$ and $\langle m_1, n_1\rangle$, and $E_{m,n}$ is a finite Boolean combination of open sets. Therefore, $D_n = \bigcup_m E_{m,n}$ is in $\mathbf{\Sigma}^0_2(X)$ for any n, hence $D = \{(n, x) \mid x \in D_n\}$ is in $\mathbf{\Sigma}^0_2(\omega \times X)$. The set D uniformizes A. $\square$

For arbitrary spaces, not much can be said about more interesting properties of the introduced hierarchies like the non-collapse property saying that any $\mathbf{\Sigma}$-level is distinct from the corresponding $\mathbf{\Pi}$-level. We come back to such non-trivial questions in Section 5.

In [Bre13] the following important characterization of quasi-Polish spaces in terms of Borel hierarchy was obtained.

Proposition 3.6. *A space is quasi-Polish iff it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $P\omega$ with the induced topology.*

The computable versions of the introduced hierarchies are defined in a straightforward way [Se06] but their non-trivial properties (like the effective Hausdorff-Kuratowski theorem) seem to be relatively well understood only for the spaces ω, $\mathcal{N}$ and $\mathcal{C}$. To my knowledge, the problem of finding a broad enough class of effective spaces with good effective DST is open.

4. Representations and Reducibilities

In this section we introduce and briefly discuss some reducibility notions which serve as tools for measuring the topological complexity of problems in DST and CA.

By a *total representation* (TR) we mean any function ν with $dom(\nu) = \mathcal{N}$. By a *total representation of a given set S* we mean a TR ν with $rng(\nu) = S$. There are several natural reducibility notions for TRs the most basic of which is the following. A TR μ is *reducible* to a TR ν (in symbols $\mu \leq \nu$) if $\mu = \nu \circ f$ for some continuous function f on $\mathcal{N}$. A TR μ is *equivalent* to ν (in symbols $\mu \equiv \nu$), if $\mu \leq \nu$ and $\nu \leq \mu$.

For any set S, we may form the preorder $(S^{\mathcal{N}}; \leq)$ which generalizes the preorder formed by the classical Wadge reducibility on subsets of $\mathcal{N}$. Indeed, for $S = 2 = \{0, 1\}$ the structures $(P(\mathcal{N}); \leq_W)$ and $(S^{\mathcal{N}}; \leq)$ are isomorphic: $A \leq_W B$ iff $c_A \leq c_B$ where $c_A : \mathcal{N} \to 2$ is the characteristic function of a set $A \subseteq \mathcal{N}$. Note that the structure $(S^{\mathcal{N}}; \leq)$ (more precisely, its quotient-structure) is an upper semilattice with the join operation induced by the binary operation $\oplus$ on $S^{\mathcal{N}}$ defined by: $(\mu\oplus\nu)(2n\cdot x)=\mu(x)$, and $(\mu\oplus\nu)((2n+1)\cdot x)=\nu(x)$. In fact, this semilattice is a *σ-semilattice* [Se07a], i.e. any countable set of elements has a supremum; the supremum operation is induced by the operation $(\bigsqcup_n \nu_n)(n\cdot x)=\nu_n(x)$ on sequences $\{\nu_n\}$ of TRs.

We will also need the unary operations p_s $(s\in S)$ on $S^{\mathcal{N}}$ introduced in [Se04] defined by: $[p_s(\nu)](a)=s$, if $a\not\in 0^*1$, and $[p_s(\nu)](a)=\nu(b)$ otherwise, where $a=0^n1b$ for some $n<\omega$. We need the following properties of the introduced operations established in [Se04]. The properties of these operations are similar to the properties of completion operations in the theory of complete numberings [Se82, Se04].

Proposition 4.1. *The quotient-structure of $(S^{\mathcal{N}};\leq,\oplus,p_s)$ is a semilattice with discrete closures, i.e.: $\oplus$ is a supremum operation for $\leq$; $\nu\leq p_s(\nu)$, $\mu\leq\nu\to p_s(\mu)\leq p_s(\nu)$, and $p_s(p_s(\nu))\leq p_s(\nu)$; $p_s(\mu)\leq p_u(\nu)\wedge s\neq u\to p_s(\mu)\leq\nu$; $p_s(\mu)\leq\nu\oplus\xi\to p_s(\mu)\leq\nu\vee p_s(\mu)\leq\xi$. Moreover, if $f:S\to T$ then $f\circ(\mu\oplus\nu)=(f\circ\mu)\oplus(f\circ\nu)$ and $f\circ p_s(\nu)=p_{f(s)}(f\circ\nu)$.*

The structure of Wadge degrees (i.e., the quotient-structure of $(P(\mathcal{N});\leq_W)$) is fairly well understood and turns our to be rather simple. In particular, $(\mathbf{\Delta}^1_1(\mathcal{N});\leq_W)$ is almost well ordered [Wad84], i.e. it has no infinite descending chain and for any $A,B\in\mathbf{\Delta}^1_1(\mathcal{N})$ we have $A\leq_W B$ or $\overline{B}\leq_W A$. Beyond the Borel sets, the structure of Wadge degrees depends on the set-theoretic axioms but under some of these axioms the whole structure remains almost well ordered. This structure includes and refines the structure of levels (more precisely, of the Wadge complete sets in these levels) of the hierarchies from the previous section (taken for the Baire space). It may serve as a nice tool to measure the topological complexity of many problems of interest in DST and CA.

In particular, we will see below that some natural classes of TRs and of spaces may be defined through the *kernel* $E_\nu=\{\langle a,b\rangle|\nu(a)=\nu(b)\}$ of a TR ν. The kernel is a subset of $\mathcal{N}$ that codes the corresponding equivalence relation on $\mathcal{N}$. Clearly, $\mu\leq\nu$ implies $E_\mu\leq_W E_\nu$ but not vice versa. Note that the kernel relation of a given numbering is rather important in numbering theory.

Already for $3\leq k<\omega$ the structures $(k^{\mathcal{N}};\leq)$ of k-partitions of $\mathcal{N}$ (i.e., of TRs of subsets of k) become much more complicated. Nevertheless, some important information on these structures is already available. For any $\mathcal{A}\subseteq P(\mathcal{N})$, let $\mathcal{A}_k$ denote the set of k-partitions $\nu\in k^{\mathcal{N}}$ such that $\nu^{-1}(i)\in\mathcal{A}$ for each $i<k$. In [EMS87] it was shown that the structure $((\mathbf{\Delta}^1_1(\mathcal{N}))_k;\leq)$ is a well preorder, i.e. it has neither infinite descending chains nor infinite antichains. In [Her93, Se07a] the quotient-structures of $((BC(\mathbf{\Sigma}^0_1))_k;\leq)$ and $((\mathbf{\Delta}^0_2)_k;\leq)$ over $\mathcal{N}$ were characterized in terms of a natural preorder $\leq_h$ on the finite and countable well-founded k-labeled forests, respectively. These characterizations clarified the corresponding structures considerably and led to deep definability theories for both structures in [KS07, KS09, KSZ09]. These results show that, similar to the structure of Wadge degrees, the structures of degrees of k-partitions may serve as tools to measure the topological complexity of natural problems. For wider classes of k-partitions like $((\mathbf{\Delta}^0_3)_k;\leq)$, the corresponding characterizations are not yet known. An impression on how they can look can be obtained in [Se07a, Se11] where the structure of Wadge degrees of regular (in the sense of automata theory) k-partitions of the Cantor space is characterized.

For a further reference we recall some details of the results in [Her93, Se07a]. A poset $(P;\leq)$ will be often shorter denoted just by P. Any subset of P may be considered as a poset with the induced partial ordering. In particular, this applies to the "cones" $\uparrow x=$

$\{y \in P \mid x \leq y\}$ and $\downarrow x = \{y \in P \mid y \leq x\}$ defined by any $x \in P$. By *a forest* we mean a finite poset in which every lower cone $\downarrow x$ is a chain. *A tree* is a forest having a smallest element (called *the root* of the tree). Note that any forest is uniquely representable as a disjoint union of trees, the roots of the trees being the minimal elements of the forest. Let $\mathcal{P}$ (resp. $\mathcal{F}$) denote the set of all finite posets (resp. forests) with $P \subseteq \omega$.

We relate to any $F \in \mathcal{F}$ the TR $\xi_F \in F^{\mathcal{N}}$ by induction on $|F|$ as follows: if $F = \{r\}$ then $\xi_F = \lambda x.r$; if F is a non-singleton tree with a root r then $\xi_F = p_r(\xi_{F\setminus\{r\}})$; if $F = T_0 \cup \cdots \cup T_n$ is a disjoint union of trees where $n > 0$ then $\xi_F = \xi_{T_0} \oplus \cdots \oplus \xi_{T_n}$. It is easy to see that ξ_F is an admissible TR of F with respect to the Scott topology on the forest F.

A *k-labeled poset* (or just a k-poset) is an object $(P; \leq, c)$ consisting of a finite poset $(P; \leq)$ and a labeling $c : P \to k$. Sometimes we simplify notation of a k-poset to (P, c) or even to P. A *morphism* $f : (P; \leq, c) \to (P'; \leq', c')$ between k-posets is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, i.e. satisfying $c = c' \circ f$. Let $\mathcal{P}_k$ (resp. $\mathcal{F}_k$) be the set of all finite k-posets (resp. k-forests) $(P; \leq, c)$ with $P \subset \omega$.

Define a preorder $\leq_h$ on $\mathcal{P}_k$ as follows: $(P, c) \leq (P', c')$, if there is a morphism from (P, c) to (P', c'). By $\equiv_h$ we denote the *h-equivalence relation* on $\mathcal{P}_k$ induced by $\leq_h$. For $T_0, \ldots, T_n \in \mathcal{F}_k$, let $F = T_0 \sqcup \cdots \sqcup T_n$ be their disjoint union, then $F \in \mathcal{F}_k$. For $F \in \mathcal{F}_k$ and $i < k$, let $p_i(F)$ be the k-tree obtained from F by joining a new bottom element (from ω) and assigning the label i to the bottom element. It is clear that any k-forest is h-equivalent to a term of signature $\{\sqcup, p_0, \ldots, p_{k-1}, 0, \ldots, k-1\}$ without free variables (the constant symbol i in the signature is interpreted as the singleton tree carrying the label i).

It is known [Her93, Se04] that the quotient-structure of $(\mathcal{F}_k; \leq_h)$, together with a new bottom element, is a distributive lattice any principal ideal of which is finite. The following assertion from [Se07a] (in which $\sqcup$ is the binary disjoint union operation) is a version of a much earlier result in [Her93].

Proposition 4.2. *The quotient-structures of the structures $(\mathcal{F}_k; \leq_h, \sqcup, p_0, \ldots, p_{k-1})$ and $(BC(\mathbf{\Sigma}^0_1(\mathcal{N})))_k; \leq, \oplus, p_0, \ldots, p_{k-1})$ are isomorphic. An isomorphism is induced by the function $(F; c) \mapsto c \cdot \xi_F$.*

Let us mention some other interesting reducibilities on TRs. A straightforward generalization of $\leq$ is the reducibility by functions in F where F is an arbitrary class of functions on $\mathcal{N}$ closed under composition and containing the identity function. In particular, let $\leq_{\mathbf{\Delta}^0_\alpha}$ (resp. $\leq_{\mathbf{\Delta}^1_1}$) be the reducibility by functions f on $\mathcal{N}$ such that $f^{-1}(A) \in \mathbf{\Delta}^0_\alpha$ for any $A \in \mathbf{\Delta}^0_\alpha$ (rep. $f^{-1}(A) \in \mathbf{\Delta}^1_1$ for any $A \in \mathbf{\Delta}^1_1$). Note that $\leq_{\mathbf{\Delta}^0_1}$ coincides with $\leq$. Some deep facts on the corresponding degree structures are known, in particular for any countable ordinal $\alpha > 1$ the quotient-structures of $(\mathbf{\Delta}^1_1(\mathcal{N}); \leq_{\mathbf{\Delta}^0_\alpha})$ and $(\mathbf{\Delta}^1_1(\mathcal{N}); \leq_W)$ are isomorphic [An06]. For recent results on similar reducibilities on arbitrary quasi-Polish spaces see [MSS12]

In [Wei92, Her93, Wei00] some notions of reducibility for functions on spaces were introduced which turned out useful for understanding the non-computability and non-continuity of interesting decision problems in computable analysis [Her96, BG11a] and constructive mathematics [BG11]. In particular, the following notions of reducibilities between functions $f : X \to Z$, $g : Y \to Z$ on topological spaces were introduced: $f \leq_1 g$ (resp. $f \leq_2 g$) iff $f = F \circ g \circ H$ for some continuous functions $H : X \to Y$ and $F : Z \to Z$, (resp. $f(x) = F(x, gH(x))$ for some continuous functions $H : X \to Y$ and $F : X \times Z \to Z$).

Deep results are known for the particular case of these relations where $X = Y = \mathcal{N}$ and $Z = k = \{0, \ldots, k-1\}$ is a discrete space with $k < \omega$ points. In this way we obtain preorders $(k^{\mathcal{N}}; \leq_1)$ and $(k^{\mathcal{N}}; \leq_2)$. In [Her93] the quotient-structures of $((BC(\mathbf{\Sigma}^0_1))_k; \leq_1)$ and

$((BC(\mathbf{\Sigma}^0_1))_k;\leq_2)$ were characterized in terms of natural preorders on the finite k-labeled forests similar to the h-preorder. These characterizations led to the proof of undecidability of first order theories of both quotient-structures in [KSZ10], for each $k\geq 3$.

In computability theory, numbering theory and CA, effective versions of $\leq$ (the reducibility by computable functions on ω and $\mathcal{N}$) and of the other reducibilities mentioned above are extensively studied. Since the corresponding degree structures become extremely complicated, they cannot serve as tools for measuring the computational complexity (in particular, the degree structures are not well-founded, hence it is not possible to assign an ordinal to an arbitrary degree). For this purpose people usually prefer to use complete sets in suitable effective hierarchies like those discussed in the previous section. Another way to "improve" the algebraic structure of, say, Weihrauch degrees is to extend the Weihrauch reducibility to multi-valued functions [Wei92, Wei00, BG11, BG11a]. In this way one obtains algebraically more regular degree structures which are applicable to the complexity of many interesting problems related to Constructive Analysis.

5. Principal Total Representations of Pointclasses

An important observation in numbering theory is that principal numberings (i.e., numberings which are the largest, w.r.t. the reducibility relation, elements in natural classes of numberings) are often interesting and have nice properties. Good examples of principal numberings are the standard computable numberings of computable partial functions and of computably enumerable sets.

This also applies to DST and CA where principal TRs appear quite naturally, as we show here and in the sequel. In particular, the TRs from the following theorem play a crucial role in proving the non-collapse property of the classical hierarchies from Section 3. Note that some relevant properties of representations were considered earlier in the context of DST and CA (see e.g. [Ke94, Mos80, Bra05]).

Let Γ be a family of pointclasses. A TR $\nu:\mathcal{N}\to\Gamma(X)$ is a Γ-*TR* if its *universal set* $U_\nu=\{(a,x)\mid x\in\nu(a)\}$ is in $\Gamma(\mathcal{N}\times X)$, and ν is a *principal* Γ-*TR* if it is a Γ-TR and any Γ-TR is reducible to ν. Note that if $\nu:\mathcal{N}\to\Gamma(X)$ is principal then it is a surjection and that $\Gamma(X)$ has at most one principal TR, up to equivalence. Note also that $\nu\mapsto U_\nu$ is a bijection between the Γ-TRs $\nu:\mathcal{N}\to\Gamma(X)$ and the sets in $\Gamma(\mathcal{N}\times X)$ because any $A\in\Gamma(\mathcal{N}\times X)$ may be considered as the universal set of the TR $a\mapsto A(a)=\{x\mid(a,x)\in A\}$. We show that the introduced notions are in a sense preserved by the operations on families in Section 2.

Lemma 5.1.
(1) *Let $\mathcal{A}\subseteq P(\omega)$ and let Γ be a family of pointclasses. If $\Gamma(X)$ has a principal Γ-TR then $\Gamma_{\mathcal{A}}(X)$ has a principal $\Gamma_{\mathcal{A}}$-TR.*
(2) *If Γ is a family of pointclasses and $\Gamma(\mathcal{N}\times X)$ has a principal Γ-TR then $\Gamma_p(X)$ has a principal Γ_p-TR.*
(3) *If $\{\Gamma_n\}$ is a sequence of families of pointclasses and $\Gamma_n(X)$ has a principal Γ_n-TR for each $n<\omega$ then $(\bigcup_n\Gamma_n)_\sigma(X)$ has a principal $(\bigcup_n\Gamma_n)_\sigma$-TR.*

Proof. We only define the corresponding TRs, it is straightforward to verify that they are indeed principal.

1. Let ν be a principal Γ-TR of $\Gamma(X)$. Define the principal $\Gamma_{\mathcal{A}}$-TR $\nu_{\mathcal{A}}$ of $\Gamma_{\mathcal{A}}(X)$ as follows: $\nu_{\mathcal{A}}\langle a_0,a_1,\ldots\rangle=\mathcal{A}(\nu(a_0),\nu(a_1),\ldots)$.

2. Let ν be a principal Γ-TR of $\Gamma(\mathcal{N}\times X)$. Define the principal Γ_p-TR ν_p of $\Gamma_p(X)$ as follows: $\nu_p(a)=pr_X(\nu(a))$.

3. Let ν_n be a principal Γ_n-TR of $\Gamma_n(X)$, for each $n<\omega$. Define the principal $(\bigcup_n \Gamma_n)_\sigma$-TR ν of $(\bigcup_n\Gamma_n)_\sigma(X)$ as follows: $\nu\langle n_0\cdot a_0, n_1\cdot a_1,\ldots\rangle=\nu_{n_0}(a_0)\cup\nu_{n_1}(a_1)\cup\cdots)$. $\square$

The following main result of this section shows that all non-self-dual levels of the hierarchies from Section 3 have principal TRs in any countably based space. Particular cases of these results for Polish spaces where known from the early days of DST [Ke94] (see also [Se92] for computable versions and [Bra05] for a study of representations of finite levels of the Borel hierarchy). Our results extend them to a wider class though the proof remains elementary.

Theorem 5.2. *Let X be a countably based space and let Γ be an arbitrary non-self-dual level of a hierarchy from Section 3. Then $\Gamma(X)$ has a principal Γ-TR.*

Proof. We consider first the level $\mathbf{\Sigma}^0_1$. Let $B_0,B_1,\ldots$ be a base in X containing the empty set, say $B_0=\emptyset$. We define the TR π of $\mathbf{\Sigma}^0_1$ by $\pi(a)=\bigcup_n B_{a(n)}$. First we have to show that U_π is open in $\mathcal{N}\times X$. For each $(a,x)\in U_\pi$ it suffices to find $V\in\mathbf{\Sigma}^0_1(\mathcal{N}\times X)$ with $(a,x)\in V\subseteq U_\pi$. Since $x\in\pi(a)$, $x\in B_{a(n)}$ for some $n<\omega$. Then $x\in\pi(b)$ for each $b\supseteq a[n+1]$. Hence we can take $V=(a[n+1]\cdot\mathcal{N})\times B_{a(n)}$.

It remains to show that TR π is a largest element in the corresponding class. Let $\nu:\mathcal{N}\to\mathbf{\Sigma}^0_1(X)$ be a $\mathbf{\Sigma}^0_1$-TR, so U_ν is open in $\mathcal{N}\times X$. Define $f:\mathcal{N}\to\mathcal{N}$ as follows: $f(a)\langle i,j\rangle=j$ if $B_j\subseteq\bigcap\nu(a[i]\cdot\mathcal{N})$, and $f(a)\langle i,j\rangle=0$ otherwise. Clearly, f is continuous (even Lipschitz). It remains to show that f reduces ν to π, i.e. $\nu(a)=\bigcup\{B_j\mid\exists i(B_j\subseteq\bigcap\nu(a[i]\cdot\mathcal{N}))\}$. Indeed, the inclusion $\supseteq$ is obvious. Conversely, let $x\in\nu(a)$, so $(a,x)\in U_\nu$. Since U_ν is open, $(a,x)\in(a[i]\cdot\mathcal{N})\times B_j\subseteq U_\nu$ for some $i,j<\omega$. Then $x\in B_j$ and $y\in\nu(b)$ for all $b\sqsupseteq a[i]$, $y\in B_j$. Then $B_j\subseteq\bigcap\nu(a[i]\cdot\mathcal{N})$ hence $x\in\bigcup\{B_j\mid\exists i(B_j\subseteq\bigcap\nu(a[i]\cdot\mathcal{N}))\}$.

For the other levels the assertion follows from Lemmas 3.4 and 5.1. $\square$

Remark 5.3. The TR π from the last proof has many other interesting properties. In particular, we will see in Section 8 that it is admissible w.r.t. some natural topologies on $\mathbf{\Sigma}^0_1(X)$.

Corollary 5.4. *Let X be a countably based space and let Γ be an arbitrary non-self-dual level of a hierarchy from Section 3. Then there is a Wadge-complete set in $\Gamma(\mathcal{N}\times X)$.*

Proof. Let ν be a principal TR of $\Gamma(X)$. We claim that U_ν is Wadge-complete in $\Gamma(\mathcal{N}\times X)$. Let $A\in\Gamma(\mathcal{N}\times X)$, we have to show $A\leq_W U_\nu$. Since A is the universal set of the TR $a\mapsto A(a)$, this TR is a Γ-TR, hence $A(a)=\nu f(a)$ for some continuous function f on $\mathcal{N}$. Then $A\leq_W U_\nu$ via the continuous function $\langle a,x\rangle\mapsto\langle f(a),x\rangle$. $\square$

Using diagonalization, one immediately derives from Proposition 5.2 the non-collapse property for all three hierarchies in the Baire space. The non-collapse property is known to hold in any uncountable Polish space [Ke94]. Recently this was extended [Bre13] (at least for the Borel and Luzin hierarchies) to any uncountable quasi-Polish space.

For Polish spaces X the following important relationships between the introduced hierarchies are known [Ke94]: $\bigcup_{\alpha<\omega_1}\mathbf{\Sigma}^0_\alpha(X)=\mathbf{\Delta}^1_1(X)$ (Suslin theorem) and $\bigcup_{\alpha<\omega_1}\mathbf{\Sigma}^{-1,\theta}_\alpha(X)=\mathbf{\Delta}^0_{\theta+1}(X)$ for all $0<\theta<\omega_1$ (Hausdorff-Kuratowski theorem). In [Bre13] these theorems were also extended to the quasi-Polish spaces.

6. Acceptability and Precompleteness

Principal TRs from Section 5 have a property similar to the corresponding property (of being principal computable) of the standard numbering of the computably enumerable sets. In this section we establish some other such properties of the principal Γ-TRs, namely those of acceptability and precompleteness.

For any set S, call a TR $\nu : \mathcal{N} \to S^{\mathcal{N}}$ *acceptable* if $rng(\nu)$ is downward closed under $\leq$, $\bigsqcup_a \nu_a \in rng(\nu)$ (where $(\bigsqcup_a \nu_a)\langle a, b\rangle = \nu_a(b)$) and $\nu_a\langle b, c\rangle = \nu_{s\langle a,b\rangle}(c)$ for some continuous function s on $\mathcal{N}$. Here ν_a is identified with $\nu(a)$ This definition applies to TRs of the form $\nu : \mathcal{N} \to P(\mathcal{N})$ if we identify $2^{\mathcal{N}}$ with $P(\mathcal{N})$ as in the beginning of Section 4.

Proposition 6.1.

(1) *Any two acceptable TRs of the same subset of $S^{\mathcal{N}}$ are equivalent.*
(2) *If $\mu \equiv \nu$ and ν is acceptable then so is μ.*

Proof. 1. Let $\mu, \nu : \mathcal{N} \to S^{\mathcal{N}}$ be acceptable TRs with the same range. We show $\mu \leq \nu$, the reduction $\nu \leq \mu$ holds then by symmetry. The TR $\bigsqcup_b \mu_b$ is in $rng(\mu)$, hence $\bigsqcup_b \mu_b = \nu_a$ for some a. Then $\mu_b(c) = \nu_a\langle b, c\rangle = \nu_{s\langle a,b\rangle}(c)$, hence the continuous function $b \mapsto s\langle a, b\rangle$ reduces μ to ν.

2. Straightforward. □

Next we show that the principal TRs of pointclasses in $\mathcal{N}$ are acceptable.

Proposition 6.2. *Let Γ be a family of pointclasses such that $\Gamma(\mathcal{N})$ has a principal Γ-TR ν. Then ν is acceptable.*

Proof. By the definition of a family of pointclassed, $rng(\nu)$ is downward closed under $\leq$. The property $\bigsqcup_a \nu_a \in rng(\nu)$ holds because $U_\nu \in \Gamma(\mathcal{N} \times \mathcal{N})$, $\langle a, b\rangle \in \bigsqcup_a \nu_a \leftrightarrow (a, b) \in U_\nu$, and $\mathcal{N} \times \mathcal{N}$ is homeomorphic to $\mathcal{N}$.

The TR μ of $\Gamma(\mathcal{N})$ defined by $\mu\langle a, b\rangle = \{c \mid (a, \langle b, c\rangle) \in U_\nu\}$, is a Γ-TR, hence $\mu \leq \nu$ via a continuous function s on $\mathcal{N}$. Then

$$c \in \nu_{s\langle a,b\rangle} \leftrightarrow c \in \mu\langle a, b\rangle \leftrightarrow \langle b, c\rangle \in \nu_a.$$

In the "characteristic functions" notation this means exactly $\nu_a\langle b, c\rangle = \nu_{s\langle a,b\rangle}(c)$, hence ν is acceptable. □

From Theorem 5.2 we now immediately obtain:

Corollary 6.3. *Let Γ be an arbitrary non-self-dual level of a hierarchy from Section 3. Then the principal TR of $\Gamma(\mathcal{N})$ is acceptable.*

Next we show that the principal TRs of the non-self-dual levels of the classical hierarchies are precomplete. The notion of precompeteness is very important in the numbering theory [Er77]. In [Wei87] the theory of precomplete numberings was extended to the context of representations where, as usual, the theory splits to the "computable" and "topological" versions. Here we consider only the topological version.

Recall from Chapter 3 of [Wei87] that a TR ν is *precomplete* if for any partial continuous function ψ on $\mathcal{N}$ there is a total continuous function g on $\mathcal{N}$ that extends ψ modulo ν, i.e. $\nu\psi(x) = \nu g(x)$ whenever $\psi(x)$ is defined (we call g a *ν-totalizer* of ψ). Precomplete TRs have several nice properties, in particular they satisfy the recursion theorem and the Rice theorem. The recursion theorem for a TR ν means the uniform fixed point property (FPP).

We say that a TR ν *has FPP* if for any continuous function f on $\mathcal{N}$ there is $c \in \mathcal{N}$ (a *fixed point* of f w.r.t. ν) such that $\nu(c) = \nu f(c)$. The uniform FPP means that the fixed point c may be found continuously from a given index for f in a natural TR ϕ of all partial continuous function ψ on $\mathcal{N}$ with a $\mathbf{\Pi}^0_2$-domain (more formally, there is a continuous function c on $\mathcal{N}$ such that $\nu(c(x)) = \nu\phi_x(c(x))$ whenever ϕ_x is total).

We show that the precompleteness property is preserved by the operations on families of pointclasses in Section 2. In the next lemma we use the notation of Lemma 5.1.

Lemma 6.4.

(1) *Let $\mathcal{A} \subseteq P(\omega)$ and let Γ be a family of pointclasses such that $\Gamma(X)$ has a principal Γ-TR ν which is precomplete. Then the TR $\nu_{\mathcal{A}}$ of $\Gamma_{\mathcal{A}}(X)$ is precomplete.*

(2) *If Γ is a family of pointclasses such that $\Gamma(\mathcal{N} \times X)$ has a principal Γ-TR ν which is precomplete then the TR ν_p of $\Gamma_p(X)$ is precomplete.*

Proof. 1. Let ψ be a partial continuous function on $\mathcal{N}$. For any $k < \omega$, let p_k be the continuous function on $\mathcal{N}$ such that $p_k(\langle a_0, a_1, \ldots\rangle) = a_n$ for all $a_i \in \mathcal{N}$. For any $n < \omega$, let g_n be a ν-totalizer of $p_n \circ \psi$. Then the continuous function $g(x) = \langle g_0(x), g_1(x), \ldots\rangle$ is a $\nu_{\mathcal{A}}$-totalizer of ψ. Therefore, $\nu_{\mathcal{A}}$ is precomplete.

2. Let ψ be a partial continuous function on $\mathcal{N}$, then there is a continuous ν-totalizer g of ψ. By the definition of ν_p, g is also a ν_p-totalizer of ψ. Therefore, ν_p is precomplete. $\square$

The following main result of this section shows that all principal TRs of non-self-dual levels of the hierarchies from Section 3 are precomplete.

Theorem 6.5. *Let X be a countably based space and let Γ be an arbitrary non-self-dual level of a hierarchy from Section 3. Then the principal Γ-TR of $\Gamma(X)$ is precomplete.*

Proof. We consider first the principal TR π of $\mathbf{\Sigma}^0_1(X)$ defined in the proof of Theorem 5.2. We have to show that π is precomplete. Let ψ be a partial continuous function on $\mathcal{N}$, we have to find a continuous π-totalizer g of ψ. As is well known [Wei87, Wei92], we can without loss of generality think that $\psi = \phi_a$ for some $a \in \mathcal{N}$, i.e., for each $x \in \mathcal{N}$, $\psi(x) = \varphi_n^{a\oplus x}$ is the n-th (where $n = a(0)$) partial computable function on ω with oracle $a \oplus x$. (Here we use standard notation from computability theory.) It is straightforward to define a continuous function g on $\mathcal{N}$ with the following properties:

- if $0 \notin dom\psi(x)$, then $g(x) = 0^\omega$;
- if $0 \in dom\psi(x)$ but $1 \notin dom\psi(x)$, then $g(x) = 0^{i_0}n_0 0^\omega$ for some $i_0 < \omega$, where $n_0 = \psi(x)(0)$;
- if $0, 1 \in dom\psi(x)$ but $2 \notin dom\psi(x)$, then $g(x) = 0^{i_0}n_0 0^{i_1}n_1 0^\omega$ for some $i_0, i_1 < \omega$, where $n_0 = \psi(x)(0)$ and $n_1 = \psi(x)(1)$;

  . . . .
- if $dom\psi(x) = \omega$ then $g(x) = 0^{i_0}n_0 0^{i_1}n_1 0^{i_2}n_2 \cdots$ for some $i_0, i_1, \ldots < \omega$, where $n_i = \psi(x)(i)$ for each $i < \omega$.

From the definition of π it follows that g is a π-totalizer of ψ.

For the other levels the assertion follows from Lemmas 3.4 and 6.4 because, as is well known [Wad84], any non-self-dual level of the Borel hierarchy and of the difference hierarchies coincides with $O_{\mathcal{A}}(X)$ for some Borel set $\mathcal{A} \subseteq P(\omega)$. $\square$

As is well known, precompleteness implies the Rice theorem. In particular, for the principal TR π of the open sets the Rice theorem looks as follows:

Proposition 6.6. *Let X be a countably based space and $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1(X)$. Then $\pi^{-1}(\mathcal{A}) \in \mathbf{\Delta}^0_1(\mathcal{N})$ iff $\mathcal{A} = \emptyset$ or $\mathcal{A} = \mathbf{\Sigma}^0_1(X)$.*

Proof. We consider the implication from left to right because implication in the opposite direction is obvious. Let $\pi^{-1}(\mathcal{A}) \in \mathbf{\Delta}^0_1(\mathcal{N})$ and suppose for a contradiction that $\emptyset \subset \mathcal{A} \subset \mathbf{\Sigma}^0_1(X)$, so $a \in \pi^{-1}(\mathcal{A}) \not\ni b$ for some $a, b \in \mathcal{N}$. Let f be the function on $\mathcal{N}$ that sends $\pi^{-1}(\mathcal{A})$ to b and the complement of $\pi^{-1}(\mathcal{A})$ to a. Since $\pi^{-1}(\mathcal{A})$ is clopen, f is continuous. By FPP for the precomplete TR π, $\pi(c) = \pi f(c)$ for some c, but this is contradictory. □

For some other levels of the classical hierarchies the Rice theorem have interesting modifications, in particular we have:

Proposition 6.7. *Let X be a countably based space, $\mathcal{A} \subseteq \mathbf{\Sigma}^{-1}_2(X)$, and let ν be a principal $\mathbf{\Sigma}^{-1}_2$-TR of $\mathbf{\Sigma}^{-1}_2(X)$. Then $\nu^{-1}(\mathcal{A}) \in \mathbf{\Delta}^{-1}_2(\mathcal{N})$ iff $\mathcal{A} = \emptyset$ or $\mathcal{A} = \mathbf{\Sigma}^{-1}_2(X)$.*

Proof. We consider the implication from left to right because implication in the opposite direction is obvious. Let $\nu^{-1}(\mathcal{A}) \in \mathbf{\Delta}^{-1}_2(\mathcal{N})$ and suppose for a contradiction that $\mathcal{A} \neq \{\emptyset, \mathbf{\Sigma}^{-1}_2(X)\}$. We may assume without loss of generality that $\emptyset \notin \mathcal{A}$ (otherwise, replace $\mathcal{A}$ by $\mathbf{\Sigma}^{-1}_2(X) \setminus \mathcal{A}$). Let $C \in \mathcal{A}$.

Let $A_0, A_1 \in \mathbf{\Sigma}^0_1(\mathcal{N})$ satisfy $\nu^{-1}(\mathcal{A}) = A_0 \setminus A_1$ and $A_0 \supseteq A_1$. Let $C_0, C_1 \in \mathbf{\Sigma}^0_1(X)$ satisfy $C = C_0 \setminus C_1$ and $C_0 \supseteq C_1$. From the definition of π it is straightforward to find continuous functions f_0, f_1 on $\mathcal{N}$ such that: $\pi f_0(x) = \emptyset$ for $x \in \mathcal{N} \setminus A_0$ and $\pi f_0(x) = C_0$ for $x \in A_0$; $\pi f_1(x) = C_1$ for $x \in \mathcal{N} \setminus A_1$ and $\pi f_1(x) = C_0$ for $x \in A_1$. Finally, let $f(x) = \langle f_0(x), f_1(x)\rangle$, then $\nu f(x) = \pi f_0(x) \setminus \pi f_1(x)$.

Note that $x \notin A_0$ implies $\nu f(x) = \emptyset$, $x \in A_0 \setminus A_1$ implies $\nu f(x) = C$, and $x \in A_1$ implies $\nu f(x) = \emptyset$. Altogether, $x \in \nu^{-1}(\mathcal{A})$ iff $f(x) \notin \nu^{-1}(\mathcal{A})$. By Theorem 6.5, ν has the FPP-property, i.e. $\nu(c) = \nu f(c)$ for some $c \in \mathcal{N}$. Then $c \in \nu^{-1}(\mathcal{A})$ iff $c \notin \nu^{-1}(\mathcal{A})$, a contradiction. □

7. Principal Continuous Total Representations

Working with a space X, it is natural to look at continuous TRs of X, hence it is instructive to ask for which spaces a principal TR in the class of continuous TRs exists. We call a TR γ of a space X *principal* if it is continuous, and any continuous TR $\nu : \mathcal{N} \to X$ is reducible to γ. In this section we show that principal continuous TRs share some basic properties of admissible representations [Wei00, Sch02, Sch03]. Our proofs are easy adaptations of the well known corresponding proofs for admissible representations.

We start with recalling some properties of sequential topologies. Let X be an arbitrary set. By a topology on X we mean the corresponding class of open sets. Let $\mathcal{T}(X)$ be the set of all topologies on X, and let $\tau \in \mathcal{T}(X)$. A sequence $\{x_n\}$ in X *τ-converges* to an element $x \in X$ if for any $U \in \tau$, the condition $x \in U$ implies that x_n is eventually in U (i.e., there is $n_0 < \omega$ such that $x_n \in U$ for all $n \geq n_0$). Let τ^s be the set of all $A \subseteq X$ such that for all $x, x_0, x_1, \ldots \in X$, if $\{x_n\}$ τ-converges to x and $x \in A$ then x_n is eventually in A. Note that our notation τ^s corresponds to notation $seq(\tau)$ in [Sch02]. The next two lemmas follow from well known facts in [En89, Sch02].

Lemma 7.1. *For any set X, $\tau \mapsto \tau^s$ is a closure operation on $(\mathcal{T}(X); \subseteq)$, i.e. $\tau^s \in \mathcal{T}(X)$ for $\tau \in \mathcal{T}(X)$, $\tau \subseteq \tau^s$, $(\tau^s)^s = \tau^s$, and $\tau \subseteq \tau_1$ implies $\tau^s \subseteq \tau^s_1$.*

For a space X, let τ_X denote the topology on X (i.e. $\tau_X = \mathbf{\Sigma}^0_1(X)$). A function $f : X \to Y$ between spaces is *sequentially continuous* if for all $x, x_0, x_1, \ldots \in X$ such that $\{x_n\}$ τ_X-converges to x, $\{f(x_n)\}$ τ_Y-converges to $f(x)$.

Lemma 7.2.

(1) *Any continuous function is sequentially continuous.*
(2) *If X is countably based then $\tau_X = \tau^s_X$.*
(3) *If $f : X \to Y$ is sequentially continuous and X is sequential (in particular, countably based) then f is continuous.*

The next result is a slight modification of the corresponding assertion for the admissible representations [Sch02]. For a TR γ of a set X, let τ_γ denote the *final topology of* γ on X consisting of all sets $A \subseteq X$ such that $\gamma^{-1}(A) \in \mathbf{\Sigma}^0_1(\mathcal{N})$

Theorem 7.3. *Let γ be a principal continuous TR of a space X.*

(1) *Any principal continuous TR of X is equivalent to γ.*
(2) *X is a T_0-space.*
(3) *For all $x, x_0, x_1, \ldots \in X$, $\{x_n\}$ τ_X-converges to x iff there exist $a \in \gamma^{-1}(x), a_0 \in \gamma^{-1}(x_0), a_1 \in \gamma^{-1}(x_1), \ldots$ such that $\{a_n\}$ $\tau_\mathcal{N}$-converges to a.*
(4) *τ_γ is the sequentialization of τ_X, i.e. $\tau^s_X = \tau_\gamma$.*

Proof. 1. Obvious.

2. Suppose not, so there are distinct $x, y \in X$ such that $\forall U \in \tau_X(x \in U \leftrightarrow y \in U)$. Then any function $\nu : \mathcal{N} \to \{x, y\}$ is continuous. Since γ is principal, $\nu \leq \gamma$ for all such ν. But this is not possible because there are hypercontinuum many of such $\nu : \mathcal{N} \to \{x, y\}$ and only continuum many of ν reducible to γ (because there are only continuum many continuous functions on $\mathcal{N}$).

3. Let $\{x_n\}$ τ_X-converges to x. Clearly, $A = \{0^\omega, 0^n1^\omega \mid n < \omega\}$ is a retract of $\mathcal{N}$, i.e. for some continuous function $r : \mathcal{N} \to A$ we have $r(a) = a$ for all $a \in A$. Moreover, $\{0^n1^\omega\}$ $\tau_\mathcal{N}$-converges to 0^ω, hence the function $g : A \to X$ is continuous where $g(0^\omega) = x$ and $g(0^n1^\omega) = x_n$ for all $n < \omega$. Since $g \circ r : \mathcal{N} \to X$ is continuous and γ is principal, $g \circ r = \gamma \circ f$ for some continuous function f on $\mathcal{N}$. Then elements $a = f(0^\omega)$, $a_n = f(0^n1^\omega)$ have the desired properties because $\{a_n\}$ $\tau_\mathcal{N}$-converges to a by continuity of f, $\gamma(a) = \gamma f(0^\omega) = gr(0^\omega) = g(0^\omega) = x$, and similarly $\gamma(a_n) = x_n$ for each $n < \omega$.

Conversely, let $a, a_0, a_1, \ldots$ have the specified properties, we have to check that $\{x_n\}$ τ_X-converges to x. Let $x \in U \in \tau_X$, then $a \in \gamma^{-1}(U)$, and $\gamma^{-1}(U) \in \tau_\mathcal{N}$ by continuity of γ. Since $\{a_n\}$ $\tau_\mathcal{N}$-converges to a, a_n is eventually in $\gamma^{-1}(U)$. Therefore, x_n is eventually in U.

4. Let $A \in \tau^s_X$. We have to show that $A \in \tau_\gamma$, i.e. $\gamma^{-1}(A) \in \tau_\mathcal{N}$, i.e. for any $a \in \gamma^{-1}(A)$ there is $n < \omega$ with $a[n] \cdot \mathcal{N} \subseteq \gamma^{-1}(A)$. Suppose for contradiction that there is $a \in \gamma^{-1}(A)$ such that $a[n] \cdot \mathcal{N} \not\subseteq \gamma^{-1}(A)$ for all $n < \omega$. For any $n < \omega$, choose $a_n \in a[n] \cdot \mathcal{N} \setminus \gamma^{-1}(A)$. Then $\{a_n\}$ $\tau_\mathcal{N}$-converges to a. By continuity of γ, $\{\gamma(a_n)\}$ τ_X-converges to $\gamma(a)$. Since $A \in \tau^s_X$ and $\gamma(a) \in A$, $\gamma(a_n)$ is eventually in A, hence $\{a_n\}$ is eventually in $\gamma^{-1}(A)$. A contradiction.

Conversely, let $A \in \tau_\gamma$, i.e. $\gamma^{-1}(A) \in \tau_\mathcal{N}$. Let $\{x_n\}$ τ_X-converge to $x \in A$; we have to show that x_n is eventually in A. Choose $a, a_0, a_1, \ldots$ as in item 3, so in particular $\{a_n\}$ $\tau_\mathcal{N}$-converges to $a \in \gamma^{-1}(A)$. Then a_n is eventually in $\gamma^{-1}(A)$, so $x_n = \gamma(a_n)$ is eventually in A. $\square$

The next important property of principal continuous TRs is again analogous to the corresponding property of admissible representations.

Theorem 7.4. *Let γ and δ be principal continuous TRs of spaces X and Y, respectively. Then $f : X \to Y$ is sequentially continuous iff there exists a continuous function $\hat{f} : \mathcal{N} \to \mathcal{N}$ with $f \circ \gamma = \delta \circ \hat{f}$. In particular, $A \mapsto \gamma^{-1}(A)$ is a homomorphism from $(P(X); \leq_W)$ into $(P(\mathcal{N}); \leq_W)$.*

Proof. Let f be sequentially continuous, then so is also $f \circ \gamma$. By Lemma 7.2, $f \circ \gamma$ is continuous. Since δ is principal, $f \circ \gamma = \delta \circ \hat{f}$ for some continuous function $\hat{f}$ on $\mathcal{N}$.

Conversely, let $\hat{f}$ be continuous with the specified property, and let $\{x_n\}$ τ_X-converge to x. Choose $a, a_0, a_1, \ldots$ as in item 3 of Proposition 7.3, so in particular $\{a_n\}$ $\tau_{\mathcal{N}}$-converges to a. Since $\delta \circ \hat{f}$ is continuous, $\{\delta \hat{f}(a_n)\}$ τ_Y-converges to $\delta \hat{f}(a)$, hence also $\{f\gamma(a_n)\}$ τ_Y-converges to $f\gamma(a)$. Therefore $\{f(x_n)\}$ τ_Y-converges to $f(x)$, as desired. □

Remarks 7.5.

1. . In numbering theory, a partial analogy to principal continuous TRs is provided by the so called approximable numberings [Er77, Se06].
2. We see that some important properties of principal continuous TRs are close to those of admissible representations which are very popular in CA. Obviously, every principal continuous TR of a space X that admits an admissible TR is already admissible. Also, every admissible TR is principal continuous. Unfortunately, currently we do not know whether the converse implication is also true. If yes, this would be a new interesting characterization of the admissible TRs (and we believe the results of this section could be useful to prove this). If no, we would obtain a new concept of interest for CA. In the next section we continue to discuss the relationships between admissible and principal continuous TRs.

8. Admissible Total Representations

A fundamental notion of CA is the notion of admissible representation, i.e. (in terminology of Section 5), principal continuous representations. This notion was introduced in [KW85] for countably based spaces and it was extensively studied by many authors. In [BH02] a close relation of admissible representations of countably based spaces to open continuous representations was established. In [Sch02] the notion was extended to non-countably based spaces and a nice characterization of the admissibly represented spaces was achieved. In [Sch95, Sch04] the admissible representations allowing a computational complexity theory in CA were identified.

As mentioned above, the previous study of admissible representations in CA paid no attention to TRs which was in a striking contrast with numbering theory where total numberings obviously dominate. But recently it became clear that the admissible TRs deserve more attention. Recall that a representation α of a space X (i.e., a partial surjection from $\mathcal{N}$ onto X) is *admissible* if it is continuous and any partial continuous function ϕ from $\mathcal{N}$ to X is reducible to α (i.e., there is a partial continuous function f on $\mathcal{N}$ such that $\phi(x) = \alpha f(x)$ for each $x \in dom(f)$).

Proposition 8.1. *If a space has a principal continuous TR then it has an admissible partial representation.*

Proof. Let γ be a principal continuous TR of a space X. By Theorem 13 in [Sch02], it suffices to show that X is a T_0-space which has a countable pseudobase. The T_0 property holds by item 2 of Theorem 7.3. A countable pseudobase for X may be constructed similarly to Lemma 11 in [Sch02]. Namely, let $\mathcal{B}$ be a countable base for $\mathcal{N}$ (say, $\mathcal{B} = \{\sigma{\cdot}\mathcal{N} \mid \sigma \in \omega^*\}$); we check that $\{\gamma(B) \mid B \in \mathcal{B}\}$ is a countable pseudobase for X. This by definition means that if $\{x_n\}$ τ_X-converges to $x \in U$ for some $U \in \tau_X$ then there is $B \in \mathcal{B}$ such that $\gamma(B) \subseteq U$, $x \in \gamma(B)$, and x_n is eventually in $\gamma(B)$ (i.e., there is $n_0 < \omega$ such that $x_n \in \gamma(B)$ for each $n \geq n_0$).

By item 3 of Theorem 7.3, there exist $a \in \gamma^{-1}(x), a_0 \in \gamma^{-1}(x_0), a_1 \in \gamma^{-1}(x_1), \ldots$ such that $\{a_n\}$ $\tau_{\mathcal{N}}$-converges to a. Since $\gamma(a) = x \in U$ and γ is continuous, $a \in \gamma^{-1}(U) \in \tau_{\mathcal{N}}$. Since $\mathcal{B}$ is a base for $\mathcal{N}$, $a \in B \subseteq \gamma^{-1}(U)$ for some $B \in \mathcal{B}$. Then a_n is eventually in B, hence x_n is eventually in $\gamma(B) \subseteq U$, and $x = \gamma(a) \in \gamma(B)$. □

From the recent paper of M. de Brecht [Bre13] it follows that admissible TRs are sufficient for treating a large and useful class of countably based spaces. The following assertion is contained among results in [Bre13], we only slightly reformulate it in order to put emphasis on total rather than partial representations.

Proposition 8.2. *For any countably based space X the following statements are equivalent:*
(1) *X is quasi-Polish.*
(2) *X has an open continuous TR.*
(3) *X has an admissible TR.*

Proof Sketch. 1→2. We reproduce the short proof from [Bre13]. By Proposition 3.6 we may assume that X is a $\mathbf{\Pi}^0_2$-subset of $P\omega$. The equation $\rho(a) = \{n \in \omega \mid n+1 \in rng(a)\}$ defines an open continuous TR of $P\omega$. Its restriction ρ' to $\rho^{-1}(X)$ is an open continuous surjection from the subspace $\rho^{-1}(X)$ of $\mathcal{N}$ onto X. Since $\rho^{-1}(X)$ is in $\mathbf{\Pi}^0_2(\mathcal{N})$, it is a Polish space by Theorem 3.11 in [Ke94]. By Exercise 7.14 in [Ke94], there is an open continuous TR f of $\rho^{-1}(X)$. Then $\rho' \circ f$ is an open continuous TR of X.

2→3. From (the proof of) Theorem 12 in [BH02] it follows that any open continuous TR is admissible.

3→1. A non-trivial result in [Bre13]. □

Remarks 8.3. 1. Note that any open continuous TR of X is automatically admissible but the converse does not hold in general [BH02].

2. From [Sch02] we know that sequential admissibly represented spaces form a cartesian closed category but, since they contain all countably based spaces, many of them have poor DST-properties (e.g., they in general do not satisfy the Hausdorff-Kuratowski theorem). From [Bre13] we know that the countably based admissibly totally representable spaces (i.e., the quasi-Polish spaces) have good DST-properties but, as recently M. Schröder has shown answering to my question, they do not form a cartesian closed category. It seems that combining the both properties (of being cartesian closed and having a good DST) is not possible for large enough classes of spaces. To my knowledge, only some rather small classes of domains are known to have both properties.
3. Let us stress that the question whether there is any space that admits a principal continuous TR but not an admissible TR remains open.

An advantage of admissible TRs (compared with partial admissible representations) is that index sets of TRs behave more “regularly” than those of the partial representations; we discuss this in more detail in Section 10. Another advantage is that it is a “more canonical” notion. We illustrate this by providing easy topological invariants for the quasi-Polish spaces in terms of admissible TRs.

Proposition 8.4. *Let α, β be admissible TRs of quasi-Polish spaces X, Y, respectively. Then the kernel relation (see Section 4) E_α is in $\mathbf{\Pi}^0_2(\mathcal{N})$, and $E_\alpha \equiv_W E_\beta$ whenever X and Y are homeomorphic. In particular, the Wadge degree of E_α is a topological invariant of X.*

Proof. The equality relation on X (as well as on arbitrary countably based T_0-space [Bre13]) is in $\mathbf{\Pi}^0_2(X \times X)$. Since $\langle a, b\rangle \in E_\alpha$ iff $\alpha(a) = \alpha(b)$, E_α is a continuous preimage of the equality relation, hence $E_\alpha \in \mathbf{\Pi}^0_2(\mathcal{N})$.

For the second assertion, assume that X, Y are homeomorphic. Then $\alpha \equiv \beta$, hence $E_\alpha \equiv_W E_\beta$. □

As is well known, the structure of Wadge degrees of $\mathbf{\Pi}^0_2(\mathcal{N})$ sets is very simple, namely it precisely corresponds to the Wadge complete sets in levels $\mathbf{\Sigma}^{-1}_\alpha$,$\mathbf{\Pi}^{-1}_\alpha$, $\mathbf{\Delta}^{-1}_{1+\alpha}$ $(\alpha < \omega_1)$ of the difference hierarchy over $\mathbf{\Sigma}^0_1(\mathcal{N})$, plus the Wadge degree of a $\mathbf{\Pi}^0_2$-complete set. Hence, the previous Proposition suggests a natural classification of quasi-Polish spaces X according to the mentioned level in which E_α is Wadge complete. Obviously, E_α cannot be in $\mathbf{\Sigma}^{-1}_0 = \{\emptyset\}$ (provided that we do not consider the empty space), $E_\alpha \in \mathbf{\Pi}^{-1}_0$ iff X is a singleton space, E_α is Wadge complete in $\mathbf{\Delta}^{-1}_1 = \mathbf{\Delta}^0_1$ iff X is a non-singleton discrete space (hence X is at most countable), and $E_\alpha \in \mathbf{\Sigma}^{-1}_1 = \mathbf{\Sigma}^0_1$ implies $E_\alpha \in \mathbf{\Delta}^0_1$. It is a nice open question to precisely characterize the quasi-Polish spaces in any class of this classification.

Although there is no such elegant classification of arbitrary admissibly represented spaces, one can use a slightly more complicated invariant. For a (partial) representation δ of a space X, let $E_\delta = \{\langle a, b\rangle \mid a, b \in dom(\delta) \wedge \delta(a) = \delta(b)\}$. Let $w(X)$ be the set of minimal Wadge degrees that contain E_δ for some admissible representation δ of X. From the structure of Wadge degrees it follows that (at least, under some more or less reasonable set-theoretic axioms) $w(X)$ always exists and it consists either of one or of two elements. A natural question is to characterize the range of the function w.

Remark. Note that the aforementioned classification of topological spaces is related to the separability axioms, in particular $E_{\alpha_X} \in \mathbf{\Pi}^0_1(\mathcal{N})$ for any Hausdorff space X (where α_X is an admissible TR of X). One could also measure the complexity of X by the complexity of singletons $\{x\}$ for $x \in X$ (or by the complexity of their index sets $\alpha^{-1}_X(\{x\})$). E.g., X is a T_1-space iff $\{x\} \in \mathbf{\Pi}^0_1(X)$ for each $x \in X$, X is a T_D-space [Bre13] iff $\{x\} \in \mathbf{\Sigma}^{-1}_2(X)$ for each $x \in X$, and if X is a countably based T_0-space then any singleton set is in $\mathbf{\Pi}^0_2(X)$ [Bre13]. Note that the last complexity measure (suggested by a referee) is related to the first one because any singleton set is a continuous preimage of the equality relation on X.

Another instructive question related to Proposition 8.2 is to investigate "natural" non-countably based spaces having an admissible TR. To see that such spaces exist consider again the principal TR π of $\mathbf{\Sigma}^0_1(X)$ for a countably based space X (see the proof of Theorem 5.2).

There are at least two natural topologies on $\mathbf{\Sigma}^0_1(X)$, for an arbitrary space X. First, this is the Scott topology σ on the complete lattice $(\mathbf{\Sigma}^0_1(X); \subseteq)$. Recall that $\mathcal{A} \in \sigma$ iff $\mathcal{A}$ is upward closed in $(\mathbf{\Sigma}^0_1(X); \subseteq)$, and $\mathcal{D} \cap \mathcal{A} \neq \emptyset$ for each directed subset $\mathcal{D}$ of $(\mathbf{\Sigma}^0_1(X); \subseteq)$ with $\bigcup \mathcal{D} \in \mathcal{A}$. Second, this is the compact-open topology κ, the basic open sets of which are of the form $O(K) = \{A \in \mathbf{\Sigma}^0_1(X) \mid K \subseteq A\}$ where K runs through the compact subsets of X.

For Polish spaces X the topologies σ and κ are known to coincide. In the general case we have:

Proposition 8.5. *Let X be an arbitrary topological space.*

(1) $\kappa \subseteq \sigma$.
(2) *If $\{A_n\}$ κ-converges to $A \in \mathbf{\Sigma}^0_1(X)$ and $K \subseteq A$ for some compact subset K of X then eventually $K \subseteq A_n$.*

Proof. 1. It suffices to show that $O(K) \in \sigma$ for each compact $K \subseteq X$. Clearly, $O(K)$ is upward closed in $(\mathbf{\Sigma}^0_1(X); \subseteq)$. It remains to show that if $\bigcup \mathcal{D} \in O(K)$ where $\mathcal{D}$ is a directed subset of $(\mathbf{\Sigma}^0_1(X); \subseteq)$ then $\mathcal{D} \cap O(K) \neq \emptyset$. Let $K \subseteq \bigcup \mathcal{D}$. Since K is compact, $K \subseteq D_0 \cup \cdots \cup D_n$ for some $n < \omega$ and $D_0, \ldots, D_n \in \mathcal{D}$. Since $\mathcal{D}$ is directed, $D_0 \cup \cdots \cup D_n \subseteq D$ for some $D \in \mathcal{D}$. Therefore $D \in \mathcal{D} \cap O(K)$.

2. Since $A \in O(K)$ and $O(K) \in \kappa$, A_n is eventually in $O(K)$, hence eventually $K \subseteq A_n$. □

Next we show that in many cases the TR π is admissible (cf. Propositions 4.4.1 and 4.4.3 in [Sch03]).

Theorem 8.6. *Let X be a countably based topological space. Then π is an admissible TR of both $(\mathbf{\Sigma}^0_1(X); \sigma)$ and $(\mathbf{\Sigma}^0_1(X); \kappa)$.*

Proof. First we show that π is continuous. Since $\kappa \subseteq \sigma$, it suffices to check that π is continuous with respect to σ, i.e. that $\pi^{-1}(\mathcal{A})$ is open in $\mathcal{N}$ for each $\mathcal{A} \in \sigma$. Let $a \in \pi^{-1}(\mathcal{A})$, i.e. $\pi(a) \in \mathcal{A}$; we have to show that $a[n] \cdot \mathcal{N} \subseteq \pi^{-1}(\mathcal{A})$ for some $n < \omega$. Let $A = \pi(a) = \bigcup_n B_{a(n)}$ and $A_n = B_{a(0)} \cup \cdots \cup B_{a(n-1)}$ for each $n < \omega$. Then $A_0 \subseteq A_1 \subseteq \cdots$ and $\bigcup_n A_n = A \in \mathcal{A}$. Since $\mathcal{A} \in \sigma$, there is $n < \omega$ with $A_n \in \mathcal{A}$. For any $b \in a[n] \cdot \mathcal{N}$ we then have $A_n \subseteq \pi(b)$, hence $\pi(b) \in \mathcal{A}$ and $b \in \pi^{-1}(\mathcal{A})$. Thus, $a[n] \cdot \mathcal{N} \subseteq \pi^{-1}(\mathcal{A})$.

It remains to show that, for each $\tau \in \{\sigma, \kappa\}$, any τ-continuous function $\nu : D \to \mathbf{\Sigma}^0_1(X)$, $D \subseteq \mathcal{N}$, is reducible to π, i.e. $\nu(d) = \pi f(d)$ for some continuous function $f : D \to \mathcal{N}$. Since $\kappa \subseteq \sigma$, it suffices to show this for $\tau = \kappa$.

First we show the auxiliary assertion that for all $d \in D$ and $x \in \nu(d)$ (i.e., $(d, x) \in U_\nu$) it holds:

$$\exists n \in \omega \exists A \in \mathbf{\Sigma}^0_1(X)(x \in A \wedge ((d[n] \cdot \mathcal{N}) \cap D) \times A \subseteq U_\nu).$$

Let $\{m \mid x \in B_m\} = \{m_0, m_1, \cdots\}$, then the formula above is equivalent to $\exists n(((d[n] \cdot \mathcal{N}) \cap D) \times A \subseteq U_\nu)$ where $A = B_{m_0} \cap \cdots \cap B_{m_n}$. Suppose that the auxiliary assertion is false, so $\forall n(((d[n] \cdot \mathcal{N}) \cap D) \times A \not\subseteq U_\nu)$. Then there are $b_0, b_1, \ldots \in \mathcal{N}$ and $x_0, x_1, \ldots \in X$ such that

$$\forall n \in \omega(d[n] \cdot b_n \in D \wedge x_n \in B_{m_0} \cap \cdots \cap B_{m_n} \wedge x_n \notin \nu(d[n] \cdot b_n)).$$

Since $\{d[n] \cdot b_n\}$ $\tau_{\mathcal{N}}$-converges to d and ν is κ-continuous, $\{\nu(d[n] \cdot b_n)\}$ κ-converges to $\nu(d)$. Since $x \in \nu(d) \in \mathbf{\Sigma}^0_1(X)$ and $\{x_n\}$ τ_X-converges to x, $K = \{x, x_{n_0}, x_{n_0+1}, \ldots\} \subseteq \nu(d)$ for some $n_0 < \omega$. Since K is compact in X, by item 2 of Proposition 8.5 there is $n_1 < \omega$ such that $\forall n \geq n_1(K \subseteq \nu(d[n] \cdot b_n))$. Then $x_n \in \nu(d[n] \cdot b_n)$ for all $n \geq n_0, n_1$ which is a contradiction.

Now we define $f : D \to \mathcal{N}$ as follows: $f(d)\langle i, j \rangle = j$, if $B_j \subseteq \bigcap \nu(d[i] \cdot \mathcal{N})$, and $f(d)\langle i, j \rangle = 0$ otherwise. Clearly, f is continuous (even Lipschitz), so it remains to check that $\nu(d) = \pi f(d)$ for each $d \in D$, i.e. $\nu(d) = \bigcup \{B_j \mid \exists i(B_j \subseteq \bigcap \nu(d[i] \cdot \mathcal{N}))\}$. The inclusion from right to left follows from $d \in d[i] \cdot \mathcal{N}$. Conversely, let $x \in \nu(d)$. By the auxiliary assertion, there are $i, j \in \omega$ such that $x \in B_j \subseteq \nu(d[i] \cdot b)$ for all $b \in \mathcal{N}$. Thus, x is in the right hand side of the equality. □

In particular, $\mathbf{\Sigma}^0_1(\mathcal{N})$ (with any of the topologies σ, κ) has an admissible TR. As is well known, the space $\mathbf{\Sigma}^0_1(\mathcal{N})$ is not countably based (see also Theorem 8.11). Thus, $\mathbf{\Sigma}^0_1(\mathcal{N})$ is a natural example of a non-countably based admissibly totally representable space.

Let τ_π be the final topology induced by π on $\mathbf{\Sigma}^0_1(X)$. From Theorem 7 in in [Sch02] we now obtain:

Corollary 8.7. *Let X be a countably based space. Then the final topology τ_π on $\mathbf{\Sigma}^0_1(X)$ coincides with the sequentialization of any of the topologies σ, κ, i.e. $\tau_\pi = \sigma^s = \kappa^s$.*

The last result may be interpreted as a topological analog of the classical Rice-Shapiro theorems in computability theory. The next result was first obtained in [HM82] for the case when X is Polish.

Corollary 8.8. *Let X be a countably based space and $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1(X)$. Then $\pi^{-1}(\mathcal{A}) \in \mathbf{\Sigma}^0_1(\mathcal{N})$ iff $\mathcal{A} \in \kappa^s$.*

Theorem 8.6 implies that π is an admissible TR of $(\mathbf{\Sigma}^0_1(X); \tau_\pi)$. Can this result be extended to other principal TRs of levels of the classical hierarchies from Section 5? The answer is no. We prove this here only for the class of differences of open sets.

Proposition 8.9. *Let X be a countably based space and let ν be a principal $\mathbf{\Sigma}^{-1}_2$-TR of $\mathbf{\Sigma}^{-1}_2(X)$. Then ν is not admissible w.r.t. τ_ν.*

Proof. By Theorem 13 in [Sch02], it suffices to show that any two points in $\mathbf{\Sigma}^{-1}_2(X)$ are not separable by sets in τ_ν (note that $\mathbf{\Sigma}^{-1}_2(X)$ has at least two elements for each non-empty X). Suppose the contrary: $A \in \mathcal{A} \not\ni B$ for some $A, B \in \mathbf{\Sigma}^{-1}_2(X)$ and some $\mathcal{A} \subseteq \mathbf{\Sigma}^{-1}_2(X)$ with $\nu^{-1}(\mathcal{A}) \in \mathbf{\Sigma}^0_1(\mathcal{N})$. By Proposition 6.7, $\mathcal{A} \in \{\emptyset, \mathbf{\Sigma}^{-1}_2(X)\}$. A contradiction. $\square$

Remark 8.10. Note that if $X = \{x\}$ is a singleton space then $(\mathbf{\Sigma}^0_1(X); \tau_\pi)$ is homeomorphic to the Sierpinski space, while the space $(\mathbf{\Sigma}^{-1}_2(X); \tau_\nu)$ consists of two points which are not separable by open sets. Note also that $(\mathbf{\Sigma}^0_1(\omega); \sigma)$ is homeomorphic to the domain $P\omega$.

We conclude this section with the following result (suggested by a referee) stating some interesting properties of the admissible TR of $\mathbf{\Sigma}^0_1(\mathcal{N})$.

Theorem 8.11. *Let π be the admissible TR of $\mathbf{\Sigma}^0_1(\mathcal{N})$. Then $E_\pi \in \mathbf{\Pi}^1_1(\mathcal{N})$, $\pi^{-1}(\{\mathcal{N}\})$ is Wadge complete in $\mathbf{\Pi}^1_1(\mathcal{N})$, and the space $\mathbf{\Sigma}^0_1(\mathcal{N})$ is not countably based.*

Proof. Let $\sigma_0, \sigma_1, \ldots$ be an enumeration without repetition of the set ω^* such that σ_0 is the empty string. Let $\{B_0, B_1, \ldots\}$ be the enumeration of a base in $\mathcal{N}$ where $B_0 = \emptyset$ and $B_{n+1} = \sigma_n \cdot \mathcal{N}$. We have

$$\pi(a) \subseteq \pi(b) \leftrightarrow \forall x \in \mathcal{N} \forall n \in \omega(x \in B_{a(n)} \to \exists m(x \in B_{b(m)})).$$

Since the predicates "$x \in B_{a(n)}$" and "$x \in B_{a(n)}$" are open in $\mathcal{N} \times \mathcal{N} \times \omega$, the predicate "$\pi(a) \subseteq \pi(b)$" is in $\mathbf{\Pi}^1_1(\mathcal{N} \times \mathcal{N})$. Therefore $E_\pi \in \mathbf{\Pi}^1_1(\mathcal{N})$.

From the previous paragraph it follows that $\pi^{-1}(\{A\}) \in \mathbf{\Pi}^1_1(\mathcal{N})$ for each $A \in \mathbf{\Sigma}^0_1(\mathcal{N})$, so in particular $\pi^{-1}(\{\mathcal{N}\}) \in \mathbf{\Pi}^1_1(\mathcal{N})$. For the second assertion of the theorem it remains to show that any $\mathbf{\Pi}^1_1(\mathcal{N})$-set is Wadge reducible to $\pi^{-1}(\{\mathcal{N}\})$.

Recall that a *tree in* ω^* is any subset T of ω^* closed under prefixes. It is well known that the closed subsets of $\mathcal{N}$ are precisely the sets $[T] = \{x \in \mathcal{N} \mid \forall n(x[n] \in T)\}$ where T ranges though the trees in ω^*, and that $[T] = \emptyset$ iff T is well founded, i.e. it contains no infinite ascending chain $\tau_0 \sqsubset \tau_1 \sqsubset \cdots$. Furthermore, $\mathcal{N} \setminus [T] = \bigcup\{\sigma \cdot \mathcal{N} \mid \sigma \in \partial T\}$ where

∂T is the set of minimal elements in $(\omega^* \setminus T; \sqsubseteq)$. For any trees T, S, we write $S \simeq T$ if there is an isomorphism φ of $(S; \sqsubseteq)$ onto $(T; \sqsubseteq)$ (note that we automatically have $|\sigma| = |\varphi(\sigma)|$ for each $\sigma \in S$).

Let W be the set of all $x \in \mathcal{N}$ such that the tree $T_x = \{\tau \mid \exists n(\tau \sqsubseteq \sigma_{x(n)})\}$ is well founded. It is well known (see e.g. Theorem 27.1 in [Ke94]) that W is Wadge complete in $\mathbf{\Pi}^1_1(\mathcal{N})$, hence it suffices to Wadge reduce W to $\pi^{-1}(\{\mathcal{N}\})$.

It is straightforward to define a continuous function g on $\mathcal{N}$ such that, for each $x \in \mathcal{N}$, $\{\sigma_{g(x)(n)} \mid n < \omega\} = \partial S_x$ where S_x is some tree with $S_x \simeq T_x$. Then the continuous function f on $\mathcal{N}$ defined by $f(x)(n) = g(x)(n) + 1$, is a desired Wadge reduction. Indeed, we have

$$\pi f(x) = \bigcup_n B_{f(x)(n)} = \bigcup_n \sigma_{g(x)(n)} \cdot \mathcal{N} = \mathcal{N} \setminus [S_x],$$

hence

$$x \in W \leftrightarrow [T_x] = \emptyset \leftrightarrow [S_x] = \emptyset \leftrightarrow \pi f(x) = \mathcal{N}.$$

For the last assertion, suppose that $\mathbf{\Sigma}^0_1(\mathcal{N})$ is countably based. By Proposition 9 in [Bre13], the equality relation on $\mathbf{\Sigma}^0_1(\mathcal{N})$ is then $\mathbf{\Pi}^0_2$. Since π is continuous, $E_\pi \in \mathbf{\Pi}^0_2(\mathcal{N})$, hence $\pi^{-1}(\{\mathcal{N}\}) \in \mathbf{\Pi}^0_2(\mathcal{N})$. This contradicts to the second assertion of the theorem. □

Remarks 8.12.

1. As noted in Section 3, for quasi-Polish spaces the class $\mathbf{\Sigma}^1_1$ coincides with the class of continuous images of Polish spaces. The last theorem implies that this characterization cannot be extended to the admissibly totally representable spaces because $\{\mathcal{N}\}$ is of course the image of a Polish spaces but it is not $\mathbf{\Sigma}^1_1$ (otherwise, we would get $\pi^{-1}(\{\mathcal{N}\}) \in \mathbf{\Sigma}^1_1(\mathcal{N})$ contradicting the third assertion of the theorem.)
2. It may be shown (as was noticed by M. de Brecht in a private communication) that any sequential admissibly represented space embeds into a sequential admissibly totally represented space (namely into the space $\mathbf{\Sigma}^0_1(X)$ for a suitable countably based space X). We hope that this result may be of use for the development of DST for non-countably based spaces, similarly to the use of the embeddability of all countably based spaces into $P\omega$ for the development of DST for quasi-Polish spaces [Bre13].
3. Although the class of sequential admissibly totally represented spaces is rather rich (by the previous remark), it does not form a cartesian closed category. This follows from results in [ScS12] where, in particular, the smallest (in some natural sense) cartesian closed category of admissibly represented spaces is identified.

9. Semilattices of $\mathbf{\Sigma}^0_1$-Total Representations

A popular field of numbering theory is the study of semilattices of computable numberings of classes of computably enumerable sets. This field is technically very complicated, even the characterization of the simplest such semilattice — the semilattice of computably enumerable m-degrees — is quite hard. A long-standing open problem [Er77, Er06] in this field is to find invariants for the isomorphism relation on the semilattices of computable numberings of finite classes of computably enumerable sets.

In this section we discuss the topological analog of this field. Again it turns out that the topological analog is much easier (though non-trivial). We resolve the topological analog of a problem related to the mentioned open problem of numbering theory. This makes use of some results mentioned in Section 4.

Simplifying notation, we denote $\mathbf{\Sigma}^0_1(\mathcal{N})$ just by $\mathbf{\Sigma}^0_1$. For $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1$, let $\mathcal{L}(\mathcal{A})$ (resp. $\mathcal{L}^*(\mathcal{A})$) be the set of all $\mathbf{\Sigma}^0_1$-TRs of $\mathcal{A}$ (resp. the set of all $\mathbf{\Sigma}^0_1$-TRs $\nu : \mathcal{N} \to \mathcal{A}$ of subsets of $\mathcal{A}$). Let $L(\mathcal{A})$ (resp. $L^*(\mathcal{A})$) denote the quotient-structure of the preorder $(\mathcal{L}(\mathcal{A}); \leq)$ (resp. $(\mathcal{L}^*(\mathcal{A}); \leq)$). Moreover, let $L^*_\perp(\mathcal{A})$ be obtained by adjoining a new bottom element $\perp$ to poset $L^*(\mathcal{A})$. We have the following topological analog of a well known simple fact about computable numberings.

Proposition 9.1.
(1) $L(\mathcal{A})$ *is an upper semilattice (in fact, a* σ*-semilattice).*
(2) $L^*_\perp(\mathcal{A})$ *is a distributive upper semilattice (in fact, a* σ*-semilattice).*

Proof. Supremums in both semilattices are obviously induced by the operation $\oplus$. Distributivity means that if $\xi \leq \mu \oplus \nu$ then $\xi \equiv \mu_1 \oplus \nu_1$ for some $\mu_1 \leq \mu, \nu_1 \leq \nu$ (the case of countable supremums is considered similarly). If $\xi = \perp$, take $\mu = \nu = \perp$. Otherwise, let f be a continuous function on $\mathcal{N}$ that reduces ξ to $\mu \oplus \nu$. Let $A_0 = \{a \in \mathcal{N} \mid \exists n(f(a)(0) = 2n)\}$ and $A_1 = \{a \in \mathcal{N} \mid \exists n(f(a)(0) = 2n + 1)\}$. Then A_0, A_1 are clopen and at least one of them is non-empty. If $A_0 = \emptyset$, take $\mu_1 = \perp, \nu_1 = \xi$. If $A_1 = \emptyset$, take $\mu_1 = \xi, \nu_1 = \perp$. If both sets A_0, A_1 are non-empty, choose for each $i < 2$ a homeomorphism f_i of $\mathcal{N}$ onto A_i and set $\mu_1 = \xi \circ f_0$ and $\nu_1 = \xi \circ f_1$. Then clearly $\mu_1 \leq \mu$ and $\nu_1 \leq \nu$, so it remains to check that $\xi \leq \mu_1 \oplus \nu_1$. Define a continuous function g on $\mathcal{N}$ as follows: $g(x) = 0 \cdot f_0^{-1}(x)$ for $x \in A_0$, and $g(x) = 1 \cdot f_1^{-1}(x)$ for $x \in A_1$. Then g reduces ξ to $\mu_1 \oplus \nu_1$. □

The semilattices $L(\mathcal{A})$ and $L^*_\perp(\mathcal{A})$ might be quite complicated even for a countable set $\mathcal{A}$. But if $\mathcal{A}$ is finite non-empty, the semilattices turn out to be finite distributive lattices. The topological analog of the mentioned problem from numbering theory is to find invariants for $L(\mathcal{A}) \simeq L(\mathcal{B})$ where $\simeq$ is the isomorphism relation. This topological question seems to be much easier than the mentioned problem (though we still do not know the exact answer). E.g., from our results it follows that there is an algorithm to answer the question $L(\mathcal{A})? \simeq L(\mathcal{B})$ if the finite posets $(\mathcal{A}; \subseteq)$ and $(\mathcal{B}; \subseteq)$ are given. The main result of this section is the following theorem that gives very simple invariants for the relation $L^*(\mathcal{A}) \simeq L^*(\mathcal{B})$.

Theorem 9.2. *Let* $\mathcal{A}, \mathcal{B}$ *be finite non-empty subsets of* $\mathbf{\Sigma}^0_1$*. Then* $L^*(\mathcal{A}) \simeq L^*(\mathcal{B})$ *iff* $(\mathcal{A}; \subseteq) \simeq (\mathcal{B}; \subseteq)$*.*

This result is a non-trivial corollary of some results in [Her93, Se04, Se07a, KS07]. In the rest of this section we recall some relevant information from those papers and deduce from them the main result. First we recall necessary information from [Se04] on k-labeled posets (see Section 4).

For a finite poset $P \in \mathcal{P}$, let $rk(P)$ denote *the rank of* P, i.e. the number of elements of the longest chain in P. For any $1 \leq i \leq rk(P)$, let $P(i) = \{x \in P \mid rk(\downarrow x) = i\}$. Then $P(1), \ldots, P(rk(P))$ is a partition of P to "levels"; note that $P(1)$ is the set of all minimal elements of P. For any $x \in P$, let $suc(x)$ denote the set of all immediate successors of x in P, i.e. $suc(x) = \{y \mid x < y \wedge \neg\exists z(x < z < y)\}$. Note that $suc(x) = \emptyset$ iff x is maximal in P. The next result is Lemma 1.1 in [Se04].

Lemma 9.3. *For any* $P \in \mathcal{P}$ *there exist* $F = F(P) \in \mathcal{F}$ *and a monotone function* f *from* F *onto* P *so that* $rk(F) = rk(P)$*,* f *establishes a bijection between* $F(1)$ *and* $P(1)$*, and for any* $x \in F$ f *establishes a bijection between* $suc(x)$ *and* $suc(f(x))$*. The forest* $F(P)$ *is obtained by a natural bottom-up unfolding of* P*.*

Now we recall some information about *minimal k-forests* from $\mathcal{F}_k$, i.e. k-forests not h-equivalent to a k-forest of lesser cardinality. The next fact is Lemma 1.3 in [Se04].

Lemma 9.4. *Any two minimal h-equivalent k-forests are isomorphic.*

The next inductive characterization of the minimal k-forests is Theorem 1.4 in [Se04].

Lemma 9.5.
(1) *Any singleton k-forest is minimal.*
(2) *A non-singleton k-tree (T, c) is minimal iff $\forall x \in T(1) \forall y \in T(2)(c(x) \neq c(y))$ and the k-forest $(T \setminus T(1), c)$ is minimal.*
(3) *A proper k-forest is minimal iff all its k-trees are minimal and pairwise incomparable under $\leq_h$.*

For any finite non-empty set $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1$, let $k = |\mathcal{A}|$, $\mathcal{A} = \{A_0, \ldots, A_{k-1}\}$, and $c(A_i) = i$ for each $i < k$. Then we may think that $(\mathcal{A}; \subseteq)$ is in $\mathcal{P}$, the unfolding $F(\mathcal{A})$ of $(\mathcal{A}; \subseteq)$ is in $\mathcal{F}$, $(\mathcal{A}; \subseteq, c)$ is in $\mathcal{P}_k$, and $(F(\mathcal{A}); c \circ f)$ is in $\mathcal{F}_k$. The next lemma follows from the previous one and the fact that the labeling $c : \mathcal{A} \to k$ is bijective.

Lemma 9.6. *For any finite non-empty set $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1$, the k-forest $(F(\mathcal{A}); c \circ f)$ is minimal.*

There is a close relation of $L^*(\mathcal{A})$ to the difference hierarchy of k-partitions over the open sets. This hierarchy developed in [KW00, Ko00, Se04, Se07a] extends from sets to k-partitions the Hausdorff difference hierarchy over the open sets. For any $P \in \mathcal{P}$, let $\mathbf{\Sigma}^0_1[P]$ be the set of functions $\nu : \mathcal{N} \to P$ *defined by P-families $\{A_p\}_{p \in P}$ of open sets*, i.e. there is a family $\{A_p\}_{p \in P}$ of open sets such that $\nu(x) = p$ iff $x \in A_p \setminus \bigcup\{A_q \mid p < q\}$, for all $p \in P, x \in \mathcal{N}$. For a k-poset $(P; d) \in \mathcal{P}_k$, define the set $\mathbf{\Sigma}^0_1[P, d]$ of k-partitions of $\mathcal{N}$ by $\mathbf{\Sigma}^0_1[P, d] = \{d \circ \nu \mid \nu \in \mathbf{\Sigma}^0_1[P]\}$.

Items 1,2 of the following lemma follow from Theorem 7.6 in [Se07a], item 3 follows from Theorem 3.1 in [Se04] (with a heavy use of the ω-reduction property of the open sets, see Theorem 3.5), and item 4 follows from Lemma 5.1 in [Se04]. For the definition of ξ_G see Section 4.

Lemma 9.7.
(1) *For any $G \in \mathcal{F}$, $\mathbf{\Sigma}^0_1[G] = \{\nu \in G^{\mathcal{N}} \mid \nu \leq \xi_G\}$, i.e. ξ_G is a complete element of $\mathbf{\Sigma}^0_1[G]$ with respect to $\leq$.*
(2) *For any $(G, d) \in \mathcal{F}_k$, $\mathbf{\Sigma}^0_1[G, d] = \{\nu \in G^{\mathcal{N}} \mid \nu \leq d \cdot \xi_G\} \subseteq (BC(\mathbf{\Sigma}^0_1))_k$.*
(3) *For any $P \in \mathcal{P}$, $\mathbf{\Sigma}^0_1[P] = \{f \circ \nu \mid \nu \in \mathbf{\Sigma}^0_1[F(P)]\}$.*
(4) *For any finite non-empty set $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1$, $\mathcal{L}^*(\mathcal{A}) = \mathbf{\Sigma}^0_1[\mathcal{A}, \subseteq]$.*

Next we establish a close relationship of $L(\mathcal{A})$ and $L^*(\mathcal{A})$ to some segments of the quotient-poset $\mathbb{F}_k$ of the preorder $(\mathcal{F}_k; \leq_h)$. For $a, b \in \mathbb{F}_k$, let $\downarrow a = \{x \mid x \leq_h a\}$ and $[b, a] = \{x \mid b \leq_h x \leq_h a\}$. For any $i < k$, let e_i be the h-equivalence class of a singleton k-forest labeled by i, so $\{e_0, \ldots, e_{k-1}\}$ is the enumeration without repetition of the minimal elements of $\mathbb{F}_k$. Let $e = e_0 \sqcup \cdots \sqcup e_{k-1}$.

Proposition 9.8. *For any finite non-empty set $\mathcal{A} \subseteq \mathbf{\Sigma}^0_1$, $L^*(\mathcal{A}) \simeq \downarrow a$ and $L(\mathcal{A}) \simeq [e, a]$, where a is the h-equivalence class of $(F(\mathcal{A}); \subseteq, c \circ f)$.*

Proof. An isomorphism between $\downarrow a$ and $L^*(\mathcal{A})$ is the restriction to $\downarrow a$ of the function induced by the map $(G, d) \mapsto A \circ d \circ \xi_G$. Indeed, if $(G, d) \leq_h (F(\mathcal{A}); \subseteq, c \circ f)$ then we subsequently deduce from Lemma 9.7 that $\xi_G \in \mathbf{\Sigma}^0_1[G]$, $d \circ \xi_G \in \mathbf{\Sigma}^0_1[G, d]$, $A \circ d \circ \xi_G \in$

$\mathbf{\Sigma}^0_1[\mathcal{A};\subseteq]=\mathcal{L}^*(\mathcal{A})$. Since $A:k\mapsto\mathcal{A}$ is a bijection, from Proposition 4.2 we obtain that $(G,d)\leq_h(G_1,d_1)$ is equivalent to $A\circ d\circ\xi_G\leq A\circ d_1\circ\xi_{G_1}$.

For the relation $L^*(\mathcal{A})\simeq\downarrow a$ we still have to show that any $\nu\in\mathcal{L}^*(\mathcal{A})$ is equivalent to $A\circ d\circ\xi_G$ for some $(G,d)\leq_h(F(\mathcal{A});\subseteq,c\circ f)$. By items 3 and 4 of Lemma 9.7, $\nu=f\circ\mu$ for some $\mu\in\mathbf{\Sigma}^0_1[F(\mathcal{A})]$, hence $c\circ\nu=c\circ f\circ\mu\in\mathbf{\Sigma}^0_1[F(\mathcal{A}),c\circ f]$. By items 1 and 2 of Lemma 9.7, $c\circ\nu\in(BC(\mathbf{\Sigma}^0_1))_k$. By Proposition 4.2, $c\circ\nu\equiv d\circ\xi_G$ for some $(G,d)\in\mathcal{F}_k$, so it remains to show that $(G,d)\leq_h(F(\mathcal{A});\subseteq,c\circ f)$. Suppose the contrary, then $d\circ\xi_G\not\in\mathbf{\Sigma}^0_1[F(\mathcal{A}),c]$ by item 2 of Lemma 9.7. By items 3 and 4 of Lemma 9.7, $\nu\not\in\mathbf{\Sigma}^0_1[\mathcal{A}]=\mathcal{L}^*(\mathcal{A})$ which is a contradiction.

It remains to show that $L(\mathcal{A})\simeq[e,a]$. For any $i<k$, let $\mu_i=\lambda x.A_i$, then clearly $\mu=\mu_0\oplus\cdots\oplus\mu_{k-1}$ is a smallest element in $(\mathcal{L}(\mathcal{A});\leq)$. Moreover, the isomorphism above sends e to the equivalence class of μ. Therefore the restriction of that isomorphism to $[e,a]$ is a desired isomorphism between $[e,a]$ and $L(\mathcal{A})$. $\square$

From the previous proposition and Proposition 9.1 we immediately obtain:

Corollary 9.9.
1. *For any finite non-empty set $\mathcal{A}\subseteq\mathbf{\Sigma}^0_1$, $L^*(\mathcal{A})$ and $L(\mathcal{A})$ are finite distributive lattices.*
2. *From given finite posets $(\mathcal{A};\subseteq)$ and $(\mathcal{B};\subseteq)$ one can compute whether $L^*(\mathcal{A})\simeq L^*(\mathcal{B})$ (or $L(\mathcal{A})\simeq L(\mathcal{B})$).*

We also need a result on automorphisms of $\mathbb{F}_k$. Let $Aut(\mathbb{F}_k)$ (resp. $Aut(k)$) denote the group of all automorphisms of $\mathbb{F}_k$ (resp. of all permutations of labels $0,\ldots,k-1$). For any $x\in\mathbb{F}_k$, let $M(x)$ be the set of minimal elements of $\mathbb{F}_k$ below x (this set is in a bijective correspondence with the set of labels in some, equivalently in any, k-forest in the h-equivalence class x). Any permutation $p\in Aut(k)$ induces the automorphism $(G,d)\mapsto(G,p\circ d)$ of $\mathbb{F}_k$ which is for simplicity denoted by the same letter p. We call elements $x,y\in\mathbb{F}_k$ *automorphic* if $g(x)=y$ for some $g\in Aut(\mathbb{F}_k)$.

Proposition 9.10. *For all $x,y\in\mathbb{F}_k$, $\downarrow x\simeq\downarrow y$ iff x,y are automorphic.*

Proof. One direction is obvious. Conversely, it suffices to show that for any isomorphism h from $\downarrow x$ onto $\downarrow y$ there is $p\in Aut(k)$ with $p(x)=h(x)$. This is checked by induction on the rank $rk(x)$ of x in $\mathbb{F}_k$. If $rk(x)=1$ then x is minimal, hence y is also minimal and the assertion is obvious. The assertion is also easy in case $|M(x)|=2$ because then $|M(y)|=2$ and the structure $\mathbb{F}_2$ is almost well ordered and of rank ω (in fact, it is isomorphic to the structure of finite levels of the difference hierarchy of sets under inclusion). So assume $|M(x)|\geq3$ and consider two cases depending on whether x is join-irreducible in the distributive lattice $\mathbb{F}_k$ enriched by a bottom element.

If x is not join-irreducible then $x=x_0\sqcup\cdots\sqcup x_n$ for some $n\geq1$ and some join-irreducible pairwise incomparable $x_0,\ldots,x_n<x$. Let h_i be the restriction of h to $\downarrow x_i$, then h_i is an isomorphism $\downarrow x_i$ onto $\downarrow y_i$ for each $i\leq n$ where $y_i=h(x_i)$. By induction, there are $p_0,\ldots,p_n\in Aut(k)$ such that $p_i(x_i)=h_i(x_i)=y_i$ for all $i\leq n$. Then $p_i(b)=h(b)=p_j(b)$ for all $i,j\leq n$ and $b\in M(x_i)\cap M(x_j)$, hence there is $p\in Aut(k)$ such that $p(b)=p_i(b)$ for all $i\leq n$ and $b\in M(x_i)$. Then $p(x_i)=p_i(x_i)=y_i$ for all $i\leq n$, hence $p(x)=p(x_0)\sqcup\cdots\sqcup p(x_n)=y_0\sqcup\cdots\sqcup y_n=y$.

Finally, let x be join-irreducible, hence y is also join-irreducible. Let $x'=\bigsqcup\{z\mid z<x\}$ and let y' be obtained similarly from y. Then $x'<x$, $\forall z<x(z\leq x')$ and similarly for y. By induction, $p(x')=h(x')$ for some $p\in Aut(k)$. By Lemma 5 in [KS07], the function

$a \mapsto a'$ on the join-irreducible elements a with $|M(a)| \geq 3$ is injective, hence $y' = h(x')$ and $p(x) = y$. □

Proof of Theorem 9.2. It is easy to see that $(\mathcal{A}; \subseteq) \simeq (\mathcal{B}; \subseteq)$ implies $L^*(\mathcal{A}) \simeq L^*(\mathcal{B})$. Conversely, let $L^*(\mathcal{A}) \simeq L^*(\mathcal{B})$. Then $|\mathcal{A}| = k = |\mathcal{B}|$ because $|\mathcal{A}|$ and $|\mathcal{B}|$ are the numbers of minimal elements in $L^*(\mathcal{A})$ and $L^*(\mathcal{B})$, respectively. By Proposition 9.8, $\downarrow a \simeq \downarrow b$ where a and b are the h-equivalence classes of the k-forests $(F(\mathcal{A}); \subseteq, c \circ f)$ and $(F(\mathcal{B}); \subseteq, c_1 \circ f_1)$, respectively. By the previous proposition, $p(a) = b$ for some $p \in Aut(k)$, i.e. $(F(\mathcal{A}); \subseteq, p \circ c \circ f) \equiv_h (F(\mathcal{B}); \subseteq, c_1 \circ f_1)$. By Lemmas 9.6 and 9.4, the last k-posets are even isomorphic via some isomorphism $\varphi : F(\mathcal{A}) \to F(\mathcal{B})$, so in particular $p \circ c \circ f = c_1 \circ f_1 \circ \varphi$. Therefore, $A_i \mapsto B_{p(i)}$ is an isomorphism of $(\mathcal{A}; \subseteq)$ onto $(\mathcal{B}; \subseteq)$. □

10. Category of Total Representations

Here we briefly discuss the category $\mathcal{N}Set$ of TRs (which is a topological version of the category of numbered sets in numbering theory [Er73a, Er75, Er77]) and its relation to the study of index sets and k-partitions.

The category $\mathcal{N}Set$ is formed by arbitrary TRs as objects and by the morphism between TRs defined as follows: a *morphism* $f : \mu \to \nu$ of TRs μ and ν is a function $f : \mu(\mathcal{N}) \to \nu(\mathcal{N})$ such that $f \circ \mu \leq \nu$ (in other words, $f \circ \mu = \nu \circ \hat{f}$ for some continuous function $\hat{f}$ on $\mathcal{N}$ called a *realizer* of f w.r.t. μ, ν).

Category $\mathcal{N}Set$ has some natural subcategories. E.g., relate to any equivalence relation E on $\mathcal{N}$ the TR $\kappa_E(x) = [x]_E = \{y \mid (x,y) \in E\}$ of the quotient-set $\mathcal{N}/E$. Let $\mathcal{N}Eq$ be the full subcategory of $\mathcal{N}Set$ with those κ_E as the objects. The proof of the next assertion is straightforward, so we give only a hint.

Proposition 10.1. *The category $\mathcal{N}Set$ has countable products and coproducts and is equivalent to the small category $\mathcal{N}Eq$.*

Proof Hint. For a sequence $\{\nu_n\}$ of TRs, let P (resp. Q) be the Cartesian product (resp. the disjoint union) of the sequence of sets $\{\nu_n(\mathcal{N})\}$. Then P consist of all sequences $(\nu_0(x_0), \nu_1(x_1), \ldots)$ where $x_n \in \mathcal{N}$. The product ν of $\{\nu_n\}$ in $\mathcal{N}Set$ is given by $\nu\langle x_0, x_1, \ldots\rangle = (\nu_0(x_0), \nu_1(x_1), \ldots)$. The set Q consists of all pairs $(n, \alpha_n(y))$ where $n < \omega, y \in \mathcal{N}$. The coproduct μ of $\{\nu_n\}$ in $\mathcal{N}Set$ is given by $\mu(n \cdot x) = (n, \nu_n(x))$.

The equivalence of categories $\mathcal{N}Set$ and $\mathcal{N}Eq$ is given by the inclusion functor $I : \mathcal{N}Eq \to \mathcal{N}Set$ and the kernel functor $K : \mathcal{N}Set \to \mathcal{N}Eq$ defined by $K(\nu) = \kappa_{E_\nu}$ on objects (where $E_\nu = \{(x,y) \mid \nu(x) = \nu(y)\}$) and by $K_f([x]_{E_\mu}) = [f(x)]_{E_\nu}$ on morphisms $f : \mu \to \nu$. □

Let $\mathcal{N}Ad$ be the full subcategory of $\mathcal{N}Set$ formed by the admissible TRs α w.r.t. the final topology on $\alpha(\mathcal{N})$. By a well known property of admissible representations [Wei00] (see also Theorem 7.4), the morphisms of $\mathcal{N}Ad$ are precisely the continuous functions. By Proposition 8.2, $\alpha \mapsto \alpha(\mathcal{N})$ is a functor from $\mathcal{N}Ad$ onto the category of sequential topological spaces having an admissible TR, with the continuous functions as morphisms.

Note that, using other reducibilities from Section 4, one can form some other categories of TRs, in particular the categories $\mathcal{N}Set(\mathbf{\Delta}^0_\alpha)$ (resp. $\mathcal{N}Set(\mathbf{\Delta}^1_1)$) which have the TRs as objects and the functions realized by the $\mathbf{\Delta}^0_\alpha$-functions (resp. by the $\mathbf{\Delta}^1_1$-functions) on the names. We would like to see some work on properties and applications of these categories.

We conclude this section by some remarks on the index sets and k-partitions in topology. For an arbitrary sequential admissibly totally representable space X, we denote by α_X an admissible TR of X. The topological complexity of subsets A of X may be measured by the Wadge degree of its index set $\alpha_X^{-1}(A)$: the structure of Wadge degrees guarantees that this complexity is essentially an ordinal. A similar situation is well known in computability theory but there, because of the complexity of the structure of m-degrees, the complexity of a set is measured not by the m-degree of its index set (which is the computable analog of the Wadge degree) but rather by the position of the index set in a suitable hierarchy. Note that in the study of index sets we again see the advantage of TRs against representations because the Wadge degree of an index set in a partial representation depends not only on the set A but also on the domain of the representation.

Note that, in contrast with computability theory, the topological complexity of $A \subseteq X$ may be in principle measured "directly" by the Wadge degree of A in the structure $(P(X); \leq_W^X)$. But here we get the obstacle that for many spaces X the structure of Wadge degrees of subsets of X is complicated (in particular, this applies to the space of reals [Her96]), so we again may have no convenient scale to measure the topological complexity. For these reasons the index set approach is often more useful. Note that $A \mapsto \alpha_X^{-1}(A)$ is a homomorphism from $(P(X); \leq_W^X)$ into $(P(\mathcal{N}); \leq_W)$.

The mentioned approach to topological complexity may be in a straightforward way extended to the study of topological complexity of k-partitions of X (and even of more complex functions on spaces). Relate to any k-partition $A : X \to k$ the k-partition $\alpha_X \circ A$ of $\mathcal{N}$. We call $\alpha_X \circ A$ the *index k-partition* of A (cf. [Se05]) because for $k = 2$ the index k-partitions essentially coincide with the index sets. The topological complexity of A is measured by the equivalence class of $\alpha_X \circ A$ in the quotient-structure of $(k^{\mathcal{N}}; \leq)$, see Section 4. This suggests a way to measure the topological complexity of k-partitions, and to compare the complexity of k-partitions of different spaces. E.g., for k-partitions $A : X \to k$ and $B : Y \to k$ of quasi-Polish spaces X, Y we say that A is *explicitly reducible* (resp. *implicitly reducible*) to B if $A = B \circ f$ for a continuous function $f : X \to Y$ (resp. if $\alpha_X \circ A \leq \alpha_Y \circ B$). Note that if A is explicitly reducible to B then it is also implicitly reducible. In [Se82] similar concepts (called there generalized index sets and reducibility by morphisms) were introduced and studied in the context of computability theory.

We give an example from [Her96] relevant to CA which illustrates the above notions. Let $\mathbb{C}$ be the space of complex numbers and, for each $n \geq 1$, let $\mathbb{P}_n$ be the set of polynomials $p = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ with complex coefficients; $\mathbb{P}_n$ may be considered as a space homeomorphic to $\mathbb{C}^n$. Define functions $c : \mathbb{C}^n \to \{1, \ldots, n\}$ and $r : \mathbb{P}_n \to \{1, \ldots, n\}$ as follows: $c(z_0, \ldots, z_{n-1})$ is the cardinality of the set $\{z_0, \ldots, z_{n-1}\}$, and $r(p)$ is the cardinality of the set of complex roots of p. Then c is explicitly reducible to r (a reduction is given by the Vieta map $(z_0, \ldots, z_{n-1}) \mapsto (z - z_0) \cdots (z - z_{n-1})$). We do not know whether r is explicitly reducible to c but certainly r is implicitly reducible to c (via a function that computes from an $\alpha_{\mathbb{P}_n}$-name of a polynomial some $\alpha_{\mathbb{C}^n}$-name of a vector of all its roots). Therefore, $\alpha_{\mathbb{C}^n} \circ c \equiv \alpha_{\mathbb{P}_n} \circ r$, hence the complexity of these problems is measured by the same element of the quotient-structure of $(k^{\mathcal{N}}; \leq)$, in fact of $((BC(\mathbf{\Sigma}_1^0))_k; \leq)$. In a sense, this shows that functions c and r have the same topological complexity (called discontinuity degree in [Her93]). By the result of P. Hertling in [Her93] mentioned in Section 4, this complexity is characterized by a finite k-labeled forest, and this forest (in fact, a linear order) is computed in [Her96]. Note that TRs $\alpha_{\mathbb{C}^n}$ and $\alpha_{\mathbb{P}_n}$ may be chosen so that the equivalence $\alpha_{\mathbb{C}^n} \circ c \equiv \alpha_{\mathbb{P}_n} \circ r$ holds even effectively, i.e. there are computable reductions in both directions.

11. Reducibilities of Equivalence Relations

A popular topic in DST is the study of some reducibilities on equivalence relations on the Baire space (see e.g. [Ka08, Gao09] for surveys). Here we note that these reducibilities fit well to our framework and answer a natural question for some of the corresponding degree structures.

The most popular reducibilities on equivalence relations are defined as follows. For equivalence relations E, F on $\mathcal{N}$, E is *continuously* (resp. *Borel*) reducible to F, in symbols $E \leq_c F$ (resp. $E \leq_B F$) if there is a continuous (resp. a Borel) function f on $\mathcal{N}$ such that for all $x, y \in \mathcal{N}$, $E(x, y)$ is equivalent to $F(f(x), f(y))$. Note that these reducibilities are closely related to (in fact, are strengthenings of) the corresponding explicit reducibilities from the previous section.

The structures $(ER(\mathcal{N}); \leq_c)$ and $(ER(\mathcal{N}); \leq_B)$ where $ER(\mathcal{N})$ is the set of all equivalence relations on $\mathcal{N}$, and especially their substructures on the set of Borel equivalence relations, were intensively studied in DST. In particular, it was shown that both structures are rather rich. But, to my knowledge, no result about the complexity of first-order theories of these structures and their natural substructures was established so far. Such results are desirable, as the history of degree structures in computability theory demonstrates.

Below we show that most of the natural substructures of the first structure have undecidable first-order theories (unfortunately, our methods do not apply to the second structure, so for the Borel reducibility the question remains open). We concentrate first on the initial segment $(ER_k; \leq_c)$ of $(ER(\mathcal{N}); \leq_c)$ formed by the set ER_k of equivalence relations which have at most k equivalence classes. We relate this substructure to the structure $(k^{\mathcal{N}}; \leq_1')$ where $\leq_1'$ is the following slight modification of the reducibility $\leq_1$ in Section 4: $\mu \leq_1' \nu$ iff $\mu = \varphi \circ \nu \circ f$ for some continuous function f on $\mathcal{N}$ and for some permutation φ of $\{0, \ldots, k-1\}$.

Proposition 11.1. *For any $2 \leq k < \omega$, the function $\nu \mapsto E_\nu$ induces an isomorphism between the quotient-structures of $(k^{\mathcal{N}}; \leq_1')$ and $(ER_k; \leq_c)$.*

Proof. First we check that $\mu \leq_1' \nu$ iff $E_\mu \leq_c E_\nu$ via f. Let $\mu \leq_1' \nu$, so $\mu = \varphi \circ \nu \circ f$ for some continuous function f on $\mathcal{N}$ and for some permutation φ of $\{0, \ldots, k-1\}$. Then $E_\mu \leq_c E_\nu$ via f.

Conversely, let $E_\mu \leq_c E_\nu$ via f. Define the function $\psi : \mu(\mathcal{N}) \to k$ by $\psi(\mu(x)) = \nu f(x)$. Since $\mu(x) = \mu(y)$ implies $\nu f(x) = \nu f(y)$, ψ is correctly defined. Since $\mu(x) \neq \mu(y)$ implies $\nu f(x) \neq \nu f(y)$, ψ is injective. Let φ be a permutation of k so that $\varphi\psi(i) = i$ for each $i \in \mu(\mathcal{N})$. Then $\varphi\nu f(x) = \varphi\psi\mu(c) = \mu(x)$, hence $\mu \leq_1' \nu$.

To complete the proof, it suffices to show that for any $E \in ER_k$ there is $\nu \in k^{\mathcal{N}}$ with $E = E_\nu$. Let $(E_0, \ldots, E_i)$ be an enumeration without repetition of the equivalence classes of E. Define $\nu : \mathcal{N} \to \{0, \ldots, i\}$ by $\nu(x) = j \leftrightarrow x \in E_j$, for all $j \leq i$ and $x \in \mathcal{N}$. Then $E = E_\nu$. $\square$

Theorem 11.2. *Let $k \geq 3$ and let A be any initial segment of $(ER(\mathcal{N}); \leq_c)$ that contains all relations in $ER_k \cap BC(\mathbf{\Sigma}^0_1(\mathcal{N}))$. Then the first-order theory of the quotient-structure of $(A; \leq_c)$ is undecidable.*

Proof. Let $B = \{\nu \in k^{\mathcal{N}} \mid E_\nu \in A\}$. By the previous proposition it suffices to show that the first-order theory of the quotient-structure of $(B; \leq'_1)$ is undecidable. By Theorem 2 in [KSZ10], the first-order theory of the quotient-structure of $(B; \leq_1)$ is undecidable. An inspection of that proof shows that it also works for the relation $\leq'_1$. □

12. Conclusion

We hope that this paper demonstrates that total representations deserve special attention because they are sufficient to represent many spaces of interest, appear naturally as the principal TRs of levels of the popular hierarchies, simplify and uniform presentation of some topics, suggest new open questions and make a much better analogy with the numbering theory than the partial representations. At the same time, there are several important topics (in particular, complexity in analysis, functionals of finite type or the study of rich enough cartesian closed categories of spaces) where partial representations are really inevitable.

Acknowledgement. I am grateful to Matthew de Brecht for providing a preliminary copy of his paper [Bre13]. I also thank Vasco Brattka, Peter Hertling and Matthias Schröder for stimulating discussions, and to the anonymous referees for many useful suggestions and bibliographical hints.

References

[An06] A. Andretta. More on Wadge determinacy. *Annals of Pure and Applied Logic*, 144(1-3), 2–32 (2006).

[Ba00] A. Bauer. *The Realizability Approach to Computable Analysis and Topology.* PhD thesis, Carnegy Mellon, 2000.

[BG11] V. Brattka and G. Gherardi. Weihrauch degrees, omniscience principles and weak computability. *Journal of Symbolic Logic*, 76(1):143–176, 2011, arXiv:1108.1445v1 [math.LO] 6 Aug 2011.

[BG11a] V. Brattka and G. Gherardi. Effective choice and boundedness principles in computable analysis. *Bulletin of Symbolic Logic*, 17(1):73–117, 2011.

[BH02] V. Brattka and P. Hertling. Topological properties of real number representations. *Theoretical Computer Science*, 284 (2002), 241–257.

[Bra05] V. Brattka, Effective Borel measurability and reducibility of functions, *Mathematical Logic Quarterly*, 51:1 (2005), 19–44.

[Bre13] M. de Brecht. Quasi-Polish spaces. *Annals of Pure and Applied Logic*, 164(3) (2013), 356–381.

[EMS87] F. van Engelen, A. Miller and J. Steel. Rigid Borel sets and better quasiorder theory. *Contemporary mathematics*, 65 (1987), 199–222.

[En89] R. Engelking. *General Topology.* Heldermann, Berlin, 1989.

[Er73a] Y.L. Ershov. Theorie der Numerierungen I. *Zeitschr. math. Logik Grundl. Math.*, 19 (1973), 289–388.

[Er75] Y.L. Ershov. Theorie der Numerierungen II. *Zeitschr. math. Logik Grundl. Math.*, 21 (1975), 473—584.

[Er77] Yu.L. Ershov. *Theory of Numberings.* Moscow, Nauka, 1977 (in Russian).

[Er06] Yu.L. Ershov. Rogers Semilattices of Finite Partially Ordered Sets. *Algebra and Logic*, 45, N 1 (2006), 26–48.

[Gao09] Su Gao. *Invariant Descriptive Set Theory.* Pure and Applied Mathematics, A Series of Monographs and Textbooks, 293. Taylor & Francis Group, 2009.

[Her93] P. Hertling. Topologische Komplexitätsgrade von Funktionen mit endlichem Bild. Informatik-Berichte 152, 34 pages, Fernuniversität Hagen, December 1993.

[Her96] P. Hertling. *Unstetigkeitsgrade von Funktionen in der effektiven Analysis.* PhD thesis, Fachbereich Informatik, FernUniversität Hagen, 1996.

[HM82] L. Hay and D Miller. A topological analog to the Rice-Shapiro index theorem. *Journal of Symbolic Logic*, 47, No 4 (1982), 824–832.

[Ka08] V. Kanovei. *Borel Equivalence Relations: Structure and Classification.* University Lecture Series, 44, American Mathematical Society, Providence, 2008.

[Ke94] A.S. Kechris. *Classical Descriptive Set Theory.* Springer, New York, 1994.

[Ko00] S. Kosub. On NP-partitions over posets with an application of reducing the set of solutions of NP problems. Proc. of MFCS-2000, Lecture Notes in Computer Science, 1893 (2000), 467–476, Berlin, Springer.

[KS07] O. V. Kudinov and V. L. Selivanov. Definability in the homomorphic quasiorder of finite labeled forests. Proc. of CiE-2007, *Lecture Notes in Computer Science*, v. 4497. Berlin: Springer, 2007, 436–445.

[KS09] O. V. Kudinov and V. L. Selivanov. A Gandy theorem for abstract structures and applications to first-order definability. Proc. CiE-2009, LNCS 5635, Springer, Berlin, 290-299, 2009.

[KSZ09] O.V. Kudinov, V. L. Selivanov, and A.V. Zhukov. Definability in the h-quasiorder of labeled forests. Annals of Pure and Applied Logic, 159(3), 318–332, 2009.

[KSZ10] O. V. Kudinov, V. L. Selivanov, and A. V. Zhukov. Undecidability in Weihrauch degrees. Proc. CiE-2010, LNCS 6158, Springer, Berlin, 256-265, 2010.

[KW85] C. Kreitz and K. Weihrauch. Theory of representations. *Theoretical Computer Science*, 38 (1985), 35–53.

[KW00] S. Kosub and K. Wagner. The boolean hierarchy of NP-partitions. Proc. of STACS-2000, Lecture Notes of Computer Science, 1770 (2000), 157—168, Berlin, Springer.

[Mos80] Y.N. Moschovakis. *Descriptive Set Theory*, North Holland, Amsterdam, 1980.

[MSS12] L. Motto Ros, P. Schlicht and V. Selivanov. Wadge-like reducibilities on arbitrary quasi-Polish spaces. To appear in *Mathematical Structures in Computer Science*, Postprocedings volume of the Dagstuhl seminar “Computing with infinite data: topological and logical foundations”, arXiv:1204.5338 v1 [math.LO] 24 Apr 2012.

[Sch95] M. Schröder. Topological spaces allowing Type-2 Complexity Theory. Proc. CCA-1995 *Informatik Berichte*, 190, (2004), FernUniversität Hagen.

[Sch02] M. Schröder. Extended admissibility. *Theoretical Computer Science*, 284 (2002), 519–538.

[Sch03] M. Schröder. *Admissible representations for continuous computations.* PhD thesis, Fachbereich Informatik, FernUniversität Hagen, 2003.

[Sch04] M. Schröder. Spaces allowing Type-2 Complexity Theory revisited. *Mathematical Logic Quarterly*, 50, No 4/5 (2004), 443–459.

[Sco76] D. Scott. Data types as lattices. *SIAM Journal on Computing*, 5, No 3 (1976), 522–587.

[ScS12] M. Schröder and V. Selivanov. Some hierarchies of QCB_0-spaces. To appear in *Mathematical Structures in Computer Science*, Postprocedings volume of the Dagstuhl seminar “Computing with infinite data: topological and logical foundations”.

[Se79] V.L. Selivanov, On the structure of degrees of index sets. *Algebra and Logic*, 18 (1979), p.286–299.

[Se82] V.L. Selivanov. On the structure of degrees of generalized index sets. *Algebra and Logic*, 21, (1982), 316–330.

[Se82a] V.L. Selivanov. On index sets in the Kleene-Mostowski hierarchy. *Trans. Inst. Math.*, Novosibirsk, v. 2 (1982), 135–158 (in Russian).

[Se84] V.L. Selivanov. Index sets in the hyperarithmetical hierarchy. *Siberian Mathematical Journal*, 25, No 3 (1984), 474–488.

[Se92] V.L. Selivanov. *Hierarchies, Numerations, Index Sets.* Handwritten notes, 1992, 300 pp.

[Se04] V.L. Selivanov. Boolean hierarchies of partitions over reducible bases. *Algebra and Logic*, 43, N 1 (2004), 44–61.

[Se04a] V.L. Selivanov. Difference hierarchy in φ-spaces. *Algebra and Logic*, 43, N 4 (2004), 238–248.

[Se05] V.L. Selivanov. Variations on the Wadge reducibility. *Siberian Advances in Mathematics*, 15, N 3 (2005), 44–80.

[Se06] V.L. Selivanov. Towards a descriptive set theory for domain-like structures. *Theoretical Computer Science*, 365 (2006), 258–282.

[Se07] V.L. Selivanov. Classifying omega-regular partitions. Preproceedings of LATA-2007, Universitat Rovira i Virgili Report Series, 35/07, 529–540.

[Se07a] V.L. Selivanov. Hierarchies of $\mathbf{\Delta}^0_2$-measurable k-partitions. *Mathematical Logic Quarterly*, 53 (2007), 446–461.

[Se08] V.L. Selivanov. On the difference hierarchy in countably based T_0-spaces. *Electronic Notes in Theoretical Computer Science*, V. 221 (2008), 257-269.

[Se11] V.L. Selivanov. A fine hierarchy of ω-regular k-partitions. Proc. CiE-2011, LNCS 6735, Springer, Berlin, 260-269, 2011.

[Sp01] D. Spreen. Representations versus numberings: on the relationship of two computability notions. *Theoretical Computer Science*, 263 (2001), 473-499.

[T79] A. Tang. Chain properties in $P\omega$. *Theoretical Computer Science*, 9 (1979), 153–172.

[Wad84] W. Wadge. *Reducibility and determinateness in the Baire space.* PhD thesis, University of California, Berkely, 1984.

[Wei87] K. Weihrauch. *Computability*, Berlin, Springer, 1987.

[Wei92] K. Weihrauch. The degrees of discontinuity of some translators between representations of the real numbers. Informatik Berichte 129, FernUniversität Hagen, Hagen, July 1992.

[Wei00] K. Weihrauch. *Computable Analysis.* Berlin, Springer, 2000.

[WK87] K. Weihrauch and C. Kreitz. Representations of the real numbers and of the open sets of the set of real numbers. *Annals of Pure and Applied Logic*, 35 (1985), 247–260.