

COMPUTABLY REGULAR TOPOLOGICAL SPACES

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ABSTRACT. This article continues the study of computable elementary topology started by the author and T. Grubba in 2009 and extends the author’s 2010 study of axioms of computable separation. Several computable T_3 - and Tychonoff separation axioms are introduced and their logical relation is investigated. A number of implications between these axioms are proved and several implications are excluded by counter examples, however, many questions have not yet been answered. Known results on computable metrization of T_3 -spaces from M. Schröder (1998) and T. Grubba, M. Schröder and the author (2007) are proved under uniform assumptions and with partly simpler proofs, in particular, the theorem that every computably regular computable topological space with non-empty base elements can be embedded into a computable metric space. Most of the computable separation axioms remain true for finite products of spaces.

1. INTRODUCTION

This article continues with the study of computable topology started in [13]. For computable topological spaces (as defined in [13]) in [12] we have introduced a number of computable versions of the topological T_0 -, T_1 - and T_2 -axioms and studied their relationship. In this article we define various computable versions of the topological T_3 -, Tychonoff- and T_4 -axioms and compare them. Furthermore, we study computable metrization. For classical topology see, for example, [3]. In addition to new material we include earlier results from [7, 4, 5, 14] and [11] (in [1]) some of which have been proved under slightly differing assumptions and give some simpler proofs.

We will use the representation approach of computable analysis [6, 9, 2]. As the basic computability structure we start with computable topological spaces as introduced in [13]. Notice that there are other slightly differing not equivalent definitions of “computable topological space” in other publications, in particular in [9]. We will use the notations and results from [13] some of which are mentioned very shortly in Section 2.

In Section 3 we introduce axioms for computable T_2 (2 axioms, which are already studied in [12]), for computable T_3 (3 axioms), for computable Tychonoff (3 axioms) and for computable T_4 and computable Urysohn. We give some examples and prove that the axioms do

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not depend on the details of the computable topological space but only on the computability concept defined by it.

In Section 4 we prove a number of implications between the introduced axioms.

In Section 5 we show by counterexamples that some implications are false. We summarize the results and list some open problems concerning the implications between the axioms. We also prove that computable T_3 and computable Tychonoff as well as their strong versions are equivalent for computable topological spaces with non-empty base elements.

In Section 6 we resume results on computable metrization from [7, 4, 5] and prove them under common weak assumptions. In particular we give a considerably simpler proof of the main theorem from [4] on the embedding computable T_3 -spaces in computable metric spaces.

Each of the introduced computable separation classes is closed under the subspace operations, and most of them are closed under Cartesian product (Section 7).

2. PRELIMINARIES

We will use the terminology and abbreviations summarized in [13, Section 2] and also results from [13]. For further details see [9, 10, 2].

Let Σ be a finite alphabet such that $0, 1 \in \Sigma$. By Σ^* we denote the set of finite words over Σ and by Σ^ω the set of infinite sequences $p : \mathbb{N} \rightarrow \Sigma$ over Σ , $p = (p(0)p(1)\dots)$. For a word $w \in \Sigma^*$ let $|w|$ be its length and let $\varepsilon \in \Sigma^*$ be the empty word. For $p \in \Sigma^\omega$ let $p^{<i} \in \Sigma^*$ be the prefix of p of length $i \in \mathbb{N}$. We use the “wrapping function” $\iota : \Sigma^* \rightarrow \Sigma^*$, $\iota(a_1a_2\dots a_k) := 110a_10a_20\dots a_k011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. Let $\langle i, j \rangle := (i + j)(i + j + 1)/2 + j$ be the bijective Cantor pairing function on \mathbb{N} . We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle \cdot \rangle$ [9, Definition 2.1.7], in particular, $\langle u_1, \dots, u_n \rangle = \iota(u_1)\dots\iota(u_n)$, $\langle u, p \rangle = \iota(u)p$, $\langle p, q \rangle = (p(0)q(0)p(1)q(1)\dots)$ and $\langle p_0, p_1, \dots \rangle \langle i, j \rangle = p_i(j)$ for $u, u_1, u_2, \dots \in \Sigma^*$ and $p, q, p_0, p_1, \dots \in \Sigma^\omega$. Consider $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$. Let $u \sqsubseteq w$ iff $\iota(u)$ is a prefix of w , $u \ll w$ iff $\iota(u)$ is a subword of w and let \widehat{w} be the longest subword $v \in 11\Sigma^*11$ of w (and the empty word if no such subword exists). Then for $u, w_1, w_2 \in \Sigma^*$, $(u \ll w_1 \vee u \ll w_2) \iff u \ll \widehat{w}_1\widehat{w}_2$.

For $Y_0, \dots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ a partial function $f : \subseteq Y_1 \times \dots \times Y_n \rightarrow Y_0$ is computable, if it is computed by a Type-2 machine. A Type-2 machine M is a Turing machine with n input tapes, one output tape and finitely many additional work tapes. A specification assigns to the input tapes $1, \dots, n$ and the output tape 0 types $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ such that the machine computes a function $f_M : \subseteq Y_1 \times \dots \times Y_n \rightarrow Y_0$ [9]. Notice that on the output tape the machine can only write and move its head to the right.

A notation of a set X is a surjective partial function $\nu : \subseteq \Sigma^* \rightarrow X$ and a representation is a surjective partial function $\delta : \subseteq \Sigma^\omega \rightarrow X$. Here, finite or infinite sequences of symbols are considered as “concrete names” of the “abstract” elements of X . Computability on X is defined by computations on names. Let $\gamma_i : \subseteq Y_i \rightarrow X_i$, $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for $i \in \{0, 1\}$ be notations or representations. A set $W \subseteq X_0$ is called γ_0 -r.e. (recursively enumerable), if there is a Type-2 machine M that halts on input $y_0 \in \text{dom}(\gamma_0)$ iff $\gamma_0(y_0) \in W$. A function $h : \subseteq Y_1 \rightarrow Y_0$ realizes a multi-function $f : X_1 \rightrightarrows X_0$, iff $\gamma_0 \circ h(y_1) \in f \circ \gamma_1(y_1)$ whenever $f \circ \gamma_1(y_1) \neq \emptyset$. The function f is called (γ_1, γ_0) -computable, if it has a computable realization. The definitions can be generalized straightforwardly to subsets of $X_1 \times \dots \times X_n$ and multi-functions $f : X_1 \times \dots \times X_n \rightrightarrows X_0$ ($(\gamma_1, \dots, \gamma_n)$ -r.e., $(\gamma_1, \dots, \gamma_n, \gamma_0)$ -computable).

In this article we study axioms of computable separation for *computable topological spaces* $\mathbf{X} = (X, \tau, \beta, \nu)$ [13, Definition 4], where τ is a T_0 -topology on the set X and $\nu : \subseteq \Sigma^* \rightarrow \beta$ is a notation of a base β of τ such that $\text{dom}(\nu)$ is recursive and there is an r.e. set $S \subseteq (\text{dom}(\nu))^3$ such that $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in S\}$. We mention expressly that in the past various spaces have been called “computable topological space”. We allow $U = \emptyset$ for $U \in \beta$ which is forbidden, for example, in [5, 14].

We define a notation ν^{fs} of the finite subsets of the base β by $\nu^{\text{fs}}(w) = W : \iff ((\forall v \ll w)v \in \text{dom}(\nu) \wedge W = \{\nu(v) \mid v \ll w\})$. Then $\bigcup \nu^{\text{fs}}$ and $\bigcap \nu^{\text{fs}}$ are notations of the finite unions and the finite intersections of base elements, respectively.

For the points of X we consider the canonical (or inner) representation $\delta : \subseteq \Sigma^\omega \rightarrow X$; $\delta(p) = x$ iff p is a list of all $\nu(u)$ (possibly padded with 1s) such that $x \in \nu(u)$ (hence $u \ll p \iff \delta(p) \in \nu(u)$). For the set of open sets, the topology τ we consider the inner representation $\theta : \subseteq \Sigma^\omega \rightarrow \tau$ defined by $u \in \text{dom}(\nu)$ if $u \ll p \in \text{dom}(\theta)$ and $\theta(p) := \bigcup \{\nu(u) \mid u \ll p\}$. For the closed sets we consider the outer representation $\psi^-(p) := X \setminus \theta(p)$ [13].

The canonical notations of the natural and the rational numbers are denoted by $\nu_{\mathbb{N}}$ and $\nu_{\mathbb{Q}}$, respectively. For the real numbers we use the canonical representation ρ (Example 3.3(1)), the lower representation $\rho_<$ and the upper representation $\rho_>$ [9].

3. AXIOMS OF COMPUTABLE SEPARATION

For a topological space $\mathbf{X} = (X, \tau)$ with set \mathcal{A} of closed sets we consider the following separation properties:

Definition 3.1 (axioms of separation).

$$T_0 : (\forall x, y \in X, x \neq y)(\exists W \in \tau)((x \in W \wedge y \notin W) \vee (x \notin W \wedge y \in W)),$$

$$T_1 : (\forall x, y \in X, x \neq y)(\exists W \in \tau)(x \in W \wedge y \notin W),$$

$$T_2 : (\forall x, y \in X, x \neq y)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge y \in V),$$

$$T_3 : (\forall x \in X, \forall A \in \mathcal{A}, x \notin A)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge A \subseteq V),$$

$$Ty : (\forall x \in X, \forall A \in \mathcal{A}, x \notin A)(\exists f : X \rightarrow \mathbb{R})$$

$$(\text{ } f \text{ is continuous, range}(f) \subseteq [0; 1], f(x) = 0 \text{ and } f[A] \subseteq \{1\})$$

$$T_4 : (\forall A, B \in \mathcal{A}, A \cap B = \emptyset)(\exists U, V \in \tau)(U \cap V = \emptyset \wedge A \subseteq U \wedge B \subseteq V).$$

$$Ur : (\forall A, B \in \mathcal{A}, A \cap B = \emptyset)(\exists f : X \rightarrow \mathbb{R})$$

$$(\text{ } f \text{ is continuous, range}(f) \subseteq [0; 1], f[A] \subseteq \{0\} \text{ and } f[B] \subseteq \{1\}.)$$

We will speak of T_2 -spaces, Ty -spaces etc.

T_2 -spaces are called *Hausdorff spaces*. Many authors, for example [3], call a space T_3 -space or regular iff $T_1 + T_3$, call a space $T_{3\frac{1}{2}}$ -space, Tychonoff space or completely regular iff $T_1 + Ty$, and call a space T_4 -space or normal iff $T_1 + T_4$. From topology [3] we know:

$$T_1 + Ur \iff T_1 + T_4 \implies T_1 + Ty \implies T_1 + T_3 \implies T_2 \implies T_1 \implies T_0$$

where the implications are proper. The first implication from the right to the left is *Urysohn's lemma*. We mention that (X, τ) is a T_1 -space, iff all sets $\{x\}$ ($x \in X$) are closed [3].

In this article we consider only computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$, which are T_0 -spaces with countable base (also called *second countable*). For such spaces $T_3 \implies T_2$ and $T_1 + T_4 \iff T_1 + T_3$ [3, Theorem 1.5.16], hence

$$T_1 + \text{Ur} \iff T_1 + T_4 \iff T_1 + T_y \iff T_3. \quad (3.1)$$

Axioms of computable separation for T_0 , T_1 and T_2 have been studied in [12]. In the following we introduce computable versions of the axioms T_3 , T_y , T_4 and Ur . The computable Hausdorff axioms CT_2 and SCT_2 are from [12]. In the direct effectivizations the existing objects must be computed. For the points we compute basic neighborhoods (w.l.o.g.) instead of general open neighborhoods. Let $C(X, \mathbb{R})$ be the set of continuous functions $f : X \rightarrow \mathbb{R}$ and let $[\delta \rightarrow \rho]$ be the canonical representation of this set [13, 9].

Definition 3.2 (axioms of computable separation).

CT_2 : The multi-function $t_2 : X \times X \rightrightarrows \beta \times \beta$ is $(\delta, \delta, [\nu, \nu])$ -computable where $(U, V) \in t_2(x, y)$ iff $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

SCT_2 : There is an r.e. set $H \subseteq \Sigma^* \times \Sigma^*$ such that $(\forall x, y, x \neq y)(\exists (u, v) \in H)(x \in \nu(u) \wedge y \in \nu(v))$ and $(\forall (u, v) \in H) \nu(u) \cap \nu(v) = \emptyset$. (3.2)

WCT_3 : The multi-function $t_3^w : X \times \beta \rightrightarrows \beta$ is (δ, ν, ν) -computable where $U \in t_3^w(x, W)$ iff $x \in U \subseteq \overline{U} \subseteq W$.

CT_3 : The multi-function $t_3 : X \times \mathcal{A} \rightrightarrows \beta \times \tau$ is $(\delta, \psi^-, [\nu, \theta])$ -computable, where $(U, V) \in t_3(x, A)$ iff $x \notin A$, $U \cap V = \emptyset$, $x \in U$ and $A \subseteq V$.

CT'_3 : The multi-function $t'_3 : X \times \beta \rightrightarrows \beta \times \mathcal{A}$ is $(\delta, \nu, [\nu, \psi^-])$ -computable where $(U, B) \in t'_3(x, W)$ iff $x \in U \subseteq B \subseteq W$.

SCT_3 : There are an r.e. set $R \subseteq \text{dom}(\nu) \times \text{dom}(\nu)$ and a computable function $r : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^\omega$ such that for all $u, w \in \text{dom}(\nu)$,

$$\nu(w) = \bigcup \{ \nu(u) \mid (u, w) \in R \}, \quad (3.4)$$

$$(u, w) \in R \implies \nu(u) \subseteq \psi^- \circ r(u, w) \subseteq \nu(w). \quad (3.5)$$

CT_y : The multi-function $t_{T_y} : X \times \mathcal{A} \rightrightarrows C(X, \mathbb{R})$ is $(\delta, \psi^-, [\delta \rightarrow \rho])$ -computable where $f \in t_{T_y}(x, A)$ iff $\text{range}(f) \subseteq [0; 1]$, $x \notin A$, $f(x) = 0$ and $f[A] \subseteq \{1\}$.

CT'_y : The multi-function $t'_{T_y} : X \times \beta \rightrightarrows \beta \times C(X, \mathbb{R})$ is $(\delta, \nu, [\nu, [\delta \rightarrow \rho]])$ -computable where $(U, f) \in t'_{T_y}(x, W)$ iff $\text{range}(f) \subseteq [0; 1]$, $x \in U \subseteq W$, $f[U] = \{0\}$ and $f[X \setminus W] \subseteq \{1\}$.

SCT_y : There are an r.e. set $T \subseteq \text{dom}(\nu) \times \text{dom}(\nu)$ and a computable function $t : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^\omega$ such that

$$\nu(w) = \bigcup \{ \nu(u) \mid (u, w) \in T \} \quad \text{for all } w \in \text{dom}(\nu) \text{ and} \quad (3.6)$$

$$f_{uw}[\nu(u)] \subseteq \{0\} \text{ and } f_{uw}[X \setminus \nu(w)] \subseteq \{1\} \quad \text{for all } (u, w) \in T, \quad (3.7)$$

where $f_{uw} := [\delta \rightarrow \rho] \circ t(u, w)$.

CT_4 : The multi-function $t_4 : \mathcal{A} \times \mathcal{A} \rightrightarrows \tau \times \tau$ is $(\psi^-, \psi^-, [\theta, \theta])$ -computable where $(U, V) \in t_4(A, B)$ iff $U \cap V = \emptyset$, $A \subseteq U$ and $B \subseteq V$.

CUr : The multi-function $t_{\text{Ur}} : \mathcal{A} \times \mathcal{A} \rightrightarrows C(X, \mathbb{R})$ is $(\psi^-, \psi^-, [\delta \rightarrow \rho])$ -computable, where $f \in t_{\text{Ur}}(A, B)$ iff $\text{range}(f) \subseteq [0; 1]$, $A \cap B = \emptyset$, $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.

The axioms CT_2 , CT_3 , CTy , CT_4 and CUr are the direct effectivizations of T_2 , T_3 , Ty , T_4 and Ur , respectively. Obviously, SCT_2 implies T_2 . WCT_3 , CT_3 , CT'_3 and SCT_3 imply T_3 . CTy , CTy' and $SCTy$ imply Ty . CT_4 implies T_4 . CUr implies Ur . In contrast to CT'_3 , in WCT_3 the function t_3^w does not compute a ψ^- -name of a closed set such that $x \in U \subseteq B \subseteq W$. The sets H from SCT_2 , R from SCT_3 and T from $SCTy$ may contain pairs (u, w) such that $\nu(u) = \emptyset$ or $\nu(w) = \emptyset$. Also, empty open or closed sets are not excluded as inputs for the separating functions.

We do not consider the numerous variants of the separation axioms where in some places the representations δ of the points, θ of the open sets and ψ^- of the closed sets are replaced by δ^- , θ^- and ψ^+ , respectively [13, Definition 5]. The following examples illustrate the definitions. Further examples are given in Section 5.

Example 3.3.

- (1) The *computable real line* is defined by $\mathbf{R} := (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ such that $\tau_{\mathbb{R}}$ is the real line topology and ν is a canonical notation of the set of all open intervals with rational endpoints. \mathbf{R} is a computable topological space. Its canonical representation is called ρ . All the axioms from Definition 3.2 are true for \mathbf{R} .
- (2) A computable metric space is a tuple $\mathbf{M} = (X, d, A, \alpha)$ such that (M, d) is a metric space and α is a notation with recursive domain of a set A which is dense in X such that the distance d restricted to $M \times M$ is (α, α, ρ) -computable [9, Definition 8.1.2]. Let ν be a canonical notation of the set β of all open balls with center from A and rational radius and let τ be the smallest topology containing β . Then $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space for which all the axioms from Definition 3.2 are true (Theorem 6.2).
- (3) (CT_0 and CT_4 but not T_1 , T_2 or T_3) A space is CT_0 iff the multi-function t_0 is (δ, δ, ν) -computable, where t_0 maps every $(x, y) \in X^2$ such that $x \neq y$ to some $U \in \beta$ such that $(x \in U$ and $y \notin U)$ or $(x \notin U$ and $y \in U)$ [12].

Let $\mathbf{Si} := (\{\perp, \top\}, \tau_{\mathbf{Si}}, \beta_{\mathbf{Si}}, \nu_{\mathbf{Si}})$ be the Sierpinski space defined by $\nu_{\mathbf{Si}}(0) = \{\perp, \top\}$ and $\nu_{\mathbf{Si}}(1) = \{\top\}$. The space is T_0 but not T_1 .

There is a machine M that on input $(p, q) \in \Sigma^\omega \times \Sigma^\omega$ writes 1 and halts. The function f_M realizes the function $(x, y) \mapsto \{\top\}$. Then for $x \neq y$, $(x = \top$ and $y = \perp)$ or $(x = \perp$ and $y = \top)$, hence for $U := \{\top\} = \nu(1)$, $(x \in U$ and $y \notin U)$ or $(x \notin U$ and $y \in U)$. Therefore, \mathbf{Si} is CT_0 .

There are computable sequences $p', q' \in \Sigma^\omega$ such that $\theta_{\mathbf{Si}}(p') = \emptyset$ and $\theta_{\mathbf{Si}}(q') = \{\perp, \top\}$. There is a machine M that on input (p, q) searches in p and q until it has found $0 \ll p$ or $0 \ll q$. In the first case it writes $\langle p', q' \rangle$ and in the second case $\langle q', p' \rangle$. Let $\psi_{\mathbf{Si}}^-(p) = A$ and $\psi_{\mathbf{Si}}^-(q) = B$ such that $A \cap B = \emptyset$. Then $A = \emptyset$ or $B = \emptyset$, hence $0 \ll p$ or $0 \ll q$. In the first case, $A = \emptyset \subseteq \theta_{\mathbf{Si}}(p')$ and $B \subseteq \theta_{\mathbf{Si}}(q') = \{\perp, \top\}$ and in the second case, $A \subseteq \theta_{\mathbf{Si}}(q') = \{\perp, \top\}$ and $B = \emptyset \subseteq \theta_{\mathbf{Si}}(p')$. Therefore, f_M realizes t_4 .

- (4) (discrete implies WCT_3) Let \mathbf{X} be a discrete computable topological space. Then every subset of X is open and closed and X is countable. The function $t_3^w : (x, W) \mapsto W$ is (δ, ν, ν) -computable. It satisfies WCT_3 since $x \in W$ implies $x \in \overline{W} \subseteq W$.

By the next lemma the above computable separation axioms are robust, that is, they do not depend on the notation ν of the base explicitly but only on the computability concept on the points induced by it. Two computable topological spaces $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{\tau}, \tilde{\beta}, \tilde{\nu})$ are called equivalent iff $(X, \tau) = (\tilde{X}, \tilde{\tau})$, $\nu \leq \tilde{\theta}$ and $\tilde{\nu} \leq \theta$, that is, there are

computable functions $g, \tilde{g} : \subseteq \Sigma^* \rightarrow \Sigma^\omega$ such that

$$\nu(u) = \tilde{\theta} \circ g(u) \quad \text{and} \quad \tilde{\nu}(u) = \theta \circ \tilde{g}(u). \quad (3.8)$$

The condition “ $\nu \leq \tilde{\theta}$ and $\tilde{\nu} \leq \theta$ ” is equivalent to $\delta \equiv \tilde{\delta}$. For equivalent topological spaces, $\theta \equiv \tilde{\theta}$, $\psi^- \equiv \tilde{\psi}^-$ and $\kappa \equiv \tilde{\kappa}$ [13, Definition 21, Theorem 22].

Lemma 3.4. *Let $\tilde{\mathbf{X}} = (X, \tau, \tilde{\beta}, \tilde{\nu})$ be a computable topological space equivalent to $\mathbf{X} = (X, \tau, \beta, \nu)$. Then each separation axiom from Definition 3.2 for \mathbf{X} is equivalent to the corresponding axiom for $\tilde{\mathbf{X}}$.*

Proof:

SCT₂: See [12].

WCT₃: Assume $\widetilde{\text{WCT}}_3$. Let $x = \delta(p)$, $W = \nu(w)$ and $x \in W$. Since $\delta \equiv \tilde{\delta}$ and $\nu \leq \tilde{\theta}$ we can compute some \tilde{p} and some \tilde{w} such that $x = \tilde{\delta}(\tilde{p}) \in \tilde{\nu}(\tilde{w}) \subseteq \nu(w)$. By $\widetilde{\text{WCT}}_3$ we can compute some \tilde{u} such that $x \in \tilde{\nu}(\tilde{u}) \subseteq \text{closure}(\tilde{\nu}(\tilde{u})) \subseteq \tilde{\nu}(\tilde{w})$. Since $\tilde{\nu} \leq \theta$, from p and \tilde{u} we can compute some u such that $x \in \nu(u) \subseteq \nu(\tilde{u})$. We obtain $x \in \nu(u) \subseteq \overline{\nu(u)} \subseteq \nu(w)$. Therefore, WCT₃ is true. By symmetry, WCT₃ \implies $\widetilde{\text{WCT}}_3$.

SCT₃: Assume SCT₃. With the functions g, \tilde{g} from (3.8) let $\tilde{R} := \{(\tilde{u}, \tilde{w}) \mid (\exists (u, w) \in R)(w \ll \tilde{g}(\tilde{w}), \tilde{u} \ll g(u))\}$. Then \tilde{R} is r.e. Suppose $(\tilde{u}, \tilde{w}) \in \tilde{R}$. Then for some $(u, w) \in R$, $\tilde{\nu}(\tilde{u}) \subseteq \tilde{\theta} \circ g(u) = \nu(u) \subseteq \nu(w) \subseteq \theta \circ \tilde{g}(\tilde{w}) = \tilde{\nu}(\tilde{w})$. On the other hand suppose, $x \in \tilde{\nu}(\tilde{w}) = \theta \circ \tilde{g}(\tilde{w})$. Then $x \in \nu(w)$ for some $w \ll \tilde{g}(\tilde{w})$. By SCT₃ there is some u such that $(u, w) \in R$ and $x \in \nu(u) = \tilde{\theta} \circ g(u)$. Then $x \in \tilde{\nu}(\tilde{u})$ for some $\tilde{u} \ll g(u)$. In summary, $x \in \tilde{\nu}(\tilde{u})$ for some \tilde{u} such that $(\tilde{u}, \tilde{w}) \in \tilde{R}$. Therefore, (3.4) holds for $\tilde{\nu}$ and \tilde{R} .

There is a computable function d translating ψ^- to $\tilde{\psi}^-$ [13]. Let M be a machine that on input (\tilde{u}, \tilde{w}) searches for $(u, w) \in R$ such that $w \ll \tilde{g}(\tilde{w})$ and $\tilde{u} \ll g(u)$ and then computes $d \circ r(u, w)$. Then $\tilde{\nu}(\tilde{u}) \subseteq \tilde{\theta} \circ g(u) = \nu(u) \subseteq \psi^- \circ r(u, w) \subseteq \nu(w) \subseteq \theta \circ \tilde{g}(\tilde{w}) = \tilde{\nu}(\tilde{w})$. Since $\psi^- \circ r(u, w) = \tilde{\psi}^- \circ d \circ r(u, w) = \tilde{\psi}^- \circ f_M(\tilde{u}, \tilde{w})$, (3.5) holds for $\widetilde{\text{SCT}}_3$ with $\tilde{r} := f_M$ and \tilde{R} . Therefore, SCT₃ \implies $\widetilde{\text{SCT}}_3$. By symmetry, $\widetilde{\text{SCT}}_3 \implies$ SCT₃.

For the other axioms the proofs are similar. Notice that $[\delta \rightarrow \rho] \equiv [\tilde{\delta} \rightarrow \rho]$ if $\delta \equiv \tilde{\delta}$. \square

4. IMPLICATIONS

In this section we prove a number of implications between the separation properties, in Section 5 we prove by counterexamples that some of the implications are proper. A topological space is discrete iff every singleton $\{x\}$ is open iff every subset $B \subseteq X$ is open. A discrete space is T_i for $i = 1, \dots, 4$. Let D be the axiom stating that the space is discrete.

Theorem 4.1.

- (1) SCT₃ \implies CTy \implies CT₃ \implies SCT₂ \implies CT₂,
- (2) D \implies WCT₃,
- (3) CT₃ \implies WCT₃
- (4) SCT₃ \implies CT₄,
- (5) SCT₃ \iff SCTy, CTy \iff CTy', CT₃ \iff CT'₃,
- (6) CT₄ \iff CUr.

The implications $\text{SCT}_3 \implies \text{CT}_4 \implies \text{CUr}$ have been already been proved in [7] for a computable topological space $T(\mathbf{Z})$ derived from a predicate space \mathbf{Z} (in the terminology of [13]). For our computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$, $\mathbf{Z} := (X, \beta, \nu)$ is a predicate space and $T(\mathbf{Z}) = (X, \tau, \tilde{\beta}, \tilde{\nu})$, where $\tilde{\nu}$ is the notation of the finite intersections of base elements canonically derived from ν , is equivalent to \mathbf{X} by [13, Lemma 23]. By Lemma 3.4, $\text{SCT}_3 \implies \text{CT}_4 \implies \text{CUr}$ for a computable topological space follows from [7]. More concise proofs are given in [4] for a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $U \neq \emptyset$ for all $U \in \beta$. This restriction, however, is unnecessary. The reader may check this in Appendix A.

Proof:

SCT₃ \implies CT₄: (cf. [3, Lemma 1.5.15, Theorem 1.5.17]) The proof from [4] is added in Appendix B.

CT₄ \implies CUr: See [3, Theorem 1.5.15]. The proof from [4] is added in the appendix.

CUr \implies CT₄: By the multi-function t_{UR} from A, B such that $A \cap B = \emptyset$ we can compute a continuous function $f : X \rightarrow \mathbb{R}$ such that $\text{range}(f) \subseteq [0; 1]$, $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Then by [13, Theorem 38] the open sets $U := f^{-1}[(-\infty; 1/2)]$ and $V := f^{-1}[(1/2; \infty)]$ can be computed. They separate A and B .

CUr+SCT₃ \implies SCTy: Let R be the set and let r be the function from SCT_3 . Define $T := R$. By r from $(u, w) \in T$ we can compute a closed set A such that $\nu(u) \subseteq A \subseteq \nu(w)$. Since $U \mapsto U^c$ for base sets is (ν, ψ^-) -computable, by t_{U^c} from A and $X \setminus \nu(w)$ we can compute some continuous function $f : X \rightarrow \mathbb{R}$ such that $\text{range}(f) \subseteq [0; 1]$, $f[A] \subseteq \{0\}$, hence $f[\nu(u)] \subseteq \{0\}$ and $f[X \setminus \nu(w)] \subseteq \{1\}$.

SCTy \implies SCT₃: Let T be the set and t be the function from SCTy . Define $R := T$. By [13, Theorem 38] the function $f \mapsto f^{-1}(1/2, \infty)$ for continuous $f : X \rightarrow \mathbb{R}$ is $([\delta \rightarrow \rho], \theta)$ -computable. Let $h : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ be a computable realization. Then for $(u, w) \in R$ and $V := ([\delta \rightarrow \rho] \circ t(u, w))^{-1}(1/2; \infty) = \theta \circ h \circ t(u, w)$, $\nu(u) \cap V = \emptyset$ and $X \setminus \nu(w) \subseteq V$. Therefore, $\nu(u) \subseteq X \setminus V = \psi^- \circ h \circ t(u, w) = X \setminus V \subseteq \nu(w)$. Define $r := h \circ t$.

CTy' \implies CTy: From (x, A) such that $x \notin A$ some $W \in \beta$ can be computed such that $x \in W \subseteq X \setminus A$. From (x, W) such that $x \in W$ by t'_{Ty} some $U \in \beta$ and some continuous function $f : X \rightarrow \mathbb{R}$ can be computed such that $x \in U \subseteq W$, $\text{range}(f) \subseteq [0; 1]$, $f(y) = 0$ for $y \in U$ and $f(y) = 1$ for $y \notin W$. For this function f , $f(x) = 0$ since $x \in U$ and $f(y) = 1$ for $y \in A$ since $W \subseteq X \setminus A$. Therefore, t_{Ty} is $(\delta, \psi^-, [\delta \rightarrow \rho])$ -computable.

CTy \implies CTy': Suppose $x \in W \in \beta$. From W , $A := X \setminus W$ can be computed. From (x, A) by t_{Ty} some continuous function g can be computed such that $\text{range}(g) \subseteq [0; 1]$, $g(x) = 0$ and $g(y) = 1$ for $y \in A$. Let $f(y) := \max(0, 2g(y) - 1)$. Then $g \mapsto f$ is $([\delta \rightarrow \rho], [\delta \rightarrow \rho])$ -computable. Obviously $\text{range}(f) \subseteq [0; 1]$, $f(y) = 0$ for $g(y) < 1/2$ and $f(y) = 1$ for $y \in A$. By [13, Theorem 38], $g \mapsto g^{-1}(-\infty, 1/2)$ is $([\delta \rightarrow \rho], \theta)$ -computable. Finally from (x, V) such that $x \in V \in \tau$ some $U \in \beta$ can be computed such that $x \in U \subseteq V$. Notice that $f(y) = 0$ for $y \in U \subseteq g^{-1}(-\infty, 1/2)$. Therefore, from (x, W) some $(U, f) \in t_{\text{Ty}}(x, W)$ can be computed.

CT'₃ \implies CT₃: Using t'_3 from (x, A) we can compute in turn $(x, X \setminus A)$, (x, W) for some $W \in \beta$ such that $x \in W \subseteq X \setminus A$, some $(U, B) \in \beta \times \mathcal{A}$ such that $x \in U \subseteq B \subseteq W$, and finally (U, V) where $V := X \setminus B$. By simple transformations, $U \cap V = \emptyset$, $x \in U$ and $A \subseteq V$.

CT₃ ⇒ CT'₃: The function $W \mapsto X \setminus W$ is (ν, ψ^-) -computable, and the function $V \mapsto X \setminus V$ is (θ, ψ^-) -computable. Then using t_3 , from (x, W) we can compute in turn (x, A) , $A := X \setminus W$, $(U, V) \in \beta \times \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $A \subseteq V$, and (U, B) , $B := X \setminus V$. By simple transformations, $x \in U \subseteq B \subseteq W$.

CT'₃ ⇒ WCT₃: Obvious.

SCTy ⇒ CTy': Let T be the set and t be the function from SCTy. Assume $x = \delta(p) \in \nu(w)$. By (3.6) there is some $u \in \text{dom}(\nu)$ such that $u \ll p$, and $(u, w) \in T$. Then $f_{uw} := [\delta \rightarrow \rho] \circ t(u, w)$ satisfies (3.7). There is a machine that on input (p, w) searches for some u such that $u \ll p$ and $(u, w) \in T$ and writes $\langle u, t(u, w) \rangle$. Then f_M realizes t'_{Ty} .

CTy ⇒ CT₃: Suppose $x \notin A$ and A is closed. By t_{Ty} we can compute some continuous function f such that $f(x) = 0$ and $f(y) = 1$ for $x \in A$. By [13, Theorem 38], the functions $f \mapsto f^{-1}(-\infty, 1/2)$ and $f \mapsto f^{-1}(1/2, \infty)$ are $([\delta \rightarrow \rho], \theta)$ -computable. Since $x \in f^{-1}(-\infty, 1/2)$ and $A \subseteq f^{-1}(1/2, \infty)$, the multi-function $(x, A) \mapsto (U, V)$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$ is $(\delta, \psi^-, [\theta, \theta])$ -computable. From q and r such that $\delta(q) \in \theta(r)$ we can compute some u such that $x \in \nu(u) \subseteq \theta(r)$. Therefore, t_3 is $(\delta, \psi^-, [\nu, \theta])$ -computable.

CT'₃ ⇒ SCT₂: Let M be a machine such that f_M realizes t'_3 . Since finite intersection is $(\nu^{\text{fs}}, \theta)$ -computable [13, Theorem 11], there is a computable function g such that $\bigcap \nu^{\text{fs}}(w) = \theta \circ g(w)$. Let H be the set of all $(u, v) \in \text{dom}(\nu) \times \text{dom}(\nu)$ with the following properties: there are words w, u_1, v_1, v_2 such that $v_1 \in \text{dom}(\nu^{\text{fs}})$, $w \ll v_1$, $u \ll g(v_1)$ and on input $(v_1 1^\omega, w)$ in $\text{length}(v_1)$ steps the machine M writes at least $\iota(u_1)v_2$ such that $u_1 \ll v_1$, and $v \ll v_2$. The set H is r.e.

Suppose $\delta(p) = x \neq y$. Since $\text{CT}'_3 \Rightarrow T_3$ and $T_3 \Rightarrow T_2$ for second countable spaces, the space is T_2 , hence there is some w such that $x \in \nu(w)$ and $y \notin \nu(w)$. Then on input (p, w) the machine M writes some $\iota(u_1)q$ such that $x \in \nu(u_1) \subseteq \psi^-(q) \subseteq \nu(w)$. Since $y \notin \nu(w)$, hence $y \in \theta(q)$, there are a prefix v_2 of q and a word v such that $v \ll v_2$ and $y \in \nu(v)$. For producing $\iota(u_1)v_2$ some prefix of p is sufficient. Since $x \in \nu(u_1) \subseteq \nu(w)$ there is a prefix v_1 of p such that $w \ll v_1$, $u_1 \ll v_1$ and on input $(v_1 1^\omega, w)$ in $\text{length}(v_1)$ steps the machine M writes at least $\iota(u_1)v_2$. Since $x \in \theta \circ g(v_1)$, there is some $u \ll g(v_1)$ such that $x \in \nu(u)$. By definition of H , $(u, v) \in H$, hence (3.2) is true.

Suppose $(u, v) \in H$. Then there are words w, u_1, v_1, v_2 with the properties listed in the definition of H . If $\nu(u) = \emptyset$, (3.3) is true.

Suppose $x \in \nu(u) \neq \emptyset$. Since $u \ll g(v_1)$ and $w \ll v_1$, $x \in \nu(u) \subseteq \theta \circ g(v_1) = \bigcap \nu^{\text{fs}}(v_1) \subseteq \nu(w)$. There is some $p' \in \Sigma^\omega$ such that $x = \delta(v_1 p') \in \bigcap \nu^{\text{fs}}(v_1)$. Since M realizes t'_3 , on input $(v_1 p', w)$ the machine M writes $\iota(u_1)q$ such that $x \in \nu(u_1) \subseteq \psi^-(q) \subseteq \nu(w)$. In $\text{length}(v_1)$ steps the machine can read only symbols from v_1 and, therefore, has the same behavior on input $(v_1 1^\omega, w)$. By assumption on u, v, w, u_1, v_1 and v_2 , v_2 is a prefix of q and $u_1 \ll v_1$, hence $\nu(u) \subseteq \nu(u_1)$. Since $v \ll v_2$, $v \ll q$, hence $\nu(v) \subseteq \theta(q)$. Since $\nu(u) \subseteq \nu(u_1) \subseteq \psi^-(q)$, $\nu(u) \cap \nu(v) \subseteq \nu(u_1) \cap \theta(q) = \emptyset$. Therefore, (3.3) is true.

SCT₂ ⇒ CT₂: See [12]

D ⇒ WCT₃: See Example 3.3(4).

The statements (1) – (6) of the theorem follow from these results. □

By [12, Theorem 7], the following statements are equivalent: \mathbf{X} is SCT_2 ; $x \neq y$ is (δ, δ) -r.e.; $x \mapsto \overline{\{x\}}$ is (δ, ψ^-) -computable. We apply this result in the next proof.

Theorem 4.2. $CT_4 + SCT_2 \Rightarrow CT_3$.

Proof: Since the space is SCT_2 it is T_1 [12, Theorem 5], hence $\{x\} = \overline{\{x\}}$ for every point x . Therefore by the above characterization, from x and A such that $x \notin A$ we can compute $\{x\}$ and A and by CT_4 we can compute disjoint open sets U, V such that $\{x\} \subseteq U$, hence $x \in U$, and $A \subseteq V$. Therefore, the space is CT_3 . \square

By [12, Theorem 7], $T_2 \Rightarrow SCT_2$ if $U \cap V = \emptyset$ is (ν, ν) -r.e.. A similar result holds for T_3 -spaces.

Theorem 4.3. $CT_3 \iff WCT_3$ if $U \cap V = \emptyset$ is (ν, ν) -r.e.

Proof: Suppose WCT_3 . Then from $x \in X$ and $W \in \beta$ such that $x \in W$ we can compute some $U \in \beta$ such that $x \in U \in \overline{U} \in W$. For showing CT'_3 it suffices to find a ψ^- -name of \overline{U} . By assumption, from $U \in \beta$ we can find a list (encoded by $q \in \Sigma^\omega$) of all $V \in \beta$ such that $U \cap V = \emptyset$. Since for open V , $U \cap V = \emptyset \iff \overline{U} \cap V = \emptyset$, q is a ψ^- -name of the closed \overline{U} . Therefore, the space is CT'_3 , hence CT_3 . \square

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ possibly $U = \emptyset$ for some $U \in \beta$.

Theorem 4.4. If the set $\{w \in \Sigma^* \mid \nu(w) \neq \emptyset\}$ is r.e. then

$$CT_3 \iff CT_y \iff SCT_3.$$

In particular, if all base elements are not empty then $CT_3 \iff SCT_y \iff SCT_3$. Of course, the space \mathbf{X} in Example 5.4 has empty base elements. The non-empty ones are not even r.e.

Proof: Suppose that $\{w \mid \nu(w) \neq \emptyset\}$ is r.e. Since finite intersection is $(\nu^{\text{fs}}, \theta)$ -computable [13, Theorem 11], there is a computable function g such that $\bigcap \nu^{\text{fs}}(w) = \theta \circ g(w)$. Therefore, the set $\{w \in \Sigma^* \mid \bigcap \nu^{\text{fs}}(w) \neq \emptyset\}$ is r.e. Suppose the space is CT_3 . By Theorem 4.1 it is CT'_3 . There is a machine M such that f_M realizes the multi-function t'_3 from CT'_3 in Definition 3.2.

Let $x_0 = \delta(p_0) \in \nu(w)$. Then for some $u \in \text{dom}(\nu)$ and $q_{p_0} \in \text{dom}(\psi^-)$, $f_M(p_0, w) = \langle u, q_{p_0} \rangle = \iota(u)q_{p_0}$ such that

$$x_0 \in \nu(u) \subseteq \psi^-(q_{p_0}) \subseteq \nu(w). \quad (4.1)$$

For computing $\iota(u)$ some prefix $u_0 \in \text{dom}(\nu^{\text{fs}}) \cap \Sigma^*11$ of p_0 suffices. Since $\delta(p_0) \in \nu(w)$ we may assume $w \ll u_0$. For all $p \in I_0 := \{p \in \text{dom}(\delta) \mid u_0 \text{ is a prefix of } p\}$, $f_M(p, w) = \iota(u)q_p$ for some q_p such that $\delta(p) \in \nu(u) \subseteq \psi^-(q_p) \subseteq \nu(w)$. Then

$$x_0 \in \nu(u) \in \bigcap_{p \in I_0} \psi^-(q_p) \subseteq \nu(w). \quad (4.2)$$

A word $u_0 \in \text{dom}(\nu^{\text{fs}})$ is a prefix of some $p \in \text{dom}(\delta)$ iff $\bigcap \nu^{\text{fs}}(u_0) \neq \emptyset$. We will define R such that $(u, w) \in R$ iff for some $u_0 \in \Sigma^*11$ such that $\bigcap \nu^{\text{fs}}(u_0) \neq \emptyset$ the machine M on input (u_01^ω, w) writes $\iota(u)$ in at most $|u_0|$ steps. From this word u_0 we will compute a sequence $q \in \Sigma^\omega$ such that $\psi^-(q) = \bigcap_{p \in I_0} \psi^-(q_p)$.

There is a machine N that works on input (u, w) as follows:

(S1) N searches for some $u_0 \in \text{dom}(\nu^{\text{fs}}) \cap \Sigma^*11$ such that $w \ll u_0$, $\bigcap \nu^{\text{fs}}(u_0) \neq \emptyset$ and the

machine M on input $(u_0 1^\omega, w)$ writes $\iota(u)$ in at most $|u_0|$ steps.

(S2) Then N writes every $\iota(v)$ such that there are words u' and v' such that

(S2a) $u' \in \text{dom}(\nu^{\text{fs}}) \cap \Sigma^* 11$, $\bigcap \nu^{\text{fs}}(u') \neq \emptyset$ and $u_0 \sqsubseteq u'$ and

(S2b) M on input $(u' 1^\omega, w)$ in $|u'|$ steps writes $\iota(u)v'$ such that $v \ll v'$.

Furthermore, N writes 11 repeatedly in order to produce an infinite sequence if only finitely many words v can be found. If no u_0 can be found the machine does not halt and writes nothing. Let $r := f_N$ and $R := \text{dom}(f_N)$. Then $R \subseteq \text{dom}(\nu) \times \text{dom}(\nu)$ and R is r.e. We must prove (3.4) and (3.5).

We show (3.4): Suppose $x = \delta(p) \in \nu(w)$. Then for some u and q , $f_M(p, w) = \iota(u)q$ such that $x \in \nu(u) \subseteq \psi^-(q) \subseteq \nu(w)$. There is a prefix $u_0 \in \Sigma^* 11$ of p such that $w \ll u_0$ and M on input $(u_0 1^\omega, w)$ writes $\iota(u)$ in at most $|u_0|$ steps, hence $(u, w) \in \text{dom}(f_N) = R$. Therefore, $x \in \nu(u)$ for some u with $(u, w) \in R$. We conclude $\nu(w) \subseteq \bigcup \{\nu(u) \mid (u, w) \in R\}$.

On the other hand, let $(u, w) \in R$. Then there is some $u_0 \in \text{dom}(\nu^{\text{fs}}) \cap \Sigma^* 11$ such that $\bigcap \nu^{\text{fs}}(u_0) \neq \emptyset$ and the machine M on input $(u_0 1^\omega, w)$ writes $\iota(u)$ in at most $|u_0|$ steps. There is some p' such that $u_0 p' \in \text{dom}(\delta)$. Then $f_M(u_0 p', w) = \iota(u)q'$ for some q' hence $\delta(u_0 p') \in \nu(u) \subseteq \nu(w)$. Therefore, $\bigcup \{\nu(u) \mid (u, w) \in R\} \subseteq \nu(w)$.

Combining the two results we obtain (3.4).

We show (3.5): Suppose $(u, w) \in R = \text{dom}(f_N)$ is the input of the machine N and let $q := f_N(u, w)$. First, N finds some u_0 with the properties listed in (S1).

Suppose, later N writes $\iota(v)$ as described in (S2). Then there are words u', v' and a sequence $p' \in \Sigma^\omega$ such that $u' p' \in \text{dom}(\delta)$ and M on input $(u' p', w)$ in at most $|u'|$ steps writes v' such that $v \ll v' \sqsubseteq q$ and $\delta(u' p') \in \nu(u) \subseteq \nu(w)$, hence $\nu(u) \cap \nu(v) = \emptyset$. Therefore, $\nu(u) \cap \nu(v) = \emptyset$ for all v such that $v \ll q$. We obtain $\nu(u) \subseteq \psi^-(q) = \psi^- \circ f_N(u, w)$.

There are some p', q' such that $u_0 p' \in \text{dom}(\delta)$ and M on input $(u_0 p', w)$ writes $\iota(u)q'$ such that $\delta(u_0 p') \in \nu(u) \subseteq \psi^-(q') \subseteq \nu(w)$. Suppose, $v \ll q'$. Then there are words u', v' such that the conditions (S2a) and (S2b) are satisfied, hence $v \ll f_N(u, w)$. Therefore, $\psi^- \circ f_N(u, w) \subseteq \psi^-(q') \subseteq \nu(w)$.

Combining the results we obtain $\nu(u) \subseteq \psi^- \circ f_N(u, w) \subseteq \nu(w)$. Therefore we have proved (3.5). \square

Notice that the proof works correctly since in (S1) we have guaranteed $\bigcap \nu^{\text{fs}}(u_0) \neq \emptyset$ hence $u_0 \sqsubseteq p$ for some $p \in \text{dom}(\delta)$. The realization f_M of t'_3 may give unreasonable results on (p, w) if $p \notin \text{dom}(\delta)$.

5. COUNTEREXAMPLES AND SUMMARY

We show by counterexamples that some of the implications from Theorem 4.1 are proper. In [12] a CT_2 -space is given that is not SCT_2 , hence $SCT_2 \implies CT_2$ is proper.

Example 5.1. (SCT_2 but not T_3) We extend [3, Example 1.5.7]. Let $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta_{\mathbb{R}}, \nu_{\mathbb{R}})$ be the computable real line from Example 3.3(1). Let $S := \{1/i \mid i \in \mathbb{Z}, i \neq 0\}$, $\sigma := \beta_{\mathbb{R}} \cup \{(-1; 1) \setminus S\}$ with canonical notation λ . Then $\mathbf{Z} = (\mathbb{R}, \sigma, \lambda)$ is a computable predicate space and $T(\mathbf{Z}) =: (\mathbb{R}, \tau, \beta, \nu)$ is a computable topological space [13, Definition 8, Lemma 9]. Since $\beta_{\mathbb{R}}$ is a subset of the topology generated by σ and $\nu_{\mathbb{R}} \leq \lambda$, “ $x \neq y$ ” is $(\delta_{\mathbf{Z}}, \delta_{\mathbf{Z}})$ -r.e., hence (δ, δ) -r.e. By [12, Theorem 7.2], $T(\mathbf{Z})$ is a SCT_2 -space. The space is not T_3 since the point 0 cannot be separated from the closed set S by disjoint open sets since $U \cap S \neq \emptyset$ for every neighborhood U of 0. \square

Since the above SCT_2 -space is not T_3 it is not WCT_3 , CT_3 or SCT_3 . The space from Example 5.3 below is T_4 and SCT_2 , but not WCT_3 . First we prove a lemma. Let us call a function $f : \subseteq \mathbb{Q} \rightarrow \mathbb{Q}$ a *lower separation function* for a real number $x > 0$ if f is computable (precisely, $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}})$ -computable) and for all rational numbers a with $0 < a < x$, ($f(a)$ exists and) $a < f(a) < x$.

Lemma 5.2. *There is a positive real number that has no lower separation function.*

Proof: We define such a number z by brute force diagonalization. Let f_1, f_2, \dots be a sequence of all computable partial functions $f : \subseteq \mathbb{Q} \rightarrow \mathbb{Q}$. Let $(a_0; b_0) := (0; 1)$ and for $i = 1, 2, \dots$ define intervals $(a_i; b_i)$ as follows. Find some rational number a such that $a_{i-1} < a < f_i(a) < b_{i-1}$ and define $(a_i; b_i) := (a, f_i(a))$, if no such a exists define $(a_i; b_i) := (a_{i-1}; b_{i-1})$. There is some positive real number $z \in \bigcap_i (a_i; b_i)$. Suppose, f_k is a lower separating function for z . Since $z \in (a_{k-1}; b_{k-1})$ there is some a such that $a_{k-1} < a < f_k(a) < z < b_{k-1}$ and a_k, b_k are chosen such that $a_{k-1} < a_k < f_k(a_k) < b_{k-1}$ and $b_k = f_k(a_k)$. If $a_k < z$ then $b_k = f_k(a_k) < z$, hence $z \notin (a_k; b_k)$, if $z \leq a_k$ then $z \notin (a_k; b_k)$ as well. But by assumption $z \in \bigcap_i (a_i; b_i)$. Therefore, f_k cannot be a lower separating function for z . \square

Example 5.3. (T_4 and SCT_2 but not WCT_3) Let $\mathbf{R} = (\mathbb{R}, \tau, \beta, \nu)$ be the computable real line from Example 3.3(1). For $c \in \mathbb{R}$ define $\mathbf{R}_c = (\mathbb{R}, \tau, \beta_c, \nu_c)$ by $\nu_c(0w) := \nu(w)$, and $\nu_c(1w) := \nu(w) \cap (-\infty; c)$. Then \mathbf{R}_c is a computable topological space. Let δ_c be the (canonical or inner) representation of \mathbb{R} for \mathbf{R}_c [13, Definition 5.1]. Since \mathbf{R}_c has the same topology as \mathbf{R} it is T_i for $i = 0, \dots, 4$. The computable real line \mathbf{R} is SCT_2 . Let H satisfy (3.2) and (3.3) for \mathbf{R} . Then $H_c := \{(0v, 0w) \mid (v, w) \in H\}$ satisfies (3.2) and (3.3) for \mathbf{R}_c . Therefore, \mathbf{R}_c is SCT_2 .

Let $c > 0$ be a real number that has no lower separation function (Lemma 5.2). Suppose \mathbf{R}_c is WCT_3 . Let t_3^w be the computable function from Definition 3.2 for \mathbf{R}_c . Then:

- the function $a \mapsto a$ for rational $0 < a < c$ is $(\nu_{\mathbb{Q}}, \delta_c)$ -computable,
- $\nu_c(w_0) = (0; c)$ for some $w_0 \in \Sigma^*$,
- for every $x \in (0; c)$, t_3^w maps $(x, (0; c))$ to some $U \in \beta_c$ such that $x \in U \subseteq \overline{U} \subseteq (0; c)$.
- $U \mapsto \sup U$ for $U \in \beta_c$ such that $\overline{U} \subseteq (0; c)$ is $(\nu_c, \nu_{\mathbb{Q}})$ -computable.

There is a $(\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}})$ -computable multi-function h mapping each rational number $0 < a < c$ to some rational number b such that $a < b < c$: From a $\nu_{\mathbb{Q}}$ -name of a compute a δ_c -name of a . By t_3^w , from a and $(0; c)$ compute some $U \in \beta_c$ such that $a \in U \subseteq \overline{U} \subseteq (0; c)$. From U compute $b := \sup U \in \mathbb{Q}$. Then $a < b < c$. Since there is an injective notation equivalent to $\nu_{\mathbb{Q}}$, the function h is single-valued. Therefore, h is a lower separation function for c . Contradiction. \square

The above space is T_4 , T_3 and SCT_2 but not WCT_3 , CT_3 or SCT_3 . Finally we separate CTy from SCT_3 . (The example in [11] in [1] for separating CT_3 from SCT_3 is not correct.)

Example 5.4. (D and CTy but not SCT_3) Let $X := \mathbb{N}$ and let $A \subseteq \mathbb{N}$ be the set defined below. Let τ be the discrete topology on \mathbb{N} and define a notation ν of a base β of τ by $\nu(12^j) := \{j\}$, $\nu(2) := A$ and $\nu(02^j) := \{j\} \cap A$. Then $\mathbf{X} := (\mathbb{N}, \tau, \beta, \nu)$ is a computable topological space. Let δ be the canonical representation of the points of \mathbf{X} [13].

We show that \mathbf{X} is CTy' .

Let $p_0, p_1 \in \Sigma^\omega$ be computable sequences such that $\rho(p_0) = 0 \in \mathbb{R}$ and $\rho(p_1) = 1 \in \mathbb{R}$. There is a machine M that in input $(p, w, q) \in \Sigma^\omega \times \Sigma^* \times \Sigma^\omega$ searches for $i, j \in \mathbb{N}$ such that $12^i \ll p$ and $12^j \ll q$ and then writes p_0 if $i = j$ and p_1 else. Then for all $p, q \in \text{dom}(\delta)$

and all $w \in \Sigma^*$, $\rho \circ f_M(p, w, q) = (0 \text{ if } \delta(p) = \delta(q) \text{ and } 1 \text{ else})$. Therefore, f_M realizes the function $f : X \times \beta \times X \rightarrow \mathbb{R}$ such that $f(x, W, y) = (0 \text{ if } x = y \text{ and } 1 \text{ else})$. By type conversion [9, Theorem 3.3.15] the function $(x, W) \mapsto f$ such that $f(y) = (0 \text{ if } x = y \text{ and } 1 \text{ else})$ is $(\delta, \nu, [\delta \rightarrow \rho])$ -computable. Furthermore, from $p \in \text{dom}(\delta)$ we can compute the (unique) i such that $12^i \ll p$. Then $\delta(p) = i$ and $\{i\} = \nu(12^i) =: U$. Therefore, from x and W such that $x \in W$ we can find U and f such that the conditions for CTy' hold true.

We define $A \subseteq \mathbb{N}$. Let $K \subseteq \mathbb{N}$ be a set with non r.e. complement. Let $A \cap (2\mathbb{N} + 1) := 2K + 1$. Define $A \cap 2\mathbb{N}$ as follows. Let γ_i be the i th computable function $f : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ ($i = 0, 1, 2, \dots$) and let λ_i be the i th computable function $f : \subseteq \Sigma^* \times \Sigma^* \rightarrow \Sigma^\omega$. For $n = \langle i, k \rangle$ ($\langle \cdot \rangle$ is the Cantor pairing function) define the position of $2n$ as follows by diagonalization.

$$\begin{array}{ll} \text{if } (12^{2n}, 2) \notin \text{dom}(\gamma_i) \text{ and } (02^{2n}, 2) \notin \text{dom}(\gamma_i) & \text{then } 2n \in A, \\ \text{if } (12^{2n}, 2) \in \text{dom}(\gamma_i) & \text{then } 2n \notin A, \\ \text{if } (12^{2n}, 2) \notin \text{dom}(\gamma_i) \text{ and } (02^{2n}, 2) \in \text{dom}(\gamma_i) & \\ \text{then} & \\ \text{if } 02^{2n} \not\ll \lambda_k(02^{2n}, 2) \text{ and } 12^{2n} \not\ll \lambda_k(02^{2n}, 2) & \text{then } 2n \notin A, \\ & \text{else } 2n \in A. \end{array}$$

Suppose, \mathbf{X} is SCT_3 . Let R be the r.e. set and let r be the computable function such that (3.4) and (3.5) hold true. Then there are $i, k \in \mathbb{N}$ such that $R = \text{dom}(\gamma_i)$ and $r = \lambda_k$.

Suppose, $(2, 2) \in \text{dom}(\gamma_i)$. Then by (3.5), $A = \nu(2) \subseteq \psi^- \circ \lambda_k(2, 2) \subseteq \nu(2)$, hence $\mathbb{N} \setminus A = \bigcup \{\nu(v) \mid v \in V\}$ for an r.e. set $V \subseteq \text{dom}(\nu)$. Since $2 \notin V$ and $\nu(02^l) = \emptyset$ for $l \notin A$, $m \notin K \iff 2m + 1 \notin A \iff 12^{2m+1} \in V$, hence the complement of K is r.e. (contradiction). Therefore, $(2, 2) \notin \text{dom}(\gamma_i)$.

We show that γ_i and λ_k cannot operate correctly for $n := \langle i, k \rangle$.

Case $(12^{2n}, 2) \notin \text{dom}(\gamma_i)$ and $(02^{2n}, 2) \notin \text{dom}(\gamma_i)$: Since $(2, 2) \notin \text{dom}(\gamma_i)$, $2n \notin \nu(2)$ by (3.4). But $2n \in A = \nu(2)$ by the definition of A . Contradiction.

Case $(12^{2n}, 2) \in \text{dom}(\gamma_i)$: Then $2n \in A = \nu(2)$ by (3.4). But $2n \notin A = \nu(2)$ by the definition of A . Contradiction.

Case $(12^{2n}, 2) \notin \text{dom}(\gamma_i)$ and $(02^{2n}, 2) \in \text{dom}(\gamma_i)$: By (3.4),

$$\{2n\} \cap A = \nu(02^{2n}) \subseteq \psi^- \circ \lambda_k(02^{2n}, 2) \subseteq \nu(2) = A. \quad (5.1)$$

Suppose $2n \in A$. Then $2n \in \psi^- \circ \lambda_k(02^{2n}, 2) \subseteq A$, hence $2n \notin \theta \circ \lambda_k(02^{2n}, 2)$. Therefore, $02^{2n} \not\ll \lambda_k(02^{2n}, 2)$ and $12^{2n} \not\ll \lambda_k(02^{2n}, 2)$. Then $2n \notin A$ by the definition of A . Contradiction.

Suppose $2n \notin A$. By (5.1), $2n \in \theta \circ \lambda_k(02^{2n}, 2)$, hence $02^{2n} \ll \lambda_k(02^{2n}, 2)$ or $12^{2n} \ll \lambda_k(02^{2n}, 2)$. Then $2n \in A$ by the definition of A . Contradiction.

Therefore, the space \mathbf{X} is not SCT_3 . \square

We summarize the counterexamples.

Theorem 5.5. *The following implications are false:*

$$CT_2 + D \implies SCT_2 \quad ([12, \text{Example 5}],) \quad (5.2)$$

$$CT_0 + CT_4 \implies T_1 \quad (\text{Example 3.3(3)}), \quad (5.3)$$

$$SCT_2 \implies T_3 \quad (\text{Example 5.1}), \quad (5.4)$$

$$SCT_2 + T_4 \implies WCT_3 \quad (\text{Example 5.3}), \quad (5.5)$$

$$CTy + D \implies SCT_3 \quad (\text{Example 5.4}). \quad (5.6)$$

Further false implications can be obtained by transitivity of “ \implies ”, for example, $CT_4 \implies SCT_3$ is false by (5.3) since $SCT_3 \implies T_1$. Figure 1 shows the positive and negative results that we have proved. “ $A \longrightarrow B$ ” means $A \implies B$, “ $A \xrightarrow{C} B$ ” means $A \wedge C \implies B$, “ $A \not\rightarrow B$ ” means that we have constructed a computable topological space for which $A \wedge \neg B$, and “ $A \xrightarrow{C} B$ ” means that we have constructed a computable topological space for which $(A \wedge C) \wedge \neg B$. EI abbreviates “ $U \cap V = \emptyset$ is (ν, ν) -r.e.” and NE abbreviates “ $U \neq \emptyset$ is ν -r.e.”.

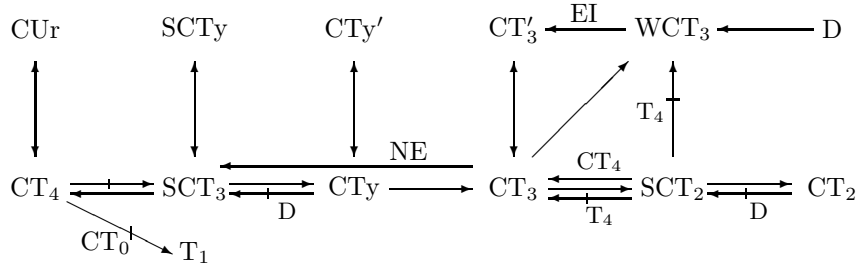


Figure 1: Logical relations between the computable separation axioms.

A number of implications have not yet been proved or disproved, for example,

$$\begin{aligned}
 WCT_3 &\implies CT_3 (SCT_2, CT_2, CT_0, WCT_0), \\
 CT_3 &\implies CTy (CT_4), \\
 CT_4 + T_3 &\implies (SCT_3, CT_3, SCT_2, CT_2, CT_0, WCT_0), \\
 CT_4 + SCT_2 &\implies SCT_3 (CTy).
 \end{aligned}$$

(The axioms CT_0 and WCT_0 are defined in [12].) A difficulty arises from the fact that for T_1 -spaces (where the singleton sets are closed) the function $x \mapsto \{x\}$ is (δ, ψ^+) -computable but in general not (δ, ψ^-) -computable. In our computable separation axioms, however, we use only the outer representation ψ^- for the closed sets.

6. COMPUTABLE METRIZATION

For a metric space (X, d) the open balls with rational radius and center from a dense set are a basis of a topology, the topology generated by it [3]. A topological space (X, τ) is metrizable, iff it is generated by a metric space. A second countable space is metrizable iff it is T_3 (remember $T_3 \implies T_2$) [3, Theorem 4.2.9]. Every second countable metrizable space has an at most countable dense subset A . Therefore, it can be enriched by a notation α of this dense set. The following definitions are essentially from [8, 9].

Definition 6.1.

- (1) An effective metric space is a tuple $\mathbf{M} = (X, d, A, \alpha)$ such that (M, d) is a metric space and α is a notation of a set $A \subseteq X$ which is dense in X .
- (2) The Cauchy representation δ_C of an effective metric space \mathbf{M} is defined by $\delta_C(p) = x$ there are words $u_0, u_1, \dots \in \text{dom}(\alpha)$ such that $p = \iota(u_0)\iota(u_1)\dots$ and $d(x, \alpha(u_i)) \leq 2^{-i}$ for all $i \in \mathbb{N}$.
- (3) The effective topological space [13, Definition 4] associated with the effective metric space is the tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $\nu, \nu\langle u, s \rangle := B(\alpha(u), \nu_{\mathbb{Q}}(s))$, is the canonical notation of the set β of all open balls with center from A and rational radius and τ is the smallest topology containing β .

- (4) An upper semi-computable (lower semi-computable) metric space is an effective metric space such that $\text{dom}(\alpha)$ is recursive and $d(a, b) < s$ ($s < d(a, b)$) is $(\alpha, \alpha, \nu_{\mathbb{Q}})$ -r.e.
- (5) A computable metric space is an effective metric space such that $\text{dom}(\alpha)$ is recursive and $r < d(a, b) < s$ is $(\nu_{\mathbb{Q}}, \alpha, \alpha, \nu_{\mathbb{Q}})$ -r.e.

Notice that $d(a, b) < s$ is $(\alpha, \alpha, \nu_{\mathbb{Q}})$ -r.e. iff the distance on A is $(\alpha, \alpha, \rho_{>})$ -computable iff d is $(\delta_C, \delta_C, \rho_{>})$ -computable [13, Example 1][9], and $r < d(a, b) < s$ is $(\nu_{\mathbb{Q}}, \alpha, \alpha, \nu_{\mathbb{Q}})$ -r.e. iff the distance on A is (α, α, ρ) -computable iff the distance is $(\delta_C, \delta_C, \rho)$ -computable (Example 3.3(1)) [9]. Since for every notation $\alpha : \subseteq \Sigma^* \rightarrow A$ with r.e. domain there is a notation $\alpha' : \subseteq \Sigma^* \rightarrow A$ with recursive domain such that $\alpha \equiv \alpha'$, allowing r.e. domains in Definition 6.1(4) and (5) is no proper generalization.

Theorem 6.2. *For every effective metric space $\mathbf{M} = (X, d, A, \alpha)$ with Cauchy representation δ_C and its associated effective topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ with canonical representation δ ,*

- (1) *if \mathbf{M} is upper semi-computable, then \mathbf{X} is a computable topological space.*
- (2) *$\delta \leq \delta_C$; $\delta_C \leq_t \delta$; $\delta_C \leq \delta$ if \mathbf{M} is upper semi-computable,*
- (3) *if \mathbf{M} is a computable metric space, then all the separation axioms from Definition 3.2 hold true for \mathbf{X} .*

The first two items of this theorem differ slightly from [9, Theorem 8.1.4] since *computable topological space* is defined differently.

Proof: (1) We must show that intersection is (ν, ν, θ) -computable. Observe that $B(a_1, r_1) \cap B(a_2, r_2) = \bigcup \{B(a, r) \mid d(a_1, a) < r_1 - r \wedge d(a_2, a) < r_2 - r\}$. Since $d(a, b) < s$ is r.e., there is an r.e. set S such that $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in S\}$.

(2) If $\delta(p) = x$, then p is a list of all $\langle u, v \rangle$ such that $d(x, \alpha(u)) < \nu_{\mathbb{Q}}(v)$. Therefore, there is a machine M which from $p \in \text{dom}(\delta)$ computes a sequence $\iota(u_0)\iota(u_1)\dots$ such that for all i , $d(x, \alpha(u_i)) \leq 2^{-i}$.

Now let \mathbf{M} be upper semi-computable. If $\delta_C(p) = x$ then $p = \iota(u_0)\iota(u_1)\dots$ such that for all i , $d(x, \alpha(u_i)) \leq 2^{-i}$. Observe that $x \in B(a, r) \iff (\exists i) d(a, \alpha(u_i)) < r - 2^{-i}$. Since $d(a, b) < s$ is r.e., from p we can compute a list of all w such that $x \in \nu(w)$.

If \mathbf{M} is not upper semi-computable then there are ‘‘oracles’’ $q, q' \in \Sigma^\omega$ such that $\text{dom}(\alpha)$ is recursive in q and $d(a, b) < s$ is r.e. in q' . Using the oracles q, q' , there is a machine translating δ_C to δ . The function f_M computed by this machine is continuous [9].

(3) By Theorem 4.1 it suffices to prove SCT_3 . Let $R := \{(\langle u, v \rangle, \langle u', v' \rangle) \mid \langle u, v \rangle, \langle u', v' \rangle \in \text{dom}(\nu), d(\alpha(u), \alpha(u')) + \nu_{\mathbb{Q}}(v) < \nu_{\mathbb{Q}}(v')\}$. Then (3.4) and $\overline{\nu(\langle u, v \rangle)} \subseteq \nu(\langle u', v' \rangle)$ for $(\langle u, v \rangle, \langle u', v' \rangle) \in R$. We compute a ψ^- -name of this closure. There is a machine that on input $(\langle u, v \rangle, \langle u', v' \rangle) \in R$ lists all $\langle w, w' \rangle \in \text{dom}(\nu)$ such that $d(\alpha(u), \alpha(w)) > \nu_{\mathbb{Q}}(v) + \nu_{\mathbb{Q}}(w')$. Then (3.5) holds true for the function $r := f_M$. \square

Since for an effective topological space (X, τ, β, ν) , δ is an admissible representation, by Theorem 6.2(2) the Cauchy representation is admissible, that is, it is continuous and $\delta \leq_t \delta_C$ for every continuous representation of X [9].

In general, we are interested in metric spaces (X, d) with representation $\delta : \subseteq \Sigma^\omega \rightarrow X$ such that the distance is at least $(\delta, \delta, \rho_{>})$ -continuous. In this case the metric space is separable and the representation $\delta : \subseteq \Sigma^\omega \rightarrow X$ is continuous [9, Lemma 8.1.1]. By adding a notation of a dense set we obtain an effective metric space with Cauchy representation δ_C . Then $\delta \leq_t \delta_C$, since the Cauchy representation is admissible.

We call a metric on a computable topological space $\mathbf{X} = (x, \tau, \beta, \nu)$ with canonical representation δ of the points lower semi-computable, if it is $(\delta, \delta, \rho_{<})$ -computable and computable, if it is (δ, δ, ρ) -computable.

Theorem 6.3. *Let \mathbf{X} be a computable topological space.*

- (1) *Suppose some lower semi-computable metric d generates the topology of \mathbf{X} . Then \mathbf{X} is SCT_2 .*
- (2) [7, 4] *Suppose \mathbf{X} is SCT_3 . Then its topology is generated by some computable metric.*

Theorem 6.3(2) has been proved in [7]. The shorter proof in [4] assumes $U \neq \emptyset$ for $U \in \beta$ but actually does not need this condition. We include a proof, since parts of it will be used in the proof of the next theorem.

Proof:

- (1) By [13, Theorem 11] there is a computable function g such that $\bigcap \nu^{\text{fs}}(w) = \theta \circ g(w)$. There is a machine M such that f_M realizes the distance function w.r.t. $(\delta, \delta, \rho_{<})$. Let H be the set of all (u, v) for which there are $v_1, v_2 \in \text{dom}(\nu^{\text{fs}})$ and v_3, v_4 such that
 - (a) the machine M on input $(v_1 1^\omega, v_2 1^\omega)$ writes in at most $\max(|v_1|, |v_2|)$ steps the word $v_3, v_4 \ll v_3$ and $\nu_{\mathbb{Q}}(v_4) > 0$, and
 - (b) $u \ll g(v_1)$ and $w \ll g(v_2)$.
 The set H is r.e. We prove (3.2) and (3.3).

Suppose $\delta(p) = x \neq y = \delta(q)$. Since $d(x, y) > 0$ there are $v_1 \sqsubseteq p, v_2 \sqsubseteq q, v_3$ and v_4 such that the machine M on input $(v_1 1^\omega, v_2 1^\omega)$ writes in at most $\max(|v_1|, |v_2|)$ steps the word $v_3, v_4 \ll v_3$ and $\nu_{\mathbb{Q}}(v_4) > 0$. Since $x \in \bigcap \nu^{\text{fs}}(v_1)$ and $y \in \bigcap \nu^{\text{fs}}(v_2)$, there are $u \ll g(v_1)$ and $w \ll g(v_2)$ such that $x \in \nu(u)$ and $y \in \nu(w)$. By the definition of H , $(u, w) \in H$. This proves (3.2).

Suppose $(u, w) \in H$. If $\nu(u) = \emptyset$ or $\nu(w) = \emptyset$, then $\nu(u) \cap \nu(w) = \emptyset$. Suppose $\nu(u) \neq \emptyset$ and $\nu(w) \neq \emptyset$. Then there are words v_1, v_2, v_3 and v_4 such that, the machine M on input $(v_1 1^\omega, v_2 1^\omega)$ writes in at most $\max(|v_1|, |v_2|)$ steps the word $v_3, v_4 \ll v_3, \nu_{\mathbb{Q}}(v_4) > 0, u \ll g(v_1)$ and $w \in g(v_2)$. Since $\nu(u) \subseteq \bigcap \nu^{\text{fs}}(v_1)$ and $\nu(w) \subseteq \bigcap \nu^{\text{fs}}(v_2)$, for every $x \in \nu(u)$ and every $y \in \nu(w)$ there are sequences p', q' such that $x = \delta(v_1 p')$ and $y = \delta(v_2 q')$. Then the machine M on input $(v_1 p', v_2 q')$ writes in at most $\max(|v_1|, |v_2|)$ steps the word v_3 such that $v_4 \ll v_3$ and $\nu_{\mathbb{Q}}(v_4) > 0$. Therefore, $d(x, y) > 0$ for every $x \in \nu(u)$ and $y \in \nu(w)$. This proves (3.3).

(2) Since $R = \text{dom}(r)$ is r.e., it has a computable numbering $(u_i, v_i)_{i \in \mathbb{N}}$. By Theorem 4.1, the Urysohn multi-function t_{U_r} has a computable $(\psi^-, \psi^-, [\delta \rightarrow \rho])$ -realization $h : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega$.

The function $U \mapsto U^c$ for $U \in \beta$ has a computable (ν, ψ^-) -realization $g : \subseteq \Sigma^* \rightarrow \Sigma^\omega$. For $i \in \mathbb{N}$ define $f_i : X \rightarrow \mathbb{R}, d_i : X \times X \rightarrow \mathbb{R}$ and $d : X \times X \rightarrow \mathbb{R}$ by

$$f_i := [\delta \rightarrow \rho] \circ h(r(u_i, v_i), g(v_i)) \quad (6.1)$$

$$d_i(x, y) := |f_i(x) - f_i(y)| \quad (6.2)$$

$$d(x, y) := \sum_i 2^{-i} d_i(x, y) \quad (6.3)$$

Then for every i , f_i is a continuous function such that $\text{range}(f_i) \subseteq [0; 1]$, $f_i(x) = 0$ for $x \in \nu(u_i)$ and $f_i(x) = 1$ for $x \notin \nu(v_i)$, and d_i is a continuous pseudometric on (X, τ) bounded by 1 such that $d_i(x, y) = 1$ for $x \in \nu(u_i)$ and $y \notin \nu(v_i)$.

Let A be closed and non-empty and $x \notin A$. Then there is some i such that $A \subseteq (\nu(v_i))^c$ and $x \in \nu(u_i) \subseteq \nu(v_i)$. Then $d_i(x, A) := \inf\{a \in A \mid d_i(x, a)\} = 1$. By [3, Lemma 4.4.6], d is a metric which generates the topology τ .

Since $i \mapsto f_i$ is $(\nu_{\mathbb{N}}, [\delta \rightarrow \rho])$ -computable, the metric d is (δ, δ, ρ) -computable. \square

The condition in the metrization theorem 6.3(2) is SCT_3 . We do not know whether STy or CT_3 are sufficient to prove the metrization theorem.

For a computable metric space a dense set of computable points is needed. In general a space with computable metric does not have computable points but its metric completion may have computable points (example: the restriction of the computable real line (Example 3.3(1)) to the non-computable real numbers). We will show that the metric space constructed in the proof of Theorem 6.3 can be completed to a computable metric space, if $\{u \in \text{dom}(\nu) \mid \nu(u) \neq \emptyset\}$ is r.e..

For a pseudo-metric d and sets A, B we define the diameter and the distance of sets as usual: $\text{dm}(A) := \sup\{d(x, y) \mid x, y \in A\}$, $d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\}$. The triangle inequality generalizes to

$$d(A, C) \leq d(A, B) + d(B, C) + \text{dm}(A) + \text{dm}(B) + \text{dm}(C). \quad (6.4)$$

We define a computable version of *homeomorphic embedding* [3, Section 2.1]. We will construct a computable metric space such that original computable topological space can be computably embedded into it.

Definition 6.4. For represented spaces (X, δ) and (X', δ') , a *computable embedding* is an injective function $f : X \rightarrow X'$ such that f is (δ, δ') -computable and f^{-1} is (δ', δ) -computable.

For computable topological spaces the standard representations are admissible, hence relatively computable functions are continuous. In this case a computable embedding is a homeomorphic embedding.

Theorem 6.5. [4] *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space such that CT_3 is true and the set $\{u \in \text{dom}(\nu) \mid \nu(u) \neq \emptyset\}$ is r.e. Then there is a computable embedding of \mathbf{X} into a computable metric space $\mathbf{M} = (M, d_M, A, \alpha)$ (where for \mathbf{X} we consider the standard representation and for \mathbf{M} the Cauchy representation).*

In [4] the theorem has been proved for SCT_3 spaces with non-empty base sets. First we show that the assumptions in Theorem 6.5 are sufficient and then present a proof that uses ideas from [4] but is more transparent and much simpler.

Proof: By [13, Lemma 25] there is a computable topological space $\mathbf{X}' = (X, \tau, \beta', \nu')$ equivalent to \mathbf{X} such that $\nu'(u) \neq \emptyset$ for all $u \in \text{dom}(\nu')$. By [13, Theorem 22] equivalent means $\delta \equiv \delta'$, hence the identity is a computable embedding of \mathbf{X} into \mathbf{X}' . By Lemma 3.4 we may assume w.l.o.g. that $\nu(u) \neq \emptyset$ for all $u \in \text{dom}(\nu)$.

By Theorem 4.4 the space \mathbf{X} is SCT_3 . For $i \in \mathbb{N}$ let f_i be the level function and d_i the pseudo-metric and let d be the metric defined in the proof of Theorem 6.3 ((6.1), (6.2), (6.3)) with diameters dm_i and dm , respectively. Remember that $i \mapsto f_i$ is $(\nu_{\mathbb{N}}, [\delta \rightarrow \rho])$ -computable.

In the following we use nested sequences of non-empty open sets instead of Cauchy-sequences of points for completion.

Proposition 6.6.

- (1) The multi-function g_1 mapping every (W, i, n) such that $W \in \beta$ and $i, n \in \mathbb{N}$ to some (U, a) such that $U \in \beta$, $a \in \mathbb{Q}$, $U \subseteq W$ and $f_i[U] \subseteq (a - 2^{-n}; a + 2^{-n})$ is computable.
- (2) The multi-function g_2 mapping every (x, W, i, n) such that $x \in W \in \beta$ and $i, n \in \mathbb{N}$ to some (U, a) such that $x \in U \in \beta$, $a \in \mathbb{Q}$, $U \subseteq W$ and $f_i[U] \subseteq (a - 2^{-n}; a + 2^{-n})$ is computable.

Proof: (Proposition 6.6)

(1) By the statement $\vec{\delta}_1 \equiv \vec{\delta}_3$ in [13, Theorem 29] and (6.1) and since intersection on open sets is computable [13, Theorem 11], there is a computable function g mapping (i, n, a, u') ($i, n \in \mathbb{N}$, $a \in \mathbb{Q}$, $u' \in \text{dom}(\nu)$) to some $q \in \Sigma^\omega$ such that $f_i^{-1}[(a - 2^{-n}; a + 2^{-n})] \cap \nu(u') = \theta(q)$. There is a machine M that on input (w, i, n) searches for some $a \in \mathbb{Q}$ and $u, u' \in \text{dom}(\nu)$ such that $(u', w) \in R$, $\nu(u) \neq \emptyset$ and $u \ll g(i, n, a, u')$ and then writes (u, a) .

There is some $y \in W = \nu(w)$. Then there is some u' such that $(u', w) \in R$ and $y \in \nu(u')$. There is some a such that $f_i(y) \in (a - 2^{-n}; a + 2^{-n})$. Since $y \in f_i^{-1}[(a - 2^{-n}; a + 2^{-n})] \cap \nu(u')$ there is some u such that $u \ll g(i, n, a, u')$. Therefore, the machine M on input (w, i, n) succeeds to write some (u, a) . In this case, $\nu(u) \subseteq \nu(u') \subseteq \nu(w)$ and $f_i[\nu(u)] \subseteq (a - 2^{-n}; a + 2^{-n})$. This proves the first statement.

(2) Let the machine from the above proof search for some u such that additionally $x \in \nu(u)$. □(Proposition 6.6)

We define a computable metric space $\mathbf{M} = (M, d_M, A, \alpha)$ as the constructive completion of a computable notated pseudometric space $\mathbf{A}' = (A', d', \alpha')$ [9, Definition 8.1.5] which will be constructed now. Let $A' := \text{dom}(\nu)$ and $\alpha'(u) := u$ for $u \in A'$. Iterating a computable realization of the multi-function g_1 from Proposition 6.6(1) for every $w \in A'$ we can compute a sequence $((u_{wk}, a_{wk}))_{k \in \mathbb{N}}$ (where $(u_{wk}, a_{wk}) \in A' \times \mathbb{Q}$) such that for $k = \langle i, n \rangle$,

$$\nu(u_{w, k+1}) \subseteq \nu(u_{wk}) \subseteq \nu(w), \quad (6.5)$$

$$f_i[\nu(u_{wk})] \subseteq (a_{wk} - 2^{-n}; a_{wk} + 2^{-n}). \quad (6.6)$$

Then for $v, w \in A'$ define

$$d'(v, w) := \sup_k d(\nu(u_{vk}), \nu(u_{wk})). \quad (6.7)$$

By (6.5) the sequence $(\text{dm} \circ \nu(u_{wk}))_{k \in \mathbb{N}}$ of diameters is decreasing and by (6.6),

$$\text{dm}_i \circ \nu(u_{w \langle i, n \rangle}) \leq 2 \cdot 2^{-n}. \quad (6.8)$$

Let $x, y \in \nu(u_{w \langle n, n \rangle})$. Then for all $j \leq n$, $\langle j, n \rangle \leq \langle n, n \rangle$, hence $x, y \in \nu(u_{w \langle j, n \rangle})$ by (6.5) and therefore, $|f_j(x) - f_j(y)| \leq 2 \cdot 2^{-n}$ by (6.8). Since $\text{range}(f_j) \subseteq [0; 1]$,

$$\begin{aligned} d(x, y) &= \sum_{j \in \mathbb{N}} 2^{-j} |f_j(x) - f_j(y)| \\ &\leq \sum_{j \leq n} 2^{-j} |f_j(x) - f_j(y)| + 2^{-n} \\ &\leq \sum_{j \leq n} 2^{-j} \cdot 2 \cdot 2^{-n} + 2^{-n} \leq 5 \cdot 2^{-n}, \end{aligned}$$

and hence,

$$\text{dm}(\nu(u_{w \langle n, n \rangle})) \leq 5 \cdot 2^{-n}. \quad (6.9)$$

Since $(\forall k)(\exists n)k \leq \langle n, n \rangle$, the sequence $(\text{dm} \circ \nu(u_{wk}))_{k \in \mathbb{N}}$ converges to 0.

For $U_k := \nu(u_{w_1k})$, $V_k := \nu(u_{w_2k})$ and $W_k := \nu(u_{w_3k})$,

$$\begin{aligned} d(U_k, W_k) &\leq d(U_k, V_k) + d(V_k, W_k) + \text{dm}(U_k) + \text{dm}(V_k) + \text{dm}(W_k) \\ &\leq d'(w_1, w_2) + d'(w_2, w_3) + \text{dm}(U_k) + \text{dm}(V_k) + \text{dm}(W_k), \end{aligned}$$

hence $d'(w_1, w_3) \leq d'(w_1, w_2) + d'(w_2, w_3)$. Therefore, d' is a pseudometric.

We will show that d' is computable. Since we have assumed that the base elements of the space \mathbf{X} are not empty, for every w, k there is some $x_{wk} \in \nu(u_{wk})$. Although we are not able to compute such points we will use their existence. For $m > n$, $x_{v(i,m)} \in \nu(u_{v(i,m)}) \subseteq \nu(u_{v(i,n)})$. Then by (6.6), $|a_{v(i,m)} - a_{v(i,n)}| \leq 2 \cdot 2^{-n}$. Therefore the sequence $(a_{v(i,n)})_{n \in \mathbb{N}}$ converges to some $b_{vi} \in \mathbb{R}$ such that $|b_{vi} - a_{v(i,n)}| \leq 2 \cdot 2^{-n}$. The function $(v, i) \mapsto b_{vi}$ is computable. Furthermore, $|a_{v(i,n)} - a_{w(i,n)}| \leq |a_{v(i,n)} - b_{vi}| + |b_{vi} - b_{wi}| + |b_{wi} - a_{w(i,n)}| \leq |b_{vi} - b_{wi}| + 4 \cdot 2^{-n}$ and correspondingly $|b_{vi} - b_{wi}| \leq |a_{v(i,n)} - a_{w(i,n)}| + 4 \cdot 2^{-n}$, hence $||b_{vi} - b_{wi}| - |a_{v(i,n)} - a_{w(i,n)}|| \leq 4 \cdot 2^{-n}$. By (6.6) for $k = \langle i, n \rangle$, $|a_{vk} - a_{wk}| - 2 \cdot 2^{-n} \leq |f_i(x_{vk}) - f_i(x_{wk})| \leq |a_{vk} - a_{wk}| + 2 \cdot 2^{-n}$, hence $||a_{vk} - a_{wk}| - d_i(x_{vk}, x_{wk})| \leq 2 \cdot 2^{-n}$. Therefore,

$$||b_{vi} - b_{wi}| - d_i(x_{v\langle i, n \rangle}, x_{w\langle i, n \rangle})| \leq 6 \cdot 2^{-n}. \quad (6.10)$$

Suppose $m \geq \langle i, n \rangle$. Since $x_{vm} \in \nu(u_{vm}) \subseteq \nu(u_{v(i,n)})$ and $x_{v(i,n)} \in \nu(u_{v(i,n)})$, $d_i(x_{v(i,n)}, x_{vm}) \leq 2 \cdot 2^{-n}$ by (6.8) and correspondingly $d_i(x_{w\langle i, n \rangle}, x_{wm}) \leq 2 \cdot 2^{-n}$, hence

$$|d_i(x_{vm}, x_{wm}) - d_i(x_{v\langle i, n \rangle}, x_{w\langle i, n \rangle})| \leq 4 \cdot 2^{-n},$$

and with (6.10),

$$||b_{vi} - b_{wi}| - d_i(x_{vm}, x_{wm})| \leq 10 \cdot 2^{-n}. \quad (6.11)$$

Then for $N \in \mathbb{N}$ and $m > \langle N+1, N+6 \rangle$,

$$\begin{aligned} &\left| d(x_{vm}, x_{wm}) - \sum_{i=0}^{N+1} 2^{-i} |b_{vi} - b_{wi}| \right| \\ &= \left| \sum_{i \in \mathbb{N}} 2^{-i} \cdot d_i(x_{vm}, x_{wm}) - \sum_{i=0}^{N+1} 2^{-i} |b_{vi} - b_{wi}| \right| \\ &\leq \left| \sum_{i=0}^{N+1} 2^{-i} \cdot d_i(x_{vm}, x_{wm}) - \sum_{i=0}^{N+1} 2^{-i} |b_{vi} - b_{wi}| \right| + 2^{-N-1} \\ &= \left| \sum_{i=0}^{N+1} 2^{-i} \cdot (d_i(x_{vm}, x_{wm}) - |b_{vi} - b_{wi}|) \right| + 2^{-N-1} \\ &\leq \sum_{i=0}^{N+1} 2^{-i} \cdot \left| d_i(x_{vm}, x_{wm}) - |b_{vi} - b_{wi}| \right| + 2^{-N-1} \\ &\leq \sum_{i=0}^{N+1} 2^{-i} \cdot 10 \cdot 2^{-N-6} + 2^{-N-1} \\ &\leq 20 \cdot 2^{-N-6} + 2^{-N-1} < 2^{-N}. \end{aligned}$$

Since $d(\nu(u_{vm}), \nu(u_{wm})) \leq d(x_{vm}, x_{wm}) \leq d(\nu(u_{vm}), \nu(u_{wm})) + \text{dm}(\nu(u_{vm})) + \text{dm}(\nu(u_{wm}))$,

$$\lim_{m \rightarrow \infty} d(\nu(u_{vm}), \nu(u_{wm})) = \lim_{m \rightarrow \infty} d(x_{vm}, x_{wm}) = d'(v, w)$$

Therefore by the above estimation,

$$\left| d'(v, w) - \sum_{i=0}^{N+1} 2^{-i} |b_{vi} - b_{wi}| \right| \leq 2^{-N}$$

for all N . Since the function $(v, w, N) \mapsto \sum_{i=0}^{N+1} 2^{-i} |b_{vi} - b_{wi}|$ is computable, the pseudometric d' on the pseudometric space $\mathbf{A}' = (A', d', \alpha')$ is (α', α', ρ) -computable.

Let $\mathbf{M} = (M, d_M, A, \alpha)$ be the constructive completion of the computable notated pseudometric space $\mathbf{A}' = (A', d', \alpha')$, see [9, Definition 8.1.5]. We summarize its definition. Define a set S , a function $d_S : S \times S \rightarrow \mathbb{R}$ and a binary relation \sim on S as follows:

$$S := \{(w_0, w_1, \dots) \mid w_i \in A', d'(w_i, w_j) \leq 2^{-i} \text{ for } j > i\}, \quad (6.12)$$

$$d_S((v_0, v_1, \dots), (w_0, w_1, \dots)) := \lim_{i \rightarrow \infty} d'(v_i, w_i), \quad (6.13)$$

$$(v_0, v_1, \dots) \sim (w_0, w_1, \dots) \iff d_S((v_0, v_1, \dots), (w_0, w_1, \dots)) = 0. \quad (6.14)$$

Then define $M := S/\sim$, $d_M := d_S/\sim$, $\alpha(w) := (w, w, w, \dots)/\sim$ for $w \in \text{dom}(\alpha) := \text{dom}(\alpha') = \text{dom}(\nu) = A'$ and $A := \text{range}(\alpha)$.

The Cauchy representation δ_C for \mathbf{M} is defined by: $p \in \text{dom}(\delta_C)$ iff there are words $w_0, w_1, \dots \in \text{dom}(\alpha)$ such that $p = \iota(w_0)\iota(w_1)\dots$ and $d'(w_i, w_j) \leq 2^{-i}$ for $j > i$, and $\delta_C(p) = (w_0, w_1, \dots)/\sim$.

We will define a function $f : X \rightarrow M$ and prove that f is well-defined, injective and (δ, δ_M) -computable and that the partial function f^{-1} is (δ_M, δ) -computable. For every $w \in \text{dom}(\nu)$ let $((u_{wk}, a_{wk}))_{k \in \mathbb{N}}$ be the sequence satisfying (6.5, 6.6) for $k = \langle i, n \rangle$ that has been used for defining the pseudometric space \mathbf{A}' .

Let $\delta(p) = x$. There is a machine N that on input $p \in \Sigma^\omega$ first finds some $w \ll p$. Using a computable realization of the multi-function g_2 from Proposition 6.6(2) from p and w it computes a sequence $((v_{pwk}, c_{pwk}))_{k \in \mathbb{N}}$ (where $(v_{pwk}, c_{pwk}) \in A' \times \mathbb{Q}$) such that for $k = \langle i, n \rangle$,

$$x \in \nu(v_{pw, k+1}) \subseteq \nu(v_{pwk}) \subseteq \nu(w), \quad (6.15)$$

$$f_i[\nu(v_{pwk})] \subseteq (c_{pwk} - 2^{-n}, c_{pwk} + 2^{-n}). \quad (6.16)$$

(compare with (6.5, 6.6)) and writes the sequence $q := \iota(v_0)\iota(v_1)\dots$ where $v_n := v_{pw\langle n+3, n+3 \rangle}$. In the same way as above from (6.5, 6.6) from (6.15, 6.16) we can conclude $\text{dm}(\nu(v_{pw\langle n, n \rangle})) \leq 5 \cdot 2^{-n}$. Then

$$x \in \nu(v_{n+1}) \subseteq \nu(v_n) \quad \text{and} \quad \text{dm}(v_n) < 2^{-n}. \quad (6.17)$$

We show $q \in \text{dom}(\delta_C)$. Suppose $i < j$ and let $n \in \mathbb{N}$. Since $\nu(v_j) \subseteq \nu(v_i)$ by (6.5), $\nu(u_{v_i\langle n, n \rangle}) \subseteq \nu(v_i)$ and $\nu(u_{v_j\langle n, n \rangle}) \subseteq \nu(v_i)$, hence $d(\nu(u_{v_i\langle n, n \rangle}), \nu(u_{v_j\langle n, n \rangle})) \leq \text{dm} \circ \nu(v_i)$. Therefore by (6.7) and (6.17),

$$d'(v_i, v_j) = \sup_n d(\nu(u_{v_i\langle n, n \rangle}), \nu(u_{v_j\langle n, n \rangle})) \leq \text{dm} \circ \nu(v_i) \leq 2^{-i}.$$

Therefore, $q \in \text{dom}(\delta_C)$.

Let $\delta(p) = x$, $\delta(p') = x'$, $f_N(p) = q = \iota(v_0)\iota(v_1)\dots$ and $f_N(p') = q' = \iota(v'_0)\iota(v'_1)\dots$. By the definition of d_M , $d_M(\delta_C(q), \delta_C(q')) = d_S((v_0, v_1, \dots), (v'_0, v'_1, \dots)) = \lim_{i \rightarrow \infty} d'(v_i, v'_i)$.

For all $i \in \mathbb{N}$, $x \in \nu(v_i)$ and $x' \in \nu(v'_i)$ by (6.17) and for all $n \in \mathbb{N}$, $\nu(u_{v_i\langle n,n \rangle}) \subseteq \nu(v_i)$ and $\nu(u_{v'_i\langle n,n \rangle}) \subseteq \nu(v'_i)$ by (6.5). For $y \in \nu(v_i)$ and $y' \in \nu(v'_i)$ by (6.17), $|d(x, x') - d(y, y')| \leq 2 \cdot 2^{-i}$. Therefore, $|d(x, x') - d(\nu(u_{v_i\langle n,n \rangle}), \nu(u_{v'_i\langle n,n \rangle}))| \leq 2 \cdot 2^{-i}$. Since by (6.7) $d'(v_i, v'_i) = \lim_{n \rightarrow \infty} d(\nu(u_{v_i\langle n,n \rangle}), \nu(u_{v'_i\langle n,n \rangle}))$, $|d(x, x') - d'(v_i, v'_i)| \leq 2 \cdot 2^{-i}$. Then by (6.13),

$$d(x, x') = d_S((v_0, v_1, \dots), (v'_0, v'_1, \dots)) = d_M(\delta_C(q), \delta_C(q')). \quad (6.18)$$

If $\delta(p) = \delta(p')$ then $\delta_C \circ f_N(p) = \delta_C \circ f_N(p')$, hence f_N realizes a single-valued function $f : X \rightarrow M$. By (6.18),

$$d(x, x') = d_M(f(x), f(x')), \quad (6.19)$$

therefore, f is a (δ, δ_C) -computable isometric function.

Finally, we show that f^{-1} is (δ_C, δ) -computable. Suppose $f(x) = y = \delta_C(q) \in \text{range}(f)$ with $q = (\iota(w_0)\iota(w_1)\dots)$. Notice that not necessarily $\nu(w_{n+1}) \subseteq \nu(w_n)$. There is some p' such that $x = \delta(p')$ and, by the definition of f , $y = f(x) = \delta_C \circ f_N(p')$. Then there are words w'_n such that $q' := f_N(p') = (\iota(w'_0)\iota(w'_1)\dots)$ and $\delta_C(q') = y$. By (6.15, 6.16), for these words, $x \in \nu(w'_{n+1}) \subseteq \nu(w'_n)$, $\text{dm} \circ \nu(w'_n) < 2^{-n}$ and hence $\lim_{n \rightarrow \infty} d'(w_n, w'_n) = 0$. Since $\delta_C(q) = y = \delta_C(q')$,

$$d'(w_n, w'_n) \leq 2 \cdot 2^{-n} \quad \text{for all } n. \quad (6.20)$$

(Still, for every $w \in \text{dom}(\nu)$ let $((u_{wk}, a_{wk}))_{k \in \mathbb{N}}$ be the sequence satisfying (6.5, 6.6) for $k = \langle i, n \rangle$ that has been used for defining the pseudometric space \mathbf{A}' .) Since $x \in \nu(w'_n)$ and $\nu(u_{w'_n\langle n,n \rangle}) \subseteq \nu(w'_n)$, $d(x, \nu(u_{w'_n\langle n,n \rangle})) \leq \text{dm}(w'_n) \leq 2^{-n}$.

Suppose $z \in \nu(u_{w_n\langle n,n \rangle})$. Then $d(\nu(u_{w_n\langle n,n \rangle}), z) = 0$, hence by (6.4, 6.9, 6.20)

$$\begin{aligned} d(x, z) &\leq d(x, \nu(u_{w'_n\langle n,n \rangle})) + d(\nu(u_{w'_n\langle n,n \rangle}), \nu(u_{w_n\langle n,n \rangle})) \\ &\quad + d(\nu(u_{w_n\langle n,n \rangle}), z) + 10 \cdot 2^{-n} \\ &\leq 2^{-n} + d'(w'_n, w_n) + 10 \cdot 2^{-n} \\ &\leq 13 \cdot 2^{-n}. \end{aligned}$$

Let $((u_i, v_i))_{i \in \mathbb{N}}$ be the computable numbering of the relation R from (3.4) for defining the functions f_i in the proof of Theorem 6.3. We prove that $x \in \nu(v)$ iff there are numbers $i, n \in \mathbb{N}$ such that

$$n \geq i + 4, \quad v = v_i, \quad \text{and} \quad \nu(u_i) \cap \nu(u_{w_n\langle n,n \rangle}) \neq \emptyset. \quad (6.21)$$

Suppose, $x \in \nu(v)$. There is some i such that $v = v_i$ and $x \in \nu(u_i) \subseteq \nu(v_i)$. There is some j such that $x \in B(x, 2^{-j}) \subseteq \nu(u_i)$. Let $n := \max(j + 4, i + 4)$. Then for all $z \in \nu(u_{w_n\langle n,n \rangle})$, $d(x, z) \leq 13 \cdot 2^{-n} < 2^{-j}$. We conclude $\nu(u_{w_n\langle n,n \rangle}) \subseteq B(x, 2^{-j}) \subseteq \nu(u_i)$, hence $\nu(u_i) \cap \nu(u_{w_n\langle n,n \rangle}) = \nu(u_{w_n\langle n,n \rangle}) \neq \emptyset$ (since $(\forall u)\nu(u) \neq \emptyset$ by assumption).

On the other hand, suppose (6.21) holds for some $i, n \in \mathbb{N}$. There is some $z \in \nu(u_i) \cap \nu(u_{w_n\langle n,n \rangle})$. Then $d(x, z) \leq 13 \cdot 2^{-n}$ as shown above. Therefore by $n \geq i + 4$,

$$|f_i(x) - f_i(z)| \leq 2^i \cdot d(x, z) \leq 2^i \cdot 13 \cdot 2^{-n} < 1.$$

Since $z \in \nu(u_i)$, $f_i(z) = 0$. Therefore, $f_i(x) < 1$ hence $x \in \nu(v_i) = \nu(v)$ (see (6.1, 6.2, 6.3)).

Since $(w, n) \mapsto u_{w, \langle n, n \rangle}$ is computable and $\nu(u) \cap \nu(v) \neq \emptyset$ is r.e., by (6.21) from $q = \iota(w_0)\iota(w_1)\dots$ we can compute a list of all v such that $x \in \nu(v)$. Therefore, f^{-1} is (δ_C, δ) -computable. \square

We mention that by (6.19) the embedding f is an isometric function from the metric space (X, d) into (M, d_M) , where d is the metrization of the original T_3 -space \mathbf{X} constructed in the proof of Theorem 6.3. Let NE abbreviate “ $U \neq \emptyset$ is ν -r.e.”. The condition in the embedding theorem 6.5 is $(CT_3 + NE)$. By Theorem 4.4 this implies SCT_3 . We do not know whether SCT_3 or STy are sufficient to prove the embedding theorem.

7. SEPARATION ON PRODUCT SPACES

For a computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ and $B \subseteq X$ the subspace $\mathbf{X}_B = (B, \tau_B, \beta_B, \nu_B)$ of \mathbf{X} to B is the computable topological space defined by $\text{dom}(\nu_B) := \text{dom}(\nu)$, $\nu_B(w) := \nu(w) \cap B$ [13, Section 8]. The separation axioms from Definition 3.2 are invariant under restriction to subspaces.

Theorem 7.1. *If a computable topological space satisfies some separation axiom from Definition 3.2 then each subspace satisfies this axiom.*

Proof: Straightforward. □

The product of two T_i -spaces is a T_i -space for $i = 0, 1, 2, 3$. This is no longer true for some of the computable separation axioms. The product $\mathbf{X}_1 \times \mathbf{X}_2 = \overline{\mathbf{X}} = (X_1 \times X_2, \overline{\tau}, \overline{\beta}, \overline{\nu})$ of two computable topological spaces $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X_2, \tau_2, \beta_2, \nu_2)$, defined by $\overline{\nu}\langle u_1, u_2 \rangle = \nu_1(u_1) \times \nu_2(u_2)$, is again a computable topological space [13, Section 8]. For the next theorem see Figure 1.

Theorem 7.2.

The SCT_2 -, WCT_3 -, CT_3 -, CTy - and SCT_3 -spaces are closed under finite products.

We consider computability w.r.t. $\nu_i, \delta_i, \psi_i^-, \overline{\nu}, \overline{\delta}$ and $\overline{\psi}^-$.

Proof: Suppose, \mathbf{X}_1 and \mathbf{X}_2 are SCT_2 . By [12, Theorem 7], $x_i \neq y_i$ is (δ_i, δ_i) -r.e. for $i = 1, 2$, hence $(x_1, x_2) \neq (y_1, y_2)$ is $([\delta_1, \delta_2], [\delta_1, \delta_2])$ -r.e., hence again by Theorem 6.3, $\mathbf{X}_1 \times \mathbf{X}_2$ is SCT_2 .

Suppose, \mathbf{X}_1 and \mathbf{X}_2 are WCT_3 . Let $(x_1, x_2) \in W_1 \times W_2$. From x_i and W_i we can find $U_i \in \beta_i$ such that $x_i \in U_i \subseteq \overline{U_i} \subseteq W_i$ (for $i = 1, 2$). Then $(x_1, x_2) \in U_1 \times U_2 \subseteq \overline{U_1} \times \overline{U_2} = \overline{U_1} \times \overline{U_2} \subseteq W_1 \times W_2$.

Suppose, \mathbf{X}_1 and \mathbf{X}_2 are CT'_3 . Suppose $(x_1, x_2) \in (W_1, W_2) \in \beta_1 \times \beta_2$. From $((x_1, x_2), (W_1, W_2))$ we can compute x_1, x_2, W_1 and W_2 . Using t'_3 for \mathbf{X}_1 and \mathbf{X}_2 we can compute (U_i, B_i) such that $U_i \in \beta_i, B_i \subseteq X_i$ is closed and $x_i \in U_i \subseteq B_i \subseteq W_i$ ($i = 1, 2$). Observe that $(x_1, x_2) \in U_1 \times U_2 \subseteq B_1 \times B_2 \subseteq W_1 \times W_2$. Form (U_1, B_1) and (U_2, B_2) we can compute $((u_1, u_2), (B_1, B_2))$.

Suppose \mathbf{X}_1 and \mathbf{X}_2 are CTy' . We show that $\mathbf{X}_1 \times \mathbf{X}_2$ is CTy' . From $(x_1, x_2) \in W_1 \times W_2$ where $W_1 \in \beta_1$ and $W_2 \in \beta_2$, we can compute x_1 and W_1 , where $x_1 \in W_1$. By CTy' for \mathbf{X}_1 , from these data we can compute some $U_1 \in \beta_1$ and a function $f_1 : X_1 \rightarrow \mathbb{R}$ such that $x_1 \in U_1 \subseteq W_1$ and f_1 is 0 inside U_1 and 1 outside W_1 . Correspondingly, we can compute U_2 and f_2 such that $x_2 \in U_2 \subseteq W_2$ and f_2 is 0 inside U_2 and 1 outside W_2 . From U_1, U_2 we can compute $\overline{U} := U_1 \times U_2$ and \overline{f} such that $\overline{f}(y_1, y_2) = \max(f_1(y_1), f_2(y_2))$. Then $(x_1, x_2) \in U_1 \times U_2 \subseteq W_1 \times W_2$, and \overline{f} is 0 inside $U_1 \times U_2$ and 1 outside $W_1 \times W_2$.

For \mathbf{X}_i ($i = 1, 2$) let R_i be the r.e. set and let r_i be the computable function for SCT_3 from Definition 3.2. By [13, Lemma 27] there is a computable function h such that

$\psi_1^-(p_1) \times \psi_2^-(p_2) = \bar{\psi}^- \circ h(p_1, p_2)$. Let

$$\begin{aligned} \bar{R} &:= \{(\langle u_1, u_2 \rangle, \langle w_1, w_2 \rangle) \mid (u_1, w_1) \in R_1 \wedge (u_2, w_2) \in R_2\}, \\ \bar{r}(\langle u_1, u_2 \rangle, \langle w_1, w_2 \rangle) &:= h(r_1(u_1, w_1), r_2(u_2, w_2)). \end{aligned}$$

A straightforward calculation shows that \bar{R} is the r.e. set and \bar{r} be the computable function for SCT_3 from the definition for the product $\mathbf{X}_1 \times \mathbf{X}_2$. \square

The CT_2 -spaces are not closed under product [12, Theorem 15]. Presumably, the CT_4 -spaces, and hence the CUR -spaces, are not closed under product.

8. FINAL REMARKS AND THANKS

The list of axioms of computable separation in Definition 3.2 is not exhaustive, there may be other ones. Applications must show which of these axioms are the most natural and useful ones. Many questions about the logical relation between the given axioms have not been answered.

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APPENDIX A (PROOF OF THEOREM 4.1(4))

SCT₃ \implies **CT₄**: Let A, B be disjoint closed sets. Suppose there are sequences of open sets V_i, W_i and of closed sets S_i, T_i ($i = 0, 1, \dots$) such that

$$V_i \subseteq S_i, \quad W_i \subseteq T_i, \quad (8.1)$$

$$A \subseteq \bigcup_{j \in \mathbb{N}} W_j, \quad B \cap T_i = \emptyset, \quad (8.2)$$

$$B \subseteq \bigcup_j V_j, \quad A \cap S_i = \emptyset. \quad (8.3)$$

For $i \in \mathbb{N}$ let

$$G_i := W_i \setminus \bigcup_{j \leq i} S_j, \quad H_i := V_i \setminus \bigcup_{j \leq i} T_j. \quad (8.4)$$

By (8.2,8.3),

$$A \subseteq O_A := \bigcup_i G_i, \quad B \subseteq O_B := \bigcup_i H_i. \quad (8.5)$$

The sets O_A and O_B are open. By (8.4) for $j \leq i$, $G_i \cap S_j = \emptyset$ and so $G_i \cap V_j = \emptyset$. Therefore, $G_i \cap H_j = \emptyset$ for $j \leq i$. Similarly, $H_i \cap G_j = \emptyset$ for $j \leq i$. Therefore, $G_i \cap H_j = \emptyset$ for $i, j \in \mathbb{N}$ and so $O_A \cap O_B = \emptyset$.

It remains to show that sets O_A and O_B can be computed from A and B . Assume $\psi^-(p) = A$ and $\psi^-(q) = B$. From p and q sequences of pairs (v_i, v_i^p) and (w_i, v_i^q) of words can be computed such that

$$\begin{aligned} \{(u, v) \in R \mid v \ll p\} &= \{(v_0, v_0^p), (v_1, v_1^p), \dots\} \\ \{(u, v) \in R \mid v \ll q\} &= \{(w_0, v_0^q), (w_1, v_1^q), \dots\}, \end{aligned}$$

where R is the r.e. set from (SCT₃). For $i \in \mathbb{N}$ let

$$\begin{aligned} V_i &:= \nu(v_i), & S_i &:= \psi^- \circ r(v_i, v_i^p), \\ W_i &:= \nu(w_i), & T_i &:= \psi^- \circ r(w_i, v_i^q). \end{aligned}$$

Then (8.1,8.2,8.3) hold true. By [13, Theorem 11] finite intersection and countable union of open sets can be computed, therefore, from the V_i, S_i, W_i and T_i the open sets sets O_A and O_B defined in (8.4,8.5) can be computed, for which $A \subseteq O_A$, $B \subseteq O_B$ and $O_A \cap O_B = \emptyset$. Therefore, the multi-function t_4 is $(\psi^-, \psi^-, [\theta, \theta])$ -computable.

APPENDIX B, PROOF OF THEOREM 4.1(6)

CUr \implies **CT₄**: See the proof of Theorem 4.1.

CT₄ \implies **CUr**: We effectivize the classical proof from [3]. Suppose the space is CT_4 . Then the multi-function

$$t : (D, U) \rightrightarrows (V, C) \text{ for open } U, V \text{ and closed } C, D \text{ such that } D \subseteq V \subseteq C \subseteq U \quad (8.6)$$

is computable. (Find $(V, W) \in t_4(D, U^c)$ and let $C := W^c$.) From closed disjoint sets A, B we compute a family (V_a, C_a) , $a \in \mathbb{Q} \cap [0; 1]$, of pairs of sets such that

$$V_a \text{ is open, } C_a \text{ is closed, } V_a \subseteq C_a, \quad (8.7)$$

$$C_a \subseteq V_b \text{ if } a < b, \quad (8.8)$$

$$A \subseteq V_0, \quad C_1 \subseteq B^c. \quad (8.9)$$

For this purpose let $i \mapsto r_i$ be a canonical bijective numbering of the rational numbers from the interval $[0; 1]$ such that $r_0 = 0$ and $r_1 = 1$. Define recursively

$$(V_0, C_0) \in t(A, B^c), \quad (V_1, C_1) \in t(C_0, B^c), \quad (V_k, C_k) \in t(C_l, V_m) \quad (8.10)$$

such that r_l is the maximum of the numbers in $\{r_0, \dots, r_{k-1}\}$ which are less than r_k and r_m is the minimum of the numbers in $\{r_0, \dots, r_{k-1}\}$ which are greater than r_k . The properties (8.7,8.8,8.9) can be verified easily. We define two real valued functions $f_<$ and $f_>$ on X as follows:

$$f_<(x) := \sup(\{a \mid x \notin C_a\} \cup \{0\}), \quad (8.11)$$

$$f_>(x) := \inf(\{a \mid x \in V_a\} \cup \{1\}). \quad (8.12)$$

If $x \in V_a$ and $b > a$ then $x \in C_b$, hence $b \leq a$ if $x \in V_a$ and $x \notin C_b$. Therefore, $f_<(x) \leq f_>(x)$. Suppose, $f_<(x) > f_>(x)$ for some x . Then there is some $c \in \mathbb{Q}$ such that $f_<(x) > c > f_>(x)$. Moreover, there are some $b > c$ such that $x \notin C_b$ and some $a < c$ such that $x \in V_a$. But $a < b$ and $x \in V_a$ implies $x \in C_b$. Contradiction. Therefore, $f := f_< = f_>$. The function f has value 0 on A and value 1 on B and is continuous [3].

We show that f can be computed from A and B . Since the function t in (8.6) is computable, the function $(A, B) \mapsto (V_{r_i}, C_{r_i})_i$ is $(\psi, \psi, [\theta, \psi]^\omega)$ -computable by (8.10). By [13, Theorem 13.2], $x \in U$ for open U is (ρ, θ) -r.e. and $x \notin C$ for closed C is (ρ, ψ) -r.e. Therefore, from $(V_{r_i}, C_{r_i})_i$ and x by (8.11) we can list all $a \in \mathbb{Q}$ such that $a < f(x)$ and by (8.12) we can list all $a \in \mathbb{Q}$ such that $a > f(x)$. Therefore, the function $((V_{r_i}, C_{r_i})_i, x) \mapsto f(x)$ is $([\theta, \psi]^\omega, \delta, \rho)$ -computable, hence by type conversion [9, Lemma 3.3.15] $(V_{r_i}, C_{r_i})_i \mapsto f$ is $([\theta, \psi]^\omega, [\delta \rightarrow \rho])$ -computable. Therefore, the space is CUr .