

## NODE REPLICATION: THEORY AND PRACTICE

DELIA KESNER <sup>a,c</sup>, LOÏC PEYROT <sup>a</sup>, AND DANIEL VENTURA <sup>b</sup>

<sup>a</sup> Université Paris Cité, CNRS, IRIF, Paris, France  
*e-mail address:* kesner@irif.fr, lpeyrot@irif.fr

<sup>b</sup> Univ. Federal de Goiás, INF, Goiânia, Brazil  
*e-mail address:* ventura@ufg.br

<sup>c</sup> Institut Universitaire de France

**ABSTRACT.** We define and study a term calculus implementing higher-order node replication. It is used to specify two different (weak) evaluation strategies: call-by-name and fully lazy call-by-need, that are shown to be observationally equivalent by using type theoretical technical tools.

### 1. INTRODUCTION

Computation in the  $\lambda$ -calculus is based on higher-order substitution, a complex operation being able to erase and copy terms during evaluation. Several formalisms have been proposed to model higher-order substitution, going from explicit substitutions (ES) [ACCL90] (see a survey in [Kes09]) and labeled systems [Lév78, BLM05] to pointer graphs [Wad71] or optimal sharing graphs [Lam90]. The operational semantics used in each formalism to implement copying (*i.e.* duplication) of terms is not the same.

Meanwhile, the Curry-Howard isomorphism [SU06] uncovers a deep connection between logical systems and term calculi. Also in this framework, different ways to normalize proofs in different logical systems correspond to different implementations of substitution. Indeed, the process of *full substitution* is somehow analogous to the normalization process in natural deduction, while *linear substitution* corresponds to cut elimination in Proof-Nets [Acc18a]. *Node-by-node replication* is based on a Curry-Howard interpretation of deep inference [GGP10, GHP13a]. Let us illustrate these different models using an example.

Indeed, suppose one wants to substitute all the free occurrences of some variable  $x$  in a term  $t = x \cdot x$  by the term  $u = y \cdot (z \cdot w)$  ( $\cdot$  denotes the application constructor). The variable substituted at each reduction step is going to be highlighted in **light blue** :

$$(x \cdot x)[x/u] \rightarrow u \cdot u \quad (1.1)$$

$$(x \cdot x)[x/u] \rightarrow (u \cdot x)[x/u] \rightarrow u \cdot u \quad (1.2)$$

$$(x \cdot x)[x/y \cdot (w \cdot z)] \rightarrow ((y \cdot x') \cdot (y \cdot x'))[x'/w \cdot z] \rightarrow u \cdot u \quad (1.3)$$

Full (or *non-linear*) substitution (1.1) is a one-shot substitution, replacing simultaneously *all* the free occurrences of  $x$  in  $t$  by the whole term  $u$ . This notion is generally defined by

induction on the structure of the term  $t$ . Linear (or *partial*) substitution (1.2) replaces *one* free occurrence of  $x$  at a time. This notion is generally defined by induction on the number of free occurrences of  $x$  in the term  $t$ . Node replication (1.3) is yet another approach which replicates *one* term-constructor of  $u$  *at a time*, instead of replicating  $u$  as a whole. This notion can be defined by induction on the structure of the term  $u$ . In the Curry-Howard interpretation of deep inference, node replication is full: all occurrences of  $x$  are replaced by a node of  $u$ . A linear version of the node replication approach can be formally defined by combining the last two models, although this does not correspond to any logical system we are aware of:

$$\begin{aligned} (x \cdot x)[x/u] &\rightarrow ((y \cdot x') \cdot x)[x/y \cdot x'][x'/w \cdot z] \rightarrow ((y \cdot x') \cdot (y \cdot x'))[x'/w \cdot z] \\ &\rightarrow ((y \cdot (w \cdot z)) \cdot (y \cdot x'))[x'/w \cdot z] \rightarrow (y \cdot (w \cdot z)) \cdot (y \cdot (w \cdot z)) = u \cdot u \end{aligned} \quad (1.4)$$

In this sense, node replication is orthogonal to the full/linear aspect of *classical* substitution, since it can be implemented by either full or linear operations, as explained above. Unsurprisingly, different notions of substitution give rise to different evaluation strategies. Indeed, linear substitution is the common model in well-known abstract machines for call-by-name and call-by-value (see *e.g.* [ABM14]), while (linear) node replication can be used to implement fully lazy sharing [Wad71]. However, node replication, originally introduced to implement optimal graph reduction in a graphical formalism, has only been studied from a Curry-Howard perspective by means of a term language known as the atomic  $\lambda$ -calculus [GHP13a].

**The Atomic Lambda-Calculus.** Logical aspects of intuitionistic deep inference are captured by the atomic  $\lambda$ -calculus  $\lambda a$  [GHP13a], where copying of terms proceeds *atomically*, *i.e.* node by node, similar to the optimal graph reduction of Lamping [Lam90]. The atomic  $\lambda$ -calculus is based on *explicit control of resources*, such as erasure and duplication. Its operational semantics explicitly handles the structural constructors of weakening and contraction, as in the calculus of resources  $\lambda 1x r$  [KL07, KR11]. As a result, understanding the meta-properties of the term-calculus, in a higher-level, and its application to concrete implementations of reduction strategies in programming languages, turn out to be quite difficult. In this paper, we take one step back, by studying the paradigm of *node replication* based on *implicit*, rather than *explicit*, weakening and contraction. This gives a new concise formulation of node replication which is simple enough to model different programming languages based on reduction strategies.

**Call-by-Name, Call-by-Value, Call-by-Need.** The theory of programming is usually based on some notion of *calculus*, which can be often seen as a higher-order rewriting system. A calculus definition results in general in a non-deterministic and unrestricted relation: a term can be reduced in different ways, some of them being more efficient than others. When implementing programming languages based on these theories, a specific (deterministic) *reduction strategy* is necessary to provide a concrete mechanism to evaluate a program.

In the specific case of functional programming, the theory is given by the  $\lambda$ -calculus, giving rise to different reduction strategies used by different functional programming languages.

*Call-by-name* is used to implement programming languages in which arguments of functions are first copied, then evaluated. This is frequently expensive, and may be improved by *call-by-value*, in which arguments are evaluated first, then consumed. The difference can be illustrated by the term  $t = \Delta(\mathbb{I}\mathbb{I})$ , where  $\Delta = \lambda x.x x$  and  $\mathbb{I} = \lambda z.z$ : call-by-name first

duplicates the argument  $\text{II}$ , so that its evaluation is also duplicated, while call-by-value first reduces  $\text{II}$  to (the value)  $\text{I}$ , so that duplications of the argument do not cause any duplicated evaluation. It is not always the best solution, though, because evaluating erasable arguments is useless. Compare for instance  $(\lambda x.z)(\text{II}) \rightarrow_{\beta} (\lambda x.z)\text{I} \rightarrow_{\beta} z$  to  $(\lambda x.z)(\text{II}) \rightarrow_{\beta} z$ .

*Call-by-need*, instead, combines the best of call-by-name and call-by-value: as in call-by-name, erasable arguments are not evaluated at all, and as in call-by-value, reduction of arguments occurs at most once. Furthermore, call-by-need implements a *demand-driven* evaluation, in which erasable arguments are never needed (so they are not evaluated), and non-erasable arguments are evaluated only if needed. Technically, some sharing mechanism is necessary, for example by extending the  $\lambda$ -calculus with explicit substitutions/let constructs [AF97]. Then, any  $\beta$ -reduction step can be decomposed in at least two steps: one creating an explicit (pending) substitution, and the other ones (linearly) substituting *values*. Thus for example,  $(\lambda x.xx)(\text{II})$  reduces to  $(xx)[x/\text{II}]$ , and the substitution argument is thus evaluated in order to find a value before performing the linear substitution.

Even when adopting this wise evaluation scheme, there are still some unnecessary copies of redexes: while only *values* (*i.e.* abstractions) are duplicated, they may contain redexes as subterms, *e.g.*  $\lambda z.z(\text{II})$  whose subterm  $\text{II}$  is a redex. Duplication of such values might cause redex duplications in *weak* (*i.e.* when evaluation is forbidden inside abstractions) call-by-need. This happens in particular in the *confluent* variant of weak reduction in [LM99]:

$$\begin{array}{lll} (\lambda x.xx)(\lambda z.z(\text{II})) & \rightarrow & (xx)[x/\lambda z.z(\text{II})] & \rightarrow & ((\lambda z.z(\text{II}))x)[x/\lambda z.z(\text{II})] \\ & & \rightarrow & & (z(\text{II}))[z/x][x/\lambda z.z(\text{II})] & \rightarrow & (x(\text{II}))[x/\lambda z.z(\text{II})] \\ & & \rightarrow & & (\text{II})(\text{II}) & \rightarrow & \text{I} \end{array}$$

**Full laziness.** Alas, it is not possible to keep all values shared forever, typically when they potentially contribute to the creation of a future  $\beta$ -reduction step. The key idea to gain efficiency is then to keep the subterm  $\text{II}$  as a *shared* redex. Therefore, the value  $\lambda z.z(\text{II})$  to be copied is split into two separate parts. The first one, called *skeleton*, contains the minimal information preserving the bound structure of the value, *i.e.* the linked structure between the binder and each of its bound variables. In our example, this is the term  $\lambda z.zy$ , where  $y$  is a fresh variable. The second one is a multiset of *maximal free expressions* (MFE), representing all the shareable expressions (here only the term  $\text{II}$ ). Only the skeleton is then copied, while the problematic redex  $\text{II}$  remains shared:

$$(\lambda x.xx)(\lambda z.z(\text{II})) \rightarrow (xx)[x/\lambda z.z(\text{II})] \rightarrow ((\lambda z.zy)x)[x/\lambda z.zy][y/\text{II}]$$

When the subterm  $\text{II}$  is needed ahead, it is first reduced inside the explicit substitution, as it is usual in call-by-need, thus avoiding to compute the redex twice. This optimization is called *fully lazy sharing* and is due to Wadsworth [Wad71].

As shown by Balabonski [Bal13], any weak strategy of the  $\lambda$ -calculus which is not using sharing is undecidable, as for example the *innermost needed reduction*. However, in the confluent weak setting evoked earlier [LM99], the fully lazy optimization is even optimal in the sense of Lévy [Lév80]. This means that the strategy reaches the weak normal form in the same number of  $\beta$ -steps as the shortest possible weak reduction sequence in the usual  $\lambda$ -calculus without sharing. Thus, fully lazy sharing turns out to be a *decidable* optimal strategy

**Quantitative Types.** Intersection types were introduced as (*denotational*) models capturing computational properties of functional programming in a broader sense [CD78, CDCV81, BCDC83]. They extend simple types with a new constructor  $\sqcap$ , thus allowing to support polymorphism in a finitary way: a program  $t$  is typable with  $\sigma \sqcap \tau$  if  $t$  is typable with both types  $\sigma$  and  $\tau$  independently. Originally, intersection enjoys associativity, commutativity, and in particular idempotency (*i.e.*  $\sigma \sqcap \sigma = \sigma$ ). By changing to a *non-idempotent* intersection constructor, one naturally comes to represent such a type by a multiset, also known as *multi-type*. Idempotent as well as non-idempotent types allow for a characterization of several operational properties of programs, *e.g.* termination of different evaluations strategies can be characterized by typability in some appropriate intersection type system. There is however a major difference between the two: while idempotent types provide *qualitative* information, non-idempotent ones also provide *quantitative* knowledge. More precisely, it is not only possible to prove that typability in one typing system characterizes termination of some particular evaluation strategy, *i.e.* that a program is terminating if and only if it is typable in the corresponding system, but also an *upper bound* or *exact measure* for the time needed for its evaluation can be derived from its typing information (see [BKV17] for a survey).

**Contributions.** The first contribution of this paper is a term calculus implementing (full) node replication (Section 2) and internally encoding skeleton extraction (Subsection 4.2). We study some of its main operational properties: termination of the substitution calculus, confluence, and its relation with the  $\lambda$ -calculus.

Our second contribution is the use of the node replication paradigm to give an alternative specification of two evaluation strategies usually described by means of full or linear substitution: call-by-name (Subsection 4.1) and weak fully lazy reduction (Subsection 4.2), based on the key notion of skeleton. The former can be related to (weak) head reduction, while the latter is a fully lazy version of (weak) call-by-need. In contrast to other implementations of fully lazy reduction relying on (external) meta-level definitions, our implementation is based on formal operations internally defined over the syntax of the calculus.

Our implementation of fully lazy reduction is based on a class of *restricted* terms, called  $\mathbb{U}$ , which simplifies the formal reasoning. This restriction is not ad-hoc in the sense it is stable through evaluations based on weak strategies, *i.e.* is an invariant property of evaluations starting from pure  $\lambda$ -terms.

Furthermore, while it is known that call-by-name and call-by-need specified by means of full/linear substitution are observationally equivalent [AF97], it was not clear at first whether the same property would hold in our case. Our third contribution is a proof of this result (Section 6) using semantical tools coming from proof theory –notably intersection types. This proof technique [Kes16] considerably simplifies other approaches [AF97, MOW98] based on syntactical tools. Moreover, the use of intersection types has another important consequence: standard call-by-name and call-by-need turn out to be observationally equivalent to call-by-name and call-by-need with node replication, as well as to the more semantical notion of neededness (see [KRV18]).

Intersection types provide quantitative information about fully lazy evaluation so that a fourth contribution of this work is a measure based on type derivations which turns out to be an upper bound to the length of reduction sequences to normal forms in a fully lazy implementation.

More generally, our work bridges the gap between the Curry-Howard theoretical understanding of node replication and concrete implementations of fully lazy sharing. Related works are presented in the concluding Section 7.

## 2. A CALCULUS FOR NODE REPLICATION

We now present the  $\lambda R$ -calculus (as in Replication). From a syntactical point of view, we add two new constructors to the  $\lambda$ -calculus: explicit substitutions and explicit distributors. From an operational point of view, we provide a rewriting system on  $\lambda R$ -terms together with a notion of *levels* which will play a key role in the next sections.

**2.1. Syntax.** Given a countably infinite set of variables  $x, y, z, \dots$ , we consider the following grammars.

$$\begin{array}{ll}
 \text{(Terms)} & t, u, r, s ::= x \mid \lambda x.t \mid tu \mid t[x/u] \mid t[x//\lambda y.u] \\
 \text{(Pure Terms)} & p, q ::= x \mid \lambda x.p \mid pq \\
 \text{(Term Contexts)} & \mathbf{C} ::= \diamond \mid \lambda x.\mathbf{C} \mid \mathbf{C}t \mid t\mathbf{C} \mid \mathbf{C}[x/t] \mid \mathbf{C}[x//\lambda y.u] \mid t[x/\mathbf{C}] \mid t[x//\lambda y.\mathbf{C}] \\
 \text{(List Contexts)} & \mathbf{L} ::= \diamond \mid \mathbf{L}[x/u] \mid \mathbf{L}[x//\lambda y.u]
 \end{array}$$

The set of terms is denoted by  $\mathbf{T}_R$  and the subset of **pure** terms is denoted by  $\mathbf{T}_P$ . We write  $\mathbf{I}$  for the identity function  $\lambda x.x$ , and  $|t|$  for the number of symbols of the term  $t$ .

The construction  $[x/u]$  is an **explicit substitution**, which allows the *sharing* of the subterm  $u$ . A term  $t[x/u]$  can be simply seen as a more concise notation for a *let-binding* let  $x = u$  in  $t$ , where  $u$  is shared across the free occurrences of the variable  $x$  in  $t$ . The second construction  $[x//\lambda y.u]$  is an **explicit distributor** (or simply *distributor*), which is used specifically in the duplication of abstractions.

An **explicit cut** is denoted by  $[x \triangleleft u]$ , which is either  $[x/u]$ , or  $[x//u]$  when  $u$  is  $\lambda y.u'$ , typically to factorize some definitions and proofs where explicit substitutions and distributors behave similarly. A characterization function  $\text{ES}(\cdot)$  on explicit cuts distinguishes these two cases:  $\text{ES}([x \triangleleft u]) = 1$  if  $[x \triangleleft u] = [x/u]$ , and 0 otherwise. The application constructor associates to the left, *i.e.*  $t_1 t_2 \dots t_n$  means  $((t_1 t_2) \dots t_n)$ . Explicit cuts affects the rightmost constructor, *i.e.*  $tu[x \triangleleft s]$  means  $t(u[x \triangleleft s])$ , otherwise we will write parenthesis like  $(tu)[x \triangleleft s]$ .

Two notions of **contexts** are used. Term contexts  $\mathbf{C}$  extend those of the  $\lambda$ -calculus to explicit cuts. A term context is said to be **pure** if it does not contain any explicit cut. List contexts  $\mathbf{L}$  denote an arbitrary list of explicit cuts. They will be used in particular to implement reduction at a *distance* (see Subsection 2.2).

**Free and bound variables** of terms are defined as expected, in particular  $\text{fv}(\lambda x.t) := \text{fv}(t) \setminus \{x\}$  and  $\text{fv}(t[x \triangleleft u]) := \text{fv}(t) \setminus \{x\} \cup \text{fv}(u)$ . We write  $|t|_x$  to denote the number of free occurrences of the variable  $x$  in the term  $t$ . These notions are extended to contexts as expected. In particular  $\text{fv}(\diamond) := \emptyset$  and  $\text{fv}(\mathbf{L}[x \triangleleft u]) := \text{fv}(\mathbf{L}) \setminus \{x\} \cup \text{fv}(u)$ . The **domain** of a list context is defined as  $\text{dom}(\diamond) := \emptyset$  and  $\text{dom}(\mathbf{L}[x \triangleleft u]) := \text{dom}(\mathbf{L}) \cup \{x\}$ .

The standard notion of  $\alpha$ -conversion [Bar85] on  $\lambda$ -terms is extended to the full set of  $\lambda R$ -terms as expected, and we systematically assume  $\alpha$ -conversion whenever necessary to avoid capture of free variables. We write  $t\{x/u\}$  for the **meta-level substitution** operation simultaneously replacing all the free occurrences of the variable  $x$  in  $t$  by the term  $u$ .

The **application of a context  $\mathbf{C}$  to a term  $t$** , written  $\mathbf{C}\langle t \rangle$ , replaces the hole  $\diamond$  of  $\mathbf{C}$  by  $t$ . Thus, for instance,  $\diamond\langle t \rangle = t$  and  $(\lambda x.\diamond)\langle t \rangle = \lambda x.t$ . This operation is not defined modulo  $\alpha$ -conversion, so that capture of variables eventually happens, such as in the second example

if  $x \in \text{fv}(t)$ . Thus, another kind of application of contexts to terms is also considered, identified by double brackets, and is only defined if there is no capture of variables. For instance,  $(\lambda y.\diamond)\langle x \rangle = \lambda y.x$  while  $(\lambda x.\diamond)\langle\langle x \rangle\rangle$  is undefined.

**2.2. Operational semantics.** Before presenting the dynamics of the calculus, let us introduce some notations and definitions concerning reduction. Let  $\rightarrow_{\mathcal{R}}$  be any reduction relation. We write  $t \rightarrow_{\mathcal{R}} t'$  when  $(t, t') \in \rightarrow_{\mathcal{R}}$ , and  $\twoheadrightarrow_{\mathcal{R}}$  (resp.  $\twoheadrightarrow_{\mathcal{R}}^+$ ) for the reflexive-transitive (resp. transitive) closure of  $\rightarrow_{\mathcal{R}}$ . A term  $t$  is said to be  **$\mathcal{R}$ -confluent** iff  $t \twoheadrightarrow_{\mathcal{R}} u$  and  $t \twoheadrightarrow_{\mathcal{R}} s$  implies there is  $t'$  such that  $u \twoheadrightarrow_{\mathcal{R}} t'$  and  $s \twoheadrightarrow_{\mathcal{R}} t'$ . The relation  $\mathcal{R}$  is **confluent** iff every term is  $\mathcal{R}$ -confluent. A term  $t$  is said to be in  **$\mathcal{R}$ -normal form** (written also  $\mathcal{R}$ -nf) iff there is no  $t'$  such that  $t \rightarrow_{\mathcal{R}} t'$ . A term  $t$  is said to be  **$\mathcal{R}$ -terminating** or  **$\mathcal{R}$ -normalizing** iff there is no infinite  $\mathcal{R}$ -sequence starting at  $t$ . The reduction  $\mathcal{R}$  is said to be **terminating** iff every term is  $\mathcal{R}$ -terminating.

Explicit substitutions may block some expected *meaningful* (i.e. non-structural) reductions. For instance,  $\beta$ -reduction is blocked in  $(\lambda x.t)[y/s]u$  because an explicit substitution lies between the function and its argument. This situation does not happen in graphical representations (e.g. [Gir96]), but it is typical in the sequential structure of *term* syntaxes.

There are at least two ways to handle this issue. The first one is based on *structural/permutation* rules, as in [GHP13a]. Therefore, in the previous example, the substitution is first pushed outside the application node by means of a permutation rule, as  $(\lambda x.t)[y/s]u \rightarrow ((\lambda x.t)u)[y/s]$ , so that  $\beta$ -reduction is finally unblocked. The second, less elementary, possibility is given by an operational semantics *at a distance* [AK10, ABKL14], where the  $\beta$ -redex can be fired by a rule like  $L\langle\lambda x.t\rangle u \rightarrow L\langle t[x/u]\rangle$ , where  $L$  is an arbitrary list context. The distant paradigm is therefore used to gather meaningful and permutation rules in only one reduction step. In  $\lambda R$ , we combine these two complementary technical tools. First, we consider the following permutation rules:

$$\begin{array}{lll} \lambda y.t[x \triangleleft u] & \mapsto_{\pi} & (\lambda y.t)[x \triangleleft u] \quad \text{if } y \notin \text{fv}(u) \\ t[x \triangleleft u]s & \mapsto_{\pi} & (ts)[x \triangleleft u] \quad \text{if } x \notin \text{fv}(s) \\ ts[x \triangleleft u] & \mapsto_{\pi} & (ts)[x \triangleleft u] \quad \text{if } x \notin \text{fv}(t) \\ t[x \triangleleft u][y \triangleleft s] & \mapsto_{\pi} & t[x \triangleleft u][y \triangleleft s] \quad \text{if } y \notin \text{fv}(t) \end{array}$$

The reduction relation  $\rightarrow_{\pi}$  is defined as the closure of the four rules  $\mapsto_{\pi}$  under *all* contexts.

**Example 2.1.** Let  $t = x[x/w[z_1//\mathbf{I}](\lambda y.y[z_2/z_3])]$ . Both inner explicit cuts  $[z_1//\mathbf{I}]$  and  $[z_2/z_3]$  are pushed outside the main explicit substitution, which results in a pure term followed by a list of explicit cuts.

$$\begin{array}{lll} t & \rightarrow_{\pi} & x[x/w[z_1//\mathbf{I}](\lambda y.y)[z_2/z_3]] & \rightarrow_{\pi} & x[x/(w[z_1//\mathbf{I}](\lambda y.y))][z_2/z_3] \\ & \rightarrow_{\pi} & x[x/w[z_1//\mathbf{I}](\lambda y.y)][z_2/z_3] & \rightarrow_{\pi} & x[x/(w(\lambda y.y))][z_1//\mathbf{I}][z_2/z_3] \\ & \rightarrow_{\pi} & x[x/w(\lambda y.y)][z_1//\mathbf{I}][z_2/z_3] & & \end{array}$$

Permutations do not hold any computational content, only a structural one. Indeed, all terms in the reduction sequence above could be naturally translated to the same graphical notation. In order to highlight the computational content of node replication, we combine distance and permutations within a single rewriting semantics. In addition, because operational semantics at a distance is directly inspired from graph formalisms, there is a better correspondence between the syntactic representation of terms and the graphical

representations of their associated reduction notion [KL07, Acc18b]. The resulting reduction relation  $\lambda R$  is given by the closure under all contexts of the following rules:

$$\begin{array}{lll}
\mathbf{L} \langle \lambda x.t \rangle u & \mapsto_{\mathbf{dB}} & \mathbf{L} \langle t[x/u] \rangle \\
t[x/\mathbf{L} \langle us \rangle] & \mapsto_{\mathbf{app}} & \mathbf{L} \langle t\{x/yz\}[y/u][z/s] \rangle \quad \text{where } y \text{ and } z \text{ are fresh} \\
t[x/\mathbf{L} \langle \lambda y.u \rangle] & \mapsto_{\mathbf{dist}} & \mathbf{L} \langle t[x//\lambda y.z[z/u]] \rangle \quad \text{where } z \text{ is fresh} \\
t[x//\lambda y.u] & \mapsto_{\mathbf{abs}} & \mathbf{L} \langle t\{x/\lambda y.p\} \rangle \quad \text{where } u \rightarrow_{\pi} \mathbf{L} \langle p \rangle \text{ and } y \notin \text{fv}(\mathbf{L}) \\
t[x/\mathbf{L} \langle y \rangle] & \mapsto_{\mathbf{var}} & \mathbf{L} \langle t\{x/y\} \rangle
\end{array}$$

where the distant contexts are highlighted in **green** to make it easier to read. The  $\lambda R$ -**calculus** is defined by the set of terms  $T_R$  equipped with this reduction relation. In the five rules just above, a list context  $L$  is pushed outside the term. We assume in all these cases that there is no capture of variables caused by this transformation, *e.g.* in rule **dB** this means that  $\text{dom}(L) \cap \text{fv}(u) = \emptyset$ . Apart from the *distant Beta* rule **dB** used to fire  $\beta$ -reduction, there are four substitution rules used to copy nodes of *pure* terms pushing outside all the cuts surrounding the node to be copied. Rule **app** copies one application node, while rule **var** copies one variable node. To copy abstractions, both rules **dist** and **abs** are needed. Notice that the (meta-level and capture-free) substitution is *full*, in the sense that it is performed simultaneously on all occurrences of the free variable  $x$  at the same time.

The reduction relation  $\mapsto_{\text{sub}}$  is defined as  $\mapsto_{\mathbf{app}} \cup \mapsto_{\mathbf{dist}} \cup \mapsto_{\mathbf{abs}} \cup \mapsto_{\mathbf{var}}$ , while the substitution relation  $\rightarrow_{\text{sub}}$  (resp. distant Beta relation  $\rightarrow_{\mathbf{dB}}$ ) is defined as the closure of  $\mapsto_{\text{sub}}$  (resp.  $\mapsto_{\mathbf{dB}}$ ) under *all* contexts, and the reduction relation  $\rightarrow_R$  is the union of  $\rightarrow_{\text{sub}}$  and  $\rightarrow_{\mathbf{dB}}$ .

**Example 2.2.** This example illustrates the use of rules **app** and **var** to replicate application and variables nodes, as well as rule **dB** to fire reduction. No distance is involved in this example.

$$\begin{array}{lll}
(\lambda x.xx)(yz) & \rightarrow_{\mathbf{dB}} (xx)[x/yz] & \rightarrow_{\mathbf{app}} ((x_1x_2)(x_1x_2))[x_1/y][x_2/z] \\
& \rightarrow_{\mathbf{var}} (yx_2)(yx_2)[x_2/z] & \rightarrow_{\mathbf{var}} (yz)(yz)
\end{array}$$

**Example 2.3.** Replication of abstractions is more involved, as illustrated below.

$$(\lambda x.xx)(\lambda y.(ww)y) \rightarrow_{\mathbf{dB}} (xx)[x/\lambda y.(ww)y] \quad (2.1)$$

$$\rightarrow_{\mathbf{dist}} (xx)[x//\lambda y.z[z/(ww)y]] \quad (2.2)$$

$$\rightarrow_{\mathbf{app}} (xx)[x//\lambda y.(z_1z_2)[z_1/ww][z_2/y]] \quad (2.3)$$

$$\rightarrow_{\mathbf{var}} (xx)[x//\lambda y.(z_1y)[z_1/ww]] \quad (2.4)$$

$$\rightarrow_{\mathbf{app}} (xx)[x//\lambda y.((z_3z_2)y) \mathbf{[z_3/w][z_2/w]}] \quad (2.5)$$

$$\rightarrow_{\mathbf{abs}} ((\lambda y.(z_3z_2)y)(\lambda y.(z_3z_2)y))[z_3/w][z_2/w] \quad (2.6)$$

$$\rightarrow_{\mathbf{var}} ((\lambda y.(wz_2)y)(\lambda y.(wz_2)y))[z_2/w] \quad (2.7)$$

$$\rightarrow_{\mathbf{var}} (\lambda y.(ww)y)(\lambda y.(ww)y) \quad (2.8)$$

The specificity in copying an abstraction  $\lambda y.u$  is due to the (binding) relation between the binder  $\lambda y$  and all the free occurrences of  $y$  in its body  $u$ . Abstractions are thus copied in two stages. The first one is implemented by the rule **dist**, which creates a distributor in which a potentially replicable abstraction is placed, while moving its body inside a new explicit substitution. Thus, in line (2.2), we create a distributor over the abstraction  $\lambda y$ , while  $(ww)y$  is placed inside an explicit substitution  $[z/(ww)y]$ . Notice that this substitution is in the scope of abstraction  $\lambda y$ . The distributor is marking the fact that the abstraction

needs to be further duplicated. There are then two kinds of potentially replicable nodes shared in the body of the corresponding abstraction.

- (1) All free occurrences of the variable bound by the main abstraction (here  $\lambda y$ ) must be replicated by means of the rule **var** (2.4), so as to keep the correct binding structure. This means that all the nodes leading to these occurrences must also be duplicated: this is why rule **app** is first used in (2.3).
- (2) All nodes which are neither a free occurrence of the bound variable nor in the path to such a node can be arbitrarily copied inside the distributor (*e.g.* the internal application node in line (2.5)), or replicated later (*e.g.* the two variable nodes  $w$  in (2.7) and (2.8)).

Components which are not replicated inside the distributor form a list of explicit cuts, which can occur at different depths inside this distributor. Indeed, in (2.5), there are two explicit substitutions  $[z_3/w]$  and  $[z_2/w]$ . The cuts can be gathered together into a list context, called **L** in the definition of rule **abs**, which is pushed outside by using permutation rules, before performing the substitution of the pure body containing all the bound occurrences of  $y$  (here  $\lambda y.(z_1 z_2)y$ ). This operation is in general hard to specify using only distance since the cuts can appear at arbitrary depth in the distributor, and this is one of the reasons to introduce the use of permutation rules in rule **abs**.

Other choices are possible, such as replicating all the nodes, or only the uppermost application and the node  $y$  (corresponding to fully lazy duplication), as long as at least all free occurrences of  $y$  are duplicated.

**Example 2.4.** This last example showcases different reduction steps with distance, highlighted in green.

$$\begin{aligned}
(\lambda x.x) [z_4/z_5] (w[z_1//\mathbf{I}](\lambda y.y[z_2/z_3])) &\rightarrow_{\mathbf{dB}} x[x/w[z_1//\mathbf{I}](\lambda y.y[z_2/z_3])][z_4/z_5] \\
&\rightarrow_{\mathbf{app}} (x_1 x_2)[x_1/w [z_1//\mathbf{I} ]][x_2/\lambda y.y[z_2/z_3]][z_4/z_5] \\
&\rightarrow_{\mathbf{var}} (w x_2)[z_1//\mathbf{I}][x_2/\lambda y.y[z_2/z_3]][z_4/z_5] \\
&\rightarrow_{\mathbf{dist}} (w x_2)[z_1//\mathbf{I}][x_2//\lambda y.x[x/y [z_2/z_3] ]][z_4/z_5] \\
&\rightarrow_{\mathbf{var}} (w x_2)[z_1//\mathbf{I}][x_2//\lambda y.y [z_2/z_3] ]][z_4/z_5] \\
&\rightarrow_{\mathbf{abs}} (w(\lambda y.y))[z_1//\mathbf{I}][z_2/z_3][z_4/z_5]
\end{aligned}$$

Notice that an **R**-step can be decomposed into some  $\pi$ -steps followed by a simpler step not involving any list context. Indeed,  $t \rightarrow_{\mathbf{R}} u$  could be simulated by  $t \rightarrow_{\pi} t' \rightarrow_{\mathbf{R}'} u$ , where  $\rightarrow_{\mathbf{R}'}$  is the closure under all contexts of the following set of rewriting rules:

$$\begin{array}{lll}
(\lambda x.t)u & \mapsto_{\mathbf{dB}'} & t[x/u] \\
t[x/us] & \mapsto_{\mathbf{app}'} & t\{x/yz\}[y/u][z/s] \\
t[x/\lambda y.u] & \mapsto_{\mathbf{dist}'} & t[x//\lambda y.z[z/u]] \\
t[x//\lambda y.p] & \mapsto_{\mathbf{abs}'} & t\{x/\lambda y.p\} \\
t[x/y] & \mapsto_{\mathbf{var}'} & t\{x/y\}
\end{array}$$

For instance, step (2.6) in Example 2.3 can be decomposed as follows, where  $r = \lambda y.(z_3 z_2)y$ :

$$(xx)[x//r[z_3/w][z_2/w]] \rightarrow_{\pi} (xx)[x//r][z_3/w][z_2/w] \rightarrow_{\mathbf{abs}'} (rr)[z_3/w][z_2/w]$$

This decomposition will be useful in some of our proofs, but we prefer to integrate distance inside the rules, as initially defined on page 7, to highlight the computational behavior and execute permutations only when strictly necessary.



**2.3. Levels.** This subsection introduces the syntactical notion of level and its associated properties. Intuitively, the level of a variable in a term indicates the maximal depth (*only* w.r.t. explicit substitutions and not w.r.t. explicit distributors) of its free occurrences. However, in order to be sound with respect to the permutation rules, levels do not consider depth in the usual sense only, but also across linked chains of explicit substitutions. For instance, the level of  $z$  in both  $(xx)[x/y[y/z]]$  and  $(xx)[x/y][y/z]$  is the same. Levels will play a key role in the next sections: they will be the combinatorial witnesses of the progress of **sub**-substitution steps, necessary to prove termination of the **sub**-relation. They will also be helpful to define a decreasing measure on typing derivations in Section 5. The **level** of a variable  $z$  in a term  $t$  is defined by induction:

$$\begin{aligned} \text{lv}_z(x) &:= 0 \\ \text{lv}_z(t_1 t_2) &:= \max(\text{lv}_z(t_1), \text{lv}_z(t_2)) \\ \text{lv}_z(\lambda x.t) &:= \text{lv}_z(t) \\ \text{lv}_z(t[x \triangleleft u]) &:= \begin{cases} \text{lv}_z(t) & \text{if } z \notin \text{fv}(u) \\ \max(\text{lv}_z(t), \text{lv}_x(t) + \text{lv}_z(u) + \text{ES}([x \triangleleft u])) & \text{otherwise} \end{cases} \end{aligned}$$

In the two last cases, we can always suppose  $z \neq x$ , because we work modulo  $\alpha$ -conversion. Notice that  $\text{lv}_z(t) = 0$  whenever  $z \notin \text{fv}(t)$  or  $t$  is pure. We illustrate the concept of level by an example. Consider  $t = x[x/z[y/w]][w/w']$ , then  $\text{lv}_z(t) = 1$ ,  $\text{lv}_{w'}(t) = 3$  and  $\text{lv}_y(t) = 0$  because  $y \notin \text{fv}(t)$ . This notion is also extended to contexts as expected, *i.e.*  $\text{lv}_\diamond(\mathbf{C}) = \text{lv}_z(\mathbf{C}\langle z \rangle)$ , where  $z$  is a fresh variable. Remark that for any variable  $x$ ,  $\text{lv}_\diamond(\mathbf{C}) \leq \text{lv}_x(\mathbf{C}\langle\langle x \rangle\rangle)$  and  $\text{lv}_x(\mathbf{C}\langle\langle p \rangle\rangle) \leq \text{lv}_x(\mathbf{C}\langle\langle x \rangle\rangle)$  for any  $p \in \mathbf{T}_P$ .

**Lemma 2.5.** *Let  $x \neq z$ ,  $t \in \mathbf{T}_R$  and  $p \in \mathbf{T}_P$ :*

- (1) *If  $z \notin \text{fv}(p)$ , then  $\text{lv}_z(t\{x/p\}) = \text{lv}_z(t)$ .*
- (2) *If  $z \in \text{fv}(p)$ , then  $\text{lv}_z(t\{x/p\}) = \max(\text{lv}_z(t), \text{lv}_x(t))$ .*

*Proof.* By induction on  $t$  (see appendix on page 44). □

**Lemma 2.6.** *Let  $t \in \mathbf{T}_R$  and  $w$  be any variable.*

- (1) *If  $t_0 \rightarrow_\pi t_1$ , then  $\text{lv}_w(t_0) \geq \text{lv}_w(t_1)$ .*
- (2) *If  $t_0 \rightarrow_{\text{sub}} t_1$ , then  $\text{lv}_w(t_0) \geq \text{lv}_w(t_1)$ .*

*Proof.* By induction on the reduction relation (see appendix on page 46). □

Notice that there are two cases when the level of a variable in a term may decrease:

- Moving an explicit cut out of another one with a permutation rule when the first cut is a void cut, *i.e.* its domain does not bind any other variable. Thus *e.g.* if  $t = x[x/z[y/w]][w/w'] \rightarrow_\pi x[x/z][y/w][w/w'] = u$ , then  $\text{lv}_{w'}(t) = 3 > 2 = \text{lv}_{w'}(u)$ .
- Using rule  $\mapsto_{\text{var}}$ . Thus *e.g.* if  $t = (xx)[x/y][y/z] \rightarrow_{\text{var}} (yy)[y/z] = u$ , then  $\text{lv}_z(t) = 2 > 1 = \text{lv}_z(u)$ .

Hence, levels alone are not enough to prove termination of  $\rightarrow_{\text{sub}}$ . We thus define a decreasing measure for  $\rightarrow_{\text{sub}}$  in which not only variables are indexed by a level, but also constructors. For instance, in the term  $t[x/\lambda y.yz]$ , we can consider that the level of *all* the constructors of  $\lambda y.yz$ , including the abstraction and the application, have level  $\text{lv}_x(t)$ . This

will ensure that the level of an abstraction will decrease when applying rule **dist**, as well as the level of an application when applying rule **app**.

### 3. OPERATIONAL PROPERTIES

We now prove three key properties of the  $\lambda R$ -calculus: termination of the reduction system  $\rightarrow_{\text{sub}}$ , relation between  $\lambda R$  and the  $\lambda$ -calculus, and confluence of the reduction system  $\rightarrow_{\text{R}}$ .

**Termination of  $\rightarrow_{\text{sub}}$ .** Some (rather informal) arguments are provided in [GHP13a] to justify termination of the substitution subrelation of their calculus. We expand these ideas into an alternative full formal proof adapted to our case, which is based on a measure being strictly decreasing w.r.t.  $\rightarrow_{\text{sub}}$ .

We consider a set  $\mathcal{O}$  of objects of the form  $a(k, n)$  or  $b(k)$  ( $k, n \in \mathbb{N}$ ), which is equipped with the following ordering  $>^{\mathcal{O}}$  ( $\geq^{\mathcal{O}}$  denotes its reflexive closure):

$$\begin{array}{ll} a(k, n) >^{\mathcal{O}} a(k', n') & \text{if } k > k', \text{ or } (k = k' \text{ and } n > n') \\ a(k, n) >^{\mathcal{O}} b(k') & \text{if } k > k' \end{array} \quad \begin{array}{ll} b(k) >^{\mathcal{O}} a(k', n) & \text{if } k \geq k' \\ b(k) >^{\mathcal{O}} b(k') & \text{if } k > k' \end{array}$$

Notice that the symbols  $a$  and  $b$  are just formal expressions, *i.e.*  $\mathcal{O}$  could be alternatively (but less clearly) defined as  $(\mathbb{N} \times \mathbb{N}) \uplus \mathbb{N}$ .

**Lemma 3.1.** *The order  $>^{\mathcal{O}}$  on the set  $\mathcal{O}$  is well-founded.*

*Proof.* Let us consider the set  $\mathbb{N}$  equipped with the standard order  $>_{\mathbb{N}}$  on natural numbers. Let us also consider the set  $\mathbb{N}_{\infty} := \mathbb{N} \uplus \{\infty\}$  equipped with the order  $>_{\infty} := >_{\mathbb{N}} \cup \{\langle \infty, n \rangle \mid n \in \mathbb{N}\}$ . Since  $>_{\mathbb{N}}$  and  $>_{\infty}$  are both WF, then the lexicographic order induced by  $\langle >_{\mathbb{N}}, >_{\infty} \rangle$  on  $\mathbb{N} \times \mathbb{N}_{\infty}$ , written  $>_{\text{LEX}}$ , is also WF. We show that  $>^{\mathcal{O}}$  is WF by projecting it into the WF order  $>_{\text{LEX}}$ , *i.e.* we define a projection function  $P$  such that  $s >^{\mathcal{O}} s'$  implies  $P(s) >_{\text{LEX}} P(s')$ , for any  $s, s' \in \mathcal{O}$ . Let us define  $P(s) = \langle \mathbb{L}(s), \mathbb{S}(s) \rangle$ , where  $\mathbb{L}(a(k, n)) := k$  and  $\mathbb{L}(b(k)) := k$  while  $\mathbb{S}(a(k, n)) := n$  and  $\mathbb{S}(b(k)) := \infty$ . We reason by cases.

- $s_0 = a(k, n) >^{\mathcal{O}} a(k', n') = s_1$ . Then  $\langle \mathbb{L}(s_0), \mathbb{S}(s_0) \rangle = \langle k, n \rangle >_{\text{LEX}} \langle k', n' \rangle = \langle \mathbb{L}(s_1), \mathbb{S}(s_1) \rangle$  holds by definition since either  $k > k'$  or  $k = k'$  and  $n > n'$ .
- $s_0 = b(k) >^{\mathcal{O}} b(k') = s_1$ . Then  $\langle \mathbb{L}(s_0), \mathbb{S}(s_0) \rangle = \langle k, \infty \rangle >_{\text{LEX}} \langle k', \infty \rangle = \langle \mathbb{L}(s_1), \mathbb{S}(s_1) \rangle$  holds by definition since  $k > k'$ .
- $s_0 = a(k, n) >^{\mathcal{O}} b(k') = s_1$ . Then  $\langle \mathbb{L}(s_0), \mathbb{S}(s_0) \rangle = \langle k, n \rangle >_{\text{LEX}} \langle k', \infty \rangle = \langle \mathbb{L}(s_1), \mathbb{S}(s_1) \rangle$  holds by definition since  $k > k'$ .
- $s_0 = b(k) >^{\mathcal{O}} a(k', n') = s_1$ . Then  $\langle \mathbb{L}(s_0), \mathbb{S}(s_0) \rangle = \langle k, \infty \rangle >_{\text{LEX}} \langle k', n' \rangle = \langle \mathbb{L}(s_1), \mathbb{S}(s_1) \rangle$  holds by definition since either  $k > k'$  or  $k = k'$  and  $\infty > n$ .  $\square$

We write  $>_{\text{MUL}}^{\mathcal{O}}$  for the multiset extension of the order  $>^{\mathcal{O}}$  on  $\mathcal{O}$ , which turns out to be well-founded [BN98] by Lemma 3.1. Some operations on multisets are needed to build the measure  $\text{CL}(\cdot)$  on terms. Indeed, let  $M$  be a multiset of objects in  $\mathcal{O}$ . Multiset sum is denoted  $\sqcup$ . Furthermore:

- (1) The **a-elements** (resp. **b-elements**) of the multiset  $M$  are all the objects of the form  $a(k, n)$  (resp.  $b(k)$ ) in  $M$ . We then may write  $M$  as  $M_a \sqcup M_b$ , where  $M_a$  (resp.  $M_b$ ) contains all the a-elements (resp b-elements) of  $M$ .
- (2) Given  $K \in \mathbb{N}$ , we write  $M^{\leq K}$  (resp.  $M^{> K}$ ) for the multiset containing all  $o \in M$  such that the first element of  $o$  is less than  $K$  (resp. strictly greater than  $K$ ). We write  $M_a^{> K}$  for  $M^{> K} \sqcap M_a$ , where  $\sqcap$  denotes multiset intersection.

- (3)  $M$  can thus be decomposed in three disjoint multisets  $M_b, M_a^{\leq K}$  and  $M_a^{>K}$ , for every  $K \in \mathbb{N}$ .
- (4) We also define the following operation on  $M$ :

$$p \cdot M := [a(p+k, n) \mid a(k, n) \in M] \sqcup [b(p+k) \mid b(k) \in M]$$

We are now ready to (inductively) define our **cuts level** measure  $\text{CL}(\cdot)$  on terms.

$$\begin{aligned} \text{CL}(x) &:= [] & \text{CL}(\lambda x.t) &:= \text{CL}(t) & \text{CL}(tu) &:= \text{CL}(t) \sqcup \text{CL}(u) \\ \text{CL}(t[x/u]) &:= \text{CL}(t) \sqcup ((\text{lv}_x(t) + 1) \cdot \text{CL}(u)) \sqcup [a(\text{lv}_x(t) + 1, |u|)] \\ \text{CL}(t[x//u]) &:= \text{CL}(t) \sqcup (\text{lv}_x(t) \cdot \text{CL}(u)) \sqcup [b(\text{lv}_x(t))] \end{aligned}$$

Intuitively, the integer  $k$  in  $a(k, n)$  and  $b(k)$  counts the level of variables bound by explicit substitutions, while  $n$  counts the size of terms to be substituted by an ES. Remark that for every pure term  $p$  we have  $\text{CL}(p) = []$ .

**Example 3.2.** Consider the following reduction sequence:

$$\begin{aligned} t_0 = (yy)[y/(\lambda z.x)w] &\rightarrow_{\text{app}} (y_1y_2)(y_1y_2)[y_1/\lambda z.x][y_2/w] &= t_1 \\ &\rightarrow_{\text{var}} (y_1w)(y_1w)[y_1/\lambda z.x] &= t_2 \\ &\rightarrow_{\text{dist}} (y_1w)(y_1w)[y_1//\lambda z.x'[x'/x]] &= t_3 \\ &\rightarrow_{\text{abs}} ((\lambda z.x')w)((\lambda z.x')w)[x'/x] &= t_4 \\ &\rightarrow_{\text{var}} ((\lambda z.x)w)((\lambda z.x)w) &= t_5 \end{aligned}$$

We have  $\text{CL}(t_0) = [a(1, 4)]$ ,  $\text{CL}(t_1) = [a(1, 1), a(1, 2)]$ ,  $\text{CL}(t_2) = [a(1, 2)]$ ,  $\text{CL}(t_3) = [a(1, 1), b(0)]$ ,  $\text{CL}(t_4) = [a(1, 1)]$  and  $\text{CL}(t_5) = []$ .

**Fact 3.3.** Some properties on multisets are straightforward:

- If  $M_1 >_{\text{MUL}}^{\mathcal{O}} M_2$ , then  $M_1 \sqcup M >_{\text{MUL}}^{\mathcal{O}} M_2 \sqcup M$ .
- If  $M_1 >_{\text{MUL}}^{\mathcal{O}} M_2$ , then  $k \cdot M_1 >_{\text{MUL}}^{\mathcal{O}} k \cdot M_2$  for any  $k \in \mathbb{N}$ .
- $k_1 \cdot k_2 \cdot M = (k_1 + k_2) \cdot M$ .

**Lemma 3.4.** If  $\text{CL}(t) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(u)$  and  $\text{lv}_x(t) \geq \text{lv}_x(u)$  holds for every  $x \in \text{dom}(\text{L})$ , then  $\text{CL}(\text{L}\langle t \rangle) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(\text{L}\langle u \rangle)$ .

*Proof.* By induction on  $\text{L}$ . The property is straightforward. □

**Lemma 3.5.** Let  $t$  be a term,  $x$  a variable and  $p$  a pure term. Let  $K = \text{lv}_x(t)$ . Then there is  $N \in \mathbb{N}$  such that  $\text{CL}(t\{x/p\}) \sqsubseteq \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>K} \sqcup [a(k, n) \mid k \leq K \text{ and } n \leq N]$ .

*Proof.* By induction on  $t$  (see appendix on page 52). □

**Lemma 3.6.** Let  $t \in \mathbf{T}_R$ . Then  $t \rightarrow_{\pi} t'$  implies  $\text{CL}(t) \geq_{\text{MUL}}^{\mathcal{O}} \text{CL}(t')$ .

*Proof.* By induction on the reduction  $t \rightarrow_\pi t'$ . We only detail here the case where  $t = s[y/r[x/u]] \mapsto_\pi s[y/r][x/u] = t'$  at root, where  $x \notin \text{fv}(t)$  (see appendix on page 53).

$$\begin{aligned}
\text{CL}(t) &= \text{CL}(s) \sqcup (\text{lv}_y(s) + 1) \cdot \text{CL}(r[x/u]) \sqcup [\text{a}(\text{lv}_y(s) + 1, |r[x/u]|)] \\
&= \text{CL}(s) \sqcup (\text{lv}_y(s) + 1) \cdot (\text{CL}(r) \sqcup (\text{lv}_x(r) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(r) + 1, |u|)]) \\
&\quad \sqcup [\text{a}(\text{lv}_y(s) + 1, |r[x/u]|)] \\
&= \text{CL}(s) \sqcup (\text{lv}_y(s) + 1) \cdot \text{CL}(r) \sqcup (\text{lv}_y(s) + \text{lv}_x(r) + 2) \cdot \text{CL}(u) \\
&\quad \sqcup [\text{a}(\text{lv}_y(s) + \text{lv}_x(r) + 2, |u|), \text{a}(\text{lv}_y(s) + 1, |r[x/u]|)] \\
&= (\text{CL}(s) \sqcup (\text{lv}_y(s) + 1) \cdot \text{CL}(r) \sqcup [\text{a}(\text{lv}_y(s) + 1, |r[x/u]|)]) \\
&\quad \sqcup (\text{lv}_y(s) + \text{lv}_x(r) + 2) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_y(s) + \text{lv}_x(r) + 2, |u|)] \\
&>_{\text{MUL}}^{\mathcal{O}} (\text{CL}(s) \sqcup (\text{lv}_y(s) + 1) \cdot \text{CL}(r) \sqcup [\text{a}(\text{lv}_y(s) + 1, |r|)]) \\
&\quad \sqcup (\text{lv}_x(s[y/r]) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(s[y/r]) + 1, |u|)] \\
&= \text{CL}(t')
\end{aligned}$$

The  $>_{\text{MUL}}^{\mathcal{O}}$  inequation is justified by the following facts:

- $|r[x/u]| > |r|$ .
- $\text{lv}_y(s) + \text{lv}_x(r) + 2 = \max(0, \text{lv}_y(s) + \text{lv}_x(r) + 1) + 1 = \text{lv}_x(s[y/r]) + 1$ .  $\square$

**Lemma 3.7.** *Let  $t \in \mathbb{T}_R$ . Then  $t \rightarrow_{\text{sub}} t'$  implies  $\text{CL}(t) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t')$ .*

*Proof.* Let  $t = \mathbf{C}(t_0) \rightarrow_{\text{sub}} \mathbf{C}(t_1) = t'$ , where  $t_0 \rightarrow_{\text{sub}} t_1$  is a reduction step at the root position. We proceed by induction on  $\mathbf{C}$ . We detail the base case which is  $\mathbf{C} = \diamond$ . In all such cases we use Lemma 3.6 to push  $\mathbf{L}$  outside, *i.e.* we can write  $t_0 \rightarrow_{\text{sub}} t_1$  as  $t_0 \rightarrow_\pi \mathbf{L}(t'_0) \rightarrow_{\text{sub}'} \mathbf{L}(t'_1) = t_1$ , where  $t'_0 \rightarrow_{\text{sub}'} t'_1$  is a root step which does not push any list context outside. We then show the property for root steps  $t'_0 \rightarrow_{\text{sub}'} t'_1$ , and we conclude with Lemma 3.6 then Lemma 3.4 by  $\text{CL}(t_0) \geq_{\text{MUL}}^{\mathcal{O}} \text{CL}(\mathbf{L}(t'_0)) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(\mathbf{L}(t'_1)) = \text{CL}(t_1)$  since  $\text{lv}_x(t'_0) \geq \text{lv}_x(t'_1)$  holds for every  $x \in \text{dom}(\mathbf{L})$  by Lemma 2.6. Let us analyse all the cases  $t'_0 \rightarrow_{\text{sub}'} t'_1$ .

(1) If  $t'_0 = t[x/us] \rightarrow_{\text{app}} t\{x/yz\}[y/u][z/s] = t'_1$ , where  $y$  and  $z$  are fresh variables, then

$$\text{CL}(t'_0) = \text{CL}(t) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(us) \sqcup [\text{a}(\text{lv}_x(t) + 1, |us|)] \text{ and}$$

$$\begin{aligned}
\text{CL}(t'_1) &= \text{CL}(t\{x/yz\}[y/u]) \sqcup (\text{lv}_z(t\{x/yz\}[y/u]) + 1) \cdot \text{CL}(s) \\
&\quad \sqcup [\text{a}(\text{lv}_z(t\{x/yz\}[y/u]) + 1, |s|)] \\
&= (\text{CL}(t\{x/yz\}) \sqcup (\text{lv}_y(t\{x/yz\}) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_y(t\{x/yz\}) + 1, |u|)]) \\
&\quad \sqcup (\text{lv}_z(t\{x/yz\}[y/u]) + 1) \cdot \text{CL}(s) \sqcup [\text{a}(\text{lv}_z(t\{x/yz\}[y/u]) + 1, |s|)] \\
&= (\text{CL}(t\{x/yz\}) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(t) + 1, |u|)]) \\
&\quad \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(s) \sqcup [\text{a}(\text{lv}_x(t) + 1, |s|)] \\
&= \text{CL}(t\{x/yz\}) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(us) \sqcup [\text{a}(\text{lv}_x(t) + 1, |u|), \text{a}(\text{lv}_x(t) + 1, |s|)]
\end{aligned}$$

By Lemma 3.5,  $\text{CL}(t\{x/yz\}) \sqsubseteq \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>\text{lv}_x(t)} \sqcup [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$  for some  $N \in \mathbb{N}$ . We also have that  $[\text{a}(\text{lv}_x(t) + 1, |us|)] >_{\text{MUL}}^{\mathcal{O}} [\text{a}(\text{lv}_x(t) + 1, |u|), \text{a}(\text{lv}_x(t) + 1, |s|)] >_{\text{MUL}}^{\mathcal{O}} [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$ . Moreover,  $\text{CL}(t) \sqsupseteq \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>\text{lv}_x(t)}$  and  $[\text{a}(\text{lv}_x(t) + 1, |us|)] >_{\text{MUL}}^{\mathcal{O}} [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$ . Thus we conclude  $\text{CL}(t'_0) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t'_1)$ .

(2) If  $t'_0 = t[x/\lambda y.u] \rightarrow_{\text{dist}} t[x//\lambda y.z[z/u]] = t'_1$ , then

$$\text{CL}(t'_0) = \text{CL}(t) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(t) + 1, |u| + 1)] \text{ and}$$

$$\begin{aligned} \text{CL}(t'_1) &= \text{CL}(t) \sqcup \text{lv}_x(t) \cdot \text{CL}(\lambda y.z[z/u]) \sqcup [\text{b}(\text{lv}_x(t))] \\ &= \text{CL}(t) \sqcup \text{lv}_x(t) \cdot \text{CL}(z[z/u]) \sqcup [\text{b}(\text{lv}_x(t))] \\ &= \text{CL}(t) \sqcup \text{lv}_x(t) \cdot (\text{CL}(z) \sqcup (\text{lv}_z(z) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_z(z) + 1, |u|)]) \sqcup [\text{b}(\text{lv}_x(t))] \\ &= \text{CL}(t) \sqcup \text{lv}_x(t) \cdot (1 \cdot \text{CL}(u) \sqcup [\text{a}(1, |u|)]) \sqcup [\text{b}(\text{lv}_x(t))] \\ &= \text{CL}(t) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(t) + 1, |u|), \text{b}(\text{lv}_x(t))] \end{aligned}$$

$\text{CL}(t'_0) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t'_1)$  because the multisets are the same except for  $\text{a}(\text{lv}_x(t) + 1, |u|)$  and  $\text{b}(\text{lv}_x(t))$  on the right which are smaller than  $\text{a}(\text{lv}_x(t) + 1, |u| + 1)$  on the left.

(3) If  $t'_0 = t[x//\lambda y.u] \rightarrow_{\text{abs}} t\{x/\lambda y.u\} = t'_1$ , then we have  $\text{CL}(t'_0) = \text{CL}(t) \sqcup [\text{b}(\text{lv}_x(t))]$ . By Lemma 3.5,  $\text{CL}(t'_1) \sqsubseteq \text{CL}(t)_b \sqcup \text{CL}(t)^{>\text{lv}_x(t)} \sqcup [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$  for some  $N \in \mathbb{N}$ . Since  $[\text{b}(\text{lv}_x(t))] >_{\text{MUL}}^{\mathcal{O}} [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$  and  $\text{CL}(t) \sqsupseteq \text{CL}(t)_b \sqcup \text{CL}(t)^{>\text{lv}_x(t)}$ , then we conclude  $\text{CL}(t'_0) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t'_1)$ .

(4) If  $t'_0 = t[x/y] \rightarrow_{\text{var}} t\{x/y\} = t'_1$ , we have  $\text{CL}(t'_0) = \text{CL}(t) \sqcup [\text{a}(\text{lv}_x(t) + 1, 1)]$ . By Lemma 3.5,  $\text{CL}(t'_1) \sqsubseteq \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>\text{lv}_x(t)} \sqcup [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$  for some  $N \in \mathbb{N}$ . Since  $[\text{a}(\text{lv}_x(t) + 1, 1)] >_{\text{MUL}}^{\mathcal{O}} [\text{a}(k, n) \mid k \leq \text{lv}_x(t), n \leq N]$  and  $\text{CL}(t) \sqsupseteq \text{CL}(t)_b \sqcup \text{CL}(t)^{>\text{lv}_x(t)}$ , we conclude  $\text{CL}(t'_0) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t'_1)$   $\square$

The sequence of Example 3.2 illustrates this phenomenon: indeed,  $\text{CL}(t_i) >_{\text{MUL}}^{\mathcal{O}} \text{CL}(t_{i+1})$  for  $0 \leq i < 5$ .

**Corollary 3.8.** *The reduction relation  $\rightarrow_{\text{sub}}$  is terminating.*

**Simulations.** We show the relation between  $\lambda R$  and  $\lambda$ , as well as the atomic  $\lambda$ -calculus  $\lambda a$ . For that, we introduce a projection from  $\mathbb{T}_R$  to  $\mathbb{T}_P$  implementing the unfolding of all the explicit cuts:

$$x^\downarrow := x \quad (\lambda x.t)^\downarrow := \lambda x.t^\downarrow \quad (tu)^\downarrow := t^\downarrow u^\downarrow \quad (t[x \triangleleft u])^\downarrow := t^\downarrow \{x/u^\downarrow\}.$$

Thus *e.g.*  $x[x/z[y/w]][w/w']^\downarrow = x\{x/z\{y/w\}\}\{w/w'\} = z$ . The previous projection can be extended from list contexts to substitutions as follows:  $\diamond^\downarrow := \{\}$  and  $(\mathbb{L}[x \triangleleft u])^\downarrow := \mathbb{L}^\downarrow \circ \{x/u^\downarrow\}$ , where  $\circ$  denotes standard composition of substitutions.

**Lemma 3.9.** *Let  $t \in \mathbb{T}_R$ . If  $t \rightarrow_R t'$ , then  $t^\downarrow \rightarrow_\beta t'^\downarrow$ . In particular, if either  $t \rightarrow_\pi t'$  or  $t \rightarrow_{\text{sub}} t'$ , then  $t^\downarrow = t'^\downarrow$ .*

*Proof.* By induction on the stated relations.  $\square$

The relation  $\rightarrow_{\text{sub}}$  enjoys **full composition** on *pure* terms. Namely:

**Lemma 3.10.** *For any  $p \in \mathbb{T}_P$ ,  $t[x/p] \rightarrow_{\text{sub}}^+ t t\{x/p\}$ .*

*Proof.* By induction on  $p$ .

- If  $p = y$ , then  $t[x/y] \rightarrow_{\text{var}} t\{x/y\}$ .

- If  $p = p_1p_2$ , then

$$\begin{aligned} t[x/p_1p_2] &\rightarrow_{\text{app}} t\{x/yz\}[y/p_1][z/p_2] \\ &\rightarrow_{i.h.}^+ t\{x/yz\}\{y/p_1\}[z/p_2] \rightarrow_{i.h.}^+ t\{x/yz\}\{y/p_1\}\{z/p_2\} \\ &= t\{x/p_1p_2\} \end{aligned}$$

- If  $p = \lambda y.q$ , then  $t[x/\lambda y.q] \rightarrow_{\text{abs}} t[x//\lambda y.z[z/q]] \rightarrow_{i.h.}^+ t[x//\lambda y.q] \rightarrow_{\text{dist}} t\{x/\lambda y.q\}$ .  $\square$

This property does not hold in general if  $p$  is not pure. Indeed, if  $t = xx$ , then  $(xx)[x/y[y/z]]$  does not **sub**-reduce to  $(y[y/z])(y[y/z])$ , but to  $(yy)[y/z]$ . However, full composition restricted to pure terms is sufficient to prove simulation of the  $\lambda$ -calculus.

**Lemma 3.11** (Simulation of the  $\lambda$ -calculus). *Let  $p_0 \in \mathbf{T}_P$ . If  $p_0 \rightarrow_{\beta} p_1$ , then  $p_0 \rightarrow_{\text{dB}} \rightarrow_{\text{sub}}^+ p_1$ .*

*Proof.* Let  $p_0 = \mathbf{C}\langle t_0 \rangle \rightarrow_{\beta} \mathbf{C}\langle t_1 \rangle = p_1$ , where  $t_0 = (\lambda x.q)p \mapsto_{\beta} q\{x/p\} = t_1$ . By Lemma 3.10,  $t_0 \rightarrow_{\text{dB}} q[x/p] \rightarrow_{\text{sub}}^+ t t_1$ . The inductive cases for  $\mathbf{C}$  are straightforward.  $\square$

The previous results have an important consequence relating the atomic  $\lambda$ -calculus and the  $\lambda R$ -calculus. Indeed, it can be shown that reduction in the atomic  $\lambda$ -calculus is captured by  $\lambda a$ , and vice-versa. More precisely, the  $\lambda R$ -calculus can be simulated into the atomic  $\lambda$ -calculus by Lemma 3.9 and [GHP13a], while the converse holds by [GHP13a] and Lemma 3.11.

A more structural correspondence between  $\lambda R$  and  $\lambda a$  could also be established. Indeed,  $\lambda R$  can be first refined into a (non-linear) calculus *without* distance, let say  $\lambda R'$ , so that permutation rules are integrated in the intermediate calculus as independent rules. Then a structural relation can be established between  $\lambda R$  and  $\lambda R'$  on one side, and  $\lambda R'$  and the atomic  $\lambda$ -calculus on the other side (as for example done in [KL07] for the  $\lambda$ -calculus).

**Confluence.** By Corollary 3.8 the reduction relation  $\rightarrow_{\text{sub}}$  is terminating. It is then not difficult to prove confluence of  $\rightarrow_{\text{sub}}$  by using the unfolding function  $\cdot^{\downarrow}$ .

**Lemma 3.12.** *Let  $t \in \mathbf{T}_R$ . Then  $t$  is in **sub-nf** if and only if  $t$  is pure.*

*Proof.* It is obvious that a pure term is **sub-normal**. Let us show the left-to-right implication and consider a **sub-normal** term  $t$ . We reason by induction on  $t$ . Suppose that  $t$  is not pure, so that  $t = \mathbf{C}\langle t_0[x \triangleleft u] \rangle$ . If the explicit cut is an explicit substitution, then one of the rules **app**, **dist**, **var** apply, which contradicts the hypothesis. Otherwise the cut is a distributor, and  $u$  is an abstraction  $\lambda y.u'$ , where  $u'$  is in particular a **sub-normal** form. By the *i.h.*  $u'$  is pure so that the rule **abs** applies, which contradicts the hypothesis again.  $\square$

**Corollary 3.13.** *Let  $t \in \mathbf{T}_R$ . If  $t$  is in **sub-nf**, then  $t^{\downarrow} = t$ .*

**Lemma 3.14.** *The reduction relation  $\rightarrow_{\text{sub}}$  is terminating and confluent.*

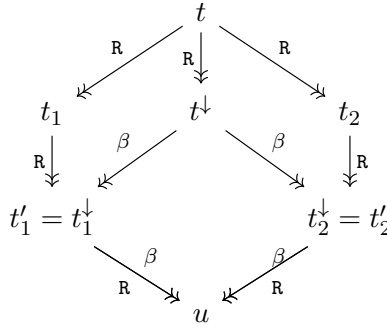
*Proof.* Termination holds by Corollary 3.8. For confluence, suppose  $t \rightarrow_{\text{sub}} t_1$  and  $t \rightarrow_{\text{sub}} t_2$ . Let  $t_1 \rightarrow_{\text{sub}} t'_1$  and  $t_2 \rightarrow_{\text{sub}} t'_2$ , where  $t'_1$  and  $t'_2$  are in **sub-nf**. Then by Corollary 3.13,  $(t'_i)^{\downarrow} = t'_i$  for both  $i = 1, 2$ . By Lemma 3.9,  $(t'_i)^{\downarrow} = t_i^{\downarrow} = t^{\downarrow}$  so that  $t'_1 = t'_2$ , closing the diagram.  $\square$

By termination of  $\rightarrow_{\text{sub}}$  any  $t \in \mathbf{T}_R$  has a **sub-nf**, and by confluence this **sub-nf** is unique. By Lemma 3.9 and Corollary 3.13 one obtains:

**Corollary 3.15.** *Let  $t \in \mathbf{T}_R$ . Then the unique **sub-nf** of  $t$  is  $t^{\downarrow}$ .*

**Theorem 3.16.** *The reduction relation  $\rightarrow_R$  is confluent.*

*Proof.* Let  $t \in \mathsf{T}_R$  such that  $t \rightarrow_R t_1$  and  $t \rightarrow_R t_2$ . By simulation (Lemma 3.9), we have  $t^\downarrow \rightarrow_\beta t_1^\downarrow$  and  $t^\downarrow \rightarrow_\beta t_2^\downarrow$ . By Lemma 3.14, there exist  $t'_1$  (resp.  $t'_2$ ) the unique **sub**-nf of  $t_1$  (resp.  $t_2$ ). By Corollary 3.15 we have  $t'_1 = t_1^\downarrow$  and  $t'_2 = t_2^\downarrow$ . Because  $\rightarrow_\beta$  is confluent, there is  $u$  such that  $t_1^\downarrow \rightarrow_\beta u$  and  $t_2^\downarrow \rightarrow_\beta u$ , and by Lemma 3.11,  $t_1^\downarrow \rightarrow_R u$  and  $t_2^\downarrow \rightarrow_R u$ . The diagram is then closed by  $t_1 \rightarrow_{\text{sub}} t'_1 = t_1^\downarrow \rightarrow_R u$  and  $t_2 \rightarrow_{\text{sub}} t'_2 = t_2^\downarrow \rightarrow_R u$ . Graphically,



#### 4. ENCODING EVALUATION STRATEGIES

In the theory of programming languages [Plo75], the notion of *calculus* is usually based on a non-deterministic rewriting relation, while the deterministic notion of *strategy* is associated to a concrete machinery being able to implement a specific evaluation procedure. Typical evaluation strategies are call-by-name, call-by-value and call-by-need, to name a few.

Although the atomic  $\lambda$ -calculus was introduced as a technical tool to implement full laziness, only its (non-deterministic) equational theory was studied. In this paper we bridge the gap between the theoretical presentation of the atomic  $\lambda$ -calculus and concrete specifications of evaluation strategies. Indeed, we use the  $\lambda R$ -calculus to investigate two concrete cases: a call-by-name strategy implementing weak head reduction, based on full substitution, and the call-by-need fully lazy strategy, which uses linear substitution.

In this work, we choose to implement full laziness for pure terms, that is, for the usual  $\lambda$ -calculus without cuts. Indeed, we see explicit cuts as a tool for a fully lazy implementation of the  $\lambda$ -calculus. We thus keep in line with the definitions found in the literature. Defining full laziness for terms with explicit cuts also brings technical difficulties, which might divert from the main point: using node replication to implement a fully lazy strategy.

We then restrict the set of terms to a subset  $\mathsf{U}$ , which simplifies the formal reasoning of explicit cuts inside distributors. Indeed, distributors will all be of the shape  $[x // \lambda y. \mathsf{LL}\langle p \rangle]$ , where  $p$  is a pure term containing the constructors that have been (symbolically) shared in the distributor, and  $\mathsf{LL}$  is a *commutative list* (defined below). We argue that this restriction is natural in a weak implementation of the  $\lambda$ -calculus: it is true on pure terms and is preserved through evaluation. We consider the following grammars.

(Linear Cut Values)	$\mathsf{T}$	$::= \lambda x. \mathsf{LL}\langle p \rangle$ where $y \in \text{dom}(\mathsf{LL}) \implies  p _y = 1$
(Commutative Lists)	$\mathsf{LL}$	$::= \diamond \mid \mathsf{LL}[x/p] \mid \mathsf{LL}[x/\mathsf{T}]$ where $ \mathsf{LL} _x = 0$
(Values)	$v$	$::= \lambda x. p$
(Restricted Terms)	$\mathsf{U}$	$::= x \mid v \mid \mathsf{U} \mathsf{U} \mid \mathsf{U}[x/\mathsf{U}] \mid \mathsf{U}[x/\mathsf{T}]$

A term  $t$  generated by any of the grammars  $G$  defined above is written  $t \in G$ . Thus *e.g.*  $\lambda x.(yz)[y/I][z/I] \in \mathbf{T}$  but  $\lambda x.(yy)[y/I] \notin \mathbf{T}$ ,  $\diamond[x/yz][x'/I] \in \mathbf{LL}$  but  $\diamond[x/yz][y/I] \notin \mathbf{LL}$ , and  $(yz)[y//I] \in \mathbf{U}$  but  $(yz)[y//\lambda x.(yy)[y/I]] \notin \mathbf{U}$ .

The set  $\mathbf{T}$  is stable by the relation  $\rightarrow_{\text{sub}}$  (Lemma 4.1), but  $\mathbf{U}$  is clearly not stable under the whole  $\rightarrow_{\mathbf{R}}$  relation, where  $\text{dB}$ -reductions may occur under abstractions. For instance, let  $t_1 = (yz)[y//\lambda x.(\lambda y.yy)I] \rightarrow_{\text{dB}} (yz)[y//\lambda x.(yy)[y/I]] = t_2$ . Then  $t_1 \in \mathbf{U}$  but  $t_2 \notin \mathbf{U}$ , since  $|yy|_y = 2$ . However,  $\mathbf{U}$  is stable under both weak strategies to be defined: call-by-name and call-by-need. We factorize the proofs by proving stability for a more general relation  $\rightarrow_{\mathbf{F}}$ , defined as the relation  $\rightarrow_{\mathbf{R}}$  with  $\text{dB}$ -reductions forbidden under abstractions and inside distributors.

**Lemma 4.1.** *If  $t \in \mathbf{T}$  and  $t \rightarrow_{\text{sub}} t'$ , then  $t' \in \mathbf{T}$ .*

*Proof.* We first show a more general statement, namely that  $t = \mathbf{LL}_0\langle p_0 \rangle$  with  $|p_0|_y = 1$  for every  $y \in \text{dom}(\mathbf{LL}_0)$ , and  $t \mapsto_{\text{sub}} t'$  imply  $t' = \mathbf{LL}_1\langle p_1 \rangle$  with  $|p_1|_y = 1$  for every  $y \in \text{dom}(\mathbf{LL}_1)$ . In the following rules  $\text{var}$ ,  $\text{app}$  and  $\text{dist}$ , there is no  $\mathbf{L}$  context inside the explicit substitutions because lists in  $\mathbf{LL}$  only contain pure terms by definition.

- $t = u[x/z] \mapsto_{\text{var}} u\{x/z\} = t'$ . This is straightforward.
- $t = \mathbf{LL}\langle p \rangle[x/q_1q_2] \mapsto_{\text{app}} \mathbf{LL}\langle p\{x/x_1x_2\} \rangle[x_1/q_1][x_2/q_2] = t'$ . Freshness of both  $x_1$  and  $x_2$  implies  $|p\{x/x_1x_2\}|_{x_1} = |p\{x/x_1x_2\}|_{x_2} = |p|_x = 1$ , and  $|q_1|_{x_2} = 0$ .
- $t = \mathbf{LL}\langle p \rangle[x/\lambda z.p'] \mapsto_{\text{dist}} \mathbf{LL}\langle p \rangle[x//\lambda z.w[w/p']] = t'$ . By hypothesis  $|p|_x = 1$ ,  $|\mathbf{LL}|_x = 0$  and  $\lambda z.p'$  is pure. Then,  $\lambda z.w[w/p'] \in \mathbf{T}$  because  $p'$  is pure and  $|w|_w = 1$ .
- $t = \mathbf{LL}\langle p \rangle[x/\lambda z.\mathbf{LL}'\langle p' \rangle] \mapsto_{\text{abs}} \mathbf{LL}'\langle \mathbf{LL}\langle p \rangle\{x/\lambda z.p'\} \rangle = t'$ . By hypothesis  $\lambda z.\mathbf{LL}'\langle p' \rangle \in \mathbf{T}$  thus  $\lambda z.p'$  and  $p\{x/\lambda z.p'\}$  are pure. We conclude since  $|\mathbf{LL}|_x = 0$  by hypothesis.

Now we can lift the property to  $\mathbf{T}$  by observing that we necessarily have  $t = \lambda x.u \rightarrow_{\text{sub}} \lambda x.u'$ , where  $u \rightarrow_{\text{sub}} u'$ . Then we conclude by the previous point.  $\square$

**Lemma 4.2.** *If  $t \in \mathbf{U}$  and  $t \rightarrow_{\mathbf{F}} t'$ , then  $t' \in \mathbf{U}$ .*

*Proof.* Straightforward by induction on the reduction relation.  $\square$

**4.1. Call-by-name.** The **call-by-name** (CBN) strategy  $\rightarrow_{\text{name}}$  (Figure 1) is defined on the set of terms  $\mathbf{U}$  as the union of the following relations  $\rightarrow_{\text{ndB}}$  and  $\rightarrow_{\text{nsub}}$ . The strategy is *weak* as there is no reduction under abstractions. It is also worth noticing (as a particular case of Lemma 4.2) that  $t \in \mathbf{U}$  and  $t \rightarrow_{\text{name}} t'$  implies  $t' \in \mathbf{U}$ .

$$\begin{array}{ccc}
\frac{t \mapsto_{\text{dB}} t'}{t \rightarrow_{\text{ndB}} t'} \text{ (DB)} & \frac{t \rightarrow_{\text{ndB}} t'}{tu \rightarrow_{\text{ndB}} t'u} \text{ (APPDB)} & \frac{t \rightarrow_{\text{ndB}} t'}{t[x \triangleleft u] \rightarrow_{\text{ndB}} t'[x \triangleleft u]} \text{ (SUBDB)} \\
\frac{t \mapsto_{\text{sub}} t'}{t \rightarrow_{\text{nsub}} t'} \text{ (S)} & \frac{t \rightarrow_{\text{nsub}} t'}{tu \rightarrow_{\text{nsub}} t'u} \text{ (APPS)} & \frac{t \rightarrow_{\text{nsub}} t'}{u[x//\lambda y.t] \rightarrow_{\text{nsub}} u[x//\lambda y.t']} \text{ (SUBS)}
\end{array}$$

Figure 1: Call-by-Name Strategy



**Example 4.3.** This example follows a call-by-name evaluation. The name of the contextual rule is written in the superscript of the arrow symbol, and the redex is underlined.

$$\begin{array}{lcl}
(\lambda x_1. \mathbf{I}(x_1 \mathbf{I}))(\lambda y. (\mathbf{II})y) & \xrightarrow{\text{dB}} & \underline{(\mathbf{I}(x_1 \mathbf{I}))} [x_1 / \lambda y. (\mathbf{II})y] \\
& \xrightarrow{\text{S}} & (\mathbf{I}(x_1 \mathbf{I})) [x_1 / \lambda y. z [z / (\mathbf{II})y]] \\
& \xrightarrow{\text{SUBS}} & (\mathbf{I}(x_1 \mathbf{I})) [x_1 / \lambda y. (z_1 z_2) [z_1 / \mathbf{II}] [z_2 / y]] \\
& \xrightarrow{\text{SUBS}} & \underline{(\mathbf{I}(x_1 \mathbf{I}))} [x_1 / \lambda y. (z_1 y) [z_1 / \mathbf{II}]] \\
& \xrightarrow{\text{S}} & (\mathbf{I}((\lambda y. z_1 y) \mathbf{I})) [z_1 / \mathbf{II}] \\
& \xrightarrow{\text{SUBDB}} & x_2 [x_2 / (\lambda y. z_1 y) \mathbf{I}] [z_1 / \mathbf{II}] \\
& \xrightarrow{+} & ((\lambda y. z_1 y) \mathbf{I}) [z_1 / \mathbf{II}] \\
& \xrightarrow{+} & \lambda y. (\mathbf{II})y
\end{array}$$

The strategy  $\rightarrow_{\text{name}}$  does not impose duplication of all nodes in the body of an abstraction inside the distributor: only the skeleton of the abstraction  $\lambda y. (\mathbf{II})y$  is replicated. But the strategy forbids dB-reductions inside explicit cuts, so that there is no benefit gained by keeping shared terms such as  $\mathbf{II}$ . Indeed, the main idea behind full laziness is that shared terms are only reduced once. The CBN strategy, on the contrary, duplicates arguments before reducing them. The absence of optimization is reflected by the fact that the strategy, although not deterministic, enjoys the remarkable *diamond* property, guaranteeing in particular that all reduction sequences starting from  $t$  and ending in a normal form have the same length.

**Proposition 4.4** (Diamond). *The CBN strategy enjoys the diamond property, i.e. for any terms  $t, u, s \in U$  such that  $t \rightarrow_{\text{name}} u$ ,  $t \rightarrow_{\text{name}} s$  and  $u \neq s$ , there exists  $t'$  such that  $u \rightarrow_{\text{name}} t'$  and  $s \rightarrow_{\text{name}} t'$ .*

*Proof.* By separate inductions on the reduction relations  $\langle \rightarrow_{\text{ndB}}, \rightarrow_{\text{ndB}} \rangle$ ,  $\langle \rightarrow_{\text{nsub}}, \rightarrow_{\text{nsub}} \rangle$  and  $\langle \rightarrow_{\text{ndB}}, \rightarrow_{\text{nsub}} \rangle$  (see appendix on page 54).  $\square$

It is worth noticing that call-by-name in the  $\lambda$ -calculus can be simulated by call-by-name in  $\lambda R$ . The former can be defined by weak-head reduction, denoted  $\rightarrow_{\text{whr}}$ , and generated by the following rules:

$$\frac{}{(\lambda y. t)u \rightarrow_{\text{whr}} t\{x/u\}} \quad \frac{t \rightarrow_{\text{whr}} t'}{tu \rightarrow_{\text{whr}} t'u}$$

There is in particular a one-to-one relation between  $\beta$ -steps and ndB-steps.

**Lemma 4.5** (Relating Call-by-Name Strategies).

- Let  $p_0 \in \mathbb{T}_P$ . If  $p_0 \rightarrow_{\text{whr}} p_1$ , then  $p_0 \rightarrow_{\text{ndB}} \rightarrow_{\text{nsub}}^+ p_1$  (thus  $p_0 \rightarrow_{\text{name}}^+ p_1$ ).
- Let  $t_0 \in U$ . If  $t_0 \rightarrow_{\text{ndB}} t_1$ , then  $t_0^\downarrow \rightarrow_{\text{whr}} t_1^\downarrow$ . If  $t_0 \rightarrow_{\text{nsub}} t_1$ , then  $t_0^\downarrow = t_1^\downarrow$ .

*Proof.*

- By induction on  $\rightarrow_{\text{whr}}$ .
  - Let  $p_0 = (\lambda x. p)q \rightarrow_{\beta} p\{x/q\} = p_1$ . Then  $(\lambda x. p)q \rightarrow_{\text{ndB}} p\{x/q\}$  and we need to verify that  $p\{x/q\} \rightarrow_{\text{nsub}}^+ p\{x/q\}$ . The proof of  $t\{x/q\} \rightarrow_{\text{nsub}}^+ t\{x/q\}$  for any  $t \in U$  and pure term  $q$  is by induction on  $q$ :
    - \* If  $q = y$  then  $t\{x/y\} \rightarrow_{\text{nsub}} t\{x/y\}$ .
    - \* If  $q = q_0 q_1$  then  $t\{x/q\} \rightarrow_{\text{nsub}} t\{x/z_0 z_1\}[z_0/q_0][z_1/q_1]$ . By the *i.h.* we have
$$\begin{aligned}
t\{x/z_0 z_1\}[z_0/q_0][z_1/q_1] &\rightarrow_{\text{nsub}}^+ (t\{x/z_0 z_1\}[z_0/q_0])\{z_1/q_1\} = t\{x/z_0 q_1\}[z_0/q_0] \\
&\text{and} \\
t\{x/z_0 q_1\}[z_0/q_0] &\rightarrow_{\text{nsub}}^+ t\{x/z_0 q_1\}\{z_0/q_0\} = t\{x/q_0 q_1\}
\end{aligned}$$

- Therefore,  $t[x/q_0q_1] \twoheadrightarrow_{\text{nsub}}^+ t\{x/q_0q_1\}$ .
- \* If  $q = \lambda y.q'$  then  $t[x/q] \rightarrow_{\text{nsub}} t[x//\lambda y.z[z/q']]$ . By the *i.h.* we have that  $z[z/q'] \twoheadrightarrow_{\text{nsub}}^+ z\{z/q'\} = q'$  thus  $t[x//\lambda y.z[z/q']] \twoheadrightarrow_{\text{nsub}}^+ t[x//\lambda y.q'] \rightarrow_{\text{nsub}} t\{x/\lambda y.q'\}$ . Therefore,  $t[x/\lambda y.q'] \twoheadrightarrow_{\text{nsub}}^+ t\{x/\lambda y.q'\}$ .
  - Let  $p_0 = pq \rightarrow_{\text{whr}} p'q = p_1$  where  $p \rightarrow_{\text{whr}} p'$ . By the *i.h.* we have that  $p \twoheadrightarrow_{\text{name}}^+ p'$  then, by (APPDB) and (APPS),  $p_0 = pq \twoheadrightarrow_{\text{name}}^+ p'q = p_1$ .
  - By case analysis on  $\rightarrow_{\text{name}}$ . If  $t_0 \rightarrow_{\text{nsub}} t_1$  then  $t_0^\downarrow = t_1^\downarrow$  by Lemma 3.9. If  $t_0 \rightarrow_{\text{ndB}} t_1$  then we prove the property by induction on  $\rightarrow_{\text{ndB}}$ .
    - Let  $t_0 = (\lambda x.t)u \rightarrow_{\text{ndB}} t[x/u] = t_1$ . Then  $t_0^\downarrow = (\lambda x.t^\downarrow)u^\downarrow \rightarrow_\beta t^\downarrow\{x/u^\downarrow\} = t_1^\downarrow$ . Note that both  $t^\downarrow$  and  $u^\downarrow$  are pure terms.
    - Let  $t_0 = tu \rightarrow_{\text{ndB}} t'u = t_1$  where  $t \rightarrow_{\text{ndB}} t'$ . Then  $t^\downarrow \rightarrow_{\text{whr}} t'^\downarrow$  by the *i.h.*, thus  $t_0^\downarrow = t^\downarrow u^\downarrow \rightarrow_{\text{whr}} t'^\downarrow u^\downarrow = t_1^\downarrow$ .
    - Let  $t_0 = t[x \triangleleft u] \rightarrow_{\text{ndB}} t'[x \triangleleft u] = t_1$  where  $t \rightarrow_{\text{ndB}} t'$ . Then  $t^\downarrow \rightarrow_{\text{whr}} t'^\downarrow$  by the *i.h.*, thus  $t_0^\downarrow = t^\downarrow\{x/u^\downarrow\} \rightarrow_{\text{whr}} t'^\downarrow\{x/u^\downarrow\} = t_1^\downarrow$ . Note that the result depends on the closure of  $\rightarrow_{\text{whr}}$  by (implicit) substitutions, which has a straightforward proof by induction on (pure) term  $t^\downarrow$ , using substitution composition.  $\square$

The following grammar  $\text{Na}$  is meant to characterize normal forms with respect to the  $\rightarrow_{\text{name}}$  strategy:

$$\begin{aligned} \text{Na} &::= \lambda x.p \mid \overline{\text{Na}} \\ \overline{\text{Na}} &::= x \mid \overline{\text{Na}}t \end{aligned}$$

Notice that all normal forms are pure terms: we unfold all explicit substitutions with sub-steps.

**Lemma 4.6.** *Let  $t \in \mathcal{U}$ . Then  $t \in \text{Na}$  iff  $t$  is in name-nf.*

*Proof.* The left-to-right implication is straightforward. The right-to-left implication proof is by induction on  $\mathcal{U}$ .

- $t = x$ . By definition,  $t \in \overline{\text{Na}}$ .
- $t = \lambda x.p$ . Then  $t \in \text{Na}$  by definition.
- $t = t'u$ , where  $t', u \in \mathcal{U}$ . By definition of  $\rightarrow_{\text{name}}$ ,  $t$  in name-nf implies  $t'$  is also in name-nf and  $t'$  is neither an explicit cut nor an abstraction. Thus  $t' \in \overline{\text{Na}}$  by the *i.h.* and we can conclude  $t \in \overline{\text{Na}}$ .
- $t = t'[x/u]$ , where  $t', u \in \mathcal{U}$ . This is not possible because there is always an applicable structural rule which would contradict  $t$  to be in name-nf.
- $t = t'[x//\lambda y.u]$ , where  $\lambda y.u = \lambda y.\text{LL}\langle p \rangle \in \mathcal{T}$ . Then either we can apply a structural rule on  $u$ , or  $u$  is pure (*i.e.*  $\text{LL} = \diamond$ ) and we can apply rule  $\rightarrow_{\text{abs}}$ . In both cases we would have a contradiction with  $t$  in name-nf.  $\square$

**4.2. Call-by-need.** We now specify a deterministic strategy  $\text{flneed}$  implementing demand-driven computations and only linearly replicating nodes of *values* (*i.e.* pure abstractions). Given a value  $\lambda x.p$ , only the piece of structure containing the paths between the binder  $\lambda x$  and all the free occurrences of  $x$  in  $p$ , named *skeleton*, will be copied. All the other components of the abstraction will remain shared, thus avoiding some future duplications of redexes, as explained in the introduction. By copying only the smallest possible substructure of the abstraction, the strategy  $\text{flneed}$  implements an optimization of call-by-need called *fully lazy sharing* [Wad71]. First, we formally define the key notions we are going to use.

A **free expression** [Jon87, Bal12a] of a *pure* term  $p$  is a strict subterm  $q$  of  $p$  such that every free occurrence of a variable in  $q$  is also a free occurrence of the variable in  $p$ . A **free expression** of  $p$  is **maximal** if it is not a subterm of another free expression of  $p$ . From now on, we will consider the (ordered) list of all maximal free expressions (MFE) of a term. Thus *e.g.* the MFEs of  $\lambda y.p$ , where  $p = (\mathbb{I}y)\mathbb{I}(\lambda z.zyw)$ , is given by the list  $[\mathbb{I}; \mathbb{I}; w]$ .

An  $n$ -**ary (pure) context** ( $n \geq 0$ ) is a (pure) context with  $n$  holes  $\diamond$ . A skeleton is an  $n$ -ary pure context where the maximal free expressions w.r.t. a variable set  $\theta$  are replaced with holes. We introduce two different yet equivalent notions of skeleton, together with a corresponding operation of skeleton extraction: we argue that they entail respectively a big-step and a small-step semantics. More precisely, in the big-step semantics the skeleton extraction process can be seen as a meta-operator, defined by operations that are external to the calculus itself, as in [AF97], while in the small-step semantics the process of extraction is defined by an explicit reduction relation encoded in the calculus itself.

**A first definition of skeleton.** The first notion of skeleton runs as follows. Given any set of variables  $\theta$ , the  $\theta$ -**skeleton** of a pure term is an  $n$ -ary pure (*i.e.* without explicit cuts) context defined as  $\{\{p\}\}^\theta := \diamond$  if  $\theta \cap \text{fv}(p) = \emptyset$ ; otherwise:

$$\{\{x\}\}^\theta := x \quad \{\{\lambda x.p\}\}^\theta := \lambda x.\{\{p\}\}^{\theta \cup \{x\}} \quad \{\{p_1 p_2\}\}^\theta := \{\{p_1\}\}^\theta \{\{p_2\}\}^\theta$$

Thus *e.g.* if  $p = (\mathbb{I}y)\mathbb{I}(\lambda z.zyw)$  as above, then  $\{\{p\}\}^{\{y\}} = (\diamond y)\diamond(\lambda z.zy\diamond)$ .

Function  $\{\{-\}\}^\theta$  is (implicitly) intended to give a context whose holes correspond to the MFE's that are abstracted out. Splitting a term into a skeleton and a multiset of MFEs is at the core of full laziness. This can naturally be implemented in the node replication model, as observed in [GHP13a]. Here, we give two different (alternative) operational semantics to achieve it. The first one (Figure 2), written  $\Downarrow^\theta$ , uses big-step semantics and implements the first definition of skeleton introduced above.

$$\begin{array}{c} \frac{x \text{ fresh}}{p \Downarrow^\theta x[x/p]} \quad \text{when } \text{fv}(p) \cap \theta = \emptyset; \text{ otherwise:} \\ \\ \frac{}{x \Downarrow^\theta x} \quad \frac{p \Downarrow^{\theta \cup \{x\}} L\langle p' \rangle}{\lambda x.p \Downarrow^\theta L\langle \lambda x.p' \rangle} \quad \frac{p \Downarrow^\theta L_1\langle p' \rangle \quad q \Downarrow^\theta L_2\langle q' \rangle}{pq \Downarrow^\theta L_2\langle L_1\langle p' q' \rangle \rangle} \end{array}$$

Figure 2: Relation  $\Downarrow^\theta$ : Splitting Skeleton and MFEs in Big-Step Semantics

Each of the rules in Figure 2 corresponds to a different case in the first definition of  $\theta$ -skeleton. In the first rule, since there is no free variable of  $p$  in  $\theta$ ,  $p$  is thus an MFE kept shared in an explicit substitution. The other three rules correspond to each possible constructor, where all the explicit cuts created during the inductive cases are pushed out.

**Example 4.7.** Let  $y, z \notin \text{fv}(t)$ , so that  $t$  is the MFE of  $\lambda y.x[x/\lambda z.(yt)z]$ . Then,

$$\frac{\frac{\frac{y \Downarrow^{\{y,z\}} y}{yt \Downarrow^{\{y,z\}} (yx)[x/t]}{\quad} \quad \frac{t \Downarrow^{\{y,z\}} x[x/t]}{z \Downarrow^{\{y,z\}} z}}{\frac{(yt)z \Downarrow^{\{y,z\}} ((yx)z)[x/t]}{\lambda z.(yt)z \Downarrow^{\{y\}} (\lambda z.(yx)z)[x/t]}}$$

**Lemma 4.8** (Correctness of  $\Downarrow^\theta$ ). *If  $p \in \mathsf{T}_P$ , then  $\exists n \geq 0$  s.t.  $p \Downarrow^\theta \{\{p\}\}^\theta \langle x_1, \dots, x_n \rangle [x_i/t_i]_{i \leq n}$ , where  $\{\{p\}\}^\theta \langle t_1, \dots, t_n \rangle = p$ , and  $(x_i)_{1 \leq i \leq n}$  are fresh pairwise distinct variables. Moreover,  $\text{fv}(t_i) \cap \theta = \emptyset$  for all  $1 \leq i \leq n$ .*

*Proof.* If  $\text{fv}(p) \cap \theta = \emptyset$ , then  $p \Downarrow^\theta x_1[x_1/p]$  and  $\{\{p\}\}^\theta = \diamond$ , so that  $\{\{p\}\}^\theta \langle p \rangle = p$  trivially holds. Otherwise, we reason by induction on  $p$ :

- If  $p = x$ , then  $\{\{x\}\}^\theta = x$ , so the property holds for  $n = 0$  because  $x \Downarrow^\theta x$ .
- If  $p = p_1 p_2$ , then  $\{\{p\}\}^\theta = \{\{p_1\}\}^\theta \{\{p_2\}\}^\theta$ . By the *i.h.* we have

$$p_1 \Downarrow^\theta \{\{p_1\}\}^\theta \langle x_1, \dots, x_k \rangle [x_i/t_i]_{i \leq k} \text{ and } p_2 \Downarrow^\theta \{\{p_2\}\}^\theta \langle x_{k+1}, \dots, x_n \rangle [x_i/t_i]_{k < i \leq n}, \text{ where}$$

$$\{\{p_1\}\}^\theta \langle t_1, \dots, t_k \rangle = p_1 \text{ and } \{\{p_2\}\}^\theta \langle t_{k+1}, \dots, t_n \rangle = p_2.$$

Hence:

$$p_1 p_2 \Downarrow^\theta (\{\{p_1\}\}^\theta \langle x_1, \dots, x_k \rangle \{\{p_2\}\}^\theta \langle x_{k+1}, \dots, x_n \rangle) [x_i/t_i]_{i \leq k} [x_i/t_i]_{k < i \leq n}$$

$$= \{\{p\}\}^\theta \langle x_1, \dots, x_n \rangle [x_i/t_i]_{i \leq n}$$

- If  $p = \lambda x.p'$ , then  $\{\{p\}\}^\theta = \lambda x.\{\{p'\}\}^{\theta \cup \{x\}}$ . By the *i.h.* we have

$$p' \Downarrow^{\theta \cup \{x\}} \{\{p'\}\}^{\theta \cup \{x\}} \langle x_1, \dots, x_n \rangle [x_i/t_i]_{i \leq n}.$$

Moreover,  $x \notin \bigcup_{i \leq n} \text{fv}(t_i)$  by definition of  $\Downarrow$  and every  $x_i$  is different from  $x$ . Hence:

$$\lambda x.p' \Downarrow^\theta (\lambda x.\{\{p'\}\}^{\theta \cup \{x\}} \langle x_1, \dots, x_n \rangle) [x_i/t_i]_{i \leq n} = \{\{\lambda x.p'\}\}^\theta \langle x_1, \dots, x_n \rangle [x_i/t_i]_{i \leq n}. \quad \square$$

The correctness lemma states in particular that  $p \Downarrow^\theta \mathsf{L}\langle p' \rangle$  implies  $p'$  is pure and  $\text{fv}(\mathsf{L}) \cap \theta = \emptyset$ .

**An alternative definition of skeleton.** An *alternative* definition of  $\theta$ -skeleton can be given by *removing* the maximal free expressions from a term. Indeed, the  $\theta$ -**skeleton**  $\{\{\{p\}\}\}^\theta$  of a pure term  $p$ , where  $\theta = \{x_1 \dots x_n\}$ , is the  $n$ -ary pure context  $\{\{\{p\}\}\}^\theta$  such that  $\{\{\{p\}\}\}^\theta \langle q_1, \dots, q_n \rangle = p$ , for  $[q_1; \dots; q_n]$  the maximal free expressions of  $\lambda x_1 \dots \lambda x_n.p$ <sup>1</sup>. It is easy to show that both notions of skeleton are equivalent, *i.e.*  $\{\{p\}\}^\theta = \{\{\{p\}\}\}^\theta$ . Thus, for the same  $p$  as before,  $\lambda y.\{\{\{p\}\}\}^{\{y\}} = \lambda y.(\diamond y)\diamond(\lambda z.z y \diamond)$ .

The second strategy to split a term into a skeleton and its MFEs is the small-step strategy  $\rightarrow_{\text{st}}$  on the set of terms  $\mathsf{T}$  (Figure 3), which is indeed a subset of the reduction relation  $\rightarrow_{\text{R}}$ . It implements the second definition of skeleton we have introduced. The relation  $\rightarrow_{\text{st}}$  makes use of four basic rules which are parameterized by the variable  $y$  upon which the skeleton is built, written  $\mapsto^y$ . There are also two contextual (inductive) rules.

**Example 4.9.** Let  $\lambda y.x[x/\lambda z.(y t)z]$  be as in Example 4.7.

$$\lambda y.x[x/\lambda z.(y t)z] \xrightarrow{y}_{\text{dist}} \lambda y.x[x/\lambda z.w[w/(y t)z]] \xrightarrow{z}_{\text{app}} \lambda y.x[x/\lambda z.(w_1 w_2)[w_1/y t][w_2/z]]$$

$$\xrightarrow{z}_{\text{var}} \lambda y.x[x/\lambda z.(w_1 z) [w_1/y t]] \xrightarrow{y}_{\text{abs}} \lambda y.(\lambda z.w_1 z)[w_1/y t]$$

$$\xrightarrow{y}_{\text{app}} \lambda y.(\lambda z.(x_1 x_2)z)[x_1/y][x_2/t] \xrightarrow{y}_{\text{var}} \lambda y.(\lambda z.(y x_2)z)[x_2/t]$$

Notice that the focused variable changes from  $y$  to  $z$ , then back to  $y$ . This is because  $\rightarrow_{\text{st}}$  constructs the innermost skeletons first. The small-step approach allows to parametrize the reduction relation by only one variable at a time, instead of a set.

**Lemma 4.10.** *If  $t \in T$  and  $t \rightarrow_{\text{st}} t'$ , then  $t' \in T$ .*

<sup>1</sup>The order of the abstractions is irrelevant.

$$\begin{array}{c}
\frac{}{t[x/y] \mapsto_{\text{var}}^y t\{x/y\}} \qquad \frac{y \in \text{fv}(p_1 p_2)}{t[x/p_1 p_2] \mapsto_{\text{app}}^y t\{x/x_1 x_2\}[x_1/p_1][x_2/p_2]} \\
\frac{y \in \text{fv}(\lambda z.p)}{t[x/\lambda z.p] \mapsto_{\text{dist}}^y t[x//\lambda z.w[w/p]]} \qquad \frac{y \in \text{fv}(\lambda z.\text{LL}\langle p \rangle) \quad z \notin \text{fv}(\text{LL})}{t[x//\lambda z.\text{LL}\langle p \rangle] \mapsto_{\text{abs}}^y \text{LL}\langle t\{x/\lambda z.p\} \rangle} \\
\frac{t \mapsto^y t' \quad y \notin \text{fv}(\text{LL})}{\lambda y.\text{LL}\langle t \rangle \rightarrow_{\text{st}} \lambda y.\text{LL}\langle t' \rangle} \text{ (CTX1)} \qquad \frac{t \rightarrow_{\text{st}} t' \quad y \in \text{fv}(t) \quad y \notin \text{fv}(\text{LL})}{\lambda y.\text{LL}\langle u[x//t] \rangle \rightarrow_{\text{st}} \lambda y.\text{LL}\langle u[x//t'] \rangle} \text{ (CTX2)}
\end{array}$$

Figure 3: Relation  $\rightarrow_{\text{st}}$ : Splitting Skeleton and MFEs in Small-Step Semantics

*Proof.* For the root  $\mapsto^y$  rules, we first show that if  $t = \text{LL}_0\langle p_0 \rangle$  with  $|p_0|_z = 1$  for all  $z \in \text{dom}(\text{LL}_0)$ , and  $t \mapsto^y t'$ , then  $t' = \text{LL}_1\langle p_1 \rangle$  with  $|p_1|_z = 1$  for all  $z \in \text{dom}(\text{LL}_1)$ .

- If  $t \mapsto_{\text{var}}^y t'$ , this is straightforward.
- If  $t = \text{LL}\langle p \rangle[x/q_1 q_2] \mapsto_{\text{app}}^y \text{LL}\langle p \rangle\{x/x_1 x_2\}[x_1/q_1][x_2/q_2] = t'$ , then, since  $x \notin \text{fv}(\text{LL})$ , we have  $\text{LL}\{x/x_1 x_2\} = \text{LL}$ . Moreover, freshness of  $x_1, x_2$  implies  $|\text{LL}\langle p' \rangle|_{x_1} = |\text{LL}\langle p' \rangle|_{x_2} = |\text{LL}\langle p \rangle|_x = 1$ , where  $p' = p\{x/x_1 x_2\}$ , and  $|q_1|_{x_2} = 0$ .
- If  $t = u[x/\lambda x'.p] \mapsto_{\text{dist}}^y u[x//\lambda x'.w[w/p]] = t'$ , this is true by hypothesis, where in particular  $|u|_x = 1$ , and  $\lambda x'.w[w/p] \in \mathbb{T}$  because  $p$  is pure and  $|w|_w = 1$ .
- If  $t = \text{LL}_1\langle p_1 \rangle[x//\lambda z.\text{LL}_2\langle p_2 \rangle] \mapsto_{\text{abs}}^y \text{LL}_2\langle \text{LL}_1\langle p_1 \rangle\{x/\lambda z.p_2\} \rangle = t'$ . By hypothesis  $|p_1|_x = 1$  and  $|\text{LL}_1|_x = 0$ , so that  $t' = \text{LL}_2\langle \text{LL}_1\langle p_1 \rangle\{x/\lambda z.p_2\} \rangle = \text{LL}'_1\langle p' \rangle$ , since for all  $z_1 \in \text{dom}(\text{LL}_1)$  and all  $z_2 \in \text{dom}(\text{LL}_2)$ ,  $|p_1|_{z_1} = |p_2|_{z_2} = 1$  and, by  $\alpha$ -conversion,  $|p_1|_{z_2} = |p_2|_{z_1} = 0$  so that  $|p_1\{x/\lambda z.p_2\}|_{z'} = 1$  for any  $z' \in \text{dom}(\text{LL}')$ .

Then, for the contextual rules, we show by induction on  $t \rightarrow_{\text{sub}} t'$ : if  $t \in \mathbb{T}$  and  $t \rightarrow_{\text{sub}} t'$ , then  $t' \in \mathbb{T}$ .

- In the case of (CTX1), we have  $t = \lambda y.\text{LL}\langle t_0 \rangle \rightarrow_{\text{sub}} \lambda y.\text{LL}\langle t_1 \rangle$ . By the hypothesis that  $t \in \mathbb{T}$  follows  $t_0 = \text{LL}_0\langle p_0 \rangle$ . By the previous case analysis,  $t_1 = \text{LL}_1\langle p_1 \rangle$ . Therefore  $t' \in \mathbb{T}$ .
- In the case of (CTX2), we have  $t = \lambda y.\text{LL}\langle u[x//t_0] \rangle \rightarrow_{\text{sub}} \lambda y.\text{LL}\langle u[x//t_1] \rangle$ . By the hypothesis that  $t \in \mathbb{T}$  follows  $t_0 \in \mathbb{T}$ . By induction hypothesis,  $t_1 \in \mathbb{T}$ . Therefore  $t' \in \mathbb{T}$ .  $\square$

**Lemma 4.11.** *The reduction relation  $\rightarrow_{\text{st}}$  is confluent and terminating.*

*Proof.* To show termination it is sufficient to notice that  $t \rightarrow_{\text{st}} t'$  implies  $t \rightarrow_{\text{sub}} t'$ . Since  $\rightarrow_{\text{sub}}$  is terminating (Corollary 3.8) then we conclude termination of  $\rightarrow_{\text{st}}$ . Next, we show that  $\rightarrow_{\text{st}}$  is confluent by observing that it is deterministic. Indeed,

- The base rules  $\mapsto^y$  only reduce the outermost cut and they are all distinct: there is one rule for an outermost distributor, and three rules for outermost explicit substitutions, one for each possible form (variable, application, abstraction).
- Because of the condition  $y \notin \text{fv}(\text{LL})$  in rules (CTX1) and (CTX2) the base rules are always applied from right to left inside an abstraction.
- Moreover, rule (CTX2) does not overlap with any other rule, in particular with  $\mapsto_{\text{abs}}^y$ . Indeed, for a term  $u[x//\lambda z.\text{LL}\langle p \rangle]$ , there are only two possibilities. Either  $z$  is a free variable of  $\text{LL}$ , and we cannot apply  $\mapsto_{\text{abs}}^y$ , or  $z$  is not a free variable of  $\text{LL}$ , and we can apply  $\mapsto_{\text{abs}}^y$ . In the latter, there is in particular no cut of  $\text{LL}$  for which  $z$  is free. Therefore, we cannot apply any base-rule recursively inside the distributor. So, we cannot apply rule (CTX2).

Since rule application is deterministic, then there is no possible diverging diagram, and thus confluence is trivial.  $\square$

Thus, from now on, we denote by  $\Downarrow_{\text{st}}$  the function relating a term of  $\mathsf{T}$  to its unique  $\text{st-nf}$ . For instance, from Example 4.9 we deduce  $\lambda y.x[x/\lambda z.(yt)z] \Downarrow_{\text{st}} \lambda y.(\lambda z.(yx_2)z)[x_2/t]$ .

**Lemma 4.12.** *If  $p$  is a pure term and  $\text{LL}$  a (commutative) list context where  $y \notin \text{fv}(\text{LL})$ , then there exists  $n$  and an  $n$ -ary pure context  $c$  such that*

$$\lambda y.\text{LL}\langle t\{z/p\} \rangle \rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/c\langle x_1, \dots, x_n \rangle\} [x_i/q_i]_{1 \leq i \leq n} \rangle$$

where the variables  $x_1, \dots, x_n$  are fresh pairwise distinct and  $[q_1; \dots; q_n]$  are the MFE of  $\lambda y.p$  such that  $c\langle q_1, \dots, q_n \rangle = p$ .

*Proof.* If  $y \notin \text{fv}(p)$ , then  $p$  is the MFE of  $\lambda y.p$  and the property is satisfied by the empty reduction, with  $n = 1$ ,  $c = \diamond$ , and  $q_1 = p$ . Otherwise, we reason by induction on  $p$ .

- If  $p = y$ , then  $\lambda y.p$  has no MFE and  $\lambda y.\text{LL}\langle t\{z/y\} \rangle \rightarrow_{\text{var}}^y \lambda y.\text{LL}\langle t\{z/y\} \rangle$ . Then the property holds for  $n = 0$  and the 0-ary context  $y$ .
- If  $p = p_1 p_2$ , then by the *i.h.* on  $p_2$  and on  $p_1$  we have:

$$\begin{aligned} \lambda y.\text{LL}\langle t\{z/p_1 p_2\} \rangle &\rightarrow_{\text{app}}^y \lambda y.\text{LL}\langle t\{z/z_1 z_2\} [z_1/p_1] [z_2/p_2] \rangle \\ &\rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/z_1 c_2\langle x_{k+1}, \dots, x_n \rangle\} [z_1/p_1] [x_i/q_i]_{k < i \leq n} \rangle \\ &\rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/c_1\langle x_1, \dots, x_k \rangle c_2\langle x_{k+1}, \dots, x_n \rangle\} [x_i/q_i]_{1 \leq i \leq k} [x_i/q_i]_{k < i \leq n} \rangle \\ &= \lambda y.\text{LL}\langle t\{z/c\langle x_1, \dots, x_n \rangle\} [x_i/q_i]_{1 \leq i \leq n} \rangle \end{aligned}$$

where  $c\langle x_1, \dots, x_n \rangle = c_1\langle x_1, \dots, x_k \rangle c_2\langle x_{k+1}, \dots, x_n \rangle$ , and the variables  $x_1, \dots, x_n$  are chosen to be pairwise distinct. To apply the *i.h.* on  $p_1$ , we take  $\text{LL}$  to be  $\text{LL}\langle \diamond [x_i/q_i]_{k < i \leq n} \rangle$ , which verifies the hypothesis of the statement since by definition of the MFEs,  $y \notin \cup_{k < i \leq n} \text{fv}(q_i)$ . We can conclude since the maximal free expressions of  $\lambda y.p_1 p_2$  can be computed by considering the MFEs of  $\lambda y.p_1$  and  $\lambda y.p_2$  respectively, *i.e.*  $[q_1; \dots; q_n]$ .

- If  $p = \lambda x.p'$ , then by the *i.h.* on  $p'$  we have:  $\lambda x.z'[z'/p'] \rightarrow_{\text{st}} \lambda x.c'\langle x_1, \dots, x_n \rangle [x_i/q_i]_{1 \leq i \leq n}$ , where the terms  $[q_1; \dots; q_n]$  are the MFEs of  $\lambda x.p'$ , so in particular  $x \notin \cup_{1 \leq i \leq n} \text{fv}(q_i)$ . We can then apply the *i.h.* on  $q_n, \dots, q_1$ , thus for  $t_0 = \lambda y.\text{LL}\langle t\{z/\lambda x.p'\} \rangle$  we have:

$$\begin{aligned} t_0 &\rightarrow_{\text{dist}}^y \lambda y.\text{LL}\langle t\{z/\lambda x.z'[z'/p']\} \rangle \\ &\rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/\lambda x.c'\langle x_1, \dots, x_n \rangle [x_i/q_i]_{1 \leq i \leq n}\} \rangle \\ &\rightarrow_{\text{abs}}^y \lambda y.\text{LL}\langle t\{z/\lambda x.c'\langle x_1, \dots, x_n \rangle\} [x_i/q_i]_{1 \leq i \leq n} \rangle \\ &\rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/\lambda x.c'\langle x_1, \dots, x_{n-1}, c_n\langle x_n^1, \dots, x_n^{m_n} \rangle \rangle\} [x_i/q_i]_{1 \leq i < n} [x_n^j/q_n^j]_{1 \leq j \leq m_n} \rangle \\ &\rightarrow_{\text{st}} \lambda y.\text{LL}\langle t\{z/\lambda x.c'\langle c_1\langle x_1^1, \dots, x_1^{m_1} \rangle, \dots, c_n\langle x_n^1, \dots, x_n^{m_n} \rangle \rangle\} [x_i^j/q_i^j]_{1 \leq j \leq m_i, 1 \leq i \leq n} \rangle \\ &= \lambda y.\text{LL}\langle t\{z/c\langle x_1^1, \dots, x_n^{m_n} \rangle\} [x_i^j/q_i^j]_{1 \leq j \leq m_i, 1 \leq i \leq n} \rangle \end{aligned}$$

where  $c\langle x_1^1, \dots, x_n^{m_n} \rangle = \lambda x.c'\langle c_1\langle x_1^1, \dots, x_1^{m_1} \rangle, \dots, c_n\langle x_n^1, \dots, x_n^{m_n} \rangle \rangle$  and the variables  $x_1^1$  to  $x_n^{m_n}$  are taken pairwise distinct. To apply the *i.h.* on  $q_k$  ( $1 \leq k \leq n$ ), we take the linear context to be  $\text{LL}\langle \diamond [x_i^j/q_i^j]_{1 \leq j \leq m_i, k < i \leq n} \rangle$ , which verifies the hypothesis of the statement since by definition of the MFEs,  $y \notin \cup_{1 \leq j \leq m_i, k < i \leq n} \text{fv}(q_i^j)$ . By the *i.h.*  $[q_i^1; \dots; q_i^{m_i}]$  are the MFEs of  $\lambda y.q_i$  for each  $i$ . Therefore, since  $[q_1; \dots; q_n]$  are the MFEs of  $\lambda x.p'$ , the terms  $[q_1^1; \dots; q_n^{m_n}]$  are also the MFEs of  $\lambda y.\lambda x.p'$ .  $\square$

**Corollary 4.13** (Correctness of  $\rightarrow_{\text{st}}$ ). *Let  $p \in \mathsf{TP}$  and  $[q_1; \dots; q_n]$  be the MFEs of  $\lambda y.p$ . Then  $\lambda y.z[z/p] \Downarrow_{\text{st}} \lambda y.\{\{\{p\}\}\}^{\{y\}} \langle x_1, \dots, x_n \rangle [x_i/q_i]_{i \leq n}$  where the variables  $x_1, \dots, x_n$  are fresh and pairwise distinct.*

*Proof.* By Lemma 4.12, there is an  $n$ -ary pure context  $c$  such that  $\lambda y.z[z/p] \rightarrow_{\text{nsub}} t = \lambda y.c \langle x_1, \dots, x_n \rangle [x_i/q_i]_{1 \leq i \leq n}$ , where  $[q_1; \dots; q_n]$  are the MFEs of  $\lambda y.p$ . Thus, by the alternative definition of skeleton,  $c$  is  $\{\{\{p\}\}\}^{\{y\}}$ . Moreover,  $t$  is the **nsub**-nf of  $\lambda y.z[z/p]$  because no more base  $\mapsto^y$ -reduction steps can be applied to the list of explicit substitutions since  $y$  is not free in  $q_1, \dots, q_n$  by definition of MFE.  $\square$

From the fact that the two definitions of skeleton are equivalent, and from both proofs of correctness (Lemma 4.8 and Corollary 4.13), we infer the equivalence between the small-step and the big-step splitting semantics (Figure 3 and Figure 2 resp.). Since the small-step semantics is contained in  $\lambda R$ , we use it to build our call-by-need strategy using node replication.

Another interesting question concerns the splitting semantics for terms with explicit cuts. It is not always clear what the maximal free expressions are, as this notion depends on the position of the explicit cuts in the term. For instance, take the term  $t = \lambda y.z_1[w/xy]z_2$ . What should be the MFEs of  $t$ ? It could be  $[z_1; x; z_2]$ , or  $[z_1z_2; x]$ , or even  $[(z_1z_2)[w/x]]$ . Similarly for the skeleton, should it be respectively (1)  $\lambda y.\diamond[w/x\diamond]\diamond$ , (2)  $\lambda y.\diamond\diamond$  or (3)  $\lambda y.\diamond$ ? Solution (1) proposes to keep explicit substitutions in the skeleton. This is not coherent with the semantics of  $\lambda R$  and  $\lambda a$ , which only substitute pure terms. Solution (2) consists in unfolding the explicit cuts, so that the skeleton is pure. This can easily be obtained by adding the following rule to the definition.

$$\{\{t[x/u]\}\}^\theta := \begin{cases} \{\{t\}\}^{\theta \cup \{x\}} \{x/\{\{u\}\}^\theta\}, & \text{if } \theta \cup \text{fv}(u) \neq \emptyset \\ \{\{t\}\}^\theta, & \text{otherwise} \end{cases}$$

Indeed, this is the definition of skeleton adopted for the atomic  $\lambda$ -calculus in [GHP13a], where the authors prove that the skeleton of a term with explicit substitutions (but without explicit distributors) can be split from the MFEs. Unfortunately, they do not provide a splitting rewriting relation.

In cases involving explicit cuts binding no variable, like  $[w/xy]$  in the term  $t$  above, this definition is a cause of inefficiency: we would prefer solution (3), which avoids duplication of the application node. More generally, many nodes can be duplicated inside a term to reach a bound variable that will finally be erased. For instance, in the term  $\lambda y.x_1[w/y]x_2x_3 \dots x_n$ ,  $n - 1$  application nodes will need to be duplicated, and the skeleton would be considered  $\lambda y.\diamond\diamond\dots\diamond$  ( $n$  times) following (2), and simply  $\lambda y.\diamond$  following (3). As another example, the skeleton of  $\lambda y.(\lambda z.z[w/y])x$  would be considered  $\lambda y.(\lambda z.z)\diamond$  following (2) and  $\lambda y.\diamond$  following (3). Unfortunately, this definition is hard to specify inductively (and therefore in a big-steps semantics) without modifying the term first by permuting the cuts. Interestingly though, giving a small-steps semantics for it simply amounts to allowing  $\rightarrow_{\text{st}}$ -reduction deep inside the distributors.

**The Call-by-Need Strategy.** The **call-by-need strategy**  $\rightarrow_{\text{flneed}}$  (Figure 4) is defined on the set of terms  $\mathsf{U}$ , by using closure under the *need contexts*, given by the grammar

$$\mathsf{N} ::= \diamond \mid \mathsf{N}t \mid \mathsf{N}[x \triangleleft t] \mid \mathsf{N}\langle\langle x \rangle\rangle[x/\mathsf{N}]$$

where  $\mathbb{N}\langle\langle\_ \rangle\rangle$  denotes capture-free application of contexts (Subsection 2.1). Like call-by-name (Subsection 4.1), the call-by-need strategy is *weak*, because no *meaningful* reduction steps are performed under abstractions.

$$\begin{array}{lll}
\mathbb{L}\langle\lambda x.p\rangle u & \mapsto_{\text{dB}} & \mathbb{L}\langle p[x/u] \rangle \\
\mathbb{N}\langle\langle x \rangle\rangle[x/\mathbb{L}\langle\lambda y.p\rangle] & \mapsto_{\text{sp1}} & \mathbb{L}\langle\mathbb{L}\langle\mathbb{N}\langle\langle x \rangle\rangle[x//\lambda y.p']\rangle\rangle \quad \text{if } \lambda y.z[z/p] \Downarrow_{\text{st}} \lambda y.\mathbb{L}\langle p' \rangle \\
\mathbb{N}\langle\langle x \rangle\rangle[x//v] & \mapsto_{\text{1s}} & \mathbb{N}\langle\langle v \rangle\rangle[x//v]
\end{array}$$

Figure 4: Call-by-Need Strategy

Rule **dB** is the same one used to define **name**. Rule **sp1** (named after splitting) only uses node replication operations to compute the skeleton of the abstraction, while rule **1s** implements one-shot *linear* substitution. There is no rule to substitute a variable, as it is usually done in call-by-need for closed terms [AF97].

Linear substitution (replacing one free occurrence of a variable at a time) as implemented by rule **1s** above is not captured by the calculus  $\lambda R$ . This shows a limitation of  $\lambda R$  and  $\lambda a$ , both using full substitution to implement fully lazy sharing. Yet, the demand-driven philosophy of call-by-need is generally understood as replacing only some desired instance of one variable [AF97]. This corresponds in particular to the behavior of abstract machines, which make explicit some of the implementation features. Nonetheless, remark that the substitution used in the small-steps semantics  $\rightarrow_{\text{st}}$  is linear, thanks to the restriction on terms. Therefore, designing **f1need** as a strategy of a linear calculus with node replication is straightforward.

Notice that as a particular case of Lemma 4.2,  $t \in \mathbb{U}$  and  $t \rightarrow_{\text{f1need}} t'$  implies  $t' \in \mathbb{U}$ . Another interesting property is that  $t \rightarrow_{\text{1s}} t'$  implies  $\text{lv}_z(t) \geq \text{lv}_z(t')$ . Moreover,  $\rightarrow_{\text{f1need}}$  is deterministic.

**Lemma 4.14** (Determinism). *The strategy  $\rightarrow_{\text{f1need}}$  is deterministic.*

*Proof.* The left hand sides of the rules **dB**, **dist** and **1s** are disjoint. On the other hand, the reduction relation  $\rightarrow_{\text{st}}$  is confluent and terminating by Lemma 4.11 so that  $\Downarrow_{\text{st}}$  defines a function, thus the relation  $\rightarrow_{\text{f1need}}$  is deterministic.  $\square$

**Example 4.15.** Let  $t_0 = (\lambda x.(\mathbb{I}(Ix)))(\lambda y.y\mathbb{I})$ . Needed variable occurrences are highlighted in orange.

$$\begin{aligned}
t_0 & \rightarrow_{\text{dB}} (\mathbb{I}(Ix))[x/\lambda y.y\mathbb{I}] \rightarrow_{\text{dB}} x_1[x_1/\mathbb{I}x][x/\lambda y.y\mathbb{I}] \\
& \rightarrow_{\text{dB}} x_1[x_1/x_2[x_2/x]][x/\lambda y.y\mathbb{I}] \rightarrow_{\text{sp1}} x_1[x_1/x_2[x_2/x]][x//\lambda y.yz_1][z_1/\mathbb{I}] \\
& \rightarrow_{\text{1s}} x_1[x_1/x_2[x_2/\lambda y.yz_1]][x//\lambda y.yz_1][z_1/\mathbb{I}] \\
& \rightarrow_{\text{sp1}} x_1[x_1/x_2[x_2//\lambda y.yz_2][z_2/z_1]][x//\lambda y.yz_1][z_1/\mathbb{I}] \\
& \rightarrow_{\text{1s}} x_1[x_1/(\lambda y.yz_2)[x_2//\lambda y.yz_2][z_2/z_1]][x//\lambda y.yz_1][z_1/\mathbb{I}] \\
& \rightarrow_{\text{sp1}} x_1[x_1//\lambda y.yz_3][z_3/z_2][x_2//\lambda y.yz_2][z_2/z_1][x//\lambda y.yz_1][z_1/\mathbb{I}] \\
& \rightarrow_{\text{1s}} (\lambda y.yz_3)[x_1//\lambda y.yz_3][z_3/z_2][x_2//\lambda y.yz_2][z_2/z_1][x//\lambda y.yz_1][z_1/\mathbb{I}]
\end{aligned}$$



In order to characterize **flneed**-nfs, we use the notion of **needed free variables**, defined as:

$$\begin{aligned} \text{ndv}(x) &:= \{x\} & \text{ndv}(t[y/u]) &:= \begin{cases} (\text{ndv}(t) \setminus \{y\}) \cup \text{ndv}(u) & \text{if } y \in \text{ndv}(t) \\ \text{ndv}(t) & \text{if } y \notin \text{ndv}(t) \end{cases} \\ \text{ndv}(tu) &:= \text{ndv}(t) & \text{ndv}(t[x//u]) &:= \text{ndv}(t) \\ \text{ndv}(\lambda x.t) &:= \emptyset \end{aligned}$$

Thus *e.g.*  $\text{ndv}(x[y//\mathbf{I}]\mathbf{I}) = \{x\}$  and  $\text{ndv}((xy_1)[x/zy_2]) = \{z\}$ . In particular,  $x \in \text{ndv}(t)$  implies  $x \in \text{fv}(t)$ .

**Lemma 4.16.** *Let  $t \in \mathcal{U}$ . Then  $x \in \text{ndv}(t)$  iff there exists a context  $\mathbf{N}$  such that  $t = \mathbf{N}\langle\langle x \rangle\rangle$ .*

*Proof.* By induction on  $t$  for the left-to-right implication and by induction on  $\mathbf{N}$  for the other one (see appendix on page 56).  $\square$

Terms of  $\mathcal{U}$  in **flneed**-nf can be characterized by the grammar  $\overline{\mathbf{Ne}}$ , defined upon the grammar of neutral terms  $\overline{\mathbf{Ne}}$ . Notice that **name**-nfs are also **flneed**-nfs.

$$\begin{aligned} \overline{\mathbf{Ne}} &::= \mathbf{L}\langle\lambda x.t\rangle \mid \overline{\mathbf{Ne}} \\ \overline{\mathbf{Ne}}, \overline{\mathbf{Ne}}_0 &::= x \mid \overline{\mathbf{Ne}} t \mid \overline{\mathbf{Ne}}[x \triangleleft u] \quad x \notin \text{ndv}(\overline{\mathbf{Ne}}) \mid \overline{\mathbf{Ne}}[x/\overline{\mathbf{Ne}}_0] \quad x \in \text{ndv}(\overline{\mathbf{Ne}}) \end{aligned}$$

**Lemma 4.17.** *Let  $t \in \mathcal{U}$ . Then  $t \in \mathbf{Ne}$  iff  $t$  is in **flneed**-nf.*

*Proof.* By induction on the grammars (see appendix on page 56).  $\square$

## 5. A TYPE SYSTEM FOR $\lambda R$

This section introduces a quantitative type system  $\cap R$  for  $\lambda R$ . Non-idempotent intersection [Gar94] has one main advantage over the idempotent model [BDS13]: it gives *quantitative* information about the length of reduction sequences to normal forms [dC07]. Indeed, not only typability and normalization can be proved to be equivalent, but a measure based on type derivations provides an *upper bound* to normalizing reduction sequences. This was extensively investigated in different logical/computational frameworks [AGL19, BKR20, CG14, Ehr12, Kes16, KV20]. However, no quantitative result based on types exists in the literature for the node replication model, including the attempts done for deep inference [GHP21]. The typing rules of our system are in themselves not surprising (see for example [KV14] where a similar system is used for a  $\lambda$ -calculus with explicit substitutions interpreting the logical cut rule), but they provide a handy quantitative characterization of fully lazy normalization (Section 6).

Types are built on the following grammar of types and (finite) multi-types, where  $\alpha$  ranges over a set of base types,  $\mathbf{a}$  is a special type constant used to type terms reducing to normal abstractions and  $I$  ranges over finite sets.

$$\begin{aligned} \text{(Types)} \quad \sigma &:= \mathbf{a} \mid \alpha \mid \mathcal{M} \rightarrow \sigma \\ \text{(Multi-Types)} \quad \mathcal{M} &:= [\sigma_i]_{i \in I} \end{aligned}$$

We write  $|\mathcal{M}|$  to denote the **size of a multi-type**  $\mathcal{M}$ . **Typing environments**, written  $\Gamma$ ,  $\Delta$ ,  $\Sigma$  are functions from variables to multi-types, assigning the empty multiset to all but a finite set of variables. The domain of  $\Gamma$  is given by  $\text{dom}(\Gamma) := \{x \mid \Gamma(x) \neq []\}$ . The **union of environments**, written  $\Gamma \uplus \Delta$ , is defined by  $(\Gamma \uplus \Delta)(x) := \Gamma(x) \sqcup \Delta(x)$ , where  $\sqcup$  denotes

multiset union. For instance,  $(x : [\sigma], y : [\tau]) \uplus (x : [\sigma], z : [\tau]) = (x : [\sigma, \sigma], y : [\tau], z : [\tau])$ . This notion is extended to several environments as expected, so that  $\uplus_{i \in I} \Gamma_i$  denotes a finite union of environments, and the empty environment when  $I = \emptyset$ . We write  $\Gamma; \Delta$  for  $\Gamma \uplus \Delta$  when  $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$ . **Type judgments** have the form  $\Gamma \vdash t : \sigma$ , where  $\Gamma$  is a typing environment,  $t$  is a term and  $\sigma$  is a type. A **(typing) derivation** is a tree obtained by

$$\begin{array}{c}
\frac{}{x : [\sigma] \vdash x : \sigma} \text{(AX)} \quad \frac{\Gamma; x : \mathcal{M} \vdash t : \sigma}{\Gamma \vdash \lambda x.t : \mathcal{M} \rightarrow \sigma} \text{(ABS)} \quad \frac{\Gamma \vdash t : \mathcal{M} \rightarrow \sigma \quad \Delta \vdash u : \mathcal{M}}{\Gamma \uplus \Delta \vdash tu : \sigma} \text{(APP)} \\
\frac{}{\emptyset \vdash \lambda x.t : \mathbf{a}} \text{(ANS)} \quad \frac{\Gamma; x : \mathcal{M} \vdash t : \sigma \quad \Delta \vdash u : \mathcal{M}}{\Gamma \uplus \Delta \vdash t[x \triangleleft u] : \sigma} \text{(CUT)} \quad \frac{(\Gamma_i \vdash t : \sigma_i)_{i \in I}}{\uplus_{i \in I} \Gamma_i \vdash t : [\sigma_i]_{i \in I}} \text{(MANY)}
\end{array}$$

Figure 5: Typing System  $\cap R$

applying the (inductive) typing rules of system  $\cap R$  (Figure 5), introduced in [KV14]. The notation  $\Phi \triangleright \Gamma \vdash t : \sigma$  means there is a derivation named  $\Phi$  of the judgment  $\Gamma \vdash t : \sigma$  in system  $\cap R$ . A term  $t$  is typable in system  $\cap R$ , or  $\cap R$ -typable, iff there is an environment  $\Gamma$  and a type  $\sigma$  such that  $\Phi \triangleright \Gamma \vdash t : \sigma$ . The **size of a type derivation**  $\text{sz}(\Phi)$  is defined as the number of its rules (ABS), (APP) and (ANS). The typing system is *relevant*:

**Lemma 5.1** (Relevance). *If  $\Phi \triangleright \Gamma \vdash t : \sigma$ , then  $\text{dom}(\Gamma) \subseteq \text{fv}(t)$ .*

*Proof.* Straightforward by induction on the typing derivation.  $\square$

**Example 5.2.** The following tree is a type derivation (called  $\Phi_u$ ) in system  $\cap R$  for the term  $u = x[x/yz]$ .

$$\frac{\frac{\frac{\frac{}{z : [\tau] \vdash z : \tau} \text{(AX)}}{z : [\tau] \vdash z : [\tau]} \text{(MANY)}}{y : [[\tau] \rightarrow \tau] \vdash y : [\tau] \rightarrow \tau} \text{(AX)} \quad \frac{}{z : [\tau] \vdash z : [\tau]} \text{(MANY)}}{z : [\tau] \vdash yz : [\tau]} \text{(APP)}}{y : [[\tau] \rightarrow \tau], z : [\tau] \vdash yz : \tau} \text{(MANY)}}{x : [\tau] \vdash x : \tau} \text{(AX)} \quad \frac{}{y : [[\tau] \rightarrow \tau], z : [\tau] \vdash yz : [\tau]} \text{(MANY)}}{y : [[\tau] \rightarrow \tau], z : [\tau] \vdash x[x/yz] : \tau} \text{(CUT)}$$

Type derivations can be measured by triples. We use a  $+$  operation on triples as pointwise addition:  $(n_1, n_2, n_3) + (m_1, m_2, m_3) = (n_1 + m_1, n_2 + m_2, n_3 + m_3)$ . These triples are computed by a **weighted derivation level** function defined on typing derivations as  $D(\Phi) := M(\Phi, 1)$ , where  $M(-, -)$  is inductively defined below. In the cases (ABS), (APP) and (CUT), we let  $\Phi_t$  (resp.  $\Phi_u$ ) be the subderivation of the type of  $t$  (resp.  $\Phi_u$ ) and in (MANY) we let  $\Phi_t^i$  be the  $i$ -th derivation of the type of  $t$  for each  $i \in I$ .

- For (AX),  $M(\Phi_x, m) = (0, 0, 1)$ ,
- For (ABS),  $M(\Phi_{\lambda x.t}, m) = M(\Phi_t, m) + (1, m, 0)$ .
- For (ANS),  $M(\Phi_{\lambda x.t}, m) = (1, m, 0)$ .
- For (APP),  $M(\Phi_{tu}, m) = M(\Phi_t, m) + M(\Phi_u, m) + (1, m, 0)$ .
- For (CUT),  $M(\Phi_{t[x \triangleleft u]}, m) = M(\Phi_t, m) + M(\Phi_u, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u]))$ .
- For (MANY),  $M(\Phi_t, m) = \sum_{i \in I} M(\Phi_t^i, m)$ .

Intuitively, the first (resp. third) component of the 3-tuple  $M(\Phi, m)$  counts the number of application/abstraction (resp. axiom) rules in the typing derivation and do not depend on

$m$ . The second one takes into account the number of application/abstraction rules as well, but *weighted* by the level of the constructor. The 3-tuples are ordered lexicographically.

**Example 5.3.** Take the derivation  $\Phi_u$  from Example 5.2. Its measure is  $D(\Phi_u) = (1, 2, 3)$ . Moreover, for  $x[x/yz] \rightarrow_{\text{app}} (x_1x_2)[x_1/y][x_2/z]$  we have  $\Phi_{u'} \triangleright y : [\sigma], z : [\tau] \vdash (x_1x_2)[x_1/y][x_2/z] : \tau$  and  $D(\Phi_{u'}) = (1, 1, 4)$ .

**Lemma 5.4.** *For all derivation  $\Phi$  and all  $m, n \in \mathbb{N}$  with  $m > n$ ,  $M(\Phi, m) = M(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0)$ .*

*Proof.* By induction on  $\Phi$  (see appendix on page 57). □

**Lemma 5.5** (Split). *Let  $\Phi \triangleright \Delta \vdash u : \mathcal{M}$  such that  $\mathcal{M} = \sqcup_{i \in I} \mathcal{M}_i$  for  $I \neq \emptyset$ . Then there are derivations  $\Phi_i \triangleright \Delta_i \vdash u : \mathcal{M}_i$  such that  $\Delta = \uplus_{i \in I} \Delta_i$  and  $M(\Phi, m) = \sum_{i \in I} M(\Phi^i, m)$ .*

*Proof.* Straightforward. □

## 6. OBSERVATIONAL EQUIVALENCE

The type system  $\cap R$  characterizes normalization of both **name** and **flneed** strategies as follows: every typable term normalizes and every normalisable term is typable. In this sense, system  $\cap R$  can be seen as a (quantitative) *model* [BE01] of our call-by-name and call-by-need strategies. We prove these results by studying the appropriate lemmas, notably weighted subject reduction and weighted subject expansion. We then deduce observational equivalence between the **name** and the **flneed** strategies from the fact that their associated normalization properties are both fully characterized by the same typing system.

**Soundness.** Soundness of system  $\cap R$  w.r.t. both  $\rightarrow_{\text{name}}$  and  $\rightarrow_{\text{flneed}}$  is investigated in this section. More precisely, we show that typable terms are normalizing for both strategies. In contrast to reducibility techniques needed to show this kind of result for simple types [GHP13b], soundness is achieved here by relatively simple combinatorial arguments based again on decreasing measures. We start by studying the interaction between system  $\cap R$  and linear as well as full substitution.

**Lemma 6.1** (Partial Substitution). *Let  $\Phi \triangleright \Gamma; x : \mathcal{M} \vdash C\langle x \rangle : \sigma$  and  $\sqsubseteq$  denote multiset inclusion. Then, there exists  $\mathcal{N} \sqsubseteq \mathcal{M}$  such that for every  $\Phi_u \triangleright \Delta \vdash u : \mathcal{N}$  we have  $\Psi \triangleright \Gamma \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash C\langle u \rangle : \sigma$  and, for every  $m \in \mathbb{N}$ ,  $M(\Psi, m) = M(\Phi, m) + M(\Phi_u, m + \text{lv}_\diamond(C)) - (0, 0, |\mathcal{N}|)$ .*

*Proof.* By induction on  $\Phi$  (see appendix on page 58). □

**Corollary 6.2** (Substitution). *If  $\Phi_t \triangleright \Gamma; x : \mathcal{M} \vdash t : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ , then  $\Phi \triangleright \Gamma \uplus \Delta \vdash t\{x/u\} : \sigma$ , and for all  $m \in \mathbb{N}$  we have  $M(\Phi, m) \leq M(\Phi_t, m) + M(\Phi_u, m + \text{lv}_x(t))$ . Moreover,  $|\mathcal{M}| > 0$  iff the inequality is strict.*

*Proof.* The proof is by induction on  $|t|_x$ . If  $|t|_x = 0$ , then by the relevance Lemma 5.1  $\mathcal{M} = []$ , so that  $\Phi_u$  necessarily comes from a (MANY) rule without any premise and thus  $\Phi = \Phi_t$ . We have  $M(\Phi, m) = M(\Phi_t, m) + M(\Phi_u, m + \text{lv}_x(t))$  because  $M(\Phi_u, m + \text{lv}_x(t)) = (0, 0, 0)$ . Otherwise,  $|t|_x > 0$  and we can write  $t$  as  $C\langle x \rangle$ . By the partial substitution Lemma 6.1, there exists  $\mathcal{N} \sqsubseteq \mathcal{M}$  such that for all  $\Phi_u^0 \triangleright \Delta_0 \vdash u : \mathcal{N}$ , there is  $\Phi' \triangleright \Gamma \uplus \Delta_0; x : \mathcal{M} \setminus \mathcal{N} \vdash C\langle u \rangle : \sigma$ . By the split Lemma 5.5, there are derivations  $\Phi_u^1 \triangleright \Delta_1 \vdash u :$

$\mathcal{N}$  and  $\Phi_u^2 \triangleright \Delta_2 \vdash u : \mathcal{M} \setminus \mathcal{N}$ , where  $\Delta = \Delta_1 \uplus \Delta_2$  so that we can apply the partial substitution Lemma to  $\Phi_t$  and  $\Phi_u^1$ , and we obtain  $\Phi' \triangleright \Gamma \uplus \Delta_1; x : \mathcal{M} \setminus \mathcal{N} \vdash \mathbf{C}\langle\langle u \rangle\rangle : \sigma$ . Since  $\text{lv}_\diamond(\mathbf{C}) \leq \text{lv}_x(t)$ , we have  $\mathbf{M}(\Phi', m) = \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u^1, m + \text{lv}_\diamond(\mathbf{C})) - (0, 0, |\mathcal{N}|) \leq_{L.5.4} \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u^1, m + \text{lv}_x(t)) - (0, 0, |\mathcal{N}|) \leq \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u^1, m + \text{lv}_x(t))$ . Because  $x \notin \text{fv}(u)$ ,  $|\mathbf{C}\langle\langle u \rangle\rangle|_x = |t|_x - 1$ . We conclude by applying the *i.h.* on  $\Phi'$  and  $\Phi_u^2$ . We get  $\Phi \triangleright \Gamma \uplus \Delta_1 \uplus \Delta_2 \vdash \mathbf{C}\langle\langle u \rangle\rangle\{x/u\} : \sigma = \Gamma \uplus \Delta \vdash t\{x/u\} : \sigma$ . For the measure, we use  $\text{lv}_x(\mathbf{C}\langle\langle u \rangle\rangle) \leq \text{lv}_x(t)$  to get  $\mathbf{M}(\Phi, m) \leq \mathbf{M}(\Phi', m) + \mathbf{M}(\Phi_u^2, m + \text{lv}_x(\mathbf{C}\langle\langle u \rangle\rangle)) \leq \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u^1, m + \text{lv}_x(t)) + \mathbf{M}(\Phi_u^2, m + \text{lv}_x(t)) =_{L.5.5} \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(t))$ . If  $\mathcal{M} \neq []$ , then either  $\mathcal{N}$  or  $\mathcal{M} \setminus \mathcal{N}$  is non-empty, so at least one of the two previous inequalities is strict.  $\square$

The key idea to show soundness is that the measure  $\mathbf{D}(\cdot)$  decreases w.r.t. the reduction relations  $\rightarrow_{\text{name}}$  and  $\rightarrow_{\text{f1need}}$ :

**Lemma 6.3** (Weighted Subject Reduction for  $\rightarrow_\pi$ ). *Let  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$  and  $t_0 \rightarrow_\pi t_1$ . Then there exists  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$  such that  $\mathbf{M}(\Phi_{t_0}, m) = \mathbf{M}(\Phi_{t_1}, m)$  for every  $m \in \mathbb{N}$ .*

*Proof.* Let  $t_0 = \mathbf{C}\langle t'_0 \rangle$  and  $t_1 = \mathbf{C}\langle t'_1 \rangle$ , where  $t'_0 \rightarrow_\pi t'_1$  is a root step. We reason by induction on  $\mathbf{C}$ . We only detail one base case (where  $\mathbf{C} = \diamond$ ) and one inductive case (see appendix on page 60).

- $t'_0 = t[x \triangleleft u[y \triangleleft s]] \mapsto_\pi t[x \triangleleft u][y \triangleleft s] = t'_1$ , where  $y \notin \text{fv}(t)$ . Let  $\Phi_i$  be

$$\frac{\frac{\Phi_u^i \triangleright \Delta_u^i; y : \mathcal{N}_i \vdash u : \rho_i}{\Delta_u^i \uplus \Delta_s^i \vdash u[y \triangleleft s] : \rho_i} \quad \frac{(\Phi_s^{i,j} \triangleright \Delta_s^{i,j} \vdash s : \delta_j)_{j \in J_i}}{\Delta_s^i \vdash s : \mathcal{N}_i} \text{ (MANY)}}{\text{ (CUT)}}$$

then the typing derivation  $\Phi$  is of the form

$$\frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x \triangleleft u[y \triangleleft s]] : \sigma} \quad \frac{(\Phi_i \triangleright \Delta_u^i \uplus \Delta_s^i \vdash u[y \triangleleft s] : \rho_i)_{i \in I}}{\Delta_u \uplus \Delta_s \vdash u[y \triangleleft s] : \mathcal{M}} \text{ (MANY)}}{\text{ (CUT)}}$$

where  $\mathcal{M} = [\rho_i]_{i \in I}$ ,  $\mathcal{N}_i = [\delta_j]_{j \in J_i}$ ,  $\Delta_u = \uplus_{i \in I} \Delta_u^i$ ,  $\Delta_s^i = \uplus_{j \in J_i} \Delta_s^{i,j}$ , and  $\Delta_s = \uplus_{i \in I} \Delta_s^i$ . Let  $\Phi_s$  be

$$\frac{(\Phi_s^{i,j} \triangleright \Delta_s^{i,j} \vdash s : \delta_j)_{j \in J_i, i \in I}}{\Delta_s \vdash s : \mathcal{N}} \text{ (MANY)}$$

We then construct the following derivation  $\Psi$ .

$$\frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma}{\Gamma' \uplus \Delta_u; y : \mathcal{N} \vdash t[x \triangleleft u] : \sigma} \quad \frac{(\Phi_u^i \triangleright \Delta_u^i; y : \mathcal{N}_i \vdash u : \rho_i)_{i \in I}}{\Delta_u; y : \mathcal{N} \vdash u : \mathcal{M}} \text{ (MANY)}}{\text{ (CUT)}} \quad \frac{\Phi_s \triangleright \Delta_s \vdash s : \mathcal{N}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x \triangleleft u][y \triangleleft s] : \sigma} \text{ (CUT)}$$

where  $\mathcal{N} = \sqcup_{i \in I} \mathcal{N}_i$ , so that  $\mathcal{N} = [\delta_j]_{j \in J_i, i \in I}$ . Moreover, because  $y \notin \text{fv}(t)$ , we have that  $\text{lv}_y(t[x \triangleleft u]) = \text{lv}_x(t) + \text{lv}_y(u) + \text{ES}([x \triangleleft u])$  if  $y \in \text{fv}(u)$ , and  $\text{lv}_y(t[x \triangleleft u]) = 0$  otherwise. Now, we show that  $\mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u]) + \text{lv}_y(u) + \text{ES}([y \triangleleft s])) = \mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_y(t[x \triangleleft u]) + \text{ES}([y \triangleleft s]))$ . If  $y \in \text{fv}(u)$ , this is immediate. Otherwise, by

the Relevance Lemma 5.1 we have  $J_i = []$  for any  $i$  thus  $s$  is necessarily typed with rule (MANY) and no premise, so that both measures are equal to  $(0,0,0)$ . Then,

$$\begin{aligned}
M(\Phi, m) &= M(\Phi_t, m) + \sum_{i \in I} M(\Phi_u^i, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\
&\quad + \sum_{i \in I} \sum_{j \in J_i} M(\Phi_s^{i,j}, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u]) + \text{lv}_y(u) + \text{ES}([y \triangleleft s])) \\
&= M(\Phi_t, m) + \sum_{i \in I} M(\Phi_u^i, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\
&\quad + \sum_{i \in I} \sum_{j \in J_i} M(\Phi_s^{i,j}, m + \text{lv}_y(t[x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\
&= M(\Psi, m)
\end{aligned}$$

- If  $\mathbf{C} = u[x \triangleleft \mathbf{C}']$ , then we have  $\Phi_u \triangleright \Delta; x : \mathcal{M} \vdash u : \sigma$  and  $\Phi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o \rangle : \mathcal{M}$ . By the *i.h.* there is  $\Psi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o' \rangle : \mathcal{M}$ , so  $\Psi \triangleright \Gamma' \uplus \Delta \vdash u[x \triangleleft \mathbf{C}' \langle o' \rangle] : \sigma$ . Moreover,

$$\begin{aligned}
M(\Phi, m) &= M(\Phi_u, m) + M(\Phi', m + \text{lv}_x(u) + \text{ES}([x \triangleleft u])) \\
&=_{i.h.} M(\Phi_u, m) + M(\Psi', m + \text{lv}_x(u) + \text{ES}([x \triangleleft u])) \\
&= M(\Psi, m)
\end{aligned}$$

□

**Lemma 6.4** (Weighted Subject Reduction for  $\rightarrow_{\text{sub}}$ ). *Let  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$ . If  $t_0 \rightarrow_{\text{sub}} t_1$ , then there exists  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$  such that  $M(\Phi_{t_0}, m) \geq M(\Phi_{t_1}, m)$  for every  $m \in \mathbb{N}$ .*

*Proof.* As remarked in Subsection 2.2,  $t_0 \rightarrow_{\text{sub}} t_1$  implies  $t_0 \rightarrow_{\pi} t' \rightarrow_{\text{sub}'} t_1$ . By Lemma 6.3, weighted subject reduction holds for  $t_0 \rightarrow_{\pi} t'$ , so it is sufficient to show the statement for the relation  $\rightarrow_{\text{sub}'}$ . We reason by induction on this relation. We show the base cases for  $\mapsto_{\text{app}'}$  and  $\mapsto_{\text{dist}'}$ , the cases  $\mapsto_{\text{abs}'}$  and  $\mapsto_{\text{var}'}$  are simply by the substitution Corollary 6.2, and the inductive cases are straightforward by the *i.h.*

- $t_0 = t[x/us] \rightarrow_{\text{app}} t\{x/yz\}[y/u][z/s] = t_1$ , where  $y$  and  $z$  are fresh variables. Let  $\Phi_i$  be of the form

$$\frac{\Phi_u^i \triangleright \Gamma_u^i \vdash u : \mathcal{N}_i \rightarrow \rho_i \quad \Phi_s^i \triangleright \Gamma_s^i \vdash s : \mathcal{N}_i}{\Gamma_u^i \uplus \Gamma_s^i \vdash us : \rho_i} \text{ (APP)}$$

then the typing derivation  $\Phi_{t_0}$  is of the form

$$\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \frac{(\Phi_i \triangleright \Gamma_u^i \uplus \Gamma_s^i \vdash us : \rho_i)_{i \in I}}{\Gamma_u \uplus \Gamma_s \vdash us : \mathcal{M}} \text{ (MANY)}}{\Gamma' \uplus \Gamma_u \uplus \Gamma_s \vdash t[x/us] : \sigma} \text{ (CUT)}$$

where  $\mathcal{M} = [\rho_i]_{i \in I}$ ,  $\Gamma_u = \uplus_{i \in I} \Gamma_u^i$  and  $\Gamma_s = \uplus_{i \in I} \Gamma_s^i$ . We have

$$\begin{aligned}
M(\Phi_{t_0}, m) &= M(\Phi_t, m) \\
&\quad + \sum_{i \in I} (M(\Phi_u^i, m + \text{lv}_x(t) + 1) + M(\Phi_s^i, m + \text{lv}_x(t) + 1) + (1, m + \text{lv}_x(t) + 1, 0))
\end{aligned}$$

Let us consider  $\Phi' \triangleright \Gamma'; y : \mathcal{N}_u; z : \mathcal{N}_s \vdash t\{x/yz\} : \sigma$ , obtained by Corollary 6.2 from  $\Phi_t$  and  $\Phi_{yz} \triangleright y : \mathcal{N}_u; z : \mathcal{N}_s \vdash yz : [\rho_i]_{i \in I}$ , where  $\mathcal{N}_u = [\mathcal{N}_i \rightarrow \rho_i]_{i \in I}$  and  $\mathcal{N}_s = \sqcup_{i \in I} \mathcal{N}_i$ . Let

$$\Phi_u = \frac{(\Phi_u^i \triangleright \Gamma_u^i \vdash u : \mathcal{N}_i \rightarrow \rho_i)_{i \in I}}{\Gamma_u \vdash u : \mathcal{N}_u} \text{ (CUT)} \quad \Phi_s = \frac{(\Phi_s^i \triangleright \Gamma_s^i \vdash s : \mathcal{N}_i)_{i \in I}}{\Gamma_s \vdash s : \mathcal{N}_s} \text{ (CUT)}$$

We construct the following derivation  $\Phi_{t_1}$  with two applications of rule (CUT).

$$\frac{\frac{\Phi' \triangleright \Gamma'; z : \mathcal{N}_s; y : \mathcal{N}_u \vdash t\{x/yz\} : \sigma \quad \Phi_u \triangleright \Gamma_u \vdash u : \mathcal{N}_u}{(\Gamma'; z : \mathcal{N}_s) \uplus \Gamma_u \vdash t\{x/yz\}[y/u] : \sigma} \quad \Phi_s \triangleright \Gamma_s \vdash s : \mathcal{N}_s}{\Gamma' \uplus \Gamma_u \uplus \Gamma_s \vdash t\{x/yz\}[y/u][z/s] : \sigma}$$

We consider two cases to conclude:

– If  $\mathcal{M} = []$ , then

$$\begin{aligned} \mathsf{M}(\Phi_{t_1}, m) &= \mathsf{M}(\Phi', m) \\ &+ \sum_{i \in I} \mathsf{M}(\Phi_u^i, m + \text{lv}_y(t\{x/yz\}) + 1) + \sum_{i \in I} \mathsf{M}(\Phi_s^i, m + \text{lv}_z(t\{x/yz\}[y/u]) + 1) \\ &= \mathsf{M}(\Phi', m) =_{L. 6.2} \mathsf{M}(\Phi_t, m) = \mathsf{M}(\Phi_{t_0}, m) \end{aligned}$$

– If  $\mathcal{M} \neq []$ , then

$$\begin{aligned} \mathsf{M}(\Phi_{t_1}, m) &= \mathsf{M}(\Phi', m) \\ &+ \sum_{i \in I} \mathsf{M}(\Phi_u^i, m + \text{lv}_y(t\{x/yz\}) + 1) + \sum_{i \in I} \mathsf{M}(\Phi_s^i, m + \text{lv}_z(t\{x/yz\}[y/u]) + 1) \\ &= \mathsf{M}(\Phi', m) \\ &+ \sum_{i \in I} \mathsf{M}(\Phi_u^i, m + \text{lv}_y(t\{x/yz\}) + 1) + \sum_{i \in I} \mathsf{M}(\Phi_s^i, m + \text{lv}_z(t\{x/yz\}) + 1) \\ &=_{L. 2.5:2} \mathsf{M}(\Phi', m) + \sum_{i \in I} (\mathsf{M}(\Phi_u^i, m + \text{lv}_x(t) + 1) + \mathsf{M}(\Phi_s^i, m + \text{lv}_x(t) + 1)) \\ &\leq_{L. 6.2} \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_{yz}, m + \text{lv}_x(t)) \\ &+ \sum_{i \in I} (\mathsf{M}(\Phi_u^i, m + \text{lv}_x(t) + 1) + \mathsf{M}(\Phi_s^i, m + \text{lv}_x(t) + 1)) \\ &= \mathsf{M}(\Phi_t, m) + (1, m + \text{lv}_x(t), 2) \\ &+ \sum_{i \in I} (\mathsf{M}(\Phi_u^i, m + \text{lv}_x(t) + 1) + \mathsf{M}(\Phi_s^i, m + \text{lv}_x(t) + 1)) \\ &< \mathsf{M}(\Phi_t, m) \\ &+ \sum_{i \in I} (\mathsf{M}(\Phi_u^i, m + \text{lv}_x(t) + 1) + \mathsf{M}(\Phi_s^i, m + \text{lv}_x(t) + 1) + (1, m + \text{lv}_x(t), 2)) \\ &< \mathsf{M}(\Phi_{t_0}, m). \end{aligned}$$

- $t_0 = t[x/\lambda y.u] \rightarrow_{\text{dist}} t[x/\lambda y.z[z/u]] = t_1$ . The typing derivation is of the form

$$\Phi_{t_0} = \frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_{\lambda y.u} \triangleright \Gamma_{\lambda y.u} \vdash \lambda y.u : \mathcal{M}}{\Gamma' \uplus \Gamma_{\lambda y.u} \vdash t[x/\lambda y.u] : \sigma} \text{ (CUT)}$$

where

$$\Phi_{\lambda y.u} = \frac{\left( \frac{\Phi^i \triangleright \Gamma_{\lambda y.u}^i; y : \mathcal{N}_i \vdash u : \rho_i}{\Gamma_{\lambda y.u}^i \vdash \lambda y.u : \mathcal{N}_i \rightarrow \rho_i} \text{ (ABS)} \right)_{i \in I} \left( \frac{}{\vdash \lambda y.u : \mathbf{a}} \text{ (ANS)} \right)^k}{\Gamma_{\lambda y.u} \vdash \lambda y.u : \mathcal{M}} \text{ (MANY)}$$

and  $\mathcal{M} = [\mathcal{N}_i \rightarrow \rho_i]_{i \in I} \sqcup \underbrace{[\mathbf{a}, \dots, \mathbf{a}]}_k$ , with  $\Gamma_{\lambda y.u} = \uplus_{i \in I} \Gamma_{\lambda y.u}^i$ . Moreover,

$$\begin{aligned} \mathbf{M}(\Phi_{t_0}, m) &= \mathbf{M}(\Phi_t, m) + k * (1, m + \text{lv}_x(t) + 1, 0) \\ &\quad + \sum_{i \in I} (\mathbf{M}(\Phi^i, m + \text{lv}_x(t) + 1) + (1, m + \text{lv}_x(t) + 1, 0)) \end{aligned}$$

We construct the following derivation

$$\Phi_{t_1} = \frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_\lambda \triangleright \Gamma_{\lambda y.u} \vdash \lambda y.z[z/u] : \mathcal{M}}{\Gamma' \uplus \Gamma_{\lambda y.u} \vdash t[x//\lambda y.z[z/u]] : \sigma} \text{ (CUT)}$$

where

$$\Phi_\lambda = \frac{\left( \frac{\Phi_\lambda^i \triangleright \Gamma_{\lambda y.u}^i; y : \mathcal{N}_i \vdash z[z/u] : \rho_i}{\Gamma_{\lambda y.u}^i \vdash \lambda y.z[z/u] : \mathcal{N}_i \rightarrow \rho_i} \text{ (ABS)} \right)_{i \in I} \left( \frac{}{\vdash \lambda y.z[z/u] : \mathbf{a}} \text{ (ANS)} \right)^k}{\Gamma_{\lambda y.u} \vdash \lambda y.z[z/u] : \mathcal{M}} \text{ (MANY)}$$

with  $\Phi_\lambda^i$  of the form

$$\frac{\frac{}{z : [\rho_i] \vdash z : \rho_i} \text{ (AX)} \quad \Phi^i \triangleright \Gamma_{\lambda y.u}^i; y : \mathcal{N}_i \vdash u : \rho_i}{\Gamma_{\lambda y.u}^i; y : \mathcal{N}_i \vdash z[z/u] : \rho_i} \text{ (CUT)}}$$

We have

$$\begin{aligned} \mathbf{M}(\Phi_{t_1}, m) &= \mathbf{M}(\Phi_t, m) + k * (1, m + \text{lv}_x(t), 0) \\ &\quad + \sum_{i \in I} (\mathbf{M}(\Phi^i, m + \text{lv}_x(t) + 1) + (0, 0, 1) + (1, m + \text{lv}_x(t), 0)) \\ &\leq \mathbf{M}(\Phi_{t_0}, m) \end{aligned} \quad \square$$

**Lemma 6.5** (Weighted Subject Reduction for  $\rightarrow_{\text{ndB}}$ ). *Let  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$ . If  $t_0 \rightarrow_{\text{ndB}} t_1$ , then there exists  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$  such that  $\mathbf{M}(\Phi_{t_0}, m) > \mathbf{M}(\Phi_{t_1}, m)$  for every  $m \in \mathbb{N}$ .*

*Proof.* We prove that  $\mathbf{M}(\Phi_{t_0}, m) > \mathbf{M}(\Phi_{t_1}, m)$  by showing in particular that it is the first component of the 3-tuple that strictly decreases.

We reason by induction on the reduction relation  $\rightarrow_{\text{ndB}}$ .

- If  $t_0 = \mathbf{L}\langle \lambda x.t \rangle u \rightarrow_{\text{dB}} \mathbf{L}\langle t[x/u] \rangle = t_1$ , then we reason by induction on  $\mathbf{L}$ . The inductive step follows from Lemma 6.3, so we only show the base case  $\mathbf{L} = \diamond$ . The typing derivation  $\Phi_{t_0}$  is of the form

$$\frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma}{\Gamma' \vdash \lambda x.t : \mathcal{M} \rightarrow \sigma} \text{ (ABS)} \quad \Phi_u \triangleright \Gamma_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Gamma_u \vdash (\lambda x.t)u : \sigma} \text{ (APP)}$$

and  $\mathbf{M}(\Phi_{t_0}, m) = \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u, m) + (2, 2 * m, 0)$ .

We construct the following derivation.

$$\Phi_{t_1} = \frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_u \triangleright \Gamma_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Gamma_u \vdash t[x/u] : \sigma} \text{ (CUT)}$$

We have

$$\begin{aligned} \mathsf{M}(\Phi_{t_1}, m) &= \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_u, m + \text{lv}_x(t) + 1) \\ &=_{L. 5.4} \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_u, m) + (0, (\text{lv}_x(t) + 1) * \text{sz}(\Phi_u), 0) \\ &< \mathsf{M}(\Phi_{t_0}, m) \end{aligned}$$

Notice that it is the first component of the first 3-tuple that strictly decreases by 2.

- If  $t_0 = tu \rightarrow_{\text{ndB}} t'u = t_1$ , where  $t \rightarrow_{\text{ndB}} t'$ , then the property trivially holds by the *i.h.*
- If  $t_0 = t[x/u] \rightarrow_{\text{ndB}} t'[x/u] = t_1$ , where  $t \rightarrow_{\text{ndB}} t'$ , then  $\Gamma = \Gamma' \setminus x \uplus \Delta$  and  $\Phi_t \triangleright \Gamma'$ ;  $x : \mathcal{M} \vdash t : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . Moreover,  $\mathsf{M}(\Phi_{t_0}, m) = \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_u, m + \text{lv}_x(t) + 1)$ . By the *i.h.* we have  $\Phi_{t'} \triangleright \Gamma'$ ;  $x : \mathcal{M} \vdash t' : \sigma$  and  $\mathsf{M}(\Phi_t, m) >_{i.h.} \mathsf{M}(\Phi_{t'}, m)$ , where in particular it is the first component of the first 3-tuple that strictly decreases. Derivation  $\Phi_{t_1}$  is then obtained by rule (CUT) from  $\Phi_{t'}$  and  $\Phi_u$ . We can conclude since:

$$\begin{aligned} \mathsf{M}(\Phi_{t_1}, m) &= \mathsf{M}(\Phi_{t'}, m) + \mathsf{M}(\Phi_u, m + \text{lv}_x(t') + 1) \\ &=_{L. 5.4} \mathsf{M}(\Phi_{t'}, m) + \mathsf{M}(\Phi_u, m) + (0, (\text{lv}_x(t') + 1) * \text{sz}(\Phi_u), 0) \\ &<_{i.h.} \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_u, m) + (0, (\text{lv}_x(t) + 1) * \text{sz}(\Phi_u), 0) \\ &=_{L. 5.4} \mathsf{M}(\Phi_t, m) + \mathsf{M}(\Phi_u, m + \text{lv}_x(t) + 1) \\ &= \mathsf{M}(\Phi_{t_0}, m) \end{aligned}$$

Note that even when  $\text{lv}_x(t') > \text{lv}_x(t)$ , the inequation  $\mathsf{M}(\Phi_{t_1}, m) < \mathsf{M}(\Phi_{t_0}, m)$  is determined by the strict relation between the first components of the 3-tuples, that is, the unweighted number of abstraction and application rules.  $\square$

**Lemma 6.6.** *Let  $\Phi \triangleright \Gamma \vdash \mathsf{N}\langle\langle x \rangle\rangle : \tau$ . Then there exists  $\Gamma'$ ,  $I \neq \emptyset$  and  $[\sigma_i]_{i \in I}$  such that  $\Gamma = \Gamma' \uplus x : [\sigma_i]_{i \in I}$  and for any variable  $z$  there is a proof  $\Phi' \triangleright \Gamma' \uplus z : [\sigma_i]_{i \in I} \vdash \mathsf{N}\langle\langle z \rangle\rangle : \tau$ . In particular, if  $z$  is fresh, then  $\Gamma' \uplus z : [\sigma_i]_{i \in I} = \Gamma'; z : [\sigma_i]_{i \in I}$ .*

*Proof.* By induction on  $\mathsf{N}$ .

- If  $\mathsf{N} = \diamond$ , this is straightforward by taking  $\Gamma' = \emptyset$  and  $[\sigma_i]_{i \in I} = [\tau]$ .
- If  $\mathsf{N} = \mathsf{N}'t$  or  $\mathsf{N} = \mathsf{N}'[x \triangleleft t]$ , then there is a derivation  $\Phi' \triangleright \Gamma_1 \vdash \mathsf{N}'\langle\langle x \rangle\rangle : \tau'$ , such that  $\Gamma = \Gamma_1 \uplus \Gamma_2$  and  $\tau' = \mathcal{M} \rightarrow \tau$  or  $\tau = \tau'$ , respectively. By the *i.h.*  $\Gamma_1 = \Gamma'_1 \uplus x : [\sigma_i]_{i \in I}$ , so that  $\Gamma' = \Gamma'_1 \uplus \Gamma_2$ .
- If  $\mathsf{N} = \mathsf{N}_1\langle\langle y \rangle\rangle[y/\mathsf{N}_2]$ , the derivation is as follows.

$$\frac{\Gamma_1; y : [\rho_j]_{j \in J} \vdash \mathsf{N}_1\langle\langle y \rangle\rangle : \tau \quad \frac{(\Gamma_j \vdash \mathsf{N}_2\langle\langle x \rangle\rangle : \rho_j)_{j \in J}}{\uplus_{j \in J} \Gamma_j \vdash \mathsf{N}_2\langle\langle x \rangle\rangle : [\rho_j]_{j \in J}} \text{ (MANY)}}{\Gamma \vdash \mathsf{N}_1\langle\langle y \rangle\rangle[y/\mathsf{N}_2\langle\langle x \rangle\rangle] : \tau} \text{ (CUT)}$$

Where  $\Gamma = \Gamma_1 \uplus \Gamma_2$  and  $\Gamma_2 = \uplus_{j \in J} \Gamma_j$ . By the *i.h.* on  $\mathsf{N}_1$ ,  $\Gamma_1; y : [\rho_j]_{j \in J} = \Gamma' \uplus y : [\rho_j]_{j \in J'}$  for some  $\emptyset \neq J' \subseteq J$ . Thus  $J \neq \emptyset$ . By the *i.h.* on  $\mathsf{N}_2$ , for every  $j \in J$  we have  $\Gamma_j = \Gamma'_j \uplus x : [\sigma_i]_{i \in I_j}$ , where  $I_j \neq \emptyset$  and a proof  $\Phi_j \triangleright \Gamma'_j \uplus x : [\sigma_i]_{i \in I_j} \vdash \mathsf{N}_2\langle\langle z \rangle\rangle : \rho_j$  for a variable  $z$ . We then take  $I = \cup_{j \in J} I_j$  and  $\Gamma' = \Gamma_1 \uplus_{j \in J} \Gamma'_j$ .  $\square$

**Lemma 6.7** (Weighted Subject Reduction for **f1need**). *Let  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$ . If  $t_0 \rightarrow_{\text{f1need}} t_1$ , then there exists  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$  such that  $\mathsf{M}(\Phi_{t_0}, m) > \mathsf{M}(\Phi_{t_1}, m)$  for every  $m \in \mathbb{N}$ .*

*Proof.* We prove that  $\mathsf{M}(\Phi_{t_0}, m) > \mathsf{M}(\Phi_{t_1}, m)$  by showing in particular that the first component of the first 3-tuple strictly decreases when the reduction is dB. We reason by induction on the reduction relation, *i.e.* by induction on the context  $\mathsf{N}$  where the root reduction takes place. We first detail the base case when  $\mathsf{N} = \diamond$ .



- $t_0 = L\langle\lambda x.t\rangle u \rightarrow_{\text{dB}} L\langle t[x/u]\rangle = t_1$ . This case is the same as for **name**.
- $t_0 = N\langle\langle x \rangle\rangle[x/\lambda y.p] \rightarrow_{\text{sp1}} LL\langle N\langle\langle x \rangle\rangle[x//\lambda y.p']\rangle = t_1$ , where  $\lambda y.z[z/p] \Downarrow_{\text{st}} \lambda y.LL\langle p'\rangle$ . The typing derivation  $\Phi_{t_0}$  is of the form

$$\frac{\Phi \triangleright \Gamma'; x : \mathcal{N} \vdash N\langle\langle x \rangle\rangle : \sigma \quad \frac{(\Phi_{\lambda y.p}^i \triangleright \Delta_i \vdash \lambda y.p : \sigma_i)_{i \in I}}{\Delta \vdash \lambda y.p : \mathcal{N}} \text{ (MANY)}}{\Gamma' \uplus \Delta \vdash N\langle\langle x \rangle\rangle[x/\lambda y.p] : \sigma} \text{ (CUT)}$$

where  $\mathcal{N} = [\sigma_i]_{i \in I}$ ,  $\Delta = \uplus_{i \in I} \Delta_i$  and  $\Gamma = \Gamma' \uplus \Delta$ . Moreover,  $I \neq \emptyset$  by Lemma 6.6. For each  $\sigma_i$  we build the following derivations  $\Phi_{p_0}^i$ :

- if  $\sigma_i = \mathcal{M}_i \rightarrow \tau_i$  then  $\Phi_{p_0}^i$  is of the form

$$\frac{\frac{z : [\tau_i] \vdash z : \tau_i \text{ (AX)} \quad \Phi_p^i \triangleright \Delta_i; y : \mathcal{M}_i \vdash p : \tau_i}{\Delta_i; y : \mathcal{M}_i \vdash z[z/p] : \tau_i} \text{ (CUT)}}{\Delta_i \vdash \lambda y.z[z/p] : \mathcal{M}_i \rightarrow \tau_i} \text{ (ABS)}$$

where  $\Phi_p^i$  is obtained from  $\Phi_{\lambda y.p}^i$  by reversing the (ABS) rule.

- if  $\sigma_i = \mathbf{a}$  then  $\Phi_{p_0}^i = \frac{}{\vdash \lambda y.z[z/p] : \mathbf{a}} \text{ (ANS)}$ .

By hypothesis,  $\lambda y.z[z/p] \rightarrow_{\text{st}} \lambda y.LL\langle p'\rangle$ . Since  $\rightarrow_{\text{st}}$  is included in  $\rightarrow_{\text{sub}}$ , then we know by Lemma 6.4 that there are derivations  $\Phi_{p_1}^i \triangleright \Delta_i \vdash \lambda y.LL\langle p'\rangle : \sigma_i$  such that  $M(\Phi_{p_0}^i, m) \geq M(\Phi_{p_1}^i, m)$ . Thus, we can build the following derivation.

$$\Phi_{t_1}' = \frac{\Phi \triangleright \Gamma'; x : \mathcal{N} \vdash N\langle\langle x \rangle\rangle : \sigma \quad \frac{(\Phi_{p_1}^i \triangleright \Delta_i \vdash \lambda y.LL\langle p'\rangle : \sigma_i)_{i \in I}}{\Delta \vdash \lambda y.LL\langle p'\rangle : \mathcal{N}} \text{ (MANY)}}{\Gamma' \uplus \Delta \vdash N\langle\langle x \rangle\rangle[x//\lambda y.LL\langle p'\rangle] : \sigma} \text{ (CUT)}$$

Let  $n = \text{lv}_x(N\langle\langle x \rangle\rangle)$ . We begin showing that  $M(\Phi_{\lambda y.p}^i, m + n + 1) > M(\Phi_{p_0}^i, m + n)$  for every  $i \in I$ . There are two cases.

- (1) If  $\sigma_i = \mathbf{a}$ , then  $M(\Phi_{\lambda y.p}^i, m + n + 1) = (1, m + n + 1, 0)$ , while  $M(\Phi_{p_0}^i, m + n) = (1, m + n, 0)$ .
- (2) If  $\sigma_i = \mathcal{M}_i \rightarrow \tau_i$ , then  $M(\Phi_{\lambda y.p}^i, m + n + 1) = (1, m + n + 1, 0) + M(\Phi_p^i, m + n + 1)$  and

$$\begin{aligned} M(\Phi_{p_0}^i, m + n) &= (1, m + n, 0) + (0, 0, 1) + M(\Phi_p^i, m + n + \text{lv}_z(z) + 1) \\ &= (1, m + n, 1) + M(\Phi_p^i, m + n + 1). \end{aligned}$$

So that  $M(\Phi_{\lambda y.p}^i, m + n + 1) > M(\Phi_{p_0}^i, m + n)$  since  $(1, m + n + 1, 0) > (1, m + n, 1)$ .

Finally, we have:

$$\begin{aligned}
\mathsf{M}(\Phi'_{t_1}, m) &= \mathsf{M}(\Phi, m) + \sum_{i \in I} \mathsf{M}(\Phi_{p_1}^i, m + n) \\
&\leq_{L. 6.4} \mathsf{M}(\Phi, m) + \sum_{i \in I} \mathsf{M}(\Phi_{p_0}^i, m + n) \\
&< \mathsf{M}(\Phi, m) + \sum_{i \in I} \mathsf{M}(\Phi_{\lambda y.p}^i, m + n + 1) \\
&= \mathsf{M}(\Phi_{t_0}, m)
\end{aligned}$$

By Lemma 6.3, we can finally construct  $\Phi_{t_1} \triangleright \Gamma' \uplus \Delta \vdash \mathbb{L}\langle \mathbb{N}\langle x \rangle \rangle[x // \lambda y.p'] : \sigma$ , where  $\mathsf{M}(\Phi_{t_1}, m) = \mathsf{M}(\Phi'_{t_1}, m)$ .

- $t_0 = \mathbb{N}\langle x \rangle[x // v] \rightarrow_{1s} \mathbb{N}\langle v \rangle[x // v] = t_1$ . The typing derivation  $\Phi_{t_0}$  is of the form

$$\frac{\Phi \triangleright \Gamma'; x : \mathcal{M} \vdash \mathbb{N}\langle x \rangle : \sigma \quad \frac{(\Phi_v^i \triangleright \Delta_i \vdash v : \tau_i)_{i \in I}}{\Phi_v \triangleright \Delta \vdash v : \mathcal{M}} \text{ (MANY)}}{\Gamma' \uplus \Delta \vdash \mathbb{N}\langle x \rangle[x // v] : \sigma} \text{ (CUT)}$$

where  $\mathcal{M} = [\tau_i]_{i \in I}$  and  $\Delta = \uplus_{i \in I} \Delta_i$ . By Lemma 6.6 we know that there is a non-empty  $\mathcal{N} \sqsubseteq \mathcal{M}$  which types the variable  $x$  in the hole of the context  $\mathbb{N}$ . We can then write  $\mathcal{M}$  as  $\mathcal{N} \sqcup \mathcal{N}'$ . By Lemma 5.5 there are two derivations  $\Phi_{v_1} \triangleright \Delta_1 \vdash v : \mathcal{N}$  and  $\Phi_{v_2} \triangleright \Delta_2 \vdash v : \mathcal{N}'$  such that  $\Delta = \Delta_1 \uplus \Delta_2$  and  $\mathsf{M}(\Phi_v, m) = \mathsf{M}(\Phi_{v_1}, m) + \mathsf{M}(\Phi_{v_2}, m)$ . Using Lemma 6.1, we can construct:

$$\Phi_{t_1} = \frac{\Psi \triangleright \Gamma' \uplus \Delta_1; x : \mathcal{N}' \vdash \mathbb{N}\langle v \rangle : \sigma \quad \Phi_{v_2} \triangleright \Delta_2 \vdash v : \mathcal{N}'}{\Gamma' \uplus \Delta; x : \mathcal{N}' \vdash \mathbb{N}\langle v \rangle[x // v] : \sigma} \text{ (CUT)}$$

We clearly have  $\text{lv}_\diamond(\mathbb{N}) \leq \text{lv}_x(\mathbb{N}\langle x \rangle)$  and, because  $x \notin \text{fv}(v)$ , we also have  $\text{lv}_x(\mathbb{N}\langle v \rangle) \leq \text{lv}_x(\mathbb{N}\langle x \rangle)$ . Then,

$$\begin{aligned}
\mathsf{M}(\Phi_{t_1}, m) &= \mathsf{M}(\Psi, m) + \mathsf{M}(\Phi_{v_2}, m + \text{lv}_x(\mathbb{N}\langle v \rangle)) \\
&=_{L. 6.1} \mathsf{M}(\Phi, m) + \mathsf{M}(\Phi_{v_1}, m + \text{lv}_\diamond(\mathbb{N})) - (0, 0, |\mathcal{N}'|) + \mathsf{M}(\Phi_{v_2}, m + \text{lv}_x(\mathbb{N}\langle v \rangle)) \\
&\leq \mathsf{M}(\Phi, m) + \mathsf{M}(\Phi_{v_1}, m + \text{lv}_x(\mathbb{N}\langle x \rangle)) - (0, 0, |\mathcal{N}'|) + \mathsf{M}(\Phi_{v_2}, m + \text{lv}_x(\mathbb{N}\langle x \rangle)) \\
&< \mathsf{M}(\Phi, m) + \mathsf{M}(\Phi_{v_1}, m + \text{lv}_x(\mathbb{N}\langle x \rangle)) + \mathsf{M}(\Phi_{v_2}, m + \text{lv}_x(\mathbb{N}\langle x \rangle)) \\
&= \mathsf{M}(\Phi_{t_0}, m)
\end{aligned}$$

Now, we analyse all the inductive cases of the form  $t_0 = \mathbb{N}\langle t'_0 \rangle \rightarrow_{f1\text{need}} \mathbb{N}\langle t'_1 \rangle = t_1$ , where  $t'_0 \rightarrow_{f1\text{need}} t'_1$ .

- (1) If  $\mathbb{N} = \mathbb{N}'u$ , then we have  $\Phi_{t'_0} \triangleright \Gamma' \vdash \mathbb{N}'\langle t'_0 \rangle : \mathcal{N} \rightarrow \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{N}$ . By the *i.h.* there is  $\Phi_{t'_1} \triangleright \Gamma' \vdash \mathbb{N}'\langle t'_1 \rangle : \mathcal{N} \rightarrow \sigma$ , so  $\Phi_{t_1} \triangleright \Gamma' \uplus \Delta \vdash \mathbb{N}'\langle t'_1 \rangle u : \sigma$ . Moreover,  $\mathsf{M}(\Phi_{t_0}, m) = \mathsf{M}(\Phi_{t'_0}, m) + \mathsf{M}(\Phi_u, m) + (1, m, 0) >_{i.h.} \mathsf{M}(\Phi_{t'_1}, m) + \mathsf{M}(\Phi_u, m) + (1, m, 0) = \mathsf{M}(\Phi_{t_1}, m)$ .
- (2) If  $\mathbb{N} = \mathbb{N}'[x \triangleleft u]$ , then we have  $\Phi_{t'_0} \triangleright \Gamma'; x : \mathcal{M} \vdash \mathbb{N}'\langle t'_0 \rangle : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . By the *i.h.* there is  $\Phi_{t'_1} \triangleright \Gamma'; x : \mathcal{M} \vdash \mathbb{N}'\langle t'_1 \rangle : \sigma$ , so  $\Phi_{t_1} \triangleright \Gamma' \uplus \Delta \vdash \mathbb{N}'\langle t'_1 \rangle[x \triangleleft u] : \sigma$ . We distinguish three different cases:

- If  $t'_0 \rightarrow_{\text{f1need}} t'_1$  is a dB-step: then we know by the *i.h.* that  $\mathbf{M}(\Phi_{t'_0}, m) > \mathbf{M}(\Phi_{t'_1}, m)$  strictly decreases the first component of the first 3-tuple. We then have

$$\begin{aligned}
\mathbf{M}(\Phi_{t_1}, m) &= \mathbf{M}(\Phi_{t'_1}, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\mathbf{N}'\langle t'_1 \rangle) + 1) \\
&=_{L. 5.4} \mathbf{M}(\Phi_{t'_1}, m) + \mathbf{M}(\Phi_u, m) + (0, (\text{lv}_x(\mathbf{N}'\langle t'_1 \rangle) + 1) * \text{sz}(\Phi_u), 0) \\
&<_{i.h.} \mathbf{M}(\Phi_{t'_0}, m) + \mathbf{M}(\Phi_u, m) + (0, (\text{lv}_x(\mathbf{N}'\langle t'_0 \rangle) + 1) * \text{sz}(\Phi_u), 0) \\
&=_{L. 5.4} \mathbf{M}(\Phi_{t'_0}, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\mathbf{N}'\langle t'_0 \rangle) + 1) \\
&= \mathbf{M}(\Phi_{t_0}, m)
\end{aligned}$$

- If  $t'_0 \rightarrow_{\text{f1need}} t'_1$  is an spl-step, then  $t'_0 \rightarrow_{\text{sub}} t'_1$ , so that  $\mathbf{N}'\langle t'_0 \rangle \rightarrow_{\text{sub}} \mathbf{N}'\langle t'_1 \rangle$ , and thus  $\text{lv}_x(\mathbf{N}'\langle t'_0 \rangle) \geq \text{lv}_x(\mathbf{N}'\langle t'_1 \rangle)$  holds by Lemma 2.6. We then conclude by:

$$\begin{aligned}
\mathbf{M}(\Phi_{t_1}, m) &= \mathbf{M}(\Phi_{t'_1}, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\mathbf{N}'\langle t'_1 \rangle) + 1) \\
&<_{i.h.} \mathbf{M}(\Phi_{t'_0}, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\mathbf{N}'\langle t'_1 \rangle) + 1) \\
&\leq \mathbf{M}(\Phi_{t'_0}, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\mathbf{N}'\langle t'_0 \rangle) + 1) \\
&= \mathbf{M}(\Phi_{t_0}, m)
\end{aligned}$$

- If  $t'_0 \rightarrow_{\text{f1need}} t'_1$  is a ls-step, then we know that  $\mathbf{N}'\langle t'_0 \rangle \rightarrow_{\text{f1need}} \mathbf{N}'\langle t'_1 \rangle$  also holds, then  $\text{lv}_x(\mathbf{N}'\langle t'_0 \rangle) \geq \text{lv}_x(\mathbf{N}'\langle t'_1 \rangle)$ . We conclude as before.
- (3) If  $\mathbf{N} = \mathbf{N}_1\langle\langle x \rangle\rangle[x/\mathbf{N}_2]$ , then we have  $\Phi_1 \triangleright \Delta; x : \mathcal{M} \vdash \mathbf{N}_1\langle\langle x \rangle\rangle : \sigma$  and  $\Phi_{t'_0} \triangleright \Gamma' \vdash \mathbf{N}_2\langle t'_0 \rangle : \mathcal{M}$ . By the *i.h.* there is  $\Phi_{t'_1} \triangleright \Gamma' \vdash \mathbf{N}_2\langle t'_1 \rangle : \mathcal{M}$ , so  $\Phi_{t_1} \triangleright \Gamma' \uplus \Delta \vdash \mathbf{N}_1\langle\langle x \rangle\rangle[x \triangleleft \mathbf{N}_2\langle t'_1 \rangle] : \sigma$ . Moreover,  $\mathbf{M}(\Phi_{t_0}, m) = \mathbf{M}(\Phi_1, m) + \mathbf{M}(\Phi_{t'_0}, m + \text{lv}_x(\mathbf{N}_1\langle\langle x \rangle\rangle) + \text{ES}([x \triangleleft \mathbf{N}_2\langle t'_0 \rangle])) >_{i.h.} \mathbf{M}(\Phi_1, m) + \mathbf{M}(\Phi_{t'_1}, m + \text{lv}_x(\mathbf{N}_1\langle\langle x \rangle\rangle) + \text{ES}([x \triangleleft \mathbf{N}_2\langle t'_1 \rangle])) = \mathbf{M}(\Phi_{t_1}, m)$ .  $\square$

**Example 6.8.** Consider the following reduction sequence:

$$(\mathbf{I}(x_1\mathbf{I}))[x_1/\lambda y.\mathbf{I}y] \rightarrow_{\text{dB}} x_2[x_2/x_1\mathbf{I}][x_1/\lambda y.\mathbf{I}y] \rightarrow_{\text{spl}} x_2[x_2/x_1\mathbf{I}][x_1//\lambda y.zy][z/\mathbf{I}]$$

We have  $\Phi_1 \triangleright \emptyset \vdash (\mathbf{I}(x_1\mathbf{I}))[x_1/\lambda y.\mathbf{I}y] : \mathbf{a}$  with  $\Phi_1$  of the form

$$\frac{\frac{\Phi_{\mathbf{I}}}{x_1 : [[\mathbf{a}] \rightarrow \mathbf{a}] \vdash \mathbf{I}(x_1\mathbf{I}) : \mathbf{a}} \quad \frac{\Phi_{x_1\mathbf{I}}}{\emptyset \vdash \lambda y.\mathbf{I}y : [\mathbf{a}] \rightarrow \mathbf{a}} \text{ (APP)}}{\emptyset \vdash (\mathbf{I}(x_1\mathbf{I}))[x_1/\lambda y.\mathbf{I}y] : \mathbf{a}} \text{ (CUT)} \quad \frac{\frac{\frac{\frac{\frac{\Phi_{\mathbf{I}}}{y : [\mathbf{a}] \vdash y : \mathbf{a}} \text{ (AX)}}{y : [\mathbf{a}] \vdash y : [\mathbf{a}]} \text{ (MANY)}}{y : [\mathbf{a}] \vdash \mathbf{I}y : \mathbf{a}} \text{ (APP)}}{\emptyset \vdash \lambda y.\mathbf{I}y : [\mathbf{a}] \rightarrow \mathbf{a}} \text{ (ABS)}}{\emptyset \vdash \lambda y.\mathbf{I}y : [[\mathbf{a}] \rightarrow \mathbf{a}]} \text{ (MANY)}}{\emptyset \vdash (\mathbf{I}(x_1\mathbf{I}))[x_1//\lambda y.zy][z/\mathbf{I}] : \mathbf{a}} \text{ (CUT)}$$

where

$$\Phi_{\mathbf{I}} = \frac{\frac{\Phi_{\mathbf{I}}}{x : [\mathbf{a}] \vdash x : \mathbf{a}} \text{ (AX)}}{\emptyset \vdash \mathbf{I} : [\mathbf{a}] \rightarrow \mathbf{a}} \text{ (ABS)}$$



contradiction because the order  $>$  on 3-tuples  $D(\cdot)$  is well-founded. Then  $t$  is necessarily name-normalizing.  $\square$

**Theorem 6.10** (Typability implies `flneed`-Normalization). *Let  $\Phi_t \triangleright \Gamma \vdash t : \sigma$ . Then  $t$  is `flneed`-normalizing. Moreover, the first element of  $D(\Phi_t)$  is an upper bound for the number of `dB`-steps to `flneed`-nf.*

*Proof.* The property trivially holds by Lemma 6.7 since the lexicographic order on 3-tuples is well-founded.  $\square$

**Completeness.** We address here completeness of system  $\cap R$  with respect to  $\rightarrow_{\text{name}}$  and  $\rightarrow_{\text{flneed}}$ . More precisely, we show that normalizing terms in each strategy are typable. The basic property in showing that consists in guaranteeing that normal forms are typable.

**Lemma 6.11** (`flneed`-nfs are Typable). *Let  $t$  be in `flneed`-nf. Then there exists a derivation  $\Phi \triangleright \Gamma \vdash t : \tau$  such that for any  $x \notin \text{ndv}(t)$ ,  $\Gamma(x) = []$ .*

*Proof.* By Lemma 4.17 we can reason by induction on the grammar `Ne` (see appendix on page 63).  $\square$

**Example 6.12.** Remember that  $\text{ndv}((xy_1)[x/z]y_1) = \{z\}$  and note that  $\text{ndv}(xy_1) = \{x\}$ .

$$\frac{\frac{\overline{x : [] \rightarrow \tau \vdash x : [] \rightarrow \tau} \quad \overline{\emptyset \vdash y_1 : []}}{x : [] \rightarrow \tau \vdash xy_1 : \tau} \quad \frac{\overline{z : [] \rightarrow [] \rightarrow \tau \vdash z : [] \rightarrow [] \rightarrow \tau} \quad \overline{\emptyset \vdash y_2 : []}}{z : [] \rightarrow [] \rightarrow \tau \vdash zy_2 : [] \rightarrow \tau}}{z : [] \rightarrow [] \rightarrow \tau \vdash (xy_1)[x/zy_2] : \tau}$$

Because `name`-nfs are also `flneed`-nfs, we infer the following corollary for free.

**Corollary 6.13** (`name`-nfs are Typable). *Let  $t$  be in `name`-nf. Then there is a derivation  $\Phi \triangleright \Gamma \vdash t : \tau$ .*

We need lemmas stating the behavior of partial and full (anti-)substitution w.r.t. typing.

**Lemma 6.14** (Partial Anti-Substitution). *Let  $C\langle\langle x \rangle\rangle$  and  $u$  be terms s.t.  $x \notin \text{fv}(u)$  and  $\Phi \triangleright \Gamma \vdash C\langle\langle u \rangle\rangle : \sigma$ . Then  $\exists \Gamma', \exists \Delta, \exists \mathcal{M}, \exists \Phi', \exists \Phi_u$  s.t.  $\Gamma = \Gamma' \uplus \Delta$ ,  $\Phi' \triangleright \Gamma' \uplus x : \mathcal{M} \vdash C\langle\langle x \rangle\rangle : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ .*

*Proof.* By induction on `C` (see appendix on page 63).  $\square$

**Corollary 6.15** (Anti-Substitution). *Let  $u$  be a term s.t.  $x \notin \text{fv}(u)$  and  $\Phi \triangleright \Gamma \vdash t\{x/u\} : \sigma$ . Then  $\exists \Gamma', \exists \Delta, \exists \mathcal{M}, \exists \Phi', \exists \Phi_u$  s.t.  $\Gamma = \Gamma' \uplus \Delta$ ,  $\Phi' \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ .*

*Proof.* The proof is by induction on  $|t|_x$ .

- If  $|t|_x = 0$  then  $t\{x/u\} = t$  and, by Lemma 5.1,  $x \notin \text{dom}(\Gamma)$  then  $\Gamma = \Gamma; x : []$ . Therefore, for  $\Gamma' := \Gamma$ ,  $\Delta := \emptyset$ ,  $\mathcal{M} = []$ ,  $\Phi' := \Phi$  and  $\Phi_u := \frac{}{\vdash u : []}$  the result holds.
- If  $|t|_x \geq 1$  then let  $C\langle\langle x \rangle\rangle$  such that  $t\{x/u\} = C\langle\langle u \rangle\rangle$ . For any fresh  $y$ , we have that  $t\{x/u\} = C\langle\langle y \rangle\rangle\{y/u\}$  where  $C\langle\langle y \rangle\rangle = t'\{x/u\}$  s.t.  $t = t'\{y/x\}$ . Note that  $|t'|_x < |t|_x$ . Then by Lemma 6.14  $\exists \Gamma'', \exists \Delta', \exists \mathcal{N}, \exists \Phi'', \exists \Phi'_u$  s.t.  $\Gamma = \Gamma'' \uplus \Delta'$ ,  $\Phi'' \triangleright \Gamma'' \uplus y : \mathcal{N} \vdash C\langle\langle y \rangle\rangle : \sigma$  and  $\Phi'_u \triangleright \Delta' \vdash u : \mathcal{N}$  where, by freshness of  $y$ ,  $\Gamma'' \uplus y : \mathcal{N} = \Gamma''; y : \mathcal{N}$ . Therefore, by the *i.h.* on  $\Phi''$   $\exists \Gamma''', \exists \Delta'', \exists \mathcal{N}', \exists \Phi''', \exists \Phi''_u$  s.t.  $\Gamma''; y : \mathcal{N} = \Gamma''' \uplus \Delta''$ ,  $\Phi''' \triangleright \Gamma'''; x : \mathcal{N}' \vdash t' : \sigma$  and  $\Phi''_u \triangleright \Delta'' \vdash u : \mathcal{N}'$ . By freshness of  $y$  and relevance, we have  $y \notin \text{dom}(\Delta'')$ . Then  $\Gamma''' = \Gamma^{iv}; y : \mathcal{N}$  where  $\Gamma'' = \Gamma^{iv} \uplus \Delta''$ . From  $\Phi'''$  and Lemma 6.1 we have  $\Phi' \triangleright (\Gamma^{iv}; x :$

$\mathcal{N}' \uplus x : \mathcal{N} \vdash t : \sigma$  while from  $\Phi'_u$  and  $\Phi''_u$  we obtain  $\Phi_u \triangleright \Delta' \uplus \Delta'' \vdash u : \mathcal{N} \sqcup \mathcal{N}'$ . Finally, for  $\Gamma' := \Gamma^{iv}$ ,  $\Delta := \Delta' \uplus \Delta''$ ,  $\mathcal{M} = \mathcal{N} \sqcup \mathcal{N}'$  the result holds, since  $(\Gamma^{iv}; x : \mathcal{N}') \uplus x : \mathcal{N} = \Gamma'; x : \mathcal{M}$  and  $\Gamma' \uplus \Delta = \Gamma^{iv} \uplus \Delta' \uplus \Delta'' = \Gamma'' \uplus \Delta' = \Gamma$ .  $\square$

To achieve completeness, we show that typing is preserved by anti-reduction.

**Lemma 6.16** (Subject Expansion). *Let  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$ . If  $t_0 \rightarrow_{\mathbf{r}} t_1$ , where  $\mathbf{r} \in \{\pi, \text{sub}, \text{ndB}, \text{flneed}\}$ , then there exists  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$ .*

*Proof.* The proof is by induction on  $\rightarrow_{\mathbf{r}}$  and uses Lemma 6.14 and Corollary 6.15. We detail some interesting cases of the proof. In all the cases shown, we suppose that the list context  $\mathbf{L}$  of the general rule is empty ( $\mathbf{L} = \diamond$ ), since we can use subject expansion for  $\rightarrow_{\pi}$  to manipulate it.

- $t_0 = t[x/us] \mapsto_{\text{sub}} t\{x/yz\}[y/u][z/s] = t_1$ . Then  $\Phi_{t_1}$  is of the form

$$\frac{\frac{\Phi \triangleright \Gamma'; z : \mathcal{N}_s; y : \mathcal{N}_u \vdash t\{x/yz\} : \sigma \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{N}_u}{(\Gamma' \uplus \Delta_u); z : \mathcal{N}_s \vdash t\{x/yz\}[y/u] : \sigma} \quad \Phi_s \triangleright \Delta_s \vdash s : \mathcal{N}_s}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t\{x/yz\}[y/u][z/s] : \sigma}$$

where  $\Gamma = \Gamma' \uplus \Delta_u \uplus \Delta_s$ . Also  $(\Gamma'; z : \mathcal{N}_s) \uplus \Delta_u = (\Gamma' \uplus \Delta_u); z : \mathcal{N}_s$  since  $z \notin \text{dom}(\Delta_u)$  by the Relevance Lemma 5.1. By Corollary 6.15  $\exists \Gamma'', \exists \Delta, \exists \mathcal{M}, \exists \Phi', \exists \Phi_{yz}$  s.t.  $\Gamma'; z : \mathcal{N}_s; y : \mathcal{N}_u = \Gamma'' \uplus \Delta$ ,  $\Phi' \triangleright \Gamma''; x : \mathcal{M} \vdash t : \sigma$  and  $\Phi_{yz} \triangleright \Delta \vdash yz : \mathcal{M}$ . By freshness of  $y, z$  and Lemma 5.1 we have that  $y, z \notin \text{dom}(\Gamma'') \cup \{x\}$ . Then  $\Gamma'' = \Gamma'$  and  $\Delta = z : \mathcal{N}_s; y : \mathcal{N}_u$ . From  $\Phi_{yz}$ ,  $\Phi_u$ ,  $\Phi_s$  and Lemma 6.1 we obtain  $\Phi_{us} \triangleright \Delta_u \uplus \Delta_s \vdash us : \mathcal{M}$  and construct  $\Phi_{t_0}$  as:

$$\Phi_{t_0} = \frac{\Phi' \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_{us} \triangleright \Delta_u \uplus \Delta_s \vdash us : \mathcal{M}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x/us] : \sigma}$$

- If  $t_0 = (\lambda x.t)u \rightarrow_{\text{dB}} t[x/u] = t_1$ . Then  $\Phi_{t_1}$  is of the form

$$\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_u \triangleright \Gamma_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Gamma_u \vdash t[x/u] : \sigma} \text{ (CUT)}$$

Therefore, we construct  $\Phi_{t_0}$  as follows:

$$\frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma}{\Gamma' \vdash \lambda x.t : \mathcal{M} \rightarrow \sigma} \text{ (ABS)} \quad \Phi_u \triangleright \Gamma_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Gamma_u \vdash (\lambda x.t)u : \sigma} \text{ (APP)}$$

- $t_0 = \mathbb{N}\langle\langle x \rangle\rangle[x/\lambda y.p] \rightarrow_{\text{sp1}} \text{LL}\langle\langle x \rangle\rangle[x/\lambda y.p'] = t_1$ , where  $\lambda y.z[z/p] \Downarrow_{\text{st}} \lambda y.\text{LL}\langle p' \rangle$ . By subject expansion for  $\rightarrow_{\pi}$ , there is  $\Phi_{t'_1} \triangleright \Gamma \vdash \mathbb{N}\langle\langle x \rangle\rangle[x/\lambda y.\text{LL}\langle p' \rangle] : \sigma$  and it is of the form

$$\frac{\Phi \triangleright \Gamma'; x : \mathcal{N} \vdash \mathbb{N}\langle\langle x \rangle\rangle : \sigma \quad \frac{(\Phi_i \triangleright \Delta_i \vdash \lambda y.\text{LL}\langle p' \rangle : \sigma_i)_{i \in I}}{\Delta \vdash \lambda y.\text{LL}\langle p' \rangle : \mathcal{N}} \text{ (MANY)}}{\Gamma' \uplus \Delta \vdash \mathbb{N}\langle\langle x \rangle\rangle[x/\lambda y.\text{LL}\langle p' \rangle] : \sigma} \text{ (CUT)}$$

where  $\Delta = \uplus_{i \in I} \Delta_i$  and  $\mathcal{N} = [\sigma_i]_{i \in I}$  where, by Lemma 6.6,  $\mathcal{N} \neq []$ . Then, for each  $i \in I$  we have by subject expansion for  $\rightarrow_{\text{sub}}$  (of which  $\rightarrow_{\text{st}}$  is a subrelation) that  $\Phi'_i \triangleright \Delta_i \vdash \lambda y.z[z/p] : \sigma_i$  which has two different shapes, depending on  $\sigma_i$ .

(1) If  $\sigma_i = \mathcal{M}_i \rightarrow \tau_i$  then  $\Phi'_i$  is of the form

$$\frac{\frac{\overline{z : [\tau_i] \vdash z : \tau_i} \text{ (AX)} \quad \Phi'_p \triangleright \Delta_i; y : \mathcal{M}_i \vdash p : \tau_i}{\Delta_i; y : \mathcal{M}_i \vdash z[z/p] : \tau_i} \text{ (CUT)}}{\Delta_i \vdash \lambda y. z[z/p] : \mathcal{M}_i \rightarrow \tau_i} \text{ (ABS)}$$

Therefore we have  $\Psi_i$  of the form

$$\frac{\Phi'_p \triangleright \Delta_i; y : \mathcal{M}_i \vdash p : \tau_i}{\Delta_i \vdash \lambda y. p : \mathcal{M}_i \rightarrow \tau_i} \text{ (ABS)}$$

(2) If  $\sigma_i = \mathbf{a}$  then  $\Delta_i = \emptyset$  and we obtain  $\Psi_i$  of the form  $\frac{}{\vdash \lambda y. p : \mathbf{a}} \text{ (ANS)}$ .

We can then construct  $\Phi_{t_0}$  as follows

$$\frac{\Phi \triangleright \Gamma'; x : \mathcal{N} \vdash \mathbf{N}\langle\langle x \rangle\rangle : \sigma \quad \frac{(\Psi_i \triangleright \Delta_i \vdash \lambda y. p : \sigma_i)_{i \in I}}{\Delta \vdash \lambda y. p : \mathcal{N}} \text{ (MANY)}}{\Gamma' \uplus \Delta \vdash \mathbf{N}\langle\langle x \rangle\rangle[x/\lambda y. p] : \sigma} \text{ (CUT)}$$

•  $t_0 = \mathbf{N}\langle\langle x \rangle\rangle[x//v] \rightarrow_{1s} \mathbf{N}\langle\langle v \rangle\rangle[x//v] = t_1$ . Then  $\Phi_{t_1}$  is of the form

$$\frac{\Phi \triangleright \Gamma'; x : \mathcal{N}' \vdash \mathbf{N}\langle\langle v \rangle\rangle : \sigma \quad \Phi'_v \triangleright \Delta' \vdash v : \mathcal{N}'}{\Gamma' \uplus \Delta' \vdash \mathbf{N}\langle\langle v \rangle\rangle[x//v] : \sigma} \text{ (CUT)}$$

By Lemma 6.14  $\exists \Gamma'', \exists \Delta'', \exists \mathcal{N}'', \exists \Phi', \exists \Phi'_v$  s.t.  $\Gamma'; x : \mathcal{N} = \Gamma'' \uplus \Delta', \Phi' \triangleright \Gamma'' \uplus x : \mathcal{N}'' \vdash \mathbf{N}\langle\langle x \rangle\rangle : \sigma$  and  $\Phi'_v \triangleright \Delta'' \vdash v : \mathcal{N}''$ . From  $x \notin \text{fv}(v)$  and the Relevance Lemma 5.1 we have that  $x \notin \text{dom}(\Delta'')$ . Thus  $\Gamma'' = \Gamma'''; x : \mathcal{N}'$  and then  $\Gamma'' \uplus x : \mathcal{N}' = \Gamma'''; x : \mathcal{N}$  where  $\mathcal{N} = \mathcal{N}' \sqcup \mathcal{N}''$ . From  $\Phi'_v$  and  $\Phi''_v$  derivations we obtain  $\Phi_v \triangleright \Delta \vdash v : \mathcal{N}$ , where  $\Delta = \Delta' \uplus \Delta''$ . Then  $\Phi_{t_0}$  is of the form

$$\frac{\Phi' \triangleright \Gamma'''; x : \mathcal{N} \vdash \mathbf{N}\langle\langle x \rangle\rangle : \sigma \quad \Phi_v \triangleright \Delta \vdash v : \mathcal{N}}{\Gamma''' \uplus \Delta \vdash \mathbf{N}\langle\langle x \rangle\rangle[x//v] : \sigma} \text{ (CUT)}$$

where  $\Gamma''' \uplus \Delta = \Gamma' \uplus \Delta'$ . □

**Theorem 6.17** (name-Normalization implies Typability). *Let  $t$  be a term. If  $t$  is name-normalizing, then  $t$  is  $\cap R$ -typable.*

*Proof.* Let  $t$  be name-normalizing. Then  $t \rightarrow_{\text{name}}^n u$  and  $u$  is a name-nf. We reason by induction on  $n$ . If  $n = 0$ , then  $t = u$  is typable by Corollary 6.13. Otherwise, we have  $t \rightarrow_{\text{name}} t' \rightarrow_{\text{name}}^{n-1} u$ . By the *i.h.*  $t'$  is typable and thus by Lemma 6.16 (because  $\rightarrow_{\text{nsub}}$  is included in  $\rightarrow_{\text{sub}}$ ),  $t$  turns out to be also typable. □

**Theorem 6.18** (flneed-Normalization implies Typability). *Let  $t$  be a term. If  $t$  is flneed-normalizing, then  $t$  is  $\cap R$ -typable.*

*Proof.* Similar to the previous proof but using Lemma 6.11 instead of Corollary 6.13. □

Summing up, Theorems 6.9, 6.17, 6.10 and 6.18 give:

**Theorem 6.19.**  $t \in \mathbf{T}_R$  is name-normalizing iff  $t$  is flneed-normalizing iff  $t$  is  $\cap R$ -typable.

All the technical tools are now available to conclude observational equivalence between our two evaluation strategies based on node replication. Let  $\mathcal{R}$  be any reduction notion on  $\mathbf{T}_R$ . Then, two terms  $t, u \in \mathbf{T}_R$  are said to be  $\mathcal{R}$ -**observationally equivalent**, written  $t \equiv_{\mathcal{R}} u$ , if for any context  $\mathbf{C}$ ,  $\mathbf{C}\langle t \rangle$  is  $\mathcal{R}$ -normalizing iff  $\mathbf{C}\langle u \rangle$  is  $\mathcal{R}$ -normalizing.

**Theorem 6.20.** *For all terms  $t, u \in \mathsf{T}_R$ ,  $t$  and  $u$  are **name-observationally equivalent** iff  $t$  and  $u$  are **flneed-observationally equivalent**.*

*Proof.* The proof uses Theorem 6.19. Indeed, we have  $t \equiv_{\text{name}} u$  iff  $(\mathsf{C}\langle t \rangle)$  is **name-normalizing** iff  $\mathsf{C}\langle u \rangle$  is **name-normalizing** for any context  $\mathsf{C}$ ) iff  $(\mathsf{C}\langle t \rangle)$  is **flneed-normalizing** iff  $\mathsf{C}\langle u \rangle$  is **flneed-normalizing** for any context  $\mathsf{C}$ ) iff  $t \equiv_{\text{flneed}} u$ .  $\square$

## 7. RELATED WORKS AND CONCLUSION

Several calculi with explicit substitutions (ES) bridge the gap between formal higher-order calculi and concrete implementations of programming languages (see a survey in [Kes07]). The first of such calculi, *e.g.* [ACCL90, BR95], were all based on *structural* substitution, in the sense that the ES operator is syntactically propagated step-by-step through the term structure until a variable is reached, when the substitution finally takes place. The correspondence between ES and Linear Logic Proof-Nets [DCKP03] led to the more recent notion of calculi *at a distance* [AK10, ABKL14, Acc18a], enlightening a natural and new application of the Curry-Howard interpretation. These calculi implement linear/partial substitution *at a distance*, where the search of variable occurrences is abstracted out with context-based rewriting rules, and thus no ES propagation rules are necessary. A third model was introduced by the seminal work of Gundersen, Heijltjes, and Parigot [GHP13a, GHP13b], introducing the atomic  $\lambda$ -calculus to implement node replication.

Inspired by the last approach we introduced the calculus  $\lambda R$ , capturing the essence of node replication. In contrast to [GHP13a], we work with an implicit (structural) mechanism of weakening and contraction, a design choice which aims at focusing and highlighting the node replication model, which is the core of our calculus, so that we obtain a rather simple and natural formalism used in particular to specify evaluation strategies. Indeed, besides the proof of the main operational meta-level properties of our calculus (confluence, termination of the substitution calculus, simulations), we use linear and non-linear versions of  $\lambda R$  to specify evaluation strategies based on node replication, namely call-by-name and call-by-need evaluation strategies. In particular, we provided simple tools to prove correctness of these reduction strategies. This was achieved in the framework of our concise calculus  $\lambda R$ , based not only on an implicit treatment of weakening and contraction, but also on the notion of commuting conversions by means of *distance*. Indeed, the treatment of weakening, contraction, and commuting conversions result in a heavy machinery for the atomic  $\lambda$ -calculus, which would make the correctness of these strategies much more involved.

Moreover, characterisation of termination of different strategies based on  $\lambda R$  were achieved with a rather standard type system and, surprisingly, no deep inference system was necessary at this point. This is of interest, since our type system is equipping full-laziness with a well-known denotational semantics. We think that this would be difficult to achieve in the framework of the atomic  $\lambda$ -calculus.

The first description of call-by-need was given by Wadsworth [Wad71], where reduction is performed on *graphs* instead of terms. Weak call-by-need on *terms* was then introduced by Ariola and Felleisen [AF97], and by Maraist, Odersky and Wadler [MOW98, MOTW99]. Reformulations were introduced by Accattoli, Barenbaum and Mazza [ABM14] and by Chang and Felleisen [CF12]. Our call-by-need strategy is inspired by the calculus in [ABM14], which uses the distance paradigm [AK10] to gather together meaningful and permutation



rules, by clearly separating *multiplicative* from *exponential* rules, in the sense of Linear Logic [Gir87].

Full laziness has been formalized in different ways. Pointer graphs [Wad71, SW10] are DAGs allowing for an elegant representation of sharing. Labeled calculi [Lév78, BLM05] implement pointer graphs by adding annotations to  $\lambda$ -terms, which makes the syntax more difficult to handle. Lambda-lifting [Hug83, Jon87] implements full laziness by resorting to translations from  $\lambda$ -terms to supercombinators. In contrast to all the previous formalisms, our calculus is defined on standard  $\lambda$ -terms with explicit cuts, without the use of any complementary syntactical tool. So is Ariola and Felleisen’s call-by-need [AF97], however, their notion of full laziness relies on external (ad-hoc) meta-level operations used to extract the skeleton. Our specification of call-by-need enables fully lazy sharing, where the skeleton extraction operation is internally encoded in the term calculus operational semantics. Last but not least, our calculus has strong links with proof-theory, notably deep inference.

Balabonski [Bal12b, Bal12a] relates many formalisms of full laziness and shows that they are equivalent when considering the number of  $\beta$ -steps to a normal form. It would then be interesting to understand if his unified approach, (abstractly) stated by means of the theory of residuals [Lév78, Lév80], applies to our own strategy.

We have also studied the calculus from a semantical point of view, by means of intersection types. Indeed, the type system can be seen as a model of our implementations of call-by-name and call-by-need, in the sense that typability and normalization turn out to be equivalent.

Intersection types go back to [CD78] and have been used to provide characterizations of qualitative [BDS13] as well as quantitative [dC07] models of the  $\lambda$ -calculus, where typability and normalization coincide. Quantitative models specified by means of non-idempotent types [Gar94, Kfo00] were first applied to the  $\lambda$ -calculus (see a survey in [BKV17]) and to several other formalisms ever since, such as call-by-value [Ehr12, CG14], call-by-need [Kes16, AGL19], call-by-push-value [GM18, BKRV20] and classical logic [KV20]. In the present work, we achieve for the first time a quantitative characterization of fully lazy normalization, which provides upper bounds for the length of reduction sequences to normal forms.

Characterizations provided by intersection type systems sometimes lead to observational equivalence results (*e.g.* [Kes16]). In this work we succeed to prove observational equivalence related to a fully lazy implementation of weak call-by-need, a result which would be extremely involved to prove by means of syntactical tools of rewriting, as done for weak call-by-need in [AF97]. Moreover, our result implies that our node replication implementation of full laziness is observationally equivalent to standard call-by-name and to weak call-by-need (see [Kes16]), as well as to the more semantical notion of neededness (see [KRV18]).

A Curry-Howard interpretation of the logical *switch* rule of deep inference is given in [She19, SHGP20] as an end-of-scope operator, thus introducing the *spinal atomic*  $\lambda$ -calculus. The calculus implements a refined optimization of call-by-need, where only the *spine* of the abstraction (tighter than the skeleton) is duplicated. It would be interesting to adapt  $\lambda R$  to spine duplication by means of an appropriate end-of-scope operator, such as the one in [HvO03]. Further optimizations might also be considered.

Finally, this paper only considers weak evaluation strategies, *i.e.* with reductions forbidden under abstractions, but it would be interesting to extend our notions to full (strong) evaluations too [GL02, BBBK17]. Extending full laziness to classical logic would be another interesting research direction, possibly taking preliminary ideas from [He18].

We would also like to investigate (quantitative) *tight* types for our fully lazy strategy, as done for weak call-by-need in [AGL19], which does not seem evident in our node replication framework.

#### ACKNOWLEDGMENT

Last author was partially supported by CNPq Universal 430667/2016-7 grant.

#### REFERENCES

- [ABKL14] Beniamino Accattoli, Eduardo Bonelli, Delia Kesner, and Carlos Lombardi. A nonstandard standardization theorem. In *POPL*, pages 659–670. ACM, 2014.
- [ABM14] Beniamino Accattoli, Pablo Barenbaum, and Damiano Mazza. Distilling abstract machines. In *ICFP*, pages 363–376. ACM, 2014.
- [Acc18a] Beniamino Accattoli. Proof nets and the linear substitution calculus. In *ICTAC*, volume 11187 of *Lecture Notes in Computer Science*, pages 37–61. Springer, 2018.
- [Acc18b] Beniamino Accattoli. Proof nets and the linear substitution calculus. In *Theoretical Aspects of Computing – ICTAC 2018*, pages 37–61. Springer International Publishing, 2018. doi:10.1007/978-3-030-02508-3\_3.
- [ACCL90] Martín Abadi, Luca Cardelli, Pierre-Louis Curien, and Jean-Jacques Lévy. Explicit substitutions. In *POPL*, pages 31–46. ACM Press, 1990.
- [AF97] Zena M. Ariola and Matthias Felleisen. The call-by-need lambda calculus. *J. Funct. Program.*, 7(3):265–301, 1997.
- [AGL19] Beniamino Accattoli, Giulio Guerrieri, and Maico Leberle. Types by need. In *ESOP*, volume 11423 of *Lecture Notes in Computer Science*, pages 410–439. Springer, 2019.
- [AK10] Beniamino Accattoli and Delia Kesner. The structural lambda-calculus. In *CSL*, volume 6247 of *Lecture Notes in Computer Science*, pages 381–395. Springer, 2010.
- [Bal12a] Thibaut Balabonski. *La plein paresse, une certain optimalité : partage de sous-termes et stratégies de réduction en réécriture d’ordre supérieur*. PhD thesis, Paris 7, 2012. URL: <http://www.theses.fr/2012PA077198>.
- [Bal12b] Thibaut Balabonski. A unified approach to fully lazy sharing. In *POPL*, pages 469–480. ACM, 2012.
- [Bal13] Thibaut Balabonski. Weak optimality, and the meaning of sharing. In *ICFP*, pages 263–274. ACM, 2013.
- [Bar85] Hendrik Pieter Barendregt. *The lambda calculus - its syntax and semantics*, volume 103 of *Studies in logic and the foundations of mathematics*. North-Holland, 1985.
- [BBBK17] Thibaut Balabonski, Pablo Barenbaum, Eduardo Bonelli, and Delia Kesner. Foundations of strong call by need. *Proc. ACM Program. Lang.*, 1(ICFP):20:1–20:29, 2017.
- [BCDC83] Henk Barendregt, Mario Coppo, and Mariangiola Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *Bulletin of Symbolic Logic*, 48:931–940, 1983.
- [BDS13] Hendrik Pieter Barendregt, Wil Dekkers, and Richard Statman. *Lambda Calculus with Types*. Perspectives in logic. Cambridge University Press, 2013.
- [BE01] Antonio Bucciarelli and Thomas Ehrhard. On phase semantics and denotational semantics: the exponentials. *Ann. Pure Appl. Log.*, 109(3):205–241, 2001.
- [BKR20] Antonio Bucciarelli, Delia Kesner, Alejandro Ríos, and Andrés Viso. The bang calculus revisited. In *FLOPS*, volume 12073 of *Lecture Notes in Computer Science*, pages 13–32. Springer, 2020.
- [BKV17] Antonio Bucciarelli, Delia Kesner, and Daniel Ventura. Non-idempotent intersection types for the lambda-calculus. *Log. J. IGPL*, 25(4):431–464, 2017.
- [BLM05] Tomasz Blanc, Jean-Jacques Lévy, and Luc Maranget. Sharing in the weak lambda-calculus. In *Processes, Terms and Cycles*, volume 3838 of *Lecture Notes in Computer Science*, pages 70–87. Springer, 2005.
- [BN98] Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge University Press, 1998.

- [BR95] Roel Bloo and Kristoffer Rose. Preservation of strong normalization in named lambda calculi with explicit substitution and garbage collection. In *Computing Science in the Netherlands*, pages 62–72. Netherlands Computer Science Research Foundation, 1995.
- [CD78] Mario Coppo and Mariangiola Dezani-Ciancaglini. A new type assignment for  $\lambda$ -terms. *Arch. Math. Log.*, 19(1):139–156, 1978.
- [CDCV81] Mario Coppo, Mariangiola Dezani-Ciancaglini, and Betti Venneri. Functional characters of solvable terms. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 27:45–58, 1981.
- [CF12] Stephen Chang and Matthias Felleisen. The call-by-need lambda calculus, revisited. In *ESOP*, volume 7211 of *Lecture Notes in Computer Science*, pages 128–147. Springer, 2012.
- [CG14] Alberto Carraro and Giulio Guerrieri. A semantical and operational account of call-by-value solvability. In *FoSSaCS*, volume 8412 of *Lecture Notes in Computer Science*, pages 103–118. Springer, 2014.
- [dC07] Daniel de Carvalho. *Sémantiques de la logique linéaire et temps de calcul*. PhD thesis, Université Aix-Marseille II, 2007. URL: <https://www.theses.fr/2007AIX22066>.
- [DCKP03] Roberto Di Cosmo, Delia Kesner, and Emmanuel Polonowski. Proof nets and explicit substitutions. *Math. Struct. Comput. Sci.*, 13(3):409–450, 2003.
- [Ehr12] Thomas Ehrhard. Collapsing non-idempotent intersection types. In *CSL*, volume 16 of *LIPICs*, pages 259–273. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2012.
- [Gar94] Philippa Gardner. Discovering needed reductions using type theory. In *TACS*, volume 789 of *Lecture Notes in Computer Science*, pages 555–574. Springer, 1994.
- [GGP10] Alessio Guglielmi, Tom Gundersen, and Michel Parigot. A proof calculus which reduces syntactic bureaucracy. In *RTA*, volume 6 of *LIPICs*, pages 135–150. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010.
- [GHP13a] Tom Gundersen, Willem Heijltjes, and Michel Parigot. Atomic lambda calculus: A typed lambda-calculus with explicit sharing. In *LICS*, pages 311–320. IEEE Computer Society, 2013.
- [GHP13b] Tom Gundersen, Willem Heijltjes, and Michel Parigot. A proof of strong normalisation of the typed atomic lambda-calculus. In *LPAR*, volume 8312 of *Lecture Notes in Computer Science*, pages 340–354. Springer, 2013.
- [GHP21] Giulio Guerrieri, Willem B. Heijltjes, and Joseph W.N. Paulus. A deep quantitative type system. In Christel Baier and Jean Goubault-Larrecq, editors, *29th EACSL Annual Conference on Computer Science Logic (CSL 2021)*, volume 183 of *Leibniz International Proceedings in Informatics (LIPICs)*, pages 24:1–24:24, Dagstuhl, Germany, 2021. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. URL: <https://drops.dagstuhl.de/opus/volltexte/2021/13458>, doi: 10.4230/LIPICs.CSL.2021.24.
- [Gir87] Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.
- [Gir96] Jean-Yves Girard. *Proof-nets: The parallel syntax for proof-theory*, volume 180 of *Lecture Notes in Pure Applied Mathematics*, page 97–124. Marcel Dekker, 1996.
- [GL02] Benjamin Grégoire and Xavier Leroy. A compiled implementation of strong reduction. In *ICFP*, pages 235–246. ACM, 2002.
- [GM18] Giulio Guerrieri and Giulio Manzonetto. The bang calculus and the two girard’s translations. In *Linearity-TLLA@FLoC*, volume 292 of *EPTCS*, pages 15–30, 2018.
- [He18] Fanny He. *The Atomic Lambda-Mu Calculus*. PhD thesis, University of Bath, January 2018.
- [Hug83] John Hughes. The design and implementation of programming languages. Technical Report PRG-40, Ouel, July 1983.
- [HvO03] Dimitri Hendriks and Vincent van Oostrom.  $\lambda$ . In *CADE*, volume 2741 of *Lecture Notes in Computer Science*, pages 136–150. Springer, 2003.
- [Jon87] Simon L. Peyton Jones. *The Implementation of Functional Programming Languages*. Prentice-Hall, 1987.
- [Kes07] Delia Kesner. The theory of calculi with explicit substitutions revisited. In *CSL*, volume 4646 of *Lecture Notes in Computer Science*, pages 238–252. Springer, 2007.
- [Kes09] Delia Kesner. A theory of explicit substitutions with safe and full composition. *Log. Methods Comput. Sci.*, 5(3), 2009.
- [Kes16] Delia Kesner. Reasoning about call-by-need by means of types. In *FoSSaCS*, volume 9634 of *Lecture Notes in Computer Science*, pages 424–441. Springer, 2016.

- [Kfo00] A. J. Kfoury. A linearization of the lambda-calculus and consequences. *J. Log. Comput.*, 10(3):411–436, 2000.
- [KL07] Delia Kesner and Stéphane Lengrand. Resource operators for lambda-calculus. *Inf. Comput.*, 205(4):419–473, 2007.
- [KR11] Delia Kesner and Fabien Renaud. A prismoid framework for languages with resources. *Theor. Comput. Sci.*, 412(37):4867–4892, 2011.
- [KRV18] Delia Kesner, Alejandro Ríos, and Andrés Viso. Call-by-need, neededness and all that. In Christel Baier and Ugo Dal Lago, editors, *Foundations of Software Science and Computation Structures - 21st International Conference, FOSSACS 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14-20, 2018, Proceedings*, volume 10803 of *Lecture Notes in Computer Science*, pages 241–257. Springer, 2018. doi: 10.1007/978-3-319-89366-2\_13.
- [KV14] Delia Kesner and Daniel Ventura. Quantitative types for the linear substitution calculus. In *IFIP TCS*, volume 8705 of *Lecture Notes in Computer Science*, pages 296–310. Springer, 2014.
- [KV20] Delia Kesner and Pierre Vial. Non-idempotent types for classical calculi in natural deduction style. *Logical Methods in Computer Science ; Volume 16*, abs/1802.05494, 2020. arXiv:1802.05494, doi: 10.23638/LMCS-16(1:3)2020.
- [Lam90] John Lamping. An algorithm for optimal lambda calculus reduction. In *POPL*, pages 16–30. ACM Press, 1990.
- [Lév78] Jean-Jacques Lévy. *Réductions correctes et optimales dans le lambda-calcul*. PhD thesis, Université Paris VII, 1978.
- [Lév80] Jean-Jacques Lévy. *Optimal Reductions in the Lambda-Calculus*, pages 159–191. Academic Press Inc, 1980.
- [LM99] Jean-Jacques Lévy and Luc Maranget. Explicit substitutions and programming languages. In *Lecture Notes in Computer Science*, pages 181–200. Springer Berlin Heidelberg, 1999. doi: 10.1007/3-540-46691-6\_14.
- [MOTW99] John Maraist, Martin Odersky, David N. Turner, and Philip Wadler. Call-by-name, call-by-value, call-by-need and the linear lambda calculus. *Theor. Comput. Sci.*, 228(1-2):175–210, 1999.
- [MOW98] John Maraist, Martin Odersky, and Philip Wadler. The call-by-need lambda calculus. *J. Funct. Program.*, 8(3):275–317, 1998.
- [Plo75] Gordon D. Plotkin. Call-by-name, call-by-value and the lambda-calculus. *Theor. Comput. Sci.*, 1(2):125–159, 1975.
- [She19] David R. Sherratt. *A lambda-calculus that achieves full laziness with spine duplication*. PhD thesis, University of Bath, March 2019.
- [SHGP20] David Sherratt, Willem Heijltjes, Tom Gundersen, and Michel Parigot. Spinal atomic lambda-calculus. In *FoSSaCS*, volume 12077 of *Lecture Notes in Computer Science*, pages 582–601. Springer, 2020.
- [SU06] M. H. Sørensen and Paweł Urzyczyn. *Lectures on the Curry-Howard Isomorphism*. Elsevier, 2006. doi: 10.1016/s0049-237x(06)x8001-1.
- [SW10] Olin Shivers and Mitchell Wand. Bottom-up beta-reduction: Uplinks and lambda-dags. *Fundam. Informaticae*, 103(1-4):247–287, 2010.
- [Wad71] Christopher P. Wadsworth. *Semantics and Pragmatics of the Lambda Calculus*. PhD thesis, Oxford University, 1971.

## APPENDIX A. PROOFS

**Lemma 2.5.** *Let  $x \neq z$ ,  $t \in \mathsf{T}_R$  and  $p \in \mathsf{T}_P$ :*

- (1) *If  $z \notin \text{fv}(p)$ , then  $\text{lv}_z(t\{x/p\}) = \text{lv}_z(t)$ .*
- (2) *If  $z \in \text{fv}(p)$ , then  $\text{lv}_z(t\{x/p\}) = \max(\text{lv}_z(t), \text{lv}_x(t))$ .*

*Proof.* If  $x \notin \text{fv}(t)$ , then  $t\{x/p\} = t$  and the property holds in both cases since  $\text{lv}_x(t) = 0$ . Let  $x \in \text{fv}(t)$ . We show the two cases.

- (1)  $z \notin \text{fv}(p)$ . We reason by induction on  $t$ .

- If  $t = x$ , then  $\text{lv}_z(x\{x/p\}) = \text{lv}_z(p) = 0 = \text{lv}_z(x)$ .
  - If  $t = \lambda y.t'$ , then  $\text{lv}_z((\lambda y.t')\{x/p\}) = \text{lv}_z(\lambda y.t'\{x/p\}) = \text{lv}_z(t'\{x/p\}) =_{i.h.} \text{lv}_z(t') = \text{lv}_z(\lambda y.t')$ .
  - If  $t = t_1 t_2$ , then  $\text{lv}_z((t_1 t_2)\{x/p\}) = \text{lv}_z(t_1\{x/p\}t_2\{x/p\})$ . Then,  $\text{lv}_z((t_1 t_2)\{x/p\}) = \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_z(t_2\{x/p\})) =_{i.h.} \max(\text{lv}_z(t_1), \text{lv}_z(t_2)) = \text{lv}_z(t_1 t_2)$ .
  - If  $t = t'[y \triangleleft u]$ , then  $t\{x/p\} = t'\{x/p\}[y \triangleleft u\{x/p\}]$ . There are two cases.
    - If  $z \notin \text{fv}(u\{x/p\})$ , that is  $z \notin \text{fv}(u)$ , then  $\text{lv}_z(t'\{x/p\}[y \triangleleft u\{x/p\}]) = \text{lv}_z(t'\{x/p\}) =_{i.h.} \text{lv}_z(t') = \text{lv}_z(t'[y \triangleleft u])$ .
    - Otherwise,  $\text{lv}_z(t'\{x/p\}[y \triangleleft u\{x/p\}]) = \max(\text{lv}_z(t'\{x/p\}), \text{lv}_y(t'\{x/p\}) + \text{lv}_z(u\{x/p\}) + \text{ES}([y \triangleleft u])) =_{i.h.} \max(\text{lv}_z(t'), \text{lv}_y(t') + \text{lv}_z(u) + \text{ES}([y \triangleleft u])) = \text{lv}_z(t'[y \triangleleft u])$ .
- (2)  $z \in \text{fv}(p)$ . By induction on  $t$ .
- If  $t = x$ , then  $t\{x/p\} = p$  and

$$\text{lv}_z(t\{x/p\}) = \text{lv}_z(p) = 0 = \max(0, 0) = \max(\text{lv}_z(x), \text{lv}_x(x)).$$

- If  $t = \lambda y.t'$ , then  $t\{x/p\} = \lambda y.t'\{x/p\}$  and

$$\text{lv}_z(\lambda y.t'\{x/p\}) = \text{lv}_z(t'\{x/p\}) =_{i.h.} \max(\text{lv}_z(t'), \text{lv}_x(t')) = \max(\text{lv}_z(\lambda y.t'), \text{lv}_x(\lambda y.t')).$$

- If  $t = t_1 t_2$  then  $t\{x/p\} = t_1\{x/p\}t_2\{x/p\}$ . For  $i \in \{1, 2\}$ , one has either  $\text{lv}_z(t_i\{x/p\}) = \max(\text{lv}_z(t_i), \text{lv}_x(t_i))$  by *i.h.* if  $x \in \text{fv}(t_i)$ , or  $\text{lv}_z(t_i\{x/p\}) = \text{lv}_z(t_i)$  by Point 1 otherwise.
  - If  $x \in \text{fv}(t_1) \cap \text{fv}(t_2)$  then,

$$\begin{aligned} \text{lv}_z(t\{x/p\}) &= \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_z(t_2\{x/p\})) \\ &=_{i.h.} \max(\max(\text{lv}_z(t_1), \text{lv}_x(t_1)), \max(\text{lv}_z(t_2), \text{lv}_x(t_2))) \\ &= \max(\text{lv}_z(t_1), \text{lv}_x(t_1), \text{lv}_z(t_2), \text{lv}_x(t_2)) \\ &= \max(\max(\text{lv}_z(t_1), \text{lv}_z(t_2)), \max(\text{lv}_x(t_1), \text{lv}_x(t_2))) \\ &= \max(\text{lv}_z(t_1 t_2), \text{lv}_x(t_1 t_2)) \end{aligned}$$

- If  $x \notin \text{fv}(t_2)$ , then

$$\begin{aligned} \text{lv}_z(t\{x/p\}) &= \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_z(t_2\{x/p\})) \\ &=_{i.h.+P.1} \max(\max(\text{lv}_z(t_1), \text{lv}_x(t_1)), \text{lv}_z(t_2)) \\ &= \max(\text{lv}_z(t_1), \text{lv}_x(t_1), \text{lv}_z(t_2)) \\ &= \max(\max(\text{lv}_z(t_1), \text{lv}_z(t_2)), \max(\text{lv}_x(t_1), 0)) \\ &= \max(\text{lv}_z(t_1 t_2), \text{lv}_x(t_1 t_2)) \end{aligned}$$

- If  $x \notin \text{fv}(t_1)$  the case is as above.

- If  $t = t_1[y \triangleleft t_2]$  then  $t\{x/p\} = t_1\{x/p\}[y \triangleleft t_2\{x/p\}]$ . By  $\alpha$ -conversion we can assume  $y \notin \text{fv}(p)$ . By *i.h.* one has  $\text{lv}_z(t_i\{x/p\}) = \max(\text{lv}_z(t_i), \text{lv}_x(t_i))$  for  $i \in \{1, 2\}$ , if  $x \in \text{fv}(t_i)$ ,  $\text{lv}_z(t_i\{x/p\}) = \text{lv}_z(t_i)$  otherwise. There are two cases.
  - (a) If  $z \notin \text{fv}(t_2\{x/p\})$  then  $z \notin \text{fv}(t_2)$  and necessarily  $x \notin \text{fv}(t_2)$  since  $z \in \text{fv}(p)$ . Therefore,

$$\begin{aligned} \text{lv}_z(t\{x/p\}) &= \text{lv}_z(t_1\{x/p\}) \\ &=_{i.h.} \max(\text{lv}_z(t_1), \text{lv}_x(t_1)) \\ &= \max(\text{lv}_z(t_1[y \triangleleft t_2]), \text{lv}_x(t_1[y \triangleleft t_2])) \end{aligned}$$

- (b) If  $z \in \text{fv}(t_2\{x/p\})$  then,

(i) If  $x \notin \text{fv}(t_1)$ , then  $x \in \text{fv}(t_2)$ .

$$\begin{aligned}
\text{lv}_z(t\{x/p\}) &= \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_y(t_1\{x/p\}) + \text{lv}_z(t_2\{x/p\}) + \text{ES}([y \triangleleft t_2])) \\
&=_{P.1} \max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2\{x/p\}) + \text{ES}([y \triangleleft t_2])) \\
&=_{i.h.} \max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \max(\text{lv}_z(t_2), \text{lv}_x(t_2)) + \text{ES}([y \triangleleft t_2])) \\
&= \max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2]), \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) \\
&= \begin{cases} \max(\max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2])), \\ \quad \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) & z \in \text{fv}(t_2) \\ \max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) & z \notin \text{fv}(t_2) \end{cases} \\
&= \max(\text{lv}_z(t_1[y \triangleleft t_2]), \text{lv}_x(t_1[y \triangleleft t_2]))
\end{aligned}$$

(ii) If  $x \notin \text{fv}(t_2)$ , then  $x \in \text{fv}(t_1)$  and  $z \in \text{fv}(t_2)$ :

$$\begin{aligned}
\text{lv}_z(t\{x/p\}) &= \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_y(t_1\{x/p\}) + \text{lv}_z(t_2\{x/p\}) + \text{ES}([y \triangleleft t_2])) \\
&=_{i.h.+P.1} \max(\text{lv}_z(t_1), \text{lv}_x(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2])) \\
&= \max(\max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2])), \text{lv}_x(t_1[y \triangleleft t_2])) \\
&= \max(\text{lv}_z(t_1[y \triangleleft t_2]), \text{lv}_x(t_1[y \triangleleft t_2]))
\end{aligned}$$

(iii) If  $x \in \text{fv}(t_1) \cap \text{fv}(t_2)$ :

$$\begin{aligned}
\text{lv}_z(t\{x/p\}) &= \max(\text{lv}_z(t_1\{x/p\}), \text{lv}_y(t_1\{x/p\}) + \text{lv}_z(t_2\{x/p\}) + \text{ES}([y \triangleleft t_2])) \\
&=_{i.h.} \max(\max(\text{lv}_z(t_1), \text{lv}_x(t_1)), \text{lv}_y(t_1) + \max(\text{lv}_z(t_2), \text{lv}_x(t_2)) + \text{ES}([y \triangleleft t_2])) \\
&= \max(\text{lv}_z(t_1), \text{lv}_x(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2]), \\
&\quad \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) \\
&= \begin{cases} \max(\text{lv}_z(t_1), \text{lv}_y(t_1) + \text{lv}_z(t_2) + \text{ES}([y \triangleleft t_2]), \\ \quad \text{lv}_x(t_1), \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) & z \in \text{fv}(t_2) \\ \max(\text{lv}_z(t_1), \text{lv}_x(t_1), \text{lv}_y(t_1) + \text{lv}_x(t_2) + \text{ES}([y \triangleleft t_2])) & z \notin \text{fv}(t_2) \end{cases} \\
&= \max(\text{lv}_z(t_1[y \triangleleft t_2]), \text{lv}_x(t_1[y \triangleleft t_2])) \quad \square
\end{aligned}$$

**Lemma 2.6.** *Let  $t \in \mathbf{T}_R$  and  $w$  be any variable.*

- (1) *If  $t_0 \rightarrow_\pi t_1$ , then  $\text{lv}_w(t_0) \geq \text{lv}_w(t_1)$ .*
- (2) *If  $t_0 \rightarrow_{\text{sub}} t_1$ , then  $\text{lv}_w(t_0) \geq \text{lv}_w(t_1)$ .*

*Proof.*

- (1) Let  $t_0 = \mathbf{C}\langle o \rangle$  and  $t_1 = \mathbf{C}\langle o' \rangle$ , where  $o \rightarrow_\pi o'$  is a root step. We reason by induction on  $\mathbf{C}$ . First we consider the base cases, where  $\mathbf{C} = \diamond$ .

- $t_0 = \lambda y.t[x \triangleleft u] \rightarrow_\pi (\lambda y.t)[x \triangleleft u] = t_1$ , where  $y \notin \text{fv}(u)$ . We have two cases:
  - (a) If  $w \notin \text{fv}(u)$ .

$$\text{lv}_w(\lambda y.t[x \triangleleft u]) = \text{lv}_w(t[x \triangleleft u]) = \text{lv}_w(t) = \text{lv}_w(\lambda y.t) = \text{lv}_w((\lambda y.t)[x \triangleleft u])$$

(b) If  $w \in \text{fv}(u)$ .

$$\begin{aligned} \text{lv}_w(\lambda y.t[x \triangleleft u]) &= \text{lv}_w(t[x \triangleleft u]) \\ &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(\lambda y.t), \text{lv}_x(\lambda y.t) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \text{lv}_w((\lambda y.t)[x \triangleleft u]) \end{aligned}$$

•  $t_0 = t[x \triangleleft u]s \rightarrow_\pi (ts)[x \triangleleft u] = t_1$ , where  $x \notin \text{fv}(s)$ . We have two cases:

(a) If  $w \notin \text{fv}(u)$ .

$$\begin{aligned} \text{lv}_w(t[x \triangleleft u]s) &= \max(\text{lv}_w(t[x \triangleleft u]), \text{lv}_w(s)) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s)) \\ &= \text{lv}_w(ts) \\ &= \text{lv}_w((ts)[x \triangleleft u]) \end{aligned}$$

(b) If  $w \in \text{fv}(u)$ .

$$\begin{aligned} &\text{lv}_w(t[x \triangleleft u]s) \\ &= \max(\text{lv}_w(t[x \triangleleft u]), \text{lv}_w(s)) \\ &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + \text{ES}([x \triangleleft u]), \text{lv}_w(s)) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \text{lv}_x(t) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \max(\text{lv}_x(t), 0) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \quad (x \notin \text{fv}(s)) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \max(\text{lv}_x(t), \text{lv}_x(s)) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(ts), \text{lv}_x(ts) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \text{lv}_w((ts)[x \triangleleft u]) \end{aligned}$$

•  $t_0 = ts[x \triangleleft u] \rightarrow_\pi (ts)[x \triangleleft u] = t_1$ , where  $x \notin \text{fv}(t)$ . We have two cases:

(a) If  $w \notin \text{fv}(u)$ .

$$\begin{aligned} \text{lv}_w(ts[x \triangleleft u]) &= \max(\text{lv}_w(t), \text{lv}_w(s[x \triangleleft u])) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s)) \\ &= \text{lv}_w(ts) \\ &= \text{lv}_w((ts)[x \triangleleft u]) \end{aligned}$$

(b) If  $w \in \text{fv}(u)$ .

$$\begin{aligned} \text{lv}_w(ts[x \triangleleft u]) &= \max(\text{lv}_w(t), \text{lv}_w(s[x \triangleleft u])) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \text{lv}_x(s) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \max(0, \text{lv}_x(s)) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(t), \text{lv}_w(s), \max(\text{lv}_x(t), \text{lv}_x(s)) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \max(\text{lv}_w(ts), \text{lv}_x(ts) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \text{lv}_w((ts)[x \triangleleft u]) \end{aligned}$$

• Let  $t_0 = t[y \triangleleft s[x \triangleleft u]] \rightarrow_\pi t[y \triangleleft s][x \triangleleft u] = t_1$ , where  $x \notin \text{fv}(t)$ . We have four cases:

(a) If  $w \notin \text{fv}(s) \cup \text{fv}(u)$ :

$$\text{lv}_w(t[y \triangleleft s[x \triangleleft u]]) = \text{lv}_w(t) = \text{lv}_w(t[y \triangleleft s]) = \text{lv}_w(t[y \triangleleft s][x \triangleleft u])$$

(b) If  $w \in \text{fv}(s)$ ,  $w \notin \text{fv}(u)$ :

$$\begin{aligned} \text{lv}_w(t[y \triangleleft s[x \triangleleft u]]) &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s[x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s) + \text{ES}([y \triangleleft s])) \\ &= \text{lv}_w(t[y \triangleleft s]) \\ &= \text{lv}_w(t[y \triangleleft s][x \triangleleft u]) \end{aligned}$$

(c) If  $w \notin \text{fv}(s)$ ,  $w \in \text{fv}(u)$ .

$$\begin{aligned} \text{lv}_w(t[y \triangleleft s[x \triangleleft u]]) &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s[x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_x(s) + \text{lv}_w(u) + \text{ES}([x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &\geq \max(\text{lv}_w(t[y \triangleleft s]), \text{lv}_x(t[y \triangleleft s]) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \text{lv}_w(t[y \triangleleft s][x \triangleleft u]) \end{aligned}$$

(d) If  $w \in \text{fv}(s) \cap \text{fv}(u)$ :

$$\begin{aligned} \text{lv}_w(t[y \triangleleft s[x \triangleleft u]]) &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s[x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &= \max(\text{lv}_w(t), \text{lv}_y(t) + \max(\text{lv}_w(s), \text{lv}_x(s) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) + \text{ES}([y \triangleleft s])) \\ &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s) + \text{ES}([y \triangleleft s]), \\ &\quad \text{lv}_y(t) + \text{lv}_x(s) + \text{lv}_w(u) + \text{ES}([x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &= \max(\max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(s) + \text{ES}([y \triangleleft s])), \\ &\quad \max(\text{lv}_x(t), \text{lv}_y(t) + \text{lv}_x(s) + \text{ES}([y \triangleleft s])) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &\geq \max(\text{lv}_w(t[y \triangleleft s]), \text{lv}_x(t[y \triangleleft s]) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \\ &= \text{lv}_w(t[y \triangleleft s][x \triangleleft u]) \end{aligned}$$

The inductive cases are the following:

- If  $\mathbf{C} = \lambda x.\mathbf{C}'$ , where  $x \neq w$ , then  $\text{lv}_w(\lambda x.\mathbf{C}'\langle o \rangle) = \text{lv}_w(\mathbf{C}'\langle o \rangle) \geq_{i.h.} \text{lv}_w(\mathbf{C}'\langle o' \rangle) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
- If  $\mathbf{C} = \mathbf{C}'u$ , then  $\text{lv}_w(\mathbf{C}'\langle o \rangle u) = \max(\text{lv}_w(\mathbf{C}'\langle o \rangle), \text{lv}_w(u)) \geq_{i.h.} \max(\text{lv}_w(\mathbf{C}'\langle o' \rangle), \text{lv}_w(u)) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
- If  $\mathbf{C} = u\mathbf{C}'$ , then  $\text{lv}_w(u\mathbf{C}'\langle o \rangle) = \max(\text{lv}_w(u), \text{lv}_w(\mathbf{C}'\langle o \rangle)) \geq_{i.h.} \max(\text{lv}_w(u), \text{lv}_w(\mathbf{C}'\langle o' \rangle)) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
- If  $\mathbf{C} = \mathbf{C}'[x \triangleleft u]$ , then
  - (a) If  $w \notin \text{fv}(u)$ , then  $\text{lv}_w(\mathbf{C}'\langle o \rangle[x \triangleleft u]) = \text{lv}_w(\mathbf{C}'\langle o \rangle) \geq_{i.h.} \text{lv}_w(\mathbf{C}'\langle o' \rangle) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
  - (b) If  $w \in \text{fv}(u)$ , then  $\text{lv}_w(\mathbf{C}'\langle o \rangle[x \triangleleft u]) = \max(\text{lv}_w(\mathbf{C}'\langle o \rangle), \text{lv}_x(\mathbf{C}'\langle o \rangle) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) \geq_{i.h.} \max(\text{lv}_w(\mathbf{C}'\langle o' \rangle), \text{lv}_x(\mathbf{C}'\langle o' \rangle) + \text{lv}_w(u) + \text{ES}([x \triangleleft u])) = \text{lv}_w(\mathbf{C}'\langle o' \rangle[x \triangleleft u]) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
- If  $\mathbf{C} = u[x \triangleleft \mathbf{C}']$ , then
  - (a) If  $w \notin \text{fv}(\mathbf{C}'\langle o \rangle)$ , then  $\text{lv}_w(u[x \triangleleft \mathbf{C}'\langle o \rangle]) = \text{lv}_w(u) = \text{lv}_w(u[x \triangleleft \mathbf{C}'\langle o' \rangle]) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .
  - (b) If  $w \in \text{fv}(\mathbf{C}'\langle o \rangle)$ , then  $\text{lv}_w(u[x \triangleleft \mathbf{C}'\langle o \rangle]) = \max(\text{lv}_w(u), \text{lv}_x(u) + \text{lv}_w(\mathbf{C}'\langle o \rangle) + \text{ES}([x \triangleleft \mathbf{C}'\langle o \rangle])) \geq_{i.h.} \max(\text{lv}_w(u), \text{lv}_x(u) + \text{lv}_w(\mathbf{C}'\langle o' \rangle) + \text{ES}([x \triangleleft \mathbf{C}'\langle o' \rangle])) = \text{lv}_w(u[x \triangleleft \mathbf{C}'\langle o' \rangle]) = \text{lv}_w(\mathbf{C}\langle o' \rangle)$ .

(2) We reason by induction on the reduction relation, *i.e.* by induction on the context  $\mathbf{C}$  where the root reduction takes place. We detail the base case which is  $\mathbf{C} = \diamond$ . In all such cases we use Point 1 to push  $\mathbf{L}$  outside, *i.e.* we can write  $t_0 \rightarrow_{\text{sub}} t_1$  as  $t_0 \rightarrow_{\pi} \mathbf{L}\langle t'_0 \rangle \rightarrow_{\text{sub}'} \mathbf{L}\langle t'_1 \rangle = t_1$ , where  $t'_0 \rightarrow_{\text{sub}'} t'_1$  does not push any list context outside. We then show the property for steps  $t'_0 \rightarrow_{\text{sub}'} t'_1$  not pushing any substitution outside



and we conclude by  $\text{lv}_w(t_0) \geq_{P.1} \text{lv}_w(\mathbf{L}\langle t'_0 \rangle) \geq \text{lv}_w(\mathbf{L}\langle t'_1 \rangle) = \text{lv}_w(t_1)$ . The inductive cases for  $\mathbf{C}$  are treated as in Point 1.

- $t'_0 = t[x/us] \rightarrow_{\text{app}} t\{x/yz\}[y/u][z/s] = t'_1$ , where  $y$  and  $z$  are fresh variables.
  - (a) If  $w \notin \text{fv}(us)$  (i.e.  $w \notin \text{fv}(u)$  and  $w \notin \text{fv}(s)$ ):

$$\begin{aligned} \text{lv}_w(t[x/us]) &= \text{lv}_w(t) \\ &=_{L.2.5:1} \text{lv}_w(t\{x/yz\}) \\ &= \text{lv}_w(t\{x/yz\}[y/u]) \\ &= \text{lv}_w(t\{x/yz\}[y/u][z/s]) \end{aligned}$$

- (b) If  $w \in \text{fv}(u)$  and  $w \notin \text{fv}(s)$ . There are two cases.
  - (i) If  $x \notin \text{fv}(t)$ :

$$\begin{aligned} \text{lv}_w(t[x/us]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\ &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1) \\ &= \max(\text{lv}_w(t), 0 + \text{lv}_w(u) + 1) \\ &= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(u) + 1) \\ &=_{L.2.5:1} \max(\text{lv}_w(t), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1) \\ &=_{L.2.5:1} \max(\text{lv}_w(t\{x/yz\}), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1) \\ &= \text{lv}_w(t\{x/yz\}[y/u]) \\ &= \text{lv}_w(t\{x/yz\}[y/u][z/s]) \end{aligned}$$

- (ii) If  $x \in \text{fv}(t)$ :

$$\begin{aligned} \text{lv}_w(t[x/us]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\ &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1) \\ &= \max(\text{lv}_w(t), \text{lv}_x(t) + 0 + \text{lv}_w(u) + 1) \\ &= \max(\text{lv}_w(t), \max(0, \text{lv}_x(t) + 0) + \text{lv}_w(u) + 1) \\ &= \max(\text{lv}_w(t), \max(\text{lv}_y(t), \text{lv}_x(t) + \text{lv}_y(yz)) + \text{lv}_w(u) + 1) \\ &=_{L.2.5:2} \max(\text{lv}_w(t), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1) \\ &=_{L.2.5:1} \max(\text{lv}_w(t\{x/yz\}), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1) \\ &= \text{lv}_w(t\{x/yz\}[y/u]) \\ &= \text{lv}_w(t\{x/yz\}[y/u][z/s]) \end{aligned}$$

- (c) If  $w \notin \text{fv}(u)$  and  $w \in \text{fv}(s)$ . There are two cases.

(i) If  $x \notin \text{fv}(t)$ :

$$\begin{aligned}
\text{lv}_w(t[x/us]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), 0 + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_z(t) + \text{lv}_w(s) + 1) \\
&=_{L.2.5:1} \max(\text{lv}_w(t), \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&=_{L.2.5:1} \max(\text{lv}_w(t\{x/yz\}), \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}[y/u]), \text{lv}_z(t\{x/yz\}[y/u]) + \text{lv}_w(s) + 1) \\
&= \text{lv}_w(t\{x/yz\}[y/u][z/s])
\end{aligned}$$

(ii) If  $x \in \text{fv}(t)$ :

$$\begin{aligned}
\text{lv}_w(t[x/us]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + 0 + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \max(0, \text{lv}_x(t) + 0) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \max(\text{lv}_z(t), \text{lv}_x(t) + \text{lv}_z(yz)) + \text{lv}_w(s) + 1) \\
&=_{L.2.5:2} \max(\text{lv}_w(t), \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&=_{L.2.5:1} \max(\text{lv}_w(t\{x/yz\}), \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}[y/u]), \text{lv}_z(t\{x/yz\}[y/u]) + \text{lv}_w(s) + 1) \\
&= \text{lv}_w(t\{x/yz\}[y/u][z/s])
\end{aligned}$$

(d) If  $w \in \text{fv}(u)$  and  $w \in \text{fv}(s)$ . There are two cases.

(i) If  $x \notin \text{fv}(t)$ :

$$\begin{aligned}
&\text{lv}_w(t[x/us]) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \max(\text{lv}_w(u), \text{lv}_w(s)) + 1) \\
&= \max(\text{lv}_w(t), \max(\text{lv}_w(u), \text{lv}_w(s)) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_w(u) + 1, \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_y(t) + \text{lv}_w(u) + 1, \text{lv}_z(t) + \text{lv}_w(s) + 1) \\
&=_{L.2.5:1} \max(\text{lv}_w(t\{x/yz\}), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}[y/u]) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}[y/u]), \text{lv}_z(t\{x/yz\}[y/u]) + \text{lv}_w(s) + 1) \\
&= \text{lv}_w(t\{x/yz\}[y/u][z/s])
\end{aligned}$$

(ii) If  $x \in \text{fv}(t)$ :

$$\begin{aligned}
& \text{lv}_w(t[x/us]) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(us) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1, \text{lv}_x(t) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1, \max(0, \text{lv}_x(t) + 0) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1, \max(\text{lv}_z(t), \text{lv}_x(t) + \text{lv}_z(yz)) + \text{lv}_w(s) + 1) \\
&=_{L.2.5.2} \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \max(0, \text{lv}_x(t) + 0) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t), \max(\text{lv}_y(t), \text{lv}_x(t) + \text{lv}_y(yz)) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&=_{L.2.5.2} \max(\text{lv}_w(t), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}), \text{lv}_y(t\{x/yz\}) + \text{lv}_w(u) + 1, \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}[y/u]), \text{lv}_z(t\{x/yz\}) + \text{lv}_w(s) + 1) \\
&= \max(\text{lv}_w(t\{x/yz\}[y/u]), \text{lv}_z(t\{x/yz\}[y/u]) + \text{lv}_w(s) + 1) \\
&= \text{lv}_w(t\{x/yz\}[y/u][z/s])
\end{aligned}$$

- $t'_0 = t[x/\lambda y.u] \rightarrow_{\text{dist}} t[x//\lambda y.z[z/u]] = t'_1$ . There are two cases.
  - (a)  $w \notin \text{fv}(\lambda y.u)$ :

$$\text{lv}_w(t[x/\lambda y.u]) = \text{lv}_w(t) = \text{lv}_w(t[x//\lambda y.z[z/u]])$$

- (b)  $w \in \text{fv}(\lambda y.u)$  (i.e.  $w \in \text{fv}(u)$  and  $w \neq y$ )

$$\begin{aligned}
\text{lv}_w(t[x/\lambda y.u]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(\lambda y.u) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(u) + 1) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \max(0, 0 + \text{lv}_w(u) + 1)) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \max(\text{lv}_w(z), \text{lv}_z(z) + \text{lv}_w(u) + 1)) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(z[z/u])) \\
&= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(\lambda y.z[z/u])) \\
&= \text{lv}_w(t[x//\lambda y.z[z/u]])
\end{aligned}$$

- $t'_0 = t[x//\lambda y.u] \rightarrow_{\text{abs}} t\{x/\lambda y.u\} = t'_1$ , where  $u$  is pure. There are two cases:
  - (a) If  $w \notin \text{fv}(\lambda y.u)$  or  $x \notin \text{fv}(t)$ :

$$\text{lv}_w(t[x//\lambda y.u]) = \text{lv}_w(t) =_{L.2.5.1} \text{lv}_w(t\{x/\lambda y.u'\}) = \text{lv}_w(\mathbf{L}\langle t\{x/\lambda y.u'\} \rangle)$$

- (b) If  $w \in \text{fv}(\lambda y.u)$  and  $x \in \text{fv}(t)$ :

$$\text{lv}_w(t[x//\lambda y.u]) = \max(\text{lv}_w(t), \text{lv}_x(t)) =_{L.2.5.2} \text{lv}_w(t\{x/\lambda y.u\})$$

- $t'_0 = t[x/y] \rightarrow_{\text{var}} t\{x/y\} = t'_1$ .
  - (a) If  $w \neq y$ :

$$\text{lv}_w(t[x/y]) = \text{lv}_w(t) =_{L.2.5.1} \text{lv}_w(t\{x/y\})$$

(b) If  $w = y$  and  $x \notin \text{fv}(t)$ :

$$\begin{aligned} \text{lv}_w(t[x/y]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(y) + 1) \\ &= \max(\text{lv}_w(t), 1) \\ &\geq \text{lv}_w(t) \\ &=_{L.2.5.1} \text{lv}_w(t\{x/y\}) \end{aligned}$$

(c) If  $w = y$  and  $x \in \text{fv}(t)$ :

$$\begin{aligned} \text{lv}_w(t[x/y]) &= \max(\text{lv}_w(t), \text{lv}_x(t) + \text{lv}_w(y) + 1) \\ &\geq \max(\text{lv}_w(t), \text{lv}_x(t)) \\ &=_{L.2.5.2} \text{lv}_w(t\{x/y\}) \end{aligned} \quad \square$$

**Lemma 3.5.** *Let  $t$  be a term,  $x$  a variable and  $p$  a pure term. Let  $K = \text{lv}_x(t)$ . Then there is  $N \in \mathbb{N}$  such that  $\text{CL}(t\{x/p\}) \sqsubseteq \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>K} \sqcup [a(k, n) \mid k \leq K \text{ and } n \leq N]$ .*

*Proof.* By induction on  $t$ . In this proof,  $\text{fst}(o)$  denotes the first element of an object  $o \in \mathcal{O}$ :  $\text{fst}(a(k, n)) = k$  and  $\text{fst}(b(k)) = k$ . If  $a(k, n) \in \mathcal{O}$  we also define  $\text{snd}(a(k, n)) = n$ .

- If  $t = y$ , then  $\text{CL}(y) = \text{CL}(y\{x/p\}) = []$  so the property is straightforward for any  $n \in \mathbb{N}$ .
- If  $t = \lambda y.u$ , then  $\text{CL}(t\{x/p\}) = \text{CL}(u\{x/p\})$  and  $\text{lv}_x(t) = \text{lv}_x(u)$ . The property trivially holds by the *i.h.*
- If  $t = u_1u_2$ , then we have  $\text{CL}(t\{x/p\}) = \text{CL}(u_1\{x/p\}) \sqcup \text{CL}(u_2\{x/p\})$  and  $\text{lv}_x(u_1u_2) = \max(\text{lv}_x(u_1), \text{lv}_x(u_2))$ . Let  $o \in \text{CL}(t\{x/p\})$  thus  $o \in \text{CL}(u_1\{x/p\}) \sqcup \text{CL}(u_2\{x/p\})$ . Suppose w.l.o.g. that  $o \in \text{CL}(u_1\{x/p\})$ . Let  $K_1 = \text{lv}_x(u_1) \leq K$ . By the *i.h.* we have either (1)  $o \in \text{CL}(u_1)_b$ , (2)  $o \in \text{CL}(u_1)_a^{>K_1}$ , or (3)  $o = a(k, n)$  where  $k \leq K_1$  and  $n \leq N_1$  for some  $N_1 \in \mathbb{N}$ . If (1) holds, then  $o \in \text{CL}(t)_b$  and we are done. Otherwise,  $o = a(k, n)$ , and we consider two cases.
  - If  $k > K$ , then (2) implies  $o \in \text{CL}(u_1)_a^{>K}$  which implies  $o \in \text{CL}(t)_a^{>K}$  while (3) implies  $k \leq K$  which leads to a contradiction.
  - If  $k \leq K$ , we are done.

Considering that  $o \in \text{CL}(u_2\{x/p\})$  we have a similar result for some  $N_2 \in \mathbb{N}$ . We thus have the result for  $N = \max(N_1, N_2)$ .

- If  $t = u_1[y/u_2]$ , then we can assume by  $\alpha$ -conversion that  $y \notin \text{fv}(p)$ . Therefore,

$$\begin{aligned} \text{CL}(t) &= \text{CL}(u_1) \sqcup (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2) \sqcup [a(\text{lv}_y(u_1) + 1, |u_2|)] \text{ and} \\ \text{CL}(t\{x/p\}) &= \text{CL}(u_1\{x/p\}) \sqcup (\text{lv}_y(u_1\{x/p\}) + 1) \cdot \text{CL}(u_2\{x/p\}) \\ &\quad \sqcup [a(\text{lv}_y(u_1\{x/p\}) + 1, |u_2\{x/p\}|)] \\ &=_{L.2.5.1} \text{CL}(u_1\{x/p\}) \sqcup (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2\{x/p\}) \sqcup [a(\text{lv}_y(u_1) + 1, |u_2\{x/p\}|)] \end{aligned}$$

There are two cases:

- (1) If  $x \notin \text{fv}(u_2)$ , then  $\text{lv}_x(t) = \text{lv}_x(u_1)$ . Moreover,  $\text{CL}(u_2\{x/p\}) = \text{CL}(u_2)$  and  $|u_2\{x/p\}| = |u_2|$ . Let  $o \in \text{CL}(t\{x/p\})$ .
  - If  $o \in \text{CL}(u_1\{x/p\})$ , then let  $K_1 = \text{lv}_x(u_1) = \text{lv}_x(t) = K$ , so that the *i.h.* gives either (1)  $o \in \text{CL}(u_1)_b$ , (2)  $o \in \text{CL}(u_1)_a^{>K_1}$ , or (3)  $o = a(k, n)$  where  $k \leq K_1$  and  $n \leq N_1$  for some  $N_1 \in \mathbb{N}$ . If (1) holds, then  $o \in \text{CL}(t)_b$  and we are done. If (2) holds, then  $o \in \text{CL}(u_1)_a^{>K}$  since  $K_1 = K$ , which implies  $o \in \text{CL}(t)_a^{>K}$  and we are done. Otherwise, (3) holds and  $k \leq K_1 = K$  as required.

- If  $o \in (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2\{x/p\}) = (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2)$ , then  $o \in \text{CL}(t) = \text{CL}(t)_b \sqcup \text{CL}(t)_a^{>K} \sqcup \text{CL}(t)_a^{\leq K}$ , which particularly implies in the last case that  $o = a(k, n)$  and  $k \leq K$ . Note that, since  $\text{CL}(t)_a^{\leq K}$  is finite, we can take  $N_2 = \max\{\text{snd}(o) \mid o \in \text{CL}(t)_a^{\leq K}\}$ .
- If  $o = a(\text{lv}_y(u_1) + 1, |u_2\{x/p\}|) = a(\text{lv}_y(u_1) + 1, |u_2|)$ , then  $o \in \text{CL}(t)$  and thus either  $o \in \text{CL}(t)^{>K}$  or  $o \in \text{CL}(t)^{\leq K}$ , which particularly implies in the last case that  $\text{fst}(o) \leq K$  and  $\text{snd}(o) \leq N_2$ , with  $N_2$  defined as above.

The result then holds for  $N = \max(N_1, N_2)$ .

- (2) If  $x \in \text{fv}(u_2)$ , then  $\text{lv}_x(t) = \max(\text{lv}_x(u_1), \text{lv}_y(u_1) + \text{lv}_x(u_2) + 1)$ . Let  $o \in \text{CL}(t\{x/p\})$ .
- If  $o \in \text{CL}(u_1\{x/p\})$ , then let  $K_1 = \text{lv}_x(u_1) = \text{lv}_x(t) \leq K$ , so that the *i.h.* gives either (1)  $o \in \text{CL}(u_1)_b$ , (2)  $o \in \text{CL}(u_1)_a^{>K_1}$ , or (3)  $o = a(k, n)$  where  $k \leq K_1$  and  $n \leq N_1$  for some  $N_1 \in \mathbb{N}$ . If (1) holds, then  $o \in \text{CL}(t)_b$  and we are done. Otherwise  $o = a(k, n)$  and we consider two cases.
    - \* If  $k > K$ , then (2) implies  $o \in \text{CL}(u_1)_a^{>K}$ , and thus  $o \in \text{CL}(t)_a^{>K}$ , while (3) implies  $k \leq K$  which leads to a contradiction.
    - \* If  $k \leq K$ , then we are done.
  - If  $o \in (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2\{x/p\})$ , then there is  $o' \in \text{CL}(u_2\{x/p\})$  such that  $\text{fst}(o) = \text{fst}(o') + (\text{lv}_y(u_1) + 1)$ . Let  $K_2 = \text{lv}_x(u_2)$ , so that the *i.h.* gives either (1)  $o' \in \text{CL}(u_2)_b$ , (2)  $o' \in \text{CL}(u_2)_a^{>K_2}$ , or (3)  $o' = a(k, n)$  where  $k \leq K_2$  and  $n \leq N_2$  for some  $N_2 \in \mathbb{N}$ . If (1) holds, then  $o \in (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2)_b$ , thus  $o \in \text{CL}(t)_b$  and we are done. If (2) holds, then  $o \in (\text{lv}_y(u_1) + 1) \cdot \text{CL}(u_2)_a^{>K_2}$  and thus  $\text{fst}(o) > K_2 + (\text{lv}_y(u_1) + 1)$ . We consider two cases.
    - \* If  $\text{fst}(o) > K \geq K_2 + \text{lv}_y(u_1) + 1$ , then (2) implies  $o \in \text{CL}(t)_a^{>K}$  while (3) leads to a contradiction.
    - \* If  $\text{fst}(o) \leq K$ , then we are done.
  - If  $o = a(\text{lv}_y(u_1) + 1, |u_2\{x/p\}|)$ , then  $\text{fst}(o) = \text{lv}_y(u_1) + 1 \leq K$  and  $\text{snd}(o) = |u_2\{x/p\}|$ .

The result then holds for  $N = \max(N_1, N_2, |u_2\{x/p\}|)$ .

- If  $t = u_1[y//u_2]$ , the analysis is similar. □

**Lemma 3.6.** *Let  $t \in \mathbf{T}_R$ . Then  $t \rightarrow_\pi t'$  implies  $\text{CL}(t) \geq_{MUL}^O \text{CL}(t')$ .*

*Proof.* Let  $t = \mathbf{C}\langle t_0 \rangle \rightarrow_\pi \mathbf{C}\langle t_1 \rangle = t'$ , where  $t_0 \rightarrow_\pi t_1$  is a reduction step at the root position. We proceed by induction on  $\mathbf{C}$ . We detail the base case where the context  $\mathbf{C}$  is  $\diamond$ , by inspecting the cases where the explicit cuts are explicit substitutions, as the remaining cases for explicit distributors are similar.

- (1) If  $t = t_0 = \lambda y.t[x/u] \rightarrow_\pi (\lambda y.t)[x/u] = t_1 = t'$ , where  $y \notin \text{fv}(u)$ :

$$\begin{aligned}
 \text{CL}(t) &= \text{CL}(t[x/u]) \\
 &= \text{CL}(t) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(u) \sqcup [a(\text{lv}_x(t) + 1, |u|)] \\
 &= \text{CL}(\lambda y.t) \sqcup (\text{lv}_x(\lambda y.t) + 1) \cdot \text{CL}(u) \sqcup [a(\text{lv}_x(\lambda y.t) + 1, |u|)] \\
 &= \text{CL}(t')
 \end{aligned}$$

(2) If  $t = t_0 = t[x/u]s \rightarrow_{\pi} (ts)[x/u] = t_1 = t'$ , where  $x \notin \text{fv}(s)$ :

$$\begin{aligned} \text{CL}(t) &= \text{CL}(t[x/u]) \sqcup \text{CL}(s) \\ &= \text{CL}(t) \sqcup (\text{lv}_x(t) + 1) \cdot \text{CL}(u) \sqcup \text{CL}(s) \\ &= \text{CL}(ts) \sqcup (\text{lv}_x(ts) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(ts) + 1, |u|)] \\ &= \text{CL}(t') \end{aligned}$$

(3) If  $t = t_0 = ts[x/u] \rightarrow_{\pi} (ts)[x/u] = t_1 = t'$ , where  $x \notin \text{fv}(t)$ :

$$\begin{aligned} \text{CL}(t) &= \text{CL}(t) \sqcup \text{CL}(s[x/u]) \\ &= \text{CL}(t) \sqcup \text{CL}(s) \sqcup (\text{lv}_x(s) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(s) + 1, |u|)] \\ &= \text{CL}(ts) \sqcup (\text{lv}_x(ts) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(ts) + 1, |u|)] \\ &= \text{CL}(t') \end{aligned}$$

(4) If  $t = t_0 = t[y/s[x/u]] \rightarrow_{\pi} t[y/s][x/u] = t_1 = t'$ , where  $x \notin \text{fv}(t)$ :

$$\begin{aligned} \text{CL}(t) &= \text{CL}(t) \sqcup (\text{lv}_y(t) + 1) \cdot \text{CL}(s[x/u]) \sqcup [\text{a}(\text{lv}_y(t) + 1, |s[x/u]|)] \\ &= \text{CL}(t) \sqcup (\text{lv}_y(t) + 1) \cdot (\text{CL}(s) \sqcup (\text{lv}_x(s) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(s) + 1, |u|)]) \\ &\quad \sqcup [\text{a}(\text{lv}_y(t) + 1, |s[x/u]|)] \\ &= \text{CL}(t) \sqcup (\text{lv}_y(t) + 1) \cdot \text{CL}(s) \sqcup (\text{lv}_y(t) + \text{lv}_x(s) + 2) \cdot \text{CL}(u) \\ &\quad \sqcup [\text{a}(\text{lv}_y(t) + \text{lv}_x(s) + 2, |u|), \text{a}(\text{lv}_y(t) + 1, |s[x/u]|)] \\ &= (\text{CL}(t) \sqcup (\text{lv}_y(t) + 1) \cdot \text{CL}(s) \sqcup [\text{a}(\text{lv}_y(t) + 1, |s[x/u]|)]) \\ &\quad \sqcup (\text{lv}_y(t) + \text{lv}_x(s) + 2) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_y(t) + \text{lv}_x(s) + 2, |u|)] \\ &>_{\text{MUL}}^{\circ} (\text{CL}(t) \sqcup (\text{lv}_y(t) + 1) \cdot \text{CL}(s) \sqcup [\text{a}(\text{lv}_y(t) + 1, |s|)]) \\ &\quad \sqcup (\text{lv}_x(t[y/s]) + 1) \cdot \text{CL}(u) \sqcup [\text{a}(\text{lv}_x(t[y/s]) + 1, |u|)] \\ &= \text{CL}(t') \end{aligned}$$

The  $>_{\text{MUL}}^{\circ}$  inequation is justified by the following facts:

- $|s[x/u]| > |s|$ .
- $\text{lv}_y(t) + \text{lv}_x(s) + 2 = \max(0, \text{lv}_y(t) + \text{lv}_x(s) + 1) + 1 = \text{lv}_x(t[y/s]) + 1$ .

The inductive cases are straightforward. □

**Proposition 4.4** (Diamond). *The CBN strategy enjoys the diamond property, i.e. for any terms  $t, u, s \in U$  such that  $t \rightarrow_{\text{name}} u$ ,  $t \rightarrow_{\text{name}} s$  and  $u \neq s$ , there exists  $t'$  such that  $u \rightarrow_{\text{name}} t'$  and  $s \rightarrow_{\text{name}} t'$ .*

*Proof.* We split the statement above in three different properties, each one proved by induction on the involved relation relations.

(1) If  $t \rightarrow_{\text{ndB}} u$  and  $t \rightarrow_{\text{ndB}} s$ , then there exists  $t'$  such that  $u \rightarrow_{\text{ndB}} t'$  and  $s \rightarrow_{\text{ndB}} t'$ . We consider the following cases:

- ((APPDB), (APPDB)) We then have  $t = t_0t_1$  such that  $t \rightarrow_{\text{ndB}} u_0t_1 = u$  and  $t \rightarrow_{\text{ndB}} s_0t_1 = s$ , where  $t_0 \rightarrow_{\text{ndB}} u_0$  and  $t_0 \rightarrow_{\text{ndB}} s_0$ . By the *i.h.* there is  $t'_0$  such that  $s_0 \rightarrow_{\text{ndB}} t'_0$  and  $u_0 \rightarrow_{\text{ndB}} t'_0$ . Therefore  $s \rightarrow_{\text{ndB}} t'_0t_1 = t'$  and  $u \rightarrow_{\text{ndB}} t'$ .
- ((SUBDB), (SUBDB)) We then have  $t = t_0[x \triangleleft t_1]$  such that  $t \rightarrow_{\text{ndB}} u_0[x \triangleleft t_1] = u$  and  $t \rightarrow_{\text{ndB}} s_0[x \triangleleft t_1] = s$ , where  $t_0 \rightarrow_{\text{ndB}} u_0$  and  $t_0 \rightarrow_{\text{ndB}} s_0$ . By the *i.h.* there is  $t'_0$  such that  $s_0 \rightarrow_{\text{ndB}} t'_0$  and  $u_0 \rightarrow_{\text{ndB}} t'_0$ . Therefore  $s \rightarrow_{\text{ndB}} t'_0[x \triangleleft t_1] = t'$  and  $u \rightarrow_{\text{ndB}} t'$ .

- ((DB), (DB)), ((DB), (APPDB)), ((DB), (SUBDB)) and ((APPDB), (SUBDB)) are impossible cases.
- (2) If  $t \rightarrow_{\text{nsb}} u$  and  $t \rightarrow_{\text{nsb}} s$ , then there exists  $t'$  such that  $u \rightarrow_{\text{nsb}} t'$  and  $s \rightarrow_{\text{nsb}} t'$ . We consider the following cases:
- ((S), (S)) Impossible since  $u$  and  $s$  are assumed to be different.
  - ((S), (SUBS)) We have  $t \in \mathbf{U}$  then  $t = t_0[x//\lambda y.\text{LL}\langle p \rangle[z \triangleleft t_1]]$ , where  $y \notin \text{fv}(\text{LL}) \cup \text{fv}(t_1)$  and such that  $t \mapsto_{\text{sub}} \text{LL}\langle t_0\{x/\lambda y.p\} \rangle[z \triangleleft t_1] = u$ . There are three cases for  $t$ .
    - If  $[z \triangleleft t_1] = [z/\text{L}\langle w \rangle]$  or  $[z \triangleleft t_1] = [z/\text{L}\langle p_1 p_2 \rangle]$ , then the only possibility in each case is (S) on term  $\text{LL}\langle p \rangle[z/t_1]$ . We then have  $t \rightarrow_{\text{nsb}} t_0[x//\lambda y.\text{L}'\langle \text{LL}\langle p \rangle\{z/q\} \rangle] = s$  for some  $\text{L}'$  and some pure term  $q$ . So that  $u \rightarrow_{\text{nsb}} \text{L}'\langle \text{LL}\langle t_0\{x/\lambda y.p\} \rangle\{z/q\} \rangle = u'$  and  $s \rightarrow_{\text{nsb}} \text{L}'\langle \text{LL}\langle t_0\{x/\lambda y.p\} \rangle\{z/q\} \rangle = s'$ . The equality  $u' = s'$  holds because we can assume  $z \neq y$  by  $\alpha$ -equivalence, and  $z \notin \text{fv}(\text{LL})$  by definition.
    - If  $[z \triangleleft t_1] = [z/\lambda w.t'_1]$ , then the only possible case is (S) on term  $\text{LL}\langle p \rangle[z/\lambda w.t'_1]$ . We then have  $t = t_0[x//\lambda y.\text{LL}\langle p \rangle[z/\lambda w.t'_1]] \rightarrow_{\text{nsb}} t_0[x//\lambda y.\text{LL}\langle p \rangle[z//\lambda w.w'[w'/t'_1]]] = s$ . And we close the diagram with  $u \rightarrow_{\text{nsb}} \text{LL}\langle t_0\{x/\lambda y.p\} \rangle[z//\lambda w.w'[w'/t'_1]] = t'$  and  $s \rightarrow_{\text{nsb}} t'$ .
    - If  $[z \triangleleft t_1] = [z//\lambda w.t'_1]$ , then we have two different cases:
      - \* If the reduction happens inside  $t'_1$ , then  $t = t_0[x//\lambda y.\text{LL}\langle p \rangle[z//\lambda w.t'_1]] \rightarrow_{\text{nsb}} t_0[x//\lambda y.\text{LL}\langle p \rangle[z//\lambda w.s_1]] = s$ , where  $t'_1 \rightarrow_{\text{nsb}} s_1$ . Then we close by  $u \rightarrow_{\text{nsb}} \text{LL}\langle t_0\{x/\lambda y.p\} \rangle[z//\lambda w.s_1] = t'$  and  $s \rightarrow_{\text{nsb}} t'$ .
      - \* Otherwise, the (S) case for  $\text{LL}\langle p \rangle[z//\lambda w.t'_1]$  gives  $t \rightarrow_{\text{nsb}} t_0[x//\lambda y.\text{L}\langle \text{LL}\langle p \rangle\{z/v\} \rangle] = s$  for some  $\text{L}$  and some value  $v$ . So that  $u \rightarrow_{\text{nsb}} \text{L}\langle \text{LL}\langle t_0\{x/\lambda y.p\} \rangle\{z/v\} \rangle = u'$  and  $s \rightarrow_{\text{nsb}} \text{L}\langle \text{LL}\langle t_0\{x/\lambda y.p\} \rangle\{z/v\} \rangle = s'$ . The equality  $u' = s'$  holds because we can assume  $y \notin \text{fv}(v) \cup \{z\}$  by  $\alpha$ -equivalence, and  $z \notin \text{fv}(\text{LL})$  by definition.
  - ((APPS), (APPS)) We then have  $t = t_0 t_1$  such that  $t \rightarrow_{\text{nsb}} u_0 t_1 = u$  and  $t \rightarrow_{\text{nsb}} s_0 t_1 = s$ , where  $t_0 \rightarrow_{\text{nsb}} u_0$  and  $t_0 \rightarrow_{\text{nsb}} s_0$ . By the *i.h.*  $s_0 \rightarrow_{\text{nsb}} t'_0$  and  $u_0 \rightarrow_{\text{nsb}} t'_0$ . Therefore  $u \rightarrow_{\text{nsb}} t'_0 t_1 = t'$  and  $s \rightarrow_{\text{nsb}} t'$ .
  - ((SUBS), (SUBS)) We have  $t = t_0[x//\lambda y.t_1]$  such that  $t \rightarrow_{\text{nsb}} t_0[x//\lambda y.u_1] = u$  and  $t \rightarrow_{\text{nsb}} t_0[x//\lambda y.s_1] = s$ , where  $t_1 \rightarrow_{\text{nsb}} u_1$  and  $t_1 \rightarrow_{\text{nsb}} s_1$ . By the *i.h.*  $s_1 \rightarrow_{\text{nsb}} t'_1$  and  $u_1 \rightarrow_{\text{nsb}} t'_1$ . Therefore  $u \rightarrow_{\text{nsb}} t_0[x//\lambda y.t'_1] = t'$  and  $s \rightarrow_{\text{nsb}} t'$ .
  - ((S), (APPS)) and ((SUBS), (APPS)) are impossible cases.
- (3) If  $t \rightarrow_{\text{ndB}} u$  and  $t \rightarrow_{\text{nsb}} s$ , then there exists  $t'$  such that  $u \rightarrow_{\text{nsb}} t'$  and  $s \rightarrow_{\text{ndB}} t'$ . We consider the following cases:
- ((DB), (APPS)) We have  $t = \text{L}\langle \lambda x.t_0 \rangle[y \triangleleft t_2]t_1$  such that  $t \rightarrow_{\text{ndB}} \text{L}\langle t_0[x/t_1] \rangle[y \triangleleft t_2] = u$ . There are three cases for  $t \rightarrow_{\text{nsb}} s$ .
    - If  $t = \text{L}\langle \lambda x.t_0 \rangle[y//\lambda z.t'_2]t_1 \rightarrow_{\text{nsb}} \text{L}\langle \lambda x.t_0 \rangle[y//\lambda z.t'_3]t_1 = s$ , where  $t_2 = \lambda z.t'_2$  and  $t'_2 \rightarrow_{\text{nsb}} t'_3$ , then  $u \rightarrow_{\text{nsb}} \text{L}\langle t_0[x/t_1] \rangle[y//\lambda z.t'_3] = t'$  and  $s \rightarrow_{\text{ndB}} t'$ .
    - If  $t = \text{L}\langle \lambda x.t_0 \rangle[y/\lambda z.t'_2]t_1 \rightarrow_{\text{nsb}} \text{L}\langle \lambda x.t_0 \rangle[y//\lambda z.w[w/t'_2]]t_1 = s$ , where  $t_2 = \lambda z.t'_2$ , then  $u \rightarrow_{\text{nsb}} \text{L}\langle t_0[x/t_1] \rangle[y//\lambda z.w[w/t'_2]] = t'$  and  $s \rightarrow_{\text{ndB}} t'$ .
    - Otherwise, we have  $t \rightarrow_{\text{nsb}} \text{L}'\langle \text{L}\langle \lambda x.t_0 \rangle\{y/p\} \rangle t_1 = s$ , for some  $\text{L}'$  and some pure term  $p$ . Therefore,  $u \rightarrow_{\text{nsb}} \text{L}'\langle \text{L}\langle t_0[x/t_1] \rangle\{y/p\} \rangle = t'$  and  $s \rightarrow_{\text{ndB}} t'$  because  $y \notin \text{fv}(t_1)$ . Note that  $y$  may be free in  $\text{L}$ .
  - ((APPDB), (APPS)) We have  $t = t_0 t_1$  such that  $t \rightarrow_{\text{ndB}} u_0 t_1 = u$  and  $t \rightarrow_{\text{nsb}} s_0 t_1 = s$ , where  $t_0 \rightarrow_{\text{ndB}} u_0$  and  $t_0 \rightarrow_{\text{nsb}} s_0$ . By *i.h.* there exists  $t'_0$  such that  $s_0 \rightarrow_{\text{ndB}} t'_0$  and  $u_0 \rightarrow_{\text{nsb}} t'_0$ . Therefore,  $u \rightarrow_{\text{nsb}} t'_0 t_1 = t'$  and  $s \rightarrow_{\text{ndB}} t'$ .
  - ((SUBDB), (S)) We have  $t = t_0[x \triangleleft t_1]$  such that  $t \rightarrow_{\text{ndB}} u_0[x \triangleleft t_1] = u$ , where  $t_0 \rightarrow_{\text{ndB}} u_0$ . If  $t = t_0[x/\text{L}\langle \lambda y.t_2 \rangle] \rightarrow_{\text{nsb}} \text{L}\langle t_0[x//\lambda y.z[z/t_2]] \rangle = s$ , where  $t_1 = \lambda y.t_2$ , then  $s \rightarrow_{\text{ndB}}$

$L\langle u_0[x//\lambda y.z[z/t_2]] \rangle = t'$  and  $u \rightarrow_{\text{nsub}} t'$ . Otherwise,  $t \rightarrow_{\text{nsub}} L\langle t_0\{x/p\} \rangle = s$  for some  $L$  and some pure term  $p$ . We show that  $t_0\{x/p\} \rightarrow_{\text{ndB}} u_0\{x/p\}$  by induction on  $t_0 \rightarrow_{\text{ndB}} u_0$ . From this, we can deduce  $s \rightarrow_{\text{ndB}} L\langle u_0\{x/p\} \rangle = t'$  and conclude because  $u \rightarrow_{\text{nsub}} t'$ .

- (a) If  $t_0 = L'\langle \lambda y.q \rangle t_2 \rightarrow_{\text{dB}} L'\langle q[y/t_2] \rangle = u_0$ , then w.l.o.g. we can assume by  $\alpha$ -conversion that  $y \notin \text{fv}(p) \cup \{x\}$ , then  $t_0\{x/p\} = L'\langle x/p \rangle \langle \lambda y.q \{x/p\} \rangle t_2\{x/p\} \rightarrow_{\text{ndB}} L'\langle x/p \rangle \langle q\{x/p\} \rangle [y/t_2\{x/p\}] = u_0\{x/p\}$ .
- (b) If  $t_0 = t'_0 t_2 \rightarrow_{\text{ndB}} u'_0 t_2 = u_0$  from  $t'_0 \rightarrow_{\text{ndB}} u'_0$ , then by the *i.h.* and (APPDB) rule we can conclude  $t_0\{x/p\} = t'_0\{x/p\} t_2\{x/p\} \rightarrow_{\text{ndB}} u'_0\{x/p\} t_2\{x/p\} = u_0\{x/p\}$ .
- (c) If  $t_0 = t'_0[y \triangleleft t_2] \rightarrow_{\text{ndB}} u'_0[y \triangleleft t_2] = u_0$  from  $t'_0 \rightarrow_{\text{ndB}} u'_0$ , then w.l.o.g. we can assume by  $\alpha$ -conversion that  $x \neq y$ , then by *i.h.* and the (SUBDB) rule we conclude  $t_0\{x/p\} = t'_0\{x/p\} [y \triangleleft t_2\{x/p\}] \rightarrow_{\text{ndB}} u'_0\{x/p\} [y \triangleleft t_2\{x/p\}] = u_0\{x/p\}$ .
- ((SUBDB), (SUBS)) We have  $t = t_0[x//\lambda y.t_1]$  such that  $t \rightarrow_{\text{ndB}} u_0[x//\lambda y.t_1] = u$  and  $t \rightarrow_{\text{nsub}} t_0[x//\lambda y.s_1] = s$ , where  $t_0 \rightarrow_{\text{ndB}} u_0$  and  $t_1 \rightarrow_{\text{nsub}} s_1$ . Therefore  $u \rightarrow_{\text{nsub}} u_0[x//\lambda y.s_1] = t'$  and  $s \rightarrow_{\text{ndB}} t'$ .
- ((DB), (S)), ((DB), (SUBS)), ((APPDB), (S)), ((APPDB), (SUBS)) and ((SUBDB), (APPS)) are impossible cases.  $\square$

**Lemma 4.16.** *Let  $t \in U$ . Then  $x \in \text{ndv}(t)$  iff there exists a context  $N$  such that  $t = N\langle\langle x \rangle\rangle$ .*

*Proof.* (1)  $x \in \text{ndv}(t)$ . By induction on  $t$ .

- $t = x$ . Then we take  $N = \diamond$ .
  - $t = t'u$ . By the *i.h.* there exists  $N'$  such that  $t' = N'\langle\langle x \rangle\rangle$ . We then take  $N = N'u$ .
  - $t = t'[y/u]$ . By  $\alpha$ -conversion we can assume  $x \neq y$ . Either  $x \in \text{ndv}(t')$  or  $(x \in \text{ndv}(u)$  and  $y \in \text{ndv}(t')$ ). In the first case, there exists by the *i.h.* on  $t'$  a context  $N'$  such that  $t' = N'\langle\langle x \rangle\rangle$ . We then take  $N = N'[y/u]$ . In the second case, there exists by the *i.h.* on  $t'$  a context  $N_1$  such that  $t' = N_1\langle\langle y \rangle\rangle$ . By the *i.h.* on  $u$  we have  $u = N_2\langle\langle x \rangle\rangle$ . We then take  $N = N_1\langle\langle y \rangle\rangle [y/N_2]$ .
  - $t = t'[x//u]$ . By the *i.h.* there exists  $N'$  such that  $t' = N'\langle\langle x \rangle\rangle$ . We then take  $N = N'[x//u]$ .
- (2)  $t = N\langle\langle x \rangle\rangle$ . By induction on  $N$ .
- $N = \diamond$ . Then  $t = x$  and  $\text{ndv}(t) = \{x\}$ .
  - $N = N'u$ . Then  $t = t'u$  and by the *i.h.*  $x \in \text{ndv}(t')$ , so  $x \in \text{ndv}(t)$  by definition.
  - $N = N'[x \triangleleft u]$ . Then  $t = t'[x \triangleleft u]$  and by the *i.h.*  $x \in \text{ndv}(t')$ , so  $x \in \text{ndv}(t)$  by definition.
  - $N = N_1\langle\langle y \rangle\rangle [y/N_2]$ . Then  $t = t'[y/u]$ , where  $y \in \text{fv}(t')$ . By the *i.h.*  $x \in \text{ndv}(u)$ , so  $x \in \text{ndv}(t)$ .  $\square$

**Lemma 4.17.** *Let  $t \in U$ . Then  $t \in Ne$  iff  $t$  is in **flneed-nf**.*

*Proof.* We first show that  $t \in \overline{Ne}$  iff  $t$  is in **flneed-nf** and  $t$  is not an answer.

$\Rightarrow$  : We reason by induction on  $t \in \overline{Ne}$ .

- $t = x$ . This case is straightforward.
- $t = t'u$  where  $t' \in \overline{Ne}$ . By the *i.h.*  $t'$  is in **flneed-nf** and is not an answer, so it is not possible to apply any **dB**-reduction at the root. Then  $t$  is in **flneed-nf**, and since it is an application it is not an answer.
- $t = t'[x \triangleleft u]$  where  $t' \in \overline{Ne}$  and  $x \notin \text{ndv}(t')$ . By the *i.h.*  $t'$  is in **flneed-nf** and it is not an answer. Moreover, we cannot apply rules  $\rightarrow_{\text{sp1}}$  nor  $\rightarrow_{\text{1s}}$  because by Lemma 4.16 there is no context  $N$  surrounding  $x$ . Then  $t$  is in **flneed-nf** and is not an answer.



- $t = t'[x/u]$  where  $t', u \in \overline{\text{Ne}}$  and  $x \in \text{ndv}(t')$ . By the *i.h.*  $t'$  and  $u$  are in **flneed-nf** and are not answers. We cannot apply rule  $\rightarrow_{\text{sp1}}$  because  $u$  is not an answer. Then  $t$  is in **flneed-nf** and is not an answer.
- $\Leftarrow$  : We reason by induction on  $t$ .
- $t = x$  is immediate.
  - $t = t'u$ . Then  $t'$  is in **flneed-nf** and is not an answer (otherwise **dB** would be applicable). By the *i.h.*  $t' \in \overline{\text{Ne}}$  and thus  $t \in \overline{\text{Ne}}$ .
  - $t = t'[x/u]$ . Then  $t'$  is in **flneed-nf** and is not an answer. By the *i.h.*  $t' \in \overline{\text{Ne}}$ . There are two cases. If  $x \notin \text{ndv}(t')$ , then  $t \in \overline{\text{Ne}}$  by definition and we are done. Otherwise  $x \in \text{ndv}(t')$ , and we get  $t' = \mathbb{N}\langle\langle x \rangle\rangle$  by Lemma 4.16. Thus  $u$  cannot be an answer because  $\rightarrow_{\text{sp1}}$  would apply. Moreover,  $u$  is in **flneed-nf** because otherwise  $t$  would not be in **flneed-nf**. Thus,  $u \in \overline{\text{Ne}}$  by the *i.h.* and we get  $t \in \overline{\text{Ne}}$  by definition.
  - $t = t'[x//u]$ . We have  $x \notin \text{ndv}(t')$ , because  $\rightarrow_{1s}$  does not apply. By the *i.h.*  $t' \in \overline{\text{Ne}}$ , so that  $t \in \overline{\text{Ne}}$ .

Neutral terms are also normal. Answers are normal because the calculus is weak and they belong to the grammar **Ne**.  $\square$

**Lemma 5.4.** *For all derivation  $\Phi$  and all  $m, n \in \mathbb{N}$  with  $m > n$ ,  $\mathbb{M}(\Phi, m) = \mathbb{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0)$ .*

*Proof.* By induction on  $\Phi$ .

- $\Phi = \frac{}{x : [\sigma] \vdash x : \sigma}$ , then,  $\mathbb{M}(\Phi, m) = (0, 0, 1) = (0, 0, 1) + (0, 0 + (m - n) * 0, 0)$ .
- If  $\Phi = \frac{\Phi_t \triangleright \Gamma; y : \mathcal{M} \vdash t : \tau}{\Gamma \vdash \lambda y. t : \mathcal{M} \rightarrow \tau}$ , then

$$\begin{aligned}
\mathbb{M}(\Phi, m) &= \mathbb{M}(\Phi_t, m) + (1, m, 0) \\
&=_{i.h.} \mathbb{M}(\Phi_t, n) + (0, (m - n) * \text{sz}(\Phi_t), 0) + (1, n, 0) + (0, m - n, 0) \\
&= \mathbb{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi_t), 0) + (0, m - n, 0) \\
&= \mathbb{M}(\Phi, n) + (0, (m - n) * (\text{sz}(\Phi_t) + 1), 0) \\
&= \mathbb{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0)
\end{aligned}$$

- If  $\Phi = \frac{}{\vdash \lambda x. t : \mathbf{a}}$ , then,  $\mathbb{M}(\Phi, m) = (1, m, 0) = \mathbb{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0)$ .
- If  $\Phi = \frac{\Phi_t \triangleright \Gamma \vdash t : \mathcal{M} \rightarrow \tau \quad \Phi_u \triangleright \Delta \vdash u : \mathcal{M}}{\Gamma \uplus \Delta \vdash tu : \tau}$ , then

$$\begin{aligned}
\mathbb{M}(\Phi, m) &= \mathbb{M}(\Phi_t, m) + \mathbb{M}(\Phi_u, m) + (1, m, 0) \\
&=_{i.h.} \mathbb{M}(\Phi_t, n) + (0, (m - n) * \text{sz}(\Phi_t), 0) \\
&\quad + \mathbb{M}(\Phi_u, n) + (0, (m - n) * \text{sz}(\Phi_u), 0) + (1, n, 0) + (0, m - n, 0) \\
&= \mathbb{M}(\Phi, n) + (0, (m - n) * (\text{sz}(\Phi_t) + \text{sz}(\Phi_u) + 1), 0) \\
&= \mathbb{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0)
\end{aligned}$$

- If  $\Phi = \frac{\Phi_t \triangleright \Gamma; x : \mathcal{M} \vdash t : \sigma \quad \Phi_u \triangleright \Delta \vdash u : \mathcal{M}}{\Gamma \uplus \Delta \vdash t[x \triangleleft u] : \tau}$ , then

$$\begin{aligned}
\mathbf{M}(\Phi, m) &= \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\
&=_{i.h.} \mathbf{M}(\Phi_t, n) + (0, (m - n) * \text{sz}(\Phi_t), 0) + \mathbf{M}(\Phi_u, n + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\
&\quad + (0, (m - n) * \text{sz}(\Phi_u), 0) \\
&= \mathbf{M}(\Phi, n) + (0, (m - n) * (\text{sz}(\Phi_t) + \text{sz}(\Phi_u)), 0) \\
&= \mathbf{M}(\Phi, n) + (0, (m - n) * \text{sz}(\Phi), 0) \quad \square
\end{aligned}$$

**Lemma 6.1** (Partial Substitution). *Let  $\Phi \triangleright \Gamma; x : \mathcal{M} \vdash \mathbf{C}\langle\langle x \rangle\rangle : \sigma$  and  $\sqsubseteq$  denote multiset inclusion. Then, there exists  $\mathcal{N} \sqsubseteq \mathcal{M}$  such that for every  $\Phi_u \triangleright \Delta \vdash u : \mathcal{N}$  we have  $\Psi \triangleright \Gamma \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash \mathbf{C}\langle\langle u \rangle\rangle : \sigma$  and, for every  $m \in \mathbb{N}$ ,  $\mathbf{M}(\Psi, m) = \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C})) - (0, 0, |\mathcal{N}|)$ .*

*Proof.* By induction on  $\Phi$ .

- If  $\Phi = \frac{}{x : [\sigma] \vdash x : \sigma}$ , then  $\mathcal{N} = [\sigma]$  and  $\Psi = \Phi_u \triangleright \Delta \vdash u : [\sigma]$ . So,  $\mathbf{M}(\Psi, m) = \mathbf{M}(\Phi_u, m) = (0, 0, 1) + \mathbf{M}(\Phi_u, m + 0) - (0, 0, 1) = \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C})) - (0, 0, |\mathcal{N}|)$ .
- If  $\Phi = \frac{\Phi' \triangleright \Gamma; x : \mathcal{M}; y : \mathcal{M}_y \vdash \mathbf{C}'\langle\langle x \rangle\rangle : \tau}{\Gamma; x : \mathcal{M} \vdash \lambda y. \mathbf{C}'\langle\langle x \rangle\rangle : \mathcal{M}_y \rightarrow \tau}$ , then we can assume by  $\alpha$ -conversion that  $y \notin \text{dom}(\Delta)$  so that  $(\Gamma; y : \mathcal{M}_y) \uplus \Delta = \Gamma \uplus \Delta; y : \mathcal{M}_y$ . By using the *i.h.* we can then construct  $\Psi = \frac{\Psi' \triangleright \Gamma \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N}; y : \mathcal{M}_y \vdash \mathbf{C}'\langle\langle u \rangle\rangle : \tau}{\Gamma \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash \lambda y. \mathbf{C}'\langle\langle u \rangle\rangle : \mathcal{M}_y \rightarrow \tau}$ . Then,

$$\begin{aligned}
\mathbf{M}(\Psi, m) &= \mathbf{M}(\Psi', m) + (1, m, 0) \\
&=_{i.h.} \mathbf{M}(\Phi', m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C}')) - (0, 0, |\mathcal{N}|) + (1, m, 0) \\
&= \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\lambda y. \mathbf{C}')) - (0, 0, |\mathcal{N}|)
\end{aligned}$$

- If  $\Phi = \frac{}{\emptyset \vdash \lambda y. \mathbf{C}'\langle\langle x \rangle\rangle : \mathbf{a}}$ , we can build  $\Psi = \frac{}{\emptyset \vdash \lambda y. \mathbf{C}'\langle\langle u \rangle\rangle : \mathbf{a}}$ . In particular, we have  $\mathcal{M} = \mathcal{N} = []$ , and thus  $\Phi_u$  comes from the application of the (MANY) rule to 0 premises, so that  $\mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C})) = (0, 0, 0)$ . We have  $\mathbf{M}(\Phi, m) = \mathbf{M}(\Psi, m) = \mathbf{M}(\Psi, m) + (0, 0, 0) - (0, 0, 0)$ .
- If  $\Phi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1 \vdash \mathbf{C}'\langle\langle x \rangle\rangle : \mathcal{M}' \rightarrow \sigma \quad \Phi_2 \triangleright \Gamma_2; x : \mathcal{M}_2 \vdash t : \mathcal{M}'}{\Gamma_1 \uplus \Gamma_2; x : \mathcal{M} \vdash \mathbf{C}'\langle\langle x \rangle\rangle t : \sigma}$ , by *i.h.* there is  $\mathcal{N} \sqsubseteq \mathcal{M}_1$  such that we can construct

$$\Psi = \frac{\Psi_1 \triangleright \Gamma_1 \uplus \Delta; x : \mathcal{M}_1 \setminus \mathcal{N} \vdash \mathbf{C}'\langle\langle u \rangle\rangle : \mathcal{M}' \rightarrow \sigma \quad \Phi_2 \triangleright \Gamma_2; x : \mathcal{M}_2 \vdash t : \mathcal{M}'}{\Gamma_1 \uplus \Gamma_2 \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash \mathbf{C}'\langle\langle u \rangle\rangle t : \sigma}$$

because  $\mathcal{M} \setminus \mathcal{N} = \mathcal{M}_1 \setminus \mathcal{N} \sqcup \mathcal{M}_2$ . We have

$$\begin{aligned}
\mathbf{M}(\Psi, m) &= \mathbf{M}(\Psi_1, m) + \mathbf{M}(\Phi_2, m) + (1, m, 0) \\
&=_{i.h.} \mathbf{M}(\Phi_1, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C}')) - (0, 0, |\mathcal{N}|) + \mathbf{M}(\Phi_2, m) + (1, m, 0) \\
&= \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathbf{C}'t)) - (0, 0, |\mathcal{N}|)
\end{aligned}$$

- If

$$\Phi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1 \vdash t : [\tau_i]_{i \in I} \rightarrow \sigma \quad \frac{(\Phi_2^i \triangleright \Gamma_2^i; x : \mathcal{M}_2^i \vdash \mathcal{C}' \langle \langle x \rangle \rangle : \tau_i)_{i \in I}}{\Gamma_2; x : \mathcal{M}_2 \vdash \mathcal{C}' \langle \langle x \rangle \rangle : [\tau_i]_{i \in I}}}{\Gamma_1 \uplus \Gamma_2; x : \mathcal{M} \vdash t \mathcal{C}' \langle \langle x \rangle \rangle : \sigma}$$

where  $\mathcal{M}_2 = \sqcup_{i \in I} \mathcal{M}_2^i$  and  $\Gamma_2 = \uplus_{i \in I} \Gamma_2^i$ . Lemma 5.5 gives  $\Phi_u^i \triangleright \Delta^i \vdash u : \mathcal{N}^i$  such that  $\mathcal{N}^i \sqsubseteq \mathcal{M}_2^i$  for all  $i \in I$  and  $\mathcal{N} = \sqcup_{i \in I} \mathcal{N}^i$ . Moreover,  $\mathbf{M}(\Phi_u, m) = \sum_{i \in I} \mathbf{M}(\Phi_u^i, m)$ . By using the *i.h.* we can construct

$$\Psi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1 \vdash t : [\tau_i]_{i \in I} \rightarrow \sigma \quad \frac{(\Psi_2^i \triangleright \Gamma_2^i + \Delta^i; x : \mathcal{M}_2^i \setminus \mathcal{N}^i \vdash \mathcal{C}' \langle \langle u \rangle \rangle : \tau_i)_{i \in I}}{\Gamma_2 \uplus \Delta; x : \mathcal{M}_2 \setminus \mathcal{N} \vdash \mathcal{C}' \langle \langle u \rangle \rangle : [\tau_i]_{i \in I}}}{\Gamma_1 \uplus \Gamma_2 \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash t \mathcal{C}' \langle \langle u \rangle \rangle : \sigma}$$

where  $\mathcal{M} \setminus \mathcal{N} = \mathcal{M}_1 \sqcup \mathcal{M}_2 \setminus \mathcal{N}$ . We have

$$\begin{aligned} \mathbf{M}(\Psi, m) &= \mathbf{M}(\Phi_1, m) + \sum_{i \in I} \mathbf{M}(\Psi_2^i, m) + (1, m, 0) \\ &=_{i.h.} \mathbf{M}(\Phi_1, m) + \sum_{i \in I} (\mathbf{M}(\Phi_2^i, m) + \mathbf{M}(\Phi_u^i, m + \text{lv}_\diamond(\mathcal{C}')) - (0, 0, |\mathcal{N}^i|)) + (1, m, 0) \\ &= \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(t \mathcal{C}')) - (0, 0, |\mathcal{N}|) \end{aligned}$$

- If  $\Phi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1; y : \mathcal{M}_y \vdash \mathcal{C}' \langle \langle x \rangle \rangle : \sigma \quad \Phi_2 \triangleright \Gamma_2; x : \mathcal{M}_2 \vdash t : \mathcal{M}_y}{\Gamma_1 \uplus \Gamma_2; x : \mathcal{M} \vdash \mathcal{C}' \langle \langle x \rangle \rangle [y \triangleleft t] : \sigma}$ , then we can assume by  $\alpha$ -conversion that  $x \notin \text{fv}(u)$  and  $y \notin \text{fv}(u)$  thus, by the Relevance Lemma 5.1,  $y \notin \text{dom}(\Delta)$  so that in particular  $(\Gamma_1; y : \mathcal{M}_y) \uplus \Delta = \Gamma_1 \uplus \Delta; y : \mathcal{M}_y$ . By using the *i.h.* we can then construct

$$\Psi = \frac{\Psi_1 \triangleright \Gamma_1 \uplus \Delta; x : \mathcal{M}_1 \setminus \mathcal{N}; y : \mathcal{M}_y \vdash \mathcal{C}' \langle \langle u \rangle \rangle : \sigma \quad \Phi_2 \triangleright \Gamma_2; x : \mathcal{M}_2 \vdash t : \mathcal{M}_y}{\Gamma_1 \uplus \Gamma_2 \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash \mathcal{C}' \langle \langle u \rangle \rangle [y \triangleleft t] : \sigma}$$

because  $\mathcal{M} \setminus \mathcal{N} = \mathcal{M}_1 \setminus \mathcal{N} \sqcup \mathcal{M}_2$ . We have:

$$\begin{aligned} \mathbf{M}(\Psi, m) &= \mathbf{M}(\Psi_1, m) + \mathbf{M}(\Phi_2, m + \text{lv}_y(\mathcal{C}' \langle \langle u \rangle \rangle) + \text{ES}([y \triangleleft t])) \\ &=_{i.h.} \mathbf{M}(\Phi_1, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathcal{C}')) - (0, 0, |\mathcal{N}|) \\ &\quad + \mathbf{M}(\Phi_2, m + \text{lv}_y(\mathcal{C}' \langle \langle x \rangle \rangle)) + \text{ES}([y \triangleleft t]) \\ &= \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(\mathcal{C}'[y \triangleleft t])) - (0, 0, |\mathcal{N}|) \end{aligned}$$

- If

$$\Phi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1; y : [\tau_i]_{i \in I} \vdash t : \sigma \quad \frac{(\Phi_2^i \triangleright \Gamma_2^i; x : \mathcal{M}_2^i \vdash \mathcal{C}' \langle \langle x \rangle \rangle : \tau_i)_{i \in I}}{\Gamma_2; x : \mathcal{M}_2 \vdash \mathcal{C}' \langle \langle x \rangle \rangle : [\tau_i]_{i \in I}}}{\Gamma_1 \uplus \Gamma_2; x : \mathcal{M} \vdash t[y \triangleleft \mathcal{C}' \langle \langle x \rangle \rangle] : \sigma}$$

where  $\mathcal{M} = \mathcal{M}_1 \sqcup \mathcal{M}_2$ ,  $\mathcal{M}_2 = \sqcup_{i \in I} \mathcal{M}_2^i$  and  $\Gamma_2 = \uplus_{i \in I} \Gamma_2^i$ . Lem. 5.5 gives  $\Phi_u^i \triangleright \Delta^i \vdash u : \mathcal{N}^i$  for all  $i \in I$ . Moreover,  $\mathbf{M}(\Phi_u, m) = \sum_{i \in I} \mathbf{M}(\Phi_u^i, m)$ . By using the *i.h.* we can construct

$$\Psi = \frac{\Phi_1 \triangleright \Gamma_1; x : \mathcal{M}_1; y : [\tau_i]_{i \in I} \vdash t : \sigma \quad \frac{(\Psi_2^i \triangleright \Gamma_2^i \uplus \Delta^i; x : \mathcal{M}_2^i \setminus \mathcal{N}^i \vdash \mathcal{C}' \langle \langle u \rangle \rangle : \tau_i)_{i \in I}}{\Gamma_2 \uplus \Delta; x : \mathcal{M}_2 \setminus \mathcal{N} \vdash \mathcal{C}' \langle \langle u \rangle \rangle : [\tau_i]_{i \in I}}}{\Gamma_1 \uplus \Gamma_2 \uplus \Delta; x : \mathcal{M} \setminus \mathcal{N} \vdash t[y \triangleleft \mathcal{C}' \langle \langle u \rangle \rangle] : \sigma}$$

because  $\mathcal{M} \setminus \mathcal{N} = \mathcal{M}_1 \sqcup \mathcal{M}_2 \setminus \mathcal{N}$ , where  $\mathcal{N} = \sqcup_{i \in I} \mathcal{N}^i$ . We have

$$\begin{aligned} \mathbf{M}(\Psi, m) &= \mathbf{M}(\Phi_1, m) + \sum_{i \in I} \mathbf{M}(\Psi_2^i, m + \text{lv}_y(t) + \text{ES}([y \triangleleft \mathbf{C}' \langle\langle u \rangle\rangle])) \\ &=_{i.h.} \mathbf{M}(\Phi_1, m) + \sum_{i \in I} (\mathbf{M}(\Phi_2^i, m + \text{lv}_y(t) + \text{ES}([y \triangleleft \mathbf{C}' \langle\langle u \rangle\rangle])) \\ &\quad + \mathbf{M}(\Phi_u^i, m + \text{lv}_y(t) + \text{ES}([y \triangleleft \mathbf{C}' \langle\langle u \rangle\rangle]) + \text{lv}_\diamond(\mathbf{C}')) - (0, 0, |\mathcal{N}^i|)) \\ &= \mathbf{M}(\Phi, m) + \mathbf{M}(\Phi_u, m + \text{lv}_\diamond(t[y \triangleleft \mathbf{C}'])) - (0, 0, |\mathcal{N}|) \end{aligned}$$

Notice that a special case is when  $y \notin \text{fv}(t)$ . Then,  $I = \emptyset$ ,  $\Gamma = \Gamma_1$ ,  $\mathcal{N} = []$  and  $\Phi_u \triangleright \emptyset \vdash u : []$  is made only of a nullary (MANY) rule. Hence,  $\Phi = \Phi_1 = \Psi$ .  $\square$

**Lemma 6.3** (Weighted Subject Reduction for  $\rightarrow_\pi$ ). *Let  $\Phi_{t_0} \triangleright \Gamma \vdash t_0 : \sigma$  and  $t_0 \rightarrow_\pi t_1$ . Then there exists  $\Phi_{t_1} \triangleright \Gamma \vdash t_1 : \sigma$  such that  $\mathbf{M}(\Phi_{t_0}, m) = \mathbf{M}(\Phi_{t_1}, m)$  for every  $m \in \mathbb{N}$ .*

*Proof.* Let  $t_0 = \mathbf{C}(t'_0)$  and  $t_1 = \mathbf{C}(t'_1)$ , where  $t'_0 \rightarrow_\pi t'_1$  is a root step. We reason by induction on  $\mathbf{C}$ . We first consider the base cases where  $\mathbf{C} = \diamond$ .

(1)  $t'_0 = \lambda y.t[x \triangleleft u] \mapsto_\pi (\lambda y.t)[x \triangleleft u] = t'_1$ , where  $y \notin \text{fv}(u)$ . There are two possible typing derivations.

(a) The typing derivation is of the form

$$\Phi = \frac{\frac{\Phi_t \triangleright \Gamma'; y : \mathcal{N}; x : \mathcal{M} \vdash t : \tau \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Delta_u; y : \mathcal{N} \vdash t[x \triangleleft u] : \tau} \text{ (CUT)}}{\Gamma' \uplus \Delta_u \vdash \lambda y.t[x \triangleleft u] : \mathcal{N} \rightarrow \tau} \text{ (ABS)}$$

We construct the following derivation.

$$\Psi = \frac{\frac{\Phi_t \triangleright \Gamma'; y : \mathcal{N}; x : \mathcal{M} \vdash t : \tau}{\Gamma'; x : \mathcal{M} \vdash \lambda y.t : \mathcal{N} \rightarrow \tau} \text{ (ABS)} \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Delta_u \vdash (\lambda y.t)[x \triangleleft u] : \mathcal{N} \rightarrow \tau} \text{ (CUT)}$$

Moreover,

$$\begin{aligned} \mathbf{M}(\Phi, m) &= \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) + (1, m, 0) \\ &= \mathbf{M}(\Phi_t, m) + (1, m, 0) + \mathbf{M}(\Phi_u, m + \text{lv}_x(\lambda y.t) + \text{ES}([x \triangleleft u])) \\ &= \mathbf{M}(\Psi, m). \end{aligned}$$

(b) The typing derivation is of the form

$$\Phi = \frac{}{\vdash \lambda y.t[x \triangleleft u] : \mathbf{a}} \text{ (ANS)}$$

We construct the following derivation that has the same measure.

$$\Psi = \frac{\frac{}{\vdash \lambda y.t : \mathbf{a}} \text{ (ANS)} \quad \frac{}{\vdash u : []} \text{ (MANY)}}{\vdash (\lambda y.t)[x \triangleleft u] : \mathbf{a}} \text{ (CUT)}$$

(2)  $t'_0 = t[x \triangleleft u]s \mapsto_\pi (ts)[x \triangleleft u] = t'_1$ , where  $x \notin \text{fv}(s)$ . The typing derivation is of the form

$$\Phi =$$

$$\frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \mathcal{N} \rightarrow \sigma \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Delta_u \vdash t[x \triangleleft u] : \mathcal{N} \rightarrow \sigma} \text{ (CUT)} \quad \Phi_s \triangleright \Delta_s \vdash s : \mathcal{N}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x \triangleleft u]s : \sigma} \text{ (APP)}$$

We construct the following derivation.

$$\Psi = \frac{\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \mathcal{N} \rightarrow \sigma \quad \Phi_s \triangleright \Delta_s \vdash s : \mathcal{N}}{\Gamma' \uplus \Delta_s; x : \mathcal{M} \vdash ts : \sigma} \text{ (APP)} \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Delta_u \uplus \Delta_s(ts)[x \triangleleft u] : \sigma} \text{ (CUT)}$$

Moreover, since  $\text{lv}_x(t) = \text{lv}_x(ts)$ ,

$$\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi_t, m) + \mathbf{M}(\Phi_s, m) + (1, m, 0) + \mathbf{M}(\Phi_u, m + \text{lv}_x(ts) + \text{ES}([x \triangleleft u])) = \mathbf{M}(\Psi, m).$$

(3)  $t'_0 = ts[x \triangleleft u] \mapsto_{\pi} (ts)[x \triangleleft u] = t'_1$ , where  $x \notin \text{fv}(t)$ . Let

$$\Phi_{s[x \triangleleft u]} = \frac{\left( \frac{\Phi_s^i \triangleright \Delta_s^i; x : \mathcal{M}_i \vdash s : \rho_i \quad \frac{(\Phi_u^{i,j} \triangleright \Delta_u^{i,j} \vdash u : \delta_j)_{j \in J_i} \text{ (MANY)}}{\Delta_u^i \vdash u : \mathcal{M}_i} \text{ (CUT)}}{\Delta_u^i \uplus \Delta_s^i \vdash s[x \triangleleft u] : \rho_i} \right)_{i \in I} \text{ (MANY)}}{\Delta_u \uplus \Delta_s \vdash s[x \triangleleft u] : \mathcal{N}}$$

The typing derivation  $\Phi$  is of the form

$$\frac{\Phi_t \triangleright \Gamma' \vdash t : \mathcal{N} \rightarrow \sigma \quad \Phi_{s[x \triangleleft u]} \triangleright \Delta_u \uplus \Delta_s \vdash s[x \triangleleft u] : \mathcal{N}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash ts[x \triangleleft u] : \sigma} \text{ (APP)}$$

where  $\mathcal{M}_i = [\delta_j]_{j \in J_i}$ ,  $\mathcal{N} = [\rho_i]_{i \in I}$ ,  $\Delta_u^i = \uplus_{j \in J_i} \Delta_u^{i,j}$ ,  $\Delta_u = \uplus_{i \in I} \Delta_u^i$ , and  $\Delta_s = \uplus_{i \in I} \Delta_s^i$ .

Now, let

$$\Phi_s = \frac{\frac{(\Phi_s^i \triangleright \Delta_s^i; x : \mathcal{M}_i \vdash s : \rho_i)_{i \in I} \text{ (MANY)}}{\Delta_s; x : \mathcal{M} \vdash s : \mathcal{N}} \text{ (MANY)}}{\Gamma' \uplus \Delta_s; x : \mathcal{M} \vdash ts : \sigma} \text{ (APP)} \quad \Phi_u = \frac{(\Phi_u^{i,j} \triangleright \Delta_u^{i,j} \vdash u : \delta_j)_{j \in J_i, i \in I} \text{ (MANY)}}{\Delta_u \vdash u : \mathcal{M}} \text{ (MANY)}$$

We construct the following derivation  $\Psi$ .

$$\frac{\Phi_t \triangleright \Gamma' \vdash t : \mathcal{N} \rightarrow \sigma \quad \Phi_s \triangleright \Gamma' \uplus \Delta_s; x : \mathcal{M} \vdash ts : \sigma \quad \Phi_u \triangleright \Delta_u \vdash u : \mathcal{M}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash (ts)[x \triangleleft u] : \sigma} \text{ (CUT)}$$

where  $\mathcal{M} = \sqcup_{i \in I} \mathcal{M}_i$ , so that  $\mathcal{M} = [\delta_j]_{j \in J_i, i \in I}$ . Moreover, because  $\text{lv}_x(s) = \text{lv}_x(ts)$ ,

$$\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi_t, m) + (1, m, 0)$$

$$+ \sum_{i \in I} \left( \mathbf{M}(\Phi_s^i, m) + \sum_{j \in J_i} \mathbf{M}(\Phi_u^{i,j}, m + \text{lv}_x(s) + \text{ES}([x \triangleleft u])) \right) \\ = \mathbf{M}(\Psi, m)$$

(4)  $t'_0 = t[x \triangleleft u][y \triangleleft s] \mapsto_{\pi} t[x \triangleleft u][y \triangleleft s] = t'_1$ , where  $y \notin \text{fv}(t)$ . Let

$$\Phi_{u[y \triangleleft s]} = \frac{\left( \frac{\Phi_u^i \triangleright \Delta_u^i; y : \mathcal{N}_i \vdash u : \rho_i \quad \frac{(\Phi_s^{i,j} \triangleright \Delta_s^{i,j} \vdash s : \delta_j)_{j \in J_i} \text{ (MANY)}}{\Delta_s^i \vdash s : \mathcal{N}_i} \text{ (CUT)}}{\Delta_u^i \uplus \Delta_s^i \vdash u[y \triangleleft s] : \rho_i} \right)_{i \in I} \text{ (MANY)}}{\Delta_u \uplus \Delta_s \vdash u[y \triangleleft s] : \mathcal{M}}$$

The typing derivation  $\Phi$  is of the form

$$\frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \Phi_{u[y \triangleleft s]} \triangleright \Delta_u \uplus \Delta_s \vdash u[y \triangleleft s] : \mathcal{M}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x \triangleleft u[y \triangleleft s]] : \sigma} \text{ (CUT)}$$

where  $\mathcal{M} = [\rho_i]_{i \in I}$ ,  $\mathcal{N}_i = [\delta_j]_{j \in J_i}$ ,  $\Delta_u = \uplus_{i \in I} \Delta_u^i$ ,  $\Delta_s^i = \uplus_{j \in J_i} \Delta_s^{i,j}$ , and  $\Delta_s = \uplus_{i \in I} \Delta_s^i$ .  
Now, let

$$\Phi_{t[x \triangleleft u]} = \frac{\Phi_t \triangleright \Gamma'; x : \mathcal{M} \vdash t : \sigma \quad \frac{(\Phi_u^i \triangleright \Delta_u^i; y : \mathcal{N}_i \vdash u : \rho_i)_{i \in I} \text{ (MANY)}}{\Delta_u; y : \mathcal{N} \vdash u : \mathcal{M}} \text{ (CUT)}}{\Gamma' \uplus \Delta_u; y : \mathcal{N} \vdash t[x \triangleleft u] : \sigma}$$

We then construct the following derivation  $\Psi$ .

$$\frac{\Phi_{t[x \triangleleft u]} \triangleright \Gamma' \uplus \Delta_u; y : \mathcal{N} \vdash t[x \triangleleft u] : \sigma \quad \frac{(\Phi_s^{i,j} \triangleright \Delta_s^{i,j} \vdash s : \delta_j)_{j \in J_i, i \in I} \text{ (MANY)}}{\Delta_s \vdash s : \mathcal{N}} \text{ (CUT)}}{\Gamma' \uplus \Delta_u \uplus \Delta_s \vdash t[x \triangleleft u][y \triangleleft s] : \sigma}$$

where  $\mathcal{N} = \sqcup_{i \in I} \mathcal{N}_i$ , so that  $\mathcal{N} = [\delta_j]_{j \in J_i, i \in I}$ . Moreover, because  $y \notin \text{fv}(t)$ , we have that  $\text{lv}_y(t[x \triangleleft u]) = \text{lv}_x(t) + \text{lv}_y(u) + \text{ES}([x \triangleleft u])$  if  $y \in \text{fv}(u)$ , and  $\text{lv}_y(t[x \triangleleft u]) = 0$  otherwise. Now, we show that  $\mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u]) + \text{lv}_y(u) + \text{ES}([y \triangleleft s])) = \mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_y(t[x \triangleleft u]) + \text{ES}([y \triangleleft s]))$ . If  $y \in \text{fv}(u)$ , this is immediate. Otherwise, by the Relevance Lemma 5.1 we have  $J_i = []$  for any  $i$  thus  $s$  is not typed, so that both measures are equal to  $(0,0,0)$ . Then,

$$\begin{aligned} \mathbf{M}(\Phi, m) &= \mathbf{M}(\Phi_t, m) + \sum_{i \in I} \mathbf{M}(\Phi_u^i, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\ &\quad + \sum_{i \in I} \sum_{j \in J_i} \mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u]) + \text{lv}_y(u) + \text{ES}([y \triangleleft s])) \\ &= \mathbf{M}(\Phi_t, m) + \sum_{i \in I} \mathbf{M}(\Phi_u^i, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \\ &\quad + \sum_{i \in I} \sum_{j \in J_i} \mathbf{M}(\Phi_s^{i,j}, m + \text{lv}_y(t[x \triangleleft u]) + \text{ES}([y \triangleleft s])) \\ &= \mathbf{M}(\Psi, m) \end{aligned}$$

Now, we analyse all the inductive cases:

- (1) If  $\mathbf{C} = \lambda x. \mathbf{C}'$ , then we have  $\sigma = \mathcal{M} \rightarrow \tau$  and  $\Phi' \triangleright \Gamma; x : \mathcal{M} \vdash \mathbf{C}' \langle o \rangle : \tau$ . By the *i.h.* there is  $\Psi' \triangleright \Gamma; x : \mathcal{M} \vdash \mathbf{C}' \langle o' \rangle : \tau$  and therefore  $\Psi \triangleright \Gamma \vdash \lambda x. \mathbf{C}' \langle o' \rangle : \tau$ . Moreover,  $\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi', m) + (1, m, 0) \stackrel{i.h.}{=} \mathbf{M}(\Psi', m) + (1, m, 0) = \mathbf{M}(\Psi, m)$ .
- (2) If  $\mathbf{C} = \mathbf{C}'u$ , then we have  $\Phi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o \rangle : \mathcal{N} \rightarrow \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{N}$ . By the *i.h.* there is  $\Psi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o' \rangle : \mathcal{N} \rightarrow \sigma$ , so  $\Psi \triangleright \Gamma' \uplus \Delta \vdash \mathbf{C}' \langle o' \rangle u : \sigma$ . Moreover,  $\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi', m) + \mathbf{M}(\Phi_u, m) + (1, m, 0) \stackrel{i.h.}{=} \mathbf{M}(\Psi', m) + \mathbf{M}(\Phi_u, m) + (1, m, 0) = \mathbf{M}(\Psi, m)$ .
- (3) If  $\mathbf{C} = u\mathbf{C}'$ , the case is similar.
- (4) If  $\mathbf{C} = \mathbf{C}'[x \triangleleft u]$ , then we have  $\Phi' \triangleright \Gamma'; x : \mathcal{M} \vdash \mathbf{C}' \langle o \rangle : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . By the *i.h.* there is  $\Psi' \triangleright \Gamma'; x : \mathcal{M} \vdash \mathbf{C}' \langle o' \rangle : \sigma$ , so  $\Psi \triangleright \Gamma' \uplus \Delta \vdash \mathbf{C}' \langle o' \rangle [x \triangleleft u] : \sigma$ . Moreover,  $\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi', m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) \stackrel{i.h.}{=} \mathbf{M}(\Psi', m) + \mathbf{M}(\Phi_u, m + \text{lv}_x(t) + \text{ES}([x \triangleleft u])) = \mathbf{M}(\Psi, m)$ .

- (5) If  $\mathbf{C} = u[x \triangleleft \mathbf{C}']$ , then we have  $\Phi_u \triangleright \Delta; x : \mathcal{M} \vdash u : \sigma$  and  $\Phi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o \rangle : \mathcal{M}$ . By the *i.h.* there is  $\Psi' \triangleright \Gamma' \vdash \mathbf{C}' \langle o' \rangle : \mathcal{M}$ , so  $\Psi \triangleright \Gamma' \uplus \Delta \vdash u[x \triangleleft \mathbf{C}' \langle o' \rangle] : \sigma$ . Moreover,  $\mathbf{M}(\Phi, m) = \mathbf{M}(\Phi_u, m) + \mathbf{M}(\Phi', m + \text{lv}_x(u) + \text{ES}([x \triangleleft u])) =_{i.h.} \mathbf{M}(\Phi_u, m) + \mathbf{M}(\Psi', m + \text{lv}_x(u) + \text{ES}([x \triangleleft u])) = \mathbf{M}(\Psi, m)$ .  $\square$

**Lemma 6.11** (flneed-nfs are Typable). *Let  $t$  be in flneed-nf. Then there exists a derivation  $\Phi \triangleright \Gamma \vdash t : \tau$  such that for any  $x \notin \text{ndv}(t)$ ,  $\Gamma(x) = []$ .*

*Proof.* First, we show that if  $t$  is an answer  $\mathbf{L}\langle \lambda x.p \rangle$ , we can type it with type  $\mathbf{a}$  and  $\Gamma = \emptyset$ . We reason by induction on  $\mathbf{L}$ . If  $\mathbf{L} = \diamond$ , this is immediate. Otherwise, using the induction hypothesis, we build:

$$\frac{\emptyset \vdash \mathbf{L}\langle \lambda x.p \rangle : \mathbf{a} \quad \overline{\emptyset \vdash u : []}}{\emptyset \vdash \mathbf{L}\langle \lambda x.p \rangle [y \triangleleft u] : \mathbf{a}} \begin{array}{l} \text{(MANY)} \\ \text{(CUT)} \end{array}$$

The statement is then trivial since  $\Gamma = \emptyset$ . For neutral terms, we use induction on  $\overline{\mathbf{N}\mathbf{e}}$  with a stronger hypothesis: there exists a derivation for any given type  $\tau$ .

- $t = x$ . We can build  $\Phi \triangleright x : [\tau] \vdash x : \tau$ . Note that  $x \in \text{ndv}(t)$ .
- $t = t'u$ , where  $t' \in \overline{\mathbf{N}\mathbf{e}}$ . By the *i.h.* there is a derivation  $\Phi' \triangleright \Gamma \vdash t' : [] \rightarrow \tau$  verifying the statement. We then build:

$$\frac{\Phi' \triangleright \Gamma \vdash t' : [] \rightarrow \tau \quad \overline{\emptyset \vdash u : []}}{\Gamma \vdash t'u : \tau} \begin{array}{l} \text{(MANY)} \\ \text{(APP)} \end{array}$$

The statement holds by the *i.h.* because  $\text{ndv}(t) = \text{ndv}(t')$ .

- $t = t'[x \triangleleft u]$ , where  $t' \in \overline{\mathbf{N}\mathbf{e}}$ . By the *i.h.* there is a derivation  $\Phi' \triangleright \Gamma_{t'} \vdash t' : \tau$  verifying the statement. Let  $\Gamma_{t'} = \Gamma'; x : [\sigma_i]_{i \in I}$ . There are two cases.
  - If  $x \notin \text{ndv}(t')$ , then by the *i.h.*  $I = []$ . We can then build the following derivation.

$$\frac{\Phi' \triangleright \Gamma' \vdash t' : \tau \quad \overline{\emptyset \vdash u : []}}{\Gamma' \vdash t'[x \triangleleft u] : \tau} \begin{array}{l} \text{(MANY)} \\ \text{(CUT)} \end{array}$$

The property holds for  $\Gamma = \Gamma'$  because  $\text{ndv}(t) = \text{ndv}(t')$ .

- Otherwise,  $t = t'[x/u]$ , and  $u \in \overline{\mathbf{N}\mathbf{e}}$ . We apply the *i.h.* on  $u$ . There are derivations  $\Phi_u^i \triangleright \Delta_i \vdash u : \sigma_i$ . We take  $\Gamma = \Gamma_{t'} \uplus_{i \in I} \Delta_i$  and we build:

$$\frac{\Phi' \triangleright \Gamma_{t'} \vdash t' : \tau \quad \frac{(\Phi_u^i \triangleright \Delta_i \vdash u : \sigma_i)}{\uplus_{i \in I} \Delta_i \vdash u : [\sigma_i]_{i \in I}} \text{(MANY)}}{\Gamma \vdash t'[x/u] : \tau} \text{(CUT)}$$

where  $\Gamma = \Gamma_{t'} \uplus_{i \in I} \Delta_i$ . Moreover,  $\text{ndv}(t) = (\text{ndv}(t') \setminus x) \cup \text{ndv}(u)$  so the second property holds on  $\Gamma$  by the two induction hypothesis.  $\square$

**Lemma 6.14** (Partial Anti-Substitution). *Let  $\mathbf{C}\langle\langle x \rangle\rangle$  and  $u$  be terms s.t.  $x \notin \text{fv}(u)$  and  $\Phi \triangleright \Gamma \vdash \mathbf{C}\langle\langle u \rangle\rangle : \sigma$ . Then  $\exists \Gamma', \exists \Delta, \exists \mathcal{M}, \exists \Phi', \exists \Phi_u$  s.t.  $\Gamma = \Gamma' \uplus \Delta$ ,  $\Phi' \triangleright \Gamma' \uplus x : \mathcal{M} \vdash \mathbf{C}\langle\langle x \rangle\rangle : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ .*

*Proof.* By induction on the structure of  $\mathbf{C}$ .

- If  $\mathbf{C} = \diamond$  then the property trivially holds taking  $\Gamma' = \emptyset$ ,  $\Delta = \Gamma$ ,  $\mathcal{M} = [\sigma]$ ,  $\Phi' \triangleright x : [\sigma] \vdash x : \sigma$  and  $\Phi_u = \Phi$ .

- If  $C = \lambda y.C'$  then  $y \notin \text{fv}(u)$  and by  $\alpha$ -conversion we can assume that  $x \neq y$ . There are two cases:

- (1) If  $\Phi = \frac{\Phi_0 \triangleright \Gamma; y : \mathcal{M}_y \vdash C' \langle\langle u \rangle\rangle : \tau}{\Gamma \vdash \lambda y.C' \langle\langle u \rangle\rangle : \mathcal{M}_y \rightarrow \tau}$  then by *i.h.* there are  $\Gamma', \Delta, \mathcal{M}, \Phi'_0$  and  $\Phi_u$  such that  $\Gamma; y : \mathcal{M}_y = \Gamma'_0 \uplus \Delta$ ,  $\Phi'_0 \triangleright \Gamma'_0 \uplus x : \mathcal{M} \vdash C' \langle\langle x \rangle\rangle : \tau$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . By the Relevance Lemma 5.1  $y \notin \text{dom}(\Delta)$  thus  $\Gamma'_0 = \Gamma'; y : \mathcal{M}_y$ . Therefore,  $\Gamma'_0 \uplus x : \mathcal{M} = (\Gamma' \uplus x : \mathcal{M}); y : \mathcal{M}_y$  and

$$\Phi' = \frac{\Phi'_0 \triangleright (\Gamma' \uplus x : \mathcal{M}); y : \mathcal{M}_y \vdash C' \langle\langle x \rangle\rangle : \tau}{\Gamma' \uplus x : \mathcal{M} \vdash \lambda y.C' \langle\langle x \rangle\rangle : \mathcal{M}_y \rightarrow \tau}$$

- (2) If  $\Phi = \frac{}{\emptyset \vdash \lambda y.C' \langle\langle u \rangle\rangle : \mathbf{a}}$  then taking  $\Gamma', \Delta = \emptyset, \mathcal{M} = []$  and  $\Phi_u \triangleright \emptyset \vdash u : []$  we have

$$\Phi' = \frac{}{\emptyset \vdash \lambda y.C' \langle\langle x \rangle\rangle : \mathbf{a}}$$

- If  $C = C't$  then  $\Phi = \frac{\Phi_1 \triangleright \Gamma_1 \vdash C' \langle\langle u \rangle\rangle : \mathcal{M}' \rightarrow \sigma \quad \Phi_2 \triangleright \Gamma_2 \vdash t : \mathcal{M}'}{\Gamma_1 \uplus \Gamma_2 \vdash C' \langle\langle u \rangle\rangle t : \sigma}$ , where  $\Gamma = \Gamma_1 \uplus \Gamma_2$ .

By *i.h.* there are  $\Gamma'_1, \Delta, \mathcal{M}, \Phi'_1$  and  $\Phi_u$  such that  $\Gamma_1 = \Gamma'_1 \uplus \Delta$ ,  $\Phi'_1 \triangleright \Gamma'_1 \uplus x : \mathcal{M} \vdash C' \langle\langle x \rangle\rangle : \mathcal{M}' \rightarrow \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . Therefore, taking  $\Gamma' = \Gamma'_1 \uplus \Gamma_2$  we have

$$\Phi' = \frac{\Phi'_1 \triangleright \Gamma'_1 \uplus x : \mathcal{M} \vdash C' \langle\langle x \rangle\rangle : \mathcal{M}' \rightarrow \sigma \quad \Phi_2 \triangleright \Gamma_2 \vdash t : \mathcal{M}'}{(\Gamma'_1 \uplus x : \mathcal{M}) \uplus \Gamma_2 \vdash C' \langle\langle x \rangle\rangle t : \sigma}$$

where  $(\Gamma'_1 \uplus x : \mathcal{M}) \uplus \Gamma_2 = \Gamma' \uplus x : \mathcal{M}$ .

- If  $C = tC'$  then  $\Phi$  is of the form

$$\frac{\Phi_1 \triangleright \Gamma_1 \vdash t : [\tau_i]_{i \in I} \rightarrow \sigma \quad \frac{(\Phi_i \triangleright \Gamma_i \vdash C' \langle\langle u \rangle\rangle : \tau_i)_{i \in I}}{\Gamma_2 \vdash C' \langle\langle u \rangle\rangle : [\tau_i]_{i \in I} :}}{\Gamma_1 \uplus \Gamma_2 \vdash tC' \langle\langle u \rangle\rangle : \sigma}$$

where  $\Gamma_2 = \uplus_{i \in I} \Gamma_i$  and  $\Gamma = \Gamma_1 \uplus \Gamma_2$ . There are two cases:

- (1) If  $I \neq \emptyset$  then by *i.h.*  $\exists \Gamma'_i, \exists \Delta_i, \exists \mathcal{M}_i, \exists \Phi'_i, \exists \Phi_u^i$  s.t.  $\Gamma_i = \Gamma'_i \uplus \Delta_i$ ,  $\Phi'_i \triangleright \Gamma'_i \uplus x : \mathcal{M}_i \vdash C' \langle\langle x \rangle\rangle : \tau_i$  and  $\Phi_u^i \triangleright \Delta_i \vdash u : \mathcal{M}_i$ , for all  $i \in I$ . Let  $\Delta = \uplus_{i \in I} \Delta_i$  and  $\mathcal{M} = \sqcup_{i \in I} \mathcal{M}_i$

then from Split Lemma 5.5 we have  $\Phi_u = \frac{(\Phi_u^i \triangleright \Delta_i \vdash u : \mathcal{M}_i)_{i \in I}}{\Delta \vdash u : \mathcal{M}}$ . Let  $\Gamma'_2 = \uplus_{i \in I} \Gamma'_i$

then  $\Gamma'_2 \uplus \Delta = \Gamma_2$  and  $\Phi'$  is defined by

$$\frac{\Phi_1 \triangleright \Gamma_1 \vdash t : [\tau_i]_{i \in I} \rightarrow \sigma \quad \frac{(\Phi'_i \triangleright \Gamma'_i \uplus x : \mathcal{M}_i \vdash C' \langle\langle x \rangle\rangle : \tau_i)_{i \in I}}{\Gamma'_2 \uplus x : \mathcal{M} \vdash C' \langle\langle x \rangle\rangle : [\tau_i]_{i \in I} :}}{(\Gamma_1 \uplus \Gamma'_2) \uplus x : \mathcal{M} \vdash tC' \langle\langle x \rangle\rangle : \sigma}$$

where  $\Gamma' = \Gamma_1 \uplus \Gamma'_2$ .

- (2) If  $I = \emptyset$ , then  $[\tau_i]_{i \in I} = [], \Gamma_2 = \emptyset$  and  $\Gamma = \Gamma_1$ . Therefore, taking  $\Gamma' = \Gamma_1, \Delta = \emptyset, \mathcal{M} = [], \Phi_u = \emptyset \vdash u : []$ , we have  $\Gamma_1 = \Gamma_1 \uplus x : [] = \Gamma' \uplus x : []$  and  $\Gamma' \uplus \Delta = \Gamma_1 \uplus \emptyset = \Gamma$ . We take

$$\Phi' = \frac{\Phi_1 \triangleright \Gamma_1 \vdash t : [] \rightarrow \sigma \quad \emptyset \vdash C' \langle\langle x \rangle\rangle : []}{\Gamma_1 \vdash tC' \langle\langle x \rangle\rangle : \sigma}.$$



- If  $\mathbf{C} = \mathbf{C}'[y \triangleleft t]$  then  $\Phi = \frac{\Phi_1 \triangleright \Gamma_1; y : \mathcal{M}_y \vdash \mathbf{C}'\langle\langle u \rangle\rangle : \sigma \quad \Phi_2 \triangleright \Gamma_2 \vdash t : \mathcal{M}_y}{\Gamma_1 \uplus \Gamma_2 \vdash \mathbf{C}'\langle\langle u \rangle\rangle[y \triangleleft t] : \sigma}$  where  $\Gamma =$

$\Gamma_1 \uplus \Gamma_2$ . Moreover,  $y \notin \text{fv}(u)$  and by  $\alpha$ -conversion we can assume that  $x \neq y$ . By *i.h.* there are  $\Gamma'_1, \Delta, \mathcal{M}, \Phi'_1$  and  $\Phi_u$  such that  $\Gamma_1; y : \mathcal{M}_y = \Gamma'_1 \uplus \Delta$ ,  $\Phi'_1 \triangleright \Gamma'_1 \uplus x : \mathcal{M} \vdash \mathbf{C}'\langle\langle x \rangle\rangle : \sigma$  and  $\Phi_u \triangleright \Delta \vdash u : \mathcal{M}$ . By the Relevance Lemma 5.1  $y \notin \text{dom}(\Delta)$  thus  $\Gamma'_1 = \Gamma''; y : \mathcal{M}_y$ ,  $\Gamma'_1 \uplus x : \mathcal{M} = (\Gamma'' \uplus x : \mathcal{M}); y : \mathcal{M}_y$  and  $\Gamma'' \uplus \Delta = \Gamma_1$ . Therefore, taking  $\Gamma' = \Gamma'' \uplus \Gamma_2$  we have

$$\Phi' = \frac{\Phi'_1 \triangleright (\Gamma'' \uplus x : \mathcal{M}); y : \mathcal{M}_y \vdash \mathbf{C}'\langle\langle x \rangle\rangle : \sigma \quad \Phi_2 \triangleright \Gamma_2 \vdash t : \mathcal{M}_y}{(\Gamma'' \uplus x : \mathcal{M}) \uplus \Gamma_2 \vdash \mathbf{C}'\langle\langle x \rangle\rangle[y \triangleleft t] : \sigma}$$

where  $(\Gamma'' \uplus x : \mathcal{M}) \uplus \Gamma_2 = \Gamma' \uplus x : \mathcal{M}$ .

- If  $\mathbf{C} = t[y \triangleleft \mathbf{C}']$  then  $\Phi$  is of the form

$$\frac{\Phi_1 \triangleright \Gamma_1; y : [\tau_i]_{i \in I} \vdash t : \sigma \quad \frac{(\Phi_i \triangleright \Gamma_i \vdash \mathbf{C}'\langle\langle u \rangle\rangle) : \tau_i)_{i \in I}}{\Gamma_2 \vdash \mathbf{C}'\langle\langle u \rangle\rangle : [\tau_i]_{i \in I}}}{\Gamma_1 \uplus \Gamma_2 \vdash t[y \triangleleft \mathbf{C}'\langle\langle u \rangle\rangle] : \sigma}$$

where  $\Gamma_2 = \uplus_{i \in I} \Gamma_i$  and  $\Gamma = \Gamma_1 \uplus \Gamma_2$ . There are two cases:

- (1) If  $I \neq \emptyset$  then by *i.h.* there are  $\Gamma'_i, \Delta_i, \mathcal{M}_i, \Phi'_i$  and  $\Phi_u^i$  such that  $\Gamma_i = \Gamma'_i \uplus \Delta_i$ ,  $\Phi'_i \triangleright \Gamma'_i \uplus x : \mathcal{M}_i \vdash \mathbf{C}'\langle\langle x \rangle\rangle : \tau_i$  and  $\Phi_u^i \triangleright \Delta_i \vdash u : \mathcal{M}_i$ , for all  $i \in I$ . Let  $\Delta = \uplus_{i \in I} \Delta_i$  and  $\mathcal{M} = \sqcup_{i \in I} \mathcal{M}_i$

then from Split Lemma 5.5 we have  $\Phi_u = \frac{(\Phi_u^i \triangleright \Delta_i \vdash u : \mathcal{M}_i)_{i \in I}}{\Delta \vdash u : \mathcal{M}}$ . Let  $\Gamma'_2 = \uplus_{i \in I} \Gamma'_i$

then  $\Gamma'_2 \uplus \Delta = \Gamma_2$  and  $\Phi'$  is defined by

$$\frac{\Phi_1 \triangleright \Gamma_1; y : [\tau_i]_{i \in I} \vdash t : \sigma \quad \frac{(\Phi'_i \triangleright \Gamma'_i \uplus x : \mathcal{M}_i \vdash \mathbf{C}'\langle\langle x \rangle\rangle) : \tau_i)_{i \in I}}{\Gamma'_2 \uplus x : \mathcal{M} \vdash \mathbf{C}'\langle\langle x \rangle\rangle : [\tau_i]_{i \in I}}}{(\Gamma_1 \uplus \Gamma'_2) \uplus x : \mathcal{M} \vdash t[y \triangleleft \mathbf{C}'\langle\langle x \rangle\rangle] : \sigma}$$

where  $\Gamma' = \Gamma_1 \uplus \Gamma'_2$ .

- (2) If  $I = \emptyset$  then  $[\tau_i]_{i \in I} = []$ ,  $\Gamma_2 = \emptyset$  and  $\Gamma = \Gamma_1$ . Moreover,  $y \notin \text{dom}(\Gamma_1)$ . Therefore, taking  $\Gamma' = \Gamma_1$ ,  $\Delta = \emptyset$ ,  $\mathcal{M} = []$ ,  $\Phi_u = \emptyset \vdash u : []$ , we have  $\Gamma_1 = \Gamma_1 \uplus x : [] = \Gamma' \uplus x : []$  and  $\Gamma' \uplus \Delta = \Gamma_1 \uplus \emptyset = \Gamma$ . We take

$$\Phi' = \frac{\Phi_1 \triangleright \Gamma_1 \vdash t : \sigma \quad \emptyset \vdash \mathbf{C}'\langle\langle x \rangle\rangle : []}{\Gamma_1 \vdash t[y \triangleleft \mathbf{C}'\langle\langle x \rangle\rangle] : \sigma} \quad \square$$