

VARIETIES OF UNARY-DETERMINED DISTRIBUTIVE ℓ -MAGMAS AND BUNCHED IMPLICATION ALGEBRAS

NATANAEL ALPAY ^a, PETER JIPSEN ^b, AND MELISSA SUGIMOTO ^b

^a University of California Irvine, California 92697, USA
e-mail address: nalpay@uci.edu

^b Chapman University, Orange, California 92866, USA
e-mail address: jipsen@chapman.edu, msugimoto@chapman.edu

ABSTRACT. A distributive lattice-ordered magma (*dl*-magma) (A, \wedge, \vee, \cdot) is a distributive lattice with a binary operation \cdot that preserves joins in both arguments, and when \cdot is associative then (A, \vee, \cdot) is an idempotent semiring. A *dl*-magma with a top \top is *unary-determined* if $x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$. These algebras are term-equivalent to a subvariety of distributive lattices with \top and two join-preserving unary operations $\mathfrak{p}, \mathfrak{q}$. We obtain simple conditions on $\mathfrak{p}, \mathfrak{q}$ such that $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$ is associative, commutative, idempotent and/or has an identity element.

This generalizes previous results on the structure of doubly idempotent semirings and, in the case when the distributive lattice is a Heyting algebra, it provides structural insight into unary-determined algebraic models of bunched implication logic. We also provide Kripke semantics for the algebras under consideration, which leads to more efficient algorithms for constructing finite models. We find all subdirectly irreducible algebras up to cardinality eight in which $\mathfrak{p} = \mathfrak{q}$ is a closure operator, as well as all finite unary-determined bunched implication chains and map out the poset of join-irreducible varieties generated by them.

1. INTRODUCTION

Idempotent semirings (A, \vee, \cdot) play an important role in several areas of computer science, such as network optimization, formal languages, Kleene algebras and program semantics. In this setting they are often assumed to have constants $0, 1$ that are the additive and multiplicative identity respectively, with 0 also being an absorbing element. However semirings are usually only assumed to have two binary operations $+, \cdot$ that are associative such that $+$ is also commutative and \cdot distributes over $+$ from the left and right [HW98]. A semiring is (additively) idempotent if $x + x = x$, hence $+$ is a (join) semilattice, and *doubly idempotent* if $x \cdot x = x$ as well. If \cdot is also commutative, then it defines a meet semilattice. The special case when these two semilattices coincide corresponds exactly to the variety of distributive lattices, which have a well understood structure theory.

In [AJ20] a complete structural description was given for finite commutative doubly idempotent semirings where either the multiplicative semilattice is a chain, or the additive

Key words and phrases: distributive lattice-ordered magmas, bunched implication algebras, idempotent semirings, enumerating finite models.

semilattice is a Boolean algebra. Here we show that the second description can be significantly generalized to the setting where the additive semilattice is a distributive lattice, dropping the assumptions of finiteness, multiplicative commutativity and idempotence in favor of the algebraic condition $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$ for two unary join-preserving operations $\mathfrak{p}, \mathfrak{q}$. While this property is quite restrictive in general, it does hold in all idempotent Boolean magmas and expresses a binary operation in terms of two simpler unary operations. A full structural description of all (finite) idempotent semirings is unlikely, but in the setting of unary-determined idempotent semirings progress is possible.

In Section 2 we provide the needed background and prove a term-equivalence between a subvariety of top-bounded dl -magmas and a subvariety of top-bounded distributive lattices with two unary operators. This is then specialized to cases where \cdot is associative, commutative, idempotent or has an identity element. In the next section we show that when the distributive lattice is a Brouwerian algebra or Heyting algebra, then \cdot is residuated if and only if both \mathfrak{p} and \mathfrak{q} are residuated. This establishes a connection with bunched implication algebras (BI-algebras) that are the algebraic semantics of bunched implication logic [OP99], used in the setting of separation logic for program verification, including reasoning about pointers [Rey02] and concurrent processes [O’H04]. Section 4 contains Kripke semantics for dl -magmas, called Birkhoff frames, and for the two unary operators $\mathfrak{p}, \mathfrak{q}$. This establishes the connection to the previous results in [AJ20] and leads to the main result (Thm. 4.9) that preorder forest P -frames capture a larger class of multiplicatively idempotent BI-algebras and doubly idempotent semirings. Although the heap models of BI-algebras used in applications are not (multiplicatively) idempotent, they contain idempotent subalgebras and homomorphic images, hence a characterization of unary-determined idempotent BI-algebras does provide insight into the general case. In the next section we define weakly conservative l -magmas and their corresponding frames. In Section 6 we apply the results from the previous sections to count the number of preorder forest P -frames up to isomorphism if their partial order is an antichain and also if it is a chain. Finally in Section 7 we calculate all subdirectly irreducible algebras up to cardinality eight in which \mathfrak{p} and \mathfrak{q} are the same closure operator, and map out the poset of join-irreducible varieties generated by them.

2. A TERM-EQUIVALENCE BETWEEN DISTRIBUTIVE LATTICES WITH OPERATORS

A *distributive lattice-ordered magma*, or *dl-magma*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot)$ such that (A, \wedge, \vee) is a distributive lattice and \cdot is a *binary operator*, which in this case means a binary operation that distributes over \vee , i.e., $x \cdot (y \vee z) = x \cdot y \vee x \cdot z$ and $(x \vee y) \cdot z = x \cdot z \vee y \cdot z$ for all $x, y, z \in A$. Throughout it is assumed that \cdot binds more strongly than \wedge, \vee , and as usual the lattice order \leq is defined by $x \leq y \iff x \wedge y = x$ ($\iff x \vee y = y$). If the distributive lattice has a top element \top or a bottom element \perp then it is called \top -*bounded* or \perp -*bounded*, or simply *bounded* if both exist. A dl -magma \mathbf{A} is *normal* and \cdot is a *normal* operation if A is \perp -bounded and satisfies $x \cdot \perp = \perp = \perp \cdot x$. Similarly, a unary operation f on A is an *operator* if it satisfies the identity $f(x \vee y) = fx \vee fy$, and it is *normal* if $f\perp = \perp$. For brevity and to reduce the number of nested parentheses, we write function application as fx rather than $f(x)$, with the convention that it has priority over \cdot hence, e.g., $fx \cdot y = (f(x)) \cdot y$ (this convention ensures unique readability). Note that since operators distribute over \vee in each argument, they are order-preserving in each argument. The operation f is said to be *inflationary* if $x \leq fx$ for all $x \in A$.

A binary operation \cdot is said to be *idempotent* if $x \cdot x = x$ for all $x \in A$, *commutative* if $x \cdot y = y \cdot x$ and *associative* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. A *semigroup* is a set with an associative operation, a *monoid* is a semigroup with an identity element denoted by 1 , a *band* is a semigroup that is also idempotent, and a *semilattice* is a commutative band. As usual, a semilattice is partially ordered by $x \sqsubseteq y \iff x \cdot y = x$, and in this case $x \cdot y$ is the meet operation with respect to \sqsubseteq . We also use this terminology with the prefix *dl*, in which case the magma operation satisfies the corresponding identities.

A *dl*-magma is called *unary-determined* if it is \top -bounded and satisfies the identity

$$x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y).$$

As examples, we mention that all doubly-idempotent semirings with a Boolean join-semilattice are unary-determined (see Lemma 2.3). Complete and atomic versions of such semirings are studied in [AJ20], and the results from that paper are generalized here to unary-determined *dl*-magmas with algebraic proofs that apply to all members of the variety, while the previous results applied only to complete and atomic algebras.

A *dlpq-algebra* is a \top -bounded distributive lattice with two unary operators $\mathfrak{p}, \mathfrak{q}$ that satisfy

$$x \wedge \mathfrak{p}\top \leq \mathfrak{q}x, \quad x \wedge \mathfrak{q}\top \leq \mathfrak{p}x.$$

We note that throughout $\mathfrak{p}, \mathfrak{q}$ denote unary operations, and they bind more strongly than \cdot, \wedge, \vee . These two (in)equational axioms are needed for our first result which shows that unary-determined *dl*-magmas and *dlpq*-algebras are term-equivalent. This means that although the two varieties are based on different sets of fundamental operations (called the signature of each class), each fundamental operation of an algebra in one variety is identical to a term-operation constructed from fundamental operations of an algebra in the other variety (and vice versa). From the point of view of category theory, term-equivalent varieties are model categories of the same Lawvere theory.

Note that the (in)equalities above are satisfied in any \top -bounded distributive lattice with inflationary operators $\mathfrak{p}, \mathfrak{q}$ since then $\mathfrak{p}\top = \top = \mathfrak{q}\top$. A *dlp*-algebra is a *dlpq*-algebra that satisfies the identity $\mathfrak{p}x = \mathfrak{q}x$.

Although unary-determined *dl*-magmas and *dlpq*-algebras seem rather special, they are simpler than general *dl*-magmas, yet include interesting idempotent semirings (as reducts).

Theorem 2.1.

- (1) Let $(A, \wedge, \vee, \top, \mathfrak{p}, \mathfrak{q})$ be a *dlpq*-algebra and define $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$. Then $(A, \wedge, \vee, \top, \cdot)$ is a unary-determined *dl*-magma and $\mathfrak{p}, \mathfrak{q}$ are given by $\mathfrak{p}x = x \cdot \top$ and $\mathfrak{q}x = \top \cdot x$.
- (2) Let $(A, \wedge, \vee, \top, \cdot)$ be a unary-determined *dl*-magma and define $\mathfrak{p}x = x \cdot \top$, $\mathfrak{q}x = \top \cdot x$. Then $(A, \wedge, \vee, \top, \mathfrak{p}, \mathfrak{q})$ is a *dlpq*-algebra and \cdot is definable from $\mathfrak{p}, \mathfrak{q}$ via $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$.

Proof. (1) Assume $\mathfrak{p}, \mathfrak{q}$ are unary operators on a \top -bounded distributive lattice (A, \wedge, \vee, \top) , and $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$. Then

$$\begin{aligned} x \cdot (y \vee z) &= (\mathfrak{p}x \wedge (y \vee z)) \vee (x \wedge \mathfrak{q}(y \vee z)) \\ &= (\mathfrak{p}x \wedge y) \vee (\mathfrak{p}x \wedge z) \vee (x \wedge \mathfrak{q}y) \vee (x \wedge \mathfrak{q}z) \\ &= (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y) \vee (\mathfrak{p}x \wedge z) \vee (x \wedge \mathfrak{q}z) \\ &= x \cdot y \vee x \cdot z. \end{aligned}$$

A similar calculation shows that $(x \vee y) \cdot z = x \cdot z \vee y \cdot z$, hence \cdot is an operator.

Since \mathbf{p}, \mathbf{q} satisfy $x \wedge \mathbf{q}\top \leq \mathbf{p}x$, it follows that $x \cdot \top = (\mathbf{p}x \wedge \top) \vee (x \wedge \mathbf{q}\top) = \mathbf{p}x \vee (x \wedge \mathbf{q}\top) = \mathbf{p}x$, and similarly $\top \cdot x = \mathbf{q}x$ is implied by $x \wedge \mathbf{p}\top \leq \mathbf{q}x$. Now the identity $x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$ holds by definition.

(2) Assume $(A, \wedge, \vee, \top, \cdot)$ is a unary-determined $d\ell$ -magma, and define $\mathbf{p}x = x \cdot \top$, $\mathbf{q}x = \top \cdot y$. Then \mathbf{p}, \mathbf{q} are unary operators and $\mathbf{p}x = x \cdot \top = (x \cdot \top \wedge \top) \vee (x \wedge \top \cdot \top) = \mathbf{p}x \vee (x \wedge \mathbf{q}\top)$, hence $x \wedge \mathbf{q}\top \leq \mathbf{p}x$. The inequality $x \wedge \mathbf{p}\top \leq \mathbf{q}x$ is proved similarly. The operation \cdot can be recovered from \mathbf{p}, \mathbf{q} since $x \cdot y = (\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)$ follows from the identity we assumed. \square

The preceding theorem shows that unary-determined $d\ell$ -magmas and $d\ell\mathbf{p}\mathbf{q}$ -algebras are “essentially the same”, and we can choose to work with the signature that is preferred in a given situation. The unary operators of $d\ell\mathbf{p}\mathbf{q}$ -algebras are simpler to handle, while the binary operator \cdot is familiar in the semiring setting. Next we examine how standard properties of \cdot are captured by identities in the language of $d\ell\mathbf{p}\mathbf{q}$ -algebras.

Lemma 2.2. *Let $(A, \wedge, \vee, \top, \mathbf{p}, \mathbf{q})$ be a $d\ell\mathbf{p}\mathbf{q}$ -algebra and define $x \cdot y = (\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)$.*

- (1) *The operator \cdot is commutative if and only if $\mathbf{p} = \mathbf{q}$.*
- (2) *If $\mathbf{p} = \mathbf{q}$, then \cdot is associative if and only if $\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}\mathbf{p}y)$.*
- (3) *The operator \cdot is idempotent if and only if \mathbf{p} and \mathbf{q} are inflationary, if and only if $\mathbf{p}\top = \top = \mathbf{q}\top$.*
- (4) *If \cdot is idempotent, then it is associative if and only if*

$$\begin{aligned} \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) &= (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{q}y) \text{ and} \\ \mathbf{q}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) &= (\mathbf{p}x \wedge y) \vee (\mathbf{q}x \wedge \mathbf{q}y). \end{aligned}$$

- (5) *The operator \cdot has an identity 1 if and only if $\mathbf{p}1 = \top = \mathbf{q}1$ and $(\mathbf{p}x \vee \mathbf{q}x) \wedge 1 \leq x$.*
- (6) *If \cdot has an identity, then \cdot is idempotent.*

Proof. (1) Assuming $x \cdot y = y \cdot x$, we clearly have $x \cdot \top = \top \cdot x$, hence $\mathbf{p}x = \mathbf{q}x$. The converse makes use of commutativity of \wedge and \vee : $x \cdot y = (\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y) = (\mathbf{p}y \wedge x) \vee (y \wedge \mathbf{p}x) = y \cdot x$.

(2) Assume $\mathbf{p} = \mathbf{q}$. If \cdot is associative then $(x \cdot y) \cdot \top = x \cdot (y \cdot \top)$, so by the previous theorem, $\mathbf{p}(x \cdot y) = x \cdot \mathbf{p}y$, which translates to

$$\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}\mathbf{p}y) \quad (*).$$

Conversely, suppose $(*)$ holds, and note that $\mathbf{p}(x \cdot y) = \mathbf{p}(y \cdot x)$ by (1), hence

$$\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (\mathbf{p}\mathbf{p}x \wedge y) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}\mathbf{p}y) \vee (\mathbf{p}\mathbf{p}x \wedge y) \quad (**).$$

It suffices to prove $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$ since then $z \cdot (y \cdot x) \leq (z \cdot y) \cdot x$ follows by commutativity. Now

$$\begin{aligned} (x \cdot y) \cdot z &= [\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)) \wedge z] \vee [((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)) \wedge \mathbf{p}z] \\ &= [((\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}\mathbf{p}y)) \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{p}z] \vee [x \wedge \mathbf{p}y \wedge \mathbf{p}z] \text{ using } (*) \\ &= [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [x \wedge \mathbf{p}\mathbf{p}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{p}z] \vee [x \wedge \mathbf{p}y \wedge \mathbf{p}z] \\ &\leq [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{p}z] \vee [x \wedge \mathbf{p}y \wedge \mathbf{p}z] \vee [x \wedge y \wedge \mathbf{p}\mathbf{p}z] \vee [x \wedge \mathbf{p}\mathbf{p}y \wedge z] \\ &= [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{p}z] \vee [x \wedge ((\mathbf{p}y \wedge \mathbf{p}z) \vee (y \wedge \mathbf{p}\mathbf{p}z) \vee (\mathbf{p}\mathbf{p}y \wedge z))] \\ &= [\mathbf{p}x \wedge ((\mathbf{p}y \wedge z) \vee (y \wedge \mathbf{p}z))] \vee [x \wedge \mathbf{p}((\mathbf{p}y \wedge z) \vee (y \wedge \mathbf{p}z))] \text{ using } (**). \\ &= x \cdot (y \cdot z). \end{aligned}$$

(3) If \cdot is idempotent, then $x = x \cdot x \leq x \cdot \top = \mathbf{p}x$ and $x \leq \top \cdot x = \mathbf{q}x$. Conversely, if \mathbf{p}, \mathbf{q} are inflationary then $x \cdot x = (\mathbf{p}x \wedge x) \vee (x \wedge \mathbf{q}x) = x \vee x = x$, hence \cdot is idempotent. For the

second equivalence, if $\mathbf{p}\top = \top = \mathbf{q}\top$, then \mathbf{p}, \mathbf{q} are inflationary since they satisfy $x \wedge \mathbf{p}\top \leq \mathbf{q}x$ and $x \wedge \mathbf{q}\top \leq \mathbf{p}x$. The reverse implication holds because $x \leq \mathbf{p}x, \mathbf{q}x$ implies $\top \leq \mathbf{p}\top, \mathbf{q}\top$.

(4) Assume \cdot is idempotent and associative. Then $(\top \cdot x) \cdot \top = \top \cdot (x \cdot \top)$, hence $\mathbf{q}\mathbf{p}x = \mathbf{p}\mathbf{q}x$. Furthermore, $\mathbf{p}\mathbf{q}x = \top \cdot x \cdot \top = \top \cdot x \cdot x \cdot \top = (\mathbf{q}x) \cdot (\mathbf{p}x) = (\mathbf{p}\mathbf{q}x \wedge \mathbf{p}x) \vee (\mathbf{q}x \wedge \mathbf{q}\mathbf{p}x)$. By (3) \mathbf{p}, \mathbf{q} are inflationary, so $\mathbf{p}x \leq \mathbf{p}\mathbf{q}x$ and $\mathbf{q}x \leq \mathbf{q}\mathbf{p}x$. Therefore $\mathbf{p}\mathbf{q}x = \mathbf{p}x \vee \mathbf{q}x$. Now we translate $(x \cdot y) \cdot \top = x \cdot (y \cdot \top)$ to obtain $\mathbf{p}(x \cdot y) = x \cdot (\mathbf{p}y)$, hence

$$\begin{aligned} \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) &= (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{q}\mathbf{p}y) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge (\mathbf{p}y \vee \mathbf{q}y)) \\ &= (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{q}y) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{q}y) \text{ since } x \leq \mathbf{p}x \text{ by (3)}. \end{aligned}$$

The identity $\mathbf{q}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) = (\mathbf{p}x \wedge y) \vee (\mathbf{q}x \wedge \mathbf{q}y)$ has a similar proof.

Conversely, assume the two identities hold. Then using distributivity

$$\begin{aligned} (x \cdot y) \cdot z &= [\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) \wedge z] \vee [((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{q}y)) \wedge \mathbf{q}z] \\ &= [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [x \wedge \mathbf{q}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{q}z] \vee [x \wedge \mathbf{q}y \wedge \mathbf{q}z] \\ &= [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{q}z] \vee [x \wedge \mathbf{q}y \wedge \mathbf{q}z] \text{ since } x \wedge \mathbf{q}y \wedge z \leq x \wedge \mathbf{q}y \wedge \mathbf{q}z \\ &= [\mathbf{p}x \wedge \mathbf{p}y \wedge z] \vee [\mathbf{p}x \wedge y \wedge \mathbf{q}z] \vee [x \wedge \mathbf{p}y \wedge z] \vee [x \wedge \mathbf{q}y \wedge \mathbf{q}z] \\ &= [\mathbf{p}x \wedge ((\mathbf{p}y \wedge z) \vee (y \wedge \mathbf{q}z))] \vee [x \wedge \mathbf{q}((\mathbf{p}y \wedge z) \vee (y \wedge \mathbf{q}z))] = x \cdot (y \cdot z). \end{aligned}$$

(5) Assume x has an identity 1 . Then $\mathbf{p}1 = 1 \cdot \top = \top = \top \cdot 1 = \mathbf{q}1$ and $x = x \cdot 1 = (\mathbf{p}x \wedge 1) \vee (x \wedge \mathbf{q}1) = (\mathbf{p}x \wedge 1) \vee x$, so $\mathbf{p}x \wedge 1 \leq x$ and similarly $\mathbf{q}x \wedge 1 \leq x$. Therefore $(\mathbf{p}x \vee \mathbf{q}x) \wedge 1 = (\mathbf{p}x \wedge 1) \vee (\mathbf{q}x \wedge 1) \leq x$.

Conversely, suppose $\mathbf{p}1 = \top = \mathbf{q}1$ and $(\mathbf{p}x \vee \mathbf{q}x) \wedge 1 \leq x$. Then $x \cdot 1 = (\mathbf{p}x \wedge 1) \vee (x \wedge \mathbf{q}1) = (\mathbf{p}x \wedge 1) \vee x = x$ since $\mathbf{p}x \wedge 1 \leq x$. Likewise $1 \cdot x = x$.

(6) This follows from (3) since $x = x \cdot 1 \leq x \cdot \top = \mathbf{p}x$ and $x = 1 \cdot x \leq \mathbf{q}x$. \square

Note that if \mathbf{A} also has a bottom bound \perp , then \mathbf{p}, \mathbf{q} are normal if and only if \cdot is normal, hence the term-equivalence preserves normality.

This term-equivalence is useful since distributive lattices with unary operators are considerably simpler than distributive lattices with binary operators. In particular, (2) and (4) show that associativity can be replaced by one or two 2-variable identities in this variety. This provides more efficient ways to construct associative operators from a (pair of) unary operator(s) on a distributive lattice. The variety of \top -bounded distributive lattices is obtained as a subvariety of $d\ell\mathbf{p}\mathbf{q}$ -algebras that satisfy $\mathbf{p}x = x = \mathbf{q}x$, or a subvariety of unary determined $d\ell$ -magmas that satisfy $x \cdot y = x \wedge y$.

For small cardinalities, Table 1 shows the number of algebras that are unary-determined (shown in the even numbered rows) for several subvarieties of normal $d\ell$ -magmas. As seen from rows 7-10, under the assumption of associativity, commutativity and idempotence of \cdot , the property of being unary-determined is a relatively mild restriction compared to the general case of normal $d\ell$ -magmas.

A *Boolean magma* is a Boolean algebra with a binary operator. The next lemma shows that if the operator is idempotent, then it is always unary-determined, hence the results in the current paper generalize the theorems about idempotent Boolean nonassociative quantales in [AJ20].

Lemma 2.3. *Every idempotent Boolean magma $(A, \wedge, \vee, \neg, \perp, \top, \cdot)$ is unary-determined, i.e., satisfies $x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$.*

Proof. Idempotence is equivalent to $x \wedge y \leq x \cdot y \leq x \vee y$ since $(x \wedge y) \cdot (x \wedge y) \leq x \cdot y \leq (x \vee y) \cdot (x \vee y)$ holds in all partially ordered algebras where \cdot is an order-preserving binary

	Cardinality $n =$	2	3	4	5	6	7	8
1	normal dl -magmas	2	20	1116				
2	normal $dlpq$ -algebras	2	6	46	3435			
3	normal comm. dl -magmas	2	10	148	3554			
4	normal dlp -algebras	2	4	15	46	183	688	
5	normal comm. dl -semigroups	2	8	57	392	3212		
6	normal assoc. dlp -algebras	2	4	13	35	109	315	998
7	normal comm. idem. dl -semigroups	1	2	8	25	97	366	
8	normal assoc. idem. dlp -algebras	1	2	7	18	57	163	521
9	normal comm. idem. dl -monoids	1	2	6	15	44	115	326
10	normal assoc. idem. $dlp1$ -algebras	1	2	5	10	24	47	108
11	distributive lattices	1	1	2	3	5	8	15

Table 1: The number of algebras of cardinality n up to isomorphism.

operation. The following calculation

$$\begin{aligned} x \cdot \top \wedge y &= x \cdot (y \vee \neg y) \wedge y = (x \cdot y \wedge y) \vee (x \cdot \neg y \wedge y) \\ &\leq x \cdot y \vee ((x \vee \neg y) \wedge y) = x \cdot y \vee (x \wedge y) \vee (\neg y \wedge y) = x \cdot y \end{aligned}$$

and a similar one for $x \wedge \top \cdot y \leq x \cdot y$ prove that $x \cdot y \geq (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$.

Using Boolean negation, the opposite inequality is equivalent to

$$x \cdot y \wedge \neg(x \cdot \top \wedge y) \leq x \wedge \top \cdot y.$$

By De Morgan's law it suffices to show $(x \cdot y \wedge \neg(x \cdot \top)) \vee (x \cdot y \wedge \neg y) \leq x \wedge \top \cdot y$. Since $x \cdot y \leq x \cdot \top$, the first meet disappears. Next, by idempotence, $x \cdot y \wedge \neg y \leq (x \vee y) \wedge \neg y = (x \wedge \neg y) \vee (y \wedge \neg y) \leq x$ and finally $x \cdot y \wedge \neg y \leq x \cdot y \leq \top \cdot y$. \square

3. BI-ALGEBRAS FROM HEYTING ALGEBRAS AND RESIDUATED UNARY OPERATIONS

We now recall some basic definitions about residuated operations, adjoints and residuated lattices. For an overview and additional details we refer to [GJKO07]. A *Brouwerian algebra* $(A, \wedge, \vee, \rightarrow, \top)$ is a \top -bounded lattice such that \rightarrow is the *residual* of \wedge , i.e.,

$$x \wedge y \leq z \iff y \leq x \rightarrow z.$$

Since \rightarrow is the residual of \wedge , we have that \wedge is join-preserving, so the lattice is distributive [GJKO07, Lem. 4.1]. The \top -bound is included as a constant since it always exists when a meet-operation has a residual: $x \wedge y \leq x$ always holds, hence $y \leq (x \rightarrow x) = \top$. A *Heyting algebra* is a bounded Brouwerian algebra with a constant \perp denoting the bottom element.

A *dual operator* is an n -ary operation on a lattice that preserves meets in each argument. A *residual* or *upper adjoint* of a unary operation \mathbf{p} on a poset $\mathbf{A} = (A, \leq)$ is a unary operation \mathbf{p}^* such that

$$\mathbf{p}x \leq y \iff x \leq \mathbf{p}^*y$$

for all $x, y \in A$. If \mathbf{A} is a lattice, then the existence of a residual guarantees that \mathbf{p} is an operator and \mathbf{p}^* is a dual operator [GJKO07, Lem. 3.5]. Moreover, if \mathbf{A} is bounded, then $\mathbf{p}\perp = \perp$ and $\mathbf{p}^*\top = \top$.

A binary operation \cdot on a poset is *residuated* if there exist a *left residual* \backslash and a *right residual* $/$ such that

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y.$$

A *residuated ℓ -magma* $(A, \wedge, \vee, \cdot, \backslash, /)$ is a lattice with a residuated binary operation. In this case \cdot is an operator and $\backslash, /$ are dual operators in the “numerator” argument. The “denominator” arguments of $\backslash, /$ map joins to meets, hence they are order reversing. A *residuated Brouwerian-magma* is a residuated ℓ -magma expanded with \rightarrow, \top such that $(A, \wedge, \vee, \rightarrow, \top)$ is a Brouwerian algebra.

A *residuated lattice* is a residuated ℓ -magma with \cdot associative and a constant 1 that is an identity element, i.e., $(A, \cdot, 1)$ is a monoid. A *generalized bunched implication algebra*, or GBI-algebra, $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \top, \cdot, 1, \backslash, /)$ is a \top -bounded residuated lattice with a residual \rightarrow for the meet operation, i.e., $(A, \wedge, \vee, \rightarrow, \top)$ is a Brouwerian algebra. A GBI-algebra is called a *bunched implication algebra* (BI-algebra) if \cdot is commutative and \mathbf{A} also has a bottom element, denoted by the constant \perp , hence a BI-algebra has a Heyting algebra reduct. These algebras are the algebraic semantics for bunched implication logic, which is the propositional part of separation logic, a Hoare logic used for reasoning about memory references in computer programs. In this setting the operation \cdot is usually denoted by $*$, the left residual \backslash is denoted \dashv , and $/$ can be omitted since $x/y = y \dashv x$.

Note that the property of being a residual can be expressed by inequalities (\mathfrak{p}^* is a residual of \mathfrak{p} if and only if $\mathfrak{p}(\mathfrak{p}^*x) \leq x \leq \mathfrak{p}^*(\mathfrak{p}x)$ for all x , and $\mathfrak{p}, \mathfrak{p}^*$ are order preserving), hence the classes of all Brouwerian algebras, Heyting algebras, residuated ℓ -magmas, residuated Brouwerian-magmas, residuated lattices, (G)BI-algebras, and pairs of residuated unary maps on a lattice are varieties (see e.g. [GJKO07, Thm. 2.7 and Lem. 3.2]). Recall also that a \top -bounded magma is unary-determined if it satisfies the identity $x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$.

We are now ready to prove a result that upgrades the term-equivalence of Theorem 2.1 to Brouwerian algebras with two pairs of residuated maps and unary-determined residuated Brouwerian-magmas.

Theorem 3.1.

- (1) Let $(A, \wedge, \vee, \rightarrow, \top, \mathfrak{p}, \mathfrak{p}^*, \mathfrak{q}, \mathfrak{q}^*)$ be a Brouwerian algebra with unary operators $\mathfrak{p}, \mathfrak{q}$ and their residuals $\mathfrak{p}^*, \mathfrak{q}^*$ such that $x \wedge \mathfrak{p}\top \leq \mathfrak{q}x$, $x \wedge \mathfrak{q}\top \leq \mathfrak{p}x$. If we define $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y)$,

$$x \backslash y = (\mathfrak{p}x \rightarrow y) \wedge \mathfrak{q}^*(x \rightarrow y) \quad \text{and} \quad x / y = \mathfrak{p}^*(y \rightarrow x) \wedge (\mathfrak{q}y \rightarrow x),$$

then $(A, \wedge, \vee, \top, \cdot, \backslash, /)$ is a unary-determined residuated Brouwerian-magma and the unary operations are recovered by $\mathfrak{p}x = x \cdot \top$, $\mathfrak{p}^*x = x / \top$, $\mathfrak{q}x = \top \cdot x$ and $\mathfrak{q}^*x = \top \backslash x$.

- (2) Let $(A, \wedge, \vee, \rightarrow, \top, \cdot, \backslash, /)$ be a unary-determined residuated Brouwerian-magma and define $\mathfrak{p}x = x \cdot \top$, $\mathfrak{p}^*x = x / \top$, $\mathfrak{q}x = \top \cdot x$ and $\mathfrak{q}^*x = \top \backslash x$. Then $(A, \wedge, \vee, \rightarrow, \top, \mathfrak{p}, \mathfrak{p}^*, \mathfrak{q}, \mathfrak{q}^*)$ is a Brouwerian algebra with a unary operators $\mathfrak{p}, \mathfrak{q}$ and dual operators $\mathfrak{p}^*, \mathfrak{q}^*$ that satisfies $x \wedge \mathfrak{p}\top \leq \mathfrak{q}x$, $x \wedge \mathfrak{q}\top \leq \mathfrak{p}x$.

Proof. (1) The following calculation shows that \cdot is residuated.

$$\begin{aligned} x \cdot y \leq z &\iff (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{q}y) \leq z &&\iff \mathfrak{p}x \wedge y \leq z \text{ and } x \wedge \mathfrak{q}y \leq z \\ &\iff y \leq \mathfrak{p}x \rightarrow z \text{ and } y \leq \mathfrak{q}^*(x \rightarrow z) &&\iff y \leq (\mathfrak{p}x \rightarrow z) \wedge \mathfrak{q}^*(x \rightarrow z) \end{aligned}$$

hence $x \backslash z = (\mathfrak{p}x \rightarrow z) \wedge \mathfrak{q}^*(x \rightarrow z)$ and similarly $z / y = \mathfrak{p}^*(y \rightarrow z) \wedge (\mathfrak{q}y \rightarrow z)$. By Theorem 2.1 it follows that $\mathfrak{p}x = x \cdot \top$, $\mathfrak{q}x = \top \cdot x$ and $x \cdot y = (x \cdot \top \wedge y) \vee (x \wedge \top \cdot y)$. Since $x \cdot \top \leq y \iff x \leq y / \top$ we obtain $\mathfrak{p}^*(x) = x / \top$, and similarly $\mathfrak{q}^*(x) = \top \backslash x$.

(2) Since \cdot is residuated it follows that \mathbf{p}^* and \mathbf{q}^* are the unary residuals of \mathbf{p} , \mathbf{q} respectively. The remaining parts hold by Theorem 2.1. \square

Recall that a closure operator \mathbf{p} is an order-preserving unary function on a poset such that $x \leq \mathbf{p}x = \mathbf{p}\mathbf{p}x$. A bounded $d\ell\mathbf{p}$ -algebra where \mathbf{p} is a normal closure operator is called a $d\ell\mathbf{p}$ -closure algebra. If \cdot is idempotent and associative then $x \cdot \top = x \cdot (\top \cdot \top) = (x \cdot \top) \cdot \top$, so $\mathbf{p}x = x \cdot \top$ is a closure operator.

Lemma 3.2. *Assume \mathbf{A} is a $d\ell\mathbf{p}$ -closure algebra and let $x \cdot y = (\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)$. Then \cdot is associative if and only if $\mathbf{p}x \wedge \mathbf{p}y \leq \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \vee \mathbf{p}y))$.*

Proof. By Lemma 2.2 \cdot is associative if and only if the identity $\mathbf{p}((\mathbf{p}x \wedge y) \vee (x \vee \mathbf{p}y)) = (\mathbf{p}x \wedge \mathbf{p}y) \vee (x \wedge \mathbf{p}y)$ holds. This is equivalent to $\mathbf{p}x \wedge \mathbf{p}y \leq \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \vee \mathbf{p}y))$ since $x \wedge \mathbf{p}y \leq \mathbf{p}x \wedge \mathbf{p}y$, $\mathbf{p}(\mathbf{p}x \wedge y) \leq \mathbf{p}\mathbf{p}x \wedge \mathbf{p}y = \mathbf{p}x \wedge \mathbf{p}y$ and similarly $\mathbf{p}(x \wedge \mathbf{p}y) \leq \mathbf{p}x \wedge \mathbf{p}y$. \square

We note that there exist non-associative $d\ell\mathbf{p}$ -algebras, as shown (later) by the algebra \mathbf{D}_{12} in Figure 4. The preceding theorems specialize to a term-equivalence for a subvariety of unary-determined BI-algebras as follows:

Corollary 3.3.

- (1) *Let $(A, \wedge, \vee, \rightarrow, \top, \perp, \mathbf{p}, \mathbf{p}^*, 1)$ be a Heyting algebra with a closure operator \mathbf{p} , residual \mathbf{p}^* and constant 1 such that $\mathbf{p}x \wedge \mathbf{p}y \leq \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y))$, $\mathbf{p}1 = \top$ and $\mathbf{p}x \wedge 1 \leq x$. If we define $x * y = (\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y)$ and $x \multimap y = (\mathbf{p}x \rightarrow y) \wedge \mathbf{p}^*(x \wedge y)$ then $(A, \wedge, \vee, \top, \rightarrow, *, \multimap, 1)$ is a unary-determined BI-algebra and $(x * \top) \wedge (y * \top) \leq (((x * \top) \wedge y) \vee (x \wedge (y * \top))) * \top$ holds.*
- (2) *Let $(A, \wedge, \vee, \rightarrow, \top, \perp, *, \multimap, 1)$ be a unary-determined BI-algebra, and define $\mathbf{p}x = x * \top$ and $\mathbf{p}^*x = \top \multimap x$. Then $(A, \wedge, \vee, \rightarrow, \top, \perp, \mathbf{p}, \mathbf{p}^*, 1)$ is a Heyting algebra with a closure operator \mathbf{p} that has \mathbf{p}^* as residual and satisfies $\mathbf{p}x \wedge \mathbf{p}y \leq \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \wedge \mathbf{p}y))$, $\mathbf{p}1 = \top$ and $\mathbf{p}x \wedge 1 \leq x$.*

Heyting algebras with a closure operator provide algebraic semantics for $\mathbf{IntS4}_\diamond$ [Doš85], an intuitionistic modal logic with an $S4$ -modality. Hence the result above establishes a connection between certain extensions of bunched implication logic and of intuitionistic modal logic.

By Lemma 2.2(6) unary-determined BI-algebras satisfy $x * x = x$, which does not hold in BI-algebras that model applications (e.g., heap storage). However, as mentioned in the introduction, they are members of the variety of BI-algebras, and understanding their properties via this term-equivalence is useful for the general theory. E.g., structural results about algebraic objects (such as rings) often start by investigating the idempotent algebras, followed by sets of idempotent elements in more general algebras. Line 10 in Table 1 also shows that finite unary-determined BI-algebras are not rare (algebras with normal join-preserving operators can be uniquely expanded with residuals in the finite case, hence expansions of the algebras counted in Line 10 are indeed term-equivalent to unary-determined BI-algebras).

4. RELATIONAL SEMANTICS FOR $d\ell$ -MAGMAS

We now briefly recall relational semantics for bounded distributive lattices with operators and then apply correspondence theory to derive first-order conditions for the equational properties of the preceding sections.

An element in a lattice is *completely join-irreducible* if it is not the supremum of all the elements strictly below it. The set of all completely join-irreducible elements of a lattice A is denoted by $J(A)$, and it is partially ordered by restricting the order of A to $J(A)$. For example, if A is a Boolean lattice, then $J(A) = At(A)$ is the antichain of *atoms*, i.e., all elements immediately above the bottom element. The set $M(A)$ of completely meet-irreducible elements is defined dually. A lattice is *perfect* if it is complete (i.e., all joins and meets exist) and every element is a join of completely join-irreducibles and a meet of completely meet-irreducibles. For a Boolean algebra, the notion of perfect is equivalent to being *complete* (i.e., joins and meets of all subsets exist) and *atomic* (i.e., every non-bottom element has an atom below it).

Recall that for a poset $\mathbf{W} = (W, \leq)$, a *downset* is a subset X such that $y \leq x \in X$ implies $y \in X$. As in modal logic, W is considered a set of “worlds” or states. We let $D(\mathbf{W})$ be the set of all downsets of \mathbf{W} , and $(D(\mathbf{W}), \cap, \cup)$ the *lattice of downsets*. The collection $D(\mathbf{W})$ is a perfect distributive lattice with infinitary meet and join given by (arbitrary) intersections and unions. The following result, due to Birkhoff [Bir67, Thm. III.3.3] for lattices of finite height, shows that up to isomorphism all perfect distributive lattices arise in this way. The poset $J(D(\mathbf{W}))$ contains exactly the principal downsets $\downarrow x = \{y \in W \mid y \leq x\}$.

Theorem 4.1 [DP02, 10.29]. *For a lattice A the following are equivalent:*

- (1) *A is distributive and perfect.*
- (2) *A is isomorphic to the lattice of downsets of a partial order.*

Note that the set of upsets of a poset is also a perfect distributive lattice, and if it is ordered by reverse inclusion then this lattice is isomorphic to the downset lattice described above. It is also well known that the maps J and D are functors for a categorial duality between the category of posets with order-preserving maps and the category of perfect distributive lattices with complete lattice homomorphisms (i.e., maps that preserve arbitrary joins and meets).

A *complete operator* on a complete lattice is an operation that, in each argument, is completely join-preserving, while a *complete dual operator* is completely meet-preserving (in each argument). A lattice-ordered algebra is called *perfect* if its lattice reduct is perfect and every fundamental operation on it is a complete operator or dual operator. The duality between the category of perfect distributive lattices and posets extends to the category of perfect distributive lattices with (a fixed signature of) complete operators and dual operators. The corresponding poset category has additional relations of arity $n + 1$ for each (dual) operator of arity n , and the relations have to be upward or downward closed in each argument. For example, a binary relation $Q \subseteq W^2$ is upward closed in the second argument if $xQy \leq z \implies xQz$. Here $xQy \leq z$ is an abbreviation for xQy and $y \leq z$.

Perfect distributive lattices with operators, their residuals and dual operators are algebraic models for many logics, including relevance logic, intuitionistic logic, Hájek’s basic logic, Łukasiewicz logic and bunched implication logic [GNV05, GJKO07]. In such an algebra \mathbf{A} , a join-preserving binary operation is determined by a ternary relation R on $J(\mathbf{A})$ given by

$$xRyz \iff x \leq yz.$$

The notation $xRyz$ is shorthand for $(x, y, z) \in R$. For $b, c \in A$ the product bc is recovered as $\bigvee \{x \in J(\mathbf{A}) \mid xRyz \text{ for some } y \leq b \text{ and } z \leq c\}$.

The relational structure $(J(\mathbf{A}), \leq, R)$ is an example of a Birkhoff frame. In general, a *Birkhoff frame* [GJ20] is a triple $\mathbf{W} = (W, \leq, R)$ where (W, \leq) is a poset, and $R \subseteq W^3$ satisfies the following three properties (downward closure in the 1st, and upward closure in the 2nd and 3rd argument):

- (R1) $u \leq xRyz \implies uRyz$
- (R2) $xRyz \ \& \ y \leq v \implies xRvz$
- (R3) $xRyz \ \& \ z \leq w \implies xRyw$.

A Birkhoff frame \mathbf{W} defines the downset algebra $\mathbf{D}(\mathbf{W}) = (D(\mathbf{W}), \cap, \cup, \cdot)$ by

$$Y \cdot Z = \{x \in W \mid xRyz \text{ for some } y \in Y \text{ and } z \in Z\}.$$

The property (R1) ensures that $Y \cdot Z \in D(\mathbf{W})$.

In relevance logic [DR02] similar ternary frames are known as Routley-Meyer frames. In that setting upsets are used to recover the distributive lattice-ordered relevance algebra, and this choice implies that $J(A)$ with the induced order from A is dually isomorphic to (W, \leq) . Another difference is that Routley-Meyer frames have a unary relation and axioms to ensure it is a left identity element of the \cdot operation.

The duality between perfect $d\ell$ -magmas and Birkhoff frames is recalled below. Here we assume that the binary operation on a complete $d\ell$ -magma is a complete operator, i.e., distributes over arbitrary joins in each argument. Such algebras are also known as *nonassociative quantales* or *prequantales* [Ros90].

Theorem 4.2 [GJ20]. (1) *If \mathbf{A} is a perfect $d\ell$ -magma and $R \subseteq J(A)^3$ is defined by $xRyz \Leftrightarrow x \leq yz$ then $J(\mathbf{A}) = (J(A), \leq, R)$ is a Birkhoff frame, and $\mathbf{A} \cong \mathbf{D}(J(\mathbf{A}))$.*
 (2) *If \mathbf{W} is a Birkhoff frame then $\mathbf{D}(\mathbf{W})$ is a perfect $d\ell$ -magma, and $\mathbf{W} \cong (J(D(\mathbf{W})), \subseteq, R_\downarrow)$, where $(\downarrow x, \downarrow y, \downarrow z) \in R_\downarrow \Leftrightarrow xRyz$.*

A ternary relation R is called *commutative* if $xRyz \implies xRzy$ for all x, y, z . The justification for this terminology is provided by the following result.

Lemma 4.3. *For any Birkhoff frame \mathbf{W} , $\mathbf{D}(\mathbf{W})$ is commutative if and only if R is commutative.*

Lemma 4.4. *Let \mathbf{W} be a Birkhoff frame. Then $\mathbf{D}(\mathbf{W})$ is idempotent if and only if $xRxx$ and $(xRyz \implies x \leq y \text{ or } x \leq z)$ for all $x, y, z \in W$.*

Proof. Assume $\mathbf{D}(\mathbf{W})$ is idempotent, and let $x \in W$. Then $\downarrow x \cdot \downarrow x = \downarrow x$ since $\downarrow x \in D(\mathbf{W})$. From $x \in \downarrow x$ we deduce $x \in \downarrow x \cdot \downarrow x$, whence it follows that $xRyz$ for some $y \in \downarrow x, z \in \downarrow x$. Therefore $xRyz$ for $y \leq x, z \leq x$, which implies $xRxx$ by (R2) and (R3).

Next assume $xRyz$ holds. Then $x \in \downarrow\{y, z\} \cdot \downarrow\{y, z\} = \downarrow\{y, z\}$ by idempotence. Hence for some $w \in \{y, z\}$ we have $x \leq w$, and it follows that $x \leq y$ or $x \leq z$.

For the converse, assume $xRxx$ and $(xRyz \implies x \leq y \text{ or } x \leq z)$ for all $x, y, z \in W$ and let $X \in D(\mathbf{W})$. From $xRxx$ we obtain $X \subseteq X \cdot X$.

For the reverse inclusion, let $x \in X \cdot X$. Then $xRyz$ holds for some $y, z \in X$. By assumption $xRyz$ implies $x \leq y$ or $x \leq z$. Since X is a downset, $x \leq y \implies x \in X$ and $x \leq z \implies x \in X$. Hence $X \cdot X = X$. \square

The previous two results are examples of correspondence theory, since they show that an equational property on a perfect $d\ell$ -magma corresponds to a first-order condition on its Birkhoff frame.

The relational semantics of a perfect $d\ell pq$ -magma is given by a PQ -frame, which is a partially-ordered relational structure (W, \leq, P, Q) such that P, Q are binary relations on W , $u \leq xPy \leq v \implies uPv$ and $u \leq xQy \leq v \implies uQv$. Relations with this property are called *weakening relations* [KV16, GJ20], and this is what ensures that if we define

$$\mathfrak{p}(Y) = \{x \mid \exists y(xPy \ \& \ y \in Y)\}$$

for a downset Y , then \mathfrak{p} is a complete normal join-preserving operator that produces a downset, and P is uniquely determined by $xPy \Leftrightarrow x \in \mathfrak{p}(\downarrow y)$. Similarly, a normal operator \mathfrak{q} is defined from Q , and uniquely determines Q . The residual \mathfrak{p}^* of \mathfrak{p} is a completely meet-preserving operator, defined by $\mathfrak{p}^*(Y) = \{x \mid \forall y(yPx \Rightarrow y \in Y)\}$, and likewise for \mathfrak{q}^* . If $P = Q$ then we omit Q and refer to (W, \leq, P) simply as a P -frame.

We now list some correspondence results for $d\ell pq$ -magmas. We begin with a theorem that restates the term-equivalence of Theorem 2.1 as a definitional equivalence on frames. A direct proof of this result is straightforward, but it also follows from Theorem 2.1 by correspondence theory.

Theorem 4.5.

- (1) Let (W, \leq, P, Q) be a PQ -frame such that $x \leq y \ \& \ xPz \Rightarrow xQy$ and $x \leq y \ \& \ xQz \Rightarrow xPy$. If we define $xRyz \Leftrightarrow (xPy \ \& \ x \leq z)$ or $(x \leq y \ \& \ xQz)$ then (W, \leq, R) is a Birkhoff frame, and P, Q are obtained from R via $xPy \Leftrightarrow \exists w(xRyw)$ and $xQy \Leftrightarrow \exists w(xRwy)$.
- (2) Let (W, \leq, R) be a Birkhoff frame that satisfies $xRyz \Leftrightarrow (\exists w(xRyw) \ \& \ x \leq z)$ or $(x \leq y \ \& \ \exists w(xRwz))$ and define $xPy \Leftrightarrow \exists w(xRyw)$, $xQy \Leftrightarrow \exists w(xRwy)$. Then (W, \leq, P, Q) is a PQ -frame in which $x \leq y \ \& \ xPz \Rightarrow xQy$ and $x \leq y \ \& \ xQz \Rightarrow xPy$ hold.

Note that the universal formula $x \leq y \ \& \ xPz \implies xQy$ corresponds to the $d\ell pq$ -magma axiom $Y \wedge \mathfrak{p}\top \leq \mathfrak{q}Y$.

A significant advantage of PQ -frames over Birkhoff frames is that binary relations have a graphical representation in the form of directed graphs (whereas ternary relations are 3-ary hypergraphs that are more complicated to draw). Equational properties from Lemma 2.2, Cor. 3.3 correspond to the following first-order properties on PQ -frames.

Lemma 4.6. Assume \mathbf{A} is a perfect $d\ell pq$ -algebra and $\mathbf{W} = (W, \leq, P, Q)$ is its corresponding PQ -frame. The constant $1 \in A$ (when present) is assumed to correspond to a downset $E \subseteq W$. Then

- (1) $a \leq \mathfrak{p}a$ holds in \mathbf{A} if and only if P is reflexive,
- (2) $\mathfrak{p}\mathfrak{p}a \leq \mathfrak{p}a$ holds in \mathbf{A} if and only if P is transitive,
- (3) $\mathfrak{p}a = \mathfrak{q}a$ holds in \mathbf{A} if and only if $P = Q$,
- (4) $\mathfrak{p}1 = \top$ holds in \mathbf{A} if and only if $\forall x \exists y (y \in E \ \& \ xPy)$ holds in \mathbf{W} ,
- (5) $\mathfrak{p}a \wedge 1 \leq a$ holds in \mathbf{A} if and only if $x \in E \ \& \ xPy \Rightarrow x \leq y$ holds in \mathbf{W} ,
- (6) $\mathfrak{p}a \wedge \mathfrak{p}b \leq \mathfrak{p}((\mathfrak{p}a \wedge b) \vee (a \wedge \mathfrak{p}b))$ holds in \mathbf{A} if and only if

$$wPx \ \& \ wPy \Rightarrow \exists v(wPv \ \& \ (vPx \ \& \ v \leq y \ \text{or} \ v \leq x \ \& \ vPy)) \quad \text{holds in } \mathbf{W}.$$

Proof. (1)–(3) These correspondences are well known from modal logic.

(4) For $x \in W$ and $E = \downarrow 1$ we have $x \leq \mathfrak{p}1$ if and only if there exists $y \in W$ such that $y \leq 1$ and $x \leq \mathfrak{p}y$, or equivalently, $y \in E$ and xPy .

(5) In the forward direction, let $a = \downarrow y$. Then it follows that $x \in \mathfrak{p}(\downarrow y) \cap E$ implies $x \in \downarrow y$, and consequently $x \in E \ \& \ xPy \implies x \leq y$.

In the reverse direction, let Y be a downset of W and assume $x \in \mathfrak{p}Y \cap E$. Then $x \in E$ and xPy for some $y \in Y$. Hence $x \leq y$, or equivalently $x \in \downarrow y \subseteq Y$. Thus, $\mathfrak{p}Y \cap E \subseteq Y$, so the algebra \mathbf{A} satisfies $\mathfrak{p}a \wedge 1 \leq a$ for all $a \in A$.

(6) In the forward direction, let $a = \downarrow x$ and $b = \downarrow y$. Then it follows from the inequality that $w \in \mathfrak{p}\downarrow x \cap \downarrow y \implies w \in \mathfrak{p}((\mathfrak{p}\downarrow x \cap \downarrow y) \cup (\downarrow x \cap \mathfrak{p}\downarrow y))$ for all $w \in W$. This in turn implies $wPx \ \& \ wPy \implies \exists v(wPv \ \& \ v \in (\mathfrak{p}\downarrow x \cap \downarrow y) \cup (\downarrow x \cap \mathfrak{p}\downarrow y))$, which translates to the given first-order condition.

In the reverse direction, let X, Y be downsets of W and assume $w \in \mathfrak{p}X \cap \mathfrak{p}Y$. Then wPx and wPy for some $x \in X$ and $y \in Y$. It follows that there exists a $v \in W$ such that $(wPv \ \& \ (vPx \ \& \ v \leq y \text{ or } v \leq x \ \& \ vPy))$, hence $v \in (\mathfrak{p}X \cap Y) \cup (X \cap \mathfrak{p}Y)$. Therefore $w \in \mathfrak{p}(\mathfrak{p}X \cap Y) \cup (X \cap \mathfrak{p}Y)$. \square

Recall that a ternary relation R is commutative if $xRyz \Leftrightarrow xRzy$ for all x, y . From Theorem 4.5 we also obtain the following result.

Corollary 4.7. *Let (W, \leq, P, Q) be a PQ -frame and define R as in Thm. 4.5(1). Then R is commutative if and only if $P = Q$.*

This corollary shows that in the commutative setting a PQ -frame only needs one of the two binary relations. Hence we define $\mathbf{W} = (W, \leq, P)$ to be a P -frame if P is a weakening relation, i.e., $u \leq xPy \leq v \implies uPv$.

We now turn to the problem of ensuring that the binary operation of a $d\ell$ -magma is associative. For Birkhoff frames the following characterization of associativity is well known from relation algebras [Mad82] (in the Boolean case) and from the Routley-Meyer semantics for relevance logic [DR02] in general.

Lemma 4.8. *Let $\mathbf{W} = (W, \leq, R)$ be a Birkhoff frame. Then $\mathbf{D}(\mathbf{W})$ is an associative ℓ -magma if and only if $\forall wxyz(\exists u(uRxy \ \& \ wRuz) \Leftrightarrow \exists v(vRyz \ \& \ wRxv))$. If R is commutative, then the equivalence can be replaced by the implication $\forall wxyz(uRxy \ \& \ wRuz \Rightarrow \exists v(vRyz \ \& \ wRxv))$.*

This lemma is another correspondence result that follows from translating $w \in (XY)Z \Leftrightarrow w \in X(YZ)$ for $X, Y, Z \in D(\mathbf{W})$. In the commutative case $(XY)Z \subseteq X(YZ)$ implies the reverse inclusion, hence only one of the implications is needed. We now show that for a large class of P -frames the 6-variable universal-existential formula for associativity can be replaced by simpler universal formulas with only three variables.

A *preorder forest P -frame* is a P -frame such that P is a preorder (i.e., reflexive and transitive) and satisfies the formula

$$xPy \text{ and } xPz \implies x \leq y \text{ or } x \leq z \text{ or } yPz \text{ or } zPy. \quad (\text{Pforest})$$

Note that since P is a weakening relation, reflexivity of P implies that $\leq \subseteq P$ because xPx and $x \leq y$ implies xPy .

It is interesting to visualize the properties that define preorder forest P -frames by implications between Hasse diagrams with \leq -edges (solid) and P -edges (dotted) as in Figure 1. However, one needs to keep in mind that dotted lines could be horizontal (if xPy and yPx) and that any line could be a loop if two variables refer to the same element.

We are now ready to state the main result. We use the algebraic characterization of associativity in Lemma 2.2.

Theorem 4.9. *Let $\mathbf{W} = (W, \leq, P)$ be a preorder forest P -frame and $\mathbf{D}(\mathbf{W})$ its corresponding downset algebra. Then the operation $x \cdot y = (\mathfrak{p}x \wedge y) \vee (x \wedge \mathfrak{p}y)$ is associative in $\mathbf{D}(\mathbf{W})$.*

$$(\text{Pforest}) \quad y \begin{array}{c} \diagup \\ \diagdown \end{array} z \implies y \begin{array}{c} \diagup \\ \diagdown \end{array} z \text{ or } y \begin{array}{c} \diagdown \\ \diagup \end{array} z \text{ or } \begin{array}{c} z \\ \vdots \\ y \\ \vdots \\ x \end{array} \text{ or } \begin{array}{c} y \\ \vdots \\ z \\ \vdots \\ x \end{array}$$

Figure 1: The (Pforest) axiom. The partial order \leq and the preorder P are denoted by solid lines and dotted lines respectively.

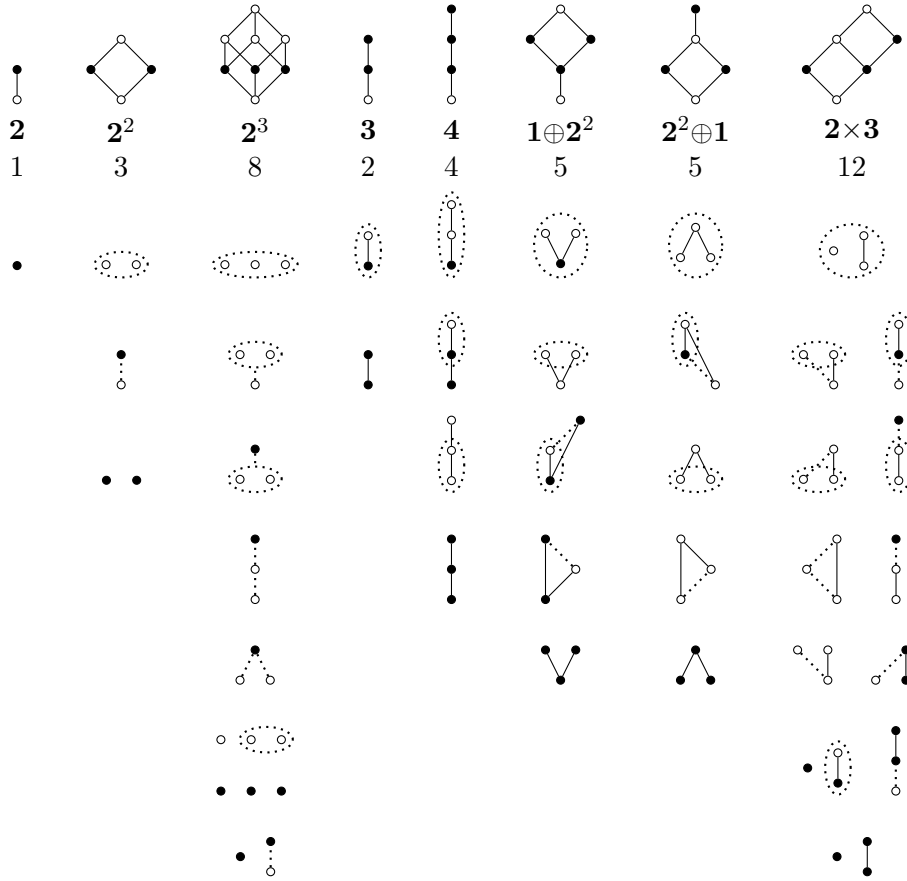


Figure 2: All 40 preorder forest P -frames (W, \leq, P) with up to 3 elements. Solid lines show (W, \leq) , dotted lines show the additional edges of P , and the identity (if it exists) is the set of black dots. The first row shows the lattice of downsets, and the Boolean quantales from [AJ20] appear in the first three columns.

Proof. Let $\mathbf{W} = (W, \leq, P)$ be a preorder forest P -frame and $\mathbf{D}(\mathbf{W})$ its $d\ell p$ -algebra of downsets with operator \mathfrak{p} . Since P is a preorder, $\mathbf{D}(\mathbf{W})$ is a $d\ell p$ -closure algebra. By Lemma 3.2, a $d\ell p$ -closure algebra is associative if and only if $\mathfrak{p}(x) \wedge \mathfrak{p}(y) \leq \mathfrak{p}(\mathfrak{p}(x) \wedge y) \vee (x \wedge \mathfrak{p}(y))$. By Lemma 4.6 this is equivalent to the frame property

$$xPy \ \& \ xPz \ \Rightarrow \ \exists w(xPw \ \& \ (wPy \ \& \ w \leq z \ \text{or} \ w \leq y \ \& \ wPz)). \quad (*)$$

We now show that this frame property holds in \mathbf{W} . We know that P is reflexive and (Pforest) holds.

Assume xPy and xPz . By (Pforest) there are four cases:

- (1) $x \leq y$: take $w = x$. Then xPx , $x \leq y$ and xPz , hence $(*)$ holds.
- (2) $x \leq z$: again take $w = x$. Then the other disjunct of $(*)$ holds.
- (3) yPz : take $w = y$. Then xPy , $y \leq z$ and yPz , hence $(*)$ holds.
- (4) zPy : take $w = z$. Then xPz , zPy and $y \leq z$, hence again $(*)$ holds. \square

The universal class of preorder forest P -frames is strictly contained in the class of all P -frames in which $x \cdot y$ is associative. In fact the latter class is not closed under substructures, hence not a universal class: $W = \{0, 1, 2, 3\}$, $\leq = id_W \cup \{(0, 1), (0, 2), (0, 3)\}$, $P = \leq \cup \{(1, 0), (1, 2), (1, 3)\}$ is a P -frame with associative \cdot (use e.g. Lemma 3.2), but restricting \leq, P to the subset $\{1, 2, 3\}$ gives a P -frame where \cdot fails to be associative, hence (Pforest) also fails.

A dl -semilattice is an associative commutative idempotent dl -magma. The point of the previous result is that it allows the construction of perfect associative commutative idempotent dl -magmas and idempotent bunched implication algebras from preorder forest P -frames. This is much simpler than constructing the ternary relation R of the Birkhoff frame of such algebras. For example the Hasse diagrams for all the preorder forest P -frames with up to 3 elements are shown in Figure 2, with the preorder P given by dotted lines and ovals. The corresponding ternary relations can be calculated from P , but would have been hard to include in each diagram.

We now examine when a P -frame will have an identity element.

Lemma 4.10. *Let \mathbf{W} be a P -frame and E a downset of W . Then the downset algebra $D(\mathbf{W})$ has E as identity element for \cdot if and only if $E = \{x \in W \mid \forall y(xPy \Rightarrow x \leq y)\}$ and $\mathfrak{p}E = W$.*

Proof. In the forward direction assume a downset E is the identity for \cdot , and let $y \in W$. It follows from Lemma 2.2(5) that $\mathfrak{p}E = W$ since W is the top element in $D(\mathbf{W})$, and moreover, $(\mathfrak{p}E \cap \downarrow y) \cup (E \cap \mathfrak{p}(\downarrow y)) = \downarrow y$. Hence $E \cap \mathfrak{p}(\downarrow y) \subseteq \downarrow y$ for all y , which shows that if $x \in E$ then $\forall y(xPy \Rightarrow x \leq y)$ holds. Now let $x \in W$ satisfy $\forall y(xPy \Rightarrow x \leq y)$. From $\mathfrak{p}E = W$ we deduce that xPz for some z , hence $x \leq z$ and, since E is a downset, $x \in E$.

Conversely, by the definition of E , if $x \in E$, then $xPy \Rightarrow x \leq y$ holds for all $y \in W$. Hence by Lemma 4.6(5) for all $X \in D(\mathbf{W})$ we have $\mathfrak{p}X \cap E \subseteq X$. Since $\mathfrak{p}E = W$ together with Lemma 2.2(5), it follows that E is an identity element in the downset algebra. \square

5. WEAKLY CONSERVATIVE PERFECT dl -MAGMAS AND BIRKHOFF FRAMES

In this section we explore a special case that arises when the relations P and Q are determined from R by $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$, i.e., the existential quantifier from the previous section is instantiated by $z = x$. We first discuss some related algebraic properties.

A binary operation \cdot is called *conservative* (or *quasitrivial*) if the output value is always one of the two inputs, i.e., it satisfies $xy = x$ or $xy = y$ for all $x, y \in A$. Note that this property implies idempotence.

In general a dl -magma is idempotent if and only if it satisfies $x \wedge y \leq xy \leq x \vee y$, since $x \wedge y = (x \wedge y)(x \wedge y) \leq xy \leq (x \vee y)(x \vee y) = x \vee y$, and conversely, identifying x, y we have $x \leq xx \leq x$.

A perfect ℓ -magma A is called *weakly conservative* if it satisfies the formula

$$xy = x \wedge y \text{ or } xy = x \text{ or } xy = y \text{ or } xy = x \vee y \text{ for all } x, y \in J(A).$$

So for completely join-irreducible elements x, y the product $xy \in \{x \wedge y, x, y, x \vee y\}$. This is a generalization of conservativity in two ways since there are additional possibilities for the value of xy and the formula only needs to hold for completely join-irreducible elements.

A typical example of a weakly conservative perfect ℓ -magma is an atomic Boolean algebra with an idempotent binary operation xy . In this case the completely join-irreducible elements are the atoms of the Boolean algebra, and for any two atoms x, y the interval $[x \wedge y, x \vee y] \subseteq \{x \wedge y, x, y, x \vee y\}$. Since we observed previously that $x \wedge y \leq xy \leq x \vee y$ it follows that xy can only take on one of the four values $x \wedge y, x, y, x \vee y$.

The notation $x \leq yRzw$ is shorthand for $x \leq y$ and $yRzw$. We also write $x \leq y, z$ as an abbreviation for $x \leq y$ and $x \leq z$. A Birkhoff frame is called *weakly conservative* if it satisfies

$$xRyz \Leftrightarrow x \leq y, z \text{ or } x \leq yRyz \text{ or } x \leq zRyz.$$

This terminology is motivated by the following result.

Lemma 5.1. *Let \mathbf{W} be a Birkhoff frame. Then $D(\mathbf{W})$ is weakly conservative if and only if \mathbf{W} is weakly conservative.*

Proof. We first note that weak conservativity for $D(\mathbf{W})$ can be written in conjunctive form as

$$x \wedge y \leq xy \leq x \vee y \text{ and } (xy \leq x \text{ or } y \leq xy) \text{ and } (xy \leq y \text{ or } x \leq xy). \quad (5.1)$$

Likewise weak conservativity for \mathbf{W} in conjunctive form (on right hand side) is

$$wRxy \Leftrightarrow (w \leq x \text{ or } w \leq y) \text{ and } (w \leq x \text{ or } yRxy) \text{ and } (w \leq y \text{ or } xRxy). \quad (5.2)$$

Now assume (1) and for $w, x, y \in W$ assume $wRxy$. To simplify notation, we identify elements of W with their principal filters in $D(\mathbf{W})$. Since w is join-irreducible and $xy \leq x \vee y$, it follows that $w \leq x$ or $w \leq y$. Next, to prove that $w \leq x$ or $yRxy$, assume $w \not\leq x$. Then $xy \not\leq x$, and again $xy \leq x \vee y$ implies $xy \leq y$. The last conjunct is proved similarly. Suppose now that the right hand side of (2) holds for $w, x, y \in W$. Using the original disjunctive form, there are 3 cases: if $w \leq x, y$ then by (1) $w = w \wedge w \leq ww \leq xy$, hence $wRxy$. If $w \leq x \leq Rxy$ then $w \leq x \leq xy$, so again we obtain $wRxy$. The third case is similar.

Conversely, assume (2) holds and let x, y be join-irreducibles of $D(\mathbf{W})$. To see that $x \wedge y \leq xy$, let w be any join-irreducible such that $w \leq x \wedge y$, in which case $wRxy$ follows from the disjunctive form of (2), hence $w \leq xy$. Since weak conservativity of $D(\mathbf{W})$ implies idempotence, $xy \leq x \vee y$ follows from Lemma 4.4. To prove that $xy \leq x$ or $y \leq xy$, assume $y \not\leq xy$, whence $yRxy$ does not hold. For any join-irreducible $w \leq xy$ we have $wRxy$, and since (2) implies $w \leq x$ or $yRxy$, we conclude that $xy \leq x$. Finally, the conjunct $xy \leq y$ or $x \leq xy$ follows by symmetry of x, y . \square

Next we show that in every weakly conservative Birkhoff frame the ternary relation R is determined by two binary relations P, Q defined by $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$. This is simpler than the previous definitions with an existential quantifier, but they need not be weakening relations, hence they do not produce a PQ -frame. Instead they are axiomatized by the following conditions.

A PQ -structure is of the form (W, \leq, P, Q) where (W, \leq) is a poset and

$$\begin{array}{ll} \text{(P0)} \ x \leq y \implies xPy & \text{(Q0)} \ x \leq y \implies xQy \\ \text{(P1)} \ x \leq y \ \& \ xPz \implies x \leq z \text{ or } yPz & \text{(Q1)} \ x \leq y \ \& \ xQz \implies x \leq z \text{ or } yQz \\ \text{(P2)} \ xPy \leq z \implies xPz & \text{(Q2)} \ xQy \leq z \implies xQz. \end{array}$$

Note that (P0) and (Q0) together with reflexivity of \leq imply that both P and Q are reflexive. The following result shows that PQ -structures and weakly conservative Birkhoff frames are definitionally equivalent. This generalizes an earlier result of [AJ20] where the partial order \leq was assumed to be $=$.

Theorem 5.2.

- (1) For a PQ -structure (W, \leq, P, Q) let $xRyz$ be defined by $x \leq y, z$ or $x \leq yQz$ or $x \leq zPy$. Then (W, \leq, R) is a weakly conservative Birkhoff frame and P, Q are recovered via $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$.
- (2) For a weakly conservative Birkhoff frame (W, \leq, R) define $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$. Then (W, \leq, P, Q) is a PQ -structure and $xRyz \Leftrightarrow x \leq y, z$ or $x \leq yQz$ or $x \leq zPy$.

Proof. (1) Assume (W, \leq, P, Q) is a PQ -structure and let R be defined as above. We need to prove that R satisfies (R1), (R2) and (R3).

(R1) Assume $xRyz$ and $w \leq x$. Then $wRyz$ follows from (the expanded form of) $xRyz$ by transitivity of \leq .

(R2) Assume $xRyz$ and $y \leq w$. By assumption $x \leq y, z$ or $x \leq yQz$ or $x \leq zPy$, so we have 3 subcases. In the first subcase $x \leq y$ implies $x \leq w$ by transitivity of \leq , hence $x \leq w, z$.

In the second subcase $x \leq yQz$. We also have $y \leq w$, hence $x \leq w$. A substitution instance of (Q1) is $yQz \ \& \ y \leq w \Rightarrow y \leq z$ or wQz , hence $y \leq z$ or wQz . From the assumption that $x \leq y$ it follows that $x \leq z$ or wQz . Since $x \leq w$ holds we obtain $x \leq w, z$ or $x \leq wQz$. Therefore $xRwz$ holds. In the third subcase $x \leq zPy$ implies $x \leq zPw$ by (P2), and again $xRyz$ holds.

(R3) The argument is symmetric to the one for (R2).

To prove that R is weakly conservative, we show that $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$. Now $xRyx$ is equivalent to $x \leq y, x$ or $x \leq yQx$ or $x \leq xPy$ which simplifies to $x \leq y$ or xPy , and by (P0) this is equivalent to xPy . Similarly $xRxy$ is equivalent to xQy .

(2) Assume (W, \leq, R) is a weakly conservative Birkhoff frame and define $xPy \Leftrightarrow xRyx$ and $xQy \Leftrightarrow xRxy$. We show that (P0–P2) hold, and the arguments for (Q0–Q2) are similar. Note that weak conservativity of R is equivalent to $xRyz \Leftrightarrow x \leq y, z$ or $x \leq yQz$ or $x \leq zPy$.

(P0) Assume $x \leq y$. From reflexivity of \leq it follows that $x \leq y, x$. This implies that $xRyx$ holds, and hence xPy .

(P1) Assume $x \leq y$ and xPz . Then $xRzx$ holds, and (R3) implies $xRzy$, or equivalently $x \leq z, y$ or $x \leq zQy$ or $x \leq yPz$. Since $x \leq y$, this disjunction simplifies to $x \leq z$ or $x \leq zQy$ or yPz . Since $x \leq z$ is a conjunct of the middle part, the formula simplifies to $x \leq z$ or yPz .

(P2) Assume $xPy \leq z$. This is equivalent to $xRyx$ and $y \leq z$, so by (R2) $xRzx$ follows. This is equivalent to $x \leq z, x$ or $x \leq zQx$ or $x \leq xPz$, which simplifies by reflexivity of \leq to $x \leq z$ or xPz , and further by (P0) to xPz . \square

Conditions (P2) and (Q2) ensure that P, Q are “half-weakening relations”. Hence a PQ -structure is a PQ -frame if and only if it satisfies the other half

$$(P2') \quad x \leq yPz \implies xPz \qquad (Q2') \quad x \leq yQz \implies xQz.$$

It follows from Theorems 4.5 and 5.2 that in a PQ -structure that is also a PQ -frame, the weakly conservative ternary relation R can be defined in two equivalent ways: as $(xPy \ \& \ x \leq z)$ or $(x \leq y \ \& \ xQz)$ and as $x \leq y, z$ or $x \leq yQz$ or $x \leq zPy$.

	cardinality	$n = 1$	2	3	4	5	6	7
preorder trees	$T_n =$	1	2	5	13	37	108	337
	$c_n =$	1	5	16	57	186	668	
preorder forests	$F_n =$	1	3	8	24	71	224	
preorder trees with singleton roots	$T_n^s =$	1	1	3	8	24	71	224
	$c_n^s =$	1	3	10	35	121	438	
preorder forests with singleton roots	$F_n^s =$	1	2	5	14	41	127	

Table 2: Number of preorder trees and forests (up to isomorphism)

6. COUNTING PREORDER FORESTS AND LINEAR P -FRAMES

In the case when the poset (W, \leq) is an antichain, a preorder forest P is simply a preorder $P \subseteq W^2$ such that xPy and xPz implies yPz or zPy . A *preorder tree* is a connected component of a preorder forest. A *rooted* preorder forest is defined to have an equivalence class of P -maximal elements in each component. For finite preorder forests this is always the case. Let F_n denote the number of preorder forests and T_n the number of preorder trees with n elements (up to isomorphism). We also let $F_0 = 1$.

A preorder forest *has singleton roots* if the P -maximal equivalence class of each component is a singleton set. The number of preorder forests and trees with singleton roots is denoted by F_n^s and T_n^s respectively.

Note that every preorder forest gives rise to a unique preorder tree with a singleton root by adding one new element r such that for all $x \in W$ we have xPr . It follows that $T_n^s = F_{n-1}$.

Every preorder tree with a non-singleton root equivalence class and n elements is obtained from a preorder tree with $n - 1$ elements by adding one more element to the root equivalence class. Hence for $n > 0$ we have $T_n = F_{n-1} + T_{n-1}$. The Euler transform of T_n is used to calculate the next value of F_n as follows:

$$c_n = \sum_{d|n} d \cdot T_n \qquad F_n = \frac{1}{n} \sum_{k=1}^n c_k \cdot F_{n-k}.$$

Since preorder forests with singleton roots are disjoint unions of preorder trees with singleton roots, F_n^s is calculated by an Euler transform from T_n^s .

Corollary 6.1. *The sequence F_n^s is the Euler transform of T_n^s .*

While it is difficult to count preorder forest P -frames in general, it is simple to count the linear ones. Note that the (Pforest) axiom is actually redundant for linearly ordered P -frames.

Theorem 6.2. *There are 2^{n-1} linearly ordered forest P -frames. In the algebraic setting, for $n > 1$, there are 2^{n-2} unary-determined commutative doubly idempotent linear semirings with n elements, and $n - 1$ of them have an identity element.*

Proof. Let \mathbf{W} be a linearly ordered P -frame with elements $W = \{1 < 2 < \dots < n\}$ such that P is transitive and (P0) holds. Then each possible relation P on W is determined by choosing a subset S of the edges $\{(2, 1), (3, 2), \dots, (n, n - 1)\}$ and defining P to be the transitive closure of $S \cup \leq$. Since there are $n - 1$ such edges to choose from, the number of P -frames is 2^{n-1} .

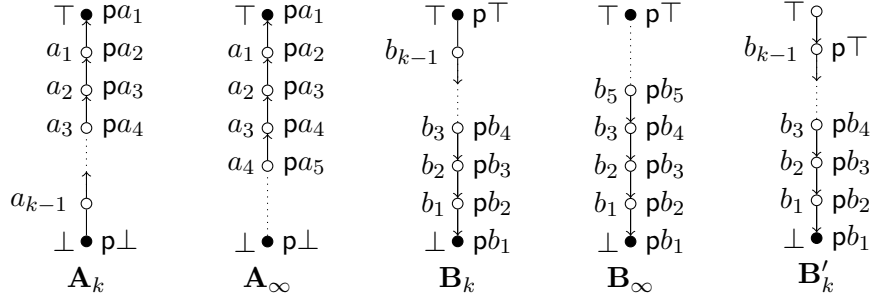


Figure 3: Subdirectly irreducible dlp -chains (black elements satisfy $px = x$).

Let \mathbf{A} be a unary-determined commutative doubly idempotent linear semiring with n elements. Then the P -frame \mathbf{W} associated with \mathbf{A} has $n - 1$ elements, is linearly ordered, and P is reflexive and transitive since \cdot is idempotent and associative. Hence there are 2^{n-2} such algebras.

By Lemma 2.2 such an algebra \mathbf{A} will have an identity 1 if and only if the operator p in the corresponding dlp -closure algebra satisfies the conditions $p1 = \top$ and $px \wedge 1 \leq x$ for every $x \in A$. The first condition means that 1 is not closed (unless it is \top), and there are no closed elements other than \top above 1. Since the partial order is a linear order and p is inflationary, the second condition is equivalent to $px = x$ or $1 \leq x$. That is to say, 1 is also the minimum non-closed element in \mathbf{A} . Hence the n -element unary-determined commutative doubly idempotent linear semirings with identity are the chains with the identity element in the k -th position, where $1 < k \leq n$, with every element below 1 closed and every element ≥ 1 either non-closed or equal to \top . Such semirings are uniquely identified by the position of the identity element, which can never be \perp . There are $n - 1$ possible positions, and hence $n - 1$ semirings with an identity element. \square

7. SUBDIRECTLY IRREDUCIBLE dlp -ALGEBRAS AND UNARY-DETERMINED BI-CHAINS

Let \mathcal{V} be a variety (= equational class) of unary-determined dl -magmas. Recall that an algebra \mathbf{A} is *subdirectly irreducible* if its congruence lattice $\text{Con}\mathbf{A}$ has a unique minimal nontrivial congruence, and \mathbf{A} is *simple* if $\text{Con}\mathbf{A}$ has exactly two elements. By Birkhoff's subdirect representation theorem every algebra is (subdirectly) embedded in a product of subdirectly irreducible factors, hence $\mathcal{V} = \text{ISP}(SI(\mathcal{V}))$ where $SI(\mathcal{V})$ is the class of all subdirectly irreducible members of \mathcal{V} and $\mathbb{I}, \mathbb{S}, \mathbb{P}$ are the class operators that return all isomorphic copies, all subalgebras and all products of members of their input class.

In [Pet96] and [Pet99] a characterization of the simple and subdirectly irreducible dlp -chains, or totally ordered modal lattices, is given. Recall that a dlp -chain is an algebra $(L, \wedge, \vee, \perp, \top, p)$ that is a linearly ordered bounded distributive lattice with normal unary operator p . We denote the following dlp -chains by \mathbf{A}_k , \mathbf{B}_k , and \mathbf{B}'_k for any integer $k \geq 1$:

Let $\top = a_0 > a_1 > a_2 > \dots > \perp$ and $\perp = b_0 < b_1 < b_2 < \dots < \top$ be bounded countable decreasing and increasing chains respectively. Then the operator p is defined in each structure as follows:

In $\mathbf{A}_k = \{\top, a_1, \dots, a_{k-1}, \perp\}$, $pa_i = a_{i-1}$ for $1 \leq i < k$, $p\top = \top$, and $p\perp = \perp$.

In $\mathbf{B}_k = \{\perp, b_1, \dots, b_{k-1}, \top\}$, $pb_i = b_{i-1}$ for $1 \leq i < k$, $p\top = \top$, and $p\perp = \perp$.

In $\mathbf{B}'_k = \{\perp, b_1, \dots, b_{k-1}, \top\}$, $pb_i = b_{i-1}$ for $1 \leq i < k$, $p\top = b_{k-1}$ and $p\perp = \perp$.

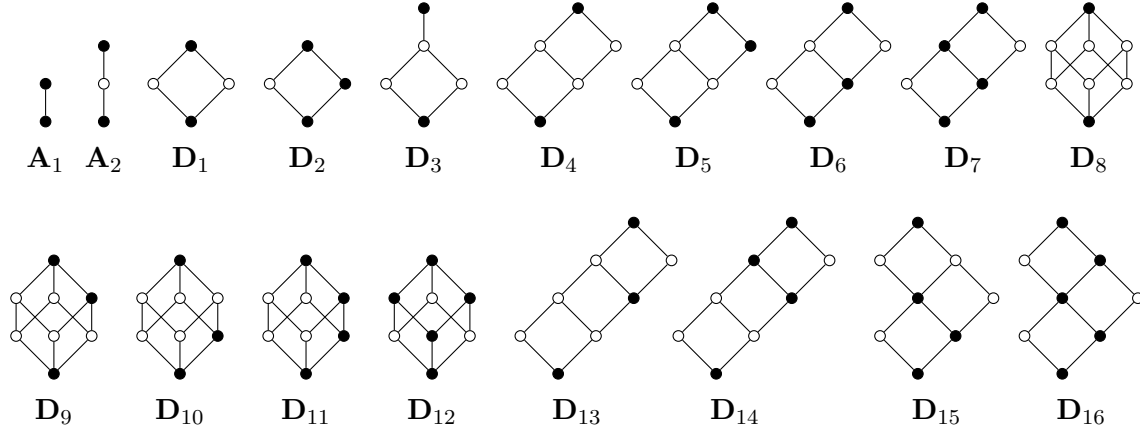


Figure 4: Subdirectly irreducible $d\ell p$ -closure algebras up to 8 elements (black = closed).

In $\mathbf{A}_\infty = \{\top, a_1, a_2, \dots, \perp\}$, $\mathbf{p}a_i = a_{i-1}$ for $1 \leq i$, $\mathbf{p}\top = \top$, and $\mathbf{p}\perp = \perp$.
 In $\mathbf{B}_\infty = \{\perp, b_1, b_2, \dots, \top\}$, $\mathbf{p}b_i = b_{i-1}$ for $1 \leq i$, $\mathbf{p}\top = \top$, and $\mathbf{p}\perp = \perp$.

Theorem 7.1 [Pet96].

- (1) The simple $d\ell p$ -chains are the algebras \mathbf{A}_1 and \mathbf{B}'_1 .
- (2) The subdirectly irreducible $d\ell p$ -chains are the algebras $\mathbf{A}_k, \mathbf{B}_k, \mathbf{B}'_k$ for every natural number k and $\mathbf{A}_\infty, \mathbf{B}_\infty$.

These chains are pictured in Figure 3. All subdirectly irreducible $d\ell p$ -closure algebras up to cardinality 8 are shown in Figure 4. Note that \mathbf{D}_{12} does not satisfy the identity $\mathbf{p}x \wedge \mathbf{p}y \leq \mathbf{p}((\mathbf{p}x \wedge y) \vee (x \vee \mathbf{p}y))$, hence by Lemma 3.2 the corresponding unary-determined magma is not associative.

Corollary 7.2. Linear $d\ell p$ -closure algebras of size $n \geq 4$ are not subdirectly irreducible.

Proof. Let \mathbf{W} be a linearly-ordered preorder-forest P -frame with corresponding linear $d\ell p$ -closure algebra $D(\mathbf{W})$. Suppose that $D(\mathbf{W})$ is subdirectly irreducible. Then $D(\mathbf{W})$ is of the form $\mathbf{A}_k, \mathbf{B}_k$, or \mathbf{B}'_k for some k in the natural numbers, or $D(\mathbf{W})$ is of the form \mathbf{A}_∞ or \mathbf{B}_∞ .

By Lemma 4.6, since P is reflexive, $X \leq \mathbf{p}X$ for all $X \in D(\mathbf{W})$. But in \mathbf{B}_∞ or \mathbf{B}_k with $k \geq 2$, there exists X such that $\mathbf{p}X < X$, so $D(\mathbf{W})$ cannot be of this form.

We also have that $\mathbf{p}\top = \top$ in all $d\ell p$ -closure algebras, so we cannot have $D(\mathbf{W}) = C_k$ for any $k > 0$.

Now suppose that $D(\mathbf{W}) = \mathbf{A}_k$ where k is a natural number or ∞ . Suppose $k \geq 3$. Then there exist $X_i, X_{i+1}, X_{i+2} \in D(\mathbf{W})$ such that $\mathbf{p}X_i = X_{i+1} \neq X_{i+2} = \mathbf{p}X_{i+1}$. Hence \mathbf{p} is not a closure operator, a contradiction. \square

Hence the only subdirectly irreducible linear $d\ell p$ -closure algebras are \mathbf{A}_1 , and \mathbf{A}_2 , pictured in Figure 4. Since $d\ell p$ -algebras have lattice reducts, the variety of all $d\ell p$ -algebras is congruence distributive, and it follows from Jónsson's Lemma [Jón67] that nonisomorphic finite subdirectly irreducible $d\ell p$ -algebras generate distinct varieties. Moreover, these varieties are completely join-irreducible elements of the lattice of all varieties. A diagram of the poset of join-irreducible varieties generated by $d\ell p$ -chains and the algebras \mathbf{D}_1 – \mathbf{D}_{16} is shown in Figure 5. The variety generated by an algebra \mathbf{A} is denoted by $\mathcal{A} = \mathbb{V}(\mathbf{A})$. Equational bases for the varieties generated by bounded $d\ell p$ -chains are given in [Pet96].

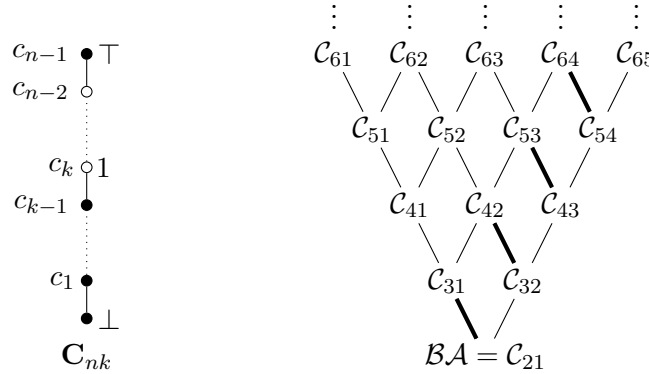


Figure 6: All finite subdirectly irreducible unary-determined BI-chains (black elements are closed) and the poset of join-irreducible varieties they generate.

Figure 6. Note that the two-element BI-chain $\mathbf{C}_{2,1}$ is term-equivalent to the two-element Boolean algebra and generates the smallest nontrivial variety.

8. CONCLUSION

We showed that unary-determined $d\ell$ -magmas have a simple algebraic structure given by two unary operators and that their relational frames are definitionally equivalent to frames with two binary relations. The complex algebras of these frames are complete distributive lattices with completely distributive operators, hence they have residuals and can be considered Kripke semantics for unary-determined bundled implication algebras and bundled implication logic. Associativity of the binary operator for idempotent unary-determined algebras can be checked by an identity with 2 rather than 3 variables, and for the frames by a 3-variable universal formula rather than a 6-variable universal-existential formula. All idempotent Boolean magmas are unary-determined, hence these results significantly extend the structural characterization of idempotent atomic Boolean quantales in [AJ20] and relate them to bundled implication logic. As an application we counted the number of preorder forest P -frames with n elements for which the partial order is an antichain, as well as the number of linearly ordered preorder P -frames. We also found all subdirectly irreducible $d\ell p$ -closure algebras up to cardinality 8, as well as all finite subdirectly irreducible unary-determined BI-chains and showed how the varieties they generate are related to each other by subclass inclusion.

Acknowledgements. The investigations in this paper made use of Prover9/Mace4 [McC10]. In particular, parts of Lemma 2.2 and Theorem 4.5 were developed with the help of Prover9 (short proofs were extracted from the output) and the results in Table 1 were calculated with Mace4. The remaining results in Sections 2–4 were proved manually, and later also checked with Prover9.

REFERENCES

[AJ20] Natanael Alpay and Peter Jipsen. Commutative doubly-idempotent semirings determined by chains and by preorder forests. In Uli Fahrenberg, Peter Jipsen, and Michael Winter, editors, *Relational*

- and *Algebraic Methods in Computer Science*, pages 1–14, Cham, 2020. Springer International Publishing. doi:10.1007/978-3-030-43520-2_1.
- [Bir67] Garrett Birkhoff. *Lattice theory*, volume Vol. XXV of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, third edition, 1967.
- [Doš85] Kosta Došen. Models for stronger normal intuitionistic modal logics. *Studia Logica*, 44(1):39–70, 1985. doi:10.1007/BF00370809.
- [DP02] Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2 edition, 2002. doi:10.1017/CB09780511809088.
- [DR02] J. Michael Dunn and Greg Restall. Relevance logic. In Dov M. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, pages 1–128. Springer Netherlands, Dordrecht, 2002. doi:10.1007/978-94-017-0460-1_1.
- [GJ20] Nikolaos Galatos and Peter Jipsen. The structure of generalized bi-algebras and weakening relation algebras. *Algebra universalis*, 81(3):35, 2020. doi:10.1007/s00012-020-00663-9.
- [GJKO07] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.
- [GNV05] Mai Gehrke, Hideo Nagahashi, and Yde Venema. A sahlqvist theorem for distributive modal logic. *Annals of Pure and Applied Logic*, 131(1):65–102, 2005. doi:10.1016/j.apal.2004.04.007.
- [HW98] Udo Hebisch and Hanns Joachim Weinert. *Semirings*. World Scientific, 1998. doi:10.1142/3903.
- [Jón67] Bjarni Jónsson. Algebras whose congruence lattices are distributive. *Mathematica Scandinavica*, 21:110–121, Dec. 1967. doi:10.7146/math.scand.a-10850.
- [KV16] Alexander Kurz and Jiří Velebil. Relation lifting, a survey. *Journal of Logical and Algebraic Methods in Programming*, 85(4):475–499, 2016. Relational and algebraic methods in computer science. doi:10.1016/j.jlamp.2015.08.002.
- [Mad82] Roger Maddux. Some varieties containing relation algebras. *Trans. Amer. Math. Soc.*, 272(2):501–526, 1982. doi:10.2307/1998710.
- [McC10] William McCune. Prover9 and mace4. <http://www.cs.unm.edu/~mccune/prover9/>, 2005–2010.
- [O’H04] Peter W. O’Hearn. Resources, concurrency and local reasoning. In Philippa Gardner and Nobuko Yoshida, editors, *CONCUR 2004 - Concurrency Theory*, pages 49–67, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg. doi:10.1007/978-3-540-28644-8_4.
- [OP99] Peter W. O’Hearn and David J. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, 1999. doi:10.2307/421090.
- [Pet96] Alejandro Petrovich. Distributive lattices with an operator. *Studia Logica*, 56(1):205–224, 1996. doi:10.1007/BF00370147.
- [Pet99] Alejandro Petrovich. Equational classes of totally ordered modal lattices. *Order*, 16(1):1–17, 1999. doi:10.1023/A:1006259631226.
- [Rey02] John C. Reynolds. Separation logic: a logic for shared mutable data structures. In *Proceedings 17th Annual IEEE Symposium on Logic in Computer Science*, pages 55–74, 2002. doi:10.1109/LICS.2002.1029817.
- [Ros90] Kimmo I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990.