

$\text{TT}_{\mathcal{C}}^{\square}$: A FAMILY OF EXTENSIONAL TYPE THEORIES WITH EFFECTFUL REALIZERS OF CONTINUITY

LIRON COHEN ^a AND VINCENT RAHLI ^b

^a Ben-Gurion University, Israel
e-mail address: cliron@cs.bgu.ac.il

^b University of Birmingham, UK
e-mail address: V.Rahli@bham.ac.uk

ABSTRACT. $\text{TT}_{\mathcal{C}}^{\square}$ is a generic family of effectful, extensional type theories with a forcing interpretation parameterized by modalities. This paper identifies a subclass of $\text{TT}_{\mathcal{C}}^{\square}$ theories that internally realizes continuity principles through stateful computations, such as reference cells. The principle of continuity is a seminal property that holds for a number of intuitionistic theories such as System T. Roughly speaking, it states that functions on real numbers only need approximations of these numbers to compute. Generally, continuity principles have been justified using semantical arguments, but it is known that the modulus of continuity of functions can be computed using effectful computations such as exceptions or reference cells. In this paper, the modulus of continuity of the functionals on the Baire space is directly computed using the stateful computations enabled internally in the theory.

1. INTRODUCTION

The framework $\text{TT}_{\mathcal{C}}^{\square}$ [CR22] is a generic family of effectful, extensional type theories with a forcing interpretation parameterized by modalities. More concretely, $\text{TT}_{\mathcal{C}}^{\square}$ uses a general possible-worlds forcing interpretation parameterized by an abstract modality \square , which, in turn, can be instantiated with simple covering relations, leading to a general sheaf model. In addition, $\text{TT}_{\mathcal{C}}^{\square}$ is parameterized by a type of time-progressing choice operators \mathcal{C} , enabling support for internal effectful computations. $\text{TT}_{\mathcal{C}}^{\square}$ is particularly suitable for studying effectful theories, and indeed, $\text{TT}_{\mathcal{C}}^{\square}$ was called an “unprejudiced” type theory since these parameters can be instantiated to obtain theories that are either “agnostic”, i.e., compatible with classical reasoning (in the sense that classical axioms, such as the Law of Excluded Middle, can be validated), or that are “intuitionistic”, i.e., incompatible with classical reasoning (in the sense that classical axioms can be proven false).

This paper uses the $\text{TT}_{\mathcal{C}}^{\square}$ framework to reason about the continuity principle which is a seminal property in intuitionistic theories which contradicts classical mathematics but is

Key words and phrases: Continuity, Stateful computations, Intuitionism, Extensional Type Theory, Constructive Type Theory, Realizability, Theorem proving, Agda .

This research was partially supported by Grant No. 2020145 from the United States-Israel Binational Science Foundation (BSF).

generally accepted by constructivists. Roughly speaking, the principle states that functions on real numbers only need approximations of these numbers to compute. Brouwer, in particular, assumed his so-called *continuity principle for numbers* to derive that all real-valued functions on the unit interval are uniformly continuous [KV65, Dum00, Bee85, BR87, TvD88]. The continuity principle for numbers, sometimes referred to as the weak continuity principle, states that all functions on the Baire space (i.e., $\mathfrak{B} \equiv \mathbf{Nat} \rightarrow \mathbf{Nat}$, the set of infinite sequences of numbers) have a modulus of continuity. More concretely, given a function F of type $\mathfrak{B} \rightarrow \mathbf{Nat}$ and a function α of type \mathfrak{B} , the principle states that $F(\alpha)$ can only depend on an initial segment of α , and the length of the smallest such segment is the modulus of continuity of F at α . This is standardly formalized as follows, where $\mathfrak{B}_n \equiv \{x : \mathbf{Nat} \mid x < n\} \rightarrow \mathbf{Nat}$ is the set of finite sequences of numbers of length n :

$$\text{WCP} = \Pi F : \mathfrak{B} \rightarrow \mathbf{Nat} . \Pi \alpha : \mathfrak{B} . \| \Sigma n : \mathbf{Nat} . \Pi \beta : \mathfrak{B} . (\alpha = \beta \in \mathfrak{B}_n) \rightarrow (F(\alpha) = F(\beta) \in \mathbf{Nat}) \|$$

A number of theories have been shown to satisfy Brouwer’s continuity principle, or uniform variants, such as N-HA $^\omega$ by Troelstra [Tro73, p.158], MLTT by Coquand and Jaber [CJ10, CJ12], System T by Escardó [Esc13b], CTT by Rahli and Bickford [RB16], BTT by Baillon, Mahboubi and Pedrot [BMP22], to cite only a few (see Sec. 5 for further details). These proofs often rely on a semantical forcing-based approach [CJ10, CJ12], where the forcing conditions capture the amount of information needed when applying a function to a sequence in the Baire space, or through suitable models that internalize (C-Spaces in [XE13]) or exhibit continuous behavior (e.g., dialogue trees in [Esc13b, BMP22]).

Not only can functions on the Baire space be proved to be continuous, but using effectful computations, as for example described in [Lon99], one can *compute* the modulus of continuity of such a function. However, as shown for example by Kreisel [Kre62, p.154], Troelstra [Tro77b, Thm.IIA], and Escardó and Xu [EX15, Xu15], continuity is not an extensional property in the sense that two equal functions might have different moduli of continuity. Therefore, to realize continuity, the existence of a modulus of continuity has to be truncated as explained, e.g., in [EX15, Xu15, RB16, RB17], which is what the $\| _ \|$ operator achieves in WCP. Following the effectful approach, continuity was shown to be realizable in [RB16, RB17] using exceptions.

Instead of using exceptions, a more straightforward way to compute the modulus of continuity of a function on the Baire space is to use reference cells. This was explained, e.g., in [Lon99], where the use of references can be seen as the programming counterparts of the more logical forcing conditions. The computation using references is more efficient than when using exceptions as it allows computing the modulus of continuity of a function F at a point α simply by executing F on α , while recording the highest argument that α is applied to, while using exceptions requires repeatedly searching for the modulus of continuity.

Following this line of work, in this paper we show how to use stateful computations to realize a continuity principle. This allows deriving constructive type theories that include continuity axioms where the modulus of continuity is internalized in the sense that it is computed by an expression of the underlying programming language. Concretely, we do so for $\text{TT}_{\mathcal{C}}^{\square}$, which is presented in more details in Sec. 2. More precisely, we prove in this paper that all $\text{TT}_{\mathcal{C}}^{\square}$ functions are continuous for some instances of \square and \mathcal{C} : namely for “non-empty” equality modalities, and reference-like stateful choice operators. Our proof is for a variant of the weak continuity principle (see Thm. 4.1), which we show to be inhabited by a program that relies on a choice operator to keep track of the modulus of continuity of a given function, following Longley’s method [Lon99]. This variant is restricted to “pure” (i.e., without side effects) functions F , α , and β , and Sec. 4.1 discusses issues arising with impure functions.

Roadmap. After presenting in Sec. 2 the main aspects of TT $_{\mathcal{C}}^{\square}$ that are relevant to the results presented in this paper, Sec. 4 validates a continuity principle using stateful computations. One key contribution of this paper, discussed in Sec. 2, is the fact that TT $_{\mathcal{C}}^{\square}$ allows computations to modify the current world, which is accounted for in its forcing interpretation. Some consequences of this fact are further discussed in Sec. 3. Another key contribution, discussed in Sec. 4, is the internalization of the modulus of continuity of functions, in the sense that it can be computed by a TT $_{\mathcal{C}}^{\square}$ expression and used to validate the continuity principle. Finally, Sec. 5 concludes and discusses the related work on continuity.

2. TT $_{\mathcal{C}}^{\square}$: SYNTAX & SEMANTICS

This section presents TT $_{\mathcal{C}}^{\square}$, a family of type theories introduced in [CR22], which is parameterized by a choice operator \mathcal{C} , and a metatheoretical modality \square , which allows typing the choice operator. The choice operators are time-progressing elements that we will in particular instantiate with references. Sec. 2 carves out a sub-family for which we can validate computationally relevant continuity rules as shown in Sec. 4.

The version presented here extends the one introduced in [CR22] in particular with the following components, which are formally defined next.

- An operator that allows making a choice ($t_1 := t_2$). Computations are performed against worlds (see Sec. 2.2), and [CR22] already provided computations to “read” choices from a world. However, even though [CR22] included metatheoretical computations to update a world in the form of the **mutability** requirement presented in Def. 2.7, it did not include corresponding object computations. Doing so has far-reaching consequences. We generalize TT $_{\mathcal{C}}^{\square}$ ’s semantics accordingly, in effect internalizing the **mutability** requirement.
- An operator to generate a “fresh” choice name ($\nu x.t$). While the version of TT $_{\mathcal{C}}^{\square}$ presented in [CR22] provided metatheoretical computations to generate new choice names in the form of the **extendability** requirement presented in Def. 2.5, it did not provide corresponding object computations. We remedy this here by extending TT $_{\mathcal{C}}^{\square}$ with a corresponding computation, which internalizes the **extendability** requirement.
- A type that states the “purity” of an expression, i.e., that the expression has no side effects. This will allow us to formalize the variant of the continuity principle we validate. Sec. 4.1 provides further details.

Moreover, the version presented here differs from the one presented in [CR23] as follows:

- Sec. 2 contains further details regarding TT $_{\mathcal{C}}^{\square}$, such as examples illustrating how effectful programs behave and are given meaning through TT $_{\mathcal{C}}^{\square}$ types, as well as a discussion of TT $_{\mathcal{C}}^{\square}$ ’s inference rules.
- The way TT $_{\mathcal{C}}^{\square}$ captures effects is simpler and more uniform. The type theory is more uniform in the sense that types are now impure by default and TT $_{\mathcal{C}}^{\square}$ provides modalities to capture different levels of purity, as opposed to [CR23] where types were what is characterized as “write-only” in Sec. 2.5, and the theory provided modalities to both make types more pure and more impure (through a complex “time truncation” type operator). Furthermore, the semantics of these new modalities (see Sec. 2.4) is simpler compared to the semantics of the “time truncation” operator used in [CR23], which was used to turn a “write-only” type into a “read & write” type (see Sec. 2.5).
- Sec. 3 discusses how extending TT $_{\mathcal{C}}^{\square}$ with computations to update the current world impacts validity results presented in [CR22] of standard axioms such as Markov’s Principle.

2.1. Metatheory. Our metatheory is Agda’s type theory [AGD]. The results presented in this paper have been formalized in Agda, and the formalization is available here: <https://github.com/vrahli/opentt/blob/lmcs24>. We will use the symbol \star to link to the corresponding definition or result in the formalization. We use $\forall, \exists, \wedge, \vee, \rightarrow, \neg$ in place of Agda’s logical connectives in this paper. Agda provides an hierarchy of types annotated with universe labels which we omit for simplicity. Following Agda’s terminology, we refer to an Agda type as a *set*, and reserve the term *type* for $\text{TT}_{\mathcal{C}}^{\square}$ ’s types. We use \mathbb{P} as the type of sets that denote propositions; \mathbb{N} for the set of natural numbers; and \mathbb{B} for the set of Booleans `true` and `false`. Induction-recursion is used to define the forcing interpretation in Sec. 2.4. We do not discuss this further here and the interested reader is referred to the Agda formalization of this forcing interpretation (\star) for further details.

2.2. Worlds. To capture the time progression notion which underlines choice operators, $\text{TT}_{\mathcal{C}}^{\square}$ is parameterized by a Kripke frame [Kri63, Kri65] defined as follows:

Definition 2.1 (\star Kripke Frame). A Kripke frame consists of a set of *worlds* \mathcal{W} equipped with a reflexive and transitive binary relation \sqsubseteq .

Let w range over \mathcal{W} . We sometimes write $w' \sqsupseteq w$ for $w \sqsubseteq w'$. Let \mathcal{P}_w be the collection of predicates on world extensions, i.e., functions in $\forall w' \sqsupseteq w. \mathbb{P}$. Note that due to \sqsubseteq ’s transitivity, if $P \in \mathcal{P}_w$ then for every $w' \sqsupseteq w$ it naturally extends to a predicate in $\mathcal{P}_{w'}$. We further define the following notations for quantifiers. $\forall_w^{\sqsubseteq}(P)$ states that $P \in \mathcal{P}_w$ is true for all extensions of w , i.e., P w' holds in all worlds $w' \sqsupseteq w$. $\exists_w^{\sqsubseteq}(P)$ states that $P \in \mathcal{P}_w$ is true at an extension of w , i.e., P w' holds for some world $w' \sqsupseteq w$. For readability, we sometime write $\forall_w^{\sqsubseteq}(w'.P)$ instead of $\forall_w^{\sqsubseteq}(\lambda w'.P)$ and $\exists_w^{\sqsubseteq}(w'.P)$ instead of $\exists_w^{\sqsubseteq}(\lambda w'.P)$.

2.3. $\text{TT}_{\mathcal{C}}^{\square}$ ’s Syntax and Operational Semantics. Fig. 1 presents $\text{TT}_{\mathcal{C}}^{\square}$ ’s syntax and call-by-name operational semantics, where the blue boxes highlight the time-related components, and where x belongs to a set of variables Var . For simplicity, numbers are considered to be primitive. The constant \star is there for convenience, and is used in place of a term, when the particular term used is irrelevant. The term `let $x = t_1$ in t_2` is a call-by-value operator that allows evaluating t_1 to a value before proceeding with t_2 . Terms are evaluated according to the operational semantics presented in Fig. 1’s lower part, which is further discussed below. In what follows, we use all letters as metavariables for terms. Let $t[x \setminus u]$ stand for the capture-avoiding substitution of all the free occurrences of x in t by u . In what follows, we use the following definitions, where x does not occur free in t_2 or t_3 :

$$\begin{aligned}
& \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \equiv \text{case } t_1 \text{ of } \text{inl}(x) \Rightarrow t_2 \mid \text{inr}(x) \Rightarrow t_3 \\
t_1; t_2 & \equiv \text{let } x = t_1 \text{ in } t_2 & \text{iszero}(t) & \equiv \text{natrec}(t, \text{tt}, \lambda m. \lambda r. \text{ff}) \\
\text{tt} & \equiv \text{inl}(\star) & \text{pred}(t) & \equiv \text{natrec}(t, \underline{0}, \lambda m. \lambda r. m) \\
\text{ff} & \equiv \text{inr}(\star) & t_1 - t_2 & \equiv \text{natrec}(t_2, t_1, \lambda m. \lambda r. \text{pred}(r)) \\
\text{neg}(t) & \equiv \text{if } t \text{ then } \text{ff} \text{ else } \text{tt} & t_1 < t_2 & \equiv \text{neg}(\text{iszero}(t_2 - t_1))
\end{aligned}$$

Types are syntactic forms that are given semantics in Sec. 2.4 via a forcing interpretation. The type system contains standard types such as dependent products of the form $\prod x:t_1. t_2$ and dependent sums of the form $\sum x:t_1. t_2$. It also includes *subsingleton* types of the form $\|t\|$, which turns a type t into a subsingleton type that equates all elements of t ; and *intersection*

Figure 1 Core syntax (above) and small-step operational semantics (below)

$v \in \mathbf{Value} ::= vt$ (type)	$\lambda x.t$ (lambda)	\star (constant)
\underline{n} (number)	$\mathbf{inl}(t)$ (left injection)	δ (choice name)
$\langle t_1, t_2 \rangle$ (pair)	$\mathbf{inr}(t)$ (right injection)	
$vt \in \mathbf{Type} ::= \mathbf{\Pi}x:t_1.t_2$ (product)	$\{x : t_1 \mid t_2\}$ (set)	$t_1 + t_2$ (disjoint union)
$\mathbf{\Sigma}x:t_1.t_2$ (sum)	$t_1 = t_2 \in t$ (equality)	\mathbf{NoRead} (no read)
\mathbb{U}_i (universe)	\mathbf{Nat} (numbers)	$\mathbf{NoWrite}$ (no write)
$t_1 \cap t_2$ (intersection)	$\ t\ $ (sub-singleton)	\mathbf{Pure} (pure)
$t \in \mathbf{Term} ::= x$ (variable)	v (value)	$!\underline{t}$ (read)
$\underline{t}_1 t_2$ (application)	$\mathbf{fix}(\underline{t})$ (fixpoint)	$\underline{t}_1 := t_2$ (choose)
$\mathbf{let} x = \underline{t}_1 \mathbf{in} t_2$ (eager)	$\mathbf{succ}(\underline{t})$ (successor)	$\nu x.t$ (fresh)
$\mathbf{natrec}(\underline{t}_1, t_2, t_3)$ (number recursor)		
$\mathbf{let} x, y = \underline{t}_1 \mathbf{in} t_2$ (pair destructor)		
$\mathbf{case} \underline{t} \mathbf{of} \mathbf{inl}(x) \Rightarrow t_1 \mid \mathbf{inr}(y) \Rightarrow t_2$ (injection destructor)		
$(\lambda x.t) u \mapsto \frac{w}{w} t[x \setminus u]$	$\mathbf{let} x, y = \langle t_1, t_2 \rangle \mathbf{in} t \mapsto \frac{w}{w} t[x \setminus t_1; y \setminus t_2]$	
$\mathbf{fix}(v) \mapsto \frac{w}{w} v \mathbf{fix}(v)$	$\mathbf{case} \mathbf{inl}(t) \mathbf{of} \mathbf{inl}(x) \Rightarrow t_1 \mid \mathbf{inr}(y) \Rightarrow t_2 \mapsto \frac{w}{w} t_1[x \setminus t]$	
$\mathbf{let} x = v \mathbf{in} t_2 \mapsto \frac{w}{w} t_2[x \setminus v]$	$\mathbf{case} \mathbf{inr}(t) \mathbf{of} \mathbf{inl}(x) \Rightarrow t_1 \mid \mathbf{inr}(y) \Rightarrow t_2 \mapsto \frac{w}{w} t_2[y \setminus t]$	
$\mathbf{succ}(\underline{n}) \mapsto \frac{w}{w} \underline{1} + \underline{n}$		$!\delta \mapsto \frac{w}{w} \mathbf{read}(w, \delta)$
$\mathbf{natrec}(\underline{0}, t_1, t_2) \mapsto \frac{w}{w} t_1$		$\delta := t \mapsto \frac{w}{w} \mathbf{write}(w, \delta, t) \star$
$\mathbf{natrec}(\underline{1} + \underline{n}, t_1, t_2) \mapsto \frac{w}{w} t_2 \underline{n} \mathbf{natrec}(\underline{n}, t_1, t_2)$		$\nu x.t \mapsto \frac{w}{w} \mathbf{start} \nu \mathcal{C}(w, x) t[x \setminus \nu \mathcal{C}(w)]$

types of the form $t_1 \cap t_2$, which is inhabited by the inhabitants of both t_1 and t_2 . For convenience we introduce the following definitions, where x does not occur free in t_2 :

$$\begin{aligned}
\mathbf{Void} &\equiv \underline{0} = \underline{1} \in \mathbf{Nat} & t_1 \rightarrow t_2 &\equiv \mathbf{\Pi}x:t_1.t_2 & \uparrow T &\equiv T = \mathbf{tt} \in \mathbf{Bool} \\
\mathbf{Unit} &\equiv \underline{0} = \underline{0} \in \mathbf{Nat} & \neg T &\equiv T \rightarrow \mathbf{Void} & \downarrow T &\equiv \{x : \mathbf{Unit} \mid T\} \\
\mathbf{Bool} &\equiv \mathbf{Unit} + \mathbf{Unit} & & & &
\end{aligned}$$

Fig. 1's lower part presents TT \mathcal{C} [□]'s small-step operational semantics, where $t_1 \mapsto_{w_2}^{w_1} t_2$ expresses that t_1 reduces to t_2 in one step of computation starting from the world w_1 and possibly updating it so that the resulting world is w_2 at the end of the computation step. Most computations do not modify the current world except $t_1 := t_2$. We omit the congruence rules that allow computing within terms such as: if $t_1 \mapsto_{w_2}^{w_1} t_2$ then $t_1(u) \mapsto_{w_2}^{w_1} t_2(u)$, and the boxed terms of the form \underline{t} in Fig. 1 indicate the arguments that have to be reduced before the outer operators can be reduced. We denote by \Rightarrow the reflexive transitive closure of \mapsto , i.e., $a \Rightarrow_{w_2}^{w_1} b$ states that a computes to b in 0 or more steps, starting from the world w_1 and updating it so that the resulting world is w_2 at the end of the computation. We write $a \Rightarrow_w b$ for $\exists \frac{w'}{w} (w'.a \Rightarrow_{w'}^w b)$. We also write $a \Rightarrow_w b$ if a computes to b in all extensions of w , i.e., if $\forall \frac{w'}{w} (w'.a \Rightarrow_{w'}^w b)$.

TT \mathcal{C} [□] includes time-progressing notions that rely on worlds to record choices and provides operators to manipulate the choices stored in a world, which we now recall. Choices are referred to through their names. A concrete example of such choices are reference cells in programming languages, where a variable name pointing to a reference cell is the name of the corresponding reference cell. To this end, TT \mathcal{C} [□]'s computation system is parameterized by a set \mathcal{N} of *choice names*, that is equipped with a decidable equality, and an operator that

given a list of names, returns a name not in the list. This can be given by, e.g., nominal sets [Pit13]. In what follows we let δ range over \mathcal{N} , and take \mathcal{N} to be \mathbb{N} for simplicity. $\text{TT}_{\mathcal{C}}^{\square}$ is further parameterized over abstract operators and properties recalled in Defs. 2.2, 2.4, 2.5 and 2.7, which we show how to instantiate in Ex. 2.8. Definitions such as Def. 2.2 provide axiomatizations of operators, and in addition informally indicate their intended use. Choices are defined abstractly as follows:

Definition 2.2 (⚙ Choices). Let $\mathcal{C} \subseteq \text{Term}$ be a set of *choices*,¹ and let c range over \mathcal{C} . We say that a computation system contains $\langle \mathcal{N}, \mathcal{C} \rangle$ -choices if there exists a partial function $\text{read} \in \mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathcal{C}$ (⚙). Given $w \in \mathcal{W}$ and $\delta \in \mathcal{N}$, the returned choice, if it exists, is meant to be the last choice made for δ according to w . \mathcal{C} is said to be *non-trivial* if it contains two values κ_0 and κ_1 , which are syntactically different (⚙).

A choice name δ can be used in a computation to access (or “read”) choices from a world as follows: $!\delta \mapsto_w^w \text{read}(w, \delta)$ (as shown in Fig. 1). This allows getting the last δ -choice from the current world w . Datatypes are by default inhabited by impure computations that can for example read choices using $!$. For example, the Nat type is the type of potentially impure natural numbers that includes expressions of the form $!\delta$, when δ ’s choices are natural numbers, in addition to expressions of the form $\underline{0}$, $\underline{1}$, etc.

Note that the above definition of read is a slight simplification of the more general notion of choices presented in [CR22]. There, the read function was of type $\mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathbb{N} \rightarrow \mathcal{C}$. The additional \mathbb{N} component enables a more general notion of choice operators, encompassing both references and choice sequences [KV65, vAvD02, Tro85, Tro77a, KT70, Vel01, Mos93], which stem from Brouwer’s intuitionistic logic, and can be seen as reference cells that record the history of all the values ever stored in the cells. In references, which is the notion of choices we especially focus on in this paper, one only maintains the latest update and so the \mathbb{N} component becomes moot. Thus, for simplicity of presentation, we elide the \mathbb{N} component in this paper, but full details are available in the Agda implementation.

The NoRead type is inhabited by expressions that when computing to a value in a world w_1 , also compute to that value in all extensions w_2 of w_1 . Intuitively, this captures expressions that “do not read” in the sense that they can only make limited use of the $!$ operator as the values they return should be the same irrespective of the world they start computing against. For example, given a choice name δ , whose choices are natural numbers, $!\delta$ does not inhabit NoRead as it could return different values in successive extensions, while $\text{let } x = !\delta \text{ in } \underline{0}$ inhabits NoRead as it always compute to $\underline{0}$, even though it reads δ . The NoRead type can be turned into a $|_|_{\mathbf{r}}$ modality as follows: let $|T|_{\mathbf{r}} \equiv T \cap \text{NoRead}$. We then obtain that $!\delta$ is not a member of $|\text{Nat}|_{\mathbf{r}}$, while it is a member of Nat , and $\text{let } x = !\delta \text{ in } \underline{0}$ is a member of both Nat and $|\text{Nat}|_{\mathbf{r}}$. More generally, $|T|_{\mathbf{r}}$ is a subtype of T in the sense that the inhabitants of $|T|_{\mathbf{r}}$ also inhabit T .

While $|T|_{\mathbf{r}}$ restricts the effects that T ’s inhabitants can have, it can still be inhabited by effectful computations (we saw that $\text{let } x = !\delta \text{ in } \underline{0}$ inhabits $|\text{Nat}|_{\mathbf{r}}$). Therefore, to capture pure expressions, $\text{TT}_{\mathcal{C}}^{\square}$ provides the term Pure , which is the type of “pure” terms, i.e. terms that do not contain choice names. This type can be turned into a modality as follows: $|T|_{\mathbf{c}} \equiv T \cap \text{Pure}$. Therefore, $|\text{Nat}|_{\mathbf{c}}$ is the type of pure natural numbers, i.e., natural numbers that do not contain choice names. Hence, $|\text{Nat}|_{\mathbf{c}}$ is a subtype of $|\text{Nat}|_{\mathbf{r}}$. For example, $\text{let } x = !\delta \text{ in } \underline{0}$ inhabits $|\text{Nat}|_{\mathbf{r}}$, while it does not inhabit $|\text{Nat}|_{\mathbf{c}}$.

¹To guarantee that $\mathcal{C} \subseteq \text{Term}$, one can for example extend the syntax to include a designated constructor for choices, or require a coercion $\mathcal{C} \rightarrow \text{Term}$. We opted for the latter in our formalization.

TT $\square_{\mathcal{C}}$ also includes the notion of a *restriction*, which allows assuming that the choices made for a given choice name all satisfy a pre-defined constraint. Here again we simplify the concept for choices without history tracking.

Definition 2.3 (⚙️ Restrictions). A restriction $r \in \mathbf{Res}$ is a pair $\langle res, d \rangle$ consisting of a function $res \in \mathcal{C} \rightarrow \mathbb{P}$ and a default choice $d \in \mathcal{C}$ such that $(res \ d)$ holds. Given such a pair r , we write $r_{\mathbf{d}}$ for d .

Intuitively, res specifies a restriction on the choices that can be made at any point in time and d provides a default choice that meets this restriction (e.g., for reference cells, this default choice is used to initialize a cell). For example, the restriction $\langle \lambda c. c \in \mathbb{N}, 0 \rangle$ requires choices to be numbers and provides 0 as a default value. To reason about restrictions, we require the existence of a “compatibility” predicate as follows.

Definition 2.4 (⚙️ Compatibility). We say that \mathcal{C} is *compatible* if there exists a predicate $\mathbf{comp} \in \mathcal{N} \rightarrow \mathcal{W} \rightarrow \mathbf{Res} \rightarrow \mathbb{P}$, intended to guarantee that restrictions are satisfied, and which is preserved by \sqsubseteq : $\forall (\delta : \mathcal{N})(w_1, w_2 : \mathcal{W})(r : \mathbf{Res}). w_1 \sqsubseteq w_2 \rightarrow \mathbf{comp}(\delta, w_1, r) \rightarrow \mathbf{comp}(\delta, w_2, r)$.

TT $\square_{\mathcal{C}}$ further requires the ability to create new choice names as follows.

Definition 2.5 (⚙️ Extendability). We say that \mathcal{C} is *extendable* if there exists a function $\nu\mathcal{C} \in \mathcal{W} \rightarrow \mathcal{N}$ (⚙️), where $\nu\mathcal{C}(w)$ is intended to return a new choice name not present in w , and a function $\mathbf{start}\nu\mathcal{C} \in \mathcal{W} \rightarrow \mathbf{Res} \rightarrow \mathcal{W}$ (⚙️), where $\mathbf{start}\nu\mathcal{C}(w, r)$ is intended to return an extension of w with the new choice name $\nu\mathcal{C}(w)$ with restriction r , satisfying the following properties:

- Starting a new choice extends the current world: $\forall (w : \mathcal{W})(r : \mathbf{Res}). w \sqsubseteq \mathbf{start}\nu\mathcal{C}(w, r)$
- Initially, the only possible choice is the default value of the given restriction, i.e.:
 $\forall (r : \mathbf{Res})(w : \mathcal{W})(c : \mathcal{C}). \mathbf{read}(\mathbf{start}\nu\mathcal{C}(w, r), \nu\mathcal{C}(w)) = c \rightarrow c = r_{\mathbf{d}}$
- A choice is initially compatible with its restriction:
 $\forall (w : \mathcal{W})(r : \mathbf{Res}). \mathbf{comp}(\nu\mathcal{C}(w), \mathbf{start}\nu\mathcal{C}(w, r), r)$

TT $\square_{\mathcal{C}}$ provides a corresponding object computation that internalizes the metatheoretical $\nu\mathcal{C}$ function, namely $\nu x.t$. Intuitively, it selects a “fresh” choice name δ and instantiate the variable x with δ . Formally, it computes as follows: $\nu x.t \mapsto_{\mathbf{start}\nu\mathcal{C}(w, r)}^w t[x \setminus \nu\mathcal{C}(w)]$ (as presented in Fig. 1). here \mathbf{r} is the restriction $\langle \lambda c. (c \in \mathbb{N}), 0 \rangle$, which constrains the choices to be numbers, with default value 0. Other restrictions could be supported, for example by adding different ν symbols to the language and by selecting during computation the appropriate restriction based on the ν operator at hand. This is however left for future work as we especially focus here on the choices presented in Ex. 2.8, i.e., natural numbers.

Remark 2.6 (Freshness). *The fresh operator used in [RB16] computes $\nu x.a$ by reducing a to b , and then returning $\nu x.b$, thereby never generating new fresh names. As opposed to that fresh operator, which was based on nominal sets, the one introduced in this paper cannot put back the “fresh” constructor at each step of the small step derivation, otherwise a multi-step computation would not be able to use a choice name to keep track of the modulus of continuity of a function across multiple computation steps by recording it in the current world. One consequence of this is that this fresh operator cannot guarantee that it generates a truly “fresh” name that does not occur anywhere else (therefore, it does not satisfy the nominal axioms). For example $(\nu x.x) \delta$ might generate the name δ because it does not occur in the local expression $\nu x.x$.*

Lastly, $\text{TT}_{\mathcal{C}}^{\square}$ requires the ability to update a choice as follows.

Definition 2.7 (⚙ Mutability). We say that \mathcal{C} is *mutable* if there exists a function $\text{write} \in \mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathcal{C} \rightarrow \mathcal{W}$ such that if $w \in \mathcal{W}$, $\delta \in \mathcal{N}$, $c \in \mathcal{C}$, then $w \sqsubseteq \text{write}(w, \delta, c)$.

$\text{TT}_{\mathcal{C}}^{\square}$ provides a corresponding object computation that internalizes the metatheoretical write function, namely $\delta := t$. Choosing a δ -choice t using $\delta := t$ results in a corresponding update of the current world, namely $\text{write}(w, \delta, t)$. The computation returns \star , which is reminiscent of reference updates in OCaml for example, which are of type `unit`. As mentioned in Def. 2.2, we require $\mathcal{C} \subseteq \text{Term}$ so that choices can be included in computations. In addition, because $\text{write} \in \mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathcal{C} \rightarrow \mathcal{W}$, for $\text{write}(w, \delta, t)$ to be well-defined for $t \in \text{Term}$, we require a coercion from Term to \mathcal{C} so that t can be turned into a choice, and write can be applied to that choice. This coercion is left implicit for readability. We further require that applying this coercion to a choice c returns c , which is used to validate the assumption `Asm3` discussed in Sec. 4.2.

The type `NoWrite` is inhabited by expressions that when computing to a value from a world w_1 to a world w_2 , satisfy $w_2 = w_1$. Intuitively, this captures expressions that “do not write” in the sense that they can only make limited use of the $:=$ operator as the resulting world at the end of the computation should be the world the computation started in. For example, given a choice name δ , whose choices are natural numbers, $\delta := (!\delta + \underline{1})$ does not inhabit `NoWrite` as the computation ends in a world different from the initial world, where δ is incremented by one, while $\text{let } x = !\delta \text{ in } ((\delta := (x + 1)); (\delta := x))$ inhabits `NoWrite`, even though it uses $:=$, as it always ends in the same world as the initial world since δ is reset to its initial value. The `NoWrite` type can be turned into a $|_|_w$ modality as follows: let $|T|_w \equiv T \cap \text{NoWrite}$. We also write $|T|_{rw}$ for $||T|_r|_w$. We then obtain that $\delta := (!\delta + \underline{1})$ is not a member of $|\text{Unit}|_w$, while it is a member of `Unit`, and $\text{let } x = !\delta \text{ in } (\delta := (x + 1)); \delta := x$ is a member of both `Unit` and $|\text{Unit}|_w$. More generally, both $|T|_w$ and $|T|_r$ are subtypes of T , and $|T|_c$ is a subtype of both $|T|_w$ and $|T|_r$.

From this point on, we will only discuss choices \mathcal{C} that are *compatible*, *extendable* and *mutable*. While the abstract notion of choice operators has many concrete instances, this paper focuses on one concrete instance, namely mutable references.

Example 2.8 (⚙ References). *Reference cells, which are values that allow a program to indirectly access a particular object, are choice operators since they can point to different objects over their lifetime. Formally, we define references to numbers, `Ref`, as follows:*

Non-trivial Choices: Let $\mathcal{N} \equiv \mathbb{N}$ and $\mathcal{C} \equiv \mathbb{N}$, which is *non-trivial*, e.g., take $\kappa_0 \equiv \underline{0}$ and $\kappa_1 \equiv \underline{1}$.

Worlds: *Worlds are lists of cells, where a cell is a quadruple of (1) a choice name, (2) a restriction, (3) a choice, and (4) a Boolean indicating whether the cell is mutable. \sqsubseteq is the reflexive transitive closure of two operations that allow (i) creating a new reference cell, and (ii) updating an existing reference cell. We define $\text{read}(w, \delta)$ so that it simply accesses the content of the δ cell in w .*

Compatible: $\text{comp}(\delta, w, r)$ states that a reference cell named δ with restriction r was created in the world w (using an operation of type (i) described above), and that the current value of the cell satisfies r .

Extendable: $\nu\mathcal{C}(w)$ returns a reference name not occurring in w ; and $\text{start}\nu\mathcal{C}(w, r)$ adds a new reference cell to w with name $\nu\mathcal{C}(w)$ and restriction r (using an operation of type (i) mentioned above).

Mutable: $\text{write}(w, \delta, c)$ updates the reference δ with the choice c if δ occurs in w , and otherwise returns w (using an operation of type (ii) mentioned above).

A coercion from **Term** to \mathcal{C} can then turn \underline{n} into n and all other terms to 0, which satisfies the requirement that choices are mapped to the same choices.

Formally, $a \Rightarrow_{w_2}^{w_1} b$ is the reflexive and transitive closure of \mapsto , i.e., it holds if a in world w_1 computes to b in world w_2 in 0 or more steps. Thanks to the properties of $\text{start}\nu\mathcal{C}$ presented in Def. 2.5, and the properties of write in Def. 2.7, computations respect \sqsubseteq :

Lemma 2.9 (⚙️ Computations respect \sqsubseteq). *If $a \Rightarrow_{w_2}^{w_1} b$ then $w_1 \sqsubseteq w_2$.*

2.4. Forcing Interpretation. TT \square 's semantics, presented in Fig. 2, while similar to the one presented in [CR22, CR23], differs in one major way, namely types are here impure by default. It is interpreted via a forcing interpretation in which the forcing conditions are worlds. This interpretation is defined using induction-recursion as follows: (1) the inductive relation $w \vDash T_1 \equiv T_2$ expresses type equality in the world w ; (2) the recursive function $w \vDash t_1 \equiv t_2 \in T$ expresses equality in a type. We further use the following abstractions:

$$\begin{aligned} w \vDash \text{type}(T) &\equiv w \vDash T \equiv T & w \vDash T &\equiv \exists(t : \text{Term}). w \vDash t \in T \\ w \vDash t \in T &\equiv w \vDash t \equiv t \in T & a \Rightarrow_{!w} b &\equiv \forall_{\bar{w}}(w'. a \Rightarrow_{w'}^{w'} b) \end{aligned}$$

$$\begin{aligned} \text{Fam}_w(A_1, A_2, B_1, B_2) &\equiv \\ w \vDash A_1 \equiv A_2 \wedge \forall_{\bar{w}}(w'. \forall(a_1, a_2 : \text{Term}). w' \vDash a_1 \equiv a_2 \in A_1 \rightarrow w' \vDash B_1[x \setminus a_1] \equiv B_2[x \setminus a_2]) \end{aligned}$$

Note that $a \Rightarrow_{!w} b$ captures the fact that the computation does not change the initial world (this is used in Thm. 2.12). Fig. 2 defines in particular the semantics of **Pure**, which is inhabited by name-free terms, where $\text{namefree}(t)$ is defined recursively over t and returns false iff t contains a choice name δ or a fresh operator of the form $\nu x.t$. This forcing interpretation is parameterized by a family of abstract modalities \square , which we sometimes refer to simply as a modality, which is a function that takes a world w to its modality $\square_w \in \mathcal{P}_w \rightarrow \mathbb{P}$. We often write $\square_w(w'.P)$ for $\square_w \lambda w'.P$. As in [CR22], to guarantee that this interpretation yields a standard type system in the sense of Thm. 2.12, we require that the modalities satisfy certain properties reminiscent of standard modal axiom schemata [CH96], which we repeat here for ease of read:

Definition 2.10 (⚙️ Equality modality). The modality \square is called an *equality modality* if it satisfies the following properties:

- \square_1 (monotonicity): $\forall(w : \mathcal{W})(P : \mathcal{P}_w). \forall w' \sqsupseteq w. \square_w P \rightarrow \square_{w'} P$.
- \square_2 (distribution): $\forall(w : \mathcal{W})(P, Q : \mathcal{P}_w). \square_w(w'.P \ w' \rightarrow Q \ w') \rightarrow \square_w P \rightarrow \square_w Q$
- \square_3 (density): $\forall(w : \mathcal{W})(P : \mathcal{P}_w). \square_w(w'. \square_{w'} P) \rightarrow \square_w P$
- \square_4 (weakening): $\forall(w : \mathcal{W})(P : \mathcal{P}_w). \forall_{\bar{w}}(P) \rightarrow \square_w P$
- \square_5 (reflexivity): $\forall(w : \mathcal{W})(P : \mathbb{P}). \square_w(w'.P) \rightarrow P$

As mentioned in [CR22], modalities can be derived from covering relations \triangleleft , where $w \triangleleft o$ captures that $o \in \mathcal{W} \rightarrow \mathbb{P}$ “covers” the world w . Covering relations are required to satisfy suitable intersection, union, top, non-emptiness, and subset properties [CR22, Def.22] to be able to derive modalities from them. We present below three examples of coverings that satisfy these properties, namely Kripke, Beth, and Open coverings, from which modalities can be derived (the result presented in Sec. 4 holds in particular for the resulting modalities):

Figure 2 Forcing Interpretation

Numbers:

- $w \Vdash \text{Nat} \equiv \text{Nat} \iff \text{True}$
- $w \Vdash t \equiv t' \in \text{Nat} \iff \Box_w(w'. \exists (n : \mathbb{N}). t \Vdash_{w'} n \wedge t' \Vdash_{w'} n)$

Products:

- $w \Vdash \Pi x:A_1. B_1 \equiv \Pi x:A_2. B_2 \iff \text{Fam}_w(A_1, A_2, B_1, B_2)$
- $w \Vdash f \equiv g \in \Pi x:A. B \iff \Box_w(w'. \forall (a_1, a_2 : \text{Term}). w' \Vdash a_1 \equiv a_2 \in A \rightarrow w' \Vdash f(a_1) \equiv g(a_2) \in B[x \setminus a_1])$

Sums:

- $w \Vdash \Sigma x:A_1. B_1 \equiv \Sigma x:A_2. B_2 \iff \text{Fam}_w(A_1, A_2, B_1, B_2)$
- $w \Vdash p_1 \equiv p_2 \in \Sigma x:A. B \iff \Box_w(w'. \exists (a_1, a_2, b_1, b_2 : \text{Term}). w' \Vdash a_1 \equiv a_2 \in A \wedge w' \Vdash b_1 \equiv b_2 \in B[x \setminus a_1] \wedge p_1 \Vdash_{w'} \langle a_1, b_1 \rangle \wedge p_2 \Vdash_{w'} \langle a_2, b_2 \rangle)$

Sets:

- $w \Vdash \{x : A_1 \mid B_1\} \equiv \{x : A_2 \mid B_2\} \iff \text{Fam}_w(A_1, A_2, B_1, B_2)$
- $w \Vdash a_1 \equiv a_2 \in \{x : A \mid B\} \iff \Box_w(w'. \exists (b_1, b_2 : \text{Term}). w' \Vdash a_1 \equiv a_2 \in A \wedge w' \Vdash b_1 \equiv b_2 \in B[x \setminus a_1])$

Disjoint unions:

- $w \Vdash A_1 + B_1 \equiv A_2 + B_2 \iff w \Vdash A_1 \equiv A_2 \wedge w \Vdash B_1 \equiv B_2$
- $w \Vdash a_1 \equiv a_2 \in A + B \iff \Box_w(w'. \exists (u, v : \text{Term}). (a_1 \Vdash_{w'} \text{inl}(u) \wedge a_2 \Vdash_{w'} \text{inl}(v) \wedge w' \Vdash u \equiv v \in A) \vee (a_1 \Vdash_{w'} \text{inr}(u) \wedge a_2 \Vdash_{w'} \text{inr}(v) \wedge w' \Vdash u \equiv v \in B))$

Equalities:

- $w \Vdash (a_1 = b_1 \in A) \equiv (a_2 = b_2 \in B) \iff w \Vdash A \equiv B \wedge \forall_w^{\square}(w'. w' \Vdash a_1 \equiv a_2 \in A) \wedge \forall_w^{\square}(w'. w' \Vdash b_1 \equiv b_2 \in B)$
- $w \Vdash a_1 \equiv a_2 \in (a = b \in A) \iff \Box_w(w'. w' \Vdash a \equiv b \in A)$

Subsingletons:

- $w \Vdash \|A\| \equiv \|B\| \iff w \Vdash A \equiv B$
- $w \Vdash a \equiv b \in \|A\| \iff \Box_w(w'. w' \Vdash a \equiv a \in A \wedge w' \Vdash b \equiv b \in A)$

Binary intersections:

- $w \Vdash A_1 \cap B_1 \equiv A_2 \cap B_2 \iff w \Vdash A_1 \equiv A_2 \wedge w \Vdash B_1 \equiv B_2$
- $w \Vdash a_1 \equiv a_2 \in A \cap B \iff \Box_w(w'. w' \Vdash a_1 \equiv a_2 \in A \wedge w' \Vdash a_1 \equiv a_2 \in B)$

No-read types:

- $w \Vdash \text{NoRead} \equiv \text{NoRead} \iff \text{True}$
- $w \Vdash a \equiv b \in \text{NoRead} \iff \Box_w(w'. \forall (v : \text{Value}). (a \Vdash_{w'} v \rightarrow a \Rightarrow_{w'} v) \wedge (b \Vdash_{w'} v \rightarrow b \Rightarrow_{w'} v))$

No-write types:

- $w \Vdash \text{NoWrite} \equiv \text{NoWrite} \iff \text{True}$
- $w \Vdash a \equiv b \in \text{NoWrite} \iff \Box_w(w'. \forall (v : \text{Value}). (a \Vdash_{w'} v \rightarrow a \Rightarrow_{w'}^w v) \wedge (b \Vdash_{w'} v \rightarrow b \Rightarrow_{w'}^w v))$

Purity:

- $w \Vdash \text{Pure} \equiv \text{Pure} \iff \text{True}$
- $w \Vdash a_1 \equiv a_2 \in \text{Pure} \iff (\exists (t : \text{Term}). a_1 \Rightarrow_{!w} t \wedge \text{namefree}(t)) \wedge (\exists (t : \text{Term}). a_2 \Rightarrow_{!w} t \wedge \text{namefree}(t))$

Modality closure:

- $w \Vdash T_1 \equiv T_2 \iff \Box_w(w'. \exists (T'_1, T'_2 : \text{Term}). T_1 \Rightarrow_{w'} T'_1 \wedge T_2 \Rightarrow_{w'} T'_2 \wedge w' \Vdash T'_1 \equiv T'_2)$
 - $w \Vdash t_1 \equiv t_2 \in T \iff \Box_w(w'. \exists (T' : \text{Term}). T \Rightarrow_{w'} T' \wedge w' \Vdash t_1 \equiv t_2 \in T')$
-

Example 2.11. The Kripke covering (\clubsuit) is defined as follows, i.e., $w \triangleleft_K o$ whenever o contains all the extensions of w :

$$w \triangleleft_K o \equiv \forall_w^{\square}(w'. o(w'))$$

The Beth covering (\star) is defined as follows, i.e., $w \triangleleft_B o$ whenever o contains at least one world from each “branch” (captured by **chain** below) starting from w :

$$w \triangleleft_B o \equiv \forall (c : \mathbf{chain}(w)). \exists (n : \mathbb{N}). \exists w' \sqsubseteq (c \ n). o(w')$$

where $\mathbf{chain}(w)$ is the set of sequences of worlds in $\mathbb{N} \rightarrow \mathcal{W}$ such that $c \in \mathbf{chain}(w)$ iff (1) $w \sqsubseteq c \ 0$, (2) for all $i \in \mathbb{N}$, $c \ i \sqsubseteq c \ (i + 1)$; and (3) c is progressing, which is formally defined in [CR22, Def.25], and informally captures that there exists two worlds $w_1 \sqsubseteq w_2$ along c where w_2 is not w_1 , and this infinitely often.

The Open covering (\star) is defined as follows, i.e., $w \triangleleft_O o$ whenever o contains a world that extends each world that extends w :

$$w \triangleleft_O o \equiv \forall_w^{\sqsubseteq} (w_1. \exists_{w_1}^{\sqsubseteq} (w_2. o(w_2)))$$

The Beth and Open coverings are in particular suitable to capture aspects of choice sequences, which as mentioned above, can be seen as reference cells that include the history of all the values ever stored in the cells. As discussed in [CR22], while the Beth covering is incompatible with classical reasoning the Open covering allows validating axioms such as the Law of Excluded Middle.

Theorem 2.12. *Given a computation system with choices \mathcal{C} and an equality modality \square , TT_C^\square is a standard type system in the sense that its forcing interpretation induced by \square satisfy the following properties (where free variables are universally quantified):*

<i>transitivity:</i>	$w \vDash T_1 \equiv T_2 \rightarrow w \vDash T_2 \equiv T_3 \rightarrow w \vDash T_1 \equiv T_3$	$w \vDash t_1 \equiv t_2 \in T \rightarrow w \vDash t_2 \equiv t_3 \in T \rightarrow w \vDash t_1 \equiv t_3 \in T$
<i>symmetry:</i>	$w \vDash T_1 \equiv T_2 \rightarrow w \vDash T_2 \equiv T_1$	$w \vDash t_1 \equiv t_2 \in T \rightarrow w \vDash t_2 \equiv t_1 \in T$
<i>computation:</i>	$w \vDash T \equiv T \rightarrow T \Rightarrow_{!w} T' \rightarrow w \vDash T \equiv T'$	$w \vDash t \equiv t \in T \rightarrow t \Rightarrow_{!w} t' \rightarrow w \vDash t \equiv t' \in T$
<i>monotonicity:</i>	$w \vDash T_1 \equiv T_2 \rightarrow w \sqsubseteq w' \rightarrow w' \vDash T_1 \equiv T_2$	$w \vDash t_1 \equiv t_2 \in T \rightarrow w \sqsubseteq w' \rightarrow w' \vDash t_1 \equiv t_2 \in T$
<i>locality:</i>	$\square_w (w'. w' \vDash T_1 \equiv T_2) \rightarrow w \vDash T_1 \equiv T_2$	$\square_w (w'. w' \vDash t_1 \equiv t_2 \in T) \rightarrow w \vDash t_1 \equiv t_2 \in T$
<i>consistency:</i>	$\neg w \vDash t \in \mathbf{Void}$	

Proof. The proof relies on the properties of the equality modality. For example: \square_1 is used to prove monotonicity (\star) when $w \vDash T_1 \equiv T_2$ is derived by closing under \square_w ; \square_2 and \square_4 are used, e.g., to prove the symmetry (\star) and transitivity (\star) of $w \vDash t \equiv t' \in \mathbf{Nat}$; \square_3 is used to prove locality (\star); and \square_5 is used to prove consistency (\star). \square

As indicated in Thm. 2.12, and as opposed to the counterpart of the theorem in [CR22], $w \vDash T_1 \equiv T_2$ and $w \vDash t_1 \equiv t_2 \in T$ are no longer closed under all computations. For example, when $T \equiv \mathbf{Nat}$, if $t \Rightarrow_w t'$ and $w \vDash t \equiv \underline{n} \in \mathbf{Nat}$, does not necessarily give us that $w \vDash t' \equiv \underline{n} \in \mathbf{Nat}$. An example is:

$$t \equiv (\delta := \underline{1}; \text{if } !\delta < \underline{1} \text{ then } \underline{0} \text{ else } \underline{1})$$

which reduces to $t' \equiv (\text{if } !\delta < \underline{1} \text{ then } \underline{0} \text{ else } \underline{1})$ and also to $\underline{1}$ in all worlds, i.e., $t \Rightarrow_w t'$ and $t \Rightarrow_w \underline{n}$ for all w , and therefore $w \vDash t \equiv \underline{1} \in \mathbf{Nat}$. However t' does not reduce to $\underline{1}$ in all worlds, and therefore $w \vDash t' \equiv \underline{1} \in \mathbf{Nat}$ does not hold, because δ could be initialized differently in different worlds, for example with $\underline{0}$, in which case t' would reduce to $\underline{0}$. This example can be adapted to show that $w \vDash T_1 \equiv T_2$ is also not closed under all computations:

$$T \equiv (\delta := \underline{1}; \text{if } !\delta < \underline{1} \text{ then } \mathbf{Unit} \text{ else } \mathbf{Void})$$

which reduces to $T' \equiv (\text{if } !\delta < \underline{1} \text{ then } \mathbf{Unit} \text{ else } \mathbf{Void})$ and also to \mathbf{Void} in all worlds, i.e., $T \Rightarrow_w T'$ and $T \Rightarrow_w \mathbf{Void}$ for all w , and therefore $w \vDash T \equiv \mathbf{Void}$. However, T' does not reduce to \mathbf{Void} in all worlds, because δ could be initialized differently in different worlds, and therefore $w \vDash T' \equiv \mathbf{Void}$ does not hold.

However, the following holds by transitivity of \Rightarrow :

$$\begin{aligned} t' \Rightarrow_w t \rightarrow w \vDash t \equiv t \in \text{Nat} \rightarrow w \vDash t \equiv t' \in \text{Nat} \\ T' \Rightarrow_w T \rightarrow w \vDash T \equiv T \rightarrow w \vDash T \equiv T' \end{aligned}$$

To summarize, $\text{TT}_{\mathcal{C}}^{\square}$ is closed under the following computations:

- $\star U \Rightarrow_w U' \rightarrow T \Rightarrow_w T' \rightarrow w \vDash T' \equiv U' \rightarrow w \vDash T \equiv U$.
- $\star U \Rightarrow_{!w} U' \rightarrow T \Rightarrow_{!w} T' \rightarrow w \vDash T \equiv U \rightarrow w \vDash T' \equiv U'$, where the restriction to $\Rightarrow_{!w}$ is due to the counterexample provided above
- $\star a \Rightarrow_{!w} a' \rightarrow b \Rightarrow_{!w} b' \rightarrow w \vDash a \equiv b \in A \rightarrow w \vDash a' \equiv b' \in A$, where the restriction to $\Rightarrow_{!w}$ is due to the counterexample provided above. In particular, as indicated in Thm. 2.12, this semantics is closed under β -reduction, as β -reduction does not modify the current world, i.e., $(\lambda x.t_1) t_2 \Rightarrow_{!w} t_1[x \setminus t_2]$, for all world w .
- $\star a \Rightarrow_{!w} a' \rightarrow b \Rightarrow_{!w} b' \rightarrow w \vDash a' \equiv b' \in A \rightarrow w \vDash a \equiv b \in A$.

2.5. Different Levels of Effects. As mentioned above, $\text{TT}_{\mathcal{C}}^{\square}$ provides types that allow capturing different levels and kinds of effects, which we summarize here using natural numbers as an example:

- *Read & write*: The type Nat according to Fig. 2 is the type of potentially “fully” effectful numbers that can both read and write, as they can modify the world they compute against (e.g., $(\delta := \underline{1});\delta$), and in addition can compute to different values in successive worlds, and therefore return different values depending on the values they read (e.g., $!\delta$).
- *Write-only*: The type $|\text{Nat}|_{\text{r}}$ according to Fig. 2 is the type of potentially effectful numbers that can write but effectively cannot read (or in a limited way) because they are constrained to compute to the same number in all extensions of the current world, which therefore limits their use of reading, as for example, reading a reference cells is likely to influence the outcome of the computation (e.g., $(\delta := \underline{1});\underline{0}$, which writes but does not read). Note that here and below, the read and write constraints are only indicative as for example $!\delta;\underline{0}$ reads but still belongs to $|\text{Nat}|_{\text{r}}$, and is considered write-only on the basis that it is observationally equivalent to $\underline{0}$.
- *Read-only*: $|\text{Nat}|_{\text{w}}$ is the type of potentially effectful numbers that can read but effectively cannot write as Nat is the type of numbers that can both read and write (e.g., $!\delta$, which reads but does not write).
- *No read or write*: The type $|\text{Nat}|_{\text{rw}}$ is the type of potentially effectful numbers that can only use effects in a limited way and effectively cannot read or write because the $|_|_{\text{w}}$ operator requires computations to end in the same worlds they started with, effectively preventing writes, while $|_|_{\text{r}}$ requires computations to compute to the same value in successive worlds, effectively preventing reads (e.g., $\underline{0}$, but also $!\delta;\underline{0}$, which reads in a limited way that does not affect the computation, or $(\delta := !\delta);\underline{0}$, which writes in a limited way that does not affect the computation).
- *Pure*: $|\text{Nat}|_{\text{c}}$ is the type of pure natural numbers, that do not contain choices, which is a syntactic restriction (e.g., $\underline{0}$ but not $!\delta;\underline{0}$ as it contains a choice name δ even though this name does not affect the computation).

We therefore obtain the following inclusions, where we write $T \subseteq U$ for T is a subtype of U , i.e., all equal members of T are equal members of U : $|\text{Nat}|_{\text{c}} \subseteq |\text{Nat}|_{\text{w}} \subseteq \text{Nat}$ and $|\text{Nat}|_{\text{c}} \subseteq |\text{Nat}|_{\text{r}} \subseteq \text{Nat}$.

2.6. Inference Rules. In TT $_{\mathcal{C}}^{\square}$, sequents are of the form $H \vdash t : T$, where H is a list of hypotheses h_1, \dots, h_n , and a hypothesis h is of the form $x:A$, where the variable x stands for the name of the hypothesis and A its type. Such a sequent denotes that, assuming h_1, \dots, h_n , the term t is a member of the type T , and that therefore T is a type. The term t in this context is called the *realizer* of T . Realizers are sometimes omitted when irrelevant to the discussion, in particular when T is an equality type, which can be realized by any term according to their semantics in Fig. 2. A rule is a pair of a conclusion sequent S and a list of premise sequents, S_1, \dots, S_n (written as usual using a fraction notation, with the premises on top). Appx. A provides a sample of TT $_{\mathcal{C}}^{\square}$'s key inference rules, while Sec. 3 provides examples of rules that only hold for some instances of TT $_{\mathcal{C}}^{\square}$, and Sec. 4.2 presents the continuity rule.

3. PRINCIPLES (IN)COMPATIBLE WITH TT $_{\mathcal{C}}^{\square}$

TT $_{\mathcal{C}}^{\square}$ enables the study of various effectful type theories and their associated theories and in particular, their (in)compatibility with classical reasoning. In fact, in [CR22] we have identified instances of TT $_{\mathcal{C}}^{\square}$ that are agnostic with respect to classical reasoning, in that they are consistent with the Law of Excluded Middle (LEM) (\clubsuit), and other instances that are incompatible with classical reasoning, in that they allow validating the *negation* of LEM (\spadesuit), the Limited Principle of Omniscience (LPO) [Bis67, p.9] (\clubsuit), and Markov's Principle (MP) (\spadesuit).

In particular, we have shown that instantiating \mathcal{C} with either references of choice sequences and \square with a Beth modality, derived from the Beth covering presented in Ex. 2.11, leads to theories that invalidate LEM, i.e., such that the following holds:

$$\forall(w : \mathcal{W}). \neg w \vDash \mathbf{IP} : \mathbb{U}_i. \downarrow (P + \neg P)$$

The following inference rule is therefore consistent with those instances of TT $_{\mathcal{C}}^{\square}$:

$$\overline{H \vdash \neg \mathbf{IP} : \mathbb{U}_i. \downarrow (P + \neg P)}$$

However, instantiating \mathcal{C} with either references of choice sequences and \square with an Open modality, derived from the Open covering presented in Ex. 2.11, leads to theories that are compatible with LEM, i.e., such that the following holds (using classical reasoning in the metatheory to validate this axiom):

$$\forall(w : \mathcal{W}). w \vDash \mathbf{IP} : \mathbb{U}_i. \downarrow (P + \neg P)$$

The following inference rule is therefore consistent with those instances of TT $_{\mathcal{C}}^{\square}$:

$$\overline{H \vdash \mathbf{IP} : \mathbb{U}_i. \downarrow (P + \neg P)}$$

While applying the $|_w$ and $|_r$ modalities to $+$ does not change the above results, this is not the case about MP and LPO. Here we demonstrate how the statements and proofs presented in [CR22] can be translated to the current modified theory. In particular, the statments are translated so that they use $|\mathbf{Nat}|_{rw}$ and $|\mathbf{Bool}|_{rw}$, where the $|_r$ modality was built in the semantics of the theory used in [CR22]. The use of the $|_w$ modality is further discussed below since there was no object computation to extend a world in [CR22]. We demonstrate this with the translation of MP:

$$\forall(w : \mathcal{W}). \neg w \vDash \mathbf{IP} f : |\mathbf{Nat}|_{rw} \rightarrow |\mathbf{Bool}|_{rw}. \neg (\Sigma n : |\mathbf{Nat}|_{rw}. \uparrow (f n)) \rightarrow \downarrow \Sigma n : |\mathbf{Nat}|_{rw}. \uparrow (f n)$$

The proof of the validity of the above statement, i.e., the negation of MP, relies crucially on the fact that $\text{TT}_{\mathcal{C}}^{\square}$'s computation system includes stateful computations that can evolve non-deterministically over time. In particular, as discussed above, $\text{TT}_{\mathcal{C}}^{\square}$ not only supports choices in the form of reference cells, but also choice sequences. As opposed to reference cells which hold a mutable choice, a choice sequence is an ever growing sequence of immutable choices. To access the n th choice, the `read` function needs to be of type $\mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathbb{N} \rightarrow \mathcal{C}$; the `!` operator needs to be updated accordingly: $!\delta(n)$; as well as its operational semantics $!\delta(n) \mapsto_w^w \text{read}(w, \delta, n)$.

This validity proof was possible in [CR22] because there was then no way for computations to extend the world they compute in. In that proof, f is instantiated with $\lambda n.!\delta(n)$ for some choice sequence name δ , which requires choices to be Booleans. We can prove that δ inhabits $|\text{Nat}|_{\text{rw}} \rightarrow |\text{Bool}|_{\text{rw}}$, and in particular we can use the no-read modality $|_|_{\text{r}}$, because choices made by choice sequences are immutable, and so for a given $n \in |\text{Nat}|_{\text{rw}}$, the term $!\delta(n)$ will always compute to the same Boolean. We then prove that even though $\neg \neg \Sigma n : |\text{Nat}|_{\text{rw}} . \uparrow (f \ n)$ holds because in any world it is always possible to find an extension where δ makes the `tt` choice, $\downarrow \Sigma n : |\text{Nat}|_{\text{rw}} . \uparrow (f \ n)$ does not hold because there is no way to prove that δ will ever make the choice `tt`. It then follows that the following inference rule is consistent with $\text{TT}_{\mathcal{C}}^{\square}$ where \mathcal{C} is instantiated with choices sequences, and \square with the Beth modality.

$$\overline{H \vdash \neg \Pi f : |\text{Nat}|_{\text{rw}} \rightarrow |\text{Bool}|_{\text{rw}} . \neg (\Sigma n : |\text{Nat}|_{\text{rw}} . \uparrow (f \ n))} \rightarrow \downarrow \Sigma n : |\text{Nat}|_{\text{rw}} . \uparrow (f \ n)$$

Let us go back to the use of the $|_|_{\text{w}}$ modality. Because computations can now update the world they compute in, it could be that a $n \in \text{Nat}$ computes to a \underline{k} while updating the world by making the choice `tt`. Therefore, while we can prove the validity of the negation of MP as formulated above, we cannot validate the following version using the same method:

$$\forall (w : \mathcal{W}) . \neg w \models \Pi f : |\text{Nat}|_{\text{r}} \rightarrow |\text{Bool}|_{\text{r}} . \neg (\Sigma n : |\text{Nat}|_{\text{r}} . \uparrow (f \ n)) \rightarrow \downarrow \Sigma n : |\text{Nat}|_{\text{r}} . \uparrow (f \ n)$$

Instantiating f with $\lambda n.!\delta(n)$ again, we can still prove that $\neg \neg \Sigma n : |\text{Nat}|_{\text{r}} . \uparrow (f \ n)$ holds because once again we can exhibit a world where the choice `tt` is made. However, we cannot prove anymore that $\downarrow \Sigma n : |\text{Nat}|_{\text{r}} . \uparrow (f \ n)$, i.e., $\downarrow \Sigma n : |\text{Nat}|_{\text{r}} . \uparrow (!\delta(n))$, does not hold because we can instantiate n with $(\delta := \text{tt}); \underline{x}$, where the choice made with $\delta := \text{tt}$ happens to be the x th choice. Using the $|_|_{\text{w}}$ modality prevents the use of $\delta := \text{tt}$.

4. PROOF OF CONTINUITY

Having covered $\text{TT}_{\mathcal{C}}^{\square}$, its support for effects, and its semantics in Sec. 2, we now state the version of Brouwer's continuity principle that we validate in this paper, along with its effectful realizer. For this we first introduce the following notation: $\Pi_{\mathcal{P}} a : A . B \equiv \Pi a : |A|_{\mathcal{C}} . B$, which quantifies over pure elements of type A . In addition, in the rest of this paper we write \mathbb{N} for $|\text{Nat}|_{\text{r}}$ for readability, and \mathfrak{B} and \mathfrak{B}_n stand for $\mathbb{N} \rightarrow \mathbb{N}$ and $\{x : \mathbb{N} \mid x < n\} \rightarrow \mathbb{N}$, respectively.

Theorem 4.1 (✿ Continuity Principle). *The following continuity principle, referred to as $\text{CONT}_{\mathcal{P}}$, is valid w.r.t. the semantics presented in Sec. 2.4 (further assuming the properties Asm_1 , Asm_2 , and Asm_3 , presented in Sec. 4.2):*

$$\Pi_{\mathcal{P}} F : \mathfrak{B} \rightarrow \mathbb{N} . \Pi_{\mathcal{P}} \alpha : \mathfrak{B} . \|\Sigma n : \mathbb{N} . \Pi_{\mathcal{P}} \beta : \mathfrak{B}_n . (\alpha = \beta \in \mathfrak{B}_n) \rightarrow (F(\alpha) = F(\beta) \in \mathbb{N})\| \quad (4.1)$$

and is inhabited by

$$\lambda F . \lambda \alpha . \langle \text{mod}(F, \alpha), \lambda \beta . \lambda e . \star \rangle \quad (4.2)$$

where $\mathbf{mod}(F, \alpha)$ is the modulus of continuity of the function $F \in \mathfrak{B} \rightarrow \mathbb{N}$ at $\alpha \in \mathfrak{B}$ and is computed by the following expression:

$$\begin{aligned} \mathbf{mod}(F, \alpha) &\equiv \nu x.((x := \underline{0}); F(\mathbf{upd}(x, \alpha)); \mathbf{succ}(!x)) \\ \mathbf{upd}(\delta, \alpha) &\equiv \lambda x.(\mathbf{let } y = x \mathbf{ in } ((\mathbf{if } !\delta < y \mathbf{ then } \delta := y \mathbf{ else } \star); \alpha(y))) \end{aligned}$$

More precisely, the following is true for any world w :

$$w \models \lambda F. \lambda \alpha. \langle \mathbf{mod}(F, \alpha), \lambda \beta. \lambda e. \star \rangle \in \mathbf{CONT}_p$$

The rest of this section describes the proof of this theorem. First, we intuitively explain how $\mathbf{mod}(F, \alpha)$ computes the modulus of continuity of a function F at a point α . This is done using the following steps:

- (1) selecting, using ν , a fresh choice name δ (the variable x gets replaced with the freshly generated name δ when computing \mathbf{mod}), with the appropriate restriction (here a restriction that requires choices to be numbers as mentioned in Sec. 2.3);
- (2) setting δ to 0 using $x := \underline{0}$ (where x is δ when this expression computes);
- (3) applying F to a modified version of α , namely $\mathbf{upd}(\delta, \alpha)$, which computes as α , except that in addition it increases δ 's value every time α is applied to a number larger than the last chosen one;
- (4) returning the last chosen number using $!x$ (again x is δ when this expression computes), increased by one in order to return a number higher than any number F applies α to.

Let us illustrate how \mathbf{mod} computes through the following example:

Example 4.2. *Given the following expressions:*

$$\begin{aligned} F &\equiv \lambda \alpha. \alpha(\alpha(\underline{2})) \\ \alpha &\equiv \lambda n. \mathbf{succ}(n) \end{aligned}$$

the expression $\mathbf{mod}(F, \alpha)$ computes as follows. First, it generates a fresh name δ , which is used to record the values to which α is applied to. Then, δ is initialized with $\underline{0}$. It then computes $F(\mathbf{upd}(\delta, \alpha))$, i.e., $t_1 \equiv \mathbf{upd}(\delta, \alpha)(\mathbf{upd}(\delta, \alpha)(\underline{2}))$. Due to \mathbf{upd} 's definition, the argument $t_2 \equiv \mathbf{upd}(\delta, \alpha)(\underline{2})$ is evaluated, which in turn results in the evaluation of $\underline{2}$. Since $\underline{2}$ is a value, δ 's value, which is $\underline{0}$, is compared to $\underline{2}$, and since $2 > 0$, δ is updated with $\underline{2}$. The computation of t_2 proceeds by applying α to $\underline{2}$, which results in $\underline{3}$. Going back to the computation of t_1 , once the argument t_2 has been evaluated to $\underline{3}$, δ 's value, which is now $\underline{2}$ is updated with $\underline{3}$ since $3 > 2$. The computation of t_1 then proceeds by applying α to $\underline{3}$, which is the result returned by $F(\mathbf{upd}(\delta, \alpha))$. Finally, \mathbf{mod} reads δ 's value, which is now $\underline{3}$ and returns $\underline{4}$, which is higher than the two values α is applied to, namely $\underline{2}$ and $\underline{3}$.

We divide the proof of the validity of the continuity principle, i.e., that it is inhabited by the expression presented in Eq. (4.2), into the following three components, where $F \in \mathfrak{B} \rightarrow \mathbb{N}$ and $\alpha \in \mathfrak{B}$:

- Proving that the modulus is a number, i.e., $\mathbf{mod}(F, \alpha) \in \mathbb{N}$;
- Proving that $\mathbf{mod}(F, \alpha)$ returns the highest number that α is applied to in the computation it performs;
- Given $\beta \in \mathfrak{B}$, proving that $F(\alpha)$ and $F(\beta)$ return the same number if α and β agree up to $\mathbf{mod}(F, \alpha)$.

4.1. Purity. The proof presented below relies, among other things, on the fact that $\alpha = \mathbf{upd}(\delta, \alpha) \in \mathfrak{B}$ given a fresh choice name δ , where $\mathbf{upd}(\delta, \alpha)$ is used to “probe” F in $\mathbf{mod}(F, \alpha)$. This equality is however not true for $\alpha \in \mathbf{Nat} \rightarrow \mathbf{Nat}$. Consider for example the function $\alpha \equiv \lambda n. \mathbf{if} \ !\gamma < \underline{1} \ \mathbf{then} \ \underline{0} \ \mathbf{else} \ \underline{1}$, where γ is a choice name. To prove $\alpha = \mathbf{upd}(\delta, \alpha) \in \mathbf{Nat} \rightarrow \mathbf{Nat}$ we have to prove that $\alpha(n) = \mathbf{upd}(\delta, \alpha)(n) \in \mathbf{Nat}$ for all $n \in \mathbf{Nat}$. Given $n \equiv (\gamma := \underline{1}); \underline{1}$, the term $\alpha(n)$ computes to $\underline{0}$ in a world where γ is $\underline{0}$ because n is not evaluated, while $\mathbf{upd}(\delta, \alpha)(n)$ first evaluates n , setting γ to $\underline{1}$, and therefore computes to $\underline{1}$ in any world. To avoid this, instead of using \mathbf{Nat} , we use \mathbf{N} , thereby limiting the effects expressions can have.

According to \mathbf{N} 's semantics, which is as follows:

$$w \models t \equiv t' \in \mathbf{N} \iff \Box_w (w'. \exists (n : \mathbf{N}). t \Rightarrow_{w'} \underline{n} \wedge t' \Rightarrow_{w'} \underline{n})$$

to prove that $\mathbf{mod}(F, \alpha) \in \mathbf{N}$ w.r.t. a world w , we have to prove that it computes to the same number in all extensions of w . However, this will not be the case if F or α have side effects. For example, if F is $\lambda f. f(!\delta_0); \underline{0}$, for some choice name δ_0 , then it could happen that f gets applied to $\underline{0}$ in some world w_1 if $!\delta_0$ returns $\underline{0}$, and to $\underline{1}$ in some world $w_2 \sqsupseteq w_1$ if $!\delta_0$ returns $\underline{1}$. As $\mathbf{mod}(F, \alpha)$ returns the highest number that F applies its argument to, then $\mathbf{mod}(F, \alpha)$ would in this instance return different numbers in different extensions, and would therefore not inhabit \mathbf{N} .

Therefore, to validate a version of continuity which requires the modulus of continuity to be “time-invariant” as in Eq. (4.1), one can require that both F and α are pure (i.e., name-free) terms. Thanks to $\mathbf{\Pi}_p$, we get to assume that both F and α are in \mathbf{Pure} and therefore are name-free. Note that it would not be enough to use the following pattern: $\mathbf{\Pi} F : \mathfrak{B} \rightarrow \mathbf{N}. (F = F \in \mathbf{Pure}) \rightarrow \dots$, because then for the continuity principle to even be a type, we would have to prove that F is name-free to prove that $F = F \in \mathbf{Pure}$ is a type, only knowing that $F \in \mathfrak{B} \rightarrow \mathbf{N}$, due to the semantics of equality types, which is not true in general, and which is why we instead use $_ \cap \mathbf{Pure}$.

Let us now discuss a potential solution to avoid such a purity requirement, as well as some difficulties it involves, which we leave investigating to future work. One could try to validate instead the following version of the continuity axiom, which mixes the uses of \mathbf{N} and \mathbf{Nat} , where $\mathfrak{B}_n^r = \{x : \mathbf{N} \mid x <^r n\} \rightarrow \mathbf{N}$, assuming the existence of some type $x <^r n$ that can relate an $x \in \mathbf{N}$ with an $n \in \mathbf{Nat}$:

$$\mathbf{\Pi} F : \mathfrak{B} \rightarrow \mathbf{N}. \mathbf{\Pi} \alpha : \mathfrak{B}. \|\mathbf{\Sigma} n : \mathbf{Nat}. \mathbf{\Pi} \beta : \mathfrak{B}. (\alpha = \beta \in \mathfrak{B}_n^r) \rightarrow (F(\alpha) = F(\beta) \in \mathbf{N})\|$$

A first difficulty with this is the type $x <^r n$, which to prove that it holds in some world w would require proving that x is equal to all possible values that n can take in extensions of w . Another related difficulty is that it is at present unclear whether this principle can be validated constructively. More precisely, proving its validity would require:

- (1) Proving that $\mathbf{mod}(F, \alpha) \in \mathbf{Nat}$, which is now straightforward.
- (2) Next, we have to prove that $\mathbf{\Pi} \beta : \mathfrak{B}. (\alpha = \beta \in \mathfrak{B}_{\mathbf{mod}(F, \alpha)}^r) \rightarrow (F(\alpha) = F(\beta) \in \mathbf{N})$, i.e., given $\beta \in \mathfrak{B}$ such that $\alpha = \beta \in \mathfrak{B}_{\mathbf{mod}(F, \alpha)}^r$, we have to prove $F(\alpha) = F(\beta) \in \mathbf{N}$. The assumption $\alpha = \beta \in \mathfrak{B}_{\mathbf{mod}(F, \alpha)}^r$ tells us that given $k \in \mathbf{N}$ such that $k <^r \mathbf{mod}(F, \alpha)$, then $\alpha(k) = \beta(k) \in \mathbf{N}$. As mentioned above, for $k <^r \mathbf{mod}(F, \alpha)$ to be true, it must be that k is less than $\mathbf{mod}(F, \alpha)$ in all extensions of the current world. However, without the purity constraint, $\mathbf{mod}(F, \alpha)$ can compute to different numbers in different extensions.

Going back to our goal $F(\alpha) = F(\beta) \in \mathbf{N}$, given the semantics of \mathbf{N} presented above, to prove this it is enough to assume that $F(\mathbf{upd}(\delta, \alpha))$ computes to some number \underline{m} in some

world w , and to prove that $F(\beta)$ also computes to \underline{m} in w . We can then inspect the computation $F(\mathbf{upd}(\delta, \alpha)) \mapsto_{w_1}^w \underline{k}$, where δ is the name generated by $\mathbf{mod}(F, \alpha)$, and show that it can be converted into a computation $F(\beta) \mapsto_{w_2}^w \underline{k}$, by replacing $\alpha(\underline{i})$ with $\beta(\underline{i})$, whenever we encounter such an expression. To do this, we need to know that $\alpha(\underline{i})$ and $\beta(\underline{i})$ compute to the same number using $\alpha = \beta \in \mathfrak{B}_{\mathbf{mod}(F, \alpha)}^r$. However, we only know that \underline{i} is less than $\mathbf{mod}(F, \alpha)$ in w , which is not enough to use this assumption, as \underline{i} might be greater than $\mathbf{mod}(F, \alpha)$ in some other world w' . We can address this issue using classical logic to prove that there exists a $w' \sqsupseteq w$ such that for all $w'' \sqsupseteq w$, the smallest number that α is applied to in the computation of $\mathbf{mod}(F, \alpha)$ w.r.t. w' is less than the number that $\mathbf{mod}(F, \alpha)$ computes to w.r.t. w'' . In the argument sketched above we can then use w' instead of w .

4.2. Assumptions. Before we prove that the continuity principle is inhabited, we will summarize here the assumptions we will be making to prove this result, where r is a restriction that requires choices to be numbers:

- (Asm $_1$ \star) $\forall(w : \mathcal{W})(P : \mathcal{P}_w).\square_w P \rightarrow \exists_w^{\square}(P)$
- (Asm $_2$ \star) $\forall(\delta : \mathcal{N})(w : \mathcal{W})(n : \mathbb{N}).\mathbf{comp}(\delta, w, r) \rightarrow \forall_{\mathbf{write}(w, \delta, \underline{n})}^{\square}(w'.\exists(k : \mathbb{N}).\mathbf{read}(w', \delta) = \underline{k})$
- (Asm $_3$ \star) $\forall(\delta : \mathcal{N})(w : \mathcal{W})(k : \mathbb{N}).\mathbf{comp}(\delta, w, r) \rightarrow \mathbf{read}(\mathbf{write}(w, \delta, \underline{k}), \delta) = \underline{k}$

Asm $_1$ requires the modality \square to be non-empty in the sense that for $\square_w P$ to be true, it has to be true for at least one extension of w . This is true about all non-empty covering relations (i.e., that satisfy the property that if $w \triangleleft o$ then $\exists_w^{\square}(w'.o(w'))$ \star), and therefore about the Kripke, Beth, and Open modalities which are derived from such coverings [CR22, Sec.6.2]. Asm $_2$ requires that the “last” choice of a r -compatible choice name δ is indeed a number. Asm $_3$ guarantees that retrieving a choice that was just made will return that choice.

The last two assumptions are true about Ref, the formalization of references to numbers presented in Ex. 2.8 (TT $_{\mathcal{C}}^{\square}$ instantiated with a Kripke modality and references satisfies these properties \star). In addition both are true about another kind of stateful computations, namely a variant of the formalization of choice sequences presented in [CR22, Ex.5], where new choices are pre-pended as opposed to being appended in [CR22] (TT $_{\mathcal{C}}^{\square}$ instantiated with a Kripke modality and choice sequences satisfies these properties \star).

It then follows from Thm. 4.1 that the following inference rule is consistent with the instances of TT $_{\mathcal{C}}^{\square}$ mentioned above:

$$\overline{H \vdash \lambda F.\lambda \alpha. \langle \mathbf{mod}(F, \alpha), \lambda \beta. \lambda e. \star \rangle : \text{CONT}_{\mathfrak{p}}}$$

4.3. The Modulus is a Number. In this section we prove that $\mathbf{mod}(F, \alpha) \in \mathbb{N}$. More precisely, we prove the following:

Theorem 4.3 (\star The Modulus is a Number). *If $\mathbf{namefree}(F)$, $\mathbf{namefree}(\alpha)$, $w \vDash F \in \mathbb{N}^{\mathfrak{B}}$, and $w \vDash \alpha \in \mathfrak{B}$, for some world w , then*

$$\square_w(w'.\exists(n : \mathbb{N}).\mathbf{mod}(F, \alpha) \Rightarrow_{w'} \underline{n}) \quad (4.3)$$

To prove the above, we will make use of the fact that $w \vDash \mathbf{upd}(\delta, \alpha) \in \mathfrak{B}$ and therefore also $w \vDash F(\mathbf{upd}(\delta, \alpha)) \in \mathbb{N}$, i.e., by the semantics of \mathbb{N} presented in Sec. 4.1, we have for some fresh name δ :

$$\square_w(w'.\exists(n : \mathbb{N}).F(\mathbf{upd}(\delta, \alpha)) \Rightarrow_{w'} \underline{n}). \quad (4.4)$$

But for this we first need to start computing $\text{mod}(F, \alpha)$ to generate a fresh name δ according to the current world. If that current world is some world $w' \sqsubseteq w$ (obtained, for example, using \square_4 from Def. 2.10 on Eq. (4.3)), then we need to be able to get that $F(\text{upd}(\delta, \alpha))$ computes to a number w.r.t. w' , which Eq. (4.4) might not provide. This is the reason for assumption Asm_1 .

Going back to the proof of Eq. (4.3), we use \square_4 , and have to prove $\exists(n : \mathbb{N}). \text{mod}(F, \alpha) \Rightarrow_{w_1} n$ for some $w_1 \sqsubseteq w$. We then:

- (A) first have to find a number n such that $\text{mod}(F, \alpha)$ computes to n w.r.t. w_1 ,
- (B) and then show that it does so also for all $w'_1 \sqsubseteq w_1$.

Let us prove (A) first. We now start computing $\text{mod}(F, \alpha)$ w.r.t. w_1 . We generate a fresh name $\delta \equiv \nu\mathcal{C}(w_1)$, and have to prove that $(\delta := \underline{0}); F(\text{upd}(\delta, \alpha)); \text{succ}(!\delta)$ computes to a number w.r.t. $w_2 \equiv \text{start}\nu\mathcal{C}(w_1, r)$ that satisfies $\text{comp}(\delta, w_2, r)$ (by the properties of $\text{start}\nu\mathcal{C}$ presented in Def. 2.5). We keep computing this expression and have to prove that $F(\text{upd}(\delta, \alpha)); \text{succ}(!\delta)$ computes to a number w.r.t. $w_3 \equiv \text{write}(w_2, \delta, \underline{0})$.

From Asm_1 and Eq. (4.4), we obtain $w_5 \sqsubseteq w$ and $n \in \mathbb{N}$ such that $F(\text{upd}(\delta, \alpha)) \Rightarrow_{w_5} n$, from which we obtain by definition that there exists a w_6 such that $F(\text{upd}(\delta, \alpha)) \Rightarrow_{w_6}^{w_5} n$. Now, because F and α are name-free, we can derive that there exists a w_4 such that $F(\text{upd}(\delta, \alpha)) \Rightarrow_{w_4}^{w_3} n$ (⚙️). It now remains to prove that $n; \text{succ}(!\delta)$, computes to a number w.r.t. w_4 . It is then enough to prove that $!\delta$ computes to a number \underline{k} w.r.t. w_4 , in which case $n; \text{succ}(!\delta)$ computes to $\underline{k+1}$ w.r.t. w_4 . To prove this we make use of Asm_2 which states that r constrains the δ -choices to be numbers. Using this and the facts that $\text{comp}(\delta, w_2, r)$ and $w_2 \sqsubseteq w_4$ (by \sqsubseteq 's transitivity since $w_3 \sqsubseteq w_4$ by Lem. 2.9 and $w_2 \sqsubseteq w_3$ by Def. 2.7), we deduce that there exists a $k \in \mathbb{N}$ such that $\text{read}(w_4, \delta) = \underline{k}$, and therefore $!\delta$ computes to \underline{k} w.r.t. w_4 , and $n; \text{succ}(!\delta)$ computes to $\underline{k+1}$ w.r.t. w_4 , which concludes the proof of (A).

To prove $\exists(n : \mathbb{N}). \text{mod}(F, \alpha) \Rightarrow_{w_1} n$, we then instantiate the formula with $\underline{k+1}$, and have to prove $\text{mod}(F, \alpha) \Rightarrow_{w_1} \underline{k+1}$. We already know that $\text{mod}(F, \alpha) \Rightarrow_{w_4}^{w_1} \underline{k+1}$, i.e., part (A) above, and we now prove part (B) above, i.e., that it does so in all extensions of w_1 too.

To prove (B) we assume a $w'_1 \sqsubseteq w_1$ and have to prove that $\text{mod}(F, \alpha)$ computes to $\underline{k+1}$ w.r.t. w'_1 . As before, we start computing $\text{mod}(F, \alpha)$ w.r.t. w'_1 , and generate a fresh name $\delta' \equiv \nu\mathcal{C}(w'_1)$, and have to prove that $F(\text{upd}(\delta', \alpha)); \text{succ}(!\delta')$ computes to $\underline{k+1}$ w.r.t. $w'_3 \equiv \text{write}(w'_2, \delta', \underline{0})$, where $w'_2 \equiv \text{start}\nu\mathcal{C}(w'_1, r)$. As F and α are name-free, $t_1 \equiv F(\text{upd}(\delta, \alpha))$ and $t_2 \equiv F(\text{upd}(\delta', \alpha))$ behave the same except that when t_1 updates δ with a number, t_2 updates δ' with that number.

Using a syntactic simulation method, we will prove that because t_1 and t_2 are “similar” (which is captured by Def. 4.4 below), $\text{read}(w_3, \delta) = \text{read}(w'_3, \delta')$, and $t_1 \Rightarrow_{w_4}^{w_3} t'_1$, then $t_2 \Rightarrow_{w'_4}^{w'_3} t'_2$ such that t'_1 and t'_2 are also “similar” and $\text{read}(w_4, \delta) = \text{read}(w'_4, \delta')$. Note that $\text{read}(w_3, \delta)$ and $\text{read}(w'_3, \delta')$ return the same choice because $\text{read}(w_3, \delta) = \text{read}(\text{write}(w_2, \delta, \underline{0}), \delta) = 0$ and $\text{read}(w'_3, \delta') = \text{read}(\text{write}(w'_2, \delta', \underline{0}), \delta') = 0$. To derive these equalities, we need assumption Asm_3 that relates read and write .

Let us now define the simulation mentioned above:

Definition 4.4 (⚙️). The similarity relation $t_1 \sim_{\delta_1, \delta_2, \alpha} t_2$ is true iff

$$\begin{aligned} & (t_1 = \mathbf{upd}(\delta_1, \alpha) \wedge t_2 = \mathbf{upd}(\delta_2, \alpha)) \\ & \vee (t_1 = x \wedge t_2 = x) \vee (t_1 = \star \wedge t_2 = \star) \vee (t_1 = \underline{n} \wedge t_2 = \underline{n}) \\ & \vee (t_1 = \lambda x.a \wedge t_2 = \lambda x.b \wedge a \sim_{\delta_1, \delta_2, \alpha} b) \\ & \vee (t_1 = (a_1 \ b_1) \wedge t_2 = (a_2 \ b_2) \wedge a_1 \sim_{\delta_1, \delta_2, \alpha} a_2 \wedge b_1 \sim_{\delta_1, \delta_2, \alpha} b_2) \\ & \vee \dots \end{aligned}$$

Most cases are omitted in this definition as they are similar to the ones presented above. Note however that crucially terms of the form δ or $\nu x.t$ are not related, and that those are the only expressions not related, thereby ruling out names except when occurring inside \mathbf{upd} through the first clause.

As discussed above, a key property of this similarity relation is as follows, where all free variables are universally quantified, and which we prove by induction on the computation $t_1 \Rightarrow_{w'_1}^{w_1} t'_1$:

Lemma 4.5 (⚙️ \sim is preserved by computations). *If $t_1 \sim_{\delta_1, \delta_2, \alpha} t_2$, $\mathbf{namefree}(\alpha)$, $\mathbf{read}(w_1, \delta_1) = \mathbf{read}(w_2, \delta_2)$, $t_1 \Rightarrow_{w'_1}^{w_1} t'_1$, $\mathbf{comp}(\delta_1, w_1, r)$, and $\mathbf{comp}(\delta_2, w_2, r)$, then there exist w'_2 and t'_2 such that $t_2 \Rightarrow_{w'_2}^{w_2} t'_2$, $t'_1 \sim_{\delta_1, \delta_2, \alpha} t'_2$, and $\mathbf{read}(w'_1, \delta_1) = \mathbf{read}(w'_2, \delta_2)$.*

We therefore obtain that there exist t'_2 and w'_4 such that $F(\mathbf{upd}(\delta', \alpha)) \Rightarrow_{w'_4}^{w'_3} t'_2$, $\underline{n} \sim_{\delta, \delta', \alpha} t'_2$ and $\mathbf{read}(w'_4, \delta') = \mathbf{read}(w_4, \delta) = \underline{k}$. Furthermore, by definition of the similarity relation, $t'_2 = \underline{n}$. We obtain that $F(\mathbf{upd}(\delta', \alpha)); \mathbf{succ}(!\delta') \Rightarrow_{w'_4}^{w'_3} \underline{n}; \mathbf{succ}(!\delta')$ and so $F(\mathbf{upd}(\delta', \alpha)); \mathbf{succ}(!\delta') \Rightarrow_{w'_4}^{w'_3} \mathbf{succ}(!\delta')$. Because $\mathbf{read}(w'_4, \delta') = \underline{k}$, we finally obtain $F(\mathbf{upd}(\delta', \alpha)); \mathbf{succ}(!\delta') \Rightarrow_{w'_4}^{w'_3} \underline{k+1}$, which concludes the proof of (B), and therefore that $\mathbf{mod}(F, \alpha) \in \mathbb{N}$.

4.4. The Modulus is the Highest Number. We now prove that $\mathbf{mod}(F, \alpha)$ returns the highest number that α is applied to in the computation it performs:

Theorem 4.6 (⚙️ The Modulus is the Highest Number). *If $\mathbf{mod}(F, \alpha) \Rightarrow_w^w \underline{n}$ such that $\mathbf{mod}(F, \alpha)$ generates a fresh name δ and $\mathbf{read}(w', \delta) = \underline{i}$, then for any world w_0 occurring along this computation, it must be that $\mathbf{read}(w_0, \delta) = \underline{j}$ such that $j \leq i$.*

As shown above, we know that for any world w_1 there exist $w_2 \in \mathcal{W}$ and $k \in \mathbb{N}$ such that $\mathbf{mod}(F, \alpha) \Rightarrow_{w_2}^{w_1} \underline{k+1}$. As in Sec. 4.3, we start computing $\mathbf{mod}(F, \alpha)$ w.r.t. the current world w_1 , and generate a fresh name $\delta \equiv \nu \mathcal{C}(w_1)$, and deduce that

$$F(\mathbf{upd}(\delta, \alpha)); \mathbf{succ}(!\delta) \Rightarrow_{w_2}^{w_1''} \underline{k+1} \quad (4.5)$$

where $w_1' \equiv \mathbf{start} \nu \mathcal{C}(w_1, r)$ and $w_1'' \equiv \mathbf{write}(w_1', \delta, \underline{0})$. Furthermore, by \mathbf{Asm}_2 , there must be a $n \in \mathbb{N}$ such that $\mathbf{read}(w_2, \delta) = \underline{n}$.

We now want to show that if $n < m$, for some $m \in \mathbb{N}$ (which we will instantiate with $k+1$), then it must also be that for any world w along the computation in Eq. (4.5), if $\mathbf{read}(w, \delta) = \underline{i}$ then $i < m$. This is not true about any computation, but it is true about the above because \mathbf{upd} only makes a choice if that choice is higher than the “current” one. To capture this, we define the property $\mathbf{Upd}_{\delta, \alpha}(t)$, which captures that the only place where δ occurs in t is wrapped inside $\mathbf{upd}(\delta, \alpha)$. That is, $\mathbf{Upd}_{\delta, \alpha}(t)$ is true iff $t \sim_{\delta, \delta, \alpha} t$. We can then prove the following result by induction on the computation $t \Rightarrow_{w_2}^{w_1} u$:

Lemma 4.7 (⚙️). *Let α be a closed name-free term, and t be a term such that $\mathbf{Upd}_{\delta,\alpha}(t)$ and $t \Rightarrow_{w_2}^{w_1} u$, and let $\mathbf{read}(w_2, \delta) = \underline{n}$, such that $n < m$, then for any world w along the computation $t \Rightarrow_{w_2}^{w_1} u$ if $\mathbf{read}(w, \delta) = \underline{i}$ then $i < m$.*

Applying this result to $F(\mathbf{upd}(\delta, \alpha)); \mathbf{succ}(!\delta) \Rightarrow_{w_2}^{w_1''} \underline{k+1}$ and instantiating m with $k+1$, we obtain that for any world w along that computation if $\mathbf{read}(w, \delta) = \underline{i}$ then $i < k+1$.

4.5. The Modulus as a Modulus. We now prove the crux of continuity, namely that F returns the same number on functions that agree up to $\mathbf{mod}(F, \alpha)$:

Theorem 4.8 (⚙️ The Modulus as a Modulus). *If $w \vDash \alpha \equiv \beta \in \mathfrak{B}_n$ then $w \vDash F(\alpha) \equiv F(\beta) \in \mathbf{N}$.*

First, we prove that $w \vDash F(\alpha) \equiv F(\mathbf{upd}(\delta, \alpha)) \in \mathbf{N}$, which follows from the semantics of $\mathbf{\Pi}$ and \mathbf{N} presented in Fig. 2, and in particular the crucial fact that $w \vDash \alpha \equiv \mathbf{upd}(\delta, \alpha) \in \mathfrak{B}$. It is therefore enough to prove that $F(\mathbf{upd}(\delta, \alpha))$ and $F(\beta)$ are equal in \mathbf{N} . Relating $F(\mathbf{upd}(\delta, \alpha))$ and $F(\beta)$ instead of $F(\alpha)$ and $F(\beta)$ allows getting access to the values α gets applied to in the computation $F(\alpha)$ as they are recorded using the choice name δ . We can then use these values to prove that $F(\mathbf{upd}(\delta, \alpha))$ and $F(\beta)$ behave similarly up to applications of α in the first computation (i.e., the computation of $F(\mathbf{upd}(\delta, \alpha))$ to a value), which are applications of β in the second (i.e., the computation of $F(\beta)$ to a value), and that these applications reduce to the same numbers because the arguments, recorded using δ , are less than $\mathbf{mod}(F, \alpha)$.

However, even though $\mathbf{upd}(\delta, \alpha)$ and α are equal in \mathfrak{B} , they behave slightly differently computationally as $\mathbf{upd}(\delta, \alpha)$ turns the call-by-name computations $\alpha(t)$ into call-by-value computations by first reducing t into an expression of the form \underline{i} . By typing, we know that $F(\mathbf{upd}(\delta, \alpha))$ and $F(\beta)$ compute to numbers, and to relate the two computations to prove that they compute to the same number, we first apply a similar transformation to $F(\beta)$. Let \mathbf{cbv} be defined as follows:

$$\mathbf{cbv}(f) \equiv \lambda x. \mathbf{let} \ y = x \ \mathbf{in} \ f(y).$$

It is straightforward to derive that $w \vDash F(\beta) \equiv F(\mathbf{cbv}(\beta)) \in \mathbf{N}$ from the semantics of $\mathbf{\Pi}$ and \mathbf{N} presented in Fig. 2, and in particular the crucial fact that $w \vDash \beta \equiv \mathbf{cbv}(\beta) \in \mathfrak{B}$. It is therefore enough to prove that $F(\mathbf{upd}(\delta, \alpha))$ and $F(\mathbf{cbv}(\beta))$ are equal in \mathbf{N} .

Because $F(\mathbf{upd}(\delta, \alpha)) \Rightarrow_{w_2}^{w_1} \underline{n}$ (as explained in part (A) in the proof of Thm. 4.3), by Lem. 4.7 for any world w_0 along this computation if $\mathbf{read}(w_0, \delta) = \underline{i}$ then $i < k+1$, where $k+1$ is the number computed by $\mathbf{mod}(F, \alpha)$.

We now prove that $F(\mathbf{upd}(\delta, \alpha))$ and $F(\mathbf{cbv}(\beta))$ both compute to \underline{n} through another simulation proof that relies on the following relation:

Definition 4.9 (⚙️). The similarity relation $t_1 \approx_{\delta,\alpha,\beta} t_2$ is true iff

$$\begin{aligned} & (t_1 = \mathbf{upd}(\delta, \alpha) \wedge t_2 = \mathbf{cbv}(\beta)) \\ \vee & (t_1 = x \wedge t_2 = x) \vee (t_1 = \star \wedge t_2 = \star) \vee (t_1 = \underline{n} \wedge t_2 = \underline{n}) \\ \vee & (t_1 = \lambda x. a \wedge t_2 = \lambda x. b \wedge a \approx_{\delta,\alpha,\beta} b) \\ \vee & (t_1 = (a_1 \ b_1) \wedge t_2 = (a_2 \ b_2) \wedge a_1 \approx_{\delta,\alpha,\beta} a_2 \wedge b_1 \approx_{\delta,\alpha,\beta} b_2) \\ \vee & \dots \end{aligned}$$

Most cases are omitted in this definition as they similar to the ones presented above. Note however that crucially terms of the form δ or $\nu x. t$ are not related, and that those are the only expressions not related, thereby ruling out names except when occurring inside \mathbf{upd} through the first clause.

A key property of this relation is as follows, which captures that $t_1 \approx_{\delta,\alpha,\beta} t_2$ is preserved by computations, and which we prove by induction on the computation $t_1 \Rightarrow_w^w t'_1$:

Lemma 4.10 (⚙️). *If $t_1 \approx_{\delta,\alpha,\beta} t_2$, α and β agree up to k , $t_1 \Rightarrow_w^w t'_1$ and for any world w_0 along this computation if $\text{read}(w_0, \delta) = \underline{i}$ then $i < k+1$, then $t_2 \Rightarrow_w^w t'_2$ such that $t'_1 \approx_{\delta,\alpha,\beta} t'_2$.*

Therefore, because $F(\text{upd}(\delta, \alpha)) \approx_{\delta,\alpha,\beta} F(\text{cbv}(\beta))$ (as F is name-free) and $F(\text{upd}(\delta, \alpha))$ computes to \underline{n} , using Lem. 4.10 it must be that $F(\text{cbv}(\beta))$ also computes to \underline{n} , which concludes our proof of Thm. 4.1, i.e., of the validity of the continuity principle for pure functions.

5. CONCLUSION AND RELATED WORKS

The TT $_{\mathcal{C}}^{\square}$ family of type theories is a uniform, flexible framework for studying effectful type theories. In this paper we focused on identifying a subset of the TT $_{\mathcal{C}}^{\square}$ family that allows for the internal validation of the continuity principle. That is, we have shown how, for this subset of theories, the modulus of continuity of functions can be computed using an expression of the underlying computation system. This internalization required stateful computations, and we discuss some of the challenges arising from such impure computations. As mentioned in the introduction, and as recalled below, this is not the first proof of continuity, however to the best of our knowledge, this is the first proof of an “internal” validity proof of continuity that relies on stateful computations. Furthermore, the proof presented above relies on an “internal” notion of probing through the use of stateful computations internal to the computation language of the type theory, while approaches such as [CJ10, CJ12, XE13] rely on a meta-theoretic (or “external”) notion of probing.

Troelstra proved in [Tro73, p.158] that every closed term $t \in \mathbb{N}^{\mathbb{B}}$ of N-HA $^{\omega}$ has a provable modulus of continuity in N-HA $^{\omega}$ —see also [Bee85] for similar proofs of the consistency of continuity with various constructive theories.

Coquand and Jaber [CJ10, CJ12] proved the *uniform* continuity of a Martin-Löf-like intensional type theory using forcing. Their method consists in adding a generic element \mathbf{f} as a constant to the language that stands for a Cohen real of type $2^{\mathbb{N}}$, and defining the forcing conditions as approximations of \mathbf{f} . They then define a suitable *computability* predicate that expresses when a term is a computable term of some type up to approximations given by the forcing conditions. The key steps are to (1) first prove that \mathbf{f} is computable and then (2) prove that well-typed terms are computable, from which they derive uniform continuity: the uniform modulus of continuity is given by the approximations.

Similarly, Escardó and Xu [XE13] proved that the definable functionals of Gödel’s system T [GTL89] are uniformly continuous on the Cantor space $\mathcal{C} \equiv \mathbb{N} \rightarrow \mathbb{B}$ (without assuming classical logic or the Fan Theorem). For that, they developed the C-Space category, which internalizes continuity, and has a *Fan functional* which computes the modulus of uniform continuity of functions in $\mathcal{C} \rightarrow \mathbb{N}$. Relating C-Space and the standard set-theoretical model of system T, they show that all System T functions on the Cantor space are uniformly continuous. Furthermore, using this model, they show how to extract computational content from proofs in HA $^{\omega}$ extended with a uniform continuity axiom, which is realized by the Fan functional.

In [Esc13b], Escardó proves that all System T functions are continuous on the Baire space and uniformly continuous on the Cantor space without using forcing. Instead, he provides an alternative interpretation of system T, where a number is interpreted by a *dialogue tree*,

which captures the computation of a function w.r.t. an oracle of type \mathfrak{B} . Escardó first proves that such computations are continuous, and then by defining a suitable relation between the standard interpretation and the alternative one, that relates the interpretations of all system T terms, derives that for all system T functions on the Baire space are continuous. While in [Esc13b], dialogue trees are constructed and leave outside of System T, in [Esc13a], Escardó showed that it is possible to derive System T definable Church-encoded dialogue trees, albeit still deriving these trees outside of System T (through a metatheoretic induction on System T terms).

Chuangjie later developed in [Xu20] a related syntactic approach, that does not rely on dialogue trees, which allows recovering the continuity of System-T functionals, as well as the fact that moduli of continuity are T-definable.

In [RB16, RB17], the authors proved that Brouwer’s continuity principle is consistent with Nuprl [CAB⁺86, ABC⁺06] by realizing the modulus of continuity of functions on the Baire space also using Longley’s method [Lon99], but using exceptions instead of references. The realizer there is more complicated than the one presented in this paper as it involves an effectful computation that repeatedly checks whether a given number is at least as high as the modulus of continuity, and increasing that number until the modulus of continuity is reached. We do not require such a loop, and can directly extract the modulus of continuity of a function.

In [BMP22] the authors prove that all BTT [PT17] functions are continuous by generalizing the method used in [Esc13b]. Their model is built in three steps as follows: an axiom model/translation adds an oracle to the theory at hand; a branching model/translation interprets types as intensional D-algebras, i.e., as types equipped with pythias; and an algebraic parametricity model/translation that relates the two previous translations by relating the calls to the pythia to the oracle. Their models allows deriving that all functions are continuous, but does not allow “internalizing” the continuity principle, which is the goal of this paper.

Another version of the continuity principle, the Inductive Continuity Principle, has also been explored recently [GHP06, GHP09b, GHP09a, Esc13b]. This principle is sometimes referred to as the Strong Continuity Principle since it implies the continuity principle discussed in this paper (and other variants). Roughly speaking, it states that for any function F from the Baire space to numbers, there exists a (dialogue) tree that contains the values of F at its leaves and such that the modulus of F at each point of the Baire space is given by the length of the corresponding branch in the tree. In [CdRPRT23] we have identified a subset of the $\text{TT}_{\mathcal{C}}^{\square}$ family that allows for the internal validation of the Inductive Continuity Principle via computations that construct such dialogue trees internally to the theories using effectful computations. To prove finiteness of the computed trees and termination of the overall program we there had to resort to (meta-)classical reasoning, and it remains to be seen if this can be avoided.

REFERENCES

- [ABC⁺06] Stuart F. Allen, Mark Bickford, Robert L. Constable, Richard Eaton, Christoph Kreitz, Lori Lorigo, and Evan Moran. Innovations in computational type theory using Nuprl. *J. Applied Logic*, 4(4):428–469, 2006. <http://www.nuprl.org/>.
- [AGD] Agda wiki. <http://wiki.portal.chalmers.se/agda/pmwiki.php>.

- [AR14] Abhishek Anand and Vincent Rahli. Towards a formally verified proof assistant. In Gerwin Klein and Ruben Gamboa, editors, *ITP 2014*, volume 8558 of *LNCS*, pages 27–44. Springer, 2014. doi:10.1007/978-3-319-08970-6_3.
- [Bee85] Michael J. Beeson. *Foundations of Constructive Mathematics*. Springer, 1985.
- [Bis67] E. Bishop. *Foundations of constructive analysis*, volume 60. McGraw-Hill New York, 1967.
- [BMP22] Martin Baillon, Assia Mahboubi, and Pierre-Marie Pédrot. Gardening with the pythia A model of continuity in a dependent setting. In Florin Manea and Alex Simpson, editors, *CSL*, volume 216 of *LIPICs*, pages 5:1–5:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICs.CSL.2022.5.
- [BR87] Douglas Bridges and Fred Richman. *Varieties of Constructive Mathematics*. London Mathematical Society Lecture Notes Series. Cambridge University Press, 1987. URL: <http://books.google.com/books?id=oN5nsPkXhhsC>.
- [CAB⁺86] Robert L. Constable, Stuart F. Allen, Mark Bromley, Rance Cleaveland, J. F. Cremer, Robert W. Harper, Douglas J. Howe, Todd B. Knoblock, Nax P. Mendler, Prakash Panangaden, James T. Sasaki, and Scott F. Smith. *Implementing mathematics with the Nuprl proof development system*. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1986.
- [CdRPRT23] Liron Cohen, Bruno da Rocha Paiva, Vincent Rahli, and Ayberk Tosun. Inductive continuity via brouwer trees. To appear in the proceedings of the 48th International Symposium on Mathematical Foundations of Computer Science, 2023.
- [CH96] M. J. Cresswell and G. E. Hughes. *A New Introduction to Modal Logic*. Routledge, 1996.
- [CJ10] Thierry Coquand and Guilhem Jaber. A note on forcing and type theory. *Fundam. Inform.*, 100(1-4):43–52, 2010. doi:10.3233/FI-2010-262.
- [CJ12] Thierry Coquand and Guilhem Jaber. A computational interpretation of forcing in type theory. In *Epistemology versus Ontology*, volume 27 of *Logic, Epistemology, and the Unity of Science*, pages 203–213. Springer, 2012. doi:10.1007/978-94-007-4435-6_10.
- [CR22] Liron Cohen and Vincent Rahli. Constructing unprejudiced extensional type theories with choices via modalities. In Amy P. Felty, editor, *FSCD*, volume 228 of *LIPICs*, pages 10:1–10:23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICs.FSCD.2022.10.
- [CR23] Liron Cohen and Vincent Rahli. Realizing continuity using stateful computations. In Bartek Klin and Elaine Pimentel, editors, *CSL*, volume 252 of *LIPICs*, pages 15:1–15:18, Dagstuhl, Germany, 2023. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPICs.CSL.2023.15.
- [Dum00] Michael A. E. Dummett. *Elements of Intuitionism*. Clarendon Press, second edition, 2000.
- [Esc13a] Martín Hötzel Escardó, 2013. Internalisation of the modulus of continuity of System T functionals. URL: <https://www.cs.bham.ac.uk/~mhe/dialogue/church-dialogue-internal.html>.
- [Esc13b] Martín Hötzel Escardó. Continuity of Gödel’s system T definable functionals via effectful forcing. *Electr. Notes Theor. Comput. Sci.*, 298:119–141, 2013. doi:10.1016/j.entcs.2013.09.010.
- [EX15] Martín H. Escardó and Chuangjie Xu. The inconsistency of a Brouwerian continuity principle with the Curry-Howard interpretation. In *TLCA*, volume 38 of *LIPICs*, pages 153–164. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. URL: <http://www.dagstuhl.de/dagpub/978-3-939897-87-3>, doi:10.4230/LIPICs.TLCA.2015.153.
- [GHP06] Neil Ghani, Peter G. Hancock, and Dirk Pattinson. Continuous functions on final coalgebras. In Neil Ghani and John Power, editors, *CMCS*, volume 164 of *Electronic Notes in Theoretical Computer Science*, pages 141–155. Elsevier, 2006. doi:10.1016/j.entcs.2006.06.009.
- [GHP09a] Neil Ghani, Peter G. Hancock, and Dirk Pattinson. Continuous functions on final coalgebras. In Samson Abramsky, Michael W. Mislove, and Catuscia Palamidessi, editors, *MFPS*, volume 249 of *Electronic Notes in Theoretical Computer Science*, pages 3–18. Elsevier, 2009. doi:10.1016/j.entcs.2009.07.081.
- [GHP09b] Neil Ghani, Peter G. Hancock, and Dirk Pattinson. Representations of stream processors using nested fixed points. *Log. Methods Comput. Sci.*, 5(3), 2009. URL: <http://arxiv.org/abs/0905.4813>.
- [GTL89] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Cambridge University Press, 1989.
- [Kre62] Georg Kreisel. On weak completeness of intuitionistic predicate logic. *J. Symb. Log.*, 27(2):139–158, 1962. doi:10.2307/2964110.

- [Kri63] Saul A. Kripke. Semantical analysis of modal logic i. normal propositional calculi. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 9(5-6):67–96, 1963. doi:10.1002/malq.19630090502.
- [Kri65] Saul A. Kripke. Semantical analysis of intuitionistic logic i. In J.N. Crossley and M.A.E. Dummett, editors, *Formal Systems and Recursive Functions*, volume 40 of *Studies in Logic and the Foundations of Mathematics*, pages 92–130. Elsevier, 1965. doi:10.1016/S0049-237X(08)71685-9.
- [KT70] Georg Kreisel and Anne S. Troelstra. Formal systems for some branches of intuitionistic analysis. *Annals of Mathematical Logic*, 1(3):229–387, 1970. doi:10.1016/0003-4843(70)90001-X.
- [KV65] Stephen C. Kleene and Richard E. Vesley. *The Foundations of Intuitionistic Mathematics, especially in relation to recursive functions*. North-Holland Publishing Company, 1965.
- [Lon99] John Longley. When is a functional program not a functional program? In *ICFP*, pages 1–7. ACM, 1999. doi:10.1145/317636.317775.
- [Mos93] Joan R. Moschovakis. An intuitionistic theory of lawlike, choice and lawless sequences. In *Logic Colloquium'90: ASL Summer Meeting in Helsinki*, pages 191–209. Association for Symbolic Logic, 1993.
- [Pit13] Andrew M Pitts. Nominal sets: Names and symmetry in computer science, volume 57 of *Cambridge tracts in theoretical computer science*, 2013.
- [PT17] Pierre-Marie Pédrot and Nicolas Tabareau. An effectful way to eliminate addiction to dependence. In *LICS*, pages 1–12. IEEE Computer Society, 2017. URL: <http://ieeexplore.ieee.org/xpl/mostRecentIssue.jsp?punumber=7999337>, doi:10.1109/LICS.2017.8005113.
- [RB16] Vincent Rahli and Mark Bickford. A nominal exploration of intuitionism. In Jeremy Avigad and Adam Chlipala, editors, *CPP*, pages 130–141. ACM, 2016. Extended version: <http://www.nuprl.org/html/Nuprl2Coq/continuity-long.pdf>. URL: <http://dl.acm.org/citation.cfm?id=2854065>, doi:10.1145/2854065.2854077.
- [RB17] Vincent Rahli and Mark Bickford. Validating brouwer’s continuity principle for numbers using named exceptions. *Mathematical Structures in Computer Science*, pages 1–49, 2017. doi:10.1017/S0960129517000172.
- [Tro73] A.S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. New York, Springer, 1973.
- [Tro77a] Anne S. Troelstra. *Choice sequences: a chapter of intuitionistic mathematics*. Clarendon Press Oxford, 1977.
- [Tro77b] A.S. Troelstra. A note on non-extensional operations in connection with continuity and recursiveness. *Indagationes Mathematicae*, 39(5):455–462, 1977. doi:10.1016/1385-7258(77)90060-9.
- [Tro85] Anne S. Troelstra. Choice sequences and informal rigour. *Synthese*, 62(2):217–227, 1985.
- [TvD88] Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics An Introduction*, volume 121 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, 1988.
- [vAvD02] Mark van Atten and Dirk van Dalen. Arguments for the continuity principle. *Bulletin of Symbolic Logic*, 8(3):329–347, 2002. URL: <http://www.math.ucla.edu/~asl/bsl/0803/0803-001.ps>.
- [Vel01] Wim Veldman. Understanding and using Brouwer’s continuity principle. In *Reuniting the Antipodes — Constructive and Nonstandard Views of the Continuum*, volume 306 of *Synthese Library*, pages 285–302. Springer Netherlands, 2001. doi:10.1007/978-94-015-9757-9_24.
- [XE13] Chuangjie Xu and Martín Hötzel Escardó. A constructive model of uniform continuity. In *TLCA*, volume 7941 of *LNCS*, pages 236–249. Springer, 2013. doi:10.1007/978-3-642-38946-7_18.
- [Xu15] Chuangjie Xu. *A continuous computational interpretation of type theories*. PhD thesis, University of Birmingham, UK, 2015. URL: <http://etheses.bham.ac.uk/5967/>.
- [Xu20] Chuangjie Xu. A syntactic approach to continuity of t-definable functionals. *Log. Methods Comput. Sci.*, 16(1), 2020. doi:10.23638/LMCS-16(1:22)2020.

APPENDIX A. INFERENCE RULES

The following provides a sample of TT $_{\mathcal{C}}^{\square}$'s key inference rules. In what follows, we write $a \in A$ for $a = a \in A$. While similar rules had been formally verified in [AR14] for a precursor of TT $_{\mathcal{C}}^{\square}$, the formal verification of the validity of these rules w.r.t. TT $_{\mathcal{C}}^{\square}$'s semantics is left for future work.

Products. The following rules are the standard Π -elimination rule, Π -introduction rule, type equality for Π types, and λ -introduction rule, respectively.

$$\frac{H, f: \mathbf{\Pi}x:A.B, J \vdash a \in A \quad H, f: \mathbf{\Pi}x:A.B, J, z: f(a) \in B[x \setminus a] \vdash e : C}{H, f: \mathbf{\Pi}x:A.B, J \vdash e[z \setminus \star] : C} \quad \frac{H, z:A \vdash b : B[x \setminus z] \quad H \vdash A \in \mathbb{U}_i}{H \vdash \lambda z. b : \mathbf{\Pi}x:A.B}$$

$$\frac{H \vdash A_1 = A_2 \in \mathbb{U}_i \quad H, y:A_1 \vdash B_1[x_1 \setminus y] = B_2[x_2 \setminus y] \in \mathbb{U}_i}{H \vdash \mathbf{\Pi}x_1:A_1. B_1 = \mathbf{\Pi}x_2:A_2. B_2 \in \mathbb{U}_i} \quad \frac{H, z:A \vdash t_1[x_1 \setminus z] = t_2[x_2 \setminus z] \in B[x \setminus z] \quad H \vdash A \in \mathbb{U}_i}{H \vdash \lambda x_1. t_1 = \lambda x_2. t_2 \in \mathbf{\Pi}x:A.B}$$

Note that the last rule requires to prove that A is a type because the conclusion requires to prove that $\mathbf{\Pi}x:A.B$ is a type, and the first hypothesis only states that B is a type family over A , but does not ensure that A is a type. Furthermore, the following rules are the standard function extensionality and β -computation rules, respectively:

$$\frac{H, z:A \vdash f_1(z) = f_2(z) \in B[x \setminus z] \quad H \vdash A \in \mathbb{U}_i}{H \vdash f_1 = f_2 \in \mathbf{\Pi}x:A.B} \quad \frac{H \vdash t[x \setminus s] = u \in T}{H \vdash (\lambda x. t) s = u \in T}$$

Sums. The following rules are the standard Σ -elimination rule, Σ -introduction rule, type equality for the Σ type, pair-introduction, and spread-computation rules, respectively:

$$\frac{H, p: |\Sigma x:A.B|_{\text{rw}}, a:A, b:B[x \setminus a], J[p \setminus \langle a, b \rangle] \vdash e : C[p \setminus \langle a, b \rangle]}{H, p: |\Sigma x:A.B|_{\text{rw}}, J \vdash \text{let } a, b = p \text{ in } e : C} \quad \frac{H \vdash a \in A \quad H \vdash b \in B[x \setminus a] \quad H, z:A \vdash B[x \setminus z] \in \mathbb{U}_i}{H \vdash \langle a, b \rangle : \Sigma x:A.B}$$

$$\frac{H \vdash A_1 = A_2 \in \mathbb{U}_i \quad H, y:A_1 \vdash B_1[x_1 \setminus y] = B_2[x_2 \setminus y] \in \mathbb{U}_i}{H \vdash \Sigma x_1:A_1. B_1 = \Sigma x_2:A_2. B_2 \in \mathbb{U}_i} \quad \frac{H, z:A \vdash B[x \setminus z] \in \mathbb{U}_i \quad H \vdash a_1 = a_2 \in A \quad H \vdash b_1 = b_2 \in B[x \setminus a_1]}{H \vdash \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle \in \Sigma x:A.B}$$

$$\frac{H \vdash u[x \setminus s; y \setminus t] = t_2 \in T}{H \vdash \text{let } x, y = \langle s, t \rangle \text{ in } u = t_2 \in T}$$

Disjoint Unions. The following rules are the disjoint union-elimination, disjoint union-introduction (left and right), type equality for disjoint unions, injection-introduction (left and right), and decide-computation (left and right) rules, respectively:

$$\frac{H, d: |A+B|_{\text{rw}}, x:A, J[d \setminus \text{inl}(x)] \vdash t : C[d \setminus \text{inl}(x)] \quad H, d: |A+B|_{\text{rw}}, y:B, J[d \setminus \text{inr}(y)] \vdash u : C[d \setminus \text{inr}(y)]}{H, d: |A+B|_{\text{rw}}, J \vdash \text{case } d \text{ of } \text{inl}(x) \Rightarrow t \mid \text{inr}(y) \Rightarrow u : C}$$

$$\frac{H \vdash a : A \quad H \vdash B \in \mathbb{U}_i}{H \vdash \text{inl}(a) : A+B} \quad \frac{H \vdash b : B \quad H \vdash A \in \mathbb{U}_i}{H \vdash \text{inr}(b) : A+B} \quad \frac{H \vdash A_1 = A_2 \in \mathbb{U}_i \quad H \vdash B_1 = B_2 \in \mathbb{U}_i}{H \vdash A_1 + B_1 = A_2 + B_2 \in \mathbb{U}_i}$$

$$\frac{H \vdash a_1 = a_2 \in A \quad H \vdash B \in \mathbb{U}_i}{H \vdash \text{inl}(a_1) = \text{inl}(a_2) \in A+B} \quad \frac{H \vdash b_1 = b_2 \in B \quad H \vdash A \in \mathbb{U}_i}{H \vdash \text{inr}(b_1) = \text{inr}(b_2) \in A+B}$$

$$\frac{H \vdash t[x \setminus s] = t_2 \in T}{H \vdash (\text{case } \text{inl}(s) \text{ of } \text{inl}(x) \Rightarrow t \mid \text{inr}(y) \Rightarrow u) = t_2 \in T} \quad \frac{H \vdash u[y \setminus s] = t_2 \in T}{H \vdash (\text{case } \text{inr}(s) \text{ of } \text{inl}(x) \Rightarrow t \mid \text{inr}(y) \Rightarrow u) = t_2 \in T}$$

Equality. The following rules are the standard equality-introduction rule, equality-elimination rule, hypothesis rule, symmetry and transitivity rules, respectively.

$$\frac{H \vdash A=B \in \mathbb{U}_i \quad H \vdash a_1=b_1 \in A \quad H \vdash a_2=b_2 \in A}{H \vdash (a_1=a_2 \in A)=(b_1=b_2 \in B) \in \mathbb{U}_i} \quad \frac{H, z:a=b \in A, J[z \setminus \star] \vdash e : C[z \setminus \star]}{H, z:a=b \in A, J \vdash e : C}$$

$$\frac{}{H, x:A, J \vdash x \in A} \quad \frac{H \vdash b=a \in T}{H \vdash a=b \in T} \quad \frac{H \vdash a=c \in T \quad H \vdash c=b \in T}{H \vdash a=b \in T}$$

The following rules allow fixing the realizer of a sequent, and rewriting with an equality in an hypothesis, respectively:

$$\frac{H \vdash t : T}{H \vdash t \in T} \quad \frac{H, x:B, J \vdash t : C \quad H \vdash A=B \in \mathbb{U}_i}{H, x:A, J \vdash t : C}$$

Universes. Let i be a lower universe than j . The following rules are the standard universe-introduction rule and the universe cumulativity rule, respectively.

$$\frac{}{H \vdash \mathbb{U}_i = \mathbb{U}_i \in \mathbb{U}_j} \quad \frac{H \vdash T \in \mathbb{U}_i}{H \vdash T \in \mathbb{U}_j}$$

Sets. The following rule is the standard set-elimination rule:

$$\frac{H, z:\{x : A \mid B\}, a:A, \boxed{b:B[x \setminus a]}, J[z \setminus a] \vdash e : C[z \setminus a]}{H, z:\{x : A \mid B\}, J \vdash e[a \setminus z] : C}$$

Note that we have used a new construct in the above rule: the *hidden* hypothesis $\boxed{b:B[x \setminus a]}$. The main feature of hidden hypotheses is that their names cannot occur in realizers (which is why we “box” those hypotheses). Intuitively, this is because the proof that B is true is discarded in the proof that the set type $\{x : A \mid B\}$ is true and therefore cannot occur in computations. Hidden hypotheses can be unhidden using the following rule:

$$\frac{H, x:T, J \vdash \star : a=b \in A}{H, \boxed{x:T}, J \vdash \star : a=b \in A}$$

which is valid since the realizer is \star and therefore does not make use of x .

The following rules are the standard set-introduction rule, type equality for the set type, and introduction rule for members of set types, respectively.

$$\frac{H \vdash a \in A \quad H \vdash B[x \setminus a] \quad H, z:A \vdash B[x \setminus z] \in \mathbb{U}_i}{H \vdash a : \{x : A \mid B\}} \quad \frac{H \vdash A_1=A_2 \in \mathbb{U}_i \quad H, y:A_1 \vdash B_1[x_1 \setminus y]=B_2[x_2 \setminus y] \in \mathbb{U}_i}{H \vdash \{x_1 : A_1 \mid B_1\}=\{x_2 : A_2 \mid B_2\} \in \mathbb{U}_i}$$

$$\frac{H, z:A \vdash B[x \setminus z] \in \mathbb{U}_i \quad H \vdash a=b \in A \quad H \vdash B[x \setminus a]}{H \vdash a=b \in \{x : A \mid B\}}$$

Intersection. The following rules are the intersection elimination and introduction rules:

$$\frac{H, x:A \cap B, y:A, z:y=x \in A, J[x \setminus y] \vdash t : C[x \setminus y]}{H, x:A \cap B, J \vdash t[y \setminus x][z \setminus \star] : C} \quad \frac{H, x:A \cap B, y:B, z:y=x \in B, J[x \setminus y] \vdash t : C[x \setminus y]}{H, x:A \cap B, J \vdash t[y \setminus x][z \setminus \star] : C}$$

$$\frac{H \vdash t : A \quad H \vdash t : b}{H \vdash t : A \cap B} \quad \frac{H \vdash A_1=A_2 \in \mathbb{U}_i \quad H \vdash B_1=B_2 \in \mathbb{U}_i}{H \vdash A_1 \cap B_1=A_2 \cap B_2 \in \mathbb{U}_i} \quad \frac{H \vdash t_1=t_2 \in A \quad H \vdash t_1=t_2 \in B}{H \vdash t_1=t_2 \in A \cap B}$$

Subsingleton. The following rules are the subsingleton elimination and introduction rules:

$$\frac{H, x:\|A\|, J \vdash T \in \mathbb{U}_i \quad H, x:\|A\|, J, y:A, z:A \vdash t[x \setminus y]=u[x \setminus z] \in T}{H, x:\|A\|, J \vdash t=u \in T}$$

$$\frac{H \vdash t : A}{H \vdash t : \|A\|} \quad \frac{H \vdash A_1=A_2 \in \mathbb{U}_i}{H \vdash \|A_1\|=\|A_2\| \in \mathbb{U}_i} \quad \frac{H \vdash t_1 \in A \quad H \vdash t_2 \in B}{H \vdash t_1=t_2 \in \|A\|}$$

Natural Numbers. The following rules are the rules for **Nat**:

$$\frac{H, x:|\mathbf{Nat}|_{rw}, J \vdash b : C[x \setminus 0] \quad H, x:|\mathbf{Nat}|_{rw}, J, y:|\mathbf{Nat}|_{rw}, r:C[x \setminus y] \vdash s : C[x \setminus \mathbf{succ}(y)]}{H, x:|\mathbf{Nat}|_{rw}, J \vdash \mathbf{natrec}(x, b, \lambda y. \lambda r. s) : C}$$

$$\frac{}{H \vdash \underline{n} : \mathbf{Nat}} \quad \frac{}{H \vdash \mathbf{Nat} = \mathbf{Nat} \in \mathbb{U}_i} \quad \frac{}{H \vdash \underline{n} = \underline{n} \in \mathbf{Nat}} \quad \frac{H \vdash t = u \in \mathbf{Nat}}{H \vdash \mathbf{succ}(t) = \mathbf{succ}(u) \in \mathbf{Nat}}$$

Effect Restrictions. The following rules are the rules for **NoWrite**, **NoRead**, **Pure**:

$$\frac{\mathbf{readfree}(t)}{H \vdash t : \mathbf{NoWrite}} \quad \frac{\mathbf{writefree}(t)}{H \vdash t : \mathbf{NoRead}} \quad \frac{\mathbf{namefree}(t)}{H \vdash t : \mathbf{Pure}}$$

$$\frac{\mathbf{writefree}(t_1) \quad \mathbf{writefree}(t_2)}{H \vdash t_1 = t_2 \in \mathbf{NoWrite}} \quad \frac{\mathbf{readfree}(t_1) \quad \mathbf{readfree}(t_2)}{H \vdash t_1 = t_2 \in \mathbf{NoRead}} \quad \frac{\mathbf{namefree}(t_1) \quad \mathbf{namefree}(t_2)}{H \vdash t_1 = t_2 \in \mathbf{Pure}}$$

$$\frac{}{H \vdash \mathbf{NoWrite} = \mathbf{NoWrite} \in \mathbb{U}_i} \quad \frac{}{H \vdash \mathbf{NoRead} = \mathbf{NoRead} \in \mathbb{U}_i} \quad \frac{}{H \vdash \mathbf{Pure} = \mathbf{Pure} \in \mathbb{U}_i}$$

Note that all these rules require syntactic checks, where **namefree**(*t*) states that no expression of the form δ or $\nu x. t_1$ occurs in *t*; **writefree**(*t*) states that no expression of the form $t_1 := t_2$ or $\nu x. t_1$ occurs in *t*; and **readfree**(*t*) states that no expression of the form $!t_1$ occurs in *t*.