

DECIDABILITY FOR STURMIAN WORDS

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In Memory of Reed Oei (1999-2022)

ABSTRACT. We show that the first-order theory of Sturmian words over Presburger arithmetic is decidable. Using a general adder recognizing addition in Ostrowski numeration systems by Baranwal, Schaeffer and Shallit, we prove that the first-order expansions of Presburger arithmetic by a single Sturmian word are uniformly ω -automatic, and then deduce the decidability of the theory of the class of such structures. Using an implementation of this decision algorithm called Pecan, we automatically reprove classical theorems about Sturmian words in seconds, and are able to obtain new results about antisquares and antipalindromes in characteristic Sturmian words.

1. INTRODUCTION

It has been known for some time that, for certain infinite words $\mathbf{c} = c_0c_1c_2 \cdots$ over a finite alphabet Σ , the first-order logical theory $\text{FO}(\mathbb{N}, <, +, 0, 1, n \mapsto c_n)$ is decidable. In the case where \mathbf{c} is a **k -automatic sequence** for $k \geq 2$, this is due to Büchi [Büc62], although his original proof was flawed. The correct statement appears, for example, in Bruyère et al. [BHMV94b, BHMV94a]. Although the worst-case running time of the decision procedure is truly formidable (and non-elementary), it turns out that an implementation can, in many cases, decide the truth of interesting and nontrivial first-order statements about automatic sequences in a reasonable length of time. Thus, one can easily reprove known results, and obtain new ones, merely by translating the desired result into the appropriate first-order statement φ and running the decision procedure on φ . For an example of the kinds of things that can be proved, see Goč, Henshall, and Shallit [GHS13].

More generally, the same ideas can be used for other kinds of sequences defined in terms of some numeration system for the natural numbers. Such a numeration system provides a unique (up to leading zeros) representation for n as a sum of terms of some other sequence $(s_n)_{n \geq 1}$. If the sequence $\mathbf{c} = c_0c_1c_2 \cdots$ can be computed by a finite automaton taking the representation of n as input, and if further, the addition of represented integers is computable by another finite automaton, then once again the first-order theory $\text{FO}(\mathbb{N}, <, +, 0, 1, n \mapsto c_n)$ is decidable. This is the case, for example, for the so-called Fibonacci-automatic sequences in Mousavi, Schaeffer, and Shallit [MSS16] and the Pell-automatic sequences in Baranwal and Shallit [BS19].

More generally, the same kinds of ideas can handle Sturmian words. For quadratic numbers, this was first observed by Hieronymi and Terry [HT18]. In this paper we extend those results to all Sturmian characteristic words. Thus, the first-order theory of Sturmian characteristic words is decidable. As a result, many classical theorems about Sturmian words, which previously required intricate proofs, can be proved automatically by a theorem-prover in a few seconds. As examples, in Section 7 we reprove basic results such as the balanced property and the subword complexity of these words.

Let $\alpha, \rho \in \mathbb{R}$ be such that α is irrational. The **Sturmian word with slope α and intercept ρ** is the infinite $\{0, 1\}$ -word $\mathbf{c}_{\alpha, \rho} = c_{\alpha, \rho}(1)c_{\alpha, \rho}(2) \cdots$ such that for all $n \in \mathbb{N}$

$$c_{\alpha, \rho}(n) = \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor - \lfloor \alpha \rfloor.$$

When $\rho = 0$, we call $\mathbf{c}_{\alpha, 0}$ the **characteristic word** of slope α . Sturmian words and their combinatorial properties have been studied extensively. We refer the reader to the survey by Berstel and Séébold [Lot02, Chapter 2]. Note that $\mathbf{c}_{\alpha, \rho}$ can be understood as a function from \mathbb{N} to $\{0, 1\}$. Let \mathcal{L} be the signature¹ of the first-order logical theory $\text{FO}(\mathbb{N}, <, +, 0, 1)$ and denote by \mathcal{L}_c the signature obtained by adding a single unary function symbol c to \mathcal{L} . Now let $\mathcal{N}_{\alpha, \rho}$ be the \mathcal{L}_c -structure $(\mathbb{N}, <, +, 0, 1, n \mapsto c_{\alpha, \rho}(n))$, where we expand Presburger arithmetic by a Sturmian word interpreted as a unary function. The main result of this paper is the decidability of the theory of the collection of such expansions. Set $\mathbf{Irr} := (0, 1) \setminus \mathbb{Q}$. Let $\mathcal{K}_{\text{sturmian}} := \{\mathcal{N}_{\alpha, \rho} : \alpha \in \mathbf{Irr}, \rho \in \mathbb{R}\}$, and let $\mathcal{K}_{\text{char}} := \{\mathcal{N}_{\alpha, 0} : \alpha \in \mathbf{Irr}\}$.

¹In model theory this is usually called (or identified with) the language of the theory. However, here this conflicts with the convention of calling an arbitrary set of words a language.

Theorem A. *The first-order logical theories² $\text{FO}(\mathcal{K}_{\text{sturmian}})$ and $\text{FO}(\mathcal{K}_{\text{char}})$ are decidable.*

So far, decidability was only known for individual $\text{FO}(\mathcal{N}_{\alpha,\rho})$, and only for very particular α . By [HT18] the logical theory $\text{FO}(\mathcal{N}_{\alpha,0})$ is decidable when α is a quadratic irrational³. Moreover, if the continued fraction of α is not computable, it can be seen rather easily that $\text{FO}(\mathcal{N}_{\alpha,0})$ is undecidable.

Theorem A is rather powerful, as it allows to automatically decide combinatorial statements about all Sturmian words. Consider the \mathcal{L}_c -sentence φ

$$\forall p (p > 0) \rightarrow \left(\forall i \exists j j > i \wedge c(j) \neq c(j+p) \right).$$

We observe that $\mathcal{N}_{\alpha,\rho} \models \varphi$ if and only if $\mathbf{c}_{\alpha,\rho}$ is not eventually periodic. Thus the decision procedure from Theorem A allows us to check that no Sturmian word is eventually periodic. Of course, it is well-known that no Sturmian word is eventually periodic, but this example indicates potential applications of Theorem A. We outline some of these in Section 7.

We not only prove Theorem A, but instead establish a vastly more general theorem of which Theorem A is an immediate corollary. To state this general result, let \mathcal{L}_m be the signature of $\text{FO}(\mathbb{R}, <, +, \mathbb{Z})$; that is, the signature of $\text{FO}(\mathbb{R}, <, +)$ together with a unary predicate for \mathbb{Z} . Let $\mathcal{L}_{m,a}$ be the extension of \mathcal{L}_m by another unary predicate. For $\alpha \in \mathbb{R}_{>0}$, we let \mathcal{R}_α denote $\mathcal{L}_{m,a}$ -structure $(\mathbb{R}, <, +, \mathbb{Z}, \alpha\mathbb{Z})$. When $\alpha \in \mathbb{Q}$, it has long been known that $\text{FO}(\mathcal{R}_\alpha)$ is decidable (arguably due to Skolem [Sko31]). Recently this result was extended to quadratic numbers.

Fact 1.1 (Hieronimi [Hie16, Theorem A]). Let α be a quadratic irrational. Then $\text{FO}(\mathcal{R}_\alpha)$ is decidable.

See also Hieronimi, Nguyen and Pak [HNP21] for a computational complexity analysis of this decision procedure. The proof of Fact 1.1 establishes that if α is quadratic, then \mathcal{R}_α is an ω -automatic structure; that is, it can be represented by Büchi automata. Since every ω -automatic structure has a decidable first-order theory, so does \mathcal{R}_α . See Khoussainov and Minnes [KM10] for a survey on ω -automatic structures. The key insight needed to prove ω -automaticity of \mathcal{R}_α is that addition in the Ostrowski-numeration system based on α is recognizable by a Büchi automaton when α is quadratic. See Section 2 for a definition of Ostrowski numeration systems.

As observed in [Hie16], there are examples of non-quadratic irrationals α such that \mathcal{R}_α has an undecidable theory and hence is not ω -automatic. However, in this paper we show that the common theory of the \mathcal{R}_α is decidable. Let \mathcal{K} denote the class of $\mathcal{L}_{m,a}$ -structures $\{\mathcal{R}_\alpha : \alpha \in \mathbf{Irr}\}$.

Theorem B. *The theory $\text{FO}(\mathcal{K})$ is decidable.*

Indeed, we will even prove a substantial generalization of Theorem B. For each $\mathcal{L}_{m,a}$ -sentence φ , we set $M_\varphi := \{\alpha \in \mathbf{Irr} : \mathcal{R}_\alpha \models \varphi\}$. Let $\mathbf{Irr}_{\text{quad}}$ be the set of all quadratic irrational real numbers in \mathbf{Irr} . Define

$$\mathcal{M} := (\mathbf{Irr}, <, (M_\varphi)_\varphi, (q)_{q \in \mathbf{Irr}_{\text{quad}}})$$

²Given a signature \mathcal{L}_0 and a class \mathcal{K} of \mathcal{L}_0 -structures, the first-order logical theory of \mathcal{K} is defined as the set of all \mathcal{L}_0 -sentences that are true in all structures in \mathcal{K} . This theory is denoted by $\text{FO}(\mathcal{K})$.

³A real number is **quadratic** if it is the root of a quadratic equation with integer coefficients.

to be the expansion of the dense linear order $(\mathbf{Irr}, <)$ by predicates for M_φ for each $\mathcal{L}_{m,a}$ -sentence φ , and constant symbols for each quadratic irrational real number in \mathbf{Irr} .

Theorem C. *The theory $\text{FO}(\mathcal{M})$ is decidable.*

Observe that Fact 1.1 and Theorem B follow immediately from Theorem C. We outline how Theorem B implies Theorem A. Note that for every irrational α , the structure \mathcal{R}_α defines the usual floor function $[\cdot] : \mathbb{R} \rightarrow \mathbb{Z}$, the singleton $\{\alpha\}$ and the successor function on $\alpha\mathbb{Z}$. Hence \mathcal{R}_α also defines the set $\{(\rho, \alpha n, c_{\alpha,\rho}(n)) : \rho \in \mathbb{R}, n \in \mathbb{N}\}$. From the definability of $\{\alpha\}$, we have that the function from $\alpha\mathbb{N}$ to $\{0, \alpha\}$ given by $\alpha n \mapsto \alpha c_{\alpha,\rho}(n)$ is definable in \mathcal{R}_α . Thus the \mathcal{L}_c -structure $(\alpha\mathbb{N}, <, +, 0, \alpha, \alpha n \mapsto \alpha c_{\alpha,\rho}(n))$ can be defined in \mathcal{R}_α , and this definition is uniform in α . Since the former structure is \mathcal{L}_c -isomorphic to $\mathcal{N}_{\alpha,\rho}$, we have that for every \mathcal{L}_c -sentence φ there is an $\mathcal{L}_{m,a}$ -formula $\psi(x)$ such that

- $\varphi \in \text{FO}(\mathcal{K}_{\text{sturmian}})$ if and only if $\forall x \psi(x) \in \text{FO}(\mathcal{K})$ and
- $\varphi \in \text{FO}(\mathcal{K}_{\text{char}})$ if and only if $\psi(0) \in \text{FO}(\mathcal{K})$.

Even Theorem C is not the most general result we prove. Its statement is more technical and we postpone it until Section 6. However, we want to point out that we can add predicates for interesting subsets of \mathbf{Irr} to \mathcal{M} without changing the decidability of the theory. Examples of such subsets are the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are powers of 2, or the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are not in some fixed finite set. This means we can not only automatically prove theorems about all characteristic Sturmian words, but also prove theorems about all characteristic Sturmian words whose slope is one of these sets. There is a limit to this technique. If we add a predicate for the set of all $\alpha \in \mathbf{Irr}$ such that the terms of continued fraction expansion of α are bounded, or add a predicate for the set of elements in \mathbf{Irr} whose continued fractions has strictly increasing terms, then our method is unable to conclude whether the resulting structure has a decidable theory. See Section 6 for a more precise statement about what kind of predicates can be added.

The proof of Theorem C follows closely the proof from [Hie16] of the ω -automaticity of \mathcal{R}_α for fixed quadratic α . Here we show that the construction of the Büchi automata needed to represent \mathcal{R}_α is actually uniform in α . See Abu Zaid, Grädel and Reinhardt [AZGR17] for a systematic study of uniformly automatic classes of structures. Deducing Theorem C from this result is then rather straightforward. The key ingredient to establish the ω -automaticity of \mathcal{R}_α is an automaton that can perform addition in Ostrowski-numeration systems. By [HT18] there is an automaton that recognizes the addition relation for α -Ostrowski numeration systems for fixed quadratic α . So for a fixed quadratic number, there exists a 3-input automaton that accepts the α -Ostrowski representations of all triples of natural numbers x, y, z with $x + y = z$. In order to prove Theorem C, we need a uniform version of such an adder. This general adder is described in Baranwal, Schaeffer, and Shallit [BSS21]. There a 4-input automaton is constructed that accepts 4-tuples consisting of an encoding of a real number α and three α -Ostrowski representations of natural numbers x, y, z with $x + y = z$. See Section 4 for details.

As mentioned above, an implementation of the decision algorithm provided by Theorem A can be used to study Sturmian words. We created a software program called Pecan [OMSH20] that includes such an implementation. Pecan is inspired by Walnut [Mou16] by Mousavi, an automated theorem-prover for deciding properties of automatic words.

The main difference is that Walnut is based on finite automata, while Pecan uses Büchi automata. In our setting it is more convenient to work with Büchi automata instead of finite automata, since the infinite families of words we want to consider—like Sturmian words—are indexed by real numbers. Section 7 provides more information about Pecan and contains further examples how Pecan is used to prove statements about Sturmian words. Pecan’s implementation is discussed in more detail in [OMSH21].

This is an extended version of the paper [HMO⁺22] presented at CSL 2022.

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2. PRELIMINARIES

Throughout, i, j, k, ℓ, m, n are used for natural numbers. Let X, Y be two sets and $Z \subseteq X \times Y$. For $x \in X$, we let Z_x denote the set $\{y \in Y : (x, y) \in Z\}$. Similarly, given a function $f : X \times Y \rightarrow W$ and $x \in X$, we write f_x for the function $f_x : Y \rightarrow W$ that maps $y \in Y$ to $f(x, y)$.

Given a (possibly infinite word) w over an alphabet Σ , we write w_i for the i -th letter of w , and $w|_n$ for $w_1 \cdots w_n$. We write $|w|$ for the length of w . We let Σ^ω denote the set of infinite words over Σ . If Σ is totally ordered by \prec , we let \prec_{lex} denote the corresponding lexicographic order on Σ^ω . Letting $u, v \in \Sigma^\omega$, we also write $u \prec_{\text{colex}} v$ if there is a maximal i such that $u_i \neq v_i$, and $u_i < v_i$ for this i . Note that while \prec_{lex} is a total order on Σ^ω , the order \prec_{colex} is only a partial order. However, for a given $\sigma \in \Sigma$, the order \prec_{colex} is a total order on the set of all words $v \in \Sigma^\omega$ such that v_j is eventually equal to σ .

We will also need to apply \prec_{lex} and \prec_{colex} to finite sequences u, v of the same length. We do this by choosing a $\sigma \in \Sigma$ (the choice does not matter) and stating that $u \prec_{\text{lex}} v$ iff $u\sigma^\omega \prec_{\text{lex}} v\sigma^\omega$, and similarly for \prec_{colex} .

A **Büchi automaton (over an alphabet Σ)** is a quintuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ where Q is a finite set of states, Σ is a finite alphabet, $\Delta \subseteq Q \times \Sigma \times Q$ is a transition relation, $I \subseteq Q$ is a set of initial states, and $F \subseteq Q$ is a set of accept states.

Let $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ be a Büchi automaton. Let $\sigma \in \Sigma^\omega$. A **run of σ from p** is an infinite sequence s of states in Q such that $s_0 = p$, $(s_n, \sigma_n, s_{n+1}) \in \Delta$ for all $n < |\sigma|$. If $p \in I$, we say s is a **run of σ** . Then σ is **accepted by \mathcal{A}** if there is a run $s_0 s_1 \cdots$ of σ such that $\{n : s_n \in F\}$ is infinite. We call this run an accepting run. We let $L(\mathcal{A})$ be the set of words accepted by \mathcal{A} .

If for every state s in \mathcal{A} there is a run of some string from an initial state through s to an accept state, where s is not the last state in the run, then we say \mathcal{A} is **trim**. Every Büchi automaton has an equivalent trim automaton, which may be obtained simply by removing

(possibly iteratively) every state failing this condition. There are other types of ω -automata with different acceptance conditions, but in this paper we only consider Büchi automata.

Let Σ be a finite alphabet. We say a subset $X \subseteq \Sigma^\omega$ is **ω -regular** if it is recognized by some Büchi automaton. Let $u_1, \dots, u_n \in \Sigma^\omega$. We define the **convolution** $c(u_1, \dots, u_n)$ of u_1, \dots, u_n as the element of $(\Sigma^n)^\omega$ whose value at position i is the n -tuple consisting of the values of u_1, \dots, u_n at position i . We say that $X \subseteq (\Sigma^\omega)^n$ is **ω -regular** if $c(X)$ is ω -regular.

Fact 2.1. The collection of ω -regular sets is closed under union, intersection, complementation and projection.

Closure under complementation is due to Büchi [Büc62]. We refer the reader to Khoussainov and Nerode [KN01] for more information and a proof of Fact 2.1. As consequence of Fact 2.1, we have that for every ω -regular subset $W \subseteq (\Sigma^\omega)^{m+n}$ the set

$$\{s \in (\Sigma^\omega)^m : \forall t \in (\Sigma^\omega)^n (s, t) \in W\}$$

is also ω -regular.

The proof of Theorem 4.1 will utilize a few other related types of automaton. A **finite automaton** has the same internal structure as a Büchi automaton i.e. is also a quintuple $\mathcal{A} = (Q, \Sigma, \Delta, I, F)$ with the same restrictions, but it takes a finite word $\sigma \in \Sigma^*$ as input. In the case of a finite automaton, runs are finite sequences instead of infinite sequences but otherwise follow the same rule on transitions. We say that σ is **accepted by** \mathcal{A} in this case if there is a run of σ such that $s_{|\sigma|} \in F$.

We will also refer to **general finite** and Büchi automata. These are the same as finite and Büchi automata, respectively, but where Σ is no longer required to be a finite alphabet. Note that Q is still finite in these cases; therefore Δ , viewed as a directed multigraph on Q , still has finitely many vertices but may have infinitely many arrows between the same pair of vertices. General finite and Büchi automata are not often considered, as they do not have the same computability properties⁴, but they may sometimes be converted into “equivalent” finite and Büchi automata, as we will see in Section 4.

2.1. ω -regular structures. Let $\mathcal{U} = (U; R_1, \dots, R_m)$ be a structure, where U is a non-empty set and R_1, \dots, R_m are relations on U . We say \mathcal{U} is **ω -regular** if its domain and its relations are ω -regular.

Büchi’s theorem [Büc62] on the decidability of the monadic second-order theory of one successor immediately gives the following well-known fact.

Fact 2.2. Let \mathcal{U} be an ω -regular structure. Then the theory $\text{FO}(\mathcal{U})$ is decidable.

In this paper, we will consider families of ω -regular structures that are uniform in the following sense. Fix $m \in \mathbb{N}$ and a map $\text{ar} : \{1, \dots, m\} \rightarrow \mathbb{N}$. Let Z be a set and for $z \in Z$ let \mathcal{U}_z be a structure $(U_z; R_{1,z}, \dots, R_{m,z})$ such that $R_{i,z} \subseteq U_z^{\text{ar}(i)}$. We say that $(\mathcal{U}_z)_{z \in Z}$ is a **uniform family of ω -regular structures** if

⁴To see why, consider e.g. a generalized Büchi automaton recognizing words over \mathbb{N} consisting of a single initial state q_0 and a single final state q_1 such that there is a noncomputable set $S \subseteq \mathbb{N}$ with $\Delta = \{(q_0, s, q_1) : s \in S\}$.

- $\{(z, y) : y \in U_z\}$ is ω -regular,
- $\{(z, y_1, \dots, y_{ar(i)}) : (y_1, \dots, y_{ar(i)}) \in R_{i,z}\}$ is ω -regular for each $i \in \{1, \dots, m\}$.

We refer the reader to [AZGR17] for an in-depth analysis of uniformity in automatic structure.

From Büchi's theorem, we immediately obtain the following.

Fact 2.3. Let $(U_z)_{z \in Z}$ be a uniform family of ω -regular structures, and let φ be a formula in the signature of these structures. Then the set

$$\{(z, u) : z \in Z, u \in U_z, U_z \models \varphi(u)\}$$

is ω -regular, and, the automaton recognizing this set can be effectively computed given φ . Moreover, the theory $\text{FO}(\{U_z : z \in Z\})$ is decidable.

Proof. When φ is an atomic formula, the statement follows immediately from the definition of a uniform family of ω -regular structures and the ω -regularity of equality. By Fact 2.1, the statement holds for all formulas. \square

Let $w \in \Sigma^\omega$. The **acceptance problem for w** is the following decision problem:

Given a Büchi automaton \mathcal{A} over Σ , is w accepted by \mathcal{A} ?

For examples of non- ω -regular words with a decidable acceptance problem, see Elgot and Rabin [ER66], Semenov [Sem83] or Carton and Thomas [CT02]. We obtain the following well-known corollary of Fact 2.3.

Fact 2.4. Let $(U_z)_{z \in Z}$ be a uniform family of ω -regular structures, and let $w \in Z$ be such that the acceptance problem for w is decidable. Then the theory $\text{FO}(U_w)$ is decidable.

2.2. Binary representations. For $k \in \mathbb{N}_{>1}$ and $b = b_0 b_1 b_2 \dots b_n \in \{0, 1, \dots, k-1\}^*$, we define $[b]_k := \sum_{i=0}^n b_i k^i$. For $N \in \mathbb{N}$ we say $b \in \{0, 1\}^*$ is a **binary representation** of N if $[b]_2 = N$.

Throughout this paper, we will often consider infinite words over the (infinite) alphabet $\{0, 1\}^*$. Let $[\cdot]_2 : (\{0, 1\}^*)^\omega \rightarrow \mathbb{N}^\omega$ be the function that maps $u = u_1 u_2 \dots \in (\{0, 1\}^*)^\omega$ to

$$[u_1]_2 [u_2]_2 [u_3]_2 \dots$$

We will consider the following different relations on $(\{0, 1\}^*)^\omega$.

Let $u, v \in (\{0, 1\}^*)^\omega$. We write $u <_{\text{lex},2} v$ if $[u]_2$ is lexicographically smaller than $[v]_2$. We write $u <_{\text{colex},2} v$ if there is a maximal i such that $[u_i]_2 \neq [v_i]_2$, and $[u_i]_2 < [v_i]_2$. Note that while $<_{\text{lex},2}$ is a total order on $(\{0, 1\}^*)^\omega$, the order $<_{\text{colex},2}$ is only a partial order. However, $<_{\text{colex},2}$ is a total order on the set of all words $v \in (\{0, 1\}^*)^\omega$ such that $[v]_j$ is eventually 0.

Let $u = u_1 u_2 \dots, v = v_1 v_2 \dots \in (\{0, 1\}^*)^\omega$. Let k be minimal such that $[u_k]_2 \neq [v_k]_2$. We write $u <_{\text{alex},2} v$ if either k is even and $[u_k]_2 < [v_k]_2$, or k is odd and $[u_k]_2 > [v_k]_2$; this is the *alternating lexicographic order* on $(\{0, 1\}^*)^\omega$.

2.3. Ostrowski representations. We now introduce Ostrowski representations based on the continued fraction expansions of real numbers. We refer the reader to Allouche and Shallit [AS03] and Rockett and Szűsz [RS92] for more details. A **finite continued fraction expansion** $[a_0; a_1, \dots, a_k]$ is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}}$$

For a real number α , we say $[a_0; a_1, \dots, a_k, \dots]$ is a **continued fraction expansion of α** if $\alpha = \lim_{k \rightarrow \infty} [a_0; a_1, \dots, a_k]$ and $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}_{>0}$ for $i > 0$. In this situation, we write $\alpha = [a_0; a_1, \dots]$. Every irrational number has precisely one continued fraction expansion, so we will usually refer to *the* continued fraction expansion of a number. We recall the following well-known fact about continued fractions.

Fact 2.5. Let $\alpha = [a_0; a_1, \dots], \alpha' = [a'_0; a'_1, \dots] \in \mathbb{R}$ be irrational. Let $k \in \mathbb{N}$ be minimal such that $a_k \neq a'_k$. Then $\alpha < \alpha'$ if and only if

- k is even and $a_k < a'_k$, or
- k is odd and $a_k > a'_k$.

For the rest of this subsection, fix a positive irrational real number $\alpha \in (0, 1)$ and let $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of α .

Let $k \geq 1$. A pair (p_k, q_k) is the **k -th convergent of α** if $p_k \in \mathbb{N}$, $q_k \in \mathbb{Z}$, $\gcd(p_k, q_k) = 1$ and

$$\frac{p_k}{q_k} = [a_0; a_1, \dots, a_k].$$

Set $p_{-1} := 1, q_{-1} := 0$ and $p_0 := a_0, q_0 := 1$. While formally a pair of integers, in practice we will think of a convergent as the quotient $\frac{p_k}{q_k}$. The convergents satisfy the following equations for $n \geq 1$:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

We now recall a numeration system due to Ostrowski [Ost22].

Fact 2.6 [RS92, Ch. II-§4]. Let $X \in \mathbb{N}$. Then X can be written uniquely as

$$X = \sum_{n=0}^N b_{n+1} q_n, \tag{2.1}$$

where $0 \leq b_1 < a_1$, $0 \leq b_{n+1} \leq a_{n+1}$ and $b_n = 0$ whenever $b_{n+1} = a_{n+1}$.

For $X \in \mathbb{N}$ satisfying (2.1) we write

$$X = [b_1 b_2 \cdots b_N b_{N+1}]_\alpha$$

and call the word $b_1 b_2 \cdots b_{N+1}$ an α -Ostrowski representation of X . This representation is unique up to trailing zeros. Let $X, Y \in \mathbb{N}$ and let $b_1 b_2 \cdots b_{N+1}$ and $c_1 c_2 \cdots c_{N+1}$ be α -Ostrowski representations of X and Y respectively. Since Ostrowski representations are obtained by a greedy algorithm, one can see easily that $X < Y$ if and only if $b_1 b_2 \cdots b_{N+1}$ is co-lexicographically smaller than $c_1 c_2 \cdots c_{N+1}$.

We now introduce a similar way to represent real numbers, also due to Ostrowski [Ost22]. The k -th difference β_k of α is defined as $\beta_k := q_k\alpha - p_k$. We use the following facts about k -th differences: for all $n \in \mathbb{N}$

- (1) $\beta_n > 0$ if and only if n is even,
- (2) $\beta_0 > -\beta_1 > \beta_2 > -\beta_3 > \beta_4 > \dots$, and
- (3) $-\beta_n = a_{n+2}\beta_{n+1} + a_{n+4}\beta_{n+3} + a_{n+6}\beta_{n+5} + \dots$.

Let I_α be the interval $[\lfloor \alpha \rfloor - \alpha, 1 + \lfloor \alpha \rfloor - \alpha)$.

Fact 2.7 (cf. [RS92, Ch. II.6 Theorem 1]). Let $x \in I_\alpha$. Then x can be written uniquely as

$$\sum_{k=0}^{\infty} b_{k+1}\beta_k, \quad (2.2)$$

where $b_k \in \mathbb{Z}$ with $0 \leq b_k \leq a_k$, and $b_{k-1} = 0$ whenever $b_k = a_k$, (in particular, $b_1 \neq a_1$), and $b_k \neq a_k$ for infinitely many odd k .

For $x \in I_\alpha$ satisfying (2.2) we write

$$x = [b_1b_2\cdots]_\alpha$$

and call the infinite word $b_1b_2\cdots$ the α -Ostrowski representation of x . This is closely connected to the integer Ostrowski representation. Note that for every real number there a unique element of I_α such that their difference is an integer. We define $f_\alpha : \mathbb{R} \rightarrow I_\alpha$ to be the function that maps x to $x - u$, where u is the unique integer such that $x - u \in I_\alpha$.

Fact 2.8 [Hie16, Lemma 3.4]. Let $X \in \mathbb{N}$ be such that $\sum_{k=0}^N b_{k+1}q_k$ is the α -Ostrowski representation of X . Then

$$f_\alpha(\alpha X) = \sum_{k=0}^{\infty} b_{k+1}\beta_k$$

is the α -Ostrowski representation of $f_\alpha(\alpha X)$, where $b_{k+1} = 0$ for $k > N$.

Since $\beta_k > 0$ if and only if k is even, the order of two elements in I_α can be determined by the Ostrowski representation as follows.

Fact 2.9 [Hie16, Fact 2.13]. Let $x, y \in I_\alpha$ with $x \neq y$ and let $[b_1b_2\cdots]_\alpha$ and $[c_1c_2\cdots]_\alpha$ be the α -Ostrowski representations of x and y . Let $k \in \mathbb{N}$ be minimal such that $b_k \neq c_k$. Then $x < y$ if and only if

- (i) $b_{k+1} < c_{k+1}$ if k is even;
- (ii) $b_{k+1} > c_{k+1}$ if k is odd.

3. #-BINARY ENCODING

In this section, we introduce #-binary coding. A similar encoding has been used in Hodgson [Hod82]. Fix the alphabet $\Sigma_\# := \{0, 1, \#\}$. Let H_∞ denote the set of all infinite $\Sigma_\#$ -words in which $\#$ appears infinitely many times. Clearly H_∞ is ω -regular.

Let $C_\# : (\{0, 1\}^*)^\omega \rightarrow H_\infty$ map an infinite word $b = b_1b_2b_3\cdots$ over $\{0, 1\}^*$ to the infinite $\Sigma_\#$ -word

$$\#b_1\#b_2\#b_3\#\cdots.$$

We note that the map $C_{\#}$ is a bijection.

Let $u = u_1u_2u_3 \cdots, v = v_1v_2v_3 \cdots \in \Sigma_{\#}^{\omega}$. We say u and v are **aligned** if for all $i \in \mathbb{N}$

$$u_i = \# \text{ if and only if } v_i = \#.$$

This defines an ω -regular equivalence relation on $\Sigma_{\#}^{\omega}$. We denote this equivalence relation by $\sim_{\#}$. We say $(w_1, \dots, w_n) \in (\Sigma_{\#}^{\omega})^n$ is **aligned** if

$$w_1 \sim_{\#} w_2 \sim_{\#} \cdots \sim_{\#} w_n.$$

We say a subset $X \subseteq (\Sigma_{\#}^{\omega})^n$ is **aligned** if every $w \in X$ is aligned.

The following fact follows easily.

Fact 3.1. The following sets are ω -regular:

- $\{(u, v) \in H_{\infty}^2 : u \sim_{\#} v \text{ and } C_{\#}^{-1}(u) <_{\text{lex},2} C_{\#}^{-1}(v)\},$
- $\{(u, v) \in H_{\infty}^2 : u \sim_{\#} v \text{ and } C_{\#}^{-1}(u) <_{\text{colex},2} C_{\#}^{-1}(v)\},$
- $\{(u, v) \in H_{\infty}^2 : u \sim_{\#} v \text{ and } C_{\#}^{-1}(u) <_{\text{alex},2} C_{\#}^{-1}(v)\}.$

3.1. #-binary coding of continued fractions. We now code the continued fraction expansions of real numbers as infinite $\Sigma_{\#}$ -words.

Definition 3.2. Let $\alpha \in (0, 1)$ be irrational such that $[0; a_1, a_2, \dots]$ is the continued fraction expansion of α . Let $u = u_1u_2 \cdots \in (\{0, 1\}^*)^{\omega}$ such that $u_i \in \{0, 1\}^*$ is a binary representation of a_i for each $i \in \mathbb{Z}_{\geq 0}$. We say that $C_{\#}(u)$ is a **-binary coding of the continued fraction of α** .

Let R be the set of elements of $\Sigma_{\#}^{\omega}$ of the form $(\#(0|1)^*1(0|1)^*)^{\omega}$. Obviously, R is ω -regular.

Lemma 3.3. *Let $w \in R$. Then there is a unique irrational number $\alpha \in [0, 1]$ such that w is a #-binary coding of the continued fraction of α .*

Proof. By the definition of R , there is $w_1w_2 \cdots \in ((0|1)^*1(0|1)^*)^{\omega}$ such that

$$w = \#w_1\#w_2\#\cdots.$$

Since $w_i \in (0|1)^*1(0|1)^*$, we have that w_i is a $\{0, 1\}$ -word containing at least one 1. Let a_i be the natural number that $a_i = [w_i]_2$. Because w_i contains a 1, we must have $a_i \neq 0$. Thus w is a #-binary coding of the infinite continued fraction of the irrational $\alpha = [0; a_1, a_2, \dots]$. Uniqueness follows directly from the fact that both binary expansions and continued fraction expansions only represent one number. \square

For $w \in R$, let $\alpha(w)$ be the real number given by Lemma 3.3. When $v = (v_1, \dots, v_n) \in R^n$, we write $\alpha(v)$ for $(\alpha(v_1), \dots, \alpha(v_n))$.

Even though continued fractions are unique, their #-binary codings are not, because binary representations can have trailing zeroes. This ambiguity is required in order to properly recognize relationships between multiple numbers, as one of the numbers involved may require more bits in a coefficient than the other(s). Occasionally we need to ensure that all possible representations of a given tuple of numbers are contained in a set. For this reason, we introduce the zero-closure of subsets of R^n .

Definition 3.4. Let $X \subseteq R^n$ be aligned. The **zero-closure** of X is

$$\{u \in R^n : u \text{ is aligned} \wedge \exists v \in X \alpha(u) = \alpha(v)\}.$$

Lemma 3.5. *Let $X \subseteq R^n$ be ω -regular and aligned. Then the zero-closure of X is also ω -regular.*

Proof. Let \mathcal{A} be a Büchi automaton recognizing X . We use Q to denote the set of states of \mathcal{A} . We create a new automaton \mathcal{A}' that recognizes the zero-closure of X , as follows:

- (Step 1) Start with the automata \mathcal{A} .
- (Step 2) For each transition on the n -tuple $(\#, \dots, \#)$ from a state p to a state q , we add a new state $\mu(p, q)$ that loops to itself on the n -tuple $(0, \dots, 0)$ and transitions to state q on $(\#, \dots, \#)$. We add a transition from p to $\mu(p, q)$ on $(0, \dots, 0)$.
- (Step 3) For every pair p, q of states of \mathcal{A} for which p has a run to q on a word of the form $(0, \dots, 0)^m(\#, \dots, \#)$ for some m , we add a transition from state p to a new state $\nu(p, q)$ on $(\#, \dots, \#)$, and for every transition out of state q , we create a copy of the transition that starts at state $\nu(p, q)$ instead. If any original run from state p to state q passes through a final state, we make $\nu(p, q)$ a final state.
- (Step 4) Denote the resulting automaton by \mathcal{A}' and its set of states by Q' .

We now show that $L(\mathcal{A}')$ is the zero-closure of X . We first show that the zero-closure is contained in $L(\mathcal{A}')$. Let $v \in X$ and $w \in R^n$ be such that w is aligned and $\alpha(v) = \alpha(w)$. Since both v and w are aligned, there are $b = b_1b_2 \dots, c = c_1c_2 \dots \in ((\{0, 1\}^n)^*)^\omega$ such that $C_\#(b) = v$ and $C_\#(c) = w$. Since $\alpha(v) = \alpha(w)$, we have that $[b_i]_2 = [c_i]_2$ for $i \in \mathbb{N}$. Therefore, for each $i \in \mathbb{N}$, the words b_i and c_i only differ by trailing (tuples of) zeroes. Let $s = s_1s_2 \dots \in Q^\omega$ be an accepting run of v on \mathcal{A} . We now transfer this run into an accepting run $s' = s'_1s'_2 \dots$ of w on \mathcal{A}' . For $i \in \mathbb{N}$, let $y(i)$ be the position of the i -th $(\#, \dots, \#)$ in v and let $z(i)$ be the position of the i -th $(\#, \dots, \#)$ in w . For each $i \in \mathbb{N}$, we define a sequence $s'_{z(i)+1} \dots s'_{z(i+1)}$ of states of \mathcal{A}' as follows:

- (1) If $|c_i| = |b_i|$, then $c_i = b_i$. We set

$$s'_{z(i)+1} \dots s'_{z(i+1)} := s_{y(i)+1} \dots s_{y(i+1)}.$$

- (2) If $|c_i| > |b_i|$, then $c_i = b_i(0, \dots, 0)^{|c_i|-|b_i|}$. We set

$$\begin{aligned} & s'_{z(i)+1} \dots s'_{z(i+1)} \\ & := s_{y(i)+1} \dots s_{y(i+1)-1} \underbrace{\mu(s_{y(i+1)-1}, s_{y(i+1)}) \dots \mu(s_{y(i+1)-1}, s_{y(i+1)})}_{(|c_i|-|b_i|)\text{-times}} s_{y(i+1)}. \end{aligned}$$

Thus the new run follows the old run up to $s_{y(i+1)-1}$ and then transitions to one of the newly added states in the Step 2. It loops on $(0, \dots, 0)$ for $|c_i| - |b_i| - 1$ -times before moving to $s_{y(i+1)}$.

- (3) If $|c_i| < |b_i|$, then $b_i = c_i(0, \dots, 0)^{|b_i|-|c_i|}$. We set

$$s'_{z(i)+1} \dots s'_{z(i+1)} := s_{y(i)+1} \dots s_{y(i)+|c_i|} \nu(s_{y(i)+|c_i|}, s_{y(i+1)}).$$

The new run utilizes one of the newly added $(\#, \dots, \#)$ transitions and corresponding states added in Step 3.

The reader can now easily check that s' is an accepting run of w on \mathcal{A}' .

We now show that $L(\mathcal{A}')$ is contained in the zero-closure of X . We prove that the only accepting runs on \mathcal{A}' are based on accepting runs on \mathcal{A} with trailing zeroes either added or removed. Let $w = w_1 w_2 \cdots \in L(\mathcal{A}')$, and let $s' = s'_1 s'_2 \cdots \in Q^\omega$ be an accepting run of w on \mathcal{A}' . We construct $v \in X$ and a run $s = s_1 s_2 \cdots \in Q^\omega$ of w_2 on \mathcal{A} such that $\alpha(v) = \alpha(w)$ and s is an accepting run of v . We start by setting $v := w_1 w_2 \cdots$ and $s := s'_1 s'_2 \cdots$. For each $i \in \mathbb{N}$, we replace w_i in v and s'_i in s as follows:

- (1) If $s'_i \in Q$, then we make no changes to s'_i and w_i .
- (2) If $s'_i = \mu(p, q)$ for some $p, q \in Q$, we delete the s'_i in s and delete w_i in v .
- (3) If $s_i = \nu(p, q)$ for some $p, q \in Q$, then we replace
 - (a) s'_i by a run $t = t_1 \cdots t_{n+1}$ of $(0, \dots, 0)^n(\#, \dots, \#)$ from p to q , and
 - (b) w_i by $(0, \dots, 0)^n(\#, \dots, \#)$.

If $\nu(p, q)$ is a final state of \mathcal{A}' , we choose t such that it passed through a final state of \mathcal{A} .

It is clear that the resulting s is in Q^ω . The reader can check s is an accepting run of v on \mathcal{A} and that $\alpha(v) = \alpha(w)$. Thus w is in the zero-closure of X . \square

Lemma 3.6. *The set*

$$\{(w_1, w_2) \in R^2 : w_1 \sim_{\#} w_2 \text{ and } \alpha(w_1) < \alpha(w_2)\}$$

is ω -regular.

Proof. Let $w_1, w_2 \in R$ be such that $w_1 \sim_{\#} w_2$. By Fact 2.5 we have that $\alpha(w_1) < \alpha(w_2)$ if only $C_{\#}^{-1}(w_1) <_{\text{alex}, 2} C_{\#}^{-1}(w_2)$. Thus ω -regularity follows from Fact 3.1. \square

Lemma 3.7. *Let $a \in [0, 1)$ be a quadratic irrational. Then*

$$\{w \in R : \alpha(w) = a\}$$

is ω -regular.

Proof. The continued fraction expansion of a is eventually periodic (see for example [HW79, Theorem 177]). Thus there is an eventually periodic $u \in (\{0, 1\}^*)^\omega$ such that $C_{\#}(u)$ is a $\#$ -binary coding of the continued fraction of a . The singleton set containing an eventually periodic string is ω -regular. It remains to expand this set to contain all representations via Lemma 3.5. \square

Lemma 3.8. *The set $\{w \in R : \alpha(w) < \frac{1}{2}\}$ is ω -regular.*

Proof. Let $\alpha(w) = [0; a_1, a_2, \dots]$. It is easy to see that $\alpha(w) < \frac{1}{2}$ if and only if $a_1 > 1$. Thus we need only check that $a_1 \neq 1$. The set of $w \in R$ for which this true is just $R \setminus Y$, where $Y \subseteq \Sigma_{\#}^\omega$ is given by the regular expression $\#10^*(\#(0 \cup 1)^*)^\omega$. \square

3.2. $\#$ -Ostrowski-representations. We now extend the $\#$ -binary coding to Ostrowski representations.

Definition 3.9. Let $v, w \in (\Sigma_{\#})^\omega$, let $x = x_1 x_2 x_3 \cdots \in \mathbb{N}^\omega$ and let $b = b_1 b_2 b_3 \cdots \in (\{0, 1\}^*)^\omega$ be such that $w = C_{\#}(b)$ and $[b_i]_2 = x_i$ for each i .

- For $N \in \mathbb{N}$, we say that w is a **$\#$ - v -Ostrowski representation of N** if v and w are aligned and x is an $\alpha(v)$ -Ostrowski representation of N .

- For $c \in I_{\alpha(v)}$, we say that w is a **#- v -Ostrowski representation of c** if v and w are aligned and x is an $\alpha(v)$ -Ostrowski representation of c .

We let A_v denote the set of all words $w \in \Sigma_{\#}^{\omega}$ such that w is a #- v -Ostrowski representation of some $c \in I_{\alpha(v)}$, and similarly, by A_v^{fin} the set of all words $w \in \Sigma_{\#}^{\omega}$ such that w is a #- v -Ostrowski representation of some $N \in \mathbb{N}$.

Lemma 3.10. *The sets*

$$A^{\text{fin}} := \{(v, w) : v \in R, w \in A_v^{\text{fin}}\}, \text{ and } A := \{(v, w) : v \in R, w \in A_v\}.$$

are ω -regular. Moreover, $A^{\text{fin}} \subseteq A$.

Proof. The statement that $A^{\text{fin}} \subseteq A$, follows immediately from the definitions of A^{fin} and A and Fact 2.8. It is left to establish the ω -regularity of the two sets.

For A^{fin} : Let $B \supseteq A^{\text{fin}}$ be the set of all pairs (v, w) such that $v \in R$ and $v \sim_{\#} w$. Note that B is ω -regular. Let $(v, w) \in B$. Since v and w have infinitely many $\#$ symbols and are aligned, there are unique $a = a_1 a_2 \cdots, b = b_1 b_2 \cdots \in (\{0, 1\}^*)^{\omega}$ such that $C_{\#}(a) = v$, $C_{\#}(b) = w$ and $|a_i| = |b_i|$ for each $i \in \mathbb{N}$. Then by Fact 2.6, $(v, w) \in A^{\text{fin}}$ if and only if

- b has finitely many 1 symbols;
- $b_1 <_{\text{colex}} a_1$;
- $b_i \leq_{\text{colex}} a_i$ for all $i > 1$;
- if $b_i = a_i$, then $b_{i-1} = 0$.

It is easy to check that all four conditions are ω -regular.

For A : As above, let $(v, w) \in B$. Since v and w have infinitely many $\#$ symbols and are aligned, there are unique $a = a_1 a_2 \cdots, b = b_1 b_2 \cdots \in (\{0, 1\}^*)^{\omega}$ such that $C_{\#}(a) = v$, $C_{\#}(b) = w$ and $|a_i| = |b_i|$ for each $i \in \mathbb{N}$. Then by Fact 2.7, $(v, w) \in A$ if and only if

- $b_1 <_{\text{colex}} a_1$;
- $b_i \leq_{\text{colex}} a_i$ for all $i > 1$;
- if $b_i = a_i$, then $b_{i-1} = 0$;
- $b_i \neq a_i$ for infinitely many odd i .

Again, it is easy to see that all four conditions are ω -regular. □

Definition 3.11. Let $v \in R$. We define $Z_v : A_v^{\text{fin}} \rightarrow \mathbb{N}$ to be the function that maps w to the natural number whose #- v -Ostrowski representation is w .

Similarly, we define $O_v : A_v \rightarrow I_{\alpha(v)}$ to be the function that maps w to the real number whose #- v -Ostrowski representation is w .

Lemma 3.12. *Let $v \in R$. Then $Z_v : A_v^{\text{fin}} \rightarrow \mathbb{N}$ and $O_v : A_v \rightarrow I_{\alpha(v)}$ are bijective.*

Proof. We first consider injectivity. By Fact 2.6 and Fact 2.7 a number in \mathbb{N} or in $I_{\alpha(v)}$ only has one $\alpha(v)$ -Ostrowski representation. So we only need to explain why such a representation will only have one encoding in A_v^{fin} (respectively A_v). This follows from the uniqueness of binary representations up to the length of the representation, and from the fact that the requirement of having the $\#$ symbols aligned with v determines the length of each binary-encoded coefficient.

For surjectivity we only need to explain why an $\alpha(v)$ -Ostrowski representation can always be encoded into a string in A_v^{fin} (respectively A_v). It suffices to show that the requirement of

having the $\#$ symbols aligned with v will never result in needing to fit the binary encoding of a number into too few symbols, i.e., that it will never result in having to encode a natural number n in binary in fewer than $1 + \lceil \log_2 n \rceil$ symbols. Since the function $1 + \lceil \log_2 n \rceil$ is monotone increasing, we can encode any natural number below n in k symbols if we can encode n in binary in k symbols. However, by Fact 2.6 and Fact 2.7, the coefficients in an $\alpha(v)$ -Ostrowski representation never exceed the corresponding coefficients in the continued fraction for $\alpha(v)$, i.e., $b_n \leq a_n$. \square

Definition 3.13. Let $v \in R$. We write $\mathbf{0}_v$ for $Z_v^{-1}(0)$, and $\mathbf{1}_v$ for $Z_v^{-1}(1)$.

Lemma 3.14. *The relations $\mathbf{0}_* = \{(v, \mathbf{0}_v) : v \in R\}$ and $\mathbf{1}_* = \{(v, \mathbf{1}_v) : v \in R\}$ are ω -regular.*

Proof. Recognizing $\mathbf{0}_*$ is trivial, as the Ostrowski representations of 0 are of the form $0 \cdots 0$ for all irrational α . Thus $\mathbf{0}_*$ is just the relation

$$\{(v, w) : v \in R, w \text{ is } v \text{ with all 1 bits replaced by 0 bits}\}.$$

This is clearly ω -regular.

We now consider $\mathbf{1}_*$. Let $\alpha = [0; a_1, a_2, \dots]$ be an irrational number. If $a_1 > 1$, the α -Ostrowski representations of 1 are of the form $10 \cdots 0$. If $a_1 = 1$, the α -Ostrowski representations of 1 are of the form $010 \cdots 0$. Thus, in order to recognize $\mathbf{1}_*$, we only need to be able to recognize if a number in binary representation is 0, 1, or greater than 1. Of course, this is easily done on a Büchi automaton. \square

Lemma 3.15. *Let $s \in A_v^{\text{fin}}$. Then $\alpha(v)Z_v(s) - O_v(s) \in \mathbb{Z}$ and*

$$O_v(\mathbf{1}_v) = \begin{cases} \alpha(v) & \text{if } \alpha(v) < \frac{1}{2}; \\ \alpha(v) - 1 & \text{otherwise.} \end{cases}$$

Proof. By Fact 2.8, $O_v(s) = f_{\alpha(v)}(\alpha(v)Z_v(s))$. Thus

$$\alpha(v)Z_v(s) - O_v(s) = \alpha(v)Z_v(s) - f_{\alpha(v)}(\alpha(v)Z_v(s)),$$

which is an integer by the definition of f . By the definition of $\mathbf{1}_v$ and by Fact 2.8, we know $O_v(\mathbf{1}_v) = f_{\alpha(v)}(\alpha(v))$ is the unique element of $I_{\alpha(v)}$ that differs from $\alpha(v)$ by an integer. If $0 < \alpha(v) < \frac{1}{2}$, then

$$-\alpha(v) < \alpha(v) < 1 - \alpha(v).$$

Thus in this case, $\alpha(v) \in I_{\alpha(v)}$ and $O_v(\mathbf{1}_v) = \alpha(v)$. When $\frac{1}{2} < \alpha(v) < 1$, then

$$-\alpha(v) < \alpha(v) - 1 < 1 - \alpha(v).$$

Therefore $\alpha(v) - 1 \in I_{\alpha(v)}$ and $O_v(\mathbf{1}_v) = \alpha(v) - 1$. \square

Lemma 3.16. *The sets*

$$\begin{aligned} \prec^{\text{fin}} &:= \{(v, s, t) \in \Sigma_{\#}^3 : s, t \in A_v^{\text{fin}} \wedge Z_v(s) < Z_v(t)\}, \\ \prec &:= \{(v, s, t) \in \Sigma_{\#}^3 : s, t \in A_v \wedge O_v(s) < O_v(t)\} \end{aligned}$$

are ω -regular.

Proof. For \prec^{fin} , first recall that for $X, Y \in \mathbb{N}$ and α irrational, we have $X < Y$ if and only if the α -Ostrowski representation of X is co-lexicographically smaller than the α -Ostrowski representation of Y . Therefore, we need only recognize co-lexicographic ordering on the list of coefficients, with each coefficient ordered according to binary. This follows immediately from Fact 3.1.

For \prec , note that by Fact 2.9 the usual order on real numbers corresponds to the alternating lexicographic ordering on real Ostrowski representations. Therefore, we need only recognize the alternating lexicographic ordering on the list of coefficients, with each coefficient ordered according to binary. This follows immediately from Fact 3.1. \square

We consider \mathbb{R}^n as a topological space using the usual order topology. For $X \subseteq \mathbb{R}^n$, we denote its topological closure by \overline{X} . This is of course defined using the product of order topologies; i.e. $x \in \overline{X}$ iff every open box containing x also contains an element of X .

Corollary 3.17. *Let $W \subseteq (\Sigma_{\#}^{n+1})^*$ ω -regular be such that*

$$W \subseteq \{(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^* : s_1, \dots, s_n \in A_v\}.$$

Then the following set is also ω -regular:

$$\overline{W} := \{(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^* : s_1, \dots, s_n \in A_v \wedge (O_v(s_1), \dots, O_v(s_n)) \in \overline{O(W_v)}\}.$$

Proof. Let $(v, s_1, \dots, s_n) \in (\Sigma_{\#}^{n+1})^*$ be such that $s_1, \dots, s_n \in A_v$. Let $X_i = O_v(s_i)$. By the definition of the topological closure, we have that $(X_1, \dots, X_n) \in \overline{O(W_v)}$ if and only if for all $Y_1, \dots, Y_n, Z_1, \dots, Z_n \in \mathbb{R}$ with $Y_i < X_i < Z_i$ for $i = 1, \dots, n$ there are $X' = (X'_1, \dots, X'_n) \in O(W_v)$ such that $Y_i < X'_i < Z_i$ for $i = 1, \dots, n$. Thus by Lemma 3.16, $(v, s_1, \dots, s_n) \in \overline{W}$ if and only if for all $t_1, \dots, t_n, u_1, \dots, u_n \in A_v$ with $t_i \prec s_i \prec u_i$, there are $s' = (s'_1, \dots, s'_n) \in W_v$ such that $t_i \prec s'_i \prec u_i$ for $i = 1, \dots, n$. The latter condition is ω -regular by Fact 2.1. \square

4. RECOGNIZING ADDITION IN OSTROWSKI NUMERATION SYSTEMS

The key to the rest of this paper is a general automaton for recognizing addition of Ostrowski representations uniformly. We will prove the following:

Theorem 4.1. *The set*

$$\oplus^{\text{fin}} := \{(v, s_1, s_2, s_3) : s_1, s_2, s_3 \in A_v^{\text{fin}} \wedge Z_v(s_1) + Z_v(s_2) = Z_v(s_3)\}$$

is ω -regular.

In order to prove this theorem, we will introduce a method to generate more complex automata for strings in H_{∞} , from general Büchi automata. For the reasons mentioned when general Büchi automata were introduced in Section 2, we will not use these automata directly. Instead, we will use the $\#$ -binary coding to convert the computation to a more familiar setting. Similarly arguments have been made before, in particular in [Hod82, Section 4].

Definition 4.2. Let $w = w_1 w_2 \dots \in (\mathbb{N}^n)^{\omega}$. A **$\#$ -binary coding of w** is a word $u = u_1 u_2 \dots \in (\Sigma_{\#}^n)^{\omega}$ such that

$$C_{\#}(u_{1,i} u_{2,i} \dots) = w_{1,i} w_{2,i} \dots,$$

where $u_{j,i}$ and $w_{j,i}$ denote the i -th component of j -th character of u and w . Let $X \subseteq (\mathbb{N}^n)^\omega$. The language of **#-binary coding of X** is the set of all #-binary codings of its elements.

Lemma 4.3. *Let $\mathcal{A} = (Q, \mathbb{N}^n, \Delta, I, F)$ be a general Büchi automaton over \mathbb{N}^n , possibly with infinitely many transitions, such that for every $s_1, s_2 \in Q$ the set*

$$\{u \in \{0, 1\}^* : (s_1, [u]_2, s_2) \in \Delta\}$$

is regular. Then the #-binary coding of the language accepted by \mathcal{A} is ω -regular.

Proof. We construct from \mathcal{A} a new Büchi automaton \mathcal{A}' over $(\Sigma_\#)^n$. It is constructed via the following procedure:

- (1) Copy the states (without their transitions) from \mathcal{A} to \mathcal{A}' . Any final states in \mathcal{A} are to remain final in \mathcal{A}' .
- (2) Add an initial state q_{start} , and endow it with transitions to every state that was an initial state in \mathcal{A} on the character $(\#, \dots, \#)$. These states are no longer initial in \mathcal{A}' , so that q_{start} is the only initial state.
- (3) For every pair $s_1, s_2 \in Q$:
 - (a) Let \mathcal{B} be a finite automaton recognizing

$$\{u \in \{0, 1\}^* : (s_1, [u]_2, s_2) \in \Delta\}.$$

Add the states and transitions of \mathcal{B} to \mathcal{A}' .

- (b) For every initial state t in \mathcal{B} , whenever t transitions to t' on a character, add a transition from s_1 to t' on the same character. Make t no longer an initial state in \mathcal{A}' .
- (c) For every final state t in \mathcal{B} , add a transition from t to s_2 on $(\#, \dots, \#)$. Make t no longer a final state in \mathcal{A}' .
- (d) If the empty word ϵ was accepted by \mathcal{B} , then add a transition from q to r on $(\#, \dots, \#)$.

One can check that the language accepted by \mathcal{A}' is the #-binary coding of the language accepted by \mathcal{A} . Indeed, if a word is accepted by \mathcal{A}' , it must begin with $\#^n$ and be followed by a sequence of binary codings that correspond to transitions in \mathcal{A} , delimited by $\#$, and visiting final states of \mathcal{A} infinitely often. \square

We will illustrate with an example. Figure 1 demonstrates the process of applying Lemma 4.3 to a simple automaton that accepts any infinite string of natural numbers containing at least one odd number.

We may now give the full proof of Theorem 4.1.

Proof of Theorem 4.1. In [BSS21, Section 2] the authors generate a general finite automaton \mathcal{A}_0 over the alphabet \mathbb{N}^4 such that a finite word $(d_1, x_1, y_1, z_1)(d_2, x_2, y_2, z_2) \cdots (d_m, x_m, y_m, z_m) \in (\mathbb{N}^4)^*$ is accepted by \mathcal{A}_0 if and only if there are $d_{m+1}, \dots \in \mathbb{N}$ and $x, y, z \in \mathbb{N}$ such that for $\alpha = [0; d_1, d_2, \dots]$ we have

$$\begin{aligned} x &= [x_1 x_2 \cdots x_m]_\alpha \\ y &= [y_1 y_2 \cdots y_m]_\alpha \\ z &= [z_1 z_2 \cdots z_m]_\alpha \\ z &= x + y. \end{aligned}$$

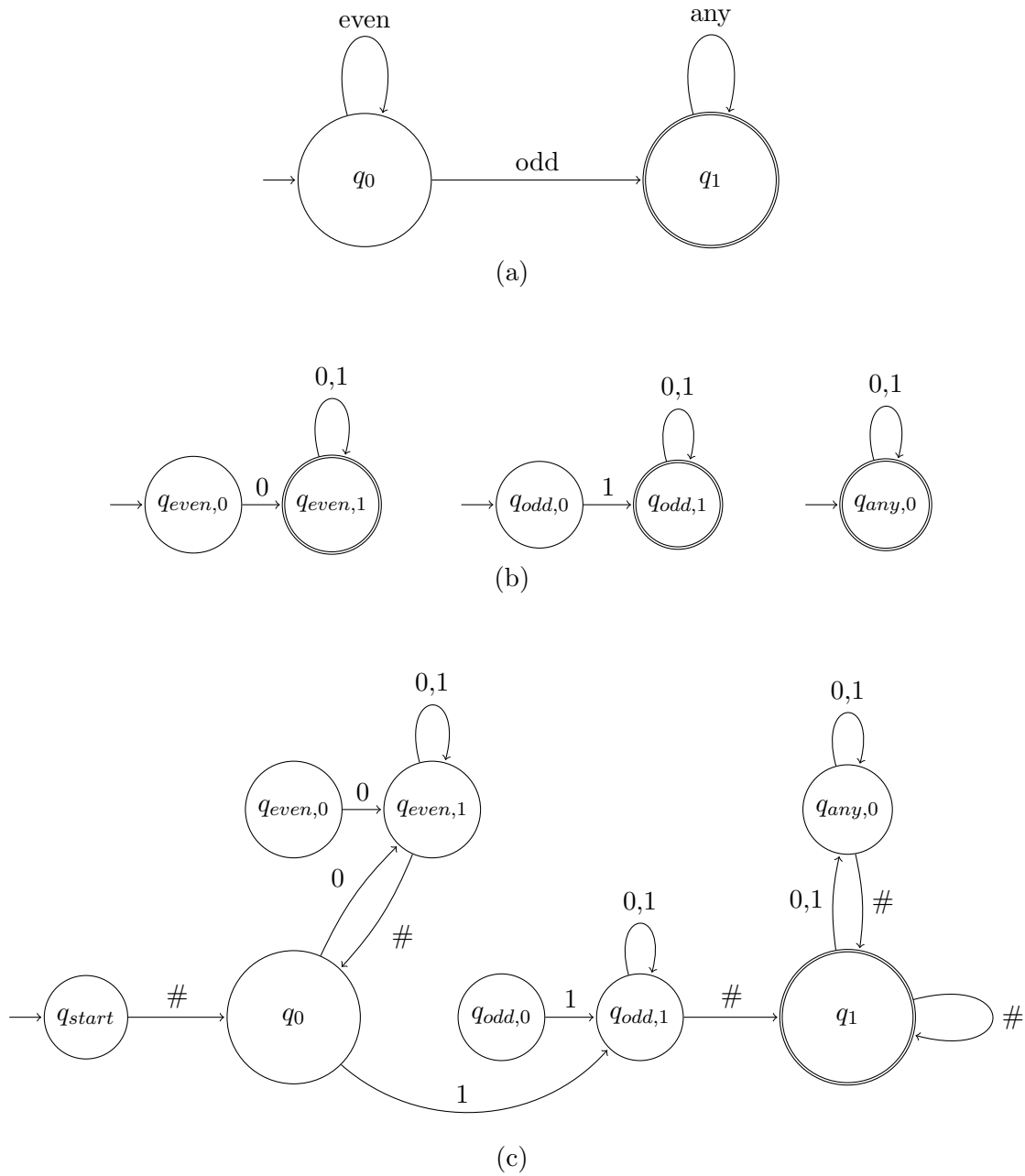


Figure 1: The procedure of Lemma 4.3. (a) The original automaton, with transitions for “any even number,” “any odd number,” and “any number.” (b) The finite automata recognizing these sets in binary encoding. (c) The combined automaton produced by Lemma 4.3.

Let \mathcal{A} be the general Büchi automaton with the same underlying quintuple as \mathcal{A}_0 . It follows immediately that if $(d_1, x_1, y_1, z_1)(d_2, x_2, y_2, z_2) \cdots \in (\mathbb{N}^4)^\omega$ is accepted by \mathcal{A} if and only if there is an infinite subset $U \subseteq \mathbb{N}$ such that for all $u \in \mathbb{N}$

$$[x_1 x_2 \cdots x_u]_\alpha + [y_1 y_2 \cdots y_u]_\alpha = [z_1 z_2 \cdots z_u]_\alpha$$

Each transition in \mathcal{A} corresponds to a linear equation with constant integer coefficients. As an example, one of the transitions in Figure 3 of [BSS21] is given as “ $-d_i + 1$,” meaning that it represents all cases where, letting $v_i, s_{1i}, s_{2i}, s_{3i}$ be the i th letter of v, s_1, s_2, s_3 respectively, we have $s_{3i} - s_{1i} - s_{2i} = -v_i + 1$. Note that the binary representation of the graph of addition and subtraction, as well as of the constant 1, are regular. Thus \mathcal{A} satisfies the conditions of Lemma 4.3. Let $X \subseteq (\Sigma_{\#}^4)^\omega$ be the $\#$ -binary coding of the language accepted by \mathcal{A} . By Lemma 4.3, we know that X is ω -regular. Observe that

$$\oplus^{\text{fin}} = \{(v, s_1, s_2, s_3) \in X : s_1, s_2, s_3 \in A_v^{\text{fin}}\}$$

and hence ω -regular. □

The automaton constructed above has 82 states⁵. Using our software Pecan, we can formally check that this automaton recognizes the set in Theorem 4.1. Following a strategy already used in Mousavi, Schaeffer, and Shallit [MSS16, Remark 2.1] we check that our adder satisfies the standard inductive definition of addition on the natural numbers; that is, for all $x, y \in \mathbb{N}$

$$\begin{aligned} 0 + y &= y \\ s(x) + y &= s(x + y) \end{aligned}$$

where $x, y \in \mathbb{N}$ and $s(x)$ denotes the successor of x in \mathbb{N} . The successor function on \mathbb{N} can be defined using only $<$ as follows:

$$s(x) = y \text{ if and only if } (x < y) \wedge (\forall z (z \leq x) \vee (z \geq y)).$$

Thus in Pecan we define `bco_succ(a,x,y)` as

```
bco_succ(a,x,y) := bco_valid(a,x) ^ bco_valid(a,y)
                  ^ bco_leq(x,y) ^ ~bco_eq(x,y)
                  ^ ~forall z. if bco_valid(a,z) then (bco_leq(z,x) v bco_leq(y,z))
```

where

- `bco_eq` recognizes $\{(x, y) : x = y\}$,
- `bco_leq` recognizes $\{(x, y) : x \leq_{\text{colex}} y\}$, and
- `bco_valid` recognizes A_{fin} .

We now confirm that our adder satisfies the above equations using the following Pecan code:

```
Let x,y,z be ostrowski(a).
Theorem ("Addition base case (0 + y = y).", {
  ~forall a. ~forall x,y,z. if bco_zero(x)
    then (bco_adder(a,x,y,z) iff bco_eq(y,z)) }).
Theorem ("Addition inductive case (s(x) + y = s(x + y)).", {
  ~forall a. ~forall x,y,z,u,v. if (bco_succ(a,u,x) ^ bco_succ(a,v,z))
    then (bco_adder(a,x,y,z) iff bco_adder(a,u,y,v)) }).
```

⁵Schmittenner [Sch23] constructs an Büchi automaton with just 24 states accepting the same language.

In the above code

- `bco_adder` recognizes \oplus^{fin} ,
- `bco_zero` recognizes $\mathbf{0}_*$, and
- `bco_succ` recognizes $\{(v, x, y) : x, y \in A_v^{\text{fin}}, Z_v(x) + 1 = Z_v(y)\}$.

Pecan confirms both statements are true. This proves Theorem 4.1 modulo correctness of Pecan and the correctness of the implementations of the automata for `bco_eq`, `bco_leq`, `bco_valid` and `bco_zero`. For more details about Pecan, see Section 7.

We need the following well-known consequence of König's Lemma (compare the proof of [BGS23, Lemma 4.3]).

Fact 4.4. Let \mathcal{A} be a Büchi automaton over Σ with all states accepting, let $w \in \Sigma^\omega$, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of words in Σ^ω such that $u_n|_n = w|_n$ for all $n \in \mathbb{N}$. If $u_n \in L(\mathcal{A})$ for every $n \in \mathbb{N}$, then $w \in L(\mathcal{A})$.

Using this result, we can extend the automaton in Theorem 4.1 to an automaton for addition modulo 1 on I_α .

Lemma 4.5. *The set*

$$\oplus := \{(v, s_1, s_2, s_3) : s_1, s_2, s_3 \in A_v \wedge O_v(s_1) + O_v(s_2) \equiv O_v(s_3) \pmod{1}\}$$

is ω -regular. Moreover, $\oplus^{\text{fin}} \subseteq \oplus$.

Proof. First, let v, s_1, s_2, s_3 be such that $s_1, s_2, s_3 \in A_v^{\text{fin}}$. We claim that on this domain, $(s_1, s_2, s_3) \in \oplus_v$ if and only if $(s_1, s_2, s_3) \in \oplus_v^{\text{fin}}$. By Fact 2.8 we know that for all $s \in A_v^{\text{fin}}$

$$\alpha(v)Z_v(s) - O_v(s) \equiv 0 \pmod{1}. \quad (4.1)$$

Let $(s_1, s_2, s_3) \in \oplus_v^{\text{fin}}$. Then by (4.1)

$$\begin{aligned} O_v(s_3) &\equiv \alpha(v)Z_v(s_3) \pmod{1} \\ &= \alpha(v)Z_v(s_1) + \alpha(v)Z_v(s_2) \\ &\equiv O_v(s_1) + O_v(s_2) \pmod{1}. \end{aligned}$$

Thus $(s_1, s_2, s_3) \in \oplus_v$.

Let \mathcal{B}^{fin} be a Büchi automaton recognizing \oplus^{fin} . Assume that \mathcal{B}^{fin} is trim. Let \mathcal{B}' be the automaton \mathcal{B}^{fin} , but with all states made accepting. Let S be the language accepted by \mathcal{B}' . We will show that $S_v \cap A_v^3 = \oplus_v$. Towards that goal, let $(v, s_1, s_2, s_3) \in (\Sigma_{\#}^\omega)^4$ be such that $(s_1, s_2, s_3) \in A_v^3$. It is left to prove that $(s_1, s_2, s_3) \in \oplus_v$ if and if $(s_1, s_2, s_3) \in S_v$.

Suppose first that $(s_1, s_2, s_3) \in \oplus_v$. Then

$$O_v(s_3) \equiv O_v(s_1) + O_v(s_2) \pmod{1}.$$

The reader can check using properties of Ostrowski representations that there is a sequence $(s_{m,1}, s_{m,2}, s_{m,3})_{m \in \mathbb{N}}$ of elements of $(A_v^{\text{fin}})^3$ such that

- (1) $O_v(s_{m,3}) \equiv O_v(s_{m,1}) + O_v(s_{m,2}) \pmod{1}$.
- (2) $s_{m,i}|_m = s_i|_m$ for $i \in \{1, 2, 3\}$; i.e., the first m letters of $s_{m,i}$ agree with the first m letters of s_i for $i \in \{1, 2, 3\}$.

By $(A_v^{\text{fin}})^3 \cap \oplus_v = \oplus_v^{\text{fin}}$ and (1), we know that $(v, s_{m,1}, s_{m,2}, s_{m,3})$ is accepted by \mathcal{B}^{fin} . By Fact 4.4 and (2), we deduce that \mathcal{B}' accepts (v, s_1, s_2, s_3) . Thus $(s_1, s_2, s_3) \in S_v$.

Suppose now that $(s_1, s_2, s_3) \in S_v$. Then (v, s_1, s_2, s_3) is accepted by \mathcal{B}' . For $m \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, let $w_{m,i} \in \Sigma_{\#}^*$ be such that $w_{m,i}$ is s_i up through the $(m+1)$ -st occurrence of $\#$. Thus $w_{m,i}$ represents the first m -th Ostrowski coefficients of $O_v(s_i)$. Since \mathcal{B}^{fin} is trim, there exist infinite extensions $s_{m,1}, s_{m,2}, s_{m,3} \in \Sigma_{\#}^{\omega}$ of $w_{m,1}, w_{m,2}, w_{m,3}$ such that \mathcal{B}^{fin} accepts $(v, s_{m,1}, s_{m,2}, s_{m,3})$. We now set

$$(x_m, y_m, z_m) := (O_v(s_{m,1}), O_v(s_{m,2}), O_v(s_{m,3})), \quad (x, y, z) := O_v(s_1, s_2, s_3).$$

It follows from Fact 2.9 that

$$\lim_{m \rightarrow \infty} (x_m, y_m, z_m) = (x, y, z).$$

Because $x_m + y_m \equiv z_m \pmod{1}$ for every $m \in \mathbb{N}$ (by definition of \mathcal{B}^{fin} and $(A_v^{\text{fin}})^3 \cap \oplus_v = \oplus_v^{\text{fin}}$), we have $x + y \equiv z \pmod{1}$. Hence $(s_1, s_2, s_3) \in \oplus_v$. \square

5. THE UNIFORM ω -REGULARITY OF \mathcal{R}_{α}

In this section, we turn to the question of the decidability of the logical first-order theory of \mathcal{R}_{α} . Recall that $\mathcal{R}_{\alpha} := (\mathbb{R}, <, +, \mathbb{Z}, \alpha\mathbb{Z})$ for $\alpha \in \mathbb{R}$. The main result of this section is the following:

Theorem 5.1. *There is a uniform family of ω -regular structures $(\mathcal{D}_v)_{v \in R}$ such that $\mathcal{D}_v \simeq \mathcal{R}_{\alpha(v)}$ for each $v \in R$.*

Theorem 5.1 then hinges on the following lemma.

Lemma 5.2. *There is a uniform family of ω -regular structures $(\mathcal{C}_a)_{a \in R}$ such that for each $a \in R$*

$$\mathcal{C}_a \simeq ([-\alpha(a), \infty), <, +, \mathbb{N}, \alpha(a)\mathbb{N}).$$

Proof of Theorem 5.1. Let $(\mathcal{C}_a)_{a \in R}$ be an uniform family of ω -regular structures as given by Lemma 5.2. Within \mathcal{C}_a , define the set $L = \{x \in [-\alpha(a), \infty) : x \geq 0\}$, where 0 is the $<$ -least element of \mathbb{N} . This is an ordered commutative monoid. Let L' be its Grothendieck group, and let $+', <'$ be the induced abelian group operation and ordering. There is a canonical inclusion map $\iota : L \hookrightarrow L'$. Let $Z' = \iota(\mathbb{N}) \cup -\iota(\mathbb{N})$ and $A' = \iota(\alpha(a)\mathbb{N}) \cup -\iota(\alpha(a)\mathbb{N})$. Observe that $(L', <', +', N', A')$ is an isomorphic copy of $\mathcal{R}_{\alpha(a)}$, defined in \mathcal{C}_a in a manner uniform in a . So let \mathcal{D}_a be this structure and conclude that $(\mathcal{D}_a)_a$ is a uniform family of ω -regular structures. \square

The proof of Lemma 5.2 itself is a uniform version of the argument given in [Hie16] that also fixes some minor errors of the original proof. By Lemma 3.16 and Theorem 4.1, we already know that

$$Z_v : (A_v^{\text{fin}}, \prec_v^{\text{fin}}, \oplus_v^{\text{fin}}) \rightarrow (\mathbb{N}, <, +)$$

is an isomorphism for every $v \in R$. As our eventual goal also requires us to define the set $\alpha\mathbb{N}$, it turns out to be much more natural to instead use the isomorphism

$$\alpha(v)Z_v : (A_v^{\text{fin}}, \prec_v^{\text{fin}}, \oplus_v^{\text{fin}}) \rightarrow (\alpha(v)\mathbb{N}, <, +)$$

and recover \mathbb{N} (and further \mathbb{Z}). We do so by following (and correcting) the argument in [Hie16].

Lemma 5.3. *Let $v \in R$, and let $t_1, t_2, t_3 \in A_v$ be such that $t_1 \oplus_v t_2 = t_3$. Then*

$$O_v(t_1) + O_v(t_2) = \begin{cases} O_v(t_3) + 1 & \text{if } \mathbf{0}_v \prec_v t_1 \text{ and } t_3 \prec_v t_2; \\ O_v(t_3) - 1 & \text{if } t_1 \prec_v \mathbf{0}_v \text{ and } t_2 \prec_v t_3; \\ O_v(t_3) & \text{otherwise.} \end{cases}$$

Proof. For ease of notation, let $\alpha = \alpha(v)$, and set $x_i = O_v(t_i)$ for $i = 1, 2, 3$. By definition of \oplus_v , we have that $x_1, x_2, x_3 \in I_{\alpha(v)}$ with $x_1 + x_2 \equiv x_3 \pmod{1}$. Note that $t_i \prec_v t_j$ if and only if $x_i < x_j$.

We first consider the case that $0 < x_1$ and $x_3 < x_2$. Thus $x_1 + x_2 > 1 - \alpha$. Note that

$$-\alpha = 1 - \alpha - 1 < x_1 + x_2 - 1 < (1 - \alpha) + (1 - \alpha) - 1 = 1 - 2\alpha < 1 - \alpha.$$

Thus $x_1 + x_2 - 1 \in I_\alpha$ and $x_3 = x_1 + x_2 - 1$.

Now assume that $x_1 < 0$ and $x_2 < x_3$. Then $x_1 + x_2 < -\alpha$, and therefore

$$1 - \alpha > x_1 + x_2 + 1 \geq (-\alpha) + (-\alpha) + 1 = (1 - \alpha) - \alpha > -\alpha.$$

Thus $x_1 + x_2 + 1 \in I_\alpha$ and hence $x_3 = x_1 + x_2 + 1$.

Finally consider that $0, x_1$ are ordered the same way as x_2, x_3 . Since $x_1 + x_2 \equiv x_3 \pmod{1}$, we know that $|x_1 - 0|$ and $|x_3 - x_2|$ differ by an integer k . If $k > 0$, would imply that one of these differences is at least 1, which is impossible within the interval I_α . Therefore $x_1 - 0 = x_3 - x_2$ and hence $x_3 = x_1 + x_2$. \square

For $i \in \mathbb{N}$, set $\mathbf{i}_v := \underbrace{\mathbf{1}_v \oplus \cdots \oplus \mathbf{1}_v}_{i \text{ times}}$.

Lemma 5.4. *The set $F := \{(v, s) \in A^{\text{fin}} : Z_v(s)\alpha(v) < 1\}$ is ω -regular, and for each $(v, s) \in F$*

$$O_v(s) = \begin{cases} \alpha(v)Z_v(s) & \text{if } (\alpha(v) + 1)Z_v(s) < 1; \\ \alpha(v)Z_v(s) - 1 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.8, we can first consider the case that $\alpha(v) > \frac{1}{2}$. In this situation, F_v is just the set $\{\mathbf{0}_v, \mathbf{1}_v\}$, and hence obviously ω -regular.

Now assume that $\alpha(v) < \frac{1}{2}$. Let w be the \prec_v^{fin} -minimal element of A_v^{fin} with $w \prec_v \mathbf{0}_v$. We will show that

$$F_v = \{s \in A_v^{\text{fin}} : s \preceq_v^{\text{fin}} w\}.$$

Then ω -regularity of F follows then immediately.

Let $n \in \mathbb{N}$ be maximal such that $n\alpha(v) < 1$. It is enough to show that $Z_v(w) = n$. By Lemma 3.15, $O_v(\mathbf{1}_v) = \alpha(v)$. Hence $1\alpha(v), 2\alpha(v), \dots, (n-1)\alpha(v) \in I_{\alpha(v)}$, but $n\alpha(v) > 1 - \alpha(v)$. Then for $i = 1, \dots, n-1$

$$O_v(\mathbf{i}_v) = i\alpha(v), \quad O_v(\mathbf{n}_v) = n\alpha(v) - 1 < 0.$$

So $\mathbf{i}_v \succeq \mathbf{0}_v$ for $i = 1, \dots, n$, but $\mathbf{n}_v \prec \mathbf{0}_v$. Thus $\mathbf{n}_v = w$ and $Z_v(w) = n$. \square

Name	Definition
A	$\{(v, w) : v \in R, w \text{ is a } \#-v\text{-Ostrowski representation}\}$
A^{fin}	$\{(v, w) : v \in R, w \text{ is a } \#-v\text{-Ostrowski representation and eventually } 0\}$
B	$\{(v, s) \in A^{\text{fin}} : s \preceq_v \mathbf{0}_v\}$
C	$\{(v, s, t) : (v, s) \in B \wedge (v, t) \in A\}$

Table 1: Definitions of sets used in the proof

Lemma 5.5. *Let $v \in R$ and $t \in A_v^{\text{fin}}$. Then there is an $s \in F_v$ and $t' \in A_v^{\text{fin}}$ such that $t' \preceq_v 0$ and $t = t' \oplus_v s$. In particular,*

$$A_v^{\text{fin}} = \{t \in A_v^{\text{fin}} : t \preceq_v \mathbf{0}_v\} \oplus_v F_v.$$

Proof. Let $n \in \mathbb{N}$ be maximal such that $n\alpha(v) < 1$. Let $t \in A_v^{\text{fin}}$. We need to find $s \in A_v^{\text{fin}}$ and $u \in F_v$ such that $t = s \oplus_v^{\text{fin}} u$. We can easily reduce to the case that $t \succ \mathbf{0}_v$ and $Z_v(t) > n$.

Let $i \in \{0, \dots, n\}$ be such that $0 \geq O_v(t) - i\alpha(v) > -\alpha(v)$. Then let $s \in A_v^{\text{fin}}$ be such that $Z_v(s) = Z_v(t) - i$. Note $t = s \oplus_v^{\text{fin}} \mathbf{i}_v$. Thus we only need to show that $s \preceq \mathbf{0}_v$.

To see this, observe that by Lemma 5.4

$$O_v(s) + \alpha(v)i \equiv O_v(s) + O_v(\mathbf{i}_v) \equiv O_v(t) \pmod{1}.$$

Since $O_v(t) - i\alpha(v) \in I_{\alpha(v)}$, we know that $O_v(s) = O_v(t) - i\alpha(v) \leq 0$.

Therefore $O_v(s) \preceq \mathbf{0}_v$. □

Proof of Lemma 5.2. Define $B \subseteq A^{\text{fin}}$ to be $\{(v, s) \in A^{\text{fin}} : s \preceq_v \mathbf{0}_v\}$. Clearly, B is ω -regular. We now define \prec^B and \oplus^B such that for each $v \in R$, the structure $(B_v, \prec_v^B, \oplus_v^B)$ is isomorphic to $(\mathbb{N}, <, +)$ under the map g_v defined as $g_v(s) = \alpha(v)Z_v(s) - O_v(s)$.

We define \prec^B to be the restriction of \prec^{fin} to B . That is, for $(v, s_1), (v, s_2) \in B$ we have

$$(v, s_1) \prec^B (v, s_2) \text{ if and only if } (v, s_1) \prec^{\text{fin}} (v, s_2).$$

It is immediate that \prec^B is ω -regular, since both B and \prec^{fin} are ω -regular.

We define \oplus^B as follows:

$$(v, s_1) \oplus^B (v, s_2) = \begin{cases} (v, s_1 \oplus_v s_2) & \text{if } s_1 \oplus_v^{\text{fin}} s_2 \preceq_v \mathbf{0}_v; \\ (v, s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) & \text{otherwise.} \end{cases}$$

We now show that $g_v(s_1 \oplus_v^B s_2) = g_v(s_1) + g_v(s_2)$ for every $s_1, s_2 \in B_v$.

Let $(v, s_1), (v, s_2) \in B$. We first consider the case that $s_1 \oplus_v s_2 \preceq_v \mathbf{0}_v$. By Lemma 5.3, $O_v(s_1 \oplus_v s_2) = O_v(s_1) + O_v(s_2)$. Thus

$$\begin{aligned} g_v(s_1 \oplus_v^B s_2) &= g_v(s_1 \oplus_v s_2) \\ &= \alpha(v)Z_v(s_1 \oplus_v s_2) - O_v(s_1 \oplus_v s_2) \\ &= \alpha Z_v(s_1) + \alpha Z_v(s_2) - O_v(s_1) - O_v(s_2) \\ &= g_v(s_1) + g_v(s_2). \end{aligned}$$

Map	Domain	Codomain
α	R	\mathbf{Irr}
O_v	A_v	$I_{\alpha(v)}$
Z_v	A_v^{fin}	\mathbb{N}
$g_v := \alpha(v)Z_v - O_v$	B_v	\mathbb{N}
$T_v := g_v + O_v$	C_v	$[-\alpha(v), \infty) \subseteq \mathbb{R}$

Table 2: A list of the maps and their domains and codomains.

Now suppose that $s_1 \oplus_v s_2 \succ_v \mathbf{0}_v$. Since $-\alpha(v) \leq O_v(s_1), O_v(s_2) \leq 0$, we get that

$$1 - \alpha(v) > O_v(s_1) + O_v(s_2) + \alpha(v) \geq -\alpha(v).$$

Thus by Lemma 3.15,

$$O_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) = O_v(s_1) + O_v(s_2) + \alpha(v).$$

We obtain

$$\begin{aligned} g_v(s_1 \oplus_v^B s_2) &= g_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) \\ &= \alpha Z_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) - O_v(s_1 \oplus_v s_2 \oplus_v \mathbf{1}_v) \\ &= \alpha(v)(Z_v(s_1) + Z_v(s_2)) + \alpha(v) - O_v(s_1) - O_v(s_2) - \alpha(v) \\ &= g_v(s_1) + g_v(s_2). \end{aligned}$$

Since $s_1 \prec_v s_2$ if and only if $Z_v(s_1) < Z_v(s_2)$, we get that g_v is an isomorphism between $(B_v, \prec_v^B, \oplus_v^B)$ and $(\mathbb{N}, <, +)$.

Let C be defined by

$$\{(v, s, t) \in (\Sigma_{\#}^{\omega})^3 : (v, s) \in B \wedge (v, t) \in A\}.$$

Clearly C is ω -regular. Let $T_v : C_v \rightarrow [-\alpha(v), \infty) \subseteq \mathbb{R}$ map $(s, t) \mapsto g_v(s) + O_v(t)$.

Note that T_v is bijective for each $v \in R$, since every real number decomposes uniquely into a sum $n + y$, where $n \in \mathbb{Z}$ and $y \in I_v$.

We define an ordering \prec_v^C on C_v lexicographically: $(s_1, t_1) \prec_v^C (s_2, t_2)$ if either

- $s_1 \prec_v^B s_2$, or
- $s_1 = s_2$ and $t_1 \prec_v t_2$.

The set

$$\{(v, s_1, t_1, s_2, t_2) : (s_1, t_1), (s_2, t_2) \in C_v \wedge (s_1, t_1) \prec_v^C (s_2, t_2)\}$$

is ω -regular. We can easily check that $(s_1, t_1) \prec_v^C (s_2, t_2)$ if and only if $T_v(s_1, t_1) < T_v(s_2, t_2)$.

Let $\mathbf{0}^B$ be $g_v^{-1}(0)$ and $\mathbf{1}^B$ be $g_v^{-1}(1)$. Let \ominus^B be the (partial) inverse of \oplus^B . We define \oplus^C for $(s_1, t_1), (s_2, t_2) \in C$ as follows:

$$(s_1, t_1) \oplus_v^C (s_2, t_2) = \begin{cases} (s_1 \oplus_v^B s_2 \ominus^B \mathbf{1}^B, t_1 \oplus_v t_2) & \text{if } t_1 \prec \mathbf{0}_v \wedge t_2 \prec_v t_1 \oplus_v t_2; \\ (s_1 \oplus_v^B s_2 \oplus_v^B \mathbf{1}^B, t_1 \oplus_v t_2) & \text{if } \mathbf{0}_v \prec t_1 \wedge t_1 \oplus_v t_2 \prec_v t_2; \\ (s_1 \oplus_v^B s_2, t_1 \oplus_v t_2) & \text{otherwise.} \end{cases}$$

(Note that \oplus^C is only a partial function, as the case where $s_1 = s_2 = \mathbf{0}^B$ and $t_1 \prec \mathbf{0}_v \wedge t_2 \prec_v t_1 \oplus_v t_2$ is outside of the domain of \oplus^B .) It is easy to check that \oplus^C is ω -regular. It follows directly from Lemma 5.3 that

$$T_v((s_1, t_1) \oplus_v^C (s_2, t_2)) = T_v((s_1, t_1)) + T_v((s_2, t_2)).$$

Thus for each $v \in R$, the function T_v is an isomorphism between $(C_v, \prec_v^C, \oplus_v^C)$ and $([-\alpha(v), \infty), <, +)$. To finish the proof, it is left to establish the ω -regularity of the following two sets:

- (1) $\{(v, s, t) \in C : T_v(s, t) \in \mathbb{N}\}$,
- (2) $\{(v, s, t) \in C : T_v(s, t) \in \alpha(v)\mathbb{N}\}$.

For (1), observe that the set $T_v^{-1}(\mathbb{N})$ is just the set $\{(s, t) \in C_v : t = \mathbf{0}_v\}$.

For (2), consider the following two sets:

- $U_1 = \{(v, s, t) \in C : s = t\}$,
- $U_2 = \{(v, \mathbf{0}_v, t) \in C : t \in F_v\}$.

Let $\mathbf{1}_v^C$ be $T_v^{-1}(1)$. Set

$$\begin{aligned} U := & \{(v, (s_1, t_1) \oplus_v^C (\mathbf{0}_v, t_2)) : (v, s_1, t_1) \in U_1, (v, \mathbf{0}_v, t_2) \in U_2, t_2 \succeq 0\} \\ & \cup \{(v, (s_1, t_1) \oplus_v^C (\mathbf{0}_v, t_2) \oplus \mathbf{1}_v^C) : (v, s_1, t_1) \in U_1, (v, \mathbf{0}_v, t_2) \in U_2, t_2 \prec 0\}. \end{aligned}$$

The set U is clearly ω -regular, since both U_1 and U_2 are ω -regular. We now show that $T_v(U) = \alpha(v)\mathbb{N}$.

Let $(v, s, s) \in U_1$ and $(v, \mathbf{0}_v, t) \in U_2$. If $t \succeq \mathbf{0}_v$, then by Lemma 5.4

$$\begin{aligned} T_v((s, s) \oplus_C (\mathbf{0}_v, t)) &= T_v(s, s) + T_v(\mathbf{0}_v, t) \\ &= \alpha(v)Z_v(s) - O_v(s) + O_v(s) + O_v(t) \\ &= \alpha(v)Z_v(s) + \alpha(v)Z_v(t) = \alpha(v)Z_v(s \oplus_v t). \end{aligned}$$

If $t \prec \mathbf{0}_v$, then by Lemma 5.4

$$\begin{aligned} T_v((s, s) \oplus_v^C (\mathbf{0}_v, t) \oplus_v^C \mathbf{1}_v^C) &= T_v(s, s) + T_v(\mathbf{0}_v, t) + 1 \\ &= \alpha(v)Z_v(s) - O_v(s) + O_v(s) + O_v(t) + 1 \\ &= \alpha(v)Z_v(s) + \alpha(v)Z_v(t) = \alpha(v)Z_v(s \oplus_v t). \end{aligned}$$

Thus $T_v(U) \subseteq \alpha(v)\mathbb{N}$. By Lemma 5.5, $T_v(U) = \alpha(v)\mathbb{N}$. □

6. DECIDABILITY RESULTS

We are now ready to prove the results listed in the introduction. We first recall some notation. Let \mathcal{L}_m be the signature of the first-order structure $(\mathbb{R}, <, +, \mathbb{Z})$, and let $\mathcal{L}_{m,a}$ be the extension of \mathcal{L}_m by a unary predicate. For $\alpha \in \mathbb{R}_{>0}$, let \mathcal{R}_α denote the $\mathcal{L}_{m,a}$ -structure $(\mathbb{R}, <, +, \mathbb{Z}, \alpha\mathbb{Z})$. For each $\mathcal{L}_{m,a}$ -sentence φ , we set

$$R_\varphi := \{v \in R : \mathcal{R}_{\alpha(v)} \models \varphi\}.$$

Theorem 6.1. *Let φ be an $\mathcal{L}_{m,a}$ -sentence. Then R_φ is ω -regular.*

Proof. By Theorem 5.1 there is a uniform family of ω -regular structures $(\mathcal{D}_v)_{v \in R}$ such that such that $\mathcal{D}_v \simeq \mathcal{R}_{\alpha(v)}$ for each $v \in R$. Then $R_\varphi = \{v \in R : \mathcal{D}_v \models \varphi\}$. This set is ω -regular by Fact 2.3. \square

Let $\mathcal{N} = (R; (R_\varphi)_\varphi, (X)_{X \subseteq R^n} \text{ } \omega\text{-regular})$ be the relational structure on R with the relations R_φ for every \mathcal{L} -sentences φ and $X \subseteq R^n$ ω -regular. Because \mathcal{N} is an ω -regular structure, we obtain the following decidability result.

Corollary 6.2. *The theory $\text{FO}(\mathcal{N})$ is decidable.*

We now proceed towards the proof of Theorem C. Recall that $\mathbf{Irr} := (0, 1) \setminus \mathbb{Q}$.

Definition 6.3. Let $X \subseteq \mathbf{Irr}^n$. Let X_R be defined by

$$X_R := \{(v_1, \dots, v_n) \in R^n : v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n \wedge (\alpha(v_1), \dots, \alpha(v_n)) \in X\}$$

We say X is **recognizable modulo $\sim_{\#}$** if X_R is ω -regular.

Lemma 6.4. *The collection of sets recognizable modulo $\sim_{\#}$ is closed under Boolean operations and coordinate projections.*

Proof. Let $X, Y \subseteq \mathbf{Irr}$ be recognizable modulo $\sim_{\#}$. It is clear that $(X \cap Y)_R = X_R \cap Y_R$. Thus $X \cap Y$ is recognizable modulo $\sim_{\#}$. Let X^c be $\mathbf{Irr}^n \setminus X$, the complement of X . For ease of notation, set $E := \{(v_1, \dots, v_n) \in R^n : v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n\}$. Then

$$\begin{aligned} (X^c)_R &= \{(v_1, \dots, v_n) \in R^n : v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n \wedge (\alpha(v_1), \dots, \alpha(v_n)) \notin X\} \\ &= E \cap \{(v_1, \dots, v_n) \in R^n : (\alpha(v_1), \dots, \alpha(v_n)) \notin X\} \\ &= E \cap \{(v_1, \dots, v_n) \in R^n : (\alpha(v_1), \dots, \alpha(v_n)) \notin X \vee \neg(v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n)\} \\ &= E \cap (R^n \setminus X_R). \end{aligned}$$

This set is ω -regular, and hence X^c is recognizable modulo $\sim_{\#}$.

For coordinate projections, it is enough to consider projections onto the first $n-1$ coordinates. Let $n > 0$ and let π be the coordinate projection onto first $n-1$ coordinates. Observe that

$$\pi(X) = \{(\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{R}^{n-1} : \exists \alpha_n \in \mathbb{R} (\alpha_1, \dots, \alpha_{n-1}, \alpha_n) \in X\}.$$

Thus $\pi(X)_R$ is equal to

$$\{(v_1, \dots, v_{n-1}) \in R^{n-1} : v_1 \sim_{\#} \dots \sim_{\#} v_{n-1} \wedge \exists \alpha_n : (\alpha(v_1), \dots, \alpha(v_{n-1}), \alpha_n) \in X\}.$$

Note that $v \mapsto \alpha(v)$ is a surjection $R \rightarrow (0, 1) \setminus \mathbb{Q}$. Thus $\pi(X)_R$ is also equal to:

$$\{(v_1, \dots, v_{n-1}) \in R^{n-1} : v_1 \sim_{\#} \dots \sim_{\#} v_{n-1} \wedge \exists v_n : (\alpha(v_1), \dots, \alpha(v_n)) \in X\}.$$

Unfortunately, this set is not necessarily equal to $\pi(X_R)$. There might be tuples (v_1, \dots, v_{n-1}) such that no v_n can be found, because it would require more bits in one of its coefficients than v_1, \dots, v_{n-1} have for that coefficient. But $\pi(X_R)$ always contains *some* representation of $\alpha(v_1), \dots, \alpha(v_{n-1})$ with the appropriate number of digits. We need only ensure that removal of trailing zeroes does not affect membership in the language. Thus $\pi(X)_R$ is just the zero-closure of $\pi(X_R)$. Thus $\pi(X)_R$ is ω -regular by Lemma 3.5. \square

Theorem 6.5. *Let X_1, \dots, X_n be recognizable modulo $\sim_{\#}$ by Büchi automata $\mathcal{A}_1, \dots, \mathcal{A}_n$, and let \mathcal{Q} be the structure $(\mathbf{Irr}; X_1, \dots, X_n)$. Then the theory of \mathcal{Q} is decidable.*

Proof. By Lemma 6.4 every set definable in \mathcal{Q} is recognizable modulo $\sim_{\#}$. Moreover, for each definable set Y the automaton that recognizes Y modulo $\sim_{\#}$, can be computed from the automata $\mathcal{A}_1, \dots, \mathcal{A}_n$. Let ψ be a sentence in the signature of \mathcal{Q} . Without loss of generality, we can assume that ψ is of the form $\exists x \chi(x)$. Set

$$Z := \{a \in \mathbf{Irr}^n : \mathcal{Q} \models \chi(a)\}.$$

Observe that $\mathcal{Q} \models \psi$ if and only if Z is non-empty. Note for every $a \in \mathbf{Irr}^n$ there are $v_1, \dots, v_n \in R$ such that $v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n$ and $(\alpha(v_1), \dots, \alpha(v_n)) = a$. Thus Z is non-empty if and only if

$$\{(v_1, \dots, v_n) \in R^n : v_1 \sim_{\#} v_2 \sim_{\#} \dots \sim_{\#} v_n \wedge (\alpha(v_1), \dots, \alpha(v_n)) \in Z\}$$

is non-empty. Thus to decide whether $\mathcal{Q} \models \psi$, we first compute the automaton \mathcal{B} that recognizes Z modulo $\sim_{\#}$, and then check whether the automaton accepts any word. \square

We are now ready to prove Theorem C; that is, decidability of the theory of the structure

$$\mathcal{M} = (\mathbf{Irr}, <, (M_{\varphi})_{\varphi}, (q)_{q \in \mathbf{Irr}_{quad}}),$$

where M_{φ} is defined for each $\mathcal{L}_{m,a}$ -formula as

$$M_{\varphi} := \{\alpha \in \mathbf{Irr} : \mathcal{R}_{\alpha} \models \varphi\}.$$

Proof of Theorem C. We just need to check that the relations we are adding are all recognizable modulo $\sim_{\#}$. By Lemma 3.6 the ordering $<$ is recognizable modulo $\sim_{\#}$. By Lemma 3.7, the singleton $\{q\}$ is recognizable modulo $\sim_{\#}$ for every $q \in \mathbf{Irr}_{quad}$. Since $M_{\varphi} = \alpha(R_{\varphi})$, recognizability of M_{φ} modulo $\sim_{\#}$ follows from Theorem 6.1. \square

We can add to \mathcal{M} a predicate for every subset of \mathbf{Irr}^n that is recognizable modulo $\sim_{\#}$, and preserve the decidability of the theory. The reader can check that examples of subsets of \mathbf{Irr} recognizable modulo $\sim_{\#}$ are the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are powers of 2, the set of all $\alpha \in \mathbf{Irr}$ such that the terms in the continued fraction expansion of α are in (or are not in) some fixed finite set, and the set of all $\alpha \in \mathbf{Irr}$ such that all even (or odd) terms in their continued fraction expansion are 1.

7. AUTOMATICALLY PROVING THEOREMS ABOUT STURMIAN WORDS

We have created an automatic theorem-prover based on the ideas and the decision algorithms outlined above, called Pecan [OMSH20], available at

<https://github.com/ReedOei/Pecan>

We use Pecan to provide proofs of known and unknown results about characteristic Sturmian words. The Pecan code for the following examples is available at

<https://github.com/ReedOei/SturmianWords>

We quote some of this code throughout this section. These code snippets should be understandable without further explanation, but interested readers can find more information and explanations in [OMSH21]. We recommend downloading the code instead of copying from this paper. In addition to the size of the automata created by Pecan, we sometimes state the runtime of Pecan on a normal laptop to indicate how quickly these statements have been proved.

7.1. Classical theorems. We begin by giving automated proofs for several classical result about Sturmian words. We refer the reader to [Lot02] for more information and traditional proofs of these results.

In the following, we assume that a is irrational and i, j, k, n, m, p, s are a -Ostrowski representations. This can be expressed in Pecan as

```
Let a is bco_standard.
Let i,j,k,n,m,p,s are ostrowski(a).
```

Here `bco_standard` is a data type for real numbers encoded using #-binary coding. Then `ostrowski(a)` determines the Ostrowski numeration system used for the variables `i, j, k, n, m, p` and `s`. Pecan allows the use of Unicode characters such as \exists , \forall , \neg and \wedge , and we will use these here for readability. Of course, Pecan also supports writing `exists`, `forall`, `!` and `and` for the same operations. We write $c_{a,0}(i)$ as `$C[i]` in Pecan.

Let w^R denote the reversal of a word w . We say a word w is a **palindrome** if $w = w^R$.

Theorem 7.1. *Characteristic Sturmian words are balanced and aperiodic.*

Proof. To show that a characteristic Sturmian word $c_{a,0}$ is balanced, it is sufficient to show that there is no palindrome w in $c_{a,0}$ such that $0w0$ and $1w1$ are in $c_{a,0}$ (see [Lot02, Proposition 2.1.3]). We encode this in Pecan as follows. The predicate `palindrome(a, i, n)` is true when $c_{a,0}[i..i+n] = c_{a,0}[i..i+n]^R$. The predicate `factor_len(a, i, n, j)` is true when $c_{a,0}[i..i+n] = c_{a,0}[j..j+n]$. Then Pecan takes 321.73 seconds to prove the following theorem:

```
Theorem ("Balanced", {
  ∀a. ¬(∃i, n. palindrome(a, i, n) ∧
      (∃j. factor_len(a, i, n, j) ∧ $C[j-1] = 0 ∧ $C[j+n] = 0) ∧
      (∃k. factor_len(a, i, n, k) ∧ $C[k-1] = 1 ∧ $C[k+n] = 1))
}).
```

Encoding the property that a word is eventually periodic is straightforward:

```
eventually_periodic(a, p) :=
  p > 0 ∧ ∃n. ∀i. if i > n then $C[i] = $C[i+p]
```

The resulting automaton has 4941 states and 35776 edges, and takes 117.78 seconds to build. We then state the theorem in Pecan, which confirms the theorem is true.

```
Theorem ("Aperiodic", {
  ∀a. ∀p. if p > 0 then ¬eventually_periodic(a, p)
}).
```

□

A word w is a **factor** of a word u if there exist words v_1, v_2 such that $u = v_1 w v_2$. A factor w of a word u **right special** if both $w0$ and $w1$ are also factors of u .

Theorem 7.2. *For each natural number n , $c_{a,0}$ contains a unique right special factor of length n , and this factor is $c_{a,0}[1..n+1]^R$.*

Proof. We first define right special factors, as above. Recall that `factor_len(a, i, n, j)` checks that $c_{a,0}[i..i+n] = c_{a,0}[j..j+n]$.

```

right_special_factor(a,i,n) :=
  (∃j. factor_len(a,i,n,j) ∧ $C[j+n] = 0) ∧
  (∃k. factor_len(a,i,n,k) ∧ $C[k+n] = 1)

```

We then define the **first** right special factor, which is the first occurrence (by index) of the right special factor in the word $c_{a,0}$. This step is purely to reduce the cost of checking the theorem: the `right_special_factor` automaton has 3375 states, but `first_right_special_factor` has only 112.

```

first_right_special_factor(a,i,n) := special_factor(a,i,n)
  ∧ ∀j. if (j>0 ∧ factor_len(a,j,n,i)) then i<=j

```

We then check that each of these right special factors is equal to $c_{a,0}[1..n+1]^R$, which also proves the uniqueness. The predicate `reverse_factor(a,i,j,l)` checks that $c_{a,0}[i..j] = c_{a,0}[k+1..l+1]^R$, where $j-i = l-k$. Then Pecan confirms:

```

Theorem ("The unique special factor of length n is C[1..n+1]^R", {
  ∀a. ∀i,n.
    if i > 0 ∧ first_right_special_factor(a,i,n) then
      reverse_factor(a,i,i+n,n)
}).

```

□

Another characterization of Sturmian words due to Droubay and Pirillo [DP99, Theorem 5] is that a word is Sturmian if and only if it contains exactly one palindrome of length n if n is even, and exactly two palindromes of length n if n is odd. We prove the forward direction below.

Theorem 7.3 [DP99, Proposition 6]. *For every $n \in \mathbb{N}$, $c_{a,0}$ contains exactly one palindrome of length n if n is even, and exactly two palindromes of length n if n is odd.*

Proof. We begin by defining a predicate defining the location of the first occurrence of each length n palindrome in $c_{a,0}$.

```

first_palindrome(a, i, n) := palindrome(a, i, n) ∧
  ∀j. if j > 0 ∧ factor_len(a,j,n,i) then i <= j

```

The resulting automaton has 247 states and 1281 edges. The following states the theorem, and Pecan proves it in 428.85 seconds.

```

Theorem ("", {
  ∀a. ∀n. (
    if even(n) ∧ n > 0 then
      ∃i. ∀k. first_palindrome(a,k,n) iff i = k ) ∧ (
    if odd(n) then
      ∃i,j. i < j ∧ ∀k. first_palindrome(a,k,n) iff (i = k ∨ j = k)
  )}).

```

□

7.2. Powers. Next, we prove the follow results about powers of Sturmian words. A finite nonempty subword x of a (finite or ω) word w is a n -**th power** if $x = y^n$ for some finite word y . We call a 2nd power a **square**, and a 3rd power a **cube**.

Using Pecan, we construct an automaton recognizing the following property, stating that there is a square of length n starting at $c_{a,0}(i)$:

```
square(a, i, n) := n > 0 ∧ i > 0
                ∧ ∀j. i ≤ j ∧ j < i + n ∧ $C[j]=$C[j+n].
```

The resulting automaton has 80 states and 400 edges. All characteristic Sturmian words contain such a square, as Pecan proves in 0.02 seconds:

```
Theorem ("", {∀a. ∃i,n. square(a, i, n)}).
```

Of course, it is easy to see all binary words of length at least four contain squares. However, it is still useful to have created an automaton for recognizing squares, because it encodes quite a bit more information than just that squares exist: it also tells us exactly where they are in the Sturmian word. This allows Pecan to prove the following result.

Theorem 7.4 (Dubickas [Dub09, Theorem 1]). *All characteristic Sturmian words start with arbitrarily long squares.*

Proof. Using Pecan and the automaton for squares that we constructed earlier, we prove the following theorem, which takes 0.40 seconds.

```
Theorem ("", {∀a. ∀j,n. ∃m. m > n ∧ square(a, j, m)}).
```

□

Furthermore, we can use an automaton recognizing squares to efficiently build automata recognizing higher-powers. Indeed, we ask Pecan to construct an automaton recognizing the following property that there is a cube of length n starting at $c_{a,0}(i)$, as follows:

```
cube(a, i, n) := square(a, i, n) ∧ square(a, i + n, n)
```

We can ask Pecan to prove the well-known fact that characteristic Sturmian words contain cubes:

```
Theorem ("", {∀a. ∃i,n. cube(a, i, n)}).
```

Pecan proves this in 0.25 seconds.

Similar to squares, we have the following property for cubic prefixes.

Theorem 7.5. *Let $a \in (0, 1)$. Then $c_{a,0}$ starts with arbitrarily long cubes if and only if the continued fraction of a is not eventually 1.*

Proof. First, we manually build an automaton recognizing a such that the continued fraction of a is not eventually one, called `eventually_one`. Pecan proves the following in 2.37 seconds:

```
Theorem ("", {
  ∀a. ((¬eventually_one(a)) iff (∀m. ∃n. n > m cube(a, 1, n)))
}).
```

□

The proof of Theorem 7.5 highlights the ability of our decision algorithm, and hence of Pecan, to not only determine whether statements hold for all irrational numbers, but also whether a statement holds for all elements of a subset that is recognizable modulo $\sim_{\#}$. Indeed, we can use Pecan to show that if the continued fraction of a is not eventually 1, then $c_{a,0}$ contains a fourth power. To do so, we construct a predicate that holds whenever there is a fourth power of length n starting at $c_{a,0}(i)$:

```
fourth_pow(a, i, n) := square(a, i, n) ^ cube(a, i + n, n)
```

Finally, Pecan proves the following in 0.56 seconds.

```
Theorem ("", {
  ∀a. if ¬eventually_one(a) then ∃i,n. forth_pow(a,i,n)
})
```

The converse is not true. Although it is easy to see without Pecan why, we can also ask Pecan for counterexamples using the following commands.

```
Restrict i, n are ostrowski(a).
has_fourth_pow(a) := ∃i,n. n > 0 ^ fourth_pow(a,i,n)
Example (ostrowskiFormat, {
  bco_standard(a) ^ eventually_one(a) ^ has_fourth_pow(a)
})
```

Pecan responds with:

```
[(a, [6] [3] ([1]) ^ ω)]
```

This means that $a = [0, 6, 3, \bar{1}]$ is a counterexample. Recall that $a \in (0, 1)$, so the first digit of the continued fraction is always 0 and therefore omitted by Pecan. For this choice of a , the characteristic Sturmian word $c_{a,0}$ starts with 000001. Thus there is a fourth power immediately at the beginning of $c_{a,0}$.

7.3. Antisquares and more. Let $w \in \{0, 1\}^*$. We let \bar{w} denote the $\{0, 1\}$ -word obtained by replacing each 1 in w by 0 and each 0 in w by 1. A word $w \in \{0, 1\}^*$ is an **antisquare** if $w = v\bar{v}$ for some $v \in \{0, 1\}^*$. We define $A_O : (0, 1) \setminus \mathbb{Q} \rightarrow \mathbb{N} \cup \{\infty\}$ to map an irrational a to the maximum order of an antisquare in $c_{a,0}$ if such a maximum exists, and to ∞ otherwise. We let $A_L : (0, 1) \setminus \mathbb{Q} \rightarrow \mathbb{N} \cup \{\infty\}$ map a to the maximum length of an antisquare in $c_{a,0}$ if such a maximum exists and ∞ otherwise. Note that $A_L(a) = 2A_O(a)$.

Recall that w^R denotes the reversal of a word w . A word $w \in \{0, 1\}^*$ is an **antipalindrome** if $w = \overline{w^R}$. We set $A_P : (0, 1) \setminus \mathbb{Q} \rightarrow \mathbb{N} \cup \{\infty\}$ to be the map that takes an irrational a to the maximum length of an antipalindrome in $c_{a,0}$ if such a maximum, and to ∞ otherwise. We will use Pecan to prove that $A_O(a)$, $A_L(a)$ and $A_P(a)$ are finite for every a . While the quantities $A_O(a)$, $A_P(a)$ and $A_L(a)$ can be arbitrarily large, we prove the new results that the length of the Ostrowski representations of these quantities is bounded, independent of a .

Let $a \in (0, 1)$ be irrational and $N \in \mathbb{N}$. Let $|N|_a$ denote the length of the a -Ostrowski representation of N , that is the index of the last nonzero digit of a -Ostrowski representation of N , or 0 otherwise.

Theorem 7.6. *For every irrational $a \in (0, 1)$*

- (i) $|A_O(a)|_a \leq 4$,
- (ii) $|A_P(a)|_a \leq 4$,
- (iii) $|A_L(a)|_a \leq 6$,
- (iv) $A_O(a) \leq A_P(a) \leq A_L(a) = 2A_O(a)$.

There are irrational numbers $a, \beta \in (0, 1)$ such that $A_O(a) = A_P(a)$ and $A_P(\beta) = A_L(\beta)$.

Proof. Using Pecan, we create automata which compute A_O , A_P , and A_L :

$$\begin{aligned} A_O(a, n) &:= \text{has_antisquare}(a, n) \wedge \forall m. \text{has_antisquare}(a, m) \implies m \leq n \\ A_P(a, n) &:= \text{has_antipalindrome}(a, n) \wedge \forall m. \text{has_antipalindrome}(a, m) \implies m \leq n \\ A_L(a, n) &:= \text{has_antisquare_len}(a, n) \wedge \forall m. \text{has_antisquare_len}(a, m) \implies m \leq n \end{aligned}$$

We build automata recognizing a -Ostrowski representations of at most 4 and 6 nonzero digits, called $\text{has_4_digits}(n)$ and $\text{has_6_digits}(n)$. Then we use Pecan to prove all the parts of the theorem by checking the following statement.

```
Theorem ("(i), (ii), (iii), and (iv)", {
  Va. has_4_digits(max_antisquare(a)) ^
      has_4_digits(max_antipalindrome(a)) ^
      has_6_digits(max_antisquare_len(a)) ^
      max_antisquare(a) <= max_antipalindrome(a) ^
      max_antipalindrome(a) <= max_antisquare_len(a)
}).
```

We also use Pecan to find examples of the equality: when $a = [0; 3, 3, \bar{1}]$, we have $A_O(a) = A_P(a) = 2$, and when $a = [0; 4, 2, \bar{1}]$, we have $A_P(a) = A_L(a) = 2$. \square

Theorem 7.7. For every irrational $a \in (0, 1)$, all antisquares and antipalindromes in $c_{a,0}$ are either of the form $(01)^*$ or of the form $(10)^*$.

Proof. We begin by creating a predicate called is_all_01 stating that a subword $c_{a,0}[i..i+n]$ is of the form $(01)^*$ or $(10)^*$. We do this simply stating that $c_{a,0}[k] \neq c_{a,0}[k+1]$ for all k with $i \leq k < i+n-1$.

```
is_all_01(a, i, n) :=
  V k. if i <= k ^ k < i+n-1 then $C[k] != $C[k+1]
```

We can now directly state both parts of the theorem; Pecan proves both in 76.1 seconds.

```
Theorem ("All antisquares are of the form (01)^* or (10)^*", {
  Va. Vi, n. if antisquare(a, i, n) then is_all_01(a, i, n)
}).
```

```
Theorem ("All antipalindromes are of the form (01)^* or (10)^*", {
  Va. Vi, n. if antipalindrome(a, i, n) then is_all_01(a, i, n)
}).
```

\square

7.4. Least periods of factors of Sturmian words. We now use Pecan to give short automatic proofs a result about the least period of factors of characteristic Sturmian words.

The *semiconvergents* $p_{n,\ell}$ and $q_{n,\ell}$ of a continued fraction $[0; a_1, a_2, \dots]$ are defined so that

$$\frac{p_{n,\ell}}{q_{n,\ell}} = \frac{\ell p_{n-1} + p_{n-2}}{\ell q_{n-2} + q_{n-2}}$$

for $1 \leq \ell < a_n$.

Theorem 7.8. *Let p be the least period of a factor of $c_{a,0}$. Then p is the denominator of a semiconvergent of a ; that is $p = q_{n,\ell}$ for some n and ℓ .*

Proof. We define when a number p is a least period of a factor of $c_{a,0}$ as an automaton `lp_occurs`, as follows:

```
least_period(a,p,i,j) := p = min{n : period(a,n,i,j)}
lp_occurs(a,p) := ∃i,j. i>0 ∧ j>0 ∧ least_period(a,p,i,j)
```

It is easy to recognize a -Ostrowski representations of denominators of semiconvergents of a , because they are simply valid representations of the form $[0 \dots 01b]_a$, where b is some valid digit.

```
Theorem ("",
  { ∀a,p. if lp_occurs(a,p) then semiconvergent_denom(p) }).
```

Pecan proves the theorem in 5016.77 seconds. □

A word w is called **unbordered** if the least period of w is $|w|$. We now are ready to reprove Lemma 8 in Currie and Saari [CS09]. This is originally due to de Luca and De Luca [dLDL06].

Theorem 7.9. *The least period of $c_{a,0}[i..j]$ is the length of the longest unbordered factor of $c_{a,0}[i..j]$.*

Proof. We have previously defined least periods, so we can easily define unbordered factors. Similarly, it is straightforward to define the longest unbordered subwords of $c_{a,0}$:

```
max_unbordered_subfactor_len(a,i,j,n) :=
  n = max{m : ∃k. i<=k ∧ k+n<=j ∧ least_period(a,n,k,k+n)}
```

Then the theorem we wish to prove is

```
Theorem ("", { ∀a,i,j,p. if i>0 ∧ j>i ∧ p>0 then
  least_period(a,p,i,j) iff max_unbordered_subfactor_len(a,i,j,p)
}).
```

Pecan confirms the theorem is true. □

7.5. Periods of the length- n prefix. In [GRS21] Gabric, Rampersand and Shallit characterize all periods of the length- n prefix of a characteristic Sturmian word in terms of the lazy Ostrowski representation. We are able to implement their argument in Pecan.

Let a be a real number with continued fraction expansion $[a_0; a_1, a_2, \dots]$ and convergents $p_k/q_k \in \mathbb{Q}$. We recall the definition of the *lazy a -Ostrowski numeration system* [EFG⁺12].

Fact 7.10. Let $X \in \mathbb{N}$. The *lazy a -Ostrowski representation of X* is the unique word $b_N \cdots b_1$ such that

$$X = \sum_{n=0}^N b_{n+1} q_n$$

where

- (1) $0 \leq b_1 < a_1$;
- (2) $0 \leq b_i \leq a_i$ for $i > 1$;
- (3) if $b_i = 0$ then $b_{i-1} = a_i$ for all $i > 2$;
- (4) if $b_2 = 0$, then $b_1 = a_1 - 1$;

Theorem 7.11 [GRS21, Theorem 6]. *Let a be an irrational real number, and define Y_n to be the length n prefix of $c_{a,0}$. Define $PER(n)$ to be the set of all periods of Y_n . Then*

- (1) *The number of periods of Y_n is equal to the sum of the digits in the lazy Ostrowski representation of n .*
- (2) *Let the lazy Ostrowski representation of n be $b_1 \cdots b_N$, and define*

$$A(n) = \left\{ iq_j + \sum_{j < k < N} b_{k+1} q_k : 1 \leq i \leq b_{j+1} \text{ and } 0 \leq j \leq N \right\}$$

Then $PER(n) = A(n)$.

Proof. As in [GRS21], we note that it is sufficient to prove only (2). We begin by defining the sets, indexed by the slope a . The set of periods of subwords of $c_{a,0}$ can be defined by the formula $p > 0 \wedge c_{a,0}[i..j-p] = c_{a,0}[i+p..j]$, allowing us to create an automaton recognizing this set, which we call $\text{period}(a, p, i, j)$. This automaton is more expressive than what we need for this theorem, so we then simply take the periods of the prefixes of $c_{a,0}$, as follows:

`p is $Per(a, n) := \exists s. s = 1 \wedge \text{period}(a, p, s, n+1)`

To define $A(n)$, we first define several auxiliary automata and notions. Earlier, we defined addition automata for the (greedy) Ostrowski numeration system, but we can also easily handle the lazy Ostrowski numeration system using an automaton recognizing

$$\left\{ (a, x, y) : x, y \in A_a^{\text{fin}}, x = \#x_1\#x_2\#\cdots, y = \#y_1\#y_2\#\cdots, \sum_{i=0}^{\infty} x_{i+1} q_i = \sum_{i=0}^{\infty} y_{i+1} q_i \right\}$$

which we call $\text{ost_equiv}(a, x, y)$. The $\text{lazy_ostrowski}(a, n)$ automaton checks whether n is a valid lazy a -Ostrowski representation. These automata allow us to convert between the two systems.

To define $A(n)$, we break it up into smaller pieces; first, we wish to recognize the set

$$B(n) = \{iq_j : 1 \leq j \leq b_{j+1} \text{ and } 0 \leq j \leq N\}.$$

For each $x \in (\#(0|1)^*)^\omega$, denote by $|x|_{\text{fin}}$ the length of the longest prefix y of x such that $x = yz$ where $z \in (\#0^*)^\omega$, or ∞ if there is no such prefix. We then create the following automata:

- `as_long_as(x,y)` recognizing the set $\{(x,y) : |x|_{\text{fin}} \geq |y|_{\text{fin}}\}$.
- `has_1_digit(x)` recognizing the set $(\#0^*)^*(\#0^*)(\#(0|1)^*1(0|1)^*)(\#0^*)^\omega$, i.e., words of the form $\#w_1\#w_2\#\dots$ such that there is at most one w_i such that $w_i \notin 0^*$.
- `bounded_by(x,y)` recognizing the set

$$\{(x,y) : x \text{ and } y \text{ are aligned, } x = \#x_1\#x_2\#\dots, y = \#y_1\#y_2\#\dots, \forall i. x_i \leq_{\text{lex}} y_i\}$$

Then we can recognize the set $B(n)$ from above by

$$i > 0 \wedge \text{has_1_digits}(i) \wedge \text{as_long_as}(n,i) \wedge \text{bounded_by}(i,n_1)$$

where `n_1` is the lazy `a`-Ostrowski representation of `n`.

The last automaton we need to create is `suffix_after(x,y,s)`, recognizing the set $\{(x,y,s) : s = 0^{|x|_{\text{fin}}} \cdot y[|x|_{\text{fin}}..]\}$. We need this to be able to recognize the set of `a`-Ostrowski representations

$$\{m : 0 \leq j \leq N, m_l = 0^j n_l[j..N], m_l \text{ is the lazy } a\text{-Ostrowski representation of } Z_a(m)\}$$

where n_l is the lazy `a`-Ostrowski representation of $Z_a(n)$.

Finally, we can put everything together and define $A(n)$, again indexed by the slope a , as:

$$\begin{aligned} p \text{ is } \$A(a,n) := & \\ & \exists n_1, m_1. \text{lazy_ostrowski}(a,n_1) \wedge \text{ost_equiv}(a,n,n_1) \wedge \\ & \exists m. \text{ost_equiv}(a,m_1,m) \wedge \\ & \exists i. i > 0 \wedge \text{has_1_digit}(i) \wedge \text{as_long_as}(n,i) \wedge \\ & \text{bounded_by}(i,n_1) \wedge \text{suffix_after}(i,n_1,m_1) \wedge \\ & i + m = p \end{aligned}$$

Finally, we can state the theorem directly, which Pecan confirms is true.

Theorem ("6 (b)", $\{ \forall a. \forall p, n. p \text{ is } \$Per(a,n) \text{ iff } p \text{ is } \$A(a,n) \}$).

□

8. CONCLUSION AND OUTLOOK

8.1. Scalar multiplication. Recall that for $\alpha \in \mathbb{R}_{>0}$ we use \mathcal{R}_α to denote the $\mathcal{L}_{m,a}$ -structure $(\mathbb{R}, <, +, \mathbb{Z}, a\mathbb{Z})$. Let $\lambda_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be the function mapping x to αx , and let \mathcal{S}_α denote the structure $(\mathbb{R}, <, +, \mathbb{Z}, \lambda_\alpha)$. It is clear that every set definable in \mathcal{R}_α is also definable in \mathcal{S}_α . The inverse is known to be true for some α : By Hieronimi [Hie19, Theorem D], the function λ_α is definable in \mathcal{R}_α if $\alpha = \sqrt{d}$ for some $d \in \mathbb{Q}$, and thus in this situation every set definable in \mathcal{S}_α is also definable in \mathcal{R}_α .

Proposition 8.1. *There is $\alpha \in \mathbb{R}$ such that \mathcal{R}_α does not define λ_α .*

Proof. By [Hie19, Theorem A] the theory $\text{FO}(\mathcal{S}_\alpha)$ is undecidable when α is not quadratic. Thus is enough to find a non-quadratic α such that the theory $\text{FO}(\mathcal{R}_\alpha)$ is decidable. To do so, it suffices by Theorem 5.1 to find some $v \in R$ such that $\text{FO}(\mathcal{D}_v)$ is decidable but $\alpha(v)$ is non-quadratic.

Let U be the set $\{i! : i \in \mathbb{N}\}$. Define $u = u_1u_2 \cdots \in \{0,1\}^\omega$ such that $u_i = 1$ if $i \in U$, and $u_i = 0$ otherwise. Let $v = v_1v_2 \cdots \in \Sigma_{\#}^\omega$ be such that

$$v_i = \begin{cases} \# & i = 1, \\ \# & i > 2 \text{ and } u_i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

That is,

$$v = \#1111\#11111111111111111111\#1 \cdots .$$

By Elgot and Rabin [ER66, Proof of Theorem 5], the acceptance problem for u is decidable. This implies that the acceptance problem for v is decidable as well. Thus the theory $\text{FO}(\mathcal{D}_v)$ is decidable by Fact 2.4. However, the coefficients of the continued fraction expansion of $\alpha(v)$ are unbounded. Since quadratic numbers have periodic continued fractions, we conclude that $\alpha(v)$ is not quadratic. \square

As argued in the proof above, it follows from Fact 2.4 that for every $\alpha \in \mathbf{Irr}$ the theory $\text{FO}(\mathcal{R}_\alpha)$ is decidable whenever there is $v \in R$ such that the acceptance problem for v is decidable and $\alpha(v) = \alpha$. We leave it as an open question whether this sufficient condition is also necessary. It would be interesting to know whether there are any natural non-quadratic numbers, like e or π , for which this condition is satisfied.

Recall that \mathcal{L}_m is the signature of $\text{FO}(\mathbb{R}, <, +)$ together with a unary predicate symbol P . Let $\mathcal{L}_{m,\lambda}$ be the extension of \mathcal{L}_m by a unary functions symbol λ . We consider \mathcal{S}_α now as an $\mathcal{L}_{m,\lambda}$ -structure. Let \mathcal{K}_λ be the class of $\mathcal{L}_{m,\lambda}$ -structures $\{\mathcal{S}_\alpha : \alpha \in \mathbf{Irr}\}$. By Proposition 8.1 there is no hope of using Theorem B to deduce the decidability of the theory $\text{FO}(\mathcal{K}_\lambda)$. Indeed, we can show the following.

Proposition 8.2. *The theory $\text{FO}(\mathcal{K}_\lambda)$ is undecidable.*

Proof. Consider the $\mathcal{L}_{m,\lambda}$ -sentence ψ

$$\forall x_1 \forall x_2 \forall x_3 \left(\bigwedge_{i=1}^3 P(x_i) \wedge \bigvee_{i=1}^3 x_i \neq 0 \rightarrow (\lambda(\lambda(x_1)) + \lambda(x_2) + x_3 \neq 0) \right).$$

Hence

$$\mathcal{S}_\alpha \models \psi \text{ if and only if } \alpha \text{ is not quadratic.}$$

Consider $U = (Q, \Sigma, \sigma_1, \delta, q_1, q_2)$ be the universal 1-tape Turing machine with 8 states and 4 symbols as defined by Neary and Woods [NW06]. By the proof of [HNP21, Theorem 7.1]⁶, given an input $x \in \Sigma^*$, there is an $\mathcal{L}_{m,\lambda}$ -sentence φ_x such that for every non-quadratic α

$$\mathcal{S}_\alpha \models \varphi_x \text{ if and only if } U \text{ halts on input } x.$$

Combining this, we have that given an input $x \in \Sigma^*$

$$\text{FO}(\mathcal{K}_\lambda) \models \psi \rightarrow \varphi_x \text{ if and only if } U \text{ halts on input } x.$$

⁶In [HNP21] it is only stated that for every non-quadratic α we can find such an $\mathcal{L}_{m,\lambda}$ -sentence φ_x . However, it is clear from the given construction that the sentence does not depend on the particular α .

Thus $\text{FO}(\mathcal{K}_\lambda)$ is undecidable. □

Let \mathcal{K}_q be the class of all $\mathcal{L}_{m,a}$ -structures \mathcal{R}_α with $\alpha \in \mathbf{Irr}$ quadratic, and similarly, let $\mathcal{K}_{\lambda,q}$ be the class of all $\mathcal{L}_{m,\lambda}$ -structures \mathcal{S}_α with $\alpha \in \mathbf{Irr}$ quadratic. We leave it as an open question whether the theories $\text{FO}(\mathcal{K}_q)$ and $\text{FO}(\mathcal{K}_{\lambda,q})$ are decidable. It is unlikely that that decidability of the latter theory could be deduced from the decidability of the theory $\text{FO}(\mathcal{K}_q)$, because the definition of multiplication by \sqrt{d} in the proof of [Hie19, Theorem D] depends on d .

8.2. Computational complexity. By [Hie16, Theorem D], the structure \mathcal{R}_α defines an isomorphic copy of the standard model of the monadic second-order theory of $(\mathbb{N}, +1)$ whenever $\alpha \in \mathbf{Irr}$. Hence there can not be a decision algorithm for $\text{FO}(\mathcal{K})$ whose computational complexity is in general lower than the complexity of the decision algorithm presented here. See [HNP21] for more detailed results for $\text{FO}(\mathcal{S}_\alpha)$ when α is quadratic. It would still be interesting to know whether improvements can be obtained for specific fragments of these theories.

If we are only interested in deciding statements about Sturmian words, we only need decidability of the less expressive theories $\mathcal{K}_{\text{sturmian}}$ and $\mathcal{K}_{\text{char}}$. Here we know very little about lower bounds for the computational complexity of these decision problems. In particular, we do not even know whether an analogue of [Hie16, Theorem D], stating the definability of an isomorphic copy of the standard model of the weak monadic second-order theory of $(\mathbb{N}, +1)$, holds for $\mathcal{N}_{\alpha,\rho}$, when $\alpha \in \mathbf{Irr}$.

Better results are likely obtainable when dropping the order relation. For $\alpha \in \mathbf{Irr}$, consider $\mathcal{Z}_\alpha := (\mathbb{Z}, +, 0, 1, f_\alpha)$, where $f_\alpha : \mathbb{Z} \rightarrow \mathbb{Z}$ is the function mapping x to $\lfloor \alpha x \rfloor$. Khani and Zarei [KZ23] and Khani, Valizadeh and Zarei [KVZ21] prove quantifier-elimination results for such structures that have the potential to produce more efficient decision algorithms (see also Günaydin and Özahakyan [GO22]). However, the usual order relation of \mathbb{Z} is unlikely to be definable in such structures, and therefore this setting might not be particularly useful to decide statements about Sturmian words.

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