

CONTROLLER SYNTHESIS FOR TIMELINE-BASED GAMES *

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ABSTRACT. In the timeline-based approach to planning, the evolution over time of a set of state variables (the timelines) is governed by a set of temporal constraints. Traditional timeline-based planning systems excel at the integration of planning with execution by handling *temporal uncertainty*. In order to handle general nondeterminism as well, the concept of *timeline-based games* has been recently introduced. It has been proved that finding whether a winning strategy exists for such games is 2EXPTIME-complete. However, a concrete approach to synthesize controllers implementing such strategies is missing. This article fills the gap by providing an effective and computationally optimal approach to controller synthesis for timeline-based games.

1. INTRODUCTION

Automated planning is the field of *artificial intelligence* that studies the development of autonomous agents able of reasoning about how to reach some goals, starting from a high-level description of their operating environment. It is one of the most studied fields of AI, with early work going several decades back [MH69, FN71]. Most of the research by the planning community focuses on the *action-based* approach, where planning problems are modeled in terms of *actions* that an agent has to perform to suitably change its *state*. The task is to devise a sequence of such actions that lead to the goal when executed starting from a given initial state [FN71, FL03].

In this paper, we focus on the alternative paradigm of *timeline-based planning*, an approach born and developed in the space sector [Mus94]. In timeline-based planning, there is no explicit separation among actions, states, and goals. Planning domains are represented as systems of independent but interacting components, whose behavior over time, the *timelines*, is governed by a set of temporal constraints, called *synchronization rules*.

Key words and phrases: Planning, automata, synthesis.

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Over the years, timeline-based planning systems have been developed and successfully exploited by space agencies on both sides of the Atlantic [CCF⁺06, CCD⁺07, FJ03, BS07, CRK⁺00], for short- to long-term mission planning [CRT⁺15] as well as on-board autonomy [FCO⁺11]. The main advantage of such a paradigm in these contexts is the ability of these systems of handling both planning and *execution* in a uniform way: by the use of *flexible timelines*, timeline-based planners can produce robust plans that, during execution, can be adapted to the current contingency.

However, flexible timelines currently employed in timeline-based systems only handle *temporal uncertainty*, where the precise timings of events in the plan are unknown, but the causal sequence of the events is determined. In particular, they cannot generate robust plans against an environment empowered with general *nondeterminism*. To overcome this limitation, the concept of *timeline-based games* was recently introduced [GMO⁺20]. In timeline-based games, state variables belong either to the controller or to the environment. The controller aims at satisfying its set of *system* rules, while the environment can make arbitrary moves, as long as the *domain* rules that define the game arena are satisfied. A controller's strategy is winning if it guarantees that the controller wins, regardless of the choices made by the environment. The moves available to the two players can determine both *what* happens and *when* it happens, thus handling temporal uncertainty and general nondeterminism in a uniform way.

Determining whether a winning strategy exists for timeline-based games has been proved to be 2EXPTIME-complete [GMO⁺20]. However, there is currently no effective way to synthesize a controller that implements such strategies. A necessary condition for synthesizing a finite-state strategy and the corresponding controller is the availability of a *deterministic* arena. Two methods to obtain such an arena have been followed in the literature, but both have limitations and turn out to be inadequate. On the one hand, the complexity result of [GMO⁺20] relies on the construction of a (doubly exponential) *concurrent game structure* used to model check some Alternating-time Temporal Logic formulas [AHK02]. Even though such a structure is deterministic and theoretically suitable to solve a reachability game and to synthesize a controller, its construction relies on theoretical nondeterministic procedures that are not realistically implementable. On the other hand, Della Monica *et al.* [DGMS18] devised an automata-theoretic solution that provides a concrete and effective way to construct an automaton that accepts a word if and only if the original planning problem has a solution plan. Unfortunately, the size of the resulting *nondeterministic* automaton is already doubly exponential, and its determinization would result in a further blowup and thus in a non-optimal procedure.

The present paper fills the gap by developing an effective and computationally optimal approach to synthesizing controllers for timeline-based games. The proposed method addresses the limitations of previous techniques by directly constructing a *deterministic* finite-state automaton of an optimal doubly-exponential size, that recognizes solution plans. Such an automaton can be turned into the arena for a reachability game, for which many controller synthesis techniques are available in the literature. The paper is a significantly revised and extended version of [AGG⁺22]. It provides a detailed account of the general framework, gives some illustrative examples, and fully works out all the proofs.

The rest of the paper is organized as follows. After discussing related work in Section 2, Section 3 introduces timeline-based planning and games. Section 4 presents the main technical contribution of the paper, namely, the construction of the deterministic automaton that recognizes solution plans. Section 5 shows how to turn such an automaton into the

arena of a suitable game from which the controller can be synthesized. Section 6 summarizes the main contributions of the work and suggest future research directions. All the technical proofs are included in the appendix.

2. RELATED WORK

The paradigm of timeline-based planning has been first introduced to plan and schedule scientific operations of the Hubble space telescope [Mus94]. In the following two decades, many timeline-based planning systems have been developed both at NASA and ESA, including EUROPA [BWMB⁺05], ASPEN [CRK⁺00], and APSI [DPC⁺08]. Such systems have been used both for short- to long-term mission planning, *e.g.*, for the renowned Rosetta mission [CRT⁺15], and for onboard autonomy [FCO⁺11]. Elements of the timeline-based and the action-based paradigm have been combined into the Action Notation Modeling Language (ANML) [SFC08], extensively used at NASA since then.

Despite the real-world success, the timeline-based planning paradigm lacked a thorough foundational understanding in contrast to the action-based one, which has been extensively studied from a theoretical perspective from the start [MH69, Byl94]. To enable theoretical investigations into timeline-based planning, Cialdea Mayer *et al.* [COU16] laid down the core features of the paradigm, describing them in a uniform formalism, which has been later studied in several contributions. The formalism was compared to traditional action-based languages like STRIPS, and it was proved that the latter are expressible by timeline-based languages [GMCO16]. The timeline-based plan existence problem was proved to be EXPSPACE-complete [GMCO17] over discrete time in the general case, and PSPACE-complete with qualitative constraints only [DGTM20]. On dense time, the problem goes from being NP-complete to undecidable, depending on the applied syntactic restrictions [BMM⁺20]. Additionally, logical [DGM⁺17] and automata-theoretic [DGMS18] counterparts have been investigated to study the expressiveness of timeline-based languages.

The above body of work focuses on *deterministic* timeline-based planning domains. However, the paradigm also fits to *uncertain* domains requiring robust plans. Current timeline-based planning systems employ the concept of *flexible timelines*, described as including uncertainty in the timings of events, representing envelopes of possible executions of the plan. Planners, when possible, produce *strongly controllable* flexible plans, whose execution is then robust for the given temporal uncertainty. In order to obtain controllers for executing strongly controllable flexible plans, the problem can be simplified by reducing it to *timed game automata* [OFCF11].

While the current approach works fairly well in handling temporal uncertainty, it does not support scenarios where the environment is fully nondeterministic. Furthermore, as pointed out in [GMO⁺20], the language of timeline-based planning as formalized in [COU16] allows one to write domains that are not solvable by strongly controllable flexible plans, but that may easily be by strategies coping with general nondeterminism. For this reason, [GMO⁺20] introduced the concept of *timeline-based game*, which is the focus of this work. Timeline-based games adopt a game-theoretic point of view, where the controller and the environment play by constructing timelines, with the controller trying to fulfill its synchronization rules independently from the choices of the environment. This setting allows one to handle both temporal uncertainty and general nondeterminism, thus strictly generalizing previous approaches based on flexible timelines. In [GMO⁺20], the problem of deciding the existence of a winning strategy for a given timeline-based game has been

proved to be 2EXPTIME-complete. The proof is based on the construction of a *concurrent game structure* where a suitable *alternating-time temporal logic* (ATL) formula is model checked [AHK02]. However, the construction relies on nondeterministic procedures that are not effectively implementable, and thus it does not solve the problem of synthesizing actual controllers for timeline-based games. This work fills the gap by providing an effective synthesis algorithm.

The devised algorithm builds on classical results in the field of *reactive synthesis*, which studies how to build correct-by-construction controllers satisfying high-level logical specifications. The original formulation of the problem of reactive synthesis is due to Church [Chu62]. The problem for *SIS* specifications was later solved by Büchi and Landweber using a non-elementary complexity algorithm [BL90]. As for Linear Temporal Logic (LTL) specifications, the problem is 2EXPTIME-complete [PR89b, Ros92], which, interestingly, is the same complexity as timeline-based games. In both cases, the core of the synthesis algorithm is the construction of a *deterministic arena*, where the game can be solved with a fix-point computation. This work focuses on constructing such an arena for timeline-based games (Sections 4 and 5).

3. PRELIMINARIES

In this section, we provide an overview of the general framework that underpins our work. We begin by introducing the general features of timeline-based planning, and then we discuss timeline-based games. Next, we introduce the reactive synthesis problem. Finally, we recall the concept of *difference bound matrices* (DBMs) [Dil89, PH07], which are the data structures that we will use to represent the temporal constraints of a system.

3.1. Timeline-based planning. The first basic notion is that of *state variable*.

Definition 3.1 (State variable). A *state variable* is a tuple $x = (V_x, T_x, D_x, \gamma)$, where:

- V_x is the *finite domain* of x ;
- $T_x : V_x \rightarrow 2^{V_x}$ is the *value transition function* of x , which maps each value $v \in V_x$ to the set of values that can immediately follow it;
- $D_x : V_x \rightarrow \mathbb{N} \times \mathbb{N}$ is the *duration function* of x , mapping each value $v \in V_x$ to a pair $(d_{min}^{x=v}, d_{max}^{x=v})$ specifying respectively the minimum and maximum duration of any interval where $x = v$;
- $\gamma : V_x \rightarrow \{c, u\}$ is the *controllability tag*, that, for each value $v \in V_x$, specifies whether it is *controllable* ($\gamma(v) = c$) or *uncontrollable* ($\gamma(v) = u$).

A state variable x takes its values from a finite domain and represents a finite state machine with a transition function T_x . The behavior over time of a state variable x is modeled by a timeline. Intuitively, a *timeline* for a state variable x is a finite sequence of *tokens*, that is, contiguous time intervals where x holds a given value.

Following the approach described in [GMO⁺20], instead of formally defining timelines in terms of tokens, we represent executions of timeline-based systems as single words, called *event sequences*, where each event describe the start/end of some token in a given time point.

To this end, we first define the notion of action.

Definition 3.2. Let SV be a set of state variables. An *action* is a term of the form $\text{start}(x, v)$ or $\text{end}(x, v)$, where $x \in SV$ and $v \in V_x$.

Actions of the form $\text{start}(x, v)$ are *starting* actions, and those of the form $\text{end}(x, v)$ are *ending* actions. We denote by A_{SV} the set of all the actions definable over a set of state variables SV .

Definition 3.3 (Event sequence [GMO⁺20]). Let SV be a set of state variables and A_{SV} be the set of all the *actions* $\text{start}(x, v)$ and $\text{end}(x, v)$, for $x \in \text{SV}$ and $v \in V_x$. An *event sequence* over SV is a sequence $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ of pairs $\mu_i = (A_i, \delta_i)$, called *events*, where $A_i \subseteq A_{\text{SV}}$ and $\delta_i \in \mathbb{N}^+$, such that, for any $x \in \text{SV}$:

- (1) for all $1 \leq i \leq n$, if $\text{start}(x, v) \in A_i$, for some $v \in V_x$, then there is no $\text{start}(x, v')$ in any μ_j before the closest event μ_k , with $k > i$, such that $\text{end}(x, v) \in A_k$ (if any);
- (2) for all $1 \leq i \leq n$, if $\text{end}(x, v) \in A_i$, for some $v \in V_x$, then there is no $\text{end}(x, v')$ in any μ_j after the closest event μ_k , with $k < i$, such that $\text{start}(x, v) \in A_k$ (if any);
- (3) for all $1 \leq i < n$, if $\text{end}(x, v) \in A_i$, for some $v \in V_x$, then $\text{start}(x, v') \in A_i$, for some $v' \in V_x$;
- (4) for all $1 < i \leq n$, if $\text{start}(x, v) \in A_i$, for some $v \in V_x$, then $\text{end}(x, v') \in A_i$, for some $v' \in V_x$.

The first two conditions guarantee correct parenthesis placement by identifying the start and the end of each token in the sequence. Condition 1 prevents a token from starting before the end of the previous one, while condition 2 prevents the occurrence of two consecutive ends not interleaved by a start. Conditions 3 and 4 ensure seamless continuity: each token's end (resp., start) is consistently followed (preceded) by the start (resp., end) of another, except for the first (resp., last) event in the sequence. These latter conditions prevent gaps in the timeline description of the represented plan.

In event sequences, a **token** for a variable x is a maximal interval with at most one occurrence of events $\mu_i = (A_i, \delta_i)$ and $\mu_j = (A_j, \delta_j)$, where $\text{start}(x, v) \in A_i$ and $\text{end}(x, v) \in A_j$, for some $v \in V_x$. We say such a token *starts* at position i and *ends* at position j . Note that Definition 3.3 implies that a token that has started is not required to end before the end of the sequence and that it can end without the corresponding starting action ever appearing. If this is the case, we say that an event sequence is *open* either to the right or to the left. Otherwise, it is said to be *closed*. An event sequence closed to the left and open to the right is called a *partial plan*. Notice that the empty event sequence ε is closed on both sides for any variable. Furthermore, in closed event sequences, the first event contains only start actions, while the last one contains only end actions, one for each variable x .

Given an event sequence $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ over a set of state variables SV , with $\mu_i = (A_i, \delta_i)$, we define $\delta(\bar{\mu})$ as $\sum_{1 < i \leq n} \delta_i$, that is, $\delta(\bar{\mu})$ is the time elapsed from the start to the end of the event sequence (its duration). For any subsequence $\langle \mu_i, \dots, \mu_j \rangle$ of $\bar{\mu}$, abbreviated $\bar{\mu}_{[i \dots j]}$, we denote by $\delta_{i,j}$ (or, equivalently, $\delta(\bar{\mu}_{[i \dots j]})$) the amount of time spanning that subsequence. Notice that $\delta_{i,j}$ is defined as $\sum_{i < k \leq j} \delta_k$. Finally, given an event sequence $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$, we define $\bar{\mu}_{< i}$ as $\langle \mu_1, \dots, \mu_{i-1} \rangle$, for each $1 < i \leq n$.

In timeline-based planning, the objective is to satisfy a set of *synchronization rules*, that specify the desired behavior of the system (constraints and goal). These rules relate tokens, possibly belonging to different timelines, through temporal relations among their endpoints. Let SV be a set of state variables and $\mathbb{N} = \{a, b, \dots\}$ be a set of *token names*.

Definition 3.4 (Atom). An atom is a temporal relation between tokens' endpoints of the form $\langle \text{term} \rangle \leq_{[l, u]} \langle \text{term} \rangle$, where $l \in \mathbb{N}$, $u \in \mathbb{N} \cup \{+\infty\}$, $l \leq u$, and a *term* is either $\text{start}(a)$ or $\text{end}(a)$, for some $a \in \mathbb{N}$.

As an example, the atom $\text{start}(a) \leq_{[3,7]} \text{end}(b)$ constrains token a to start at least 3 and at most 7 time units before the end of token b , while the atom $\text{start}(a) \leq_{[0,+\infty]} \text{start}(b)$ simply constrains token a to start before token b .

Definition 3.5 (Synchronization rule). A synchronization rule R has one of the following two forms:

$$\begin{aligned} \langle \text{rule} \rangle &:= a_0[x_0 = v_0] \rightarrow \langle \text{body} \rangle \\ \langle \text{rule} \rangle &:= \top \rightarrow \langle \text{body} \rangle \\ \langle \text{body} \rangle &:= E_1 \vee E_2 \vee \dots \vee E_k \\ E_j &:= \exists a_1[x_1 = v_1] a_2[x_2 = v_2] \dots a_n[x_n = v_n] . C_j, \text{ for } 1 \leq j \leq k, \end{aligned}$$

where $a_i \in \mathbf{N}$, $x_i \in \mathbf{SV}$, $v_i \in V_{x_i}$, and C_j is a conjunction of atoms, for $0 \leq i \leq n$.

Terms $a_i[x_i = v_i]$ are referred to as *quantifiers*. The term $a_0[x_0 = v_0]$ is called the *trigger*. The disjuncts in the body are called *existential statements*. Quantifiers refer to tokens with the corresponding variable and value. The intuitive semantics of a synchronization rule can be given as follows: for every token satisfying the trigger, at least one of the existential statements must be satisfied as well. Each existential statement E_j requires the existence of tokens that satisfy the quantifiers in its prefix and the clause C_j . A token that satisfies the trigger of a rule is said to *trigger* that rule. The trigger of a rule can be empty (\top). In such a case, the rule is referred to as *triggerless* and it requires the satisfaction of its body without any precondition.

Let a and b be token names. Here are two examples of synchronization rules (relations $=$ and \leq are syntactic sugar for $\leq_{[0,0]}$ and $\leq_{[0,+\infty]}$, respectively):

$$\begin{aligned} a[x_s = \text{Comm}] &\rightarrow \exists b[x_g = \text{Available}] . \text{start}(b) \leq \text{start}(a) \wedge \text{end}(a) \leq \text{end}(b) \\ a[x_s = \text{Science}] &\rightarrow \exists b[x_s = \text{Slewing}] c[x_s = \text{Earth}] d[x_s = \text{Comm}] . \\ &\text{end}(a) = \text{start}(b) \wedge \text{end}(b) = \text{start}(c) \wedge \text{end}(c) = \text{start}(d) \end{aligned}$$

where variables x_s and x_g represent the state of a spacecraft and the visibility of the communication ground station, respectively. The first synchronization rule requires the satellite and the ground station to coordinate their communications so that when the satellite is transmitting, the ground station is available for reception. The second one instructs the system to send data to Earth after every measurement session, interleaved by the required slewing operation. Triggerless rule can be used to state the *goal* of the system. As an example, the following rule ensures that the spacecraft performs some scientific measurement:

$$\top \rightarrow \exists a[x_s = \text{Science}]$$

Triggerless rules only require the existence of tokens specified by the existential statements, being their universal quantification trivial. In fact, they are syntactic sugar, as it is possible to translate them into triggered rules, as shown in [GMO⁺20]. From now on, we will not consider them anymore.

We now formalise the above intuitive account of the semantics of synchronization rules.

Definition 3.6 (Matching functions [Gig19]). Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be a (possibly open) event sequence, $E \equiv \exists a_1[x_1 = v_1] \dots a_k[x_k = v_k] . C$ be one of the existential statements of a synchronization rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \vee \dots \vee E_m$, and V be a set of terms such that $\text{start}(a) \in V$ or $\text{end}(a) \in V$ only if $a \in \{a_0, \dots, a_k\}$. A *matching function* $\gamma : V \rightarrow [1, \dots, n]$ maps each term $T \in V$ to an event $\mu_{\gamma(T)}$ in $\bar{\mu}$, such that:

- (1) for each $T \in V$, with $T = \text{start}(a)$ (resp., $T = \text{end}(a)$), if a is quantified as $a[x = v]$ in \mathbf{E} , then the event $\mu_{\gamma(T)} = (A_T, \delta_T)$ is such that $\text{start}(x, v) \in A_T$ (resp., $\text{end}(x, v) \in A_T$);
- (2) if both $T = \text{start}(a)$ and $T' = \text{end}(a)$ belong to V for some token name $a \in \mathbf{N}$, then $\gamma(T)$ and $\gamma(T')$ identify the endpoints of the same token.

As a matter of fact, in [Gig19], matching functions are defined in terms of *rule graphs*, a data structure that we do not use here. For this reason, we reformulated the original definition in terms of event sequences.

The following definition gives a formal account of the semantics of synchronization rules.

Definition 3.7 (Semantics of synchronization rules). Let $\mathbf{R} \equiv a_0[x_0 = v_0] \rightarrow \mathbf{E}_1 \vee \dots \vee \mathbf{E}_m$ and let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence. We say that \mathbf{R} is *satisfied* by $\bar{\mu}$ if, for each event $\mu_i = (A_i, \delta_i)$ such that $\text{start}(x_0, v_0) \in A_i$, there exist an existential statement $\mathbf{E}_j \equiv \exists a_1[x_1 = v_1] \dots a_k[x_k = v_k] . \mathbf{C}$ and a matching function γ such that if $T \leq_{[l,u]} T'$ appears in \mathbf{C} , then $l \leq \gamma(T') - \gamma(T) \leq u$, for any pair of terms T and T' .

Timeline-based planning problems can be defined as follows.

Definition 3.8 (Timeline-based planning problem). A *timeline-based planning problem* is a pair $P = (\mathbf{SV}, \mathbf{S})$, where \mathbf{SV} is a set of state variables and \mathbf{S} is a set of synchronization rules over \mathbf{SV} . An event sequence $\bar{\mu}$ over \mathbf{SV} is a solution plan for P if all the rules in \mathbf{S} are satisfied by $\bar{\mu}$.

3.2. Timeline-based games. We are now ready to introduce the notion of *timeline-based game*, that subsumes that of *timeline-based planning with uncertainty* given in [COU16].

Definition 3.9 (Timeline-based game). A *timeline-based game* is a tuple $G = (\mathbf{SV}_C, \mathbf{SV}_E, \mathbf{S}, \mathbf{D})$, where \mathbf{SV}_C and \mathbf{SV}_E are the sets of *controlled* and *external* state variables, respectively, and \mathbf{S} and \mathbf{D} are the sets of *system* and *domain* synchronization rules, respectively, both involving variables from \mathbf{SV}_C and \mathbf{SV}_E .

A partial plan for G is a partial plan over the variables $\mathbf{SV}_C \cup \mathbf{SV}_E$. Let Π_G be the set of all possible partial plans for G , simply Π when there is no ambiguity. Since the empty event sequence ε is closed and $\delta(\varepsilon) = 0$, the *empty* partial plan ε is a good starting point for the game. Players incrementally build onto a partial plan, starting from ε , by playing actions that specify which tokens to start and (or) to end, adding an event that extends the event sequence, or complementing the existing last one.

Formally, we partition the set of all the available actions $\mathbf{A}_{\mathbf{SV}}$ into those that are playable by either of the two players.

Definition 3.10 (Partition of player actions). Let $\mathbf{SV} = \mathbf{SV}_C \cup \mathbf{SV}_E$. The set $\mathbf{A}_{\mathbf{SV}}$ of available actions over \mathbf{SV} is partitioned into the sets \mathbf{A}_C of *Charlie's* actions and \mathbf{A}_E of *Eve's* actions, which are defined as follows:

$$\mathbf{A}_C = \underbrace{\{\text{start}(x, v) \mid x \in \mathbf{SV}_C, v \in V_x\}}_{\text{start tokens on Charlie's timelines}} \cup \underbrace{\{\text{end}(x, v) \mid x \in \mathbf{SV}, v \in V_x, \gamma_x(v) = \mathbf{c}\}}_{\text{end controllable tokens}} \quad (1)$$

$$\mathbf{A}_E = \underbrace{\{\text{start}(x, v) \mid x \in \mathbf{SV}_E, v \in V_x\}}_{\text{start tokens on Eve's timelines}} \cup \underbrace{\{\text{end}(x, v) \mid x \in \mathbf{SV}, v \in V_x, \gamma_x(v) = \mathbf{u}\}}_{\text{end uncontrollable tokens}} \quad (2)$$

Hence, players can start tokens for owned variables and end them for values that they control. Let $d = \max(L, U) + 1$, where L and U are the maximum lower and (finite) upper bounds appearing in any rule of G . Note that, by Definition 3.10, we may have $x \in \text{SV}_E$ and $\gamma_x(v) = c$ for some $v \in V_x$. This means that Charlie may control the duration of a variable that belongs to Eve. This situation is symmetrical to the more common one where Eve controls the duration of a variable that belongs to Charlie, that is, uncontrollable tokens. As an example, Charlie may decide to start a task, without being able to foresee how long it will take. Similarly, the environment may trigger the start of a process, *e.g.*, fixing a plant fault, but Charlie may be able to control, to some extent, how long it will take to end it, *e.g.*, we can decide to fix it today or tomorrow.

Actions combine into *moves* starting (resp., ending) multiple tokens simultaneously.

Definition 3.11 (Move). A *move* μ_C for *Charlie* is a term of the form $\text{wait}(\delta_C)$ or $\text{play}(A_C)$, where $1 \leq \delta_C \leq d$ and $\emptyset \neq A_C \subseteq \mathbf{A}_C$ is either a set of *starting* actions or a set of *ending* actions. A *move* μ_E for *Eve* is a term of the form $\text{play}(A_E)$ or $\text{play}(\delta_E, A_E)$, where $1 \leq \delta_E \leq d$ and $A_E \subseteq \mathbf{A}_E$ is either a set of *starting* actions or a set of *ending* actions.

By Definition 3.11, moves like $\text{play}(A_C)$ and $\text{play}(\delta_E, A_E)$ can play either $\text{start}(x, v)$ actions only or $\text{end}(x, v)$ actions only. A move of the former kind is called a *starting* move, while a move of the latter kind is called an *ending* move. We consider wait moves as *ending* moves. Starting and ending moves must alternate during the game.

Let us denote the sets of *Charlie's* and *Eve's* moves by \mathcal{M}_C and \mathcal{M}_E , respectively. A round of the game is defined as follows.

Definition 3.12 (Round). A *round* ρ is a pair $(\mu_C, \mu_E) \in \mathcal{M}_C \times \mathcal{M}_E$ of moves such that:

- (1) μ_C and μ_E are either both *starting* or both *ending* moves;
- (2) either $\rho = (\text{play}(A_C), \text{play}(A_E))$, or $\rho = (\text{wait}(\delta_C), \text{play}(\delta_E, A_E))$, with $\delta_E \leq \delta_C$;

A *starting* (resp., *ending*) round is one made of starting (resp., ending) moves. Since *Charlie* cannot play empty moves and wait moves are ending moves, each round is unambiguously either a starting or an ending round. Moreover, since $\text{play}(\delta_E, A_E)$ moves are always paired with $\text{wait}(\delta_C)$ ones, which are ending moves, then $\text{play}(\delta_E, A_E)$ moves are necessarily ending moves (item 1 of Definition 3.12).

We can now specify how to apply a round to the current partial plan to obtain the new one. The game always starts with a single starting round.

Definition 3.13 (Outcome of rounds). Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence, with $\mu_n = (A_n, \delta_n)$ ($\mu_n = (\emptyset, 0)$ if $\bar{\mu} = \varepsilon$). Let $\rho = (\mu_C, \mu_E)$ be a round, A_E and A_C be the sets of actions of the two moves (A_C is empty if μ_C is a wait move), and δ_E and δ_C be the time increments of the moves. We define $\delta_C = 1$ (resp., $\delta_E = 1$) for $\text{play}(A_C)$ (resp., $\text{play}(A_E)$).

The *outcome* of the application of ρ on $\bar{\mu}$ is the event sequence $\rho(\bar{\mu})$ defined as follows:

- (1) if ρ is a starting round, then $\rho(\bar{\mu}) = \bar{\mu}_{<n} \mu'_n$, where $\mu'_n = (A_n \cup A_C \cup A_E, \delta_n)$;
- (2) if ρ is an ending round, then $\rho(\bar{\mu}) = \bar{\mu} \mu'$, where $\mu' = (A_C \cup A_E, \delta_E)$;

We say that ρ is *applicable* to $\bar{\mu}$ if:

- a) $\rho(\bar{\mu})$ complies with Definition 3.3;
- b) ρ is an ending round if and only if $\bar{\mu}$ is open for all variables that appear in the moves.

A single move by either player is applicable to $\bar{\mu}$ if there is a move for the other player such that the resulting round is applicable to $\bar{\mu}$. The game starts from the empty partial plan ε , and players play in turn, composing a round from the move of each one, which is

applied to the current partial plan to obtain the new one. We can now define the notion of *strategy* for each player and that of *winning strategy* for *Charlie*.

Definition 3.14 (Strategy). A *strategy for Charlie* is a function $\sigma_C : \Pi \rightarrow \mathcal{M}_C$ that maps any given partial plan $\bar{\mu}$ into a move μ_C applicable to $\bar{\mu}$. A *strategy for Eve* is a function $\sigma_E : \Pi \times \mathcal{M}_C \rightarrow \mathcal{M}_E$ that maps a partial plan $\bar{\mu}$ and a move $\mu_C \in \mathcal{M}_C$ applicable to $\bar{\mu}$ into a move μ_E such that the round $\rho = (\mu_C, \mu_E)$ is applicable to $\bar{\mu}$.

A sequence $\bar{\rho} = \langle \rho_0, \dots, \rho_n \rangle$ of rounds is called a *play* of the game. A play is said to be *played according to* some strategy σ_C for *Charlie*, if, starting from the initial partial plan $\bar{\mu}_0 = \varepsilon$, it holds that $\rho_i = (\sigma_C(\Pi_{i-1}), \mu_E^i)$, for some μ_E^i , for all $0 < i \leq n$, and to be played according to some strategy σ_E for *Eve* if $\rho_i = (\mu_C^i, \sigma_E(\Pi_{i-1}, \mu_C^i))$, for all $0 < i \leq n$. It can be easily seen that for any pair of strategies (σ_C, σ_E) and any $n \geq 0$, there is a unique play $\bar{\rho}_n(\sigma_C, \sigma_E)$ of length n played according to both σ_C and σ_E .

Then, we say that a partial plan $\bar{\mu}$ and the play $\bar{\rho}$ such that $\bar{\mu} = \bar{\rho}(\varepsilon)$ are *admissible*, if the partial plan satisfies the domain rules, and that they are *successful* if the partial plan satisfies the system rules.

Definition 3.15 (Admissible strategy for *Eve*). A strategy σ_E for *Eve* is *admissible* if for each strategy σ_C for *Charlie*, there is $k \geq 0$ such that the play $\bar{\rho}_k(\sigma_C, \sigma_E)$ is admissible.

Charlie wins if, *assuming* that domain rules are respected, he manages to satisfy the system rules no matter how *Eve* plays.

Definition 3.16 (Winning strategy for *Charlie*). Let σ_C be a strategy for *Charlie*. We say that σ_C is a *winning strategy for Charlie* if for any *admissible* strategy σ_E for *Eve*, there exists $n \geq 0$ such that the play $\bar{\rho}_n(\sigma_C, \sigma_E)$ is successful.

We say that *Charlie wins* the game G if he has a winning strategy, while *Eve wins* the game if a winning strategy for *Charlie* does not exist.

3.3. Synthesis. The synthesis problem is the problem of devising an implementation that satisfies a formal specification of an input-output relation [PR89a]. Such an implementation may be a transducer, a Mealy machine, a Moore machine, a circuit, or the like. In the following, we give a short account of the roles of games and strategies in game-based synthesis.

Definition 3.17 (Game Graph). A finite game graph G is a triple (Q, Q_C, E) , where Q is a finite set of nodes, $Q_C \subseteq Q$ is the subset of *Charlie's* nodes, and $E \subseteq Q \times Q$ is a transition relation. The relation E must satisfy the condition: $\forall q \exists q' : (q, q') \in E$ (totality).

A *play* on a game graph G starting from the initial state q_0 is an infinite sequence $p = q_0 q_1 q_2 \dots$, where $(q_i, q_{i+1}) \in E$, for all $i \geq 0$. A game is a pair (G, \mathcal{W}) , where G is a game graph and \mathcal{W} is the winning condition of the game. In the general case, \mathcal{W} consists of the set of plays won by *Charlie*.

Here, we focus on reachability winning conditions, which are expressed as $\mathcal{W} := \{R \subseteq Q \mid R \cap F \neq \emptyset\}$, for a given set $F \subseteq Q$. A play p is said to satisfy \mathcal{W} if the set of states visited by p , denoted by $occ(p) = \{q \in Q \mid \exists i . p(i) = q\}$, intersects \mathcal{W} , that is, *Charlie* wins the play p if p visits at least one state in F .

Definition 3.18 (Reachability game). A reachability game is a pair (G, \mathcal{W}) , where $G = (Q, Q_C, E)$ is a game graph and \mathcal{W} is a reachability winning condition.

A strategy for *Charlie* is a function $f : Q^* \cdot Q_C \rightarrow Q$. A play p adheres to strategy f if, for each $q_i \in Q_C$, $q_{i+1} = f(q_0 \dots q_i)$. Given an initial state q , a strategy for *Charlie* is a winning strategy if *Charlie* wins any play from q that follows the strategy f . The same holds for *Eve*. *Charlie* (resp., *Eve*) wins if a winning strategy exists from q .

Given a game (G, \mathcal{W}) , with $G = (Q, Q_C, E)$, the winning region of *Charlie* is defined as $W_C := \{q \in Q \mid \text{Charlie wins from } q\}$. The winning region W_E for *Eve* is defined in an analogous way. The two sets are clearly disjoint ($W_C \cap W_E = \emptyset$). The game is said to be *determined* if $W_C \cup W_E = Q$. It is well known that reachability games are determined [Tho08].

The next step is to build a Controller starting from a winning strategy f such that the specification is met. We use Moore machines as *Charlie* plays first.

Definition 3.19 (Moore machine). A Moore machine is a tuple $M = (Q, \Sigma, \Gamma, q_0, \delta, \tau)$, where Q is a finite set of states, Σ is a finite input alphabet, Γ is a finite output alphabet, $q_0 \in Q$ is the initial state, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, and $\tau : Q \rightarrow \Gamma$ is the output function.

By suitably tying δ and τ to f , one can effectively implement f . We refer the reader to Definition 5.3 for the details on how we do it.

3.4. Difference Bound Matrices. *Difference bound matrices* (DBMs) were introduced by Dill [Dil89] as a pragmatic representation of constraints ($x - y \leq c$). Later on, Péron et al. [PH07] suitably expanded the formalism. The following short account of the formalism is basically borrowed from the latter work,

Let $Var = \{v_0, v_1, \dots, v_n\}$ be a finite set of variables, $\bar{V} = \mathbb{Z} \cup \{+\infty\}$ be a set of values that variables and constants can take, and C be a set of constraints of the form $v_i - v_j \leq c$, where $v_i, v_j \in Var$ and $c \in \bar{V}$. The DBM that represents C is an $(n+1) \times (n+1)$ matrix defined as follows:

$$M_{ij} = \inf\{c \mid (v_i - v_j \leq c) \in C\},$$

where $\inf(\emptyset) = +\infty$.

M_{ij} equals the tightest value of c if there is some constraint $(v_i - v_j \leq c)$ in C ; otherwise, it is $+\infty$. The variable $v_0 \in Var$ is always valued to 0, and it is used to express bounds on variables, that is, $v_i \leq c$ is written as $v_i - v_0 \leq c$. In Section 4, we use DBMs to conveniently represent atoms (see Definition 3.4).

4. A DETERMINISTIC AUTOMATON FOR TIMELINE-BASED PLANNING

In this section, we define an encoding of timeline-based planning problems into *deterministic* finite state automata (DFA). Given a timeline-based planning problem, the corresponding automaton recognizes all and only those *event sequences* that represent solution plans for the problem. In the next section, we will use such an automaton as the game arena for a timeline-based game.

4.1. Plans as words. Let $P = (\text{SV}, S)$ be a timeline-based planning problem and, as already stated in the previous section, let $d = \max(L, U) + 1$, where L and U are the maximum lower and (finite) upper bounds appearing in any rule of P . We restrict our attention to event sequences where the distance between two consecutive events is at most d . Such a restriction guarantees us the finiteness of the considered alphabet, and it does not cause any loss in generality, as proved by Lemma 4.8 of [Gig19]. Moreover, it agrees with the notion of move of a timeline-based game (see Definition 3.11).

We define the symbols of the alphabet Σ as *events* of the form $\mu = (A, \delta)$, where $A \subseteq \text{ASV}$ and $1 \leq \delta \leq d$. Formally, $\Sigma = 2^{\text{ASV}} \times [d]$, where $[d] = \{1, \dots, d\}$. Note that the size of Σ is exponential in the size of the problem. Moreover, we define $\text{window}(P)$ as the sum of all the coefficients appearing as upper bounds in the rules of P . This value represents the maximum amount of time a rule can “count” far away from the occurrence of the quantified tokens. Consider, for instance, the following rule:

$$a_0[x_0 = v_0] \rightarrow \exists a_1[x_1 = v_1]a_2[x_2 = v_2]a_3[x_3 = v_3] \cdot \quad (3)$$

$$\text{start}(a_1) \leq_{[4,14]} \text{end}(a_0) \wedge \text{end}(a_0) \leq_{[0,+\infty]} \text{end}(a_2) \wedge \text{start}(a_2) \leq_{[0,3]} \text{end}(a_3)$$

In this case, assuming the above rule to be the only one in the problem, $\text{window}(P)$ would be $3 + 14 = 17$. Thus, the rule can account for what happens at most 17 time points from the occurrence of its quantified tokens. For instance, if the token a_1 appears at a specific distance from a_0 , it has to be within less than 17 time points, and any modification of the plan that alters this distance can break the rule’s satisfaction. However, what occurs further away from a_0 only affects the fulfillment of the rule *qualitatively*. Suppose that the tokens a_2 and a_3 are, together, at 100 time points from a_0 . Changing this distance while maintaining the qualitative order between tokens does not break the rule’s satisfaction. For $\text{window}(P)$ ’s properties refer to [Gig19].

4.2. Matching structures. A key insight underlying the construction we are going to outline is that every atomic temporal relation $T \leq_{[l,u]} T'$ can be rewritten as the conjunction of two upper bound constraints $T' - T \leq u$ and $T - T' \leq -l$, where we represent a lower bound constraint $T' - T \geq l$ as an upper bound one. This allows us to rewrite the clause C of an existential statement E as a constraint system $\nu(C)$ with constraints of the form $T - T' \leq n$, for $n \in \mathbb{Z} \cup \{+\infty\}$.

The constraint system $\nu(C)$ can be represented by a difference bound matrix D indexed by terms, where the entry $D[T, T']$ gives the upper bound n on $T - T'$. In building D , we ensure the right duration of tokens by augmenting the system with constraints of the kind $\text{start}(a_i) - \text{end}(a_i) \leq -d_{\min}^{x_i=v_i}$ and $\text{end}(a_i) - \text{start}(a_i) \leq d_{\max}^{x_i=v_i}$, for any quantified token $a_i[x_i = v_i]$ of E . As an example, the constraint system and the DBM for the above rule are the ones in Figs. 1 and 2, respectively.

On top of DBMs, we define the concept of *matching structure*, a data structure that allows us to monitor and update the fulfillment of atomic temporal relations among terms throughout the execution of the plan. More precisely, it allows us to manipulate and reason about existential statements of which only a portion of the requests has been satisfied by the word read so far, while the rest is potentially satisfiable in the future.

Definition 4.1 (Matching Structure). Let $E \equiv \exists a_1[x_1 = v_1] \dots a_m[x_m = v_m] \cdot C$ be an existential statement of a synchronization rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \vee \dots \vee E_k$ over the set of state variables SV . The *matching structure* for E is a tuple $\mathbf{M}_E = (V, D, M, t)$, where:

$$\begin{cases} \text{end}(a_0) - \text{start}(a_1) \leq 14 \\ \text{start}(a_1) - \text{end}(a_0) \leq -4 \\ \text{end}(a_0) - \text{end}(a_2) \leq 0 \\ \text{end}(a_3) - \text{start}(a_2) \leq 3 \\ \text{start}(a_2) - \text{end}(a_3) \leq 0 \end{cases}$$

Figure 1: The constraint system of Eq. (3).

	start(a_0)	end(a_0)	start(a_1)	end(a_1)	start(a_2)	end(a_2)	start(a_3)	end(a_3)
start(a_0)								
end(a_0)			14			0		
start(a_1)		-4						
end(a_1)								
start(a_2)								0
end(a_2)								
start(a_3)								
end(a_3)					3			

Figure 2: DBM of Eq. (3). Missing entries are intended to be $+\infty$.

- V is the set of terms $\text{start}(a)$ and $\text{end}(a)$, for $a \in \{a_0, \dots, a_m\}$;
- D is a DBM of size $|V| \times |V|$, indexed by terms of V , whose entries take value over $\mathbb{Z} \cup \{+\infty\}$, where

$$\begin{cases} D[T, T'] = n & \text{if } T - T' \leq n \in \nu(\mathbb{C}), \\ D[T, T'] = 0 & \text{if } T = T', \\ D[T, T'] = +\infty & \text{otherwise;} \end{cases}$$

- $M \subseteq V$ and $0 \leq t \leq \text{window}(P)$.

The set M contains the set of terms from V correctly seen in the sequence so far. We say these terms have been *matched* by the matching structure. We use $\overline{M} = V \setminus M$ to refer to terms yet to be matched. We say a matching structure \mathbf{M} to be *closed* if $M = V$, *initial* if $M = \emptyset$, and *active* if $\text{start}(a_0) \in M$ and it is not closed. The component t represents the time elapsed since matching $\text{start}(a_0)$. As time progresses, we update a matching structure as follows.

In the DBMs of a matching structure, the bounds between any pair of terms T and T' , with one in M while the other not, are tightened by the elapsing of time. When $T \in M$ and $T' \in \overline{M}$, $D[T, T']$ is a lower bound loosened by adding the elapsed time δ . When $T \in \overline{M}$ and $T' \in M$, $D[T, T']$ is an upper bound tightened by subtracting δ . Consider the DBM in Figure 2 and the pair of terms $\text{start}(a_1)$ and $\text{end}(a_0)$. We have $D[\text{start}(a_1), \text{end}(a_0)] = -4$, implying that $\text{start}(a_1) - \text{end}(a_0) \leq -4$ must hold. Suppose that $\text{start}(a_1) \in M$ (it has been matched), and that $\text{end}(a_0) \in \overline{M}$ (it needs to be matched). Now, in a time step, the entry in the DBM is incremented and updated to $-4 + 1 = -3$ reflecting the fact that we now have 3 time steps left to match $\text{end}(a_0)$. A similar analysis leads us to the conclusion that

the entry $D[\text{end}(a_0), \text{start}(a_1)] = 14$ has to be decremented by 1 and updated to $14 - 1 = 13$. This intuition is formalized as follows.

Definition 4.2 (Time shifting). Let $\delta > 0$ be a positive amount of time, and let $M = (V, D, M, t)$ be a matching structure. The result of shifting M by δ time units, written $M + \delta$, is a matching structure $M' = (V, D', M, t')$, where:

- for all $T, T' \in V$:

$$D'[T, T'] = \begin{cases} D[T, T'] + \delta & \text{if } T \in M \text{ and } T' \in \overline{M} \\ D[T, T'] - \delta & \text{if } T \in \overline{M} \text{ and } T' \in M \\ D[T, T'] & \text{otherwise} \end{cases}$$

- and

$$t' = \begin{cases} t + \delta & \text{if } M \text{ is } \textit{active} \\ t & \text{otherwise} \end{cases}$$

Definition 4.2 specifies how to update the entries of D and how to update t to the trigger occurrence of an active matching structure.

Definition 4.3 (Matching). Let $M = (V, D, M, t)$ be a matching structure and $I \subseteq \overline{M}$ a set of matched terms. A matching structure $M' = (V, D, M', t)$ is the result of matching the set I , written $M \cup I$, with $M' = M \cup I$.

To correctly match an existential statement while reading an event sequence, a matching structure is updated only as long as one witnesses no violation of temporal constraints. As such, we deem an event as *admissible* or not.

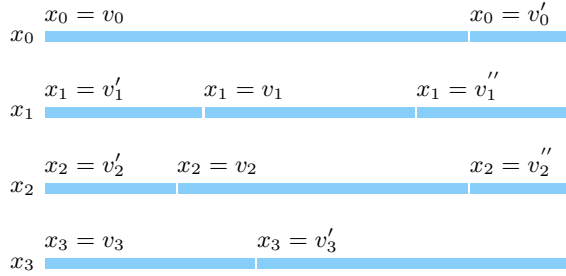
Definition 4.4 (Admissible Event). An event $\mu = (A, \delta)$ is *admissible* for a matching structure $M_E = (V, D, M, t)$ if and only if, for every $T \in M$ and $T' \in \overline{M}$, $\delta \leq D[T', T]$, *i.e.*, the elapsing of δ time units does not exceed the upper bound of some term T' not yet matched by M_E .

Each admissible event μ that is read can be matched with a subset of terms from the matching structure. However, there can be multiple ways to match events and terms. To make this choice explicit, we introduce the following definition.

Definition 4.5 (*I*-match Event). Let $M_E = (V, D, M, t)$ be a matching structure and $I \subseteq \overline{M}$. An *I-match event* is an admissible event $\mu = (A, \delta)$ for M_E such that:

- (1) for all token names $a \in \mathbb{N}$ quantified as $a[x = v]$ in E we have that:
 - (a) if $\text{start}(a) \in I$, then $\text{start}(x, v) \in A$;
 - (b) $\text{end}(a) \in I$ if and only if $\text{start}(a) \in M$ and $\text{end}(x, v) \in A$;
- (2) and for all $T \in I$ it holds that:
 - (a) for every other term $T' \in V$, if $D[T', T] \leq 0$, then $T' \in M \cup I$;
 - (b) for all $T' \in M$, $\delta \geq -D[T', T]$, *i.e.*, all the lower bounds on T are satisfied;
 - (c) for each other term $T' \in I$, either $D[T', T] = 0$, $D[T, T'] = 0$, or $D[T', T] = D[T, T'] = +\infty$.

We consider an event μ an *I-match event* if its actions correspond to the terms in I . The definition in Item 1 ensures the correct matching of each term to an action it represents and that the endpoints of a quantified token precisely identify the endpoints of a token in the event sequence. Meanwhile, Item 2 guarantees that matching the terms in I does not

Figure 3: Example of timelines for variables x_0, x_1, x_2, x_3 .

violate any atomic temporal relation. In addition, Item 2a deals with the qualitative aspect of a “happens before” relation, while Items 2b and 2c address the quantitative aspects of the lower bounds of these relations. It is worth noting that an \emptyset -event is also considered admissible.

Let \mathbb{M}_P denote the set of all matching structures for a planning problem P , and let \mathbb{I} be the set of all possible terms built from token names in \mathbf{N} . To describe the evolution of a matching structure, we define a quaternary relation $S \subseteq \mathbb{M}_P \times \Sigma \times \mathbb{I} \times \mathbb{M}_P$ as $(M, \mu, I, M') \in S$, for an event $\mu = (A, \delta)$, if and only if μ is an I -match event for M , and $M' = (M + \delta) \cup I$. We also write $M \xrightarrow{\mu, I} M'$ in place of $(M, \mu, I, M') \in S$. Note that, from Definition 4.5, a single event can represent multiple I -match events for a matching structure. Therefore, given a matching structure M and an event μ , automaton states will collect all the matching structures M' resulting from the relation S , for some set of terms I . Given a set of matching structures Υ , this notion is best described by the function $\text{step}_\mu(\Upsilon) = \{M' \mid (M, \mu, I, M') \in S, \text{ for some } M \in \Upsilon \text{ and } I \in \mathbb{I}\}$. Furthermore, we define $\Upsilon_t^R \subseteq \Upsilon$ as the set of all the *active* matching structures $M \in \Upsilon$ with timestamp t , associated with any existential statement of R . Matching structures in Υ_t^R contribute to fulfilling the same triggering event of R , regardless of their existential statement. We also define $\Upsilon_\perp \subseteq \Upsilon$ as the set of *non-active* matching structures of Υ . Lastly, we say that Υ is *closed* if there exists $M \in \Upsilon$ such that M is *closed*.

We conclude this section by providing an example of updating a matching structure $M = (V, D, M, t)$ for the rule discussed at the beginning of the section. Consider the set of timelines in Fig. 3. Before matching any term M is initial with $M = \emptyset$, $t = 0$, D as the DBM in Fig. 2, and V as the set of term $\text{start}(a)$ and $\text{end}(a)$ for $a \in \{a_0, a_1, a_2, a_3\}$. We begin by matching the terms $\text{start}(a_0)$ and $\text{start}(a_3)$ from the event $\mu = (\{\text{start}(x_0, v_0), \text{start}(x_3, v_3)\}, 0)$ (we do not consider $\text{start}(x_1, v'_1)$ and $\text{start}(x_2, v'_2)$ since they are not in V). Such event is an I -match event for $I = \{\text{start}(a_0), \text{start}(a_3)\}$: it is an admissible event (Definition 4.4), Item 1a holds, for both $\text{start}(a_0)$ and $\text{start}(a_3)$, there are no terms that should appear before them (Item 2a), there are no related lower bounds (Item 2b), and $D[\text{start}(a_0), \text{start}(a_3)] = D[\text{start}(a_3), \text{start}(a_0)] = +\infty$ (Item 2c). Hence, we update $M = M \cup I = \{\text{start}(a_0), \text{start}(a_3)\}$ and $t = t + \delta = 0$; now M is active. The next term to consider is $\text{start}(a_2)$, which occurs after $\delta = 5$ time steps.

First, we ensure that the event $\mu = (\text{start}(x_2, v_2), 5)$ is admissible. We show that by examining the DBM in Fig. 2, we see that the elapsing of time δ does not exceed any upper bound related to terms $T \in M$ and $T' \in \overline{M}$. Next, the set I in the current state appears as $I = \{\text{start}(a_2)\}$. Notice that we are in the case of Item 1a, and Item 2 holds because no constraint involves the term $\text{start}(a_2)$ (Item 2a), no lower bounds are

related to $\text{start}(a_2)$ (Item 2b), and $\text{start}(a_2)$ is the only term in I (Item 2c). Therefore, from Definitions 4.2 and 4.3, we update M as follows: $M = (M + \delta) \cup I$. Each entry of the DBM will remain unchanged since the third update case of Definition 4.2 applies, $M = M \cup I = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2)\}$, and $t = t + \delta = 5$.

Similarly, for the next event is $\mu = (\text{start}(x_1, v_1), 1)$, we check if such an event is admissible, and indeed it is since the upper bound $D[\text{end}(a_0), \text{start}(a_1)] = 9 \geq \delta$. It is also an I -match event for $I = \{\text{start}(a_1)\}$, since it respects Item 1a and all the relations in Item 2; thus we update M . We decrement $D[\text{end}(a_3), \text{start}(a_2)]$ and increment $D[\text{start}(a_2), \text{end}(a_3)]$ by 1 (see Definition 4.2), update M like follows $M = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2), \text{start}(a_1)\}$, and $t = t + \delta = 5 + 1 = 6$.

The next event is $\mu = (\text{end}(x_3, v_3), 3)$ after 2 time steps. Note that it is an admissible event and also an I -match event for $I = \{\text{end}(a_3)\}$. In this case, we emphasize that Items 1b and 2 are respected. We update the DBM as follows: $D[\text{end}(a_0), \text{start}(a_1)] = 14 - 2 = 12$, $D[\text{start}(a_1), \text{end}(a_0)] = -4 + 2 = -2$, $D[\text{end}(a_3), \text{start}(a_2)] = 2 - 2 = 0$, $D[\text{start}(a_2), \text{end}(a_3)] = 1 + 2 = 3$. Then, we update $M = M \cup I = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2), \text{start}(a_1), \text{end}(a_3)\}$ and $t = t + \delta = 6 + 2 = 8$. Notice that if we did not match $\text{end}(a_3)$ now, at the next time step, the timeline would have violated the rule above because the upper bound $D[\text{end}(a_3), \text{start}(a_2)] = 0$.

The subsequent event is $\mu = (\text{end}(x_1, v_1), \text{start}(x_1 = v'_1), 6)$ for which $I = \text{end}(a_1)$. Since there is no constraint involving $\text{end}(a_1)$, this event is admissible and an I -match event. The DBM is shifted by 6 time steps, and $M = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2), \text{start}(a_1), \text{end}(a_1)\}$.

The last event $\mu = (\{\text{start}(x_0, v'_0), \text{start}(x_2, v''_2), \text{end}(x_0, v_0), \text{end}(x_2, v_2)\}, 2)$ is admissible and an I -match for $I = \{\text{end}(a_0), \text{end}(a_2)\}$, note that there is not an upper bound between $\text{end}(a_0)$ and $\text{end}(a_2)$ and that Items 1b and 2 of the definition of I -match event are respected.

4.3. Building the automaton. We can now define the automaton. First, given an existential statement E , let \mathbb{E}_E be the set of all existential statements in the same rule of E . Next, let \mathbb{F}_P be the set of functions that map each existential statement of P to a set of existential statements and let \mathbb{D}_P be the set of functions that map each existential statement to a set of matching structures Υ . An automaton \mathbb{T}_P that checks the transition functions of the variables is easy to define. Then, given a timeline-based planning problem $P = (\text{SV}, S)$, we can characterize the corresponding automaton as $A_P = \mathbb{T}_P \cap \mathbb{S}_P$. Here, \mathbb{S}_P checks the fulfillment of the synchronization rules, and we define it as $\mathbb{S}_P = (Q, \Sigma, q_0, F, \tau)$ where

- (1) $Q = 2^{\mathbb{M}_P} \times \mathbb{D}_P \times \mathbb{F}_P \cup \{\perp\}$ is the finite set of states, *i.e.*, states are tuples of the form $\langle \Upsilon, \Delta, \Phi \rangle \in 2^{\mathbb{M}_P} \times \mathbb{D}_P \times \mathbb{F}_P$, plus a sink state \perp ;
- (2) Σ is the input alphabet defined above;
- (3) the initial state $q_0 = \langle \Upsilon_0, \Delta_0, \Phi_0 \rangle$ is such that Υ_0 is the set of initial matching structures of the existential statements of P and, for all existential statements E of P , we have $\Delta_0(E) = \emptyset$ and $\Phi_0(E) = \mathbb{E}_E$;
- (4) $F \subseteq Q$ is the set of final states defined as:

$$F = \left\{ \langle \Upsilon, \Delta, \Phi \rangle \in Q \mid \begin{array}{l} \text{M is not active for all } M \in \Upsilon \\ \text{and } \Delta(E) = \emptyset \text{ for all } E \text{ of } P \end{array} \right\}$$

- (5) $\tau : Q \times \Sigma \rightarrow Q$ is the transition function that given a state $q = \langle \Upsilon, \Delta, \Phi \rangle$ and a symbol $\mu = (A, \delta)$ computes the new state $\tau(q, \mu)$. Let $\text{step}_\mu^E(\Upsilon_t^R) = \{M_E \mid M_E \in \text{step}_\mu(\Upsilon_t^R)\}$. Moreover, let $\Psi_t^R = \{E \mid M_E \in \text{step}_\mu(\Upsilon_t^R)\}$. Then, the updated components of the state

are based on what follows, where $W = \text{window}(P)$:

$$\begin{aligned} \Upsilon' &= \text{step}_\mu(\Upsilon_\perp) \cup \bigcup \left\{ \text{step}_\mu(\Upsilon_t^R) \mid t \leq W - \delta \text{ and } \text{step}_\mu(\Upsilon_t^R) \text{ is not } \textit{closed} \right\} \\ \Delta'(\mathbf{E}) &= \begin{cases} \text{step}_\mu^{\mathbf{E}}(\Upsilon_t^R) & \text{where } t \text{ is the minimum such that } t > W - \delta \text{ and } \text{step}_\mu^{\mathbf{E}}(\Upsilon_t^R) \neq \emptyset \\ \text{step}_\mu(\Delta(\mathbf{E})) & \text{if such } t \text{ does not exist} \end{cases} \\ \Phi'(\mathbf{E}) &= \begin{cases} \mathbb{E}_{\mathbf{E}} & \text{if } \mathbf{E} \in \Psi(\mathbf{E}') \text{ for some } \mathbf{E}' \text{ such that } \Delta'(\mathbf{E}') \text{ is } \textit{closed} \\ \Phi(\mathbf{E}) \setminus \{\mathbf{E}' \mid \exists t > W - \delta. \mathbf{E}' \in \Psi_t^R \wedge \mathbf{E} \notin \Psi_t^R\} & \text{otherwise} \end{cases} \end{aligned}$$

Let $\Delta''(\mathbf{E}) = \Delta'(\mathbf{E})$ unless there is an \mathbf{E}' with $\mathbf{E} \in \Phi'(\mathbf{E}')$ such that $\Delta'(\mathbf{E}')$ is *closed*, in which case $\Delta''(\mathbf{E}) = \emptyset$. Then, $\tau(q, \mu) = \langle \Upsilon', \Delta'', \Phi' \rangle$ if the following holds:

- (a) for every Υ_t^R , $\text{step}_\mu(\Upsilon_t^R) \neq \emptyset$, and
- (b) for every synchronization rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \vee \dots \vee E_n$ in S , if $\text{start}(x_0, v_0) \in A$, then there exists $M_{E_i} = (V, D, M, 0) \in \Upsilon'$, with $i \in \{1 \dots n\}$, such that $\text{start}(a_0) \in M$;

Otherwise, $\tau(q, \mu) = \perp$.

The first component Υ of an automaton's state q is a set of matching structures that keeps track of the occurred events in the last $\text{window}(P)$ time points. The timestamp t of any matching structure in Υ satisfies $t < \text{window}(P)$. These matching structures evolve using the step_μ function until they become closed or their timestamp reaches $\text{window}(P)$.

Matching structures that reach $\text{window}(P)$ get promoted to a new role where they record the pieces of existential statements not yet matched to satisfy all the trigger events of R that occurred before the last $\text{window}(P)$ time points. However, the automaton does not store these matching structures in Υ . Instead, it uses the function Δ mapping each existential statement \mathbf{E} of a rule R to the set of matching structures for \mathbf{E} with $t = \text{window}(P)$. Thus, effectively summarizing events happening before this window to keep size under control.

When a set Υ_t^R exceeds the bound $\text{window}(P)$, the Δ function needs to be updated by merging the information from Υ_t^R with the information already stored in Δ . However, closing a set $\Delta(\mathbf{E})$ does not necessarily mean that every event that triggered R satisfies R . This is because there may be other sets, say $\Delta(\mathbf{E}')$, responsible for fulfilling the same rule R , but for different trigger events. Therefore, closing $\Delta(\mathbf{E})$ alone does not imply that R has been satisfied. Conversely, there may be cases where $\Delta(\mathbf{E})$ and $\Delta(\mathbf{E}')$ contribute to match the same trigger events, and closing either set is enough to satisfy R .

To address the issue of lost information when adding a set of matching structures to Δ , we introduce the Φ function, mapping existential statements to sets of existential statements, as the third component of the automaton states. For an existential statement \mathbf{E} and for every existential statement $\mathbf{E}' \in \Phi(\mathbf{E})$, it holds that the set of matching structures $\Delta(\mathbf{E}')$ tracks the satisfaction of the same trigger events as the set $\Delta(\mathbf{E})$. This way, when a set $\Delta(\mathbf{E})$ is closed, we can discard its matching structures as well as the matching structures in $\Delta(\mathbf{E}')$.

In Section 4.4 we state and prove soundness and completeness of the automaton construction. Now, instead, let us address the size of the automaton.

Let us recall that we assumed that the timestamp of each event in an event sequence is bounded. However, it is worth noting that since events may have an empty set of actions, Theorem 4.14 can handle arbitrary event sequences as well, provided that we add suitable empty events. Let us now analyze the size of the automaton.

Theorem 4.6 (Size of the automaton). *Let $P = (\mathcal{SV}, S)$ be a timeline-based planning problem and let A_P be the associated automaton. Then, the size of A_P is at most doubly-exponential in the size of P .*

Proof. We define E as the overall number of existential statements in P , which is linear in the size of P . We can then observe that $|\mathbb{D}_P| \in \mathcal{O}((2^{|\mathbb{M}_P|})^E) = \mathcal{O}(2^{E \cdot |\mathbb{M}_P|})$, thus the number of Δ functions is doubly exponential in the size of P . Next, note that $|\mathbb{F}_P| \in \mathcal{O}((2^E)^E) = \mathcal{O}(2^{E^2})$. Then, $|\mathcal{S}_P| \in \mathcal{O}(|\Sigma| \cdot 2^{|\mathbb{M}_P|})$ indicating that the size of \mathcal{S}_P is at most exponential in the number of possible matching structures. To bound this number, we define N as the largest finite constant appearing in P in any atom or value duration and L as the length of the longest existential prefix of an existential statement occurring inside a rule of P . Note that N is exponential in the size of P since constants are expressed in binary, while $L \in \mathcal{O}(|P|)$. We can then observe that the entries of a DBM for P , of which the number is quadratic in L , are constrained to take values within the interval $[-N, N]$ (excluding the value $+\infty$), which size is linear in N . By Definition 4.1, it follows that $|\mathbb{M}_P| \in \mathcal{O}(N^{L^2} \cdot 2^L \cdot \text{window}(P))$ indicating that the number of matching structures is at most exponential in the size of P . \square

Note that our automaton is the same size as the automaton built by Della Monica et al. in [DGMS18]. However, while their automaton is nondeterministic, ours is deterministic: an essential property to achieve the 2EXPTIME optimal asymptotic complexity for the synthesis procedure.

4.4. Soundness and Completeness. In the following, we present auxiliary notation, definitions, and essential lemmas for establishing the soundness and completeness of the automaton construction. For readability, we have included proofs in the appendix.

Definition 4.7 (Run of a matching structure). Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be a (possibly open) event sequence, and let M_E be the initial matching structure of an existential statement E . A *run* of M_E on $\bar{\mu}$ yielding a matching structure M_n is a sequence $\bar{I} = \langle I_1, \dots, I_n \rangle$ of I -match events for the matching structures $\langle M_E, M_1, \dots, M_{n-1} \rangle$, such that for every $i \in [1, \dots, n]$, $M_{i-1} \xrightarrow{\mu_i, I_i} M_i$. We write $M_E \xrightarrow{\bar{\mu}, \bar{I}} M_n$ when such run exists, or $M_E \xrightarrow{\bar{\mu}} M_n$, if \bar{I} is not relevant.

To link matching structures with the semantics of synchronization rules we establish a connection between matching functions (Definition 3.6) and runs.

Lemma 4.8 (Correspondence between runs and matching functions). *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be a (possibly open) event sequence, and let M_E be the initial matching structure of an existential statement $E \equiv \exists a_1[x_1 = v_1] \dots a_k[x_k = v_k] \cdot C$, with C augmented with atoms $\text{start}(a_i) \leq_{[d_{\min}^{x_i=v_i}, d_{\max}^{x_i=v_i}]} \text{end}(a_i)$, for every $0 \leq i \leq k$. Then, there exists a run $\bar{I} = \langle I_1, \dots, I_n \rangle$ of M_E on $\bar{\mu}$, yielding a matching structure $M_n = (V, D_n, M_n, t_n)$, if and only if there exists a matching function $\gamma : M_n \rightarrow [1, \dots, n]$ such that, for every atom of the form $T \leq_{[l, u]} T'$ in C :*

- (I) if $T' \in M_n$, then also $T \in M_n$, $\gamma(T) \leq \gamma(T')$, and $l \leq \delta(\bar{\mu}_{[\gamma(T) \dots \gamma(T')]} \leq u$;
- (II) if $T' \notin M_n$, but $T \in M_n$, then $\delta(\bar{\mu}_{[\gamma(T) \dots n]}) \leq u$.

Furthermore, γ and \bar{I} are such that for every $T \in M_n$, $T \in I_{\gamma(T)}$, i.e., they agree on the matching of the terms of M_n . We write $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$, if γ corresponds to a run of M_E , or $\bar{\mu}, \gamma \models M_n$, if M_E is clear from the context.

Observation 4.9. *Note that the existence of the matching function γ stated by Lemma 4.8, if the corresponding matching structure is closed, implies the satisfaction of the given existential statement, and vice versa.*

We now state the core technical result of the completeness proof, which ensures no important details are lost when matching structures are discarded.

Lemma 4.10. *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence, let M_E be the initial matching structure of some existential statement E of a rule R , and let M_r be an active matching structure resulting from a run $M_E \xrightarrow{\bar{\mu}, \gamma_r} M_r$, such that $\gamma_r(\text{start}(a_0)) = r$. If there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma_s} M_s$, such that $\gamma_s(\text{start}(a_0)) < r$, then there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma} M$, such that $\gamma(\text{start}(a_0)) = \gamma_s(\text{start}(a_0))$ and M matches at least as many tokens as M_r .*

The last needed notion is that of *residual* matching structure, which is an active matching structure with only infinite bounds.

Definition 4.11 (Residual matching structure). A matching structure $M = (V, D, M, t)$ is *residual* if it is *active* and for every $T \in M$ and $T' \in \bar{M}$, $D[T', T] = +\infty$.

In other words, M does not impose any finite upper bound on the distance at which terms yet to be matched may appear relative to those already matched. The definition implies that for any residual matching structure, denoted as $\hat{M} = (V, D, M, t)$, every event $\mu = (A, \delta)$ is admissible. Additionally, it is never the case that $\text{start}(a) \in M$ and $\text{end}(a) \in \bar{M}$ for any quantified token $a[x = v]$ of E , given that such terms always have a finite upper bound in D that is at least as strict as the value $d_{max}^{x=v}$. As a result, the “if” direction of Item 1b in the Definition 4.5 of I -match never applies to \hat{M} for any event μ . Therefore, every event is a valid \emptyset -match event for \hat{M} .

Observation 4.12. *Let $M_E \xrightarrow{\bar{\mu}_1, \bar{I}_1} \hat{M}$ be a run of the initial matching structure M_E , on an event sequence $\bar{\mu}_1$, yielding a residual matching structure \hat{M} . Then, for any event sequence $\bar{\mu}_2$, there exists a run $M_E \xrightarrow{\bar{\mu}_1 \bar{\mu}_2, \bar{I}_1 \bar{I}_2} \hat{M}'$ such that every I -match event in \bar{I}_2 is an \emptyset -match event and \hat{M}' differs from \hat{M} by at most the value of the component t .*

Consequently, whenever a residual matching structure appears in a run, it has the potential to remain there indefinitely, which is why it is called *residual*.

Lemma 4.13 (Existence of residual matching structure). *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence, and let M_n be an active matching structure such that $\bar{\mu}, \gamma \models M_n$ and $\delta(\bar{\mu}_{[\gamma(\text{start}(a_0)) \dots n]}) > \text{window}(P)$. If we consider the intermediate matching structures $\langle M_1, \dots, M_{n-1} \rangle$ of the run $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$, then there exists a position $\gamma(\text{start}(a_0)) \leq k < n$ such that M_k is a residual matching structure.*

We are now ready to prove the final result.

Theorem 4.14 (Soundness and completeness). *Let $P = (SV, S)$ be a timeline-based planning problem and let A_P be the associated automaton. Then, any event sequence $\bar{\mu}$ is a solution plan for P if and only if $\bar{\mu}$ is accepted by A_P .*

5. CONTROLLER SYNTHESIS

We leverage the deterministic automaton constructed in the previous section to establish a deterministic arena that enables us to solve a reachability game and determine whether a controller exists. If a controller exists, our procedure allows its synthesis.

5.1. From the automaton to the arena. Let $G = (\text{SV}_C, \text{SV}_E, \text{S}, \text{D})$ be a timeline-based game. The automaton construction we used considered a planning problem with a single set of synchronization rules, while in G , we have to account for the roles of both S and D .

To address this, we define A_S and A_D as the deterministic automata constructed over the timeline-based planning problems $P_S = (\text{SV}_C \cup \text{SV}_E, \text{S})$ and $P_D = (\text{SV}_C \cup \text{SV}_E, \text{D})$, respectively. We then construct the automaton A_G by taking the union of A_S with the complement of A_D ($\overline{A_D}$). Note that these are standard automata-theoretic operations over DFAs. An accepting run of A_G represents either a plan that violates the domain rules or a plan that adheres to domain and system rules, according to the definition of winning strategy in Definition 3.16. Furthermore, A_G is deterministic, and its size only polynomially increases when built from A_D and A_S .

The A_G automaton is not immediately applicable as a game arena since its transitions' labels only reflect events, not game moves. In A_G , a single transition can correspond to various combinations of rounds due to the absence of $\text{wait}(\delta)$ moves in the transition's label. For example, an event $\mu = (A, 5)$ can arise from either a $\text{wait}(5)$ move by *Charlie*, followed by a $\text{play}(5, A)$ move by *Eve*, or any $\text{wait}(\delta)$ move with $\delta > 5$ followed by a $\text{play}(5, A)$ move. To obtain a suitable game arena, we need to modify A_G further.

Let $A_G = (Q, \Sigma, q_0, F, \tau)$ be the automaton constructed as described above. Formally, we define a new automaton $A'_G = (Q, \Sigma, q_0, F, \tau')$ where τ' is a partial transition function, meaning that the automaton is now incomplete. The function τ' agrees with τ on all transitions except those of the form $\tau(q, (A, \delta))$, where $\delta > 1$ and A contains a $\text{end}(x, v)$ action with $x \in \text{SV}_C$. In such cases, the transition is undefined in A'_G . An example is shown in Figure 4 (left). Note that this removal does not alter the set of plans accepted by the automaton since for each transition $\tau(q, (A, \delta)) = q'$ with $\delta > 1$, there exist two transitions $\tau(q, (\emptyset, \delta - 1)) = q''$ and $\tau(q'', (A, 1)) = q'$ in A'_G .

To make the game rounds and moves explicit, we can transform the automaton by splitting each transition into four transitions representing the four moves of the two rounds. Starting from the incomplete automaton $A'_G = (Q, \Sigma, q_0, F, \tau')$, we define a new automaton $A_G^a = (Q^a, \Sigma^a, q_0^a, F^a, \tau^a)$ as the game arena.

- (1) The set of states Q^a is given by $Q^a = Q \cup \{q_\delta \mid 1 \leq \delta \leq d\} \cup \{q_{\delta, A} \mid 1 \leq \delta \leq d, A \subseteq \mathbf{A}\}$.
- (2) The alphabet Σ^a is defined as $\Sigma^a = \mathcal{M}_C \cup \mathcal{M}_E$, which corresponds to the set of moves of the two players.
- (3) The initial and final states of A_G^a are $q_0^a = q_0$ and $F^a = F$, respectively.
- (4) The partial transition function τ^a is defined as follows. Let $w = \tau(q, \mu)$ with $\mu = (A, \delta)$. We distinguish the cases where $\delta = 1$ or $\delta > 1$.
 - (a) if $\delta = 1$, let $A_C \subseteq A$ and $A_E \subseteq A$ be the set of actions in A playable by *Charlie* and by *Eve*, respectively. Then:
 - (i) $\tau(q, \text{play}(A_C^e)) = q_{1, A_C^e}$, where A_C^e is the set of *ending* actions in A_C ;
 - (ii) $\tau(q_{1, A_C^e}, \text{play}(A_E^e)) = q_{1, A_C^e \cup A_E^e}$, where A_E^e is the set of *ending* actions in A_E ;
 - (iii) $\tau(q_{1, A_C^e \cup A_E^e}, \text{play}(A_C^s)) = q_{1, A_C^e \cup A_E^e \cup A_C^s}$, where A_C^s is the set of *starting* actions in A_C ;

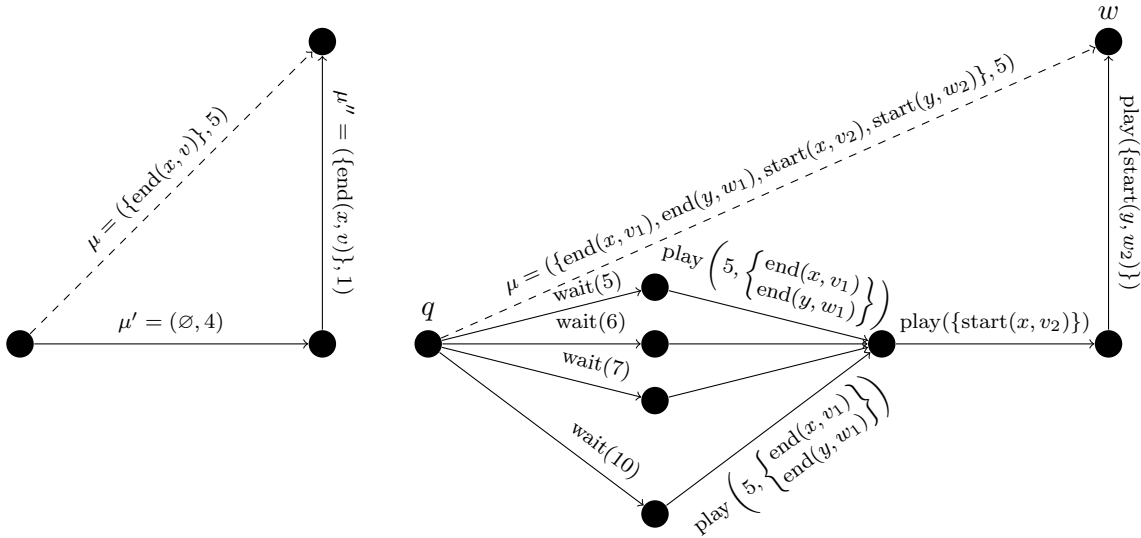


Figure 4: On the left, the removal of transitions $\mu = (A, \delta)$ with $\delta > 1$ and ending actions of controllable tokens in A . On the right, the transformation of a transition of A_G into a sequence of transitions in A_G^a , with $x \in \text{SV}_C$, $y \in \text{SV}_E$, and $\gamma_x(v_1) = \gamma_y(w_1) = u$.

- (iv) $\tau(q_{1, A_C^e \cup A_E^e \cup A_C^s}, \text{play}(A_E^s)) = w$, where A_E^s is the set of *starting* actions in A_E ; Here, the states mentioned are added to Q^a as needed.
- (b) if $\delta > 1$, let $A_C \subseteq A$ and $A_E \subseteq A$ be the set of actions in A playable by *Charlie* and by *Eve*, respectively. Note that by construction, A_C only contains *starting* actions.

Then:

- (i) $\tau(q, \text{wait}(\delta_C)) = q_{\delta_C}$ for all $\delta \leq \delta_C \leq d$;
- (ii) $\tau(q_{\delta_C}, \text{play}(\delta, A_E^e)) = q_{\delta, A_E^e}$ where A_E^e is the set of *ending* actions in A_E ;
- (iii) $\tau(q_{\delta, A_E^e}, \text{play}(A_C)) = q_{\delta, A_E^e \cup A_C}$;
- (iv) $\tau(q_{\delta, A_E^e \cup A_C}, \text{play}(A_E^s)) = w$ where A_E^s is the set of *starting* actions in A_E ; where the mentioned states are added to Q^a as needed.

All the transitions not explicitly defined above are undefined.

We present a graphical illustration of the above construction in Fig. 4. It is worth noting that the automaton preserves the structure of the original automaton A_G . For any state, $q \in Q$ and event $\mu = (A, \delta)$, any sequence of moves that would result in appending μ to the partial plan (see Definition 3.13) reaches the same state w in A_G^a as it does in A_G by reading μ . Therefore, we can consider A_G^a as being able to read event sequences, even though its alphabet is different. We use the notation $[\bar{\mu}]$ to represent the state $q \in Q^a$ reached by reading $\bar{\mu}$ in A_G^a . Furthermore, note that, with a slight abuse of notation, any play \bar{p} in the game G is a readable word by the automaton A_G^a . Thus, we can establish the following result.

Theorem 5.1. *If G is a timeline-based game, for any play \bar{p} for G , \bar{p} is successful if and only if it is accepted by A_G^a .*

5.2. Computing the Winning Strategy and Building the Controller. Let us define $Q_C^a \subset Q^a$ as the set of states in which *Charlie* can make a move, and $Q_E^a = Q^a \setminus Q_C^a$ as the set of states where *Eve* can make a move. Additionally, we define $E = \{(q, q') \in Q^a \times Q^a \mid \exists \mu . \tau^a(q, \mu) = q'\}$ as the set of edges in A_G^a . By solving the reachability game (G_R, \mathcal{W}) , where $G_R = (Q^a, Q_C^a, E^a)$ and $\mathcal{W} = \{R \subseteq Q^a \mid R \cap F^a \neq \emptyset\}$, we aim to determine the winning region W_C and the winning strategy s_C for *Charlie*, provided they exist. In the following discussion, we will show that the winning strategy σ_C for the timeline-based game G is derivable from strategy s_C when $q_0^a \in W_C$.

To determine the winning region W_C , we use the well-known *attractor* construction. We are interested to the attractor set of F^a for *Charlie*, written $Attr_C(F^a)$, thus given $i \geq 0$ we compute the set of states from which *Charlie* can reach a state $q \in F^a$ within i moves, defined as $Attr_C^i(F^a)$:

$$\begin{aligned} Attr_C^0(F^a) &= F^a \\ Attr_C^{i+1}(F^a) &= Attr_C^i(F^a) \\ &\quad \cup \{q^a \in Q_C^a \mid \exists r ((q^a, r) \in E \wedge r \in Attr_C^i(F^a))\} \\ &\quad \cup \{q^a \in Q_E^a \mid \forall r ((q^a, r) \in E \rightarrow r \in Attr_C^i(F^a))\}. \end{aligned}$$

The sequence $Attr_C^0(F^a) \subseteq Attr_C^1(F^a) \subseteq Attr_C^2(F^a) \subseteq \dots$ eventually becomes stationary for some index $k \leq |Q^a|$, hence we can define $Attr_C(F^a) = \bigcup_{i=0}^{|Q^a|} Attr_C^i(F^a)$ as the attractor set. Note that $W_C = Attr_C(F^a)$ is a known fact for which proof is available in [Tho08]. Next, we want that $q_0^a \in W_C$ since we are interested in a winning strategy σ_C for the timeline-based game G . If it is the case, by defining $s_C(q) = \mu$ for any μ such that $\tau^a(q, \mu) = q'$ with $q, q' \in W_C$, which is guaranteed to exist by the attractor construction, we can define σ_C for *Charlie* in G as $\sigma_C(\bar{\mu}) = s_C([\bar{\mu}])$ for any event sequence $\bar{\mu}$. We prove this claim in the following:

Theorem 5.2. *Given $A_G^a = (Q^a, \Sigma^a, q_0^a, F^a, \tau^a)$, $q_0^a \in W_C$ if and only if σ_C is a winning strategy for *Charlie* for G .*

Proof. (\leftarrow). From the definition of a winning strategy for *Charlie* in G (Definition 3.16), we know that for every admissible strategy σ_E for *Eve*, there exists $n \geq 0$ such that the play $\bar{\rho}_n(\sigma_C, \sigma_E)$ is successful. By the soundness of the arena construction (Theorem 5.1), we know that the event sequence $\bar{\mu}_n$ representing $\bar{\rho}_n(\sigma_C, \sigma_E)$, when seen as a word over Σ^a , is accepted by A_G^a . Therefore, $\bar{\mu}_n$ reaches a state in the set F^a starting from q_0^a . By the definition of the reachability game, this means that $q_0^a \in W_C$. Thus, we have proved that if σ_C is a winning strategy *Charlie* in G , then $q_0^a \in W_C$.

(\rightarrow). If $q_0^a \in W_C$, then by definition, s_C is a winning strategy for *Charlie* in the reachability game over the arena A_G^a . Hence, any word over Σ^a obtained by playing with s_C is accepted by A_G^a , and therefore, by the soundness of the arena construction (Theorem 5.1), any corresponding play $\bar{\rho}$ is successful in G . Now, recall that $\sigma_C(\bar{\mu}) = s_C([\bar{\mu}])$ for any event sequence $\bar{\mu}$. Hence, $\bar{\rho} = \bar{\rho}(\sigma_C, \sigma_E)$ for some strategy σ_E of *Eve*. As a result, we can conclude that σ_C is a winning strategy for *Charlie* in G . \square

Finally, we build a Controller that implements the winning strategy σ_C , provided it exists. First, by Theorem 5.2, the existence of σ_C implies that $q_0^a \in W_C$. Next, we define the following Moore machine (Definition 3.19) based on s_C :

Definition 5.3 (Controller). Given $A_G^a = (Q^a, \Sigma^a, q_0^a, F^a, \tau^a)$, we define a Controller as $\mathcal{M} = (Q, \Sigma, \Gamma, q_0, \delta, \tau)$, where $Q = Q_C^a \cap W_C$ represents the set of states, $q_0 = q_0^a$ is the initial state, $\Sigma = \mathcal{M}_E$ is the input alphabet, $\Gamma = \mathcal{M}_C$ is the output alphabet, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, and $\tau : Q \rightarrow \Gamma$ is the output function. The transition function δ and the output function τ are defined as follows:

$$\begin{aligned}\delta(q_C, \mu_E) &= \tau^a(s_C(q_C), \mu_E) \\ \tau(q_C) &= s_C(q_C).\end{aligned}$$

Note that by construction the states of \mathcal{M} belong to the winning region W_C of A_G^a , and δ follows the transition function τ^a of A_G^a . Hence, the output of \mathcal{M} after reading a word $\bar{\mu}$ is exactly $\sigma_C(\bar{\mu}) = s_C([\bar{\mu}])$ and \mathcal{M} implements σ_C , which is a winning strategy by Theorem 5.2.

6. CONCLUSIONS AND FUTURE WORK

Our article presents an effective procedure for synthesizing controllers for timeline-based games, whereas previously, only a proof of the 2EXPTIME-completeness of the problem of determining the existence of a strategy was available in the literature. We use a novel construction of a *deterministic* automaton of doubly-exponential (thus optimal) size, which is then adapted to serve as the arena for the game. Then, with standard methods, we solve a reachability game on the arena to effectively compute the winning strategy for the game, if it exists.

This work paves the way for future developments. First, the procedure provided in this article can be realistically implemented and tested. It is conceivable, though, that to avoid the state explosion problem due to the doubly-exponential size of the automaton, it will be necessary to apply *symbolic techniques*. Moreover, an implementation would also need a concrete syntax to specify timeline-based games. Existing languages supported by timeline-based systems (*e.g.*, NDDL [CO96] or ANML [SFC08]) might be inadequate for this purpose. Next, as in the case of LTL, the high complexity makes one wonder whether simpler but still expressive fragments can be found. One possibility might be restricting the synchronization rules to only talk about the *past* concerning the rule's trigger. For co-safety properties (*i.e.*, properties expressing the fact that something good will eventually happen) expressed in pure-past LTL, the realizability problem goes down to being EXPTIME-complete, and by analogy, this might happen to pure-past timeline-based games as well.

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APPENDIX A.

Lemma 4.8 (Correspondence between runs and matching functions). *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be a (possibly open) event sequence, and let M_E be the initial matching structure of an existential statement $E \equiv \exists a_1[x_1 = v_1] \dots a_k[x_k = v_k] . C$, with C augmented with atoms $\text{start}(a_i) \leq_{[d_{\min}^{x_i=v_i}, d_{\max}^{x_i=v_i}]} \text{end}(a_i)$, for every $0 \leq i \leq k$. Then, there exists a run $\bar{I} = \langle I_1, \dots, I_n \rangle$ of M_E on $\bar{\mu}$, yielding a matching structure $M_n = (V, D_n, M_n, t_n)$, if and only if there exists a matching function $\gamma : M_n \rightarrow [1, \dots, n]$ such that, for every atom of the form $T \leq_{[l, u]} T'$ in C :*

- (I) *if $T' \in M_n$, then also $T \in M_n$, $\gamma(T) \leq \gamma(T')$, and $l \leq \delta(\bar{\mu}_{[\gamma(T) \dots \gamma(T')]}) \leq u$;*
- (II) *if $T' \notin M_n$, but $T \in M_n$, then $\delta(\bar{\mu}_{[\gamma(T) \dots n]}) \leq u$.*

Furthermore, γ and \bar{I} are such that for every $T \in M_n$, $T \in I_{\gamma(T)}$, i.e., they agree on the matching of the terms of M_n . We write $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$, if γ corresponds to a run of M_E , or $\bar{\mu}, \gamma \models M_n$, if M_E is clear from the context.

Proof. (\leftarrow). We proceed by induction on the length of the event sequence $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$.

Base case. If $n = 0$, the only well defined function on an empty codomain is the function $\gamma_0 : \emptyset \rightarrow \emptyset$ with an empty domain, which vacuously satisfies the definition of matching function and Items (I) and (II). Then, the only run of $M_E = (V, D, \emptyset, 0)$ on an empty event sequence $\bar{\mu}$ is the empty run \bar{I} yielding M_E itself, which vacuously satisfies the definition of run.

Inductive step. Let $\gamma : M_n \rightarrow [1, \dots, n]$ be a matching function satisfying Items (I) and (II), and let $\gamma|^{<n} : M_{n-1} \rightarrow [1, \dots, n-1]$ be the restriction of γ on the domain M_{n-1} defined as the inverse image of $[1, \dots, n-1]$ under γ , i.e., $M_{n-1} = \gamma^{-1}([1, \dots, n-1])$. $\gamma|^{<n}$ is a matching function for the event sequence $\bar{\mu}_{[1 \dots n-1]}$ and satisfies Items (I) and (II). By the inductive hypothesis, there exists a run $\langle I_1, \dots, I_{n-1} \rangle$ of M_E on $\bar{\mu}_{[1 \dots n-1]}$, yielding a matching structure $M_{n-1} = (V, D_{n-1}, M_{n-1}, t_{n-1})$. Let $I_n = \gamma^{-1}(n)$, and note that $I_n \subseteq \overline{M_{n-1}}$. We show that $\mu_n = (A_n, \delta_n)$ is an I_n -match event for M_{n-1} by breaking the proof in steps.

(Step: μ_n is an *admissible* event for M_{n-1}). Let $T \in M_{n-1}$ and $T' \notin M_{n-1}$. If $D_{n-1}[T', T] = +\infty$, $\delta_n \leq D_{n-1}[T', T]$ trivially holds. Otherwise, there exists an atom $T \leq_{[l, u]} T'$ in C and $D_{n-1}[T', T] = u - \delta(\bar{\mu}_{[\gamma|^{<n}(T) \dots n-1]})$. We consider two cases based on whether T' belongs to the domain of γ , or not. In the first case, $\gamma(T') = n$ and $\delta(\bar{\mu}_{[\gamma(T) \dots n-1]}) + \delta_n = \delta(\bar{\mu}_{[\gamma(T) \dots \gamma(T')]}) \leq u$, by Item (I). In the second case, $\delta(\bar{\mu}_{[\gamma(T) \dots n-1]}) + \delta_n = \delta(\bar{\mu}_{[\gamma(T) \dots n]}) \leq u$, by Item (II). In either case, $\delta_n \leq u - \delta(\bar{\mu}_{[\gamma(T) \dots n-1]}) = D_{n-1}[T', T]$.

(Step: Item 1a of Definition 4.5). Let $a[x = v]$ be a quantified token of \mathbf{E} . If $\text{start}(a) \in I_n$, then $\gamma(\text{start}(a)) = n$ and by definition of matching function $\text{start}(x, v) \in A_n$.

(Step: Item 1b of Definition 4.5).(\leftarrow). Let $\text{end}(a) \notin M_{n-1}$ be a possible candidate for inclusion in I_n . If $\text{start}(a) \in M_{n-1}$ and $\text{end}(x, v) \in A_n$, then $\text{end}(x, v)$ ends the token started at $\mu_{\gamma|<n}(\text{start}(a))$; otherwise, there would exist $\mu_i = (A_i, \delta_i)$ prior to μ_n such that $\text{end}(x, v) \in A_i$, contradicting that $\gamma|<n$ is undefined on $\text{end}(a)$. By definition of matching function, since $\text{end}(x, v) \in A_n$ ends the token started at $\mu_{\gamma(\text{start}(a))}$, we have $\gamma(\text{end}(a)) = n$ and $\text{end}(a) \in I_n$.

(Step: Item 1b of Definition 4.5).(\rightarrow). If $\text{end}(a) \in I_n$, then by definition of matching function $\text{end}(x, v) \in A_n$. Furthermore, since $\text{end}(a) \in M_n$, Item (I) gives $\gamma(\text{start}(a)) \leq \gamma(\text{end}(a))$ for the atom $\text{start}(a) \leq_{[l,u]} \text{end}(a)$ in \mathbf{C} . By definition of event sequence, $\text{start}(x, v)$ and $\text{end}(x, v)$ cannot appear in the same event; hence, $\gamma(\text{start}(a)) < \gamma(\text{end}(a)) = n$ and $\text{start}(a) \in M_{n-1}$.

(Step: Item 2a of Definition 4.5). Let T be a term in I_n , and let $T' \in V$ be any other term such that $D_{n-1}[T', T] \leq 0$. Then, $D_{n-1}[T', T]$ can either be the lower bound of an atom $T' \leq_{[l,u]} T$, or the upper bound of an atom $T \leq_{[l,u]} T'$ in \mathbf{C} . In the first case, we can directly conclude that $T' \in M_{n-1} \cup I_n$, because $T' \in M_n$ by Item (I) of γ and $M_n = M_{n-1} \cup I_n$ by definition of M_{n-1} and I_n . In the second case, note that $D_{n-1}[T', T] = u$, *i.e.*, it has never been decremented because $T \notin M_{n-1}$, and that upper bounds u can never be negative. Thus, u is equal to 0 and γ satisfies $0 \leq \delta(\bar{\mu}_{[\gamma(T) \dots \gamma(T')]} \leq 0$ (Item (I)), meaning that $\gamma(T') = \gamma(T)$ and $T' \in I_n$.

(Step: Item 2b of Definition 4.5). Let $T \in I_n$ and $T' \in M_{n-1}$. $D_{n-1}[T', T]$ cannot be the upper bound of an atom $T \leq_{[l,u]} T'$; otherwise, Item (I) would imply $T \in M_{n-1}$, contradicting $T \in I_n$. Thus, $D_{n-1}[T', T]$ must either represent the lower bound of an atom $T' \leq_{[l,u]} T$ in \mathbf{C} , or be equal to $+\infty$. In the latter case, $\delta_n \geq -D_{n-1}[T', T]$ trivially holds. In the former case, $D_{n-1}[T', T] = -l + \delta(\bar{\mu}_{[\gamma|<n}(T') \dots n-1])$. Since $\gamma(T) = n$, we have $\delta(\bar{\mu}_{[\gamma(T') \dots \gamma(T)]}) = \delta(\bar{\mu}_{[\gamma(T') \dots n]}) = \delta(\bar{\mu}_{[\gamma(T') \dots n-1]}) + \delta_n \geq l$. Hence, $\delta_n \geq l - \delta(\bar{\mu}_{[\gamma(T') \dots n-1]}) = -D_{n-1}[T', T]$.

(Step: Item 2c of Definition 4.5). Let $T, T' \in I_n$ be two distinct terms. Then, $\gamma(T') = \gamma(T)$ and $\delta(\bar{\mu}_{[\gamma(T') \dots \gamma(T)]}) = 0$. If $T \leq_{[l,u]} T'$ (resp., $T' \leq_{[l,u]} T$) belongs to \mathbf{C} , then $D_{n-1}[T, T']$ (resp., $D_{n-1}[T', T]$) is the lower bound l and equals 0 by Item (I). Otherwise, $D_{n-1}[T, T'] = D_{n-1}[T', T] = +\infty$.

Hence, $M_{n-1} \xrightarrow{\mu_n, I_n} M_n$ is well defined and $\langle I_1, \dots, I_n \rangle$ is a run of \mathbf{M}_E on $\bar{\mu}$ yielding \mathbf{M}_n .

(\rightarrow). We proceed by induction on the length of the event sequence $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$.

Base case. An empty run \bar{I} yields $\mathbf{M}_E = (V, D, \emptyset, 0)$ itself. Then the function $\gamma_0 : \emptyset \rightarrow \emptyset$ vacuously satisfies the definition of matching function and Items (I) and (II).

Inductive step. Let $\bar{I} = \langle I_1, \dots, I_n \rangle$ be a run of \mathbf{M}_E on $\bar{\mu}$, yielding a matching structure $\mathbf{M}_n = (V, D_n, M_n, t_n)$. Note that $\bar{I}_{[1 \dots n-1]}$ is a run of \mathbf{M}_E on $\bar{\mu}_{[1 \dots n-1]}$ yielding a matching structure $\mathbf{M}_{n-1} = (V, D_{n-1}, M_{n-1}, t_{n-1})$. By the inductive hypothesis, there exists a matching function $\gamma_{<n} : M_{n-1} \rightarrow [1, \dots, n-1]$ satisfying Items (I) and (II). Let $\gamma : M_n \rightarrow [1, \dots, n]$ be the extension of $\gamma_{<n}$ to M_n , such that $\gamma(T) = n$, for all $T \in I_n$.

(Step: γ is a matching function). Items 1 and 2 hold for all the terms already present in the domain of $\gamma_{<n}$. For every term in I_n , Item 1 for γ follows from Item 1 of I_n -match event. Let $\text{start}(a), \text{end}(a) \in M_n$ be two terms not both already present in M_{n-1} , meaning that $\text{start}(a) \in M_{n-1}$ and $\text{end}(a) \in I_n$, for some quantified token $a[x = v]$ in \mathbf{E} . By definition of I_n -match event, $\mu_n = (A_n, \delta_n)$ is such that $\text{end}(x, v) \in A_n$ and no other event in

$\mu_{[\gamma_{<n}(T)\dots n-1]}$ contains an action $\text{end}(x, v)$, otherwise $\text{end}(a)$ would already belong to M_{n-1} (by Item 1b of I -match event). Thus, $\text{end}(x, v) \in A_n$ ends the token started at $\mu_{\gamma(\text{start}(a))}$, and $\gamma(\text{start}(a))$ and $\gamma(\text{end}(a))$ correctly identify the endpoints of such token.

(Step: Item (I) of Lemma 4.8). Let $T \leq_{[l,u]} T'$ be an atom in \mathbf{C} , and note that γ already satisfies Item (I) for every $T' \in M_{n-1}$. If $T' \in I_n$ instead, consider the entry $D_{n-1}[T, T']$ representing the lower bound l of the aforementioned atom. If $D_{n-1}[T, T'] \leq 0$, Item 2a of I -match event gives $T \in M_{n-1} \cup I_n = M_n$. If $D_{n-1}[T, T'] > 0$, $D_{n-1}[T, T']$ no longer stores its initial value $-l \leq 0$, meaning that T must have been previously matched and $T \in M_{n-1} \subseteq M_n$. In either case, $T \in M_n$ and $\gamma(T) \leq \gamma(T')$, because $\gamma(T) \leq n$. If $T \in I_n$, then $\delta(\bar{\mu}_{[\gamma(T)\dots \gamma(T')]}) = 0 \leq u$, is trivially satisfied by any upper bound u . Furthermore, by Item 2c of I -match event, either the lower bound $D_{n-1}[T, T'] = 0$ or the upper bound $D_{n-1}[T', T] = 0$, and they both equal their initial values l and u . Note that the former case is also implied by the latter, so that $l = 0$ and $l \leq \delta(\bar{\mu}_{[\gamma(T)\dots \gamma(T')]})$. If $T \in M_{n-1}$, by Item 2b of I -match event, $\delta_n \geq -D[T, T'] = l - \delta(\bar{\mu}_{[\gamma(T)\dots n-1]})$. Hence, $l \leq \delta(\bar{\mu}_{[\gamma(T)\dots n-1]}) + \delta_n = \delta(\bar{\mu}_{[\gamma(T)\dots \gamma(T')]})$. While $\delta_n \leq D_{n-1}[T', T] = u - \delta(\bar{\mu}_{[\gamma(T)\dots n-1]})$, since μ_n is an admissible event for M_{n-1} . Hence, $\delta(\bar{\mu}_{[\gamma(T)\dots n-1]}) + \delta_n = \delta(\bar{\mu}_{[\gamma(T)\dots \gamma(T')]}) \leq u$.

(Step: Item (II) of Lemma 4.8). Let $T \leq_{[l,u]} T'$ be an atom in \mathbf{C} such that $T \in M_n$ and $T' \notin M_n$. Since μ_n is an admissible event for M_{n-1} , $\delta_n \leq D_{n-1}[T', T] = u - \delta(\bar{\mu}_{[\gamma(T)\dots n-1]})$. Hence, $\delta(\bar{\mu}_{[\gamma(T)\dots n-1]}) + \delta_n = \delta(\bar{\mu}_{[\gamma(T)\dots n]}) \leq u$. \square

Lemma 4.10. *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence, let M_E be the initial matching structure of some existential statement E of a rule R , and let M_r be an active matching structure resulting from a run $M_E \xrightarrow{\bar{\mu}, \gamma_r} M_r$, such that $\gamma_r(\text{start}(a_0)) = r$. If there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma_s} M_s$, such that $\gamma_s(\text{start}(a_0)) < r$, then there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma} M$, such that $\gamma(\text{start}(a_0)) = \gamma_s(\text{start}(a_0))$ and M matches at least as many tokens as M_r .*

Proof. Let $M_E \xrightarrow{\bar{\mu}, \gamma_r} M_r = (V, D_r, M_r, t_r)$ and $M_E \xrightarrow{\bar{\mu}, \gamma_s} M_s = (V, D_s, M_s, T_s)$, with $\gamma_s(\text{start}(a_0)) \leq \gamma_r(\text{start}(a_0))$. Let $M = M_r \cup M_s$ and $\gamma : M \rightarrow [1, \dots, n]$ be a function defined as:

$$\gamma(T) = \begin{cases} \gamma_s(T) & \text{if } T \in M_s \cap M_r \text{ and } \gamma_s(T) \leq \gamma_r(T) \\ \gamma_r(T) & \text{if } T \in M_s \cap M_r \text{ and } \gamma_s(T) > \gamma_r(T) \\ \gamma_s(T) & \text{if } T \in M_s \setminus M_r \\ \gamma_r(T) & \text{if } T \in M_r \setminus M_s \end{cases}$$

(Step: γ is a matching function). Item 1 of Definition 3.6 for γ follows from our hypothesis on γ_s and γ_r . Regarding Item 2, let $\text{start}(a), \text{end}(a) \in M$ for some quantified token $a[x = v]$ in E . If γ_s and γ_r map the endpoints of a to the same token in $\bar{\mu}$, then $\gamma(\text{start}(a))$ and $\gamma(\text{end}(a))$ correctly identify the endpoints of that token. If instead γ_s and γ_r map a to two distinct tokens in $\bar{\mu}$, then γ would match a according to the function whose token comes first, correctly identifying the endpoints of such token.

(Step: γ satisfies Items (I) and (II) of Lemma 4.8). Let $T \leq_{[l,u]} T'$ be an atom in \mathbf{C} . If $T' \in M$, then either $T' \in M_s$, and $T \in M_s \subseteq M$, or $T' \in M_r$, and $T \in M_r \subseteq M$. If γ maps both terms with either γ_s or γ_r , then $\gamma(T) \leq \gamma(T')$ and $l \leq \delta(\bar{\mu}_{[\gamma(T)\dots \gamma(T')]}) \leq u$ immediately follows. If instead $\gamma(T) = \gamma_s(T)$ and $\gamma(T') = \gamma_r(T')$, then $T' \in M_r$ and $T \in M_s \cap M_r$. By definition of γ , $\gamma_s(T) \leq \gamma_r(T)$, and, by Item (I) for γ_r , $\gamma_r(T) \leq \gamma_r(T')$. Hence, $\gamma(T) \leq \gamma(T')$.

If $T' \in M_s$, then $\gamma_s(T') > \gamma_r(T')$, and:

$$\begin{array}{ll}
l \leq \delta_{\gamma_r(T), \gamma_r(T')} & \text{Item (I) for } \gamma_r \\
\leq \delta_{\gamma_s(T), \gamma_r(T')} & \gamma_s(T) \leq \gamma_r(T) \\
< \delta_{\gamma_s(T), \gamma_s(T')} & \gamma_s(T') > \gamma_r(T') \\
\leq u & \text{Item (I) for } \gamma_s
\end{array}$$

otherwise:

$$\begin{array}{ll}
l \leq \delta_{\gamma_r(T), \gamma_r(T')} & \text{Item (I) for } \gamma_r \\
< \delta_{\gamma_s(T), \gamma_r(T')} & \gamma_s(T) \leq \gamma_r(T) \\
\leq \delta_{\gamma_s(T), n} & \gamma_r(T') \leq n \\
\leq u & \text{Item (II) for } \gamma_s
\end{array}$$

The case for $\gamma(T) = \gamma_r(T)$ and $\gamma(T') = \gamma_s(T')$ is completely symmetrical.

Lastly, if $T' \notin M$, but $T \in M$, then either $\gamma(T) = \gamma_s(T)$ or $\gamma(T) = \gamma_r(T)$, and Item (II) for γ follows from Item (II) for γ_s and γ_r . \square

Lemma 4.13 (Existence of residual matching structure). *Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence, and let M_n be an active matching structure such that $\bar{\mu}, \gamma \models M_n$ and $\delta(\bar{\mu}_{[\gamma(\text{start}(a_0)) \dots n]}) > \text{window}(P)$. If we consider the intermediate matching structures $\langle M_1, \dots, M_{n-1} \rangle$ of the run $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$, then there exists a position $\gamma(\text{start}(a_0)) \leq k < n$ such that M_k is a residual matching structure.*

Proof. Let $\gamma(\text{start}(a_0)) = s$, assuming there is no residual matching structure M_k in the sequence $\langle M_s, \dots, M_{n-1} \rangle$, then for every matching structure $M_i = (V, D_i, M_i, t_i)$, where $s \leq i < n$, there exists a pair of terms (T, T') such that $T \in M_i$ and $T' \notin M_i$, and their distance $D_i[T', T]$ has a finite upper bound. Let $E \subseteq V \times V$ be the set that collects all pairs (T, T') for the matching structures M_i . We define $\delta_{T, T'}$ as $\delta(\bar{\mu}_{[\gamma(T) \dots \gamma(T')]})$ if $\gamma(T')$ is defined, or as $\delta(\bar{\mu}_{[\gamma(T) \dots n]})$ otherwise. Let $\delta_E = \sum_{(T, T') \in E} \delta_{T, T'}$ and note that $\delta_E \geq \delta(\bar{\mu}_{[\gamma(\text{start}(a_0)) \dots n]})$, because every position in $\bar{\mu}_{[\gamma(\text{start}(a_0)) \dots n]}$ is covered by some distance $\delta_{T, T'}$. Moreover, each pair $(T, T') \in E$ corresponds to an atom of the form $T \leq_{[l, u]} T'$ in \mathcal{C} . According to Lemma 4.8, we have $\delta_{T, T'} \leq u$, and therefore, $\delta_E \leq \text{window}(P)$. Hence, we have $\delta(\bar{\mu}_{[\gamma(\text{start}(a_0)) \dots n]}) \leq \delta_E \leq \text{window}(P)$: a contradiction hence proving the existence of a residual matching structure M_k . \square

Theorem 4.14 (Soundness and completeness). *Let $P = (\text{SV}, S)$ be a timeline-based planning problem and let A_P be the associated automaton. Then, any event sequence $\bar{\mu}$ is a solution plan for P if and only if $\bar{\mu}$ is accepted by A_P .*

Proof. (\longrightarrow). Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be a solution plan for P , and let $\bar{q} = \langle q_0, \dots, q_n \rangle$ be the run of A_P on $\bar{\mu}$. We first show that the sink state is never reached, and then that q_n is a final state.

Let $\mu_s = (A_s, \delta_s)$ be the trigger event of a rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \vee \dots \vee E_m$, i.e., $\text{start}(x_0, v_0) \in A_s$. Since $\bar{\mu}$ is a solution plan, there exist tokens satisfying an existential statement E of R for the trigger μ_s . Hence, by Lemma 4.8 and Observation 4.9 there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$, yielding a closed matching structure M_n , such that $\gamma(\text{start}(a_0)) = s$.

Let $\bar{M} = \langle M_E, M_1, \dots, M_n \rangle$ be the sequence of all the matching structures involved in such run. Note that, by construction (Section 4.3), the states of \bar{q} induce all the possible

runs for the *initial* matching structures of P that can be defined on $\bar{\mu}$. In particular, the run γ must be one of them. However, only a subsequence of the matching structures \bar{M} will appear in the states of the run \bar{q} . Indeed, we can identify three key points for the sequence \bar{M} : the least position s such that M_s is *active* (corresponding to $\gamma(\text{start}(a_0))$), the least position h following s such that M_h no longer belongs to the component Υ of the states in $\bar{q}_{[h\dots n]}$, either because M_h is *closed* or because $\delta(\bar{\mu}_{[s\dots h]}) > \text{window}(P)$, and the least position k following h such that M_k no longer belongs to the component Δ of the states in $\bar{q}_{[k\dots n]}$, either because M_k is *closed* or because it gets discarded in favour of the matching structures of a later trigger event.

Every matching structure in $\bar{M}_{[1\dots h-1]}$ belongs to the component Υ of a corresponding state in $\bar{q}_{[1\dots h-1]}$, so the set Υ of the state q_{s-1} is such that $M_s \in \text{step}_{\mu_s}(\Upsilon_{\perp})$, satisfying condition 5b of Section 4.3 for the trigger event μ_s . Matching structures $\bar{M}_{[s+1\dots h]}$ instead belong to the set $\text{step}_{\mu}(\Upsilon_t^R)$, for the partition Υ_t^R tracking the satisfaction of the trigger event μ_s of every state $\bar{q}_{[s\dots h-1]}$. Hence, all such states satisfy condition 5a of Section 4.3.

We now show that no *active* matching structures for the trigger event μ_s exists after some state q_h , following q_s in \bar{q} . Note that the run γ yields a *closed* matching structure, and if it does so within $\text{window}(P)$ time units from the event $\mu_{\text{start}(a_0)}$, we identified such position as the closed matching structure M_h . So that the state q_{h-1} is such that $M_h \in \text{step}_{\mu_h}(\Upsilon_t^R)$, for the partition Υ_t^R tracking the trigger event μ_s , and $\text{step}_{\mu_h}(\Upsilon_t^R)$ is discarded from q_h .

If instead γ yields a *closed* matching structure after $\text{window}(P)$ time units from the event $\mu_{\text{start}(a_0)}$, lets identify such position as M_j , with $j \leq n$. If M_{j-1} belongs to the set $\Delta(E)$ of the state q_{j-1} , then $M_j \in \text{step}_{\mu_j}(\Delta(E))$, so that $\text{step}_{\mu_j}(\Delta(E))$ is *closed* and discarded from q_j , alongside all the other matching structures in $\Delta(E')$, for every $E' \in \Phi(E)$, *i.e.*, for every other existential statement E' of R still tracking the trigger μ_s . If instead M_{j-1} for the trigger event μ_s does not belong to the set $\Delta(E)$ of the state q_{j-1} , by construction (Section 4.3), there exist a state q_h in which the matching structures tracking μ_s have been replaced by those of a later event, and they no longer appear in $\Delta(E)$ from q_h onwards.

Since $\bar{\mu}$ is a solution plan, the previous argument holds for all the trigger events in $\bar{\mu}$ of any rules in S . Hence, conditions 5a and 5b are always met, *i.e.*, the sink state is never reached, and no active matching structures belong to q_n , making it a final state.

(\leftarrow). Let $\bar{\mu} = \langle \mu_1, \dots, \mu_n \rangle$ be an event sequence accepted by A_P and let $\rho = \langle q_0, \dots, q_n \rangle$ be its accepting run. We have to show that the plan corresponding to $\bar{\mu}$ is a solution plan for P , *i.e.*, for every event triggering a rule R in S , at least one of the existential statements of R is satisfied by $\bar{\mu}$.

Let $\mu_s = (A_s, \delta_s)$ be an event in $\bar{\mu}$ triggering a rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \vee \dots \vee E_m$, *i.e.*, $\text{start}(x_0, v_0) \in A_s$. Since the sink state is never visited in an accepting run, the state q_s , reached upon reading the event μ_s , is such that the partition Υ_0^R , tracking the satisfaction of the trigger event μ_s , is not empty. For the same reason, the partition Υ_t^R tracking μ_s in every state following q_s can never be empty as a result of the function step_{μ} . However, since the final state q_n does not contain any *active* matching structure, there must exists a state q_h in \bar{q} whose partition $\text{step}_{\mu_{h+1}}(\Upsilon_t^R)$ gets discarded from q_{h+1} . This can happen either because $\text{step}_{\mu_{h+1}}(\Upsilon_t^R)$ is a *closed* set, or because the matchings structures in $\text{step}_{\mu_{h+1}}(\Upsilon_t^R)$ get promoted to the component Δ . In the first case, we can conclude that there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma} M_n$ for the initial matching structure M_E of an existential statement E of R

such that M_n is *closed* and $\gamma(\text{start}(a_0)) = s$, hence, by Observation 4.9 and Lemma 4.8, the trigger event μ_s satisfies R.

In the second case, let Ψ be the set of existential statements having an active matching structure in $\text{step}_{\mu_{h+1}}(\Upsilon_t^R)$, so that we can identify them as the sets $\Delta(E)$, for $E \in \Psi$, in the states from q_{h+1} onwards. By Lemma 4.13, every such set contains a *residual* matching structure. Hence, by Observation 4.12, they can become empty only if, at some state q_k following q_h , $\text{step}_{\mu_{k+1}}(\Delta(E))$ contains a *closed* matching structure for some existential statement $E \in \Psi$. Note that the run \bar{q} is an accepting run, so every non-empty set $\Delta(E)$ must become empty before the end of the run. Hence, q_k is guaranteed to exist.

However, it may be the case that, by the time $\text{step}_{\mu_{k+1}}(\Delta(E))$ is *closed*, $\Delta(E)$ no longer contains the matching structures for the trigger event μ_s , but those for a later trigger event μ_r of R. Since the sets $\Delta(E)$ store only the matching structures tracking the most recent trigger event older than $\text{window}(P)$. Thus, if $\text{step}_{\mu_{k+1}}(\Delta(E))$ contains a *closed* matching structure for μ_s , we can directly assert the existence of a run for M_E implying the satisfaction of R for the trigger event μ_s . If instead $\text{step}_{\mu_{k+1}}(\Delta(E))$ contains a *closed* matching structure M_r for a later event μ_r , there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma_r} M_r$, such that $\gamma_r(\text{start}(a_0)) = r$. Furthermore, by a previous consideration on q_{h+1} , there exists a run $M_E \xrightarrow{\bar{\mu}_{[1\dots h+1]}, \gamma_s} \hat{M}_{h+1}$, yielding a *residual* matching structure \hat{M}_{h+1} , and, by Observation 4.12, such run can be extended on the entire event sequence $M_E \xrightarrow{\bar{\mu}, \gamma_s} \hat{M}_n$, to yield a *residual* matching structure \hat{M}_n . Given $M_E \xrightarrow{\bar{\mu}, \gamma_r} M_r$ and $M_E \xrightarrow{\bar{\mu}, \gamma_s} \hat{M}_n$, with $\gamma_s(\text{start}(a_0)) \leq \gamma_r(\text{start}(a_0))$, by Lemma 4.10, there exists a run $M_E \xrightarrow{\bar{\mu}, \gamma_s} M_n$ yielding a matching structure M_n matching as many terms as M_r and such that $\gamma_s(\text{start}(a_0)) = s$. Hence, M is a *closed* matching structure for the existential statement E , and, by Observation 4.9 and Lemma 4.8, R satisfies the trigger event μ_s .

Furthermore, all the value duration functions are satisfied by the tokens in $\bar{\mu}$, being encoded as synchronisation rules by the automaton S_P . Meanwhile, the automaton T_P guarantees the fulfilment of the value transition functions. Hence, we can conclude that $\bar{\mu}$ is a solution plan for P , because every rule in S is satisfied, as well as the value duration and value transition functions of every state variable. \square