ADVANCED MODEL CONSISTENCY RESTORATION WITH HIGHER-ORDER SHORT-CUT RULES

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Abstract. Sequential model synchronisation is the task of propagating changes from one model to another correlated one to restore consistency. It is challenging to perform this propagation in a least-changing way that avoids unnecessary deletions (which might cause information loss). From a theoretical point of view, so-called short-cut (SC) rules have been developed that enable provably correct propagation of changes while avoiding information loss. However, to be able to react to every possible change, an infinite set of such rules might be necessary. Practically, only small sets of pre-computed basic SC rules have been used, severely restricting the kind of changes that can be propagated without loss of information. In this work, we close that gap by developing an approach to compute more complex required SC rules on-the-fly during synchronisation. These higher-order SC rules allow us to cope with more complex scenarios when multiple changes must be handled in one step. We implemented our approach in the model transformation tool eMoflon. An evaluation shows that the overhead of computing *higher-order SC rules* on-the-fly is tolerable and at times even improves the overall performance. Above that, completely new scenarios can be dealt with without the loss of information.

1. INTRODUCTION

Model-Driven Engineering (MDE) [\[BCW17\]](#page-30-0) provides the necessary means to tackle the challenges of modern software systems, which become more and more complex and distributed. Using MDE, a system can be described by various models that describe different aspects and provide specific views onto the system itself, where each view may overlap to some degree with other views of the same system. Consequently, if one view changes, we have to propagate these changes to other views that share the same information. This is necessary degree with other
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to ensure that all models together consistently represent the overall system state. The propagation process is often called model synchronisation, where we distinguish between sequential and concurrent model synchronisation. In this paper, we will focus on the former, where only one model is changed at a time, while the latter case would also incorporate changes to multiple models at once.

As our methodology of choice, we employ *Triple Graph Grammars* (TGGs) [\[Sch94\]](#page-33-0), which are a declarative way to specify a consistency relationship between possibly heterogeneous models by means of a set of graph grammar rules. By transforming these rules, we can automatically derive different consistency restoring operators that served as central ingredients for various model management processes, such as batch translators [\[Sch94,](#page-33-0) $EEE⁺07, HEGO14$ $EEE⁺07, HEGO14$, consistency checkers [\[Leb18\]](#page-32-1) or sequential [\[GW09,](#page-32-2) [LAVS12,](#page-32-3) [LAF](#page-32-4)⁺17, [FKST21\]](#page-31-1) and concurrent synchronisers [\[OPN20,](#page-33-1)[FKM](#page-31-2)+20,[WFA20\]](#page-33-2). In general, synchronisers are required to avoid information loss. Most TGG-based approaches, in fact, do not meet this requirement: They propagate changes by deleting (parts of) the model that is to be synchronised and then retranslating (the missing parts) from the updated model [\[GW09,](#page-32-2) [LAVS12,](#page-32-3) $HEO⁺15$, [Leb18,](#page-32-1) [LAF](#page-32-4)⁺17. This procedure is both inefficient and prone to loss of information that is private to the synchronised model. For this purpose, we introduced Short-Cut Rules [\[FKST18,](#page-31-3)[FKST21,](#page-31-1) [KT23\]](#page-32-6), which implement consistency- and informationpreserving modifications. Compared to former approaches, Short-Cut Rules reduce (or even completely avoid) the unnecessary deletion of elements in the model that is to be synchronised but are able to reuse these elements and integrate them consistently. In this way, deleting elements just to re-create them is avoided and information is preserved instead. Even more, we can provide conditions under which the application of a *Short-Cut Rule* reestablishes or at least improves consistency [\[Kos22,](#page-32-7) [KT23\]](#page-32-6).

The translation and, thus, revocation of former translation steps is based on basic TGG rules that describe atomic consistency-preserving model changes. The same holds for our first works on *Short-Cut Rules* [\[FKST18,](#page-31-3) [FKST21\]](#page-31-1), where we created them by combining pairs of basic TGG rules. While the resulting *Short-Cut Rules* describe useful repairs in many situations, we found that there are cases where we have to repair multiple inconsistencies at once to yield a better result. Hence, we here propose to concatenate TGG rules that describe a sequence of atomic model changes and derive *Short-Cut Rules* from these to repair multiple TGG rule applications at once. Since there are usually infinitely many possibilities of concatenating TGG rules and thus infinitely many Short-Cut Rules, we propose an analysis that determines at runtime which TGG rules to concatenate to construct a Short-Cut Rule for a particular scenario. This allows us to also reduce information loss during synchronisation in scenarios where the *Short-Cut Rules* we computed so far did not suffice.

In more details, we make the following contributions in this paper:

- We introduce a runtime analysis for the derivation of what we call *higher-order Short-*Cut Rules; their use promises to reduce the amount of lost information during model synchronisation.
- We show how our proposed runtime analysis can be presented as an integer linear programming (ILP) problem [\[LY84\]](#page-32-8).
- We integrate our approach into eMoflon [\[WFA20\]](#page-33-2), a state-of-the-art graph transformation tool that implements TGGs.
- We evaluate our approach on a synthetic example; our key findings are that we can indeed preserve more information in certain cases and that while our implementation

This paper extends our previous contribution [\[FKMS23\]](#page-31-4), where we were only able to convey the general concept behind finding *Short-Cut Rules* that repair multiple steps at once. In this extension, we provide a detailed description of the construction process of Short-Cut Rules by presenting it as an ILP problem and, as promised in our previous work, evaluate our approach for another scenario that is particularly challenging for it.

This paper is structured as follows: We introduce TGGs and *Short-Cut Rules* together with our running example in Section [2.](#page-2-0) In Section [3,](#page-7-0) we will present our novel analysis to determine non-trivial consistency-restoring operators in the form of *Short-Cut Rules* at runtime. The construction process of *Short-Cut Rules* using optimisation techniques is presented in Section [4.](#page-14-0) Section [5](#page-24-0) presents our evaluation investigating how much information is preserved and whether it comes with additional costs. Finally, in Section [6,](#page-28-0) we discuss related works and summarise open challenges in Section [7.](#page-30-1)

2. Fundamentals

This section provides an informal introduction to the state-of-the-art of TGG-based information-preserving model synchronisation. We begin with introducing TGGs along with our running example. Then, we illustrate the need for synchronisation and exemplify precedence graphs, which are the central data structure we use to keep track of inconsistencies between models and to direct our synchronisation process. Finally, based on a small yet non-trivial synchronisation example, we will motivate the need for specific repair rules, namely Short-Cut Rules.

2.1. Running Example & Triple Graph Grammars. In our running example throughout this paper, we will define consistency between terrace house and construction planning models to which we will refer in the following as source and target, respectively. Due to the complexity of the derived repair operations later in this paper, we had to choose a rather small example. Yet, TGGs have been successfully applied in industrial applications [\[GHNW12,](#page-31-5) $BW03, AWO⁺20$ $BW03, AWO⁺20$ $BW03, AWO⁺20$. As a first step, we introduce a third correspondence model between source and target, which connects elements from both sides and thus makes corresponding information traceable. Figure [1](#page-3-0) depicts the three metamodels with the source metamodel on the left, the target metamodel on the right and the correspondence metamodel in between. The source metamodel consists solely of the House class and HouseType enum. Each House contains information about its HouseType and has a reference to the next House (neighbour). The HouseType determines the architecture of a House on the target side, namely which construction steps are needed to build that House. This relation will be expressed in the following using a set of graph grammar rules. Additionally, each building contains information about its architect, which has no representation in the target model. On the target side, there are six classes. There is a (construction) Plan for every row of houses that contains the corresponding Constructions. Each Construction contains the name of its assigned (construction) company and a sequence of Construction Steps consisting of Cellar, Floor and Saddle Roof elements that have to be processed in the given order. Finally, we connect both metamodels using the correspondence type depicted as a hexagon, which maps Houses to their corresponding Construction.

Figure 1: Source, Target and Correspondence Metamodel

To define a consistency relationship between source and target models, we employ Triple Graph Grammars (TGGs) [\[Sch94\]](#page-33-0), which are a declarative and rule-based way to achieve this. As the name indicates, the rules of a TGG define the language of all consistent triple graphs, where our source, target and correspondence models are interpreted as graphs. Figure [2](#page-4-0) shows the TGG rule set of our running example, consisting of the three rules on the left. Before a rule can be applied, its precondition must be met, meaning that all context elements in black must exist and all attribute conditions hold. When the rule is applied, all elements depicted in green and annotated with $++$ are created. The first rule is Nook Rule, which can be applied without any precondition. It creates a Nook House together with a corresponding Construction on the target side and a connecting correspondence link. Since this will create the first House of a row, it also creates a Plan on the target side. Finally, each Nook House must have a Floor but no Cellar or Saddle Roof. The two rules Cube Rule and Villa Rule are very similar to each other in that they both require a House in the row that must already exist together with a corresponding Construction and Plan. Given that, they create a House with the Cube or Villa HouseType as the next House in line and a corresponding Construction, where a Cube has a Cellar and Floor, while a Villa will have a Floor and Saddle Roof. For better readability, we do not display edge types in the rules; they can be unambiguously inferred from the metamodels.

Using these rules, we can create consistent models from scratch. More interestingly, we can transform these rules to obtain forward translation rules as is depicted on the right in Figure [2.](#page-4-0) The main difference to our original rules is that their created source elements have now become part of the precondition. Assuming that we want to translate a source to a target model, this makes sense as a source model must already exist beforehand. To avoid translating elements more than once, we introduce the annotations $\Box \to \Box$ and \Box ^{[1](#page-3-1)}. Source elements that were created in the original TGG rule are annotated with $\Box \to \Box$ in the derived forward rule. This indicates that they mark elements upon translation such that the rule is only applicable to elements that have not been marked yet. Other source elements that were already black are annotated with \mathbb{Z} , which means that these elements must have already been marked; different formalisations of such a marking mechanism are available [\[HEGO14,](#page-32-0) $LAF+17$, [Kos22\]](#page-32-7). Also, attribute assignments now turn to attribute constraints. In our example this means that instead of setting the HouseType each forward rule can only be used to translate a specific **HouseType**. *Backward rules* that allow translating construction plans to the corresponding row of houses can be derived completely analogously.

¹These markings are not needed for the original TGG rules.

Figure 2: TGG Rules

2.2. Precedence Graphs & Synchronisation. We want to incrementally synchronise changes from one model to another. Figure [3](#page-5-0) depicts a small model, which was consistent w.r.t. our TGG but then was changed on the source side. On the left, we see a Nook House h_1 , followed by a Villa h_2 and a Cube h_3 , while on the right side there are the expected corresponding elements in the form of a Plan p_1 and three Constructions c_1 , c_2 and c_3 , each with its Construction Steps. As a change, we deleted the second House h_2 , which means that the third House h_3 now succeeds h_1 . Furthermore, we changed the type of h_3 from Cube to Villa.

On the right side of Figure [3,](#page-5-0) we can see a so-called precedence graph, which is of particular interest for the rest of this paper. Some of its nodes correspond to the rule applications that created the original triple graph; others represent rule applications that could have possibly been used to create the newly added elements on the source side. Edges denote sequential dependence between rule applications. Each precedence node contains the initials of the corresponding TGG rule together with a small index, which is the same as the one of the created elements of this rule application. Apart from that, each node has two boxes where the left and right depict the consistency state of the source and the target side, respectively. Blank boxes indicate that the elements on this side are still intact w.r.t. the corresponding TGG rule, which means that they have not been tampered with or that the changes had no consistency violating effect, e.g., by changing the architect. Green boxes containing a "+" indicate that new elements have been detected, which can be translated by a TGG rule, while "*" would mean that there are new elements but they cannot be

Figure 3: Synchronisation Example

translated because some elements that would need to be marked together with them are already marked, i.e., they are still part of another (possibly inconsistent) rule application. In both cases, the right box for the target side is annotated with "u", expressing our lack of information on whether the corresponding target elements already exist. In contrast, red boxes containing a "-" indicate that all translatable elements on this side were deleted, meaning that created elements on the opposite side should be removed as well. A red box with "/" states that some but not all translatable elements were deleted, which means that the remaining elements must be translated differently than before. Finally, an orange box with " $\#$ " means that an attribute was changed such that a rule application has become invalid, e.g., the HouseType. A formalisation of precedence graphs via (partial) comatchs can be found in [\[Kos22\]](#page-32-7).

Regarding our example, we have an intact Nook Rule application N_1 on top. The Villa Rule application V_2 depends on N_1 as this provides the previous House, Construction and Plan. V_2 itself is no longer intact as indicated by "-" on the source side due to the deletion of House h_2 . Cube Rule application C_3 , which depends on V_2 is also no longer intact due to an attribute change within House h_3 from HouseType Cube to Villa, which is denoted as "#". C_3 is also inconsistent due to the deletion of the edge between House h_2 and h_3 , which is a partial deletion meaning that not all created source elements of this rule application were removed. Partial deletions are denoted with "/". Due to this attribute change and the new edge between House h_1 and h_3 , we could retranslate h_3 using Villa Rule as represented via the precedence node V_3 . However, since h_3 is still part of the former rule application C_3 , we find an "*" annotation on the source side of V_3 ; C_3 has to be revoked before it becomes possible to retranslate h_3 .

Those annotations and the actual precedence graph are constructed using the results of an incremental graph pattern matcher (IGPM) that tracks all rules' pre- and postconditions. If a postcondition is violated, then the IGPM engine will notify us that the postcondition can no longer be matched and we can analyse which model change caused this. For preconditions, we also track whether the source (or target) side are matched for forward (or backward) rules, which we refer to as source (or target) matches. This gives us all possible translation steps even if the necessary context has not yet been created on the target side that would enable the translation to be executed. Note, however, that some steps may be mutually

2.3. Preserving Information using Short-Cut Rules. One sees that the above-described change of moving the third house and changing an attribute should only trigger a change of Construction Steps while the information about the architect should persist. To achieve this, we use so-called *Short-Cut Rules* [\[FKST21\]](#page-31-1), which describe consistency-preserving operations, e.g., moving a house in the row without losing the target side information. Basically, a *Short-Cut Rule* revokes a rule application and applies another one instead, while preserving those elements that would be deleted just to be recreated. For our example, a Short-Cut Rule could exchange the **HouseType** while adding the missing and removing the now superfluous Construction Steps. A TGG *Short-Cut Rule* is created by overlapping an inversed TGG rule (the replaced rule that revokes a former rule application) with another TGG rule (the *replacing* rule). Deleted elements from the replaced rule that are overlapped with created elements from the replacing rule are preserved and become context as they would otherwise be unnecessarily deleted and then recreated. Consequently, deleted elements from the replaced rule that are not in the overlap must be removed and, analogously, created elements from the replacing rule that are not part of the overlap must be created. Finally, only the attribute conditions from the replacing rule must hold after applying the *Short-Cut* Rule as we want to revoke the former replaced rule application.

exclusive either because they would translate the same elements or because they rely on

other rules to create necessary context on the opposite side.

The precedence graph and the dependency links between precedence nodes tell us which TGG rules to overlap with each other. In our example (Figure [3\)](#page-5-0), there is an invalid Villa Rule application, while there is a new Cube Rule application that could be applied instead to make the now necessary transformation steps for Cube h₃. We, thus, overlap Villa Rule with Cube Rule such that the parent House is not overlapped as it has changed, while the created Houses are assumed to be the same as well as their corresponding Constructions, the Plans and Floors. Note that generally, there are many ways to overlap rules, which means that there may be many possible *Short-Cut Rules*. As discussed in Section [4,](#page-14-0) we use optimisation techniques to encode the space of all overlaps and calculate a replacing rule that matches the given situation. Note that this process is completely automated. Figure [4](#page-7-1) depicts the resulting *Short-Cut Rule* on the left. It tells us that we can move a **House** and change its HouseType from Villa to Cube at the same time if, on the opposite side, we remove the superfluous Cellar and add a new Saddle Roof.

As with TGG rules, *Short-Cut Rules* can be transformed to yield forward and backward operationalised versions. On the right of Figure [4,](#page-7-1) we can see its forward operationalisation. Intuitively, we have to make sure that a user made the same changes as the *Short-Cut Rule* on the source side so that applying the forward operationalisation will propagate these changes correctly. Similar to before, formerly created elements on the source side must be marked because they are new, while black elements remain marked. Deleted elements on

Figure 4: Cube-To-Villa Short-Cut Rule

the source side must have been deleted by a user change. Hence, we must ensure that these elements do no longer exist, which is expressed as a *negative application condition* [\[EEPT06\]](#page-31-7) depicted in blue and annotated with "nac". Also note that some context elements are omitted from the rule that would stem from the replaced rule, e.g., the context construction. We can leave some of them out, if they are not needed to perform the *Short-Cut Rule*. This is the case for the House h_1 , which is needed to check whether the edge to h_2 has been deleted. The "rel" annotation on the right side indicates that h_1 is relaxed. Relaxed means that this element does not necessarily need to exist, because if it was deleted together with its adjacent edges, then the "nac" is satisfied. Of course, this is only reasonable if h_1 is indeed the House that was used as context in the replacing rule application that we are going to replace. We ensure this by only applying *Short-Cut Rules* at locations with formerly valid replaced rule applications and using the information about these rule applications to construct those parts of our *Short-Cut Rule* match that overlap with the replaced rule.

Using Cube-To-Villa *forward Short-Cut Rule*, we can now resolve the situation from before by first processing the deleted second House and deleting its corresponding parts and then repairing the target side of the third House by using this rule. While the result looks very similar to the one of translating the whole source model from scratch, we still have the information about each Constructions' architect, which means that this information is no longer lost during synchronisation. Another advantage is that moving a House in a long row of terrace Houses can be very expensive as all succeeding rule applications would have to be revoked. In many cases, *Short-Cut Rules* can also help with this issue by preserving the consistency of subsequent steps and thereby boost the performance.

3. Higher-Order Short-Cut Rules

Formally, a short-cut rule is sequentially composed from a rule that only deletes structure and another one that only creates structure [\[FKST18\]](#page-31-3). (We have generalised that kind of sequential composition to arbitrary rules in [\[KT23\]](#page-32-6).) In practical applications so far, we made use of a static, finite set of short-cut rules; we composed each inverse of a rule from

Figure 5: Synchronisation Example

the given TGG with every rule from the TGG [\[FKM](#page-31-2)+20,[FKST21,](#page-31-1)[Fri22\]](#page-31-8). Intuitively, such a Short-Cut Rule replaces the action of the inverse rule by the one of its second input rule (while preserving common elements). Note that the *Short-Cut Rule* from Section [2](#page-2-0) is created in that way and that such *Short-Cut Rules* enable repairs in many situations. But certain complex model changes are not supported yet, namely situations in which the common effect of several TGG rules should be replaced by the effect of another set of TGG rules. For this, one needs *Short-Cut Rules* that are not just computed from the TGG rules (and their inverses) but from (arbitrary long) *concurrent rules* [\[EEPT06\]](#page-31-7), i.e., sequential compositions, of the TGG rules (and their inverses). Yet, in contrast to before, we cannot precalculate this set as there are usually infinitely many ways to concatenate an arbitrarily large set of TGG rules and, thus, infinitely many *Short-Cut Rules*. Hence, we must investigate each inconsistency and deduce what TGG rules to concatenate at runtime to create a helpful higher-order Short-Cut Rule that repairs multiple rule applications at once. In this chapter, we will explain the process of deriving these new rules using our running example and conclude with a discussion on its correctness.

3.1. Exemplifying the Need for Higher-Order Short-Cut Rules. Figure [5](#page-8-0) shows another example model together with its precedence graph. We can see two Houses, where one was of HouseType Nook and the other of HouseType Cube together with their corresponding Constructions. Then, the first House's HouseType was changed from Nook to Cube and another House of HouseType Nook was added at the beginning of this row. Similar to Figure [3,](#page-5-0) this results in an inconsistency shown in the precedence graph on the right. There, we see that the Nook Rule application N_2 has become invalid due to an attribute constraint violation. Intuitively, we would expect that a new Construction is created for the new Nook House, which is then added to the already existing Plan. However, our consistency specification makes this rather hard to achieve as the Nook Rule creates this Plan together with the first House in a row. By handling one rule application at a time, we have to translate the new Nook House, create a corresponding Construction and another Plan. Then, we would either

Figure 6: Construction Process of Higher-Order Short-Cut Rules

have to revoke the former Nook Rule application and retranslate the Cube House or use a Short-Cut Rule to transform a Nook Rule to a Cube Rule application. Since Cube Rule does not create a Plan but requires one, either way we would have to delete the old Plan and then try to connect the newly created or preserved Constructions with the new Plan. While this restores consistency, this procedure has two disadvantages. First, any information stored in the original Plan would be lost and, second, we would have to fix all succeeding rule applications to connect them with the new Plan. Hence, we want to create a *Short-Cut Rule* that preserves the original Plan by translating the new Nook House and repairing the Cube House in one step. Since also the order of the two original Houses was changed, this repair step necessarily needs to involve reacting to that and performing the required repair for the former Cube House that now is a Villa. Hence, a higher-order *Short-Cut Rule* is needed.

3.2. Deriving Higher-Order Short-Cut Rules On-the-fly. Figure [6](#page-9-0) shows our proposed process on how to obtain the necessary higher-order Short-Cut Rule using the information from an annotated precedence graph. Remember, to create a *Short-Cut Rule*, we need two ingredients: a *higher-order replaced* and *higher-order replacing* rule; i.e., one concurrent rule. built from inverses of the TGG rules, and one built from the TGG rules. The first is trivial to obtain as we can directly derive it from neighboured inconsistencies in our precedence graph (2) (2) (2) by forming sets of inconsistent precedence graph nodes that are connected (1) (1) (1) . Such a set is found by picking an inconsistent precedence node and exploring its adjacent neighbours. Every neighbour that is also inconsistent is added to the set and we explore its neighbours recursively. Note that here, and also for the synthesis of the replacing rule, the order in which we compose more than two rules is irrelevant since sequential rule composition is associative [\[BK21\]](#page-31-9). In contrast, nodes that are still intact or depict alternative rule applications are ignored and not explored further. We, thereby, identify all inconsistencies that should be repaired in one step.

In our case, we only have one set consisting of the precedence graph nodes N_2 and C3. These nodes represent formerly valid TGG rule applications for which we know what elements were created and which were used as precondition. We also know what changes invalidated them. Thus, we can use this information to concatenate Nook Rule and Cube Rule (3) (3) (3) because we know that the Plan, House and Construction required by C_3 were created by N_2 . In general, the resulting *replaced* rule can have context elements that stem from

Figure 7: Inverse Higher-Order Replaced Rule & Higher-Order Replacing Rule

rule applications that are still considered intact. The resulting inverse *replaced* rule (4) (4) (4) is depicted on the left of Figure [7](#page-10-0) and can delete all elements created by the corresponding rules.

The much more challenging task is to synthesise the *replacing* rule. While we used some inconsistencies to deduce our inverse *replaced* rule, we now need information that tells us how to restore consistency (5) (5) (5) . We, thus, need to use the precedence graph nodes with " $+$ " annotations, which indicate new translation options using entirely new elements, and those with "*" that describe translation alternatives for elements that are still part of an older rule application. Yet, not all of these nodes are relevant as some translations could just be carried out without using *Short-Cut Rules* or they might be needed for fixing another inconsistency. We can identify the relevant nodes as those that (partially) overlap with nodes from our set of connected inconsistencies. Intuitively, these overlapped nodes are the ones we would like to reuse and, thus, make part of a new rule application that is again consistent. Naturally, "+" annotated nodes cannot overlap with any inconsistencies as their source elements are entirely new. However, at times it is necessary to repair elements and, simultaneously, translate new elements as we have seen in Section [3.1.](#page-8-1) We, thus, also transitively add precedence nodes annotated with "+" if they are connected with a "*" or "+" annotated node that already belongs to the set of relevant nodes.

In general, we try to identify clusters of precedence nodes that are connected and only consist of nodes with a "*" or "+" annotation. The intuition behind this step is that all these nodes represent rule applications that are related and might have to be repaired in one step to yield a better result, where each cluster is used to create a specific replacing rule. Having a set of possible rule applications, we now have to construct our *replacing* rule (6) (6) (6) . However, we only have a partial knowledge on how to concatenate the identified rules because these changes have not yet been propagated. Regarding our example, we know that we have to concatenate Nook Rule, Villa Rule and Cube Rule, but only on the source side we know exactly how.

In general, there are more challenges that we have to overcome. First, we have to make sure that after the repair, all elements on the source side are again part of an intact rule application. This is a hard task because we may find that there are multiple ways to translate a specific element, e.g., by different rules that compete with each other. Then, selection of an appropriate set of rule applications is required. In some cases, this choice can influence how much information on the target side is preserved. Furthermore, we must select a set of rule applications such that the precondition of every selected rule application holds – through other rules selected for this replacing rule or through original rule applications that are still intact. Note that for sake of brevity, our example is rather small and does not show such a scenario. However, we discuss this case in Section [4.](#page-14-0) Finally, we must ensure that the newly repaired rule applications will not introduce a cyclic dependency, i.e., a situation where rule applications would mutually guarantee their preconditions; in our example, for instance, a cyclic row of houses.

To guarantee these conditions, we automated the encoding of the selection of a suitable subset of to be concatenated rules and the calculation of their overlap as an ILP problem. A solution to such a problem will essentially describe an applicable sequence of rule applications that do not lead to a dead-end. We encode competing rule applications as mutually exclusive binary variables and define constraints ensuring that a rule is only chosen if (one or many in combination) other rules create the elements needed by its precondition. Hence, we have to ensure via constraints that context elements are mapped to created elements of compatible type (the same or a sub-type) and that edges can only be mapped if their corresponding source and target nodes are also mapped accordingly. We also make sure that all remaining source elements that are part of the inconsistency must be handled by the replacing rule application. This encoding is inspired by former approaches that present the finding of applicable TGG rule sequences as an ILP problem, e.g., [\[Leb18,](#page-32-1)[WA20\]](#page-33-3). Regarding our example, there are no competing rule applications but considering the precondition of Villa Rule and Cube Rule, we know that the Plan must be the one created by the Nook Rule application. The Construction could be taken from the Nook Rule or in case of Cube Rule from Villa Rule. Due to our knowledge that the needed House of Cube Rule stems from a Villa Rule application, the constraints will make it infeasible for Cube Rule to take the Nook Rule's Construction as there is no correspondence node between them. The identified rules are concatenated (6) (6) (6) and form the *replacing* rule as depicted on the right of Figure [7.](#page-10-0)

Using both the inverse replaced and replacing rule as well as the information of the actual inconsistency (7) (7) (7) , we can now construct the *Short-Cut Rule*. As with the search for a replacing rule, we know what elements to overlap on the source side. This means that we only have to calculate the overlap for the correspondence and target side. While there are still many possible overlaps, e.g., between all Constructions of both rules, we usually want to find the one(s) that preserve more elements. To support least surprise, for target elements that have a corresponding source element we prefer solutions that only identify these target elements if their corresponding source elements are also identified. We also encode the overlapping as an ILP problem to find an optimal solution w.r.t. preserving elements. Currently, we do not differentiate between elements but the optimisation function can be customised to favour elements that, e.g., contain attributes, which have no representation

Figure 8: Higher-Order Short-Cut Rule and Forward Higher-Order Short-Cut Rule

on the other side and should be prioritised. In our computation of overlaps, we also support node type inheritance.

The resulting higher-order *Short-Cut Rule* is depicted in Figure [8](#page-12-0) on the left. It results from overlapping the Houses h_2 and h_3 from the replaced rule with h_3 and h_2 from the replacing rule. We also overlap p, c_2 , f_2 , c_3 and f_3 from the replaced rule's target side with p, c_3, f_3, c_2 and f_2 from the replacing rule. Finally, we also overlap correspondence links and edges if their corresponding source and target nodes are also mapped onto another. By applying this rule, we can preserve the two Houses on the left together with their corresponding Constructions and Floors. We also preserve the Plan and create a new Nook House with corresponding target structures. Since h_2 was a Nook House, we now assign the new HouseType Cube, which means that we have to add a new Cellar on the target side and make it the first Construction Step. Similarly, h_3 was a Cube House but is now transformed to a Villa House, which means that the former Cellar is now superfluous and thus removed. Instead, we add a new Saddle Roof as a final Construction Step. As before, we can operationalise this rule $(\langle 8 \rangle)$ $(\langle 8 \rangle)$ $(\langle 8 \rangle)$ to obtain a forward (and likewise backward) variant as depicted in Figure [8](#page-12-0) on the right. Applying it to our example from Figure [5,](#page-8-0) we can now translate the new Nook, while simultaneously repairing the other two Houses and preserving the Plan with its content.

3.3. Discussion. From a technical point of view, there are some aspects worth mentioning. As a result of the construction process, we do not only get a higher-order *Short-Cut Rule* but also (implicitly) its match w.r.t. the analysed inconsistency. Hence, we can directly apply the rule and repair the inconsistencies it addresses. Besides that, however, we do

not store the generated rule for applying it at other locations, which has two reasons. First, a higher-order *Short-Cut Rule* is quite specific to a certain inconsistency that is characterised by a set of related broken rule applications and how exactly each of them has been invalidated. To efficiently determine whether an existing rule can be applied to another set of inconsistencies can, thus, be very challenging and iterating and trying out all stored rules will probably become more expensive than simply constructing the rule from scratch. Second, as a higher-order *Short-Cut Rule* targets multiple inconsistencies at once, the space of related inconsistencies is generally very large. Hence, we argue that encountering the exact same constellation of inconsistencies is less likely. Yet, this has to be investigated further for real-world scenarios.

Regarding our choice to use ILP solving, by applying a Short-Cut Rule, we want to preserve as much information as possible that would otherwise be lost when unnecessarily deleting and recreating elements. Hence, we face an optimisation problem where we want to overlap as many created elements from both the replaced and replacing rule as these elements end up being preserved. For our current implementation, we use ILP solving separately for (5) and (7) . In (5) , we want to maximise the number of mappings between context and created elements, which are the locations where the rules are glued together. Selecting more such mappings means that less created elements must stem from outside of the computed Short-Cut Rule and, thus, it is more likely to be actually applicable. In 7 , we want to maximise the number of mappings between created elements in both rules, reducing the number of elements that must be deleted, potentially reducing information loss. In the future, we intend to combine both problems (5) and (7) into a single optimisation problem because the choice of how to construct the replaced and replacing rule may affect the resulting mapping and, thus, the final Short-Cut Rule.

In the following, we shortly discuss applicability and correctness of the individual process steps. In (1) (1) (1) , we identify clusters of related inconsistencies by analysing the precedence graph. This information is then used to create the *replaced* and *replacing* rule. Choosing the cluster too small, e.g., to save performance, might lead to a *Short-Cut Rule* that is not applicable if necessary context has been altered and this change was not considered. In that case, we can still fall back to restoring consistency by revoking the rule applications and applying alternative translation steps instead, although, this has the risk of losing some information. In contrast, if the set is chosen bigger than necessary, then the resulting ILP problems also tend to become harder to solve as the search space increases. In our implementation, these clusters contain all related inconsistencies that are (transitively) connected with each other without any consistent steps in between. We, thereby, ensure that the cluster is not chosen too small as inconsistencies that are (remotely) related are repaired together. However, at the moment, we cannot guarantee that our chosen clusters are minimal because some inconsistencies may be related but do not need to be handled together, i.e., could be repaired by standard *Short-Cut Rules* or smaller *higher-order Short-Cut Rules*. In the future, we would like to investigate ways to identify situations where these clusters could be broken up to yield less complex ones, which has the potential of improving the performance of our approach. All (2) (2) (2) , (3) (3) (3) and (4) (4) (4) are trivial as we use information about formerly valid rule applications for which we rely on a pattern matcher to identify these locations.

Regarding (5) (5) (5) , we may encounter multiple alternative translation options for one element, which may lead to an exponentially increasing number of possible translation sequences and, thus, performance issues. However, under certain technical (and not too

Figure 9: Double Cube Rule

strict) circumstances, any choice can be part of a synchronisation process that finally restores consistency [\[Kos22,](#page-32-7) Theorem 6.17] (also compare [\[Leb18,](#page-32-1) Theorem 4]). Yet, as discussed for (1) (1) (1) , different choices may lead to different amounts of preserved model elements. Creating the inverse of the resulting rule in (6) (6) (6) is again trivial. Regarding the underlying theory of (higher-order) Short-Cut Rules, we refer to the literature for details $[FKST18, Kos22,KT23]$ $[FKST18, Kos22,KT23]$ $[FKST18, Kos22,KT23]$. Importantly, for our purposes in (7) (7) (7) and (8) (8) (8) , both *replaced* and *replacing rule* are composed from the rules of the given TGG, through our computation of Short-Cut Rules and the information provided by the IGPM, we ensure consistent matching of the replaced and the replacing rule, and we prevent the introduction of cyclic dependencies. This means that we meet criteria that guarantee language-preserving applications of *Short-Cut Rules* (cf. [\[Kos22,](#page-32-7) Theorem 4.16] or [\[KT23,](#page-32-6) Theorem 17]). The according application of their operationalised versions during synchronisation then incrementally improves consistency (compare [\[Kos22,](#page-32-7) Proposition 6.15]).

4. ILP-based Construction

This chapter extends our previous work [\[FKMS23\]](#page-31-4), where we could only convey the abstract concept of how to obtain higher-order Short-Cut Rules. Our new contribution in this regard is a description on how to implement the concept using optimisation techniques and how to constrain the solution space to get desirable results.

As discussed in the previous chapter, we employ ILP solving for two subsequent tasks. First, we use it to find out how to concatenate a set of TGG rules to obtain the replacing rule. Second, we use it to overlap both the replaced and the replacing rule to obtain the final *Short-Cut Rule*. Note that the latter case was already presented in [\[Fri22\]](#page-31-8), which is why we will focus on creating the replacing rule. For that purpose, we slightly extend our example from before so that creating the replacing rule is no longer straightforward and different solutions are possible from which the most promising one should be chosen. Also note that the entire ILP problem for this example can be found in Appendix [A.](#page-33-4)

Figure [9](#page-14-1) shows an additional TGG rule and its forward operationalised form that extend our rule sets from Figure [2.](#page-4-0) In contrast to Cube Rule, the new Double Cube Rule creates two Cubes on the source side that both correspond to one newly created Construction on the target side meaning that both Cubes are planned as a semi-detached House. The purpose of this new rule is to introduce a kind of variability during the translation and likewise synchronisation of a terrace house model because there may be multiple ways on how to restore consistency now. Translating two Cubes in a row, for example, we are now able to choose between applying Cube Forward Rule twice or Double Cube Forward Rule once. Ultimately, this is something a modeller would have to decide on their own for two reasons: First, there is no information on the source side on whether there should be one or two Constructions for both Cubes in the target model. Second, choosing either option will let us still translate all other Houses that come after both Cubes. This means that this is no scenario where the translation may lead to a dead-end for certain translation sequences and, thus, neither of the two possibilities is wrong. Of course, this means that for synchronisation scenarios, we also must account for this kind of variability when we create the replacing rule. Specifically, we may now encounter situations with alternative TGG rule concatenations forming different replacing rules that will lead to different (but equally valid) Short-Cut Rules in the end. For our example, this might present us with the choice between a *Short-Cut Rule* that uses a Double Cube Rule and another using two concatenated Cube Rules to repair translation steps for two former Villa Houses. Using the *Short-Cut Rule* based on the Double Cube Rule would remove one of the Constructions on the other side, which could mean an unintended loss of information. Hence, these rules mainly differ in what and how many elements are preserved and for that reason, it is important to choose the replacing rule that will be a better candidate for preserving information.

Figure [10](#page-16-0) depicts a scenario in which we will encounter such a situation. As can be seen, we start with a row of three Houses beginning with a Nook followed by two Villa Houses. For each House, there is a corresponding Construction as well as a Plan containing them on the target side. As before, however, we change the source side and introduce an inconsistency. This time, we change the type of h_2 and h_3 from Nook and Villa to Cube. Additionally, we add a new Nook house h_1 before h_2 . As before, the most apparent inconsistency is that the Constructions c_2 and c_3 no longer contain the necessary building steps since a Cube has a Floor but no Saddle Roof and a Nook House has neither. Besides that, there are no corresponding elements for the newly added Nook as it has not yet been translated to the target side. To resolve this issue and restore consistency, we must investigate the precedence graph that is shown on the right side of Figure [10.](#page-16-0) We see that there is a new Nook Rule application as well as an invalidated one. The first N_1 is annotated with "+" meaning that its source elements are entirely new and thus free for translation and the latter is N_2 , which became inconsistent due to an attribute change ("#"). Since the Nook is now a Cube, we find a new Cube Rule application C_2 as an alternative translation for the House h_2 . Besides that, we have an inconsistent Villa Rule application V_3 , which was also invalidated due to an attribute change and which caused the appearance of the new Cube Rule application C_3 . In combination, changing both h_2 and h_3 to being Cube Houses lead to a Double Cube Rule application $DC_{1,2}$. Notably, the rule applications C_2 , C_3 and $DC_{1,2}$ can not be applied yet, as indicated by their "*"-annotation, because they would mark elements that are still marked by both N_2 and V_3 . Hence, our goal in the following is to create a replacing rule (followed by a *Short-Cut Rule*) that will let us translate the new Nook House and synchronise the attribute change, while trying to preserve elements on the target side.

Figure 10: Synchronisation Example

We, therefore, have to consider several aspects. (1) We know of all elements that were part of a replaced rule application, i.e., the rule applications that were once consistent and should now be repaired. Yet, the same does not hold for the candidates that will form the replacing rule. This is because we only know of elements from the source domain[2](#page-16-1) but not the target domain, which in the precedence graph is expressed with a "u". This annotation stands for undefined and indicates that we do not know yet, whether there already are any elements on this side that we could reuse or whether we should create them all from scratch. Hence, we have to figure out whether context elements from one replacing rule candidate are created by another candidate or whether they come from yet another still consistent rule application. For example, we would have to figure out for C_3 what rule application created the Plan. (2) Some candidates may exclude each other. In our case, we know that C_2 and C_3 cannot coexist with $DC_{2,3}$ because they mark the same elements on the source side. Hence, we have to make sure that choosing a candidate means to exclude these other candidates from the solution. (3) Several *dependencies* must be considered. There are, for example, dependencies between replacing rule candidates such as between C_2 and C_3 because the first has to be chosen (and applied) before the latter (if the latter is chosen at all). Similar dependencies exist between rule elements, e.g., by mapping a context node to the created

 2 For a backward synchronisation case this would be the target domain.

node of another rule because we assume that this rule creates the necessary context. In our example, this could be the context Plan needed by C_3 and found created by N_1 . Edges and correspondence links are a special case in that regard as they are only allowed to be mapped when their source and target elements were also mapped onto each other, ensuring that this is indeed the same edge. (4) While there may be multiple possible replacing rules, not all candidates are equal in how much information can be preserved in the end. Thus, we have to find a metric for assessing the value of a solution that indicates the usefulness of a specific replacing rule w.r.t. the final *Short-Cut Rule*.

4.1. Concatenation of rule candidates. As mentioned before, we will use ILP to solve the above described problem. In general, when using ILP, we abstractly encode search spaces by means of optimisation variables that in combination with an objective function must adhere to certain constraints. In our case, we are only interested in 0-1 or pseudo-boolean ILP of the form:

$$
\begin{aligned}\n\max(w^T v) \\
Av \ge b, \\
v \in \{0, 1\}^m\n\end{aligned} \tag{4.1}
$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ being a vector of weights and v being the vector of our boolean optimisation variables $v_1, ..., v_m$. The variables can represent entities such as rule application candidates and whether we consider them for our replacing rule or mappings between rule elements, e.g., whether we assume that a certain context element of one rule application will be created by a certain other one. In the following, we will write $v[\ldots]$ to reference a specific variable in our vector v , where the value between the brackets represents a unique index. Maximising the function should ultimately yield a valid concatenation of our replacing rule candidates, where weights can be used to favor the selection of specific variables. In our example, all weights will usually be set to 1. By choosing to customise these weights, however, a user can encode restoration preferences, e.g., to give certain elements a higher priority to be preserved or even penalise others that should be avoided. Due to the various aspects that we have to consider, we will start with a small set of optimisation variables and constraints and extend them step-by-step solving the above-described sub-problems along the way.

Encoding rule applications and their dependencies. Constructing the replacing rule means to determine, which rule candidates to concatenate. Therefore, for each replacing rule candidate from our precedence graph, there is one variable in v that states whether this candidate is chosen for the replacing rule or not. For our example, we would have the following variables: $v[N_1], v[C_1], v[C_2]$ and $v[DC_{2,3}]$. Then, we encode their dependencies that we can read from the precedence graph, e.g., we can see that C_2 depends on N_1 , which means that $v[C_2]$ may only be set to 1 if $v[N_1]$ is 1 as well. This can be expressed via the constraint $v[C_2] \le v[N_1]$. Consequently, we introduce the same constraint for the other candidates: $v[C_3] \le v[C_2]$, $v[DC_{2,3}] \le v[N_1]$. Some rule candidates are mutually exclusive because they would mark the same elements, which must be avoided because each element can only be translated once. Hence, we need to make sure that C_2 and C_3 cannot be chosen alongside $DC_{2,3}$ and vice versa because the latter marks the same elements as both Cube Rule applications. We, therefore, add the constraints $v[C_2] + v[DC_{2,3}] \leq 1$ and

Figure 11: Context-Create Mappings

 $v[C_3] + v[DC_{2,3}] \leq 1$, which ensure that at most one of the terms can become 1 while the other has to be 0.

Encoding possible dependencies of target context. In the next step, we need to figure out how to concatenate our replacing rule candidates. While we already know how rule application candidates can be concatenated on the source side of our example, it is not clear where target elements come from, i.e., which rule application would create them. Hence, we have to determine whether a context element from one rule candidate might be the created element of another candidate. For that purpose and for each candidates context (black) element, we search for created $(++')$, green) elements in candidates that the current candidate (transitively) depends on and that have the compatible type and no conflicting attribute assignments/conditions. We will call these pairs of context and created rule elements in the following *context-create mappings*. Figure [11](#page-18-0) depicts the context-create mappings for our rule candidates in blue, e.g., the context Construction $c_{2,3}^r$ of $DC_{2,3}$ could be created as nc_1^r from N_1 , which corresponds to the context-create mapping $c_{2,3}^r \rightarrow nc_1^r$. Note that the source side elements are model elements from Figure [10,](#page-16-0) while the correspondence and target side depict rule elements, which is why we draw the latter with dashed lines and added r as a superscript to emphasise the difference. In general, there may exist several possible mappings for each context element as is the case for the context Construction c_3^r of candidate C_3 that could stem from C_2 or N_1 . While it is impossible that N_1 would create it because ultimately Construction c_2^r will be the one that corresponds to the Cube House h_2 , we have no static analysis to determine this beforehand and rely on the optimiser to find a reasonable solution in compliance with all constraints. For that reason, it is possible that no context-create mapping is chosen, which means that we assume that the context comes from outside our current scope. In general, we cannot be sure whether our candidate set is actually able to create all necessary context or if the context stems from a still valid rule application. Yet, incorporating all rule applications would severely impact the applicability of this approach, which would scale with the size of the model rather than the size of the change. That being said, we also create context-create mappings between edges. In comparison with context-create mappings between nodes, however, edges can only be mapped if their respective source and target nodes are also mapped onto another. Hence, we constrain these edge context-create mapping variables to ensure that they are only 1 if the correct node mappings are 1, too. For mapping edge e_2^r from candidate C_2 to e_1^r from N_1 , we would need to make sure that p_2^r is mapped to p_1^r and likewise c_2^r to nc_1^r . This can be expressed as $v[e_2^r \to e_1^r] \le v[p_2^r \to p_1^r]$ and $v[e_2^r \to e_1^r] \le v[e_2^r \to n e_1^r]$. Finally, we also have to create context-create mappings between correspondence links. Note that these mappings were omitted from Figure [11](#page-18-0) for readability reasons. These context-create mappings are handled in a similar way as edges with the slight difference that we already know whether their source side matches, e.g., mapping the context correspondence link from C_2 to the created one from N_1 , we can see that both are compatible on the source side because they both use the same model element h_1 . If this is not the case, we can directly drop the mapping and continue.

We must also add exclusions to circumvent that more than one context-create mapping is chosen for each context element. Choosing more than one would mean that the context element is created by more than one rule application, which cannot be the case. As before, we can achieve this by formulating an exclusion, which ensures that the sum of all contextcreate mapping variables associated with the context element is less or equal than 1, e.g., $v[c_3^r \to nc_1^r] + v[c_3^r \to nc_2^r] \leq 1$, which states that c_3^r must be the element created as nc_1^r , nc_2^r or neither.

Combining rule applications and context dependency. Beyond that, we combine our rule candidate variables from before with these new context-create mapping variables to make sure that using a context-create mapping also implies that the respective rule candidates are chosen as well. We can express this implication by stating that the variable of a context-create mapping must be less or equal than the variable of a rule candidate, which means that the first can only become 1 if the latter is 1 as well. In our example, this must hold when setting the mapping variable $v[c_3^r \to n c_2^r]$ to 1, which means that both $v[C_3]$ and $v[C_2]$ must be 1 as well because they use elements from these candidates. Hence, we add the constraints $v[c_3^r \to n c_2^r] \le v[C_2]$ and $v[c_3^r \to n c_2^r] \le v[C_3]$, where the context-create mapping can only be 1 if both $v[C_2]$ and $v[C_3]$ are 1, too.

Optimisation goal. Assuming that we would now maximise the number of chosen contextcreate mappings, we would get a valid replacing rule that overlaps context and created elements as much as possible. Our constraints ensure that no competing rule candidates may be chosen at the same time, while our constraints for edge and correspondence context-create mappings ensure that a solution is favored that does not blindly map each context node to any possible creating node. Instead, it should always give a higher reward when also considering how each node is connected. Thereby, the edge and correspondence constraints

Figure 12: Mapping Rule Candidate to Model Elements

become the glue that pushes us towards a concatenation that is not only valid but also reasonable and applicable.

4.2. Incorporating former target side elements. The result can still be improved by incorporating information about target side elements that were created by our formerly valid replaced rule applications. When being presented with different alternatives as in our example, incorporating knowledge about formerly created elements can help us making a better decision, e.g., to preserve more elements. For example, considering our example from above, it could be better to use a replacing rule that includes both Cube Rule candidates instead of the Double Cube Rule because we want to repair a Nook Rule and a Villa Rule application that both created a Construction. Choosing the Double Cube Rule would result in a Short-Cut Rule that removes one Construction. Even if this is what we wanted, we need the means to state that either solution should be preferred. We can achieve this by introducing rule-model mappings between created rule elements from our rule candidates and model elements that were created previously by rule applications that are no longer intact and which we are trying to repair with the final *Short-Cut Rule*. The intuition behind this step is that we try to find out whether we can reuse and, thus, preserve elements created by prior rule applications that would otherwise be deleted. Figure [12](#page-20-0) depicts these rule-model mappings for our example, although, the rule candidate $D_{1,2}$ has been omitted for readability reasons. Similar to the context-create mappings, for each context element in our rule candidates, we search for compatible model elements. This time, however, they must have the same type (i.e., we exclude subtyping) or else the created rule element would not be the same as the one we already have. Again, correspondence mappings are omitted due to readability reasons but we discuss them in the following, too. Also note that, as

explained previously, we do not need mappings to source side elements, because we already know these mappings.

Having a look at Figure [12,](#page-20-0) we can see that there is only one available rule application candidate on the right side that creates a Plan, which means that we can only map p_1^r to p_2 . In contrast, each of the three Constructions on the right side could be mapped to any of the two existing Constructions in our model, which results in six mappings. This also holds for the edges between the Plan and a Construction for which we know that there must be three in the end but for which there are only two candidates in our model. We also map created correspondence links for which there are also six possible mappings. Creating all these mappings is an important difference between context-create and rule-model mappings. For the first, we also considered dependencies between rule candidates and (implicitly) filtered some mappings, while for the latter, we are not sure which rule application would be a good candidate for preserving the information. This is because the order in which elements are translated may have changed as is the case for the example in Section [3.](#page-7-0) Hence, we create mappings from each created rule element to all matching model elements and let the optimiser find a solution that reuses these elements in a reasonable way. Comparing the former rule applications with the new rule application candidates and incorporating this knowledge to improve performance is left to future work.

Encoding dependencies between rule-context mappings. Since each model element can only be created by one rule element, we must ensure that mappings pointing to the same model element are mutually exclusive. Similar to Section [4.1,](#page-17-0) we enforce this by adding a constraint that ensures that only one mapping variable can be set to one, e.g., for mappings to c_3 we add $v[nc_1^r \to c_3] + v[nc_2^r \to c_3] + v[nc_3^r \to c_3] \leq 1$. Of course, these mappings may only be chosen if the rule application candidate on the right side is chosen as well. As with the rule concatenation constraints, we can express this by stating that the rule-model mapping variable may only be 1 if the rule application candidate variable is 1, e.g., $v[nc_3^r \to c_3] \le v[C_3]$.

Finally, we have to make sure that edges and correspondence links are not mapped if their sources and targets are not mapped onto each other as well since edges can not be the same if they point to different elements. In comparison to the Section [4.1,](#page-17-0) where we mapped context to created rule elements, we are now only interested in mapping created rule elements to formerly created model elements. However, most of our created rule edges have one context node as source, e.g., e_3^r . This means that we can only take such an edge if its context rule node is mapped to a created rule node of another rule application candidate, which is then mapped to the corresponding element in the model. For mapping e_5^r to e_2 , we would add the following constraint: $3v[e_5^r \to e_2] \le v[nc_3^r \to c_3] + v[p_1^r \to p_2] + v[p_3^r \to p_1^r]$. It states that $v[e_5^r \to e_2]$ can only be 1 if the three other variables are set to 1 and sum up to 3, which means that nc_3^r and p_1^r were mapped to c_3 and p_2 , respectively, while the context Plan from C_3 is assumed to be created by N_1 . Only then, we can be certain that e_5^r could indeed create e_2 if C_3 is chosen. As in Section [4.1,](#page-17-0) correspondence links are handled similarly as edges with the slight difference that we already know what element they point to on the source side. Hence, we only have to encode constraints for the target side elements of correspondence links.

Incorporating edges and correspondence links is crucial as this will ensure that nodes are not reused randomly. Instead, those solutions will be preferred where relations between model nodes are preserved. In our example, we could map the created Constructions of C_2

Figure 13: Replaced and Replacing Rule

and C_3 freely but when mapping nc_3^r to c_2 , we won't be able to map the correspondence link and, thus, get a smaller reward. Another interesting case is that of N_1 . We could not only map its Plan p_1^r to p_2 in our model but also map nc_1^r to c_2 . Doing so, however, would leave us with no option but to remove the correspondence link between h_2 and c_3 because we cannot (and probably also do not want to) repurpose the edge to point to h_1 . This means that edges and likewise correspondence links are our means of reasoning on whether elements are assumed to be the same and, thus, should be preserved. Maximising the number of chosen mappings, finally, tells us how to concatenate some of the rule candidates in order to obtain our final replacing rule. For our example and without choosing specific weights, we would choose N_1 , C_2 and C_3 as our candidates, where the context Plan of the latter two stems from N_1 as well as the context Construction from C_2 , while C_2 creates the Construction that is needed by C_3 . This means that the Constructions from C_2 and C_3 were mapped to the model Constructions c_2 and c_3 , respectively. In contrast, when choosing $DC_{1,2}$, we would only be able to map one Construction from our rule application candidate to our model and there are less context-create mappings from context to created elements, which in total gives a smaller reward. If we would like to prefer the $DC_{1,2}$ nonetheless, then we can choose specific weights to make it more rewarding to choose it. An obvious choice would be to add a large weight to $v[DC_{1,2}]$ so that the reward of choosing a Double Cube Rule is always higher than choosing two Cube Rule candidates. As a rule of thumb, we must choose a weight that is greater than the difference between the number of mappings a Double Cube Rule and two Cube Rules may have. Another, more fine-granular, option is to add weightings to rule-model mappings such that choosing mappings between a Double Cube Rule candidate and a model is worth more than choosing similar mappings for two subsequent Cube Rules. The difference between the first and second option is that the latter lets us prioritise specific elements over others.

Figure 14: Higher-Order Short-Cut Rule and Forward Higher-Order Short-Cut Rule

Figure [13](#page-22-0) shows the resulting replaced and replacing rule. Both are used to create the Short-Cut Rule shown in Figure [14](#page-23-0) (cf. [\[FKST21\]](#page-31-1)) by overlapping both rules and thereby identifying common elements. Here, the overlap would be between elements with the same name. Overlapped elements are preserved when transforming from our formerly valid replaced rule application to the replacing rule application. Finally, applying this rule on our changed model from Figure [10](#page-16-0) would create the missing Construction for House $h1$, while preserving the already existing Plan and changing the other Constructions to have a Cellar instead of a Saddle Roof.

4.3. Discussion. We would like to highlight and discuss some details of our approach. First, we would like to point out that, in general, there are multiple possible *Short-Cut Rules* as there are often various possible overlaps between a replaced and a replacing rule. While we do not search for all possible *Short-Cut Rules* due to performance reasons and because many of them are not useful, we limit ourselves to finding useful ones, according to a custom (hand-crafted) metric, i.e., a specific choice of weightings (cf. [\[FKMS23\]](#page-31-4)). This means two things. First, in order to preserve specific elements, we need to encode this via weights such that a solution that focuses on these elements is preferred. Second, since we do not search and try all *Short-Cut Rules*, our approach is an heuristic where in its basic form we choose all weights to be 1 due to our experience with examples from our repository of test scenarios^{[3](#page-23-1)}.

 $\rm ^3$ <www.github.com/eMoflon/emoflon-ibex-tests>

Of course, since our approach is a heuristic, it can happen that we are not able to find a Short-Cut Rule that can be used in that particular scenario. As discussed in the previous chapter, however, we can guarantee that each created higher-order *Short-Cut Rule* is indeed correct, meaning that these rules cannot be used to perform synchronisation steps that inflict new inconsistencies. If no rule can be found, there is still the option to retranslate elements that were part in prior rule applications, which is guaranteed to restore consistency [\[Leb18\]](#page-32-1).

With the current ILP formulation, mapping an element from a rule to a model gives some reward. In contrast, we do not penalise if not all elements from a rule have a matching counterpart in the model, which in some cases would be reasonable. A future extension could thus introduce such a penality term, e.g., adding another rule-model mapping variable for each rule element that has a negative weight and enforcing that for each node exactly one rule-model mapping variable has to be chosen. If there are not enough elements in the model to map all rule elements for a rule candidate, we would have to set variables with a negative weight to 1, which reduces the overall reward. Then, it would become more likely that another alternative candidate is chosen that explains more model elements.

Another important aspect is that after finding the replacing rule, we create the *Short-Cut* Rule using ILP solving in a subsequent step with our former *Short-Cut Rule* framework presented in [\[FKST21\]](#page-31-1). In fact, having found mappings from our replacing rule candidates to both the replaced rule and the model, we could already create the *Short-Cut Rule*. Currently, we rely on the former construction mechanism of *Short-Cut Rules* but we plan to replace it with one that creates replaced, replacing and *Short-Cut Rules* in one step in the future.

5. Evaluation

We implemented our approach and integrated it into the state-of-the-art graph transforma-tion tool eMoflon^{[4](#page-24-1)}. Our approach extends our previous work on *Short-Cut Rules* [\[FKST18,](#page-31-3) [FKST21\]](#page-31-1) and while the approach has the potential to preserve more information during synchronisation, we have to investigate how our original Short-Cut Rules perform in comparison. Also, since our approach analyses dependencies and creates *Short-Cut Rules* on-demand at runtime, we have to investigate whether it introduces a noteworthy offset. Thus, we pose the following research questions:

- (RQ1) Does the new runtime analysis introduce any additional cost in comparison with using a precalculated set of Short-Cut Rules?
- (RQ2) Does our new approach indeed preserve more information in certain cases?

Evaluation scenarios. To answer our research questions, we investigate five different scenarios of our running example with three different resolution techniques: The *legacy* algorithm [\[Leb18\]](#page-32-1) revokes invalid rule applications and retranslates parts of the model, while SC stands for the original Short-Cut Rule framework with precalculcated repair rules [\[FKST21\]](#page-31-1) and *HO SC* stands for our new *higher-order Short-Cut Rules*. We compare both SC and HO SC with legacy to show the impact of repairing models instead of retranslating them (partially). Note that in both SC and HO SC our algorithm falls back to using the legacy algorithm in case no Short-Cut Rule was found.

In the first and second scenario, we have a fixed number of 2 000 rows of Houses with around 26 000 nodes on both source and target side and we increase the number of applied

⁴<www.emoflon.org/>

changes in steps of 50 from 50 to 250. Our primary goal for these scenarios is to investigate how our approach copes with linearly scaling a number of different kinds of changes. In contrast, the third and fourth scenario comprise two long rows of Houses that we grow iteratively by adding new Houses in steps of 20 from 20 to 200. This change is expected to become more expensive to resolve as the rows grow larger but should also be able to preserve more information than the other two algorithms. The fifth scenario was not part of our previous work and extends our former evaluation. It features one long row of 1 000 Houses, where the first House is of type Nook and all following ones are of type Villa. Then, we change an increasing number of Houses in this row to become of type Cube starting from 10 to 60 subsequent Houses in steps of 10. Both the former *Short-Cut Rule* and the newer higher *higher-order Short-Cut Rule* framework should be able to preserve the same amount of information, the latter is expected to be more expensive due to the prior analysis needed to find a suitable rule. We, thus, want to measure the effect that subsequently dependent changes have when using our new approach.

The specific changes for each scenario and their expected outcomes are as follows: In the first and third scenario, we add new Nook Houses at the start of each row. HO SC can synthesise a *higher-order Short-Cut Rule* to react to this change, while *legacy* has to re-translate each affected row and SC has to separately translate the newly created Nook Houses and then move the affected Constructions to the newly translated Plan (using precalculated Short-Cut Rules). In the second and fourth scenario, we relocate subrows. This is a worst-case scenario for *HO SC*. In this case, every moved **House's Construction** must be repaired and moved to the new Plan. While *legacy* has to perform this via retranslation, both SC and HO SC can use the same repair rules – only SC does not have the offset of creating them on-the-fly. In the fifth scenario, we change an increasing number of Villa Houses to being Cube, starting from the front. As a result, HO SC will have to construct larger Short-Cut Rules, which constitutes another worst-case scenario, while legacy will simply retranslate the Houses and SC can repair them step-by-step.

Experimental setup. We measure the time it takes to resolve the inconsistencies as well as how many elements are deleted in the process. Every measurement was repeated five times and we show the average runtime while the number of deleted elements did not fluctuate. The evaluation was executed on a system with an AMD Ryzen 9 3900x with 64GB RAM. It can be replicated using our prepared VM^5 VM^5 , which comes with a detailed explanation on how to reproduce the results. The current version v1.1 of our VM needs a license of the Gurobi^{[6](#page-25-1)} ILP solver to execute the evaluation but the former version is still available and implements the first four test scenarios without this dependency.

Results. Figure [15](#page-26-0) shows the measured times of the all scenarios. For the first and third scenario, where *legacy* and SC have to transitively retranslate/repair, creating the needed higher-order *Short-Cut Rules* on-the-fly even has performance benefits. For the second and fourth case, the results indicate that for one of the worst-case scenarios where SC will have the same effect as HO SC, constructing higher-order Short-Cut Rules and applying them instead of performing many smaller repair steps only introduces a small cost. However, the fifth scenario shows there are cases where the construction of large *Short-Cut Rules* can be very expensive, at least when compared to applying smaller *Short-Cut Rules* instead. For

 5 <www.zenodo.org/record/10377155>

⁶<www.gurobi.com>

Figure 15: Time Comparison

changing 60 Houses, $HO\ SC$ takes over 40 seconds to construct a *Short-Cut Rule*, where over 90% of the time is needed to find a solution for the ILP problem.

Figure [16](#page-27-0) shows the largest number of deleted elements (nodes and edges) for all scenarios. As can be seen, using $HO\,SC$ has the potential of preserving more information than SC and legacy in the right cases ($RQ2$). In the others, both SC and HO SC outperform the legacy algorithm; this cannot preserve any information at all.

In sum, the research questions can be answered as follows: RQ2 can be answered with a clear yes because there is no case where our new approach preserved less information than the former *Short-Cut Rule* framework and for two scenarios it even preserved more. Regarding $(RQ1)$, we can state that there are cases where our new approach introduces a significant overhead compared to our previous Short-Cut Rule framework. Yet, at the same time, there are cases where using our new framework can even save time in situations where we cannot repair a model using smaller *Short-Cut Rules* and large parts have to be retranslated to restore consistency. For several cases, both our new and the old Short-Cut

Figure 16: Comparison of deleted elements

Rule framework even performed similarly w.r.t. the performance. Considering that using higher-order Short-Cut Rules can indeed preserve more information, one can argue that it is advisable to use *higher-order Short-Cut Rules* and switch to the former algorithm in case that a specific scenario only consists of worst-case scenarios such as the fifth scenario of our evaluation.

Discussion. There are several points worth mentioning. First, the current evaluation features only one example with synthetic data. Our approach, thus, has to be evaluated further on other (preferably real-world) scenarios with non-synthetic data and user changes. Yet, based on our experience with our test suite of 22 TGGs^{[7](#page-27-1)}, we argue that the investigated scenarios are representative and similar results are to be expected there as well.

Second, while using $HO\,SC$ has the potential to preserve more information than the other two approaches, there are cases where calculating a Short-Cut Rule would take too much time. This issue could be resolved by not creating one large *Short-Cut Rule* to handle all changes at once but rather splitting them up. Therefore, we need to find some criteria to judge whether a split could cause information loss. Apart from that, we could also reduce the size of the ILP problem by filtering those mapping variables that cannot be part of the final solution. Currently, we encode most mappings and rely on our ILP constraints to filter these ones out.

⁷<https://github.com/eMoflon/emoflon-ibex-tests>

6. Related Work

In this section, we relate our approach to other sequential model synchronisation approaches. We will first focus on TGG-based approaches and then discuss other related works with a different methodology.

TGG-based approaches. There have been several works that implement model synchronisation by transitively revoking invalid rule applications and retranslating possibly large parts of a model [\[GW09,](#page-32-2) [LAVS12,](#page-32-3) [LAF](#page-32-4)+17]. While these approaches come with proofs of termination, correctness and completeness, they may lead to an unnecessary information loss and at times a decrease in performance.

In contrast, Hermann et al. $[\text{HEO}^+15]$ tried to solve this issue by calculating the maximal still-consistent sub-model, which is used as a starting point to propagate the model changes. Their approach is shown to be correct but cannot guarantee that all changes are carried out in a least-changing way. Above that, calculating the sub-model is computationally very expensive and their approach is limited to a deterministic TGG rule set only.

A very similar approach to ours is given by Giese and Hildebrandt [\[GHNW12\]](#page-31-5) where they propose repairing rules that closely resemble *Short-Cut Rules*. These also save nodes instead of unnecessarily deleting them and are able to propagate relocations of model elements. However, neither a construction scheme to obtain these rules is provided nor is the preservation of information proven to be correct. Yet, using our approach from [\[FKST21\]](#page-31-1), we are indeed able to construct the same rules as mentioned in their work. Similarly, Blouin et al. [\[BPD](#page-31-10)+14] also propose to use custom repair rules.

Another approach was initially proposed by Greenyer et al. [\[GPR11\]](#page-32-9) and later formalised by Orejas and Pino [\[OP14\]](#page-33-5). They propose that elements are never directly deleted but rather marked. Thereby, these elements are still free to be re-used during synchronisation, e.g., by letting a forward rule transformation take elements from the set of these elements instead of creating new ones and, thus, prevent them from being deleted. Yet, to the best of our knowledge, their approach was never implemented.

Finally, Anjorin et al. $[ALK^+15]$ $[ALK^+15]$ discussed design guidelines for TGGs that lead to less transitive dependencies between rule applications and can, thus, be synchronised without triggering a large cascade of retranslation steps. Yet in many cases, remodelling a TGG to comply with these guidelines means to change the defined language and may for that reason not be desired.

In summary, there are several TGG-based approaches that acknowledge the limitations of propagating changes by retranslating possibly large parts of a model. Some approaches follow a similar approach as ours, yet without a formal foundation or construction concept. Others have, to the best of our knowledge, never been implemented. While we provide both, we extend our previous works [\[FKST18,](#page-31-3) [FKST21\]](#page-31-1) by broadening the practical applicability and preserving information in more complex scenarios than before.

Bidirectional transformations. Another popular formal framework for bidirectional transformations are lenses. While there are many works related to lenses and least-changing incremental synchronisation [\[DXC10,](#page-31-11) [HB19,](#page-32-10) [HPW12,](#page-32-11) [HPC18,](#page-32-12)[WGW11\]](#page-33-6), most are only theoretical and were not implemented. Most closely related to our work are the works of Hofmann et al. [\[HPW12\]](#page-32-11) and Wang et al. [\[WGW11\]](#page-33-6). Hofmann et al. introduced so-called edit lenses that focus on the changes and how to propagate and preserve them instead of taking a state-based approach. This is similar to our approach that works incrementally

based on the perceived change to the model. Wang et al. too derive functions to propagate changes between models. However, their approach is limited to tree-like structures only.

Another approach by Macedo and Cunha [\[MC16\]](#page-32-13) was influenced by the lenses framework and is based on ATL and SAT solving via Alloy. In their work, they describe how to encode the search space of all feasible synchronisation solutions as an optimisation problem. Their approach rates changes based on one of two metrics, namely the graph edit distance and the operation-based distance. Using these ratings, a least-changing propagation is calculated, interpreting least-change either as the minimal set of (propagation) operations needed or the minimal set of atomic graph edit steps to be taken. In comparison to our work, they incorporate more advanced metamodel constraints, e.g., OCL, and, generally, their approach should yield good results. The major drawback is the scalability as it is applicable to rather small models only.

Finally, Anjorin et al. $[ADJ^+17]$ $[ADJ^+17]$ compared three existing state-of-the art tools for bidirectional transformations of different methodologies against each other. Specifically, they compared the rule- and TGG-based tool eMoflon [\[WAF](#page-33-7)+19], the constraint-based tool mediniQVT 8 8 and BiGUL [\[KZH16\]](#page-32-14), which is a bidirectional programming language and related to lenses. Comparing these tools, they showed that eMoflon tends to perform better than the other two approaches w.r.t. performance. Additionally, eMoflon only failed on one scenario, while mediniQVT failed in four and BiGUL in two. Using our previous smaller *Short-Cut Rules* [\[FKST18\]](#page-31-3), we were already able to solve the last case and as shown in our evaluation, the new approach is able to preserve information in even more cases. This benchmark was later extended [\[ABW](#page-30-5)+20], featuring more scenarios and being implemented by a more diverse set of tools. There, eMoflons performance was showing a worse runtime performance than most other approaches and was not able to solve 6 out of 34 cases. However, an older version was used that did not incorporate *Short-Cut Rules* and worked with another incremental pattern matcher that has since been replaced by a more efficient one.

Model repair. Model repair $[MTC17]$ is the task of restoring the consistency of a model with regard to some specification. Since modelling languages like UML [\[Obj17\]](#page-33-9) can be used to describe a single system using a set of models of different types, e.g., class and sequence diagrams, there are several approaches to model repair that do not only restore intra-model but also inter-model consistency, i.e., that propagate changes on one kind of model to related models of other kinds. Recent such approaches, in particular, with support for synchronising class and sequence diagrams, include $[OPK^+21, KKLE21, BHR122, MKA^+23]$ $[OPK^+21, KKLE21, BHR122, MKA^+23]$. In these cases, consistency is defined via a modelling language like UML or the requirements of a modelling framework like the Eclipse Modeling Framework [\[Ecl24\]](#page-31-13), possibly complemented by constraints expressed in a language like OCL [\[OMG14\]](#page-33-12). Generally, TGGs could also be applied for such repair tasks. For this, it would first be necessary to develop grammars that express the desired consistency relationships between the different kinds of models. However, not every consistency requirement expressible in OCL can be expressed using a TGG since for TGGs the language inclusion problem is decidable. Having their formal foundation in the theory of graph transformation, TGG-based approaches, including ours, often come with far more comprehensive formal guarantees than offered in approaches like [\[OPK](#page-33-10)+21, [KKLE21,](#page-32-15) [BHRI22,](#page-31-12) [MKA](#page-33-11)+23]. On the other hand, whereas TGG-based approaches usually use repair operations that are derived from the rules of the grammar,

 ${}^{8}\rm{medianiQVT}$ is not maintined and no longer available.

approaches to model repair are often more flexible. For instance, Barriga et al. employ machine learning [\[BHRI22\]](#page-31-12) and Ohrndorf et al. $[OPK^+21]$ $[OPK^+21]$ and Kretschmer et al. $[KKLE21]$ use information on the edit history of a model. Furthermore, Ohrndorf et al. $[OPK^+21]$ $[OPK^+21]$ as well as Marchezan et al. $[MKA^+23]$ $[MKA^+23]$ do not automatically repair a model but provide the user with a set of (relevant) repair options from which they can choose; in this way, it is guaranteed that user intent will be met. Our approach could be extended in this direction by not computing a single higher-order Short-Cut Rule but different interesting ones, letting the user select between the different outcomes.

7. Conclusion

In this paper, we presented a novel approach to construct non-trivial repair rules in the form of higher-order *Short-Cut Rules* from a given consistency specification. These repair rules are constructed on demand by analysing a precedence graph that is annotated based on user changes. Our approach was fully implemented into eMoflon, a state-of-the-art graph transformation tool, which forms the base of our evaluation. In the evaluation, we showed that higher-order *Short-Cut Rules* can preserve information in more complex cases, while introducing only a small overhead to the runtime and at times even outperforming other strategies in eMoflon. Next, we would like to investigate whether we can identify situations statically where a higher-order *Short-Cut Rule* is needed to improve performance when smaller *Short-Cut Rules* suffice. For concurrent synchronisation scenarios, these rules could also be used to check whether sequences of user changes to both models correspond to or contradict each other.

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Appendix A.

This is the entire ILP formulation of our example from Section [4.](#page-14-0) Note that we added variables for correspondence links, where co_i stands for the correspondence link in between the model elements with the same index *i*. The correspondence link from Nook Rule is nco_i^r and the ones from Cube Rule are denoted co_i^r and nco_i^r , where the first represents the context and the other the created correspondence link. The correspondence links in Double Cube Rule are named similar to Cube Rule but the second created correspondence link connecting House h_2 with the Construction is called $nnco_i^r$. Note that all weights are equal 1.

Objective Function.

$$
max(v[N_1] + v[C_2] + v[C_3] + v[DC_{1,2}] + v[p_2^r \rightarrow p_1^r] + v[p_3^r \rightarrow p_1^r] + v[p_{2,3}^r \rightarrow p_1^r] + v[c_{2,3}^r \rightarrow n c_1^r] + v[c_{2,3}^r \rightarrow n c_2^r] + v[p_1^r \rightarrow p_2] + v[n c_1^r \rightarrow c_2] + v[n c_1^r \rightarrow c_3] + v[n c_2^r \rightarrow c_2] + v[n c_2^r \rightarrow c_3] + v[n c_3^r \rightarrow c_2] + v[n c_3^r \rightarrow c_2] + v[c_3^r \rightarrow c_2] + v[nc_3^r \rightarrow c_2] + v[n c_2^r \rightarrow c_2] + v[n c_2
$$

Rule Candidate Constraints.

$$
v[C_2] \le v[N_1]
$$

\n
$$
v[C_3] \le v[DC_{1,2}]
$$

\n
$$
v[DC_{1,2}] \le v[N_1]
$$

\n
$$
v[C_2] + v[DC_{1,2}] \le 1
$$

\n
$$
v[C_3] + v[DC_{1,2}] \le 1
$$

Context-Create Mapping Constraints.

$$
v[p_2^r \to p_1^r] \leq v[N_1] \qquad v[p_3^r \to p_1^r] \leq v[C_2]
$$

\n
$$
v[p_3^r \to p_1^r] \leq v[N_1] \qquad v[p_3^r \to p_1^r] \leq v[C_3]
$$

\n
$$
v[e_2^r \to nc_1^r] \leq v[N_1] \qquad v[e_2^r \to nc_1^r] \leq v[DC_{1,2}]
$$

\n
$$
v[c_3^r \to nc_1^r] \leq v[N_1] \qquad v[c_3^r \to nc_1^r] \leq v[C_2]
$$

\n
$$
v[c_3^r \to nc_1^r] \leq v[N_1] \qquad v[c_3^r \to nc_1^r] \leq v[C_3]
$$

\n
$$
v[e_2^r \to e_1^r] \leq v[N_1] \qquad v[e_3^r \to nc_1^r] \leq v[C_3]
$$

\n
$$
v[e_2^r \to e_1^r] \leq v[N_1] \qquad v[e_3^r \to nc_1^r] \leq v[C_3]
$$

\n
$$
v[e_4^r \to e_1^r] \leq v[N_1] \qquad v[e_4^r \to e_1^r] \leq v[C_2]
$$

\n
$$
v[e_4^r \to e_1^r] \leq v[N_1] \qquad v[e_4^r \to e_1^r] \leq v[C_3]
$$

\n
$$
v[e_4^r \to e_3^r] \leq v[C_2] \qquad v[e_4^r \to e_3^r] \leq v[DC_{1,2}]
$$

\n
$$
v[co_2^r \to nco_1^r] \leq v[N_1] \qquad v[co_2^r \to nco_1^r] \leq v[C_2]
$$

\n
$$
v[co_3^r \to nco_1^r] \leq v[N_1] \qquad v[co_3^r \to nco_1^r] \leq v[C_2]
$$

\n
$$
v[co_3^r \to nco_1^r] \leq v[N_1] \qquad v[co_3^r \to nco_1^r] \leq v
$$

$$
v[c_3^r \to nc_1^r] + v[c_2^r \to nc_2^r] \le 1
$$

$$
v[e_4^r \to e_1^r] + v[e_4^r \to e_3^r] \le 1
$$

$$
v[co_3^r \to nco_1^r] + v[co_3^r \to nc_2^r] \le 1
$$

 $v[p_1^r \to p_2] \le v[N_1]$ $v[nc_1^r]$ $v[nc_1^r \to c_2] \le v[N_1]$ $v[nc_1^r \rightarrow c_3] \le v[N_1]$ $v[nc_2^r$ $v[nc_2^r \rightarrow c_2] \le v[C_2]$ $v[nc_2^r \rightarrow c_3] \le v[C_2]$ $v[nc_3^r$ $v[nc_3^r \rightarrow c_2] \le v[C_3]$ $v[nc_3^r \rightarrow c_3] \le v[C_3]$ $v[e]$ $r_1 \to e_1 \leq v[N_1]$ $v[e_1^r \to e_2] \le v[N_1]$ $v[e_2^r \to e_2]$ $r_3 \to e_1 \leq v[C_2]$ $v[e_3^r \to e_2] \le v[C_2]$ $v[e_3^r \to e_2]$ $r_5 \to e_1 \leq v[C_3]$ $v[e_5^r \to e_2] \le v[C_3]$ $v[nco_1^r]$ $v[nco_1^r \rightarrow co_2] \le v[N_1]$ $v[nco_2^r \to co_2] \le v[C_2]$ $v[nco_3^r$ $v[nco_3^r \rightarrow co_2] \le v[C_3]$ $v[nco_{1,2}^r \to co_2] \le v[DC_{1,2}]$ $v[nco_1^r]$ $v[nco_{1,2}^r] \to co_2] \le v[DC_{1,2}]$ $v[nco_1^r \to co_3] \le v[N_1]$ $v[nco_2^r$ $v[nco_2^r \rightarrow co_3] \le v[C_2]$ $v[nco_3^r \rightarrow co_3] \le v[C_3]$ $v[nco_1^r$ $v[nco_{1,2}^r] \to co_3] \le v[DC_{1,2}]$ $v[nnco_{1,2}^r \to co₃]] \leq v[DC_{1,2}]$ $v[nc_1^r \to c_2] + v[nc_1^r \to c_3] \leq 1$ $v[nc_2^r \to c_2] + v[nc_2^r \to c_3] \leq 1$ $v[nc_3^r \to c_2] + v[nc_3^r \to c_3] \leq 1$ $v[e_1^r \to e_1] + v[e_1^r \to e_2] \leq 1$ $v[e_3^r \to e_1] + v[e_3^r \to e_2] \leq 1$ $v[e_5^r \to e_1] + v[e_5^r \to e_2] \leq 1$ $v[nco_1^r \rightarrow co_2] + v[nco_1^r \rightarrow co_3] \leq 1$ $v[nco_2^r \rightarrow co_2] + v[nco_2^r \rightarrow co_3] \leq 1$ $v[nco_3^r \rightarrow co_2] + v[nco_3^r \rightarrow co_3] \leq 1$ $v[nco_{1,2}^r \to co_2] + v[nco_{1,2}^r \to co_3] \leq 1$ $v[nnco_{1,2}^T \to co₂] + v[nnco_{1,2}^T \to co₃] \le 1$ $3v[e_3^r \to e_1] = v[nc_2^r \to c_2] + v[p_1^r \to p_2] + v[p_2^r \to p_1^r]$ $3v[e_3^r \to e_2] = v[nc_2^r \to c_3] + v[p_1^r \to p_2] + v[p_2^r \to p_1^r]$ $3v[e_5^r \to e_1] = v[nc_3^r \to c_2] + v[p_1^r \to p_2] + v[p_3^r \to p_1^r]$ $3v[e_5^r \to e_2] = v[nc_3^r \to c_3] + v[p_1^r \to p_2] + v[p_3^r \to p_1^r]$ $v[e_1^r \to e_1] \le v[nc_1^r \to c_2]$ $v[e_1^r \to e_1]$ $r_1 \rightarrow e_1 \leq v[p_1^r \rightarrow p_2]$ $v[e_1^r \to e_2] \le v[nc_1^r \to c_3]$ $v[e_1^r \to e_2]$ $r_1 \rightarrow e_2 \leq v[p_1^r \rightarrow p_2]$ $v[nco_2^r \rightarrow co_2] \le v[nc_2^r]$ $v[nc_3^r \rightarrow co_3] \le v[nc_3^r \rightarrow c_3]$ $v[nco_{1,2}^r \rightarrow co_2] \le v[nc_3^r]$ $v[nnco_{1,2}^r \rightarrow co₃] \le v[nc₃^r \rightarrow c₃]$