

A CONE-THEORETIC BARYCENTER EXISTENCE THEOREM

JEAN GOUBAULT-LARRECQ ^a AND XIAODONG JIA ^b

^a Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France
e-mail address: jgl@lmf.cnrs.fr

^b School of Mathematics, Hunan University, Changsha, Hunan, 410082, China
e-mail address: jiaxiaodong@hnu.edu.cn (Corresponding author)

ABSTRACT. We show that every continuous valuation on a locally convex, locally convex-compact, sober topological cone \mathcal{C} has a barycenter. This barycenter is unique, and the barycenter map β is continuous, hence is the structure map of a \mathbf{V}_w -algebra, i.e., an Eilenberg-Moore algebra of the extended valuation monad on the category of T_0 topological spaces; it is, in fact, the unique \mathbf{V}_w -algebra that induces the cone structure on \mathcal{C} .

1. INTRODUCTION

It is well-known that every probability measure μ defined on a compact convex subset of a Hausdorff topological vector space has a unique barycenter $\beta(\mu)$, and that the barycenter map β is continuous [Cho69, Chapter 6, 26.3]. Our purpose is to establish a similar theorem in the setting of topological cones, in the sense of Reinhold Heckmann [Hec96] and Klaus Keimel [Kei08b]; those are definitely not limited to cones that one can embed in vector spaces.

Applications. Such a barycenter existence theorem is strongly linked to the question of the elucidation of the algebras of the monad \mathbf{V}_w of continuous valuations on the category \mathbf{Top}_0 of T_0 topological spaces, as well as of the monads of probability and subprobability valuations. (See [CEK06, GJ19]. We will introduce \mathbf{V}_w below.) As a result, we will be able to show that a rich collection of topological cones give rise to such algebras.

Investigating the structure of algebras of monads is not just an interesting, and important, mathematical question, but also has application in semantics. For example, knowing that the category of algebras of the subprobability valuation monad on the category of continuous dekos is isomorphic to the category of so-called continuous Kegelspitzen is crucial to give an adequate semantics to the variational quantum programming language of [JKL⁺22], where barycenter maps are used to link the category of some special Von-Neumann algebras to the Kleisli category of the so-called minimal valuations monad.

Key words and phrases: valuation monad, Eilenberg-Moore algebra, barycenter, locally convex, locally convex-compact, topological cone .

Related work. Ben Cohen, Martín Escardó and Klaus Keimel [CEK06] first asked the question of characterizing the algebras of \mathbf{V}_w on \mathbf{Top}_0 , and then on the subcategory \mathbf{SComp} of stably compact spaces and continuous maps. They also observed how the question was deeply linked to a notion of barycenter adapted from Choquet’s work. There seems to be a gap in their Proposition 2, however (see Proposition 2.5 below), and [GJ19] is probably now a better reference. Both papers only give *examples* of spaces where such barycenters exist, as well as of algebras of \mathbf{V}_w , but no general existence theorem.

Keimel also fully characterized the algebras of the monad of probability measures over compact ordered spaces [Kei08a]. This can be seen as a close result, because of the tight relationship between compact ordered spaces and stably compact spaces. But the category of compact ordered spaces and continuous maps is equivalent to the category of stably compact spaces and *perfect* maps, not continuous maps. Keimel’s result itself extends Świrszcz’s characterization of the algebras of the monad of probability measures on compact Hausdorff spaces [Świ74].

Returning to categories of spaces that are not necessarily Hausdorff, with continuous, not necessarily perfect maps, we do have some results for certain submonads of \mathbf{V}_w : this includes the monads of so-called simple valuations and of point-continuous valuations [GJ19, Hec96], for which we have general barycenter existence theorems, and complete characterizations of the corresponding algebras. Since point-continuous valuations and continuous valuations coincide on continuous dpos, those results encompass the case of continuous Kegelspitzen mentioned earlier.

Our purpose here is to give a barycenter existence theorem that would need assumptions that are as weak as possible. We initially looked for variants of arguments due to Choquet [Cho60], Edwards [Edw63], and Roberts [Rob78], among others. These arguments work as follows. Define a partial ordering between probability distributions by $\mu_1 \preceq \mu_2$ if and only if $\int h d\mu_1 \geq \int h d\mu_2$ for every convex map h in some class (typically, bounded, continuous maps); this formalizes the idea that μ_2 has a more concentrated distribution of mass than μ_1 , while keeping the same center of mass. Then, build a continuous map F such that $\mu \preceq F(\mu)$ for all probability distributions μ , and such that $F(\mu) = \mu$ if and only if μ is a Dirac measure δ_x . Finally, using a compactness argument, extract a convergent subsequence from the sequence $(F^n(\mu))_{n \in \mathbb{N}}$: this must converge to some Dirac measure δ_x , and by construction x will be a barycenter of μ . Details may vary. While there are immediate obstructions to generalizing that form of argument to a non-Hausdorff setting (e.g., limits are not unique), it is possible to do so, but the result we obtained by following this path was disappointing: a long, complicated proof, which required very strong assumptions in the end.

In comparison, we will use a much simpler strategy: we reduce the question to the already known construction of barycenters of point-continuous valuations, by embedding our cone into a larger one, the upper powercone introduced by Keimel [Kei08b, Section 11]. We only need two additional lemmata: one to actually show that there is such a cone embedding, Lemma 3.1, and one to transfer any barycenter in the larger cone back to the smaller cone, Lemma 3.3.

2. CONES, CONTINUOUS VALUATIONS, BARYCENTERS

We refer the reader to [GHK⁺03, Gou13] for more information on domain theory and topology, especially non-Hausdorff topology.

A *dcpo* (short for directed-complete partial order) is a poset (X, \leq) such that every directed family D has a supremum $\sup D$; we write $\sup^\uparrow D$ to stress the fact that D is directed. A subset A of D is *upwards-closed* if and only if $x \in A$ and $x \leq y$ imply $y \in A$. We will also write $\uparrow A$ for the upward closure $\{y \in X \mid x \leq y \text{ for some } x \in A\}$, and $\uparrow x$ instead of $\uparrow\{x\}$, and similarly for the downward closure $\downarrow A$, and $\downarrow x$. The *Scott topology* on (X, \leq) consists of Scott-open subsets of X , which are defined to be upwards-closed and inaccessible by directed suprema. That is, an upwards-closed subset U is *Scott-open* if and only if every directed family D intersects U if $\sup^\uparrow D \in U$. A Scott-continuous map between dcpos is a monotonic map that preserves directed suprema; equivalently, a continuous map, provided the given dcpos are given the Scott topology.

We write \mathbb{R}_+ for the set of non-negative real numbers, and $\overline{\mathbb{R}}_+$ for $\mathbb{R}_+ \cup \{\infty\}$, where ∞ is larger than any element of \mathbb{R}_+ . \mathbb{R}_+ is a dcpo in the natural ordering \leq on real numbers, $\overline{\mathbb{R}}_+$ is not. We equip the dcpo $\overline{\mathbb{R}}_+$ with the Scott topology; its non-empty open subsets are the half-open intervals $]a, \infty]$, $a \in \mathbb{R}_+$. (And we give \mathbb{R}_+ the subspace topology.) With that topology, the continuous maps from a topological space to $\overline{\mathbb{R}}_+$ are usually called the *lower semicontinuous maps*.

The way-below relation \ll on a dcpo X is defined by $x \ll y$ if and only if for every directed family D such that $y \leq \sup^\uparrow D$, there is a $z \in D$ such that $x \leq z$. We write $\downarrow y$ for $\{x \in X \mid x \ll y\}$. A *continuous dcpo* is a dcpo in which $\downarrow y$ is directed and $\sup^\uparrow \downarrow y = y$ for every point y .

The *specialization preordering* \leq of a topological space X is defined by $x \leq y$ if and only if every open neighborhood of x contains y . The space X is T_0 if and only if \leq is antisymmetric. The *saturation* of a subset A of X is the intersection of its open neighborhoods, and coincides with $\uparrow A$. The closure of a singleton $\{x\}$ is equal to $\downarrow x$, where \downarrow is relative to the specialization preordering. Any continuous map between topological spaces is monotonic with respect to the underlying specialization preorderings.

A subset K of a topological space X is *compact* if and only if every open cover of K contains a finite subcover; no separation axiom is intended here. If K is compact, then $\uparrow K$ is both compact and saturated. A topological space is *locally compact* if every point in it has a neighborhood base of compact subsets, namely if for every point x and every open neighborhood U of x , there is a compact set Q such that $x \in \text{int}(Q) \subseteq Q \subseteq U$. (We write $\text{int}(K)$ for the *interior* of K .) We may equivalently require Q to be compact *saturated*, by replacing Q by $\uparrow Q$. A space is *sober* if every nonempty irreducible closed subset in it is equal to $\downarrow x$ for some unique point x . Every continuous dcpo is locally compact and sober in its Scott topology [GHK⁺03, Corollary II-1.13].

Following Heckmann [Hec96] and Keimel [Kei08b], a *cone* \mathfrak{C} is an additive commutative monoid with an action $a, x \mapsto a \cdot x$ of the semi-ring \mathbb{R}_+ on \mathfrak{C} (*scalar multiplication*). The latter means that the following laws are satisfied:

$$\begin{aligned} 0 \cdot x &= 0 & (ab) \cdot x &= a \cdot (b \cdot x) & 1 \cdot x &= x \\ a \cdot 0 &= 0 & a \cdot (x + y) &= a \cdot x + a \cdot y & (a + b) \cdot x &= a \cdot x + b \cdot x. \end{aligned}$$

We will often write ax instead of $a \cdot x$. In other words, a cone satisfies the same axioms as a vector space, barring those mentioning subtraction. \mathbb{R}_+ is a cone, and embeds (as a cone) into the real vector space \mathbb{R} . $\overline{\mathbb{R}}_+$ is a cone, but embeds in no real vector space, because it fails the cancellation law $x + y = x + z \Rightarrow y = z$. None of the other cones we will consider embed in any real vector space, for the same reason.

In this paper, a *topological cone* is a cone with a T_0 topology that makes both addition and scalar multiplication jointly continuous—where we recall that the topology on \mathbb{R}_+ is induced by the Scott topology on $\overline{\mathbb{R}}_+$, which is not Hausdorff. A *semitopological cone* is defined similarly, except that addition is only required to be separately continuous. Since continuous maps are monotonic with respect to the specialization preordering \leq , in a semitopological cone, $ax \leq bx$ whenever $a \leq b$; in particular, 0 is the least element. Hence a non-trivial semitopological cone \mathfrak{C} is never T_1 , but is always compact. Compactness comes from the fact that every open cover $(U_i)_{i \in I}$ will be such that $0 \in U_i$ for some $i \in I$, and then $U_i = \mathfrak{C}$, since U_i is upwards-closed and 0 is the least element of \mathfrak{C} .

A *d-cone* is a cone with a partial ordering that turns it into a dcpo, and such that addition and scalar multiplication are both Scott-continuous. A continuous d-cone is a d-cone that is also a continuous dcpo. Every continuous d-cone is a topological cone in the Scott topology. For example, $\overline{\mathbb{R}}_+$ is a continuous d-cone.

A subset A of a cone \mathfrak{C} is *convex* if and only if $ax + (1 - a)y \in A$ for all $x, y \in A$ and $a \in [0, 1]$. A semitopological cone is *weakly locally convex*, resp. *locally convex*, resp. *locally convex-compact*, if and only if every point has a base of convex, resp. convex open, resp. convex compact, neighborhoods. Every continuous d-cone is not only topological, but also locally convex and locally convex-compact [Kei08b, Lemma 6.12].

Example 2.1. Here are a few examples to bear in mind. We will give a few others below. The point of (2) and (3) is twofold: to show that cones can be very different from real vector spaces; and to show that there are sober, locally convex, locally convex-compact topological cones that are not continuous d-cones.

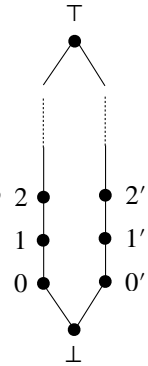
- (1) For every space X , let us consider the dcpo $\mathcal{L}X$ of all lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$, namely, those continuous maps from X to $\overline{\mathbb{R}}_+$ equipped with the Scott topology. With the obvious addition and scalar multiplication, and the Scott topology of the pointwise ordering, $\mathcal{L}X$ is a semitopological cone.

If X is core-compact, namely, if its lattice of open sets is a continuous dcpo (every locally compact space is core-compact, and the two notions are equivalent for sober spaces), then $\mathcal{L}X$ is a continuous d-cone [GJ19, Example 3.5]. In particular, it is topological, sober, locally convex, and locally convex-compact.

- (2) This one is due to Heckmann [Hec96, Section 6.1], see also [GJ19, Example 3.6]. Every sup-semilattice L with a least element gives rise to a semitopological cone in its Scott topology by defining $x + y$ as $x \vee y$, 0 as the least element \perp , and $a \cdot x$ as \perp if $a = 0$, and as x otherwise; it is a continuous d-cone (hence topological, locally convex, and locally convex-compact) if L is a continuous complete lattice.

For general L , it remains that every upwards-closed subset is convex. Hence each such sup-semilattice L is a locally convex semitopological cone in its Scott topology. It is locally convex-compact if L is also locally compact in the Scott topology. In that case, L is also a topological cone. This is because on a core-compact poset L , the Scott topology on $L \times L$ is the same as the product topology of the Scott topology on L [GHK⁺03, Theorem II-4.13].

In particular, consider any locally compact sup-semilattice L with bottom that is not a continuous dcpo. Then L is topological, locally convex, and locally convex-compact, but not a continuous d-cone. The simplest example is given by putting two copies of \mathbb{N} side by side, with extra bottom and top elements, as shown on the right. Its nonempty proper Scott open subsets are those of the form $\uparrow\{m, n'\}$ where $m, n \in \mathbb{N}$, and they are all compact, so L is locally compact. One can even check that it is a quasi-continuous dcpo [GHK⁺03, Definition III-3.2], which not only implies that it is locally compact, but also sober [GHK⁺03, Proposition III-3.7]. However, L is not a continuous dcpo, since the way-below relation is characterized by $x \ll y$ if and only if $x = \perp$.



- (3) Here is a variant on the previous example. We consider a complete lattice L with its *upper topology*, which is the coarsest topology that makes the sets $\downarrow x$ closed. The specialization ordering is the ordering \leq of L . By a result of Schalk [Sch93, Proposition 1.7], L is sober. We equip L with the same cone structure as in (2). Then L is topological, since $(\cdot + \cdot)^{-1}(\downarrow x) = \downarrow x \times \downarrow x$ is closed in the product topology for every $x \in L$, and similarly for $(\cdot \cdot)^{-1}(\downarrow x) = (\mathbb{R}_+ \times \downarrow x) \cup (\{0\} \times L)$. As in (2), every upwards-closed subset of L is convex, so L is locally convex, and L is locally convex-compact if and only if it is locally compact.

In particular, the lattice $\Gamma(X)$ of closed subsets of any topological space X , with the upper topology of inclusion (also known as the *lower Vietoris topology*), union as addition, and empty set as zero, is a sober, locally convex topological cone. It is locally compact, hence locally convex-compact, if X is itself locally compact [Sch93, Proposition 6.11]. But, say, $\Gamma(\mathbb{R})$, where \mathbb{R} is taken with its usual metric topology (a locally compact space), is not a continuous dcpo (or a quasi-continuous dcpo) with its Scott topology; the point is that the upper topology differs from the Scott topology: by a result of Chen, Kou and Lyu [CKL22, Proposition 3.15], the only T_1 spaces X such that the upper and Scott topologies coincide on $\Gamma(X)$ are discrete.

We write \mathfrak{C}^* for the cone of linear lower semicontinuous maps $\Lambda: \mathfrak{C} \rightarrow \overline{\mathbb{R}}_+$, where linear means preserving sums and scalar products. Every locally convex T_0 semitopological cone is *linearly separated*, in the sense that for every pair of points x, y such that $x \not\leq y$, there is a map $\Lambda \in \mathfrak{C}^*$ such that $\Lambda(x) < \Lambda(y)$ [Kei08b, Corollary 9.3]. By multiplying by an appropriate constant, we may require $\Lambda(x) \leq 1$ and $\Lambda(y) > 1$.

A *continuous valuation* ν on a space X is a map from the lattice of open subsets of X to $\overline{\mathbb{R}}_+$ such that $\nu(\emptyset) = 0$, $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all open sets U and V , and such that ν is Scott-continuous [Jon90]. This is a close cousin of the notion of measure: every continuous valuation extends to a τ -smooth measure on the Borel σ -algebra of X provided that X is LCS-complete [DBGJL19, Theorem 1], and that includes the case of all locally compact sober spaces. Additionally, there is a notion of integral of maps $h \in \mathcal{L}X$ with respect to continuous valuations ν on X , with the usual properties. We will use the following four:

- $\int_{x \in X} h(x) d\delta_{x_0} = h(x_0)$, where δ_{x_0} is the *Dirac valuation* at x_0 , defined by $\delta_{x_0}(U) \stackrel{\text{def}}{=} 1$ if $x_0 \in U$, 0 otherwise;
- $\int_{x \in X} \chi_U(x) d\nu = \nu(U)$, where χ_U is the characteristic function of an open set U , defined as $\chi_U(x) = 1$ if $x \in U$, 0 otherwise;

- the *change of variable* formula: for every continuous map $f: X \rightarrow Y$, $\int_{y \in Y} h(y) df[\nu] = \int_{x \in X} h(f(x)) d\nu$; here $f[\nu]$ is the *image valuation* defined by $f[\nu](V) \stackrel{\text{def}}{=} \nu(f^{-1}(V))$;
- integration is linear in the integrated function h .

We write $\mathbf{V}X$ for the set of all continuous valuations on X . The *weak topology* on $\mathbf{V}X$ is the coarsest one that makes $\nu \mapsto \nu(U)$ continuous for every open set U . In other words, a subbase of the weak topology is given by the sets $[U > r] \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}X \mid \nu(U) > r\}$, where U ranges over the open subsets of X and $r \in \mathbb{R}_+$. We write $\mathbf{V}_w X$ for $\mathbf{V}X$ with the weak topology. The specialization ordering on $\mathbf{V}_w X$ is the so-called *stochastic ordering*: for all continuous valuations μ and ν , $\mu \leq \nu$ if and only if $\mu(U) \leq \nu(U)$ for every open subset U of X . \mathbf{V}_w extends to an endofunctor on the category \mathbf{Top}_0 of T_0 topological spaces, and its action on morphisms $f: X \rightarrow Y$ is given by $\mathbf{V}_w f(\nu) \stackrel{\text{def}}{=} f[\nu]$.

Example 2.2. Here are few more examples of semitopological cones.

- (4) With the obvious addition and scalar multiplication, $\mathbf{V}_w X$ is a locally convex, sober, topological cone [GJ19, Proposition 3.13]. This is really a special case of the next item, considering that $\mathbf{V}_w X$ is isomorphic to the dual cone $(\mathcal{L}X)^*$ (see next item) as semitopological cones.
- (5) We equip \mathfrak{C}^* with the *weak* upper topology*, namely the coarsest one that makes the functions $\text{ev}_x: \Lambda \mapsto \Lambda(x)$ lower semicontinuous from \mathfrak{C}^* to $\overline{\mathbb{R}}_+$, for each $x \in \mathfrak{C}$; $\overline{\mathbb{R}}_+$ again has the Scott topology of its ordering. (The weak* upper topology is also known as the *weak*-Scott topology* [Plo06].) The cone \mathfrak{C}^* is the *dual cone* of \mathfrak{C} [Kei08b]; see also [CEK06, Section 3] or Example 3.7 (ii) of [GJ19]. A dual cone is always topological [GJ19, Example 3.7 (ii)]. It is also locally convex, since it has a base of open sets of the form $\{\Lambda \in \mathfrak{C}^* \mid \Lambda(x_1) > 1, \dots, \Lambda(x_n) > 1\}$, where x_1, \dots, x_n ranges over the finite lists of points of \mathfrak{C} , and such sets are convex. Finally, \mathfrak{C}^* is sober, using the following argument, adapted from an argument of Heckmann's [Hec96, Proposition 5.1]. For every sober space Y , the space $[X \rightarrow Y]_p$ of all continuous maps from X to Y , with the pointwise topology, is sober by [Tix95, Lemma 5.8]; the pointwise topology is the coarsest that makes $h \mapsto h(x)$ continuous for every $x \in X$. By letting $Y \stackrel{\text{def}}{=} \overline{\mathbb{R}}_+$ with the Scott topology, $[X \rightarrow \overline{\mathbb{R}}_+]_p$ is sober. Then \mathfrak{C}^* occurs as a T_0 -equalizer subspace of $[X \rightarrow \overline{\mathbb{R}}_+]_p$, namely as the equalizer of the continuous maps $h \mapsto ((h(ax))_{a \in \mathbb{R}_+, x \in \mathfrak{C}}, (h(x+y))_{x,y \in \mathfrak{C}})$ and $h \mapsto ((ah(x))_{a \in \mathbb{R}_+, x \in \mathfrak{C}}, (h(x) + h(y))_{x,y \in \mathfrak{C}})$ with codomain the T_0 space $\overline{\mathbb{R}}_+^{\mathbb{R}_+ \times \mathfrak{C}} \times \overline{\mathbb{R}}_+^{\mathfrak{C}^2}$. And any T_0 -equalizer subspace of a sober space is sober [Tix95, Lemma 5.9].

\mathbf{V}_w is the functor part of a monad on \mathbf{Top}_0 , whose unit η_X maps every point $x \in X$ to δ_x , and the multiplication map at X sends a continuous valuation $\varphi \in \mathbf{V}_w \mathbf{V}_w X$ to $(U \mapsto \int_{\nu \in \mathbf{V}_w X} \nu(U) d\varphi)$. We are interested in its algebras $\alpha: \mathbf{V}_w X \rightarrow X$. By Lemma 4.6 of [GJ19], any such algebra induces a cone structure on X by $x+y \stackrel{\text{def}}{=} \alpha(\delta_x + \delta_y)$, $a \cdot x \stackrel{\text{def}}{=} \alpha(a\delta_x)$, which turns X into a weakly locally convex, sober, topological cone [GJ19, Proposition 4.9]. Additionally, with this induced cone structure, α is a linear map, and for every $\nu \in \mathbf{V}_w X$, $\alpha(\nu)$ is a barycenter of ν .

Following [CEK06, GJ19], we define a *barycenter* of $\nu \in \mathbf{V}_w \mathfrak{C}$, where \mathfrak{C} is a semitopological cone, as any point x_0 of \mathfrak{C} such that $\Lambda(x_0) = \int_{x \in \mathfrak{C}} \Lambda(x) d\nu$ for every $\Lambda \in \mathfrak{C}^*$. If \mathfrak{C} is linearly separated and T_0 , then barycenters are unique if they exist. This definition is

similar to Choquet's classic definition [Cho69, Chapter 6, 26.2]. The main difference is that our linear maps are only required to be lower semicontinuous; also, we have replaced vector spaces by cones, and the definition applies to all continuous valuations, not just probability valuations (namely, valuations ν such that $\nu(\mathfrak{C}) = 1$).

The following example should explain how barycenters generalize the usual notion of barycenters of points x_i with weights a_i .

Example 2.3. A *simple valuation* is any finite linear combination $\nu \stackrel{\text{def}}{=} \sum_{i=1}^n a_i \delta_{x_i}$ of Dirac valuations, with $a_i \in \mathbb{R}_+$. If all the points x_i are taken from a semitopological cone \mathfrak{C} , then $\sum_{i=1}^n a_i \cdot x_i$ is a barycenter of ν , by the linearity of integration and of the maps Λ involved in the definition [GJ19, Example 4.3].

Example 2.4. Here are additional, more sophisticated examples.

- (1) For every core-compact space X , every continuous valuation on $\mathcal{L}X$ has a barycenter. This is really a consequence of Proposition 3.2 below, plus the fact that $\mathcal{L}X$ is a continuous d-cone in that case. Explicitly, one can check that the map $x \mapsto \int_{h \in \mathcal{L}X} h(x) d\nu$ is that barycenter [GJ19, Proposition 4.15].
- (2) If L is a complete lattice with its Scott topology (see Example 2.1 (2)), then every continuous valuation μ on L has a barycenter, which is the supremum of the elements of its support $\text{supp } \mu$ [GJ19, Example 4.4]; the *support* $\text{supp } \mu$ of μ is defined as the complement of the largest open set U such that $\mu(U) = 0$.
- (3) Let L be a complete lattice with its upper topology (see Example 2.1 (3)). Just like in the previous item, every continuous valuation μ on L has a barycenter, which is the supremum of $\text{supp } \mu$. The argument follows the same lines as [GJ19, Examples 3.15, 3.20, 4.4]. Each non-empty convex subset of L is directed. Every non-empty closed convex subset C must then be of the form $\downarrow x$ for some $x \in L$: namely, $x = \sup^\uparrow C$, which must be in C since C is closed, hence Scott-closed (note that the upper topology is coarser than the Scott topology). The elements Λ of L^* must be such that $\Lambda^{-1}([0, 1])$ is closed, non-empty, and convex, hence of the form $\downarrow x_0$ for some $x_0 \in L$. For every $x \in L$, the fact that $\Lambda(x) = \Lambda(r \cdot x) = r\Lambda(x)$ for every $r > 0$ implies that $\Lambda(x)$ can only be equal to 0 or to ∞ . Hence $\Lambda = \infty \cdot \chi_{L \setminus \downarrow x_0}$. Conversely, any map of this form is in L^* . Using the definition of barycenters, a point $x \in L$ is a barycenter of μ if and only if for every $x_0 \in L$, the inequality $x \leq x_0$ is equivalent to that $\infty \cdot \mu(L \setminus \downarrow x_0) = 0$, to that $\text{supp } \mu \subseteq \downarrow x_0$, and to that $\text{sup}(\text{supp } \mu) \leq x_0$, whence the conclusion.
- (4) The barycenters of continuous valuations ν on dual cones \mathfrak{C}^* do not exist in general. However, if Λ_0 is a barycenter of a continuous valuation ν on \mathfrak{C}^* , then $\varphi(\Lambda_0) = \int_{\Lambda \in \mathfrak{C}^*} \varphi(\Lambda) d\nu$ for every $\varphi \in \mathfrak{C}^{**}$ by definition of barycenters. For each point $x \in \mathfrak{C}$, the map $\varphi: \Lambda \mapsto \Lambda(x)$ is an element of \mathfrak{C}^{**} , so $\Lambda_0(x) = \int_{\Lambda \in \mathfrak{C}^*} \Lambda(x) d\nu$ for every $x \in \mathfrak{C}$. In other words, if ν has a barycenter, then it must be the map $x \mapsto \int_{\Lambda \in \mathfrak{C}^*} \Lambda(x) d\nu$. However, it is unknown whether that map is a barycenter of ν . Also, that map may fail to be an element of \mathfrak{C}^* at all, as it is a map that may well fail to be lower semicontinuous. We will return to this in Example 3.5 (4).
- (5) Every continuous valuation φ on $\mathbf{V}_w X$, for any space X whatsoever, has a barycenter, and it is obtained by applying the multiplication of the \mathbf{V}_w monad to φ . Indeed, by general category theory, that multiplication is a \mathbf{V}_w -algebra (in fact, the free \mathbf{V}_w -algebra), and we have seen that every \mathbf{V}_w -algebra is a barycenter map.

We have seen that every \mathbf{V}_w -algebra α induces a cone structure on its codomain, and that α maps every continuous valuation to one of its barycenters. Conversely, we have:

Proposition 2.5 (Proposition 4.10 of [GJ19]). *If every continuous valuation ν on a semitopological cone \mathfrak{C} has a unique barycenter $\beta(\nu)$, and if β is continuous, then β is a \mathbf{V}_w -algebra, and is in fact the unique \mathbf{V}_w -algebra that induces the given cone structure on \mathfrak{C} .*

The continuity of β is essential, and seems to have been forgotten in [CEK06, Proposition 2].

3. THE BARYCENTER EXISTENCE THEOREM

We start with a topological cone \mathfrak{C} . We use a construction due to Keimel [Kei08b, Section 11]: the *upper powercone* $\mathcal{S}\mathfrak{C}$ of \mathfrak{C} consists of all non-empty convex, compact saturated subsets of \mathfrak{C} . (A *saturated* set is one that is the intersection of its open neighborhoods, or equivalently an upwards-closed set, with respect to the specialization preordering.) Addition is defined by $Q_1 \dot{+} Q_2 \stackrel{\text{def}}{=} \uparrow\{x_1 + x_2 \mid x_1 \in Q_1, x_2 \in Q_2\}$, scalar multiplication by $a \cdot Q \stackrel{\text{def}}{=} \uparrow\{a \cdot x \mid x \in Q\}$, and the zero is $\underline{0} \stackrel{\text{def}}{=} \mathfrak{C}$. The cone $\mathcal{S}\mathfrak{C}$ is given the *upper Vietoris topology*, which is generated by the basic open sets $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{S}\mathfrak{C} \mid Q \subseteq U\}$, where U ranges over the open subsets of \mathfrak{C} . Its specialization ordering is *reverse* inclusion \supseteq , and every open set is Scott-open (with respect to \supseteq) when \mathfrak{C} is sober. If additionally \mathfrak{C} is locally convex and locally convex-compact, then $\mathcal{S}\mathfrak{C}$ is a continuous d-cone, and the Scott and upper Vietoris topologies coincide [Kei08b, Theorem 12.6].

The construction \mathcal{S} is the functor part of a monad on the category of sober topological cones and linear continuous maps as homomorphisms. This is a consequence of the results in Sections 11 and 12 of [Kei08b]. We will use its unit, which we write as $\eta_{\mathfrak{C}}^{\mathcal{S}}: \mathfrak{C} \rightarrow \mathcal{S}\mathfrak{C}$, and which maps every x to its upward closure $\uparrow x$.

For every $\Lambda \in \mathfrak{C}^*$, for every $Q \in \mathcal{S}\mathfrak{C}$, we let $\bar{\Lambda}(Q) \stackrel{\text{def}}{=} \min_{x \in Q} \Lambda(x)$.

Lemma 3.1. *Let \mathfrak{C} be a topological cone. For every $\Lambda \in \mathfrak{C}^*$, $\bar{\Lambda}$ is an element of $(\mathcal{S}\mathfrak{C})^*$, and $\bar{\Lambda} \circ \eta_{\mathfrak{C}}^{\mathcal{S}} = \Lambda$.*

Proof. $\bar{\Lambda}$ is continuous because $\bar{\Lambda}^{-1}(]r, \infty]) = \square \Lambda^{-1}(]r, \infty])$ for every $r \in \mathbb{R}_+$. For every $a \in \mathbb{R}_+$, for every $Q \in \mathcal{S}\mathfrak{C}$, $\bar{\Lambda}(a \cdot Q) = \min_{x \in Q} \Lambda(a \cdot x)$, since Λ is continuous hence monotonic, and taking the minimum of a monotonic map over the upward-closure of a set is the same as taking the minimum on that set. Hence $\bar{\Lambda}(a \cdot Q) = a \cdot \min_{x \in Q} \Lambda(x) = a \bar{\Lambda}(Q)$. Similarly, for all $Q_1, Q_2 \in \mathcal{S}\mathfrak{C}$, $\bar{\Lambda}(Q_1 \dot{+} Q_2) = \min_{x_1 \in Q_1, x_2 \in Q_2} \Lambda(x_1 + x_2) = \bar{\Lambda}(Q_1) + \bar{\Lambda}(Q_2)$. Finally, $\bar{\Lambda} \circ \eta_{\mathfrak{C}}^{\mathcal{S}}$ maps every point x to $\min_{y \in \uparrow x} \Lambda(y) = \Lambda(x)$, because Λ is monotonic. \square

There is a subspace $\mathbf{V}_p X$ of $\mathbf{V}_w X$, for every space X , consisting of so-called *point-continuous valuations*, and due to Heckmann [Hec96]. It does not really matter to us what they are exactly. We will use the following facts:

- \mathbf{V}_p defines a monad on \mathbf{Top}_0 , with similarly defined units and multiplication, and its algebras are exactly the weakly locally convex, sober topological cones [GJ19, Theorem 5.11]. This would solve the question of finding the \mathbf{V}_w -algebras if all continuous valuations were point-continuous, but that is not the case, even for continuous valuations on dcpos [GLJ21].

- \mathbf{V}_w restricts to a monad on the subcategory \mathbf{Cont} of continuous dcpos. ($\mathbf{V}_w X$ coincides with $\mathbf{V}X$ with the Scott topology of the stochastic ordering, for every continuous dcpo X , by a result of Kirch [Kir93], and Jones showed that \mathbf{V} is a monad on \mathbf{Cont} [Jon90]—although, technically, Kirch was the first one to deal with continuous valuations with values in $\overline{\mathbb{R}}_+$.)
- On the subcategory of continuous dcpos, the \mathbf{V}_p and \mathbf{V}_w monads coincide [Hec96, Theorem 6.9].

This gives us the following result for free. We remember that every continuous d-cone is locally convex, sober, and topological.

Proposition 3.2. *For every continuous d-cone \mathfrak{D} , there is a unique \mathbf{V}_w -algebra $\alpha: \mathbf{V}_w \mathfrak{D} \rightarrow \mathfrak{D}$ that induces the given cone structure on \mathfrak{D} .*

The final ingredient we will need is Lemma 3.3 below. Given any non-empty compact saturated subset Q of a space X , we let $\text{Min } Q$ be its set of minimal elements—as usual, with respect to the specialization preordering \leq of X . It is well-known that $Q = \uparrow \text{Min } Q$, namely that every element of Q is above some minimal element. The quick proof proceeds by applying Zorn’s Lemma to Q with the opposite ordering \geq , noting that this is an inductive poset: for every chain $(x_i)_{i \in I}$ of elements of Q , $(\downarrow x_i)_{i \in I}$ is a chain of closed subsets (since $\downarrow x_i$ is the closure of $\{x_i\}$) that intersect Q , and since Q is compact, their intersection must also intersect Q , say at x ; then $x \leq x_i$ for every $i \in I$.

Lemma 3.3. *Let \mathfrak{C} be a linearly separated T_0 semitopological cone and Q be a non-empty compact saturated subset of \mathfrak{C} . If the map $\varphi: \Lambda \mapsto \min_{x \in Q} \Lambda(x)$ is linear from \mathfrak{C}^* to $\overline{\mathbb{R}}_+$, then $Q = \uparrow x$ for some $x \in \mathfrak{C}$.*

Proof. Let us assume that Q cannot be written as $\uparrow x$ for any $x \in \mathfrak{C}$, or equivalently, that $\text{Min } Q$ is not reduced to just one point. (Also, $\text{Min } Q$ is non-empty, since Q is non-empty.) For every $x \in \text{Min } Q$, $\text{Min } Q$ contains a distinct, hence incomparable, point \bar{x} .

Since \mathfrak{C} is linearly separated, we can find a $\Lambda_x \in \mathfrak{C}^*$ such that $\Lambda_x(x) > 1$ and $\Lambda_x(\bar{x}) \leq 1$, for each $x \in \text{Min } Q$. Let $H_x \stackrel{\text{def}}{=} \Lambda_x^{-1}([1, \infty])$: when x varies over $\text{Min } Q$, those open sets cover $\text{Min } Q$, hence also its upward closure, which is Q . Since Q is compact, there is a finite subset E of $\text{Min } Q$ such that $Q \subseteq \bigcup_{x \in E} H_x$.

Let $\Lambda \stackrel{\text{def}}{=} \sum_{x \in E} \Lambda_x$, another element of \mathfrak{C}^* . Since φ is linear by assumption, $\varphi(\Lambda) = \sum_{x \in E} \varphi(\Lambda_x)$.

Relying on the compactness of Q , $\varphi(\Lambda) = \min_{z \in Q} \Lambda(z)$ is equal to $\Lambda(x_0)$ for some $x_0 \in Q$. Since $Q \subseteq \bigcup_{x \in E} H_x$, there is an $x \in E$ such that $x_0 \in H_x$, namely such that $\Lambda_x(x_0) > 1$. By the choice of Λ_x , $\Lambda_x(\bar{x}) \leq 1$, and therefore $\varphi(\Lambda_x) = \min_{z \in Q} \Lambda_x(z) \leq 1 < \Lambda_x(x_0)$. For all the other points y of E , $\varphi(\Lambda_y) = \min_{z \in Q} \Lambda_y(z) \leq \Lambda_y(x_0)$, so, summing up, we obtain that $\varphi(\Lambda) = \varphi(\Lambda_x) + \sum_{y \in E \setminus \{x\}} \varphi(\Lambda_y) < \Lambda_x(x_0) + \sum_{y \in E \setminus \{x\}} \Lambda_y(x_0) = \Lambda(x_0) = \varphi(\Lambda)$, a contradiction. \square

Albeit natural, Lemma 3.3 is surprising, for the following reason. We recall from Example 2.2 (5) that \mathfrak{C}^* is given the weak* upper topology, a.k.a. the weak*-Scott topology. If one knows that $Q = \uparrow x$, then the map φ coincides with $\text{ev}_x: \Lambda \mapsto \Lambda(x)$, and must therefore be continuous. However, we are *not* assuming that φ is continuous, only linear, in Lemma 3.3. For general non-empty convex compact saturated sets Q , we have no reason to believe that $\varphi: \Lambda \mapsto \min_{x \in Q} \Lambda(x)$ should be lower semicontinuous from \mathfrak{C}^* , with the weak*-Scott topology, to $\overline{\mathbb{R}}_+$. But Lemma 3.3 implies that it is, provided it is linear.

Theorem 3.4. *Let \mathfrak{C} be a locally convex, locally convex-compact, sober topological cone \mathfrak{C} . Every continuous valuation ν on \mathfrak{C} has a unique barycenter $\beta(\nu)$, and the barycenter map β is continuous; hence β is the structure map of a \mathbf{V}_w -algebra on \mathfrak{C} , and the unique one that induces the given cone structure on \mathfrak{C} .*

Proof. Under the given assumptions on \mathfrak{C} , $\mathcal{S}\mathfrak{C}$ is a continuous d-cone, so Proposition 3.2 applies: there is a unique \mathbf{V}_w -algebra structure $\alpha: \mathbf{V}_w(\mathcal{S}\mathfrak{C}) \rightarrow \mathcal{S}\mathfrak{C}$ that induces the given cone structure on $\mathcal{S}\mathfrak{C}$.

Let us consider any continuous valuation ν on \mathfrak{C} . Since the unit $\eta_{\mathfrak{C}}^{\mathcal{S}}$ of the \mathcal{S} monad is, in particular, continuous, we can form the image valuation $\eta_{\mathfrak{C}}^{\mathcal{S}}[\nu]$ on $\mathcal{S}\mathfrak{C}$. We let $Q \stackrel{\text{def}}{=} \alpha(\eta_{\mathfrak{C}}^{\mathcal{S}}[\nu])$.

For every $\Lambda \in \mathfrak{C}^*$, by Lemma 3.1, $\bar{\Lambda}$ is in $(\mathcal{S}\mathfrak{C})^*$. Since α is a barycenter map, we must have $\bar{\Lambda}(Q) = \int_{Q' \in \mathcal{S}\mathfrak{C}} \bar{\Lambda}(Q') d\eta_{\mathfrak{C}}^{\mathcal{S}}[\nu]$. By the change of variable formula, the latter is equal to $\int_{x \in \mathfrak{C}} \bar{\Lambda}(\eta_{\mathfrak{C}}^{\mathcal{S}}(x)) d\nu$, which is equal to $\int_{x \in \mathfrak{C}} \Lambda(x) d\nu$ by Lemma 3.1. Also, $\bar{\Lambda}(Q) = \min_{x \in Q} \Lambda(x) = \varphi(\Lambda)$, taking the notation φ from Lemma 3.3. Since Λ is arbitrary in \mathfrak{C}^* , this shows that φ is equal to the map $\Lambda \in \mathfrak{C}^* \mapsto \int_{x \in \mathfrak{C}} \Lambda(x) d\nu$, which is linear. Hence, by Lemma 3.3 (which applies since \mathfrak{C} is locally convex T_0 hence linearly separated; it is T_0 since sober), $Q = \uparrow x_\nu$ for some $x_\nu \in \mathfrak{C}$.

For every $\Lambda \in \mathfrak{C}^*$, the equation $\bar{\Lambda}(Q) = \int_{x \in \mathfrak{C}} \Lambda(x) d\nu$ then simplifies to $\Lambda(x_\nu) = \int_{x \in \mathfrak{C}} \Lambda(x) d\nu$, showing that x_ν is a barycenter of ν . Since \mathfrak{C} is linearly separated and T_0 , it is unique [CEK06, GJ19].

Let us define β by $\beta(\nu) \stackrel{\text{def}}{=} x_\nu$, for every $\nu \in \mathbf{V}_w\mathfrak{C}$. We claim that β is continuous. For every open subset U of \mathfrak{C} , for every $\nu \in \mathbf{V}_w\mathfrak{C}$, $\beta(\nu) \in U$ if and only if $\alpha(\eta_{\mathfrak{C}}^{\mathcal{S}}[\nu]) \in \square U$. In other words, $\beta^{-1}(U) = (\alpha \circ \mathbf{V}_w(\eta_{\mathfrak{C}}^{\mathcal{S}}))^{-1}(\square U)$, which is open since α and $\eta_{\mathfrak{C}}^{\mathcal{S}}$ are continuous, and since \mathbf{V}_w is a functor.

We conclude by Proposition 2.5. □

Example 3.5. We compare the strength of Theorem 3.4 with what we already know about barycenters in special cases (see Example 2.4).

- (1) For every core-compact space X , $\mathcal{L}X$ is a continuous d-cone, hence a sober, locally convex and locally convex-compact topological cone. Therefore Theorem 3.4 applies; but Example 2.4(1) suffices in this case.
- (2) Let L be a sup-semilattice with the Scott topology and the cone structure defined in Example 2.1 (2). If L is locally compact and sober in the Scott topology, then L is a sober, locally convex, locally convex-compact topological cone. Hence the barycenter map on L exists, and the induced cone structure on L is the original one on L . In this case, L is actually a complete lattice, as sobriety of L guarantees that directed suprema on L exist. One can describe the barycenter map concretely, see Example 2.4 (2).
- (3) Similarly with sup-semilattices with their upper topology (Example 2.4 (3)).
- (4) A dual cone \mathfrak{C}^* is always topological, sober and locally convex (see Example 2.2 (5)). Theorem 3.4 informs us that it suffices that \mathfrak{C}^* be locally convex-compact to have a \mathbf{V}_w -algebra structure inducing the cone structure on \mathfrak{C}^* , and therefore for continuous valuations ν on \mathfrak{C}^* to have barycenters $\beta(\nu)$. This barycenter is the map $x \mapsto \int_{\Lambda \in \mathfrak{C}^*} \Lambda(x) d\nu$, see Example 2.4 (4). We will see in Example 4.2 that not all continuous valuations ν on a dual cone \mathfrak{C}^* have barycenters, hence not all \mathfrak{C}^* are locally convex-compact.

- (5) For every topological space X , $\mathbf{V}_w X$ is topological, sober, and locally convex, see Example 2.2 (4). Hence Theorem 3.4 also gives us a \mathbf{V}_w -algebra structure and barycenters if $\mathbf{V}_w X$ is locally convex-compact. We will see below that this holds true when X is stably locally compact, notably (Proposition 4.1). But the local convex-compactness assumption is unneeded, as we have seen in Example 2.4 (5); examining this gap is the purpose of Section 4.

4. ARE THE ASSUMPTIONS NECESSARY?

One would like to know whether the assumptions of Theorem 3.4 are needed. If a semi-topological cone \mathfrak{C} has a \mathbf{V}_w -algebra structure that induces the given cone structure, then \mathfrak{C} has to be sober, topological, and weakly locally convex [GJ19, Proposition 4.9]. We do not know whether it has to be locally convex. But does it have to be locally convex-compact? Example 3.5 (5) suggests that the answer is no. However, we will give an example illustrating that Theorem 3.4 may fail if \mathfrak{C} is not locally convex-compact.

But first let us look at the special case of \mathbf{V}_w -algebras in the category of stably compact spaces, as Cohen, Escardó and Keimel did [CEK06]. In that setting, we will show that it is indeed required that \mathfrak{C} be locally convex-compact. We write **SComp** for the category of stably compact spaces and continuous maps. A space is *stably locally compact* if and only if it is sober, locally compact, and the intersection of any two compact saturated subsets is compact; it is *stably compact* if and only if it is also compact. On a stably compact space Y , the complements of compact saturated subsets form a topology called the *cocompact topology*. The set Y with the cocompact topology is then also stably compact [Gou13, Section 9.1.2].

In the proof of the following, we will use some results due to Gordon Plotkin [Plo06]. Plotkin uses the weak*-Scott topology on $\mathbf{V}X$, equating it with \mathfrak{C}^* where $\mathfrak{C} \stackrel{\text{def}}{=} \mathcal{L}X$; that is the same thing as the weak topology on $\mathbf{V}X$, by [AMJK04, Proposition 34]. Let us write $[Q \geq r]$ for the collection of continuous valuations ν such that $\nu(U) \geq r$ for every open neighborhood U of Q . Then Corollary 1 of [Plo06], together with subsequent comments, states that, for every stably locally compact space X , $\mathbf{V}_w X$ is stably compact, and a subbase of closed subsets of the cocompact topology is given by the sets $[Q \geq r]$, where Q ranges over the compact saturated subsets of X and $r \in \mathbb{R}_+$. In particular, all such sets $[Q \geq r]$ are compact in $\mathbf{V}_w X$. They are clearly saturated.

A *linear retract* of a semitopological cone is a retract whose retraction (not the section) is linear [Hec96, Proposition 6.6].

Proposition 4.1. *For every stably locally compact space X , $\mathbf{V}_w X$ is a locally convex-compact topological cone. Each of those properties is preserved by linear retracts. Hence, for every \mathbf{V}_w -algebra $\alpha: \mathbf{V}_w X \rightarrow X$ in **SComp**, X is a locally convex-compact topological cone in the induced cone structure.*

X is also sober, since stably locally compact. As a topological cone, it will even be stably compact. Hence only local convexity is missing.

Proof. We already know that $\mathbf{V}_w X$ is a topological cone. We use the following fact about locally compact spaces such as X : for every compact saturated subset Q_0 of X and every open subset U of X that contains Q_0 , there is a compact saturated subset Q of X such that

$Q_0 \subseteq \text{int}(Q) \subseteq Q \subseteq U$ [Gou13, Proposition 4.8.14]. As a consequence, the family \mathcal{F}_U of interiors $\text{int}(Q)$, where Q ranges over the compact saturated subsets of U , is directed: it is nonempty by the previous observation with $Q_0 \stackrel{\text{def}}{=} \emptyset$, and for any two elements $\text{int}(Q_1)$ and $\text{int}(Q_2)$ of \mathcal{F}_U , the same observation with $Q_0 \stackrel{\text{def}}{=} Q_1 \cup Q_2$ gives a larger element $\text{int}(Q)$ of \mathcal{F}_U . Additionally, U is the union of that family, since every point $x \in U$ is in some element of \mathcal{F}_U , by definition of local compactness.

Given $\nu \in \mathbf{V}_w X$ and an open neighborhood \mathcal{U} of ν in the weak topology, \mathcal{U} contains an open neighborhood of ν of the form $\bigcap_{i=1}^n [U_i > r_i]$, where each U_i is open in X and $r_i > 0$. Since X is locally compact, each U_i is the union of the directed family \mathcal{F}_{U_i} . Since ν is Scott-continuous, $\nu(\text{int}(Q_i)) > r_i$ for some compact saturated subset Q_i of U_i . Let $\epsilon > 0$ be such that $\nu(\text{int}(Q_i)) > r_i + \epsilon$ for every i . Then ν is in $\bigcap_{i=1}^n [\text{int}(Q_i) > r_i + \epsilon]$. The latter is open and included in $\bigcap_{i=1}^n [Q_i \geq r_i + \epsilon]$, which is compact saturated, being a finite intersection of compact saturated sets in a stably compact space. Then, $\bigcap_{i=1}^n [Q_i \geq r_i + \epsilon]$ is included in $\bigcap_{i=1}^n [U_i > r_i] \subseteq \mathcal{U}$. Finally, $\bigcap_{i=1}^n [Q_i \geq r_i + \epsilon]$ is convex, as an intersection of convex sets; any set of the form $[Q \geq r]$ is convex, since for all $\mu, \nu \in [Q \geq r]$, for every $a \in [0, 1]$, for every open neighborhood U of Q , $(a \cdot \mu + (1-a) \cdot \nu)(U) = a\mu(U) + (1-a)\nu(U) \geq ar + (1-a)r = r$. Therefore $\mathbf{V}_w X$ is locally convex-compact.

Let us consider a linear retraction $r: \mathfrak{C} \rightarrow \mathfrak{D}$, with associated section $s: \mathfrak{D} \rightarrow \mathfrak{C}$. If \mathfrak{C} is locally convex-compact, then for every $y \in \mathfrak{D}$ and every open neighborhood V of y , $s(y)$ has a convex compact neighborhood Q included in $r^{-1}(V)$. Let U be the interior of Q . The image of Q under r is convex, compact, contains the open neighborhood $s^{-1}(U)$ of y , and is included in V . Hence \mathfrak{D} is locally convex-compact. (This argument is the same as [Hec96, Proposition 6.6].) Also, if \mathfrak{C} is topological, then addition and scalar multiplication on \mathfrak{D} are such that $x + y = r(s(x) + s(y))$ and $a \cdot x = r(a \cdot s(x))$, hence are jointly continuous, so \mathfrak{D} is topological.

The final part follows from the fact that any \mathbf{V}_w -algebra is a linear retraction, with the unit η_X as section. \square

We now give an example to show that Theorem 3.4 may fail if local convex-compactness of \mathfrak{C} is not assumed, even when \mathfrak{C} is a locally convex, sober topological cone.

Example 4.2. Let \mathbb{R} be the set of reals with the usual metric topology and $\mathfrak{C} = \mathbf{V}_p \mathbb{R}$, the set of point-continuous valuations on \mathbb{R} . Similar to $\mathbf{V}_w \mathbb{R}$, the space $\mathbf{V}_p \mathbb{R}$, with the relative weak topology from $\mathbf{V}_w \mathbb{R}$, also is a locally convex sober topological cone [Hec96].

The cone $\mathbf{V}_p \mathbb{R}$ is actually strictly contained in $\mathbf{V}_w \mathbb{R}$. The Lebesgue valuation λ on \mathbb{R} , obtained by restricting the Lebesgue measure on opens of \mathbb{R} is a continuous valuation that is not in $\mathbf{V}_p \mathbb{R}$; see for example [Hec96, Section 4.1]. The map $\eta_p: \mathbb{R} \rightarrow \mathbf{V}_p \mathbb{R}: x \mapsto \delta_x$ is a well-defined continuous map, as δ_x is point-continuous for each $x \in \mathbb{R}$. Hence $\eta_p[\lambda]$, the image valuation defined as $\mathbf{V}_w(\eta_p)(\lambda)$, is a continuous valuation in $\mathbf{V}_w \mathbf{V}_p \mathbb{R}$. However, we claim that $\eta_p[\lambda]$ does not have a barycenter in $\mathbf{V}_p \mathbb{R}$.

Assume that $\eta_p[\lambda]$ does have a barycenter in $\mathbf{V}_p \mathbb{R}$ and we denote it by ν . Then by definition of barycenters, $\Lambda(\nu) = \int_{\mu \in \mathbf{V}_p \mathbb{R}} \Lambda(\mu) d\eta_p[\lambda]$ for each continuous linear map $\Lambda: \mathbf{V}_p \mathbb{R} \rightarrow \overline{\mathbb{R}}_+$. In particular, each open subset U in \mathbb{R} determines such a continuous linear

map $\Lambda_U: \mu \mapsto \mu(U)$. So for each open U in \mathbb{R} ,

$$\begin{aligned}
 \nu(U) &= \Lambda_U(\nu) \\
 &= \int_{\mu \in \mathbf{V}_p\mathbb{R}} \Lambda_U(\mu) \, d\eta_p[\lambda] \\
 &= \int_{\mu \in \mathbf{V}_p\mathbb{R}} \mu(U) \, d\eta_p[\lambda] \\
 &= \int_{x \in \mathbb{R}} \delta_x(U) \, d\lambda && \text{(by the change of variable formula)} \\
 &= \int_{x \in \mathbb{R}} \chi_U(x) \, d\lambda \\
 &= \lambda(U).
 \end{aligned}$$

So $\nu = \lambda$. But this would contradict to the fact that λ is not point-continuous, and hence $\eta_p[\lambda]$ does not have any barycenters in $\mathbf{V}_p\mathbb{R}$. As a byproduct, we know that $\mathbf{V}_p\mathbb{R}$ cannot be locally convex-compact by Theorem 3.4.

We also realize that $\mathbf{V}_p\mathbb{R}$ is isomorphic to the dual cone $(\mathcal{L}\mathbb{R})^*$, where $\mathcal{L}\mathbb{R}$ consists of all lower semi-continuous maps on \mathbb{R} , and both $\mathcal{L}\mathbb{R}$ and $(\mathcal{L}\mathbb{R})^*$ are endowed with the topology of pointwise convergence [Hec96, Theorem 8.2]. Hence, continuous valuations on a dual cone may fail to have barycenters.

5. CONCLUSION

Theorem 3.4 is a pretty general barycenter existence theorem, which also serves to show that a large collection of topological cones has a (unique) \mathbf{V}_w -algebra structure inducing its cone structure. Through the examples we have given, we hope to have demonstrated that this theorem encompasses most of the cases where we already knew that barycenters existed, at the very least.

We sum up the main remaining questions as follows.

- Which dual cones \mathfrak{C}^* are \mathbf{V}_w -algebras? We mentioned this in Example 2.4 (4), and we said that those that are locally convex-compact fit the bill in Example 3.5, and not all dual cones are locally convex-compact in Example 4.2.
- Given a \mathbf{V}_w -algebra $\alpha: \mathbf{V}_w X \rightarrow X$, we know that X has an induced cone structure that is weakly locally convex. Need it be locally convex? Both local convexity and local convex-compactness are stronger properties than weak local convexity, and none is known to imply the other. One should probably focus on the case of stably locally compact X , where we know at least that local convex-compactness is needed, by Proposition 4.1.
- In the converse direction, what weaker conditions than local convexity and local convex-compactness would be enough to deduce the same conclusions as those of Theorem 3.4? Those conditions need to be at least as strong as weak local convexity.

ACKNOWLEDGMENTS

Both authors would like to thank the anonymous referees for their careful reading of our manuscript and pointing out several areas where arguments were not clear. The final version

has benefitted a lot from their comments. Xiaodong Jia acknowledges support from NSFC (No. 12371457, No. 12231007).

REFERENCES

- [AMJK04] Mauricio Alvarez-Manilla, Achim Jung, and Klaus Keimel. The probabilistic powerdomain for stably compact spaces. *Theoretical Computer Science*, 328(3):221–244, 2004.
- [CEK06] Ben Cohen, Martín Hötzel Escardó, and Klaus Keimel. The extended probabilistic powerdomain monad over stably compact spaces. In *Proceedings of the 3rd International Conference on Theory and Applications of Models of Computation (TAMC'06)*, pages 566–575, Beijing, China, 2006. Extended abstract.
- [Cho60] Gustave Choquet. Le théorème de représentation intégrale dans les ensembles convexes compacts. *Annales de l'institut Fourier*, 10:333–344, 1960.
- [Cho69] Gustave Choquet. *Lectures on Analysis*, volume II. Representation Theory. W. A. Benjamin, Inc., 1969.
- [CKL22] Yu Chen, Hui Kou, and Zhenchao Lyu. Two topologies on the lattice of Scott closed subsets. *Topology and its Applications*, 306:107918, 2022.
- [dBGJL19] Matthew de Brecht, Jean Goubault-Larrecq, Xiaodong Jia, and Zhenchao Lyu. Domain-complete and LCS-complete spaces. *Electronic Notes in Theoretical Computer Science*, 345:3–35, 2019. Proc. 8th International Symposium on Domain Theory (ISDT'19).
- [Edw63] David Albert Edwards. On the representation of certain functionals by measures on the Choquet boundary. *Annales de l'institut Fourier*, 13(1):111–121, 1963.
- [GHK⁺03] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana Stewart Scott. *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2003.
- [GJ19] Jean Goubault-Larrecq and Xiaodong Jia. Algebras of the extended probabilistic powerdomain monad. *Electronic Notes in Theoretical Computer Science*, 345:37–61, 2019. Proc. 8th International Symposium on Domain Theory (ISDT'19).
- [GLJ21] Jean Goubault-Larrecq and Xiaodong Jia. Separating minimal valuations, point-continuous valuations, and continuous valuations. *Mathematical Structures in Computer Science*, 31(6):614–632, 2021. doi:10.1017/S0960129521000384.
- [Gou13] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
- [Hec96] Reinhold Heckmann. Spaces of valuations. In S. Andima, R. C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, and P. Misra, editors, *Papers on General Topology and its Applications*, volume 806, pages 174–200. Annals of the New York Academy of Science, Dec 1996.
- [JKL⁺22] Xiaodong Jia, Andre Kornell, Bert Lindenhovius, Michael Mislove, and Vladimir Zamdzhev. Semantics for variational quantum programming. *Proc. ACM Program. Lang.*, 6 (POPL), Jan 2022. doi:10.1145/3498687.
- [Jon90] Claire Jones. *Probabilistic Non-Determinism*. PhD thesis, University of Edinburgh, 1990. Technical Report ECS-LFCS-90-105.
- [Kei08a] Klaus Keimel. The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras. *Topology and its Applications*, 156(2):227–239, 2008.
- [Kei08b] Klaus Keimel. Topological cones: Functional analysis in a T_0 -setting. *Semigroup Forum*, 77(1):109–142, 2008.
- [Kir93] Olaf Kirch. *Bereiche und Bewertungen*. Master's thesis, Technische Hochschule Darmstadt, 1993. 77pp.
- [Plo06] Gordon Plotkin. A domain-theoretic Banach-Alaoglu theorem. *Mathematical Structures in Computer Science*, 16:299–311, 2006.
- [Rob78] James W. Roberts. The embedding of compact convex sets in locally convex spaces. *Canadian Journal of Mathematics*, XXX(3):449–454, 1978.
- [Sch93] Andrea Schalk. *Algebras for Generalized Power Constructions*. PhD thesis, Technische Hochschule Darmstadt, 1993. Available from <http://www.cs.man.ac.uk/~schalk/publ/diss.ps.gz>.
- [Świ74] Tadeusz Świrszcz. Monadic functors and convexity. *Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques*, 22:39–42, 1974.

- [Tix95] Regina Tix. *Stetige Bewertungen auf topologischen Räumen*. Diplomarbeit, TH Darmstadt, June 1995.