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ON THE HOME-SPACE PROBLEM FOR PETRI NETS AND ITS ACKERMANNIAN COMPLEXITY *

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Abstract. A set of configurations H is a home-space for a set of configurations X of a Petri net if every configuration reachable from (any configuration in) X can reach (some configuration in) H. The semilinear home-space problem for Petri nets asks, given a Petri net and semilinear sets of configurations X, H, if H is a home-space for X. In 1989, David de Frutos Escrig and Colette Johnen proved that the problem is decidable when X is a singleton and H is a finite union of linear sets with the same periods. In this paper, we show that the general (semilinear) problem is decidable. This result is obtained by proving a duality between the reachability problem and the non-home-space problem. In particular, we prove that for any Petri net and any semilinear set of configurations H we can effectively compute a semilinear set C of configurations, called a non-reachability core for H, such that for every set X the set H is not a home-space for X if, and only if, C is reachable from X. We show that the established relation to the reachability problem yields the Ackermann-completeness of the (semilinear) home-space problem. For this we also show that, given a Petri net with an initial marking, the set of minimal reachable markings can be constructed in Ackermannian time.

1. Introduction

On an abstract level, various practical systems and theoretical models can be viewed as instances of transition systems $(S, \to)$ where S is a (possibly infinite) set of configurations and $\to \subseteq S \times S$ is a relation capturing when one configuration can change into another by an atomic step; the reachability relation $\xrightarrow{*}$ is then the reflexive and transitive closure of $\to$.

Given a system $(S, \to)$ and two sets $X, H \subseteq S$, we say that H is a *home-space for* X if from every configuration reachable from (any configuration in) X we can reach (some configuration in) H. The *home-space problem* asks, given $(S, \to)$, X, H, whether H is a home-space for X. For instance, the home-space problem can ask whether the system can always return to an initial configuration. This paper focuses on the *semilinear home-space problem for Petri nets*, in which the respective sets X, H are semilinear sets (consisting of nonnegative integer vectors of a given dimension).

Key words and phrases: Petri net, home-space property, semilinear set.
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We recall that Petri nets provide a popular formal method for modelling and analyzing parallel processes. The standard model is not Turing-complete, and many analyzed properties are decidable; we can refer to [EN94] as to one of the first survey papers on this issue.

A central algorithmic problem for Petri nets is reachability: given a Petri net A and two configurations $\mathbf{x}$ and $\mathbf{y}$, decide whether there exists an execution of A from $\mathbf{x}$ to $\mathbf{y}$. In fact, many important computational problems in logic and complexity reduce or are even equivalent to this problem (we can refer, e.g., to [Sch16b, Hac75] to exemplify this). It was nontrivial to show that the reachability problem is decidable [May84], and recently the complexity of this problem was proved to be extremely high, namely Ackermann-complete (see [LS19] for the upper-bound and [CLL+21, Ler21, CO21] for the lower-bound).

The reachability problem for Petri nets can be generalized to semilinear sets, a class of geometrical sets that coincides with the sets definable in Presburger arithmetic [GS66]. The semilinear reachability problem for Petri nets asks, given a Petri net A and (presentations of) semilinear sets of configurations $\mathbf{X}$,$\mathbf{Y}$, if there exists an execution from a configuration in $\mathbf{X}$ to a configuration in $\mathbf{Y}$. Denoting by $\textsc{post}^*_A(\mathbf{X})$ the set of configurations reachable from $\mathbf{X}$ and by $\textsc{pre}^*_A(\mathbf{Y})$ the set of configurations that can reach a configuration in $\mathbf{Y}$, the semilinear reachability problem thus asks, in fact, if the intersection $\textsc{post}^*_A(\mathbf{X}) \cap \textsc{pre}^*_A(\mathbf{Y})$ is nonempty (which is equivalent to the non-emptiness of $X \cap \textsc{pre}^*_A(\mathbf{Y})$ or $\textsc{post}^*_A(\mathbf{X}) \cap \mathbf{Y}$). This problem can be easily reduced to the classical reachability problem for Petri nets (where $\mathbf{X}$ and $\mathbf{Y}$ are singletons).

The semilinear home-space problem is a problem that seems to be similar to the semilinear reachability problem at first sight. This problem asks, given a Petri net A, and two semilinear sets $\mathbf{X}, \mathbf{H}$, if every configuration reachable from $\mathbf{X}$ can reach $\mathbf{H}$, hence if $\textsc{post}^*_A(\mathbf{X}) \subseteq \textsc{pre}^*_A(\mathbf{H})$. In 1989, David de Frutos Escrig and Colette Johnen [dFEJ89] proved that the semilinear home-space problem is decidable for instances where $\mathbf{X}$ is a singleton set and $\mathbf{H}$ is a finite union of linear sets using the same periods; they left the general case open. In fact, the general problem seems close to the decidability/undecidability border, since the reachability set inclusion problem, which can be viewed as asking if $\textsc{post}^*_A(\mathbf{x}) \subseteq \textsc{pre}^*_B(\mathbf{y})$ where A, B are Petri nets of the same dimension (i.e., with the same sets of places), is undecidable [Bak73, Hac76], even when the dimension of A, B is fixed to a small value [Jan95].

Our contribution. In this paper, we show that the general semilinear home-space problem is decidable. This result is obtained by proving a duality between the reachability problem and the non-home-space problem. A crucial point consists in proving that for any Petri net A and for any linear set of configurations $\mathbf{L}$, we can effectively compute a semilinear "non-reachability core" $\mathbf{C}$ such that for every set $\mathbf{X}$ the set $\mathbf{L}$ is not a home-space for $\mathbf{X}$ if, and only if, $\mathbf{C}$ is reachable from $\mathbf{X}$. By a technical analysis using the known complexity results for reachability we show that the (semilinear) home-space problem is Ackermann-complete. As an ingredient, we also show that, given a Petri net with an initial marking, the set of minimal reachable markings can be constructed in Ackermannian time. Moreover, by using the results on inductive semilinear invariants [Ler10] we also show that a semilinear, and moreover inductive, non-reachability core can be computed for any semilinear (not only linear) set $\mathbf{S}$. This yields a modification of the decidability proof but without complexity bounds. We remark that only recently it has turned out that Ackermannian upper bounds could be derived in this way as well, due to the enhancement [Ler24] of [Ler10]. Finally, we also discuss the form of positive and negative witnesses of the home-space property.

Organization of the paper. Section 2 describes an idea of our approach in the context of general transition systems. Section 3 states our main results for (transition systems generated by) Petri nets, after providing necessary preliminaries. Section 4 shows the hardness results, yielding the complexity lower bounds, and Sections 5 and 6 give a decidability proof. Sections 7 and 8 contain the complexity analysis, yielding the Ackermannian upper bounds. Section 9 provides a proof that any semilinear set admits an effectively computable inductive semilinear non-reachability core. In Section 10 we discuss the question of positive and negative witnesses of the home-space property. We conclude by a few remarks in Section 11.

2. A General Approach to the Home-Space Problem

In this section we provide an overview of the way the home-space problem can be solved via the so-called *non-reachability cores.* Though we apply this approach to Petri nets, we start with presenting it for a general transition system given as a pair $(S, \rightarrow)$ where S is a (possibly infinite) set of *states* (or *configurations*) and $\rightarrow \subseteq S \times S$ is a *transition relation.* The *reachability relation* $\xrightarrow{*} \subseteq S \times S$ is then the reflexive and transitive closure of $\rightarrow$. For sets $X \subseteq S$, we introduce the following notions and notation:

- by $\overline{X}$ we denote the complement of X (hence $S \smallsetminus X$);
- $\textsc{Pre}^*(X) = \{s \in S \mid \exists s' \in X : s \xrightarrow{*} s'\}$;
- $\textsc{Post}^*(X) = \{s \in S \mid \exists s' \in X : s' \xrightarrow{*} s\}$;
- X is *inductive* (or *closed w.r.t.* $\rightarrow$) if $\textsc{Post}^*(X) = X$;
- $H \subseteq S$ is a *home-space* for X if $\textsc{Post}^*(X) \subseteq \textsc{Pre}^*(H)$.

We might implicitly use simple observations like the following ones:

- $X \subseteq \textsc{Pre}^*(X) = \textsc{Pre}^*(\textsc{Pre}^*(X))$,
- $\textsc{Pre}^*(X_1 \cup X_2) = \textsc{Pre}^*(X_1) \cup \textsc{Pre}^*(X_2)$,
- if both X_1 and X_2 are inductive, then $X_1 \cap X_2$ is inductive.

We also observe that H is a home-space for X iff it is a home-space for every $s \in X$ (implicitly viewed as the singleton $\{s\}$).

Figure 1 depicts the set S of states of a system, and a subset $H \subseteq S$ as a potential "home-space" in which we are interested. The set $\overline{\textsc{Pre}^*(H)}$ consists of the states from which H is not reachable, which entails that $\overline{\textsc{Pre}^*(H)}$ is inductive (i.e., $\textsc{Post}^*(\overline{\textsc{Pre}^*(H)}) = \overline{\textsc{Pre}^*(H)}$). We observe that

$$\textsc{Pre}^*(\overline{\textsc{Pre}^*(H)}) = \{s \in S \mid H \text{ is not a home-space for } s\};$$

hence H is a home-space for X iff $X \cap \textsc{Pre}^*(\overline{\textsc{Pre}^*(H)}) = \emptyset$.

We also note that H and $\overline{\textsc{Pre}^*(H)}$ are disjoint, but $\textsc{Pre}^*(\overline{\textsc{Pre}^*(H)})$ might intersect H. Since $S = \textsc{Pre}^*(H) \cup \overline{\textsc{Pre}^*(H)}$, we have

$$S = \textsc{Pre}^*(H \cup \overline{\textsc{Pre}^*(H)}).$$

Non-Reachability Cores. For some (infinite-state) systems it might be hard to construct (a description of) the set $\overline{\textsc{Pre}^*(H)}$ and/or to decide for $s \in S$ whether $s \in \textsc{Pre}^*(\overline{\textsc{Pre}^*(H)})$. Surely, for Turing-powerful systems such problems are not algorithmically solvable. But in the case of Petri nets it has turned out useful to introduce the notion of a *non-reachability core*, or just a *core*, *for* H: it is a set $C \subseteq S$ (also depicted in Figure 1) such that

$$C \subseteq \overline{\textsc{Pre}^*(H)} \subseteq \textsc{Pre}^*(C),$$

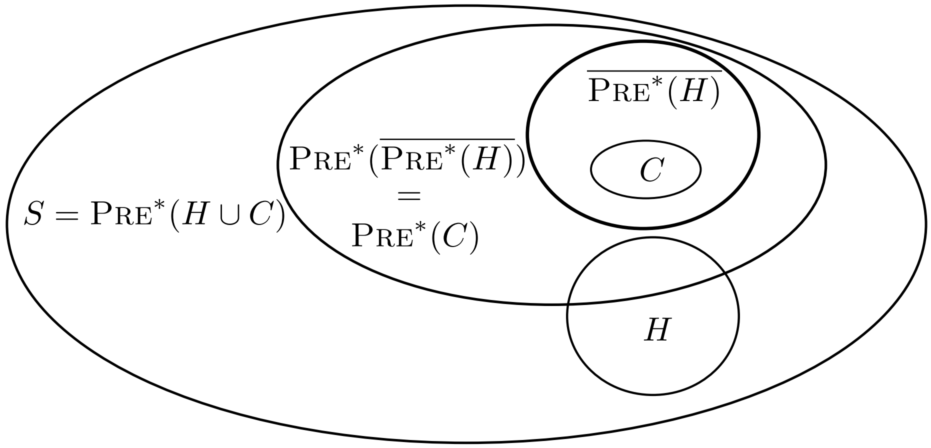

Figure 1: C is a non-reachability core for H.

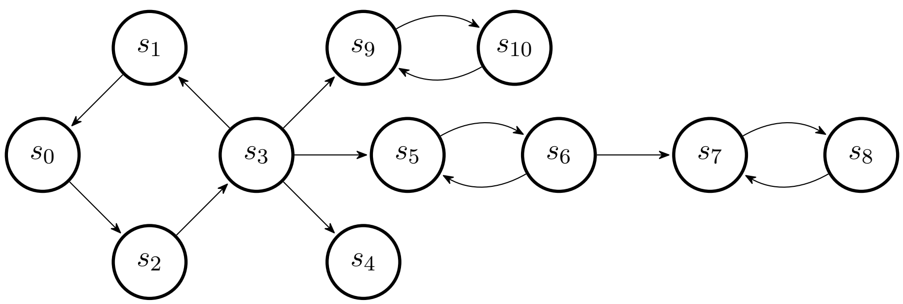

Figure 2: Let $H = \{s_3, s_4\}$. We have $\textsc{Pre}^*(H) = \{s_0, s_1, s_2, s_3, s_4\}$, and the bottom SCCs of $\overline{\textsc{Pre}^*(H)}$ are $\{s_7, s_8\}$ and $\{s_9, s_{10}\}$. Hence $C = \{s_7, s_9\}$ is one non-reachability core for H.

which entails that $S = \textsc{Pre}^*(H \cup C)$ (since $S = \textsc{Pre}^*(H) \cup \overline{\textsc{Pre}^*(H)}$); in other words, C is a subset of $\overline{\textsc{Pre}^*(H)}$ that is its home-space (i.e., C is a home-space for $\overline{\textsc{Pre}^*(H)}$). Hence if C is a core for H, then

$$\textsc{Pre}^*(\overline{\textsc{Pre}^*(H)}) = \textsc{Pre}^*(C);$$

therefore H is not a home-space for X iff C is reachable from some $s \in X$.

Of course, such a notion can help us only if there are cores C for H that are somehow simpler than $\overline{\textsc{Pre}^*(H)}$ itself. We have noted that $\overline{\textsc{Pre}^*(H)}$ is inductive; the cores $C \subseteq \overline{\textsc{Pre}^*(H)}$ do not need to be inductive, but inductive non-reachability cores will be of special interest for us.

Non-Reachability Cores in Finite-State Systems. It is straightforward to characterize the non-reachability cores in finite-state systems, which are exemplified by the system in Figure 2. We can partition $\overline{\textsc{Pre}^*(H)}$ into the strongly connected components (SCCs), and observe that a set C is a core if, and only if, it is included in $\overline{\textsc{Pre}^*(H)}$ and contains at least one state in each bottom SCC of $\overline{\textsc{Pre}^*(H)}$ (from which no other SCC is reachable).

Non-Reachability Cores for Unions of Sets. A "home-space" set $H \subseteq S$ can be sometimes naturally given as the union of smaller sets (in the case of Petri nets we are interested in semilinear home-space sets, which are defined as finite unions of linear sets). For instance, in Figure 2 we have $H = H_1 \cup H_2$ where $H_1 = \{s_3\}$ and $H_2 = \{s_4\}$. We can consider $C_1 = \{s_9, s_7, s_4\}$ as a core for H_1 and $C_2 = \{s_{10}, s_8\}$ as a core for H_2.

Having some cores $C_1, C_2, \ldots, C_m$ for sets $H_1, H_2, \ldots, H_m$, respectively, it is natural to ask if we can combine these cores to get a core C for the set $H = H_1 \cup H_2 \cdots \cup H_m$. Proposition 2.2 gives a simple answer if the cores C_i are inductive: then the intersection of the cores C_i is such a core C, which is, moreover, inductive. It will turn out that this fact is sufficient for developing a decidability proof for the semilinear home-space problem for Petri nets; in particular we will show that any semilinear set has an effectively constructible semilinear non-reachability core that is inductive. Nevertheless, for deriving the complexity upper bound we will use Proposition 2.1 that does not require the cores to be inductive.

Proposition 2.1. *Given $(S, \to)$ and $H \subseteq S$, let $H = H_1 \cup H_2 \cdots \cup H_m$ for some $m \geq 1$, and let $C_1, C_2, \ldots, C_m$ be non-reachability cores for $H_1, H_2, \ldots, H_m$, respectively. For each $X \subseteq S$ we have that H is not a home-space for X if, and only if, there is an execution*

$$s_0 \xrightarrow{*} s_1 \xrightarrow{*} s_2 \cdots \xrightarrow{*} s_m \tag{2.1}$$

where $s_0 \in X$, and $s_1 \in C_1$, $s_2 \in C_2$, $\ldots$, $s_m \in C_m$.

Proof. Given an execution (2.1), the facts that $s_i \in C_i$ and C_i is a non-reachability core for H_i (hence $C_i \subseteq \overline{\text{PRE}^*(H_i)}$) entail $s_i \stackrel{*}{\not\to} H_i$, for all $i \in \{1, 2, \ldots, m\}$. The facts $s_i \stackrel{*}{\not\to} H_i$ and $s_i \xrightarrow{*} s_m$ entail that $s_m \stackrel{*}{\not\to} H_i$ (for all $i \in \{1, 2, \ldots, m\}$). Hence $s_m \stackrel{*}{\not\to} H$ (where $H = H_1 \cup H_2 \cdots \cup H_m$), and the facts $s_0 \in X$ and $s_0 \xrightarrow{*} s_m \stackrel{*}{\not\to} H$ entail that H is not a home-space for X.

Conversely, we consider a set $X \subseteq S$ for which H is not a home-space. Hence there exist configurations s_0, s'_0 such that $s_0 \in X$ and $s_0 \xrightarrow{*} s'_0 \stackrel{*}{\not\to} H$. In particular $s'_0 \stackrel{*}{\not\to} H_1$, and thus H_1 is not a home-space for s'_0. Since C_1 is a non-reachability core for H_1, we have $s'_0 \xrightarrow{*} s_1$ for some $s_1 \in C_1$. Since $s'_0 \stackrel{*}{\not\to} H$ and $s'_0 \xrightarrow{*} s_1$, we have $s_1 \stackrel{*}{\not\to} H$, and in particular $s_1 \stackrel{*}{\not\to} H_2$. Since H_2 is not a home-space for s_1 and C_2 is a non-reachability core for H_2, we get $s_1 \xrightarrow{*} s_2$ for some $s_2 \in C_2$. Continuing in this way, we successively derive the existence of an execution (2.1). □

Proposition 2.2. *Given $(S, \to)$ and $H \subseteq S$, let $H = H_1 \cup H_2 \cdots \cup H_m$ for some $m \geq 1$, and let $C_1, C_2, \ldots, C_m$ be inductive non-reachability cores for $H_1, H_2, \ldots, H_m$, respectively. Then $C_1 \cap C_2 \cdots \cap C_m$ is an inductive non-reachability core for H.*

Proof. By induction on m. The case $m = 1$ is trivial, so we now suppose $m = 2$, hence $H = H_1 \cup H_2$. Since C_1 and C_2 are inductive, the intersection $C = C_1 \cap C_2$ is inductive as well. Since $C_1 \subseteq \overline{\text{PRE}^*(H_1)}$ and $C_2 \subseteq \overline{\text{PRE}^*(H_2)}$, we have

$$C_1 \cap C_2 \subseteq \overline{\text{PRE}^*(H_1)} \cap \overline{\text{PRE}^*(H_2)} = \overline{\text{PRE}^*(H_1) \cup \text{PRE}^*(H_2)} = \overline{\text{PRE}^*(H_1 \cup H_2)},$$

hence $C \subseteq \overline{\text{PRE}^*(H)}$.

Let us show that $\overline{\text{PRE}^*(H)} \subseteq \text{PRE}^*(C)$; we recall that $\overline{\text{PRE}^*(H)}$ is inductive. If $s \in \overline{\text{PRE}^*(H)}$, then

$$\text{POST}^*(\{s\}) \subseteq \overline{\text{PRE}^*(H)} = \overline{\text{PRE}^*(H_1 \cup H_2)} = \overline{\text{PRE}^*(H_1)} \cap \overline{\text{PRE}^*(H_2)}.$$

Since $s \in \overline{\text{PRE}^*(H_1)}$, there is $s_1 \in C_1$ such that $s \xrightarrow{*} s_1$. Using the fact that C_1 is inductive, we deduce that

$$\text{POST}^*(\{s_1\}) \subseteq C_1 \cap \overline{\text{PRE}^*(H_1)} \cap \overline{\text{PRE}^*(H_2)}.$$

Since $s_1 \in \overline{\text{PRE}^*(H_2)}$, there is $s_2 \in C_2$ such that $s_1 \xrightarrow{*} s_2$; we thus have $s_2 \in C_1 \cap C_2$. Since $s \xrightarrow{*} s_2$, we have shown that $\overline{\text{PRE}^*(H)} \subseteq \text{PRE}^*(C)$, which finishes the proof that $C = C_1 \cap C_2$ is a non-reachability core for $H = H_1 \cup H_2$.

The claim for $m \geq 3$ follows by the induction hypothesis, since $H_1 \cup H_2 \cdots \cup H_m$ can be viewed as $H_1 \cup H_2 \cdots \cup H_{m-2} \cup (H_{m-1} \cup H_m)$ where we consider $C_{m-1} \cap C_m$ as the inductive core for the set $(H_{m-1} \cup H_m)$. □

3. Basic Notions, and Main Results

In this section we state the main results, which deal with transitions systems $(S, \rightarrow)$ generated by (unmarked place/transition) Petri nets. We start with introducing basic notions and notation.

By $\mathbb{N}$ we denote the set $\{0, 1, 2, \dots\}$ of nonnegative integers. For $i, j \in \mathbb{N}$, by $[i, j]$ we denote the set $\{i, i{+}1, \dots, j\}$ (which is empty if $i > j$).

Notation for Vectors of Nonnegative Integers. For (a dimension) $d \in \mathbb{N}$, the elements of $\mathbb{N}^d$ are called (d-dimensional) *vectors*; they are denoted in bold face, and for $\mathbf{x} \in \mathbb{N}^d$ we write

$$\mathbf{x} = (\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(d))$$

so that we can refer to the vector components. We use the component-wise sum $\mathbf{x} + \mathbf{y}$ of vectors, and their component-wise order $\mathbf{x} \leq \mathbf{y}$. For $c \in \mathbb{N}$, we write

$$c \cdot \mathbf{x} = (c \cdot \mathbf{x}(1), c \cdot \mathbf{x}(2), \dots, c \cdot \mathbf{x}(d)).$$

By the *norm of* $\mathbf{x}$, denoted $\|\mathbf{x}\|$, we mean the sum of components, i.e., $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$.

By $\mathbf{0}$ we denote the zero vector whose dimension is always clear from its context. Occasionally we slightly abuse notation by presenting a vector as a mix of subvectors and integers; in particular, given $\mathbf{x} \in \mathbb{N}^d$ and $y_1, y_2, \dots, y_m \in \mathbb{N}$, we might write $(\mathbf{x}, y_1, y_2, \dots, y_m)$ to denote the $(d{+}m)$-dimensional vector $(\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(d), y_1, y_2, \dots, y_m)$.

Given a set $\mathbf{X} \subseteq \mathbb{N}^d$, by $\overline{\mathbf{X}}$ we denote its complement, i.e., $\overline{\mathbf{X}} = \mathbb{N}^d \setminus \mathbf{X}$.

Linear and Semilinear Sets of Vectors, and their Presentations. A *set* $\mathbf{L} \subseteq \mathbb{N}^d$ is *linear* if there are d-dimensional vectors $\mathbf{b}$, the *basis*, and $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k$, the *periods* (for $k \in \mathbb{N}$), such that

$$\mathbf{L} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \mathbf{b} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k \text{ for some } \mathbf{u} \in \mathbb{N}^k\}.$$

In this case, by a *presentation of* $\mathbf{L}$ we mean the tuple $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k)$.

A *set* $\mathbf{S} \subseteq \mathbb{N}^d$ is *semilinear* if it is a finite union of linear sets, i.e.

$$\mathbf{S} = \mathbf{L}_1 \cup \mathbf{L}_2 \cdots \cup \mathbf{L}_m$$

where $\mathbf{L}_i$ are linear sets (for all $i \in [1, m]$). In this case, by a *presentation of* $\mathbf{S}$ we mean the sequence of presentations of $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m$. When we say that a *semilinear set* $\mathbf{S}$ is given, we mean that we are given a presentation of $\mathbf{S}$; when we say that $\mathbf{S}$ is *effectively constructible* in some context, we mean that there is an algorithm computing its presentation (in the respective context).

Semilinear sets and Presburger arithmetic. We recall that a set $\mathbf{S} \subseteq \mathbb{N}^d$ is semilinear if, and only if, it is expressible in Presburger arithmetic [GS66]; the respective transformations between presentations and formulas are effective. Hence if $\mathbf{S} \subseteq \mathbb{N}^d$ is semilinear, then also its complement $\overline{\mathbf{S}}$ is semilinear, and $\overline{\mathbf{S}}$ is effectively constructible when (a presentation of) $\mathbf{S}$ is given.

Petri Nets. We use a concise definition of (unmarked place/transition) Petri nets. By a *d-dimensional Petri-net action* we mean a pair $a = (\mathbf{a}_-, \mathbf{a}_+) \in \mathbb{N}^d \times \mathbb{N}^d$. With $a = (\mathbf{a}_-, \mathbf{a}_+)$ we associate the binary relation $\xrightarrow{a}$ on the set $\mathbb{N}^d$ by putting $(\mathbf{x} + \mathbf{a}_-) \xrightarrow{a} (\mathbf{x} + \mathbf{a}_+)$ for all $\mathbf{x} \in \mathbb{N}^d$. The relations $\xrightarrow{a}$ are naturally extended to the relations $\xrightarrow{\sigma}$ for finite sequences σ of (d-dimensional Petri net) actions.

A *Petri net A of dimension d* (with d places in more traditional definitions) is a finite set of d-dimensional Petri-net actions (transitions). Here the vectors $\mathbf{x} \in \mathbb{N}^d$ are also called *configurations* (markings). On the set $\mathbb{N}^d$ of configurations we define the *reachability relation* that we now denote by $\xrightarrow{A^*}$ (instead of $\xrightarrow{*}$), to highlight the underlying Petri net A: we write $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ if there is $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$. For $\mathbf{x} \in \mathbb{N}^d$ and $\mathbf{X} \subseteq \mathbb{N}^d$ we put

$$\textsc{post}^*_A(\mathbf{x}) = \{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{A^*} \mathbf{y}\}, \text{ and } \textsc{post}^*_A(\mathbf{X}) = \bigcup_{\mathbf{x}\in\mathbf{X}} \textsc{post}^*_A(\mathbf{x}).$$

Symmetrically, for $\mathbf{y} \in \mathbb{N}^d$ and $\mathbf{Y} \subseteq \mathbb{N}^d$ we put

$$\textsc{pre}^*_A(\mathbf{y}) = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} \xrightarrow{A^*} \mathbf{y}\} \text{ and } \textsc{pre}^*_A(\mathbf{Y}) = \bigcup_{\mathbf{y}\in\mathbf{Y}} \textsc{pre}^*_A(\mathbf{y}).$$

By $\mathbf{X} \xrightarrow{A^*} \mathbf{Y}$ we denote that $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ for some $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{Y}$, i.e. that $\textsc{post}^*_A(\mathbf{X}) \cap \mathbf{Y} \neq \emptyset$, or equivalently $\mathbf{X} \cap \textsc{pre}^*_A(\mathbf{Y}) \neq \emptyset$.

(Semilinear) Reachability Problem. By the (semilinear) *reachability problem* we mean the following decision problem:

> *Instance:* a d-dimensional Petri net A and presentations of two semilinear sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{N}^d$, which we refer to as the triple $A, \mathbf{X}, \mathbf{Y}$.
> *Question:* does $\mathbf{X} \xrightarrow{A^*} \mathbf{Y}$ hold?

In the standard definition of the reachability problem the sets $\mathbf{X}, \mathbf{Y}$ are singletons; the problem is decidable [May84], and it has been recently shown to be Ackermann-complete [LS19, Ler21, CO21]. It is well-known (and easy to show) that the above more general version (the semilinear reachability problem) is tightly related to the standard version, and has thus the same complexity.

Remark 3.1. We can sketch this tight relation as follows. If $\mathbf{X}$ and $\mathbf{Y}$ are linear, with presentations $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k)$ and $(\mathbf{b}', \mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_{k'})$ respectively, then it suffices to ask whether $\mathbf{b} \xrightarrow{(A')^*} \mathbf{b}'$ where A' arises from A by adding the actions $(\mathbf{0}, \mathbf{p}_i)$ for all $i \in [1, k]$ and $(\mathbf{p}'_i, \mathbf{0})$ for all $i \in [1, k']$. Now if $\mathbf{X} = \mathbf{L}_1 \cup \mathbf{L}_2 \cdots \cup \mathbf{L}_m$ and $\mathbf{Y} = \mathbf{L}'_1 \cup \mathbf{L}'_2 \cdots \cup \mathbf{L}'_{m'}$, then it suffices to check if $\mathbf{L}_i \xrightarrow{A^*} \mathbf{L}'_j$ for some $i \in [1, m]$ and $j \in [1, m']$. (In fact, there is also a polynomial reduction of the general version to the standard one, which increases the dimension.)

Semilinear Home-Space Problem. For a Petri net A of dimension d and two sets $\mathbf{X}, \mathbf{H} \subseteq \mathbb{N}^d$, by following the definitions introduced in the previous section we call $\mathbf{H}$ a *home-space for* $(A, \mathbf{X})$, or just for $\mathbf{X}$ when A is clear from the context, if $\text{POST}_A^*(\mathbf{X}) \subseteq \text{PRE}_A^*(\mathbf{H})$. We note that the above (semilinear) reachability problem in fact asks, given $A, \mathbf{X}, \mathbf{Y}$, if $\text{POST}_A^*(\mathbf{X}) \cap \text{PRE}_A^*(\mathbf{Y}) \neq \emptyset$. The *semilinear home-space problem* is defined as follows:

> *Instance:* a triple $A, \mathbf{X}, \mathbf{H}$ where A is a Petri net, of dimension d, and $\mathbf{X}$, $\mathbf{H}$ are two (finitely presented) semilinear subsets of $\mathbb{N}^d$.
> *Question:* is $\text{POST}_A^*(\mathbf{X}) \subseteq \text{PRE}_A^*(\mathbf{H})$ (i.e., is $\mathbf{H}$ a home-space for $\mathbf{X}$) ?

Main Results. Our main result is stated by Theorem 3.3. Nevertheless, we first prove the weaker claim, Theorem 3.2, that answers an open question from [dFEJ89] and does not need the technicalities related to the complexity analysis.

Theorem 3.2. *The semilinear home-space problem is decidable.*

Theorem 3.3. *The semilinear home-space problem is Ackermann-complete.*

We remark that by [dFEJ89] we know that the home-space problem is decidable for the instances A, $\mathbf{X}$, $\mathbf{H}$ where $\mathbf{X}$ is a singleton set, and $\mathbf{H}$ is a finite union of linear sets with the same periods; this was established by a Turing reduction to the reachability problem. The decidability in the case where $\mathbf{H}$ is a general semilinear set was left open in [dFEJ89]; this more general problem indeed looks more subtle but we manage to provide a solution here. Before doing this, we note in Section 4 that the problem has also a high computational complexity, and can be naturally viewed as residing at the decidability/undecidability border.

Remark 3.4. Some intermediate results that help us to derive Theorems 3.2 and 3.3 seem to be interesting on their own. In particular we name Theorem 9.3 showing that for each semilinear set we can effectively construct its inductive semilinear non-reachability core. Another example is an Ackermannian-time algorithm constructing the minimal elements in the reachability set of a given Petri net with an initial configuration (i.e., in the set $\text{POST}_A^*(\mathbf{x})$), which is given in Section 7.2.

4. The Home-Space Problem is Hard

We first note that even a simple version of the home-space problem is at least as hard as (non)reachability, and thus Ackermann-hard. We use a polynomial reduction that increases the Petri net dimension, by additional vector components that can be viewed as control states. (It would be natural to use the model of *vector addition systems with states* but we do not introduce them formally in this paper.)

Proposition 4.1. *The non-reachability problem is polynomially reducible to the home-space problem restricted to the instances $A, \mathbf{X}, \mathbf{H}$ where $\mathbf{X}$ and $\mathbf{H}$ are singletons.*

Proof. Let us consider a Petri net A of dimension d and two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, as an instance of the (non)reachability problem. We create the $(d{+}3)$-dimensional Petri net A' so that each action $a = (\mathbf{a}_-, \mathbf{a}_+)$ of A is transformed to the action $a' = ((\mathbf{a}_-, 1, 0, 0), (\mathbf{a}_+, 1, 0, 0))$ of A', and A' has also the additional actions $((\mathbf{y}, 1, 0, 0), (\mathbf{0}, 0, 1, 0))$, $((\mathbf{0}, 1, 0, 0), (\mathbf{0}, 0, 0, 1))$, and the actions $((\mathbf{i}_j, 0, 1, 0), (\mathbf{0}, 0, 0, 1))$, $((\mathbf{i}_j, 0, 0, 1), (\mathbf{0}, 0, 0, 1))$ for all $j \in [1, d]$, where $\mathbf{i}_j \in \mathbb{N}^d$ satisfies $\mathbf{i}_j(j) = 1$ and $\mathbf{i}_j(i) = 0$ for all $i \neq j$.

We verify that $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ iff $\{(\mathbf{0}, 0, 0, 1)\}$ is not a home-space for $(A', \{(\mathbf{x}, 1, 0, 0)\})$:

- if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$, then $(\mathbf{x}, 1, 0, 0) \xrightarrow{(A')^*} (\mathbf{y}, 1, 0, 0) \xrightarrow{(A')^*} (\mathbf{0}, 0, 1, 0)$, and $(\mathbf{0}, 0, 0, 1)$ is not reachable from $(\mathbf{0}, 0, 1, 0)$;
- if $\mathbf{x} \not\xrightarrow{A^*} \mathbf{y}$, then any configuration reachable from $(\mathbf{x}, 1, 0, 0)$ in A' is in one of the forms $(\mathbf{y}', 1, 0, 0)$, $(\mathbf{z}, 0, 1, 0)$, $(\mathbf{z}', 0, 0, 1)$ where $\mathbf{y}' \neq \mathbf{y}$ and $\mathbf{z} \neq \mathbf{0}$, and $(\mathbf{0}, 0, 0, 1)$ is clearly reachable from all of them. □

Now we note that a slight generalization of the semilinear home-space problem is undecidable; it is the case when instead of semilinear sets $\mathbf{H}$ in the instances $A, \mathbf{X}, \mathbf{H}$ we allow $\mathbf{H}$ to be reachability sets of Petri nets (that are a special case of so called *almost semilinear sets* [Ler12]).

Proposition 4.2. *Given Petri nets A, B of the same dimension d, and two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, it is undecidable if $\text{POST}_B^*(\mathbf{y})$ is a home-space for $(A, \{\mathbf{x}\})$.*

Proof. We recall that the reachability set inclusion problem is undecidable for Petri nets (and for the equivalent model of vector addition systems); see [Bak73, Hac76, Jan95]. Hence it is undecidable, given Petri nets A, B of the same dimension d and $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, whether $\text{POST}_A^*(\mathbf{x}) \subseteq \text{POST}_B^*(\mathbf{y})$. If A' arises from A by replacing each action $a = (\mathbf{a}_-, \mathbf{a}_+)$ with $a' = ((\mathbf{a}_-, 1), (\mathbf{a}_+, 1))$ and by adding the action $((\mathbf{0}, 1), (\mathbf{0}, 0))$, and B' arises from B by replacing each $b = (\mathbf{b}_-, \mathbf{b}_+)$ with $b' = ((\mathbf{b}_-, 0), (\mathbf{b}_+, 0))$, then we obviously have that $\text{POST}_{B'}^*((\mathbf{y}, 0))$ is a home-space for $(A', (\mathbf{x}, 1))$ if, and only if, $\text{POST}_A^*(\mathbf{x}) \subseteq \text{POST}_B^*(\mathbf{y})$. □

Remark 4.3. Since [Jan95] shows, in fact, that the reachability set inclusion (or equality) problem is undecidable even for some fixed five-dimensional vector addition systems with states (VASSs), we could appropriately strengthen Proposition 4.2; but we do not pursue this technical issue here.

We can note that the undecidability of the question whether $\text{POST}_B^*(\mathbf{x}) \subseteq \text{POST}_A^*(\mathbf{y})$ entails that the question whether $\text{POST}_B^*(\mathbf{x}) \subseteq \text{PRE}_A^*(\mathbf{y})$ is also undecidable (since $\text{POST}_A^*(\mathbf{y})$ is equal to $\text{PRE}_{A_{rev}}^*(\mathbf{y})$ where A_{rev} arises from A by reversing each action $(\mathbf{a}_-, \mathbf{a}_+)$ to $(\mathbf{a}_+, \mathbf{a}_-)$). On the other hand, in the next sections we show that the question whether $\text{POST}_A^*(\mathbf{x}) \subseteq \text{PRE}_A^*(\mathbf{y})$ is decidable. We will show that, given a d-dimensional Petri net A and $\mathbf{y} \in \mathbb{N}^d$, we can effectively construct a semilinear non-reachability core $C \subseteq \mathbb{N}^d$ for $\{y\}$, where $\text{POST}_A^*(\mathbf{x}) \not\subseteq \text{PRE}_A^*(\mathbf{y})$ if, and only if, $\text{POST}_A^*(\mathbf{x})$ intersects C. The equality of the nets on both sides is crucial, since if $\text{POST}_B^*(\mathbf{x})$ does not intersect C, then this does not entail $\text{POST}_B^*(\mathbf{x}) \subseteq \text{PRE}_A^*(\mathbf{y})$.

5. Decidability of Home-Space via Semilinear Non-Reachability Cores

Now we start to discuss how to decide the semilinear home-space problem. We consider a fixed Petri net A of dimension d if not said otherwise.

Since a semilinear set is a finite union of linear sets, Proposition 2.1 shows that the semilinear home-space problem can be reduced to a form of the reachability problem as soon as semilinear non-reachability cores can be computed for linear sets:

Lemma 5.1. *Given a Petri net A of dimension d, and (a presentation of) a linear set $\mathbf{L} \subseteq \mathbb{N}^d$, there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{L}$.*

This crucial lemma will be proved in the next section (Section 6). Here we show the decidability of the semilinear home-space problem when assuming the lemma. We note that the semilinear non-reachability core claimed by the lemma is not necessarily inductive; that's why we use Proposition 2.1, and not Proposition 2.2.

The next proposition (related to Proposition 2.1) gives us the final ingredient for showing an algorithm deciding the semilinear home-space problem.

Proposition 5.2. *Given a Petri net A of dimension d, and (presentations of) semilinear subsets $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_m$ of $\mathbb{N}^d$, the existence of an execution*

$$\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2 \cdots \xrightarrow{A^*} \mathbf{x}_m \tag{5.1}$$

where $\mathbf{x}_i \in \mathbf{X}_i$ for each $i \in [0,m]$ is decidable (by a reduction to reachability).

Proof. By a standard construction, we can build a Petri net with a bigger dimension and an initial configuration that first generates m copies of some $\mathbf{x}_0 \in \mathbf{X}_0$, then performs an execution of A from $\mathbf{x}_0$ on all these copies, while at some moment it freezes some configuration $\mathbf{x}_1$ reached in the first copy, later it freezes some $\mathbf{x}_2$ reached in the second copy, etc.; at the end it starts a "testing part" that enables to reach the zero configuration if, and only if, $\mathbf{x_1} \in \mathbf{X}_1$, $\mathbf{x}_2 \in \mathbf{X}_2$, ..., $\mathbf{x}_m \in \mathbf{X}_m$. □

We note that a proof of Theorem 3.2 is now clear: Given a Petri net A of dimension d and two semilinear sets $\mathbf{X}, \mathbf{H} \subseteq \mathbb{N}^d$, we use that $\mathbf{H} = \mathbf{H}_1 \cup \mathbf{H}_2 \ldots \cup \mathbf{H}_m$ where $\mathbf{H}_i$ are linear sets, and by Lemma 5.1 we can construct a semilinear non-reachability core $\mathbf{C}_i$ for $\mathbf{H}_i$, for each $i \in [1,m]$. Then we ask if there is an execution (2.1) from Proposition 2.1; this can be decided effectively by Proposition 5.2.

6. Effective Semilinear Non-Reachability Core for Linear Set

In Section 6.1 we recall an important ingredient dealing with computing the minimal elements in some set $\mathbf{X} \subseteq \mathbb{N}^d$; its use in Petri nets originates in the work by Valk and Jantzen [VJ84]. This will enable us to prove Lemma 5.1 in Section 6.2.

6.1. Computing $\min(\mathbf{X})$ for $\mathbf{X} \subseteq \mathbb{N}^d$.

For $\mathbf{X} \subseteq \mathbb{N}^d$ we call a vector $\mathbf{m} \in \mathbf{X}$ *minimal in* $\mathbf{X}$ if there is no vector $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x} \leq \mathbf{m}$ and $\mathbf{x} \neq \mathbf{m}$. (We recall that $\mathbf{x} \leq \mathbf{y}$ denotes that $\mathbf{x}(i) \leq \mathbf{y}(i)$ for all $i \in [1,d]$.) By $\min(\mathbf{X})$ we denote *the set of minimal elements in* $\mathbf{X}$. Since $\leq$ is a well-partial-order on $\mathbb{N}^d$ (by Dickson's lemma), the set $\min(\mathbf{X})$ is finite and for every $\mathbf{x} \in \mathbf{X}$ there exists (at least one) $\mathbf{m} \in \min(\mathbf{X})$ such that $\mathbf{m} \leq \mathbf{x}$.

As a basis for computing $\min(\mathbf{X})$ (for special sets $\mathbf{X} \subseteq \mathbb{N}^d$), it is useful to extend the ordered set $(\mathbb{N}, \leq)$ with an extra element $\omega \notin \mathbb{N}$ so that $x \leq \omega$ for every $x \in \mathbb{N}_\omega$, where $\mathbb{N}_\omega$ denotes $\mathbb{N} \cup \{\omega\}$. By $\mathbb{N}_\omega^d$ we denote the set of d-dimensional vectors over $\mathbb{N}_\omega$; the (component-wise) order $\leq$ on $\mathbb{N}^d$ is naturally extended to $\mathbb{N}_\omega^d$. For $\mathbf{v} \in \mathbb{N}_\omega^d$ we put $\downarrow\mathbf{v} = \{\mathbf{y} \in \mathbb{N}^d \mid \mathbf{y} \leq \mathbf{v}\}$. Hence even when $\mathbf{v}$ has some ω-components, $\mathbf{y} \in \downarrow\mathbf{v}$ has none.

For $\mathbf{X} \subseteq \mathbb{N}^d$ we trivially have $\min(\mathbf{X}) = \min(\mathbf{X} \cap \downarrow(\omega, \omega, \ldots, \omega))$. If we want to describe $\min(\mathbf{X} \cap \downarrow\mathbf{v})$, for $\mathbf{v} \in \mathbb{N}_\omega^d$, and we have some $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$, then we observe that

$$\min(\mathbf{X} \cap \downarrow\mathbf{v}) = \min\Big(\{\mathbf{y}\} \cup \min\big(\mathbf{X} \cap (\downarrow\mathbf{v} \setminus \{\mathbf{x} \mid \mathbf{y} \leq \mathbf{x}\})\big)\Big).$$

To write this more concretely, by $\mathbf{v}[i \leftarrow k]$, where $i \in [1,d]$ and $k \in \mathbb{N}$, we denote the vector $\mathbf{v}' \in \mathbb{N}_\omega^d$ coinciding with $\mathbf{v}$ except that we have $\mathbf{v}'(i) = k$, and we put

$$\delta_{\mathbf{y}}(\mathbf{v}) = \{\mathbf{w} \in \mathbb{N}_\omega^d \mid \mathbf{w} = \mathbf{v}[i \leftarrow (\mathbf{y}(i) - 1)], i \in [1,d], \mathbf{y}(i) > 0\}.$$

Observation 6.1. For all $\mathbf{v} \in \mathbb{N}_\omega^d$ and $\mathbf{y} \in \downarrow\mathbf{v}$ we have:

(1) Each $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$ is strictly less than $\mathbf{v}$ (i.e., $\mathbf{w} \leq \mathbf{v}$ and $\mathbf{w} \neq \mathbf{v}$).
(2) $\downarrow\mathbf{v} \smallsetminus \{\mathbf{x} \mid \mathbf{y} \leq \mathbf{x}\} = \bigcup_{\mathbf{w}\in\delta_{\mathbf{y}}(\mathbf{v})} \downarrow\mathbf{w}$.

Observation 6.2. For all $\mathbf{X} \subseteq \mathbb{N}^d$, $\mathbf{v} \in \mathbb{N}^d_\omega$, and $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$ we have:

$$\min(\mathbf{X} \cap \downarrow\mathbf{v}) = \min\left(\{\mathbf{y}\} \cup \bigcup_{\mathbf{w}\in\delta_{\mathbf{y}}(\mathbf{v})} \min(\mathbf{X} \cap \downarrow\mathbf{w})\right).$$

Since each strictly decreasing sequence $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots$ of vectors in $\mathbb{N}^d_\omega$ is finite, we easily observe that there is an algorithm stated in the next lemma. Its inputs are special algorithms that we call *set-related algorithms*. Each set-related algorithm is related to some set $\mathbf{X} \subseteq \mathbb{N}^d$ (for some $d \in \mathbb{N}$); given $\mathbf{v} \in \mathbb{N}^d_\omega$, the algorithm decides if $(\mathbf{X} \cap \downarrow\mathbf{v})$ is nonempty, and in the positive case returns some $\mathbf{y} \in (\mathbf{X} \cap \downarrow\mathbf{v})$.

Lemma 6.3. *There is an algorithm that, given a set-related algorithm related to* $\mathbf{X} \subseteq \mathbb{N}^d$*, computes the set* $\min(\mathbf{X})$*.*

Remark 6.4. In fact, the algorithm claimed by the lemma does not require to get a code of a set-related algorithm; it suffices to get (black-box) access to such an algorithm.

6.2. **Proof of Lemma 5.1.** Now we prove the lemma whose statement is repeated here:

> *Given a Petri net A of dimension d, and (a presentation of) a linear set $\mathbf{L} \subseteq \mathbb{N}^d$, there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{L}$.*

We consider a fixed Petri net A of dimension d, and we first prove the claim for the case where $\mathbf{L}$ is a singleton; hence $\mathbf{L} = \{\mathbf{b}\}$ (there is a basis $\mathbf{b} \in \mathbb{N}^d$, but no periods). We observe that if $\|\mathbf{x}\| > \|\mathbf{b}\|$ (where $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$), then a necessary condition for reachability of $\mathbf{b}$ from $\mathbf{x}$ is that $\mathbf{x}$ belongs to the set

$$\mathrm{DC} = \{\mathbf{x} \in \mathbb{N}^d \mid \text{there is } \mathbf{x}' \text{ such that } \mathbf{x} \xrightarrow{A^*} \mathbf{x}' \text{ and } \|\mathbf{x}\| > \|\mathbf{x}'\|\}.$$

For $\mathbf{x} \in \mathrm{DC}$ we say that $\mathbf{x}$ *can Decrease the token-Count*. Since there is no infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{N}^d$ where $\|\mathbf{x}_1\| > \|\mathbf{x}_2\| > \|\mathbf{x}_3\| > \cdots$, for $\mathrm{NDC} = \overline{\mathrm{DC}}$ (the complement of DC, i.e. $\mathbb{N}^d \smallsetminus \mathrm{DC}$) we note the following trivial fact:

Observation 6.5. NDC is a home-space for every $\mathbf{X} \subseteq \mathbb{N}^d$.

Proposition 6.6 is a crucial ingredient for Proposition 6.7 that finishes the proof of Lemma 5.1 in the special case when $\mathbf{L}$ is a singleton.

Proposition 6.6. *The set* DC *is upward closed and the set* $\min(\mathrm{DC})$ *is effectively constructible. Hence both* DC *and* NDC *are effectively constructible semilinear sets.*

Proof. If $\mathbf{x} \xrightarrow{\sigma} \mathbf{x}'$, then $\mathbf{x} + \mathbf{y} \xrightarrow{\sigma} \mathbf{x}' + \mathbf{y}$ (by the monotonicity property of Petri nets). Since $\|\mathbf{x}\| > \|\mathbf{x}'\|$ entails $\|\mathbf{x} + \mathbf{y}\| > \|\mathbf{x}' + \mathbf{y}\|$, it is clear that DC is upward closed (i.e., if $\mathbf{x} \in \mathrm{DC}$ and $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} \in \mathrm{DC}$).

Regarding the effective constructability of min(DC), we recall Lemma 6.3. The question whether $(\mathrm{DC} \cap \downarrow\mathbf{v})$ is nonempty, for a given $\mathbf{v} \in \mathbb{N}^d_\omega$, can be reduced to the reachability problem in a standard way (recall the technique sketched for Proposition 5.2): We construct a net of bigger dimension from the original net, that first generates some $\mathbf{y} \in \mathbb{N}^d$ belonging to $\downarrow\mathbf{v}$ that is frozen, and then some $\mathbf{y}'$ reachable from $\mathbf{y}$ in the original net that is also frozen, and in the final phase a particular place can reach zero if, and only if, $\|\mathbf{y}\| > \|\mathbf{y}'\|$. Hence in the positive case a witness of the respective reachability also yields some $\mathbf{y} \in (\mathrm{DC} \cap \downarrow\mathbf{v})$.

The effective semilinearity of DC and NDC follows trivially. ∎

Proposition 6.7. *Given a Petri net A of dimension d and a vector $\mathbf{b} \in \mathbb{N}^d$, the set*

$$\mathbf{C} = \mathrm{NDC} \cap \big(\{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| > \|\mathbf{b}\|\} \cup \{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| \leq \|\mathbf{b}\| \text{ and } \mathbf{x} \overset{A^*}{\not\longrightarrow} \mathbf{b}\}\big)$$

is an effectively constructible semilinear non-reachability core for $\{\mathbf{b}\}$.

Proof. We first show that $\mathbf{C}$ is a core for $\{\mathbf{b}\}$, i.e., $\mathbf{C} \subseteq \overline{\mathrm{PRE}^*_A(\{\mathbf{b}\})} \subseteq \mathrm{PRE}^*_A(\mathbf{C})$:

(1) We have $\mathbf{C} \overset{A^*}{\not\longrightarrow} \{\mathbf{b}\}$, since $\mathbf{b}$ is clearly not reachable from any element of $\mathbf{C}$.

(2) For each $\mathbf{x} \in \mathbb{N}^d$, if $\mathbf{x} \overset{A^*}{\not\longrightarrow} \mathbf{b}$, then $\mathbf{x} \overset{A^*}{\longrightarrow} \mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{b}$ for some $\mathbf{x}' \in \mathrm{NDC}$ (recall Observation 6.5); the facts $\mathbf{x}' \in \mathrm{NDC}$ and $\mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{b}$ obviously entail $\mathbf{x}' \in \mathbf{C}$, and thus $\mathbf{x} \overset{A^*}{\longrightarrow} \mathbf{C}$.

The effective semilinearity of $\mathbf{C}$ follows from Proposition 6.6 and from the fact that the finite set $\{\mathbf{x} \in \mathbb{N}^d \mid \|\mathbf{x}\| \leq \|\mathbf{b}\| \text{ and } \mathbf{x} \overset{A^*}{\not\longrightarrow} \mathbf{b}\}$ can be constructed by repeatedly using an algorithm deciding reachability. □

Now we proceed to prove Lemma 5.1 in general. We have a Petri net A of dimension d, and a linear set $\mathbf{L}$ presented by a basis $\mathbf{b} \in \mathbb{N}^d$ and periods $\mathbf{p}_1, \mathbf{p}_2 \ldots, \mathbf{p}_k \in \mathbb{N}^d$; we aim to construct a semilinear non-reachability core for $\mathbf{L}$. We would like to generalize the above special-case proof (which is, in fact, closely related to the approach in [dFEJ89]), with the upward closed set DC. But here is a subtle problem that leads us to not working with configurations $\mathbf{x} \in \mathbb{N}^d$ directly but rather via their $\mathbf{L}$-like presentations.

We note that each configuration $\mathbf{x} \in \mathbb{N}^d$ can be presented as

$$\mathbf{x} = \mathbf{y} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k$$

for at least one (but often more) pairs $(\mathbf{y}, \mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k$. For $\mathbf{y} \in \mathbb{N}^d$ and $\mathbf{u} \in \mathbb{N}^k$ we put

$$\mathrm{CONF}(\mathbf{y}, \mathbf{u}) = \mathbf{y} + \mathbf{u}(1) \cdot \mathbf{p}_1 + \mathbf{u}(2) \cdot \mathbf{p}_2 \cdots + \mathbf{u}(k) \cdot \mathbf{p}_k.$$

Hence $\mathbf{L} = \{\mathrm{CONF}(\mathbf{b}, \mathbf{u}) \mid \mathbf{u} \in \mathbb{N}^k\}$.

Let DCB-PR (determined by the Petri net A and the sequence of periods of $\mathbf{L}$) be the set of presentation pairs that present configurations that *can Decrease the token-Count in the presentation Basis*:

$$\mathrm{DCB\text{-}PR} = \{(\mathbf{y}, \mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k \mid \exists(\mathbf{y}', \mathbf{u}') : \|\mathbf{y}\| > \|\mathbf{y}'\|, \mathrm{CONF}(\mathbf{y}, \mathbf{u}) \overset{A^*}{\longrightarrow} \mathrm{CONF}(\mathbf{y}', \mathbf{u}')\}.$$

We note that if $\mathbf{y} \geq \mathbf{p}_i$, for some $i \in [1, k]$, then we trivially have $(\mathbf{y}, \mathbf{u}) \in \mathrm{DCB\text{-}PR}$ since $\mathrm{CONF}(\mathbf{y}, \mathbf{u}) = \mathrm{CONF}(\mathbf{y} - \mathbf{p}_i, \mathbf{u}')$ where $\mathbf{u}'$ arises from $\mathbf{u}$ by adding 1 to $\mathbf{u}(i)$. (As expected, we assume that all $\mathbf{p}_i$ are nonzero vectors.)

Proposition 6.8. DCB-PR *is upward closed and the set* $\min(\mathrm{DCB\text{-}PR})$ *is effectively constructible.*

Proof. As expected, we compare the elements of DCB-PR component-wise. To show that DCB-PR is upward closed, we assume that $(\mathbf{y}_1, \mathbf{u}_1) \in \mathrm{DCB\text{-}PR}$ and $(\mathbf{y}_1, \mathbf{u}_1) \leq (\mathbf{y}_2, \mathbf{u}_2)$. To demonstrate that $(\mathbf{y}_2, \mathbf{u}_2) \in \mathrm{DCB\text{-}PR}$ as well, we again use monotonicity of Petri nets: Since $\mathrm{CONF}(\mathbf{y}_1, \mathbf{u}_1) \overset{\sigma}{\to} \mathrm{CONF}(\mathbf{y}'_1, \mathbf{u}'_1)$ (for some sequence σ) where $\|\mathbf{y}_1\| > \|\mathbf{y}'_1\|$, and $\mathrm{CONF}(\mathbf{y}_1, \mathbf{u}_1) \leq \mathrm{CONF}(\mathbf{y}_2, \mathbf{u}_2)$, we have $\mathrm{CONF}(\mathbf{y}_2, \mathbf{u}_2) \overset{\sigma}{\to} \mathrm{CONF}(\mathbf{y}'_1 + (\mathbf{y}_2 - \mathbf{y}_1), \mathbf{u}'_1 + (\mathbf{u}_2 - \mathbf{u}_1))$; $\|\mathbf{y}_1\| > \|\mathbf{y}'_1\|$ entails $\|\mathbf{y}_2\| > \|\mathbf{y}'_1 + (\mathbf{y}_2 - \mathbf{y}_1)\|$.

The effective constructability of $\min(\mathrm{DCB\text{-}PR})$ is again based on Lemma 6.3, when we identify $\mathbb{N}^d \times \mathbb{N}^k$ with $\mathbb{N}^{d+k}$. It is again a technical routine to show that the question whether $(\mathrm{DCB\text{-}PR} \cap \downarrow \mathbf{v})$ is nonempty, for a given $\mathbf{v} \in \mathbb{N}^{d+k}_\omega$, can be reduced to the reachability problem, so that in the positive case a witness of this reachability also yields some $(\mathbf{y},\mathbf{u}) \in (\text{DCB-PR} \cap {\downarrow}\mathbf{v})$. □

We now define the set of configurations with presentations in which the basis cannot be decreased:

$$\text{NDCB} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \text{CONF}(\mathbf{y},\mathbf{u}) \text{ for some } (\mathbf{y},\mathbf{u}) \notin \text{DCB-PR}\}.$$

Observation 6.9. NDCB is a home-space for every $\mathbf{X} \subseteq \mathbb{N}^d$.

Proof. Suppose there is some $\mathbf{x} \in \mathbb{N}^d$ such that $\mathbf{x} \overset{A^*}{\not\longrightarrow} \text{NDCB}$; we fix one such $\mathbf{x}$ that can be written as $\mathbf{x} = \text{CONF}(\mathbf{y},\mathbf{u})$ for $\mathbf{y}$ with the least norm $\|\mathbf{y}\|$. Since $\mathbf{x} \notin \text{NDCB}$, we have $(\mathbf{y},\mathbf{u}) \in \text{DCB-PR}$, which entails a contradiction by the definition of DCB-PR. □

Proposition 6.10. NDCB *is an effectively constructible semilinear set.*

Proof. By Proposition 6.8, DCB-PR is an effectively constructible semilinear set. Since semilinear sets (effectively) coincide with the sets definable in Presburger arithmetic, the claim is clear. □

The next proposition finishes a proof of Lemma 5.1, and thus also of Theorem 3.2.

Proposition 6.11. *Given a Petri net A of dimension d and a linear set $\mathbf{L} \subseteq \mathbb{N}^d$ presented by $(\mathbf{b},\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_k)$, the set*

$$\mathbf{C} = \{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{x} = \text{CONF}(\mathbf{y},\mathbf{u}) \textit{ where } (\mathbf{y},\mathbf{u}) \notin \text{DCB-PR} \textit{ and}$$
$$\textit{either } \|\mathbf{y}\| > \|\mathbf{b}\|, \textit{ or } \|\mathbf{y}\| \leq \|\mathbf{b}\| \textit{ and } \text{CONF}(\mathbf{y},\mathbf{u}) \overset{A^*}{\not\longrightarrow} \mathbf{L}\}$$

is an effectively constructible semilinear non-reachability core for $\mathbf{L}$.

Proof. We note that $\mathbf{C}$ is a subset of NDCB, and we recall that $\mathbf{x} \in \mathbf{L}$ iff $\mathbf{x} = \text{CONF}(\mathbf{b},\mathbf{u})$ for some $\mathbf{u} \in \mathbb{N}^k$. We verify that $\mathbf{C}$ is a core for $\mathbf{L}$, i.e., $\mathbf{C} \subseteq \overline{\text{PRE}^*_A(\mathbf{L})} \subseteq \text{PRE}^*_A(\mathbf{C})$:

(1) By definition of $\mathbf{C}$ we clearly have $\mathbf{C} \overset{A^*}{\not\longrightarrow} \mathbf{L}$.
(2) For each $\mathbf{x} \in \mathbb{N}^d$, if $\mathbf{x} \overset{A^*}{\not\longrightarrow} \mathbf{L}$, then $\mathbf{x} \overset{A^*}{\longrightarrow} \mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{L}$ for some $\mathbf{x}' \in \text{NDCB}$ (recall Observation 6.9); the facts $\mathbf{x}' \in \text{NDCB}$ and $\mathbf{x}' \overset{A^*}{\not\longrightarrow} \mathbf{L}$ obviously entail $\mathbf{x}' \in \mathbf{C}$, and thus $\mathbf{x} \overset{A^*}{\longrightarrow} \mathbf{C}$.

Now we aim to show that $\mathbf{C}$ is an effectively constructible semilinear set. We recall Propositions 6.10 and 6.8, and the fact that for any concrete $\mathbf{y}$ and $\mathbf{u}$ we can decide if $\text{CONF}(\mathbf{y},\mathbf{u}) \overset{A^*}{\longrightarrow} \mathbf{L}$. Though there are only finitely many $\mathbf{y}$ to consider, namely those satisfying $\|\mathbf{y}\| \leq \|\mathbf{b}\|$, we are not done: it is not immediately obvious how to express $\text{CONF}(\mathbf{y},\mathbf{u}) \overset{A^*}{\not\longrightarrow} \mathbf{L}$ in Presburger arithmetic, even when $\mathbf{y}$ is fixed. To this aim, for any fixed $\mathbf{y} \in \mathbb{N}^d$ we define the set

$$\mathbf{U_y} = \{\mathbf{u} \in \mathbb{N}^k \mid \text{CONF}(\mathbf{y},\mathbf{u}) \overset{A^*}{\longrightarrow} \mathbf{L}\} = \{\mathbf{u} \in \mathbb{N}^k \mid \exists \mathbf{u}' \in \mathbb{N}^k : \text{CONF}(\mathbf{y},\mathbf{u}) \overset{A^*}{\longrightarrow} \text{CONF}(\mathbf{b},\mathbf{u}')\}.$$

For each fixed $\mathbf{y} \in \mathbb{N}^d$, the set $\mathbf{U_y}$ is clearly upward closed (by monotonicity of Petri nets). Moreover, the set $\min(\mathbf{U_y})$ is effectively constructible, again by using Lemma 6.3: Given a fixed $\mathbf{y}$, for each $\mathbf{v} \in \mathbb{N}^k_\omega$ we can decide whether $(\mathbf{U_y} \cap {\downarrow}\mathbf{v})$ is nonempty by a reduction to the reachability problem, so that in the positive case a witness of this reachability also yields some $\mathbf{u} \in (\mathbf{U_y} \cap {\downarrow}\mathbf{v})$.

Now it is clear that we can effectively construct a Presburger formula defining $\mathbf{C}$; hence $\mathbf{C}$ is a semilinear set for which we can effectively construct a presentation. □

7. Minimal Reachable Configurations

We have proven the decidability (Theorem 3.2), and now we aim to analyze the presented approach to get some complexity upper bounds that will enable us to prove Theorem 3.3. This aim leads us to show several Ackermannian-time algorithms in this section.

The first algorithm gets a Petri net A of dimension d and a configuration $\mathbf{x} \in \mathbb{N}^d$ as input, and computes the set $\min(\text{POST}^*_A(\mathbf{x}))$, i.e. the set of minimal configurations in the respective reachability set. The second algorithm computes $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ when it gets (a presentation of) a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ besides A and $\mathbf{x}$. The third algorithm gets $A, \mathbf{x}$, and (a presentation of) a semilinear predicate $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$ (for some $h \in \mathbb{N}$), and computes the set

$$\min(\{\mathbf{x} \in \mathbb{N}^h \mid \exists \alpha, \beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{x}, \alpha, \beta) \in P\}).$$

The complexity of computing the above mentioned minimal configurations can be derived by using the approach by Hsu-Chun Yen and Chien-Liang Chen in [YC09]; they observed that complexity bounds on a set-related algorithm related to some set $\mathbf{X} \subseteq \mathbb{N}^d$ (recall the definition before Lemma 6.3) allow us to derive complexity bounds on the computation of $\min(\mathbf{X})$. As a crucial ingredient here, we recall the known complexity upper bound for reachability in Section 7.1. In Section 7.2 we derive an Ackermannian bound on the size of minimal configurations in Petri net reachability sets, and we extend this bound in Section 7.3 and in Section 7.4 to obtain the mentioned second algorithm and the third algorithm, respectively.

Remark 7.1. Mayr and Meyer described in [MM81] a family of Petri nets that exhibits finite reachability sets whose size grows as the Ackermann function; hence also the size of the maximal configurations in these sets grows similarly. Concerning the size of minimal configurations, we cannot deduce any interesting size properties using the same family. However, by using the family of Petri nets recently introduced in [Ler21, CO21, Las22] for proving that the reachability problem is Ackermann-hard, we can observe that the maximal size of minimal configurations in Petri net reachability sets grows at least as the Ackermann function.

7.1. Petri Net Reachability Problem in Fixed Dimension.

Here we recall some definitions in order to state that the Petri net reachability problem is primitive-recursive when restricted to a fixed dimension, and Ackermannian in general.

The *fast-growing functions* $F_d \colon \mathbb{N} \to \mathbb{N}$, $d \in \mathbb{N}$, are defined inductively as follows:

$$F_0(n) = n + 1, \text{ and } F_{d+1}(n) = F_d^{(n+1)}(n);$$

where by $f^{(n)}$, for a function $f \colon \mathbb{N} \to \mathbb{N}$, we mean the respective iteration of f (i.e., $f^{(n+1)} = f^{(n)} \circ f$). Following [Sch16a], we introduce the class $\mathbf{F}_d$ of functions computable in time $O(F_d(F_{d-1}^{(c)}(n)))$ where n is the size of the input and $c \in \mathbb{N}$ is any constant. We recall that $\bigcup_{d \in \mathbb{N}} \mathbf{F}_d$ is the class of *primitive-recursive functions*. We also introduce the function $F_\omega \colon \mathbb{N} \to \mathbb{N}$ defined by $F_\omega(n) = F_n(n)$, which is a variant of the Ackermann function; by $\mathbf{F}_\omega$ we denote the class of functions computable in time $O(F_\omega(F_d(n)))$ where $d \in \mathbb{N}$ is any constant and n is the size of the input. A function in $\mathbf{F}_\omega$ is said to be computable in *Ackermannian time*. (We note that Ackermannian time coincides with Ackermannian space.)

For $\mathbf{x} \in \mathbb{N}^d$ we have defined the norm of $\mathbf{x}$ as $\|\mathbf{x}\| = \sum_{i=1}^{d} \mathbf{x}(i)$. Now we extend the notion of norm to other objects. For a *Petri net action* $a = (\mathbf{a}_-, \mathbf{a}_+)$, by its *norm* we mean $\|a\| = \max\{\|\mathbf{a}_-\|, \|\mathbf{a}_+\|\}$. For a *Petri net* A, by its *norm* we mean $\|A\| = \max_{a \in A} \|a\|$. The *norm* of a linear set $\mathbf{L} \subseteq \mathbb{N}^d$ implicitly given by a presentation $(\mathbf{b}, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k)$ is defined by $\|\mathbf{L}\| = \max\{\|\mathbf{b}\|, \|\mathbf{p}_1\|, \|\mathbf{p}_2\|, \ldots, \|\mathbf{p}_k\|\}$. The *norm* of a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ implicitly given by a sequence of presentations of $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$ is defined by $\|\mathbf{S}\| = \max_{1 \le n \le m} \|\mathbf{L}_n\|$.

Now we recall a result showing that the reachability problem restricted to Petri nets of dimension d is in $\mathbf{F}_{d+4}$, and that the general Petri net reachability problem is in $\mathbf{F}_\omega$. (We view a decision problem as a function with the co-domain $\{0, 1\}$.) This result is crucial for us to derive the upper bound in Theorem 3.3.

Theorem 7.2 [LS19]**.** *There is a constant $c > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{y}$ where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x}, \mathbf{y} \in \mathbb{N}^d$, and the norms of $A, \mathbf{x}, \mathbf{y}$ are bounded by n, we have that if $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$, then $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ for a word $\sigma \in A^*$ such that $|\sigma| \le F_{d+4} \circ F_{d+3}^{(c)}(n)$.*

We remark that in what follows we formulate some results in the form
"There is a constant $c' > 0$ such that..."

Naturally we could replace c' with c without changing the meaning of the respective statements, but we prefer keeping the difference in order to highlight the special role of the constant c introduced in Theorem 7.2.

7.2. **Minimal Reachable Configurations.** We provide an algorithm computing the set of minimal reachable configurations, by following the approach of [YC09]. To ease notation, we introduce the functions $f_d = F_{d+4} \circ F_{d+3}^{(c)}$ ($d \in \mathbb{N}$) where c is the constant introduced in Theorem 7.2, and we first prove the following proposition; for $\mathbf{v} \in \mathbb{N}_\omega^d$, by its *norm* we mean $\|\mathbf{v}\| = \sum_{i:\mathbf{v}(i) \ne \omega} \mathbf{v}(i)$.

Proposition 7.3. *For all d, n, A, $\mathbf{x}$, $\mathbf{v}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{v} \in \mathbb{N}_\omega^d$, and the norms of $A, \mathbf{x}, \mathbf{v}$ are bounded by n, we have that if $(\mathrm{POST}_A^*(\mathbf{x}) \cap {\downarrow}\mathbf{v})$ is nonempty, then there is $\mathbf{y} \in (\mathrm{POST}_A^*(\mathbf{x}) \cap {\downarrow}\mathbf{v})$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ for some $\sigma \in A^*$ where $|\sigma| \le f_d(n)$.*

Proof. For $n = 0$ the claim is trivial, so we assume $n \ge 1$.

For each $j \in [1, d]$ we define the Petri net action $b_j = (\mathbf{i}_j, \mathbf{0})$ where $\mathbf{i}_j(j) = 1$ and $\mathbf{i}_j(i) = 0$ for all $i \in [1, d] \smallsetminus \{j\}$; this action decrements the jth component of configurations. We put $I_\omega = \{j \mid j \in [1, d], \mathbf{v}(j) = \omega\}$, and by B we denote the Petri net $\{b_j \mid j \in I_\omega\}$. Since $n \ge 1$, we derive $\|A \cup B\| \le n$.

Let us now consider a configuration $\mathbf{z} \in (\mathrm{POST}_A^*(\mathbf{x}) \cap {\downarrow}\mathbf{v})$. Let $\mathbf{c}$ be the configuration arising from $\mathbf{z}$ by replacing the components in I_ω with zero; we thus have $\|\mathbf{c}\| \le \|\mathbf{v}\| \le n$ (using the fact that $\mathbf{c} \le \mathbf{z}$, and thus $\mathbf{c} \in {\downarrow}\mathbf{v}$).

From $\mathbf{x} \xrightarrow{A^*} \mathbf{z}$ and $\mathbf{z} \xrightarrow{B^*} \mathbf{c}$ we derive $\mathbf{x} \xrightarrow{(A \cup B)^*} \mathbf{c}$. By Theorem 7.2 we deduce that $\mathbf{x} \xrightarrow{u} \mathbf{c}$ for some word $u \in (A \cup B)^*$ for which $|u| \le f_d(n)$. Since Petri net actions in B only decrease some components, we can assume that all these actions in u are at the end; hence $u = \sigma v$ where $\sigma \in A^*$ and $v \in B^*$, and we have $\mathbf{x} \xrightarrow{\sigma} \mathbf{y} \xrightarrow{v} \mathbf{c}$ for a configuration $\mathbf{y} \in \mathrm{POST}_A^*(\mathbf{x})$. Since $\mathbf{c} \le \mathbf{z}$, $\mathbf{z} \in {\downarrow}\mathbf{v}$, and $\mathbf{y} \xrightarrow{v} \mathbf{c}$ only decreases the components that are ω in $\mathbf{v}$, we deduce that $\mathbf{y} \in {\downarrow}\mathbf{v}$. ∎

To ease the formulation of the next proposition, for all $d \in \mathbb{N}$ we define the functions $g_d \colon \mathbb{N} \to \mathbb{N}$ by

$$g_d(n) = n \cdot (\, 2 + f_d(n)\,).$$

Proposition 7.4. *For all $d, n, A, \mathbf{x}, \mathbf{v}, \mathbf{m}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{v} \in \mathbb{N}^d_\omega$, $\mathbf{m}$ belongs to $\min(\textsc{post}^*_A(\mathbf{x}) \cap \downarrow \mathbf{v})$, and the norms of $A, \mathbf{x}, \mathbf{v}$ are bounded by n, there exists a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{m}$ and $|\sigma| \leq f_d \circ g_d^{(k)}(n)$ where $k = |\{i \mid \mathbf{v}(i) = \omega\}|$.*

Proof. The strict version $<$ of the relation $\leq$ on $\mathbb{N}^d_\omega$ (defined by $\mathbf{w} < \mathbf{v}$ if $\mathbf{w} \leq \mathbf{v}$ and $\mathbf{w} \neq \mathbf{v}$) is clearly well-founded. We use this property for an inductive proof.

We aim to show the claim for a considered tuple $d, n, A, \mathbf{x}, \mathbf{v}, \mathbf{m}$, while we can assume that the claim is valid for $d, n', A, \mathbf{x}, \mathbf{w}, \mathbf{m}'$ for all $\mathbf{w} < \mathbf{v}$ and all $\mathbf{m}' \in \min(\textsc{post}^*_A(\mathbf{x}) \cap {\downarrow}\mathbf{w})$.

Since $\mathbf{m}$ is in $(\textsc{post}^*_A(\mathbf{x}) \cap \downarrow \mathbf{v})$, we deduce from Lemma 7.3 that we can fix $\mathbf{y} \in (\textsc{post}^*_A(\mathbf{x}) \cap \downarrow \mathbf{v})$ and a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ and $|\sigma| \leq f_d(n)$; we thus have $\|\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{A}\| \cdot |\sigma| \leq g_d(n) - n$. If $\mathbf{m} = \mathbf{y}$, then the claim is proved; so we assume that $\mathbf{m} \neq \mathbf{y}$.

By Observation 6.2 we can fix $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$ such that $\mathbf{m} \in \min(\textsc{post}^*_A(\mathbf{x}) \cap {\downarrow}\mathbf{w})$; since $\mathbf{w} \in \delta_{\mathbf{y}}(\mathbf{v})$, we have $\mathbf{w} < \mathbf{v}$. By the induction hypothesis, there is a word $\sigma' \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma'} \mathbf{m}$ and $|\sigma'| \leq f_d \circ g_d^{(k')}(n')$ where $n' = \max\{\|A\|, \|\mathbf{x}\|, \|\mathbf{w}\|\}$ and $k' = |\{i \mid \mathbf{w}(i) = \omega\}|$. Putting $k = |\{i \mid \mathbf{v}(i) = \omega\}|$, we observe that $k' = k$ or $k' = k-1$. If $k' = k$, then $\|\mathbf{w}\| < \|\mathbf{v}\|$ and we are done by monotonicity of f_d and g_d. Otherwise $k' = k - 1$ and in that case $\|\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{y}\| \leq g_d(n)$ since in that case $\mathbf{w}$ is obtained from $\mathbf{v}$ by replacing component i of $\mathbf{v}$ for some i such that $\mathbf{v}(i) = \omega$ and $\mathbf{y}(i) > 0$ by $\mathbf{y}(i) - 1$. It follows that $n' \leq g_d(n)$ and we are done also in that case by monotonicity of f_d and g_d. □

Finally, by instantiating the previous proposition with $\mathbf{v} = (\omega, \omega, \ldots, \omega)$, and by bounding $f_d \circ g_d^{(d)}(n)$ as provided by the next proposition, we deduce the following two corollaries.

Proposition 7.5. *For every d, n, we have $f_d \circ g_d^{(d)}(n) \leq F_{d+5}((d+1+n)(c+2))$.*

Proof. As $F_2(x) = 2^x(x+1) - 1$ we deduce that $F_2(x) \geq x(2+x)$ for every $x \geq 0$. It follows from $F_{d+4}(x) \geq F_2(x)$ that $F_{d+4}(x) \geq x(2+x)$ for every x. Now, let $y \geq 0$ and let us put $x = F_{d+4}^{(c+1)}(y)$. We have $F_{d+4}^{(c+2)}(y) = F_{d+4}(x) \geq x(2+x) \geq y(2+x)$. Since $x = F_{d+4} \circ F_{d+4}^{(c)}(y) \geq F_{d+4} \circ F_{d+3}^{(c)}(y) = f_d(y)$, we deduce that $x \geq f_d(y)$. Combined with $F_{d+4}^{(c+2)}(y) \geq y(2+x)$ we get $F_{d+4}^{(c+2)}(y) \geq y(2 + f_d(y)) = g_d(y)$. We have proved that $g_d(y) \leq F_{d+4}^{(c+2)}(y)$ for every y. In particular $f_d \circ g_d^{(d)}(n) \leq F_{d+4}^{(c+1)} \circ F_{d+4}^{(d(c+2))}(n) \leq F_{d+4}^{((d+1)(c+2))}(n)$ for every n. It follows that $f_d \circ g_d^{(d)}(n) \leq F_{d+4}^{((d+1)(c+2))}(n) \leq F_{d+4}^{((d+1+n)(c+2))}((d+1+n)(c+2)) = F_{d+5}((d+1+n)(c+2))$. □

Corollary 7.6. *There is a constant $c > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{m}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{m}$ belongs to $\min(\textsc{post}^*_A(\mathbf{x}))$, and the norms of $A, \mathbf{x}$ are bounded by n, there exists a word $\sigma \in A^*$ such that $\mathbf{x} \xrightarrow{\sigma} \mathbf{m}$ and $|\sigma| \leq F_{d+5}((d+1+n)(c+2))$.*

Corollary 7.7. *There is a constant $c > 0$ such that for all $d, n, A, \mathbf{x}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, and the norms of $A, \mathbf{x}$ are bounded by n, the set $\min(\textsc{post}^*_A(\mathbf{x}))$ is computable in time exponential in $F_{d+5}((d+1+n)(c+2))$ and the norms of vectors in that set are bounded by $n \cdot (1 + F_{d+5}((d+1+n)(c+2)))$.*

Proof. In fact, the set of minimal reachable configurations can be obtained by exploring configurations reachable from $\mathbf{x}$ by sequences of at most $F_{d+5}((d+1+n)(c+2))$ actions in A. We note that the norms of configurations reachable in this way are bounded by $\|\mathbf{x}\| + F_{d+5}((d+1+n)(c+2)) \cdot \|A\| \leq n \cdot (1 + F_{d+5}((d+1+n)(c+2)))$. □

7.3. **Extension to Semilinear Sets.** The algorithm computing minimal reachable configurations can be also simply used for computing the set $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ where $\mathbf{S}$ is a semilinear set; we thus formulate this fact as a corollary (though with a proof). We recall that the norm of a semilinear set is the maximum norm of vectors occurring in its (implicitly assumed) presentation.

Corollary 7.8. *There is a constant $c > 0$ such that for all $d, n, A, \mathbf{x}, \mathbf{S}$, where $d, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, $\mathbf{S}$ is (a presentation of) a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$, and the norms of $A, \mathbf{x}, \mathbf{S}$ are bounded by n, the set $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ is computable in time exponential in $F_{2d+6}(n(c+2))$ and the norms of vectors in that set are bounded by $n \cdot (1 + F_{2d+6}((2d+2+n)(c+2)))$.*

Proof. Let us consider a d-dimensional Petri net A, an initial configuration $\mathbf{x}$, and a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$ given as the union of linear sets $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$. Since $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S}) = \min(\bigcup_{j=1}^m \min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L}_j))$ we can reduce the problem of computing $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{S})$ to the special case of a linear set $\mathbf{S}$, denoted as $\mathbf{L}$ in the sequel. So, let $\mathbf{L}$ be a linear set presented by a basis $\mathbf{b} \in \mathbb{N}^d$ and a sequence of periods $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k \in \mathbb{N}^d$, and let us provide an algorithm for computing $\min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L})$.

To do so, we build from A a new Petri net B of dimension $2d+1$ defined as follows and an initial configuration $(\mathbf{x}, 1, \mathbf{0})$. We associate to each Petri net action $a \in A$ of the form $(\mathbf{a}_-, \mathbf{a}_+)$ the action $((\mathbf{a}_-, 1, \mathbf{0}), (\mathbf{a}_+, 1, \mathbf{0}))$ in B that intuitively executes a on the first d counters and check that the middle counter (the counter $d+1$) is at least 1. We also add in B for each $j \in [1, k]$ an action $((\mathbf{p}_j, 0, \mathbf{0}), (\mathbf{0}, 0, \mathbf{p}_j))$ that removes the period $\mathbf{p}_j$ on the first d counters and adds it on the last d counters. Finally, we add to B the action $((\mathbf{b}, 1, \mathbf{0}), (\mathbf{0}, 0, \mathbf{b}))$ that decrements the middle counter and simultaneously removes $\mathbf{b}$ from the first d counters, and adds $\mathbf{b}$ on the last d counters. Since for any set $\mathbf{X} \subseteq \mathbb{N}^d$ and any set $I \subseteq [1, d]$, the set $\min(\{\mathbf{x} \in \mathbf{X} \mid \bigwedge_{i \in I} \mathbf{x}(i) = 0\})$ is equal to $\{\mathbf{m} \in \min(\mathbf{X}) \mid \bigwedge_{i \in I} \mathbf{m}(i) = 0\})$, one can observe that $\{\mathbf{0}\} \times \{0\} \times \min(\text{POST}^*_A(\mathbf{x}) \cap \mathbf{L})$ is equal to $\min(\text{POST}^*_B(\mathbf{x}, 1, \mathbf{0})) \cap (\{\mathbf{0}\} \times \{0\} \times \mathbb{N}^d)$. □

7.4. **Extension to Semilinear Predicates.** By another corollary (with a proof) we also note that the algorithm computing minimal reachable configurations can be used for computing minimal vectors in sets of the following form

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{N}^h \mid \exists \alpha, \beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{x}, \alpha, \beta) \in P\} \tag{7.1}$$

where $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$ is a semilinear predicate given by a presentation. Notice that we use Greek letters α and β in the definition of $\mathbf{X}$ in order to emphasise vectors that act as configurations of the Petri net A.

Corollary 7.9. *There is a constant $c > 0$ such that for all d, h, n, A, P, where $d, h, n \in \mathbb{N}$, A is a Petri net of dimension d, $\mathbf{x} \in \mathbb{N}^d$, P is (a presentation of) a semilinear predicate $P \subseteq \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d$, and the norms of $A, \mathbf{x}, P$ are bounded by n, the set of minimal elements of* *the set* $\mathbf{X}$ *denoted by equation (7.1) is computable in time exponential in* $F_{2h+4d+6}((2h+4d+2+n)(c+2))$ *and the norms of these minimal elements are bounded by* $n\cdot(1+F_{2h+4d+6}(n(c+2)))$.

Proof. We first introduce the set Y defined as $Z \cap P$ where

$$Z = \{(\mathbf{x},\alpha,\beta) \in \mathbb{N}^h \times \mathbb{N}^d \times \mathbb{N}^d \mid \alpha \xrightarrow{A^*} \beta\}.$$

Since $\min(\mathbf{X}) = \min\{\mathbf{x} \in \mathbb{N}^k \mid \exists \alpha,\beta \in \mathbb{N}^d : \ (\mathbf{x},\alpha,\beta) \in \min(Y)\}$ it is sufficient to provide an algorithm computing $\min(Y)$.

Our algorithm is based on the fact that Z is the reachability set of a $(h+2d)$-dimensional Petri net B starting from the zero configuration and defined as follows from A. By $\mathbf{i}_i$ we denote the vector in $\mathbb{N}^h$ defined by $\mathbf{i}_i(i)=1$ and $\mathbf{i}_i(j)=0$ if $j \in [1,h]\backslash\{i\}$. The Petri net B is defined as the actions $((\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{i}_j,\mathbf{0},\mathbf{0}))$ where $j\in[1,h]$ that increment the counters corresponding to $\mathbf{x}$, actions $((\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{i}_j,\mathbf{i}_j))$ that increment simultaneously by the same amount the counters corresponding to α and β, and actions obtained from A that simulate the computation of A on the counters β and defined for each action a of A of the form $(\mathbf{a}_-,\mathbf{a}_+)$ by the action $((\mathbf{0},\mathbf{0},\mathbf{a}_-),(\mathbf{0},\mathbf{0},\mathbf{a}_+))$ in B. Notice that $Z = \textsc{post}^*_B(\mathbf{0},\mathbf{0},\mathbf{0})$ and we are done by Corollary 7.8. □

8. Complexity of the Semilinear Home-Space Problem

In this section we provide an Ackermannian complexity upper-bound for deciding the semilinear home-space problem; Theorem 3.3 will thus be proven.

So let $A,\mathbf{X},\mathbf{H}$ be an instance of the semilinear home-space problem where A is a Petri net, of dimension d, and $\mathbf{X},\mathbf{H}$ are two (presentations of) semilinear subsets of $\mathbb{N}^d$. Since $\mathbf{H}$ can be decomposed, in elementary time, into a finite union of linear sets using presentations with at most d periods [GS64, Lemma 6.6], we can assume that each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$ satisfies this constraint. We put $n = d+\max\{\|A\|,\|\mathbf{X}\|,\|\mathbf{H}\|\}$.

We first consider the problem of computing a semilinear non-reachability core for each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$. Such a linear set $\mathbf{L}$ is presented with a basis $\mathbf{b}$ and a sequence of k periods $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_k$ with $k\leq d$. As previously shown, this computation reduces to the computation of the minimal elements of the upward closed set $\textsc{dcb-pr}$ and the upward-closed sets $\mathbf{U}_\mathbf{y}$ where $\mathbf{y}$ belongs to the finite set of vectors in $\mathbb{N}^d$ satisfying $\|\mathbf{y}\|\leq\|\mathbf{b}\|$. The computation of those minimal elements can be obtained by rewriting the definitions of $\textsc{dcb-pr}$ and $\mathbf{U}_\mathbf{y}$ to match the statement of Corollary 7.9. To do so, we note that $\textsc{dcb-pr}$ and $\mathbf{U}_\mathbf{y}$ can be described in the following way:

$$\textsc{dcb-pr} = \{(\mathbf{y},\mathbf{u}) \in \mathbb{N}^d \times \mathbb{N}^k \mid \exists \alpha,\beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{y},\mathbf{u},\alpha,\beta) \in P\}$$

$$\mathbf{U}_\mathbf{y} = \{\mathbf{u} \in \mathbb{N}^k \mid \exists \alpha,\beta \in \mathbb{N}^d : \alpha \xrightarrow{A^*} \beta \wedge (\mathbf{u},\alpha,\beta) \in P_\mathbf{y}\}$$

where:

$$P = \left\{ (\mathbf{y},\mathbf{u},\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^k \times \mathbb{N}^d \times \mathbb{N}^d \mid \exists (\mathbf{y}',\mathbf{u}') \in \mathbb{N}^d \times \mathbb{N}^k : \begin{array}{l} \|\mathbf{y}\| > \|\mathbf{y}'\| \wedge \\ \alpha = \textsc{conf}(\mathbf{y},\mathbf{u}) \wedge \\ \beta = \textsc{conf}(\mathbf{y}',\mathbf{u}') \end{array} \right\}$$

$$P_\mathbf{y} = \{(\mathbf{u},\alpha,\beta) \in \mathbb{N}^k \times \mathbb{N}^d \times \mathbb{N}^d \mid \alpha = \textsc{conf}(\mathbf{y},\mathbf{u}) \wedge \beta \in \mathbf{L}\}.$$

Since the sets P and $P_\mathbf{y}$ are clearly expressible by formulas in Presburger arithmetic, we can effectively construct, in elementary time, semilinear presentations of those sets [GS66]. We introduce an elementary function E (independent of any instance) corresponding to that computation. We deduce that for some constant $c' > 0$, independent of any input, we can compute, in time exponential in $F_{8d+6}(c'E(n))$, the sets $\min(\text{DCB-PR})$ and $\min(\mathbf{U}_\mathbf{y})$ for $\|\mathbf{y}\| \leq \|\mathbf{b}\|$. Moreover, the norms of vectors in those sets are bounded by $F_{8d+6}(c'E(n))$. It follows from the proof of Proposition 6.11 that there exists an elementary function E' (independent of any instance) such that we can compute, in time $E'(F_{8d+6}(c'E(n)))$, a (presentation of a) semilinear non-reachability core $\mathbf{C}$ for each linear set $\mathbf{L}$ of the presentation of $\mathbf{H}$.

Let $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$ be the presentation sequence of $\mathbf{H}$, and let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$ be the respective semilinear non-reachability cores computed for $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_m$, respectively, as shown in the previous paragraph. Proposition 2.1 shows that $\mathbf{H}$ is not a home-space for $\mathbf{X}$ if, and only if, there is an execution

$$\mathbf{x}_0 \xrightarrow{A^*} \mathbf{x}_1 \xrightarrow{A^*} \mathbf{x}_2 \cdots \xrightarrow{A^*} \mathbf{x}_m \tag{8.1}$$

where $\mathbf{x}_0 \in \mathbf{X}$, and $\mathbf{x}_i \in \mathbf{C}_i$ for each $i \in [1, m]$.

The existence of such an execution can be decided by Proposition 5.2, by a reduction to the reachability problem for a Petri net of a dimension that is elementary in $\max\{d, m, n\}$. Theorem 7.2 thus entails that the semilinear home-space problem is decidable in Ackermannian time, which finishes the proof of Theorem 3.3.

9. Semilinear Inductive Cores for Semilinear Sets

In Lemma 5.1 we proved that for any Petri net A of dimension d and (a presentation of) a linear set $\mathbf{L} \subseteq \mathbb{N}^d$ there is an effectively constructible semilinear non-reachability core $\mathbf{C}$ for $\mathbf{L}$. A natural question is if we can compute a semilinear core for any semilinear set. By Proposition 2.2, this is the case if we can extend Lemma 5.1 so that the respective semilinear cores $\mathbf{C}$ for linear sets $\mathbf{L}$ are, moreover, inductive. We can indeed achieve this, by using the following known result and its corollary.

Theorem 9.1 [Ler10, Theorem 8.3]**.** *Given a Petri net A and two semilinear sets $\mathbf{X}, \mathbf{Y}$ of configurations, we have $\mathbf{X} \subseteq \overline{\text{PRE}^*_A(\mathbf{Y})}$ if, and only if, there exists an effectively constructible semilinear inductive set $\mathbf{I}$ such that $\mathbf{X} \subseteq \mathbf{I} \subseteq \overline{\mathbf{Y}}$.*

Corollary 9.2. *Given a Petri net A and two semilinear sets $\mathbf{C}, \mathbf{H}$ of configurations where $\mathbf{C}$ is a non-reachability core for $\mathbf{H}$, there is an effectively constructible semilinear inductive non-reachability core $\mathbf{C}' \supseteq \mathbf{C}$ for $\mathbf{H}$.*

Proof. For the considered $A, \mathbf{C}, \mathbf{H}$ we have $\mathbf{C} \subseteq \overline{\text{PRE}^*_A(\mathbf{H})} \subseteq \text{PRE}^*_A(\mathbf{C})$. By Theorem 9.1 there is an effectively constructible semilinear inductive set $\mathbf{C}'$ such that $\mathbf{C} \subseteq \mathbf{C}' \subseteq \overline{\mathbf{H}}$. Since $\mathbf{C}'$ is inductive, i.e. $\text{POST}^*_A(\mathbf{C}') = \mathbf{C}'$, we also have $\text{PRE}^*_A(\overline{\mathbf{C}'}) = \overline{\mathbf{C}'}$. Hence $\mathbf{C}' \subseteq \overline{\mathbf{H}}$, i.e. $\mathbf{H} \subseteq \overline{\mathbf{C}'}$, entails $\text{PRE}^*_A(\mathbf{H}) \subseteq \text{PRE}^*_A(\overline{\mathbf{C}'}) = \overline{\mathbf{C}'}$, i.e. $\mathbf{C}' \subseteq \overline{\text{PRE}^*_A(\mathbf{H})}$. We thus have $\mathbf{C} \subseteq \mathbf{C}' \subseteq \overline{\text{PRE}^*_A(\mathbf{H})} \subseteq \text{PRE}^*_A(\mathbf{C}) \subseteq \text{PRE}^*_A(\mathbf{C}')$. □

We can thus deduce the following theorem.

Theorem 9.3. *Given a Petri net A of dimension d, and (a presentation of) a semilinear set $\mathbf{H} \subseteq \mathbb{N}^d$, there is an effectively constructible semilinear inductive non-reachability core $\mathbf{C}$ for $\mathbf{H}$.*

Proof. If $\mathbf{H} = \emptyset$, then we can take $\mathbf{C} = \mathbb{N}^d$.

Now we assume that $\mathbf{H} = \mathbf{H}_1 \cup \mathbf{H}_2 \cdots \cup \mathbf{H}_m$ for some $m \geq 1$, where $\mathbf{H}_i$ is a linear set for each $i \in [1, m]$. By Lemma 5.1 we can construct semilinear sets $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_m$ that are non-reachability cores for $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_m$, respectively (hence $\mathbf{C}_i \subseteq \overline{\text{PRE}^*_A(\mathbf{H}_i)} \subseteq \text{PRE}^*_A(\mathbf{C}_i)$).

By Corollary 9.2, for each $i \in [1, m]$ we can construct a semilinear inductive core $\mathbf{C}'_i$ for $\mathbf{H}_i$. By Proposition 2.2 we deduce that $\mathbf{C}'_1 \cap \mathbf{C}'_2 \cdots \cap \mathbf{C}'_m$ is a semilinear inductive non-reachability core for $\mathbf{H}$ (using the fact that the intersection of semilinear sets is effectively semilinear). □

Remark 9.4. Theorem 9.3 also yields the decidability of the semilinear home-space problem, since the problem if $\mathbf{X} \xrightarrow{A^*} \mathbf{C}$ (i.e., if $\mathbf{x} \xrightarrow{A^*} \mathbf{c}$ for some $\mathbf{x} \in \mathbf{X}$ and $\mathbf{c} \in \mathbf{C}$) for semilinear sets $\mathbf{X}, \mathbf{C}$ of configurations of a given Petri net is decidable; we have already recalled that the semilinear reachability problem is easily reducible to the standard reachability problem, thus being also Ackermann-complete. Theorem 9.1 from [Ler10] gives us no complexity bound for constructing the inductive semilinear set $\mathbf{I}$; therefore we could not derive any complexity bound in this way. In fact, this would be possible now; we could show that the inductive semilinear cores for semilinear sets are computable in Ackermannian time, by using the results of a new paper [Ler24]. Such a complexity proof would be thus based on an involved result about semilinear inductive invariants, whereas the complexity proof presented in this paper is independent of this.

10. Home-Space Witnesses

We recall that the reachability problem for Petri nets is decidable but extremely hard, namely Ackermann-complete. Nevertheless there are positive witnesses of reachability that are easily verifiable: given a Petri net A and two configurations $\mathbf{x}, \mathbf{y}$, a witness of the fact $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ is simply a word $w \in A^*$ such that $\mathbf{x} \xrightarrow{w} \mathbf{y}$. Verifying the validity of $\mathbf{x} \xrightarrow{w} \mathbf{y}$ is trivial; of course, the size of such a witness w is another issue. A negative witness, meaning a witness of the fact $\mathbf{x} \stackrel{A^*}{\not\longrightarrow} \mathbf{y}$, is a more involved question; a solution is provided by Theorem 9.1: we have $\mathbf{x} \stackrel{A^*}{\not\longrightarrow} \mathbf{y}$ iff there is an inductive semilinear $\mathbf{I}$ such that $\mathbf{x} \in \mathbf{I}$ and $\mathbf{y} \notin \mathbf{I}$. Verifying if a given semilinear set $\mathbf{I}$ is inductive and satisfies $\mathbf{x} \in \mathbf{I}$ and $\mathbf{y} \notin \mathbf{I}$ is much easier than solving the reachability problem (we can refer, e.g., to [Haa18] for complexity details); again, the size of such a witness $\mathbf{I}$ is another issue.

When looking for similar witnesses in the case of the semilinear home-space problem, the following lemma provides a solution in terms of semilinear inductive invariants.

Lemma 10.1. *Given a Petri net A of dimension d, and two semilinear sets $\mathbf{X}, \mathbf{H} \subseteq \mathbb{N}^d$, we have $\text{POST}^*_A(\mathbf{X}) \subseteq \text{PRE}^*_A(\mathbf{H})$ (i.e., $\mathbf{H}$ is a home-space for $\mathbf{X}$) iff there is an inductive semilinear set $\mathbf{I}$ such that $\text{POST}^*_A(\mathbf{X}) \subseteq \mathbf{I} \subseteq \text{PRE}^*_A(\mathbf{H})$.*

Proof. The "if" direction is trivial.

Now we show the "only if" direction. Let us assume that $\text{POST}^*_A(\mathbf{X}) \subseteq \text{PRE}^*_A(\mathbf{H})$, and let $\mathbf{C}$ be a semilinear non-reachability core for $\mathbf{H}$ guaranteed by Theorem 9.3 (while here we do not need $\mathbf{C}$ to be inductive); we thus have $\mathbf{C} \subseteq \overline{\text{PRE}^*_A(\mathbf{H})} \subseteq \text{PRE}^*_A(\mathbf{C})$. The assumption $\text{POST}^*_A(\mathbf{X}) \subseteq \text{PRE}^*_A(\mathbf{H})$ thus entails that $\mathbf{X} \cap \text{PRE}^*_A(\mathbf{C}) = \emptyset$, i.e., $\mathbf{X} \subseteq \overline{\text{PRE}^*_A(\mathbf{C})}$. Hence by Theorem 9.1 there is an inductive semilinear set $\mathbf{I}$ such that $\mathbf{X} \subseteq \mathbf{I} \subseteq \overline{\mathbf{C}}$. Since $\mathbf{I}$ is inductive (hence $\text{POST}^*_A(\mathbf{I}) = \mathbf{I}$ and $\text{PRE}^*_A(\bar{\mathbf{I}}) = \bar{\mathbf{I}}$), $\mathbf{X} \subseteq \mathbf{I}$ entails $\text{POST}^*_A(\mathbf{X}) \subseteq \mathbf{I}$, and $\mathbf{I} \subseteq \overline{\mathbf{C}}$, i.e. $\mathbf{C} \subseteq \bar{\mathbf{I}}$, entails $\textsc{pre}^*_A(\mathbf{C}) \subseteq \bar{\mathbf{I}}$, i.e. $\mathbf{I} \subseteq \overline{\textsc{pre}^*_A(\mathbf{C})}$; moreover, $\overline{\textsc{pre}^*_A(\mathbf{H})} \subseteq \textsc{pre}^*_A(\mathbf{C})$ entails $\overline{\textsc{pre}^*_A(\mathbf{C})} \subseteq \textsc{pre}^*_A(\mathbf{H})$, hence $\mathbf{I} \subseteq \textsc{pre}^*_A(\mathbf{H})$. □

Let us look at the question of verifying the validity of a witness $\mathbf{I}$ suggested by Lemma 10.1. Given a Petri net A and two semilinear sets $\mathbf{X}, \mathbf{H}$ of its configurations, for a given semilinear $\mathbf{I}$ we can "easily" (see [Haa18]) decide if $\mathbf{I}$ is inductive and subsumes $\mathbf{X}$ (which entails that $\textsc{post}^*_A(\mathbf{X}) \subseteq \mathbf{I}$). For deciding if $\mathbf{I} \subseteq \textsc{pre}^*_A(\mathbf{H})$ we also try to avoid solving the (semilinear) reachability problem; we achieve this by the following extension of (positive) witnesses $\mathbf{I}$.

Given a d-dimensional Petri net A, a *positive home-space witness for a pair* $(\mathbf{X}, \mathbf{H})$ of semilinear subsets of $\mathbb{N}^d$ is a pair

$$(\mathbf{I}, (w_1, w_2, \ldots, w_k))$$

(for some $k \in \mathbb{N}$) where $\mathbf{I} \subseteq \mathbb{N}^d$ is an inductive semilinear set that contains $\mathbf{X}$, and $w_1, w_2, \ldots, w_k$ are words from A^+ satisfying the following formula:

$$(\forall \mathbf{y} \in \mathbf{I})(\exists n_1, n_2, \ldots, n_k \in \mathbb{N})(\exists \mathbf{h} \in \mathbf{H})\, \mathbf{y} \xrightarrow{w_1^{n_1} w_2^{n_2} \cdots w_k^{n_k}} \mathbf{h}. \tag{10.1}$$

From [Ler13, Theorem XIII.2] we deduce that there exists a sequence $(w_1, w_2, \ldots, w_k)$ satisfying (10.1) precisely when $\mathbf{I} \subseteq \textsc{pre}^*_A(\mathbf{H})$ (since $\mathbf{I}$ and $\mathbf{H}$ are semilinear).

Corollary 10.2. *Given a Petri net A and two semilinear sets $\mathbf{X}, \mathbf{H}$ of its configurations, the set $\mathbf{H}$ is a home-space for $\mathbf{X}$ iff there is a positive home-space witness $(\mathbf{I}, (w_1, w_2, \ldots, w_k))$ for $(\mathbf{X}, \mathbf{H})$.*

We note that by compiling the relation $\mathbf{y} \xrightarrow{w_1^{n_1} w_2^{n_2} \cdots w_k^{n_k}} \mathbf{h}$ into a Presburger formula over the free variables $\mathbf{y}, n_1, \ldots, n_k, \mathbf{h}$ (see [FO97] for details), we deduce that formula (10.1) can be efficiently transformed into a Presburger formula. Since the complexity of Presburger arithmetic is at most 3-exponential [Opp78], checking if a tuple $(\mathbf{I}, (w_1, w_2, \ldots, w_k))$ is a positive home-space witness for a pair $(\mathbf{X}, \mathbf{H})$ is elementary (while the general reachability problem is nonelementary, namely Ackermann-complete).

Remark 10.3. A negative home-space witness for a pair $(\mathbf{X}, \mathbf{H})$ of semilinear sets of configurations, which exists precisely when $\mathbf{H}$ is not a home space for $\mathbf{X}$, can be defined as a tuple $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$, and $\mathbf{y} \not\xrightarrow{A^*} \mathbf{H}$. To avoid requirements to solve instances of the reachability problem, we can define such a negative witness as a tuple $(\mathbf{Y}, \mathbf{x}, w, \mathbf{y})$ where $\mathbf{Y}$ is an inductive semilinear set disjoint from $\mathbf{H}$, and we have $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, and $\mathbf{x} \xrightarrow{w} \mathbf{y}$.

Hence deciding if $\mathbf{H}$ is a home-space for $\mathbf{X}$ can be performed by simultaneously searching for a positive or a negative witness. This also yields the decidability of the semilinear home-space problem.

11. Concluding Remarks

There are various issues that can be elaborated on and added to the presented material. One such issue was mentioned in Remark 4.3, dealing with strengthening the lower bound.

We also leave open the complexity of deciding if an inductive semilinear set $\mathbf{C}$ is a non-reachability core for a semilinear set $\mathbf{H}$. Let us recall that this problem is equivalent to prove that $\mathbf{C}$ is disjoint from $\mathbf{H}$, and $\textsc{pre}^*_A(\mathbf{C} \cup \mathbf{H})$ is the full set of configurations $\mathbb{N}^d$. This last problem is related to the *semilinear universal problem* for Petri nets defined as follows:

Instance: a Petri net A, of dimension d, and a semilinear set $\mathbf{S} \subseteq \mathbb{N}^d$.
Question: is $\text{POST}^*_A(\mathbf{S}) = \mathbb{N}^d$?

The paper [JLS19] shows that the problem is decidable, and expspace-complete when the problem is restricted to singleton sets $\mathbf{S}$. In general, the complexity of the problem is still open.

Best and Esparza [BE16] consider the "existential" home-space problem that asks, given a Petri net A of dimension d and an initial configuration $\mathbf{x}$, if there exists a singleton home-space for $\{\mathbf{x}\}$; the main result of [BE16] shows that this existential problem is decidable. We can consider a related problem that asks, given A and $\mathbf{x}$, if there is a semilinear home-space included in $\text{POST}^*_A(\mathbf{x})$; currently we have no answer to the respective decidability question.

References

[Bak73] Henry G. Baker, Jr. Rabin's proof of the undecidability of the reachability set inclusion problem of vector addition systems. Massachusetts Institute of Technology, Project MAC, Computation Structures Group Memo 79, July 1973.

[BE16] Eike Best and Javier Esparza. Existence of home states in Petri nets is decidable. *Inf. Process. Lett.*, 116(6):423–427, 2016. `doi:10.1016/j.ipl.2016.01.011`.

[CLL+21] Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux, and Filip Mazowiecki. The reachability problem for Petri nets is not elementary. *J. ACM*, 68(1):7:1–7:28, 2021. `doi:10.1145/3422822`.

[CO21] Wojciech Czerwinski and Lukasz Orlikowski. Reachability in vector addition systems is Ackermann-complete. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1229–1240. IEEE, 2021. `doi:10.1109/FOCS52979.2021.00120`.

[dFEJ89] David de Frutos Escrig and Colette Johnen. Decidability of home space property. Univ. de Paris-Sud, Centre d'Orsay, Laboratoire de Recherche en Informatique, LRI-503, July 1989.

[EN94] Javier Esparza and Mogens Nielsen. Decidability issues for Petri nets - a survey. *Bulletin of the European Association for Theoretical Computer Science*, 52:245–262, 1994.

[FO97] Laurent Fribourg and Hans Olsén. Proving safety properties of infinite state systems by compilation into Presburger arithmetic. In Antoni W. Mazurkiewicz and Józef Winkowski, editors, *CONCUR '97: Concurrency Theory, 8th International Conference, Warsaw, Poland, July 1-4, 1997, Proceedings*, volume 1243 of *Lecture Notes in Computer Science*, pages 213–227. Springer, 1997. `doi:10.1007/3-540-63141-0\_15`.

[GS64] Seymour Ginsburg and Edwin H. Spanier. Bounded Algol-like languages. *Transactions of the American Mathematical Society*, 113(2):333–368, 1964. `doi:10.2307/1994067`.

[GS66] Seymour Ginsburg and Edwin H. Spanier. Semigroups, Presburger formulas and languages. *Pacific Journal of Mathematics*, 16(2):285–296, 1966. `doi:10.2140/pjm.1966.16.285`.

[Haa18] Christoph Haase. A survival guide to Presburger arithmetic. *ACM SIGLOG News*, 5(3):67–82, 2018. `doi:10.1145/3242953.3242964`.

[Hac75] Michel Hack. *Decidability questions for Petri nets.* PhD thesis, MIT, 1975. URL: `http://publications.csail.mit.edu/lcs/pubs/pdf/MIT-LCS-TR-161.pdf`.

[Hac76] Michel Hack. The equality problem for vector addition systems is undecidable. *Theor. Comput. Sci.*, 2(1):77–95, 1976. `doi:10.1016/0304-3975(76)90008-6`.

[Jan95] Petr Jančar. Undecidability of bisimilarity for Petri nets and some related problems. *Theor. Comput. Sci.*, 148(2):281–301, 1995. `doi:10.1016/0304-3975(95)00037-W`.

[JLS19] Petr Jančar, Jérôme Leroux, and Grégoire Sutre. Co-finiteness and co-emptiness of reachability sets in vector addition systems with states. *Fundam. Informaticae*, 169(1-2):123–150, 2019. `doi:10.3233/FI-2019-1841`.

[Las22] Slawomir Lasota. Improved Ackermannian lower bound for the Petri nets reachability problem. In Petra Berenbrink and Benjamin Monmege, editors, *39th International Symposium on Theoretical Aspects of Computer Science, STACS 2022, March 15-18, 2022, Marseille, France (Virtual*

Conference), volume 219 of *LIPIcs*, pages 46:1–46:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.STACS.2022.46.

[Ler10] Jérôme Leroux. The general vector addition system reachability problem by Presburger inductive invariants. *Log. Methods Comput. Sci.*, 6(3), 2010. doi:10.2168/LMCS-6(3:22)2010.

[Ler12] Jérôme Leroux. Vector addition systems reachability problem (A simpler solution). In Andrei Voronkov, editor, *Turing-100 - The Alan Turing Centenary, Manchester, UK, June 22-25, 2012*, volume 10 of *EPiC Series in Computing*, pages 214–228. EasyChair, 2012. doi:10.29007/bnx2.

[Ler13] Jérôme Leroux. Presburger vector addition systems. In *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013*, pages 23–32. IEEE Computer Society, 2013. doi:10.1109/LICS.2013.7.

[Ler21] Jérôme Leroux. The reachability problem for Petri nets is not primitive recursive. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1241–1252. IEEE, 2021. doi:10.1109/FOCS52979.2021.00121.

[Ler24] Jérôme Leroux. Ackermannian completion of separators. In Naoki Kobayashi and James Worrell, editors, *Foundations of Software Science and Computation Structures - 27th International Conference, FoSSaCS 2024, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2024, Luxembourg City, Luxembourg, April 6-11, 2024, Proceedings, Part I*, volume 14574 of *Lecture Notes in Computer Science*, pages 3–10. Springer, 2024. doi:10.1007/978-3-031-57228-9_1.

[LS19] Jérôme Leroux and Sylvain Schmitz. Reachability in vector addition systems is primitive-recursive in fixed dimension. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–13. IEEE, 2019. doi:10.1109/LICS.2019.8785796.

[May84] Ernst W. Mayr. An algorithm for the general Petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984. doi:10.1137/0213029.

[MM81] Ernst W. Mayr and Albert R. Meyer. The complexity of the finite containment problem for Petri nets. *J. ACM*, 28(3):561–576, 1981. doi:10.1145/322261.322271.

[Opp78] Derek C. Oppen. A 2ˆ2ˆ2ˆpn upper bound on the complexity of Presburger arithmetic. *J. Comput. Syst. Sci.*, 16(3):323–332, 1978. doi:10.1016/0022-0000(78)90021-1.

[Sch16a] Sylvain Schmitz. Complexity hierarchies beyond elementary. *TOCT*, 8(1):3:1–3:36, 2016. URL: http://doi.acm.org/10.1145/2858784.

[Sch16b] Sylvain Schmitz. The complexity of reachability in vector addition systems. *SIGLOG News*, 3(1):4–21, 2016. URL: https://dl.acm.org/citation.cfm?id=2893585.

[VJ84] Rüdiger Valk and Matthias Jantzen. The residue of vector sets with applications to decidability problems in Petri nets. In Grzegorz Rozenberg, Hartmann J. Genrich, and Gérard Roucairol, editors, *Advances in Petri Nets 1984*, volume 188 of *Lecture Notes in Computer Science*, pages 234–258. Springer, 1984. doi:10.1007/3-540-15204-0_14.

[YC09] Hsu-Chun Yen and Chien-Liang Chen. On minimal elements of upward-closed sets. *Theor. Comput. Sci.*, 410(24-25):2442–2452, 2009. doi:10.1016/j.tcs.2009.02.036.