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INTEGRATION IN CONES

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Abstract. Measurable cones, with linear and measurable functions as morphisms, are a model of intuitionistic linear logic and of call-by-name probabilistic PCF which accommodates "continuous data types" such as the real line. So far however, they lacked a major feature to make them a model of more general probabilistic programming languages (notably call-by-value and call-by-push-value languages): a theory of integration for functions whose codomain is a cone, which is the key ingredient for interpreting the sampling programming primitives. The goal of this paper is to develop such a theory: our definition of integrals is an adaptation to cones of Pettis integrals in topological vector spaces. We prove that such integrable cones, with integral-preserving linear maps as morphisms, form a model of Linear Logic for which we develop two exponential comonads: the first based on a notion of stable and measurable functions introduced in earlier work and the second based on a new notion of integrable analytic function on cones.

Introduction

There are several approaches in the denotational semantics of functional probabilistic programming languages that we can summarize as follows:

- quasi-Borel spaces (QBSs) [VKS19] which are, roughly speaking, separated presheaves on the cartesian category of measurable spaces and measurable functions (or on a full cartesian sub-category thereof), and the considered category of QBSs must be given together with a well behaved probability monad (*à la* Giry);
- probabilistic games [DH00] which are similar to deterministic games apart that now strategies are probability distributions on plays;
- models based on categories of domains, possibly equipped with a probabilistic monad, and where morphisms are Scott continuous functions;
- probabilistic coherence spaces [DE11] (PCSs) which are a refinement of the relational model of Linear Logic (LL). In the PCS model, an object is a set equipped with a collection of "valuations", which are functions[1] from this set to $\mathbb{R}_{\geq 0}$, and a morphism is a linear functions on these valuations, or analytic functions in the CCC used for interpreting the programming languages. This approach can be understood as extending to higher

[1]For objects corresponding to ground types, these valuations are the subprobability distributions.

types the basic idea of [Koz81] which is to interpret programs as probability distribution transformers.

Main motivation. Modern probabilistic programming languages deal with probability distributions on continuous data-types such as the real line, and PCSs are not able to represent such types: PCSs are fundamentally of a discrete nature. On the other hand, QBS-based models accept continuous data-types by construction, and give rise to cartesian closed categories for a very general reason — they are essentially categories of presheaves —. This also means that these models are not very informative about morphisms: they are essentially only required to satisfy a hereditary measurability condition and, accordingly, they have in general no clear underlying linear structure (in the sense of the categorical semantics of Linear Logic). The benefit of such a linear structure is that it allows to take into account in a modular way the various options in the design of a programming language, and in particular the choice of operational semantics (call-by-name or call-by-value, typically). Also the linear structure provides tools — versions of the Taylor expansion of analytic functions — allowing to analyze the resource usage of programs.

In contrast to QBSs, PCSs are natively a model of LL whose associated cartesian closed category can be used as a model of probabilistic functional languages. In this CCC the morphisms are quite regular: they are analytic functions described by generalized power series with nonnegative coefficients. This feature allowed the first author to prove, for instance, two full abstraction results [EPT18a, ET19] wrt. the PCS semantics.

The main purpose of the model presented in this paper is to extend to the continuous probability setting these two main features of PCSs: the model has a linear underlying structure and the programs are interpreted as functions which are analytic in some generalized sense. One essential feature of our semantics is that a functional program M of type ρ (the type of real numbers) with only one variable x of type ρ will be interpreted as a function f from the set $\mathcal{R}$ of subprobability measures on $\mathbb{R}$ to $\mathcal{R}$. With this intuition in mind, it is easier to understand what linearity can mean for such a function (very roughly: commutation with existing linear combinations of measures), and also what analyticity can mean: the function g which maps a subprobability distribution μ on $\mathbb{R}$ to $\mu * \mu$ (convolution product of measures) is clearly not linear, but it is polynomial of degree 2. More precisely, the addition program on real numbers will typically be represented as a function $a : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ which will be bilinear: it maps a pair $(\mu, \nu) \in \mathcal{R}^2$ of subprobability measures to $\alpha_*(\mu \times \nu)$ where $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the addition function, $\mu \times \nu$ is the usual product of μ and ν, which is a subprobability measure on $\mathbb{R} \times \mathbb{R}$, and α_* is the push-forward operation on measures associated with the measurable function α. The function g is polynomial of degree 2 because $g(\mu) = a(\mu, \mu)$.

Types as cones. In recent works [EPT18b, Ehr20] we have developed such a continuous extension of the PCS semantics, using quite a suitable notion of *positive cone* introduced by Selinger in [Sel04] (we will often drop the adjective "positive"). Cones are similar to real Banach spaces, with the difference that, in a cone, "everything is positive"; for instance the coefficients are taken in $\mathbb{R}_{\geq 0}$ and not in $\mathbb{R}$ and $x + y = 0$ is possible only if $x = y = 0$. For that reason cones are naturally ordered and are required to satisfy a completeness property expressed *à la* Scott, in terms of the norm and of this order relation. This notion of completeness is very different from the standard Cauchy-completeness of ordinary Banach spaces. It has the benefit of making the interpretation of recursive programs quite straightforward (no need for contractivity assumptions).

In this setting, the ground type ρ of real numbers of our programming language is interpreted as the set $\mathcal{R}$ of finite nonnegative measures on the real line equipped with its Borel σ-algebra, this set $\mathcal{R}$ has indeed an obvious structure of cone. Cones are naturally equipped with a notion of linear morphisms, which are also assumed to be Scott continuous, and with a notion of non-linear morphism introduced in [EPT18b], called stable functions and characterized by a *total monotonicity* condition (plus Scott continuity) which allow to define a cartesian closed category where fixpoint operators are available at all types. With these morphisms, cones are a conservative extension of the category of PCSs and analytic functions as shown in [Cru18].

Integration and sampling. The most essential feature of a probabilistic programming language is the possibility of *sampling* a value according to a given probability distribution. In our semantical setting and in the presence of continuous data-types this requires some form of integration and therefore the morphisms (here, the linear or the stable functions between cones) must satisfy a suitable measurability condition. Consider indeed a functional program M such that $x : \rho \vdash M : \sigma$ for some type σ, where we recall that ρ is the type of real numbers, and a program N such that $\vdash N : \rho$. Then N will be interpreted as an element μ of $\mathcal{R}$ (a subprobability measure on $\mathbb{R}$ actually) and M as an analytic function $g : \mathcal{R} \to P$ where P is the cone interpreting the type σ. Then we typically would like to write a program $R = \mathtt{sample}(x, N, M)$ which should satisfy $\vdash R : \sigma$. The semantics of R should then be

$$\int g(\boldsymbol{\delta}(r))\mu(dr)$$

because the Dirac probability measure at $r \in \mathbb{R}$, $\boldsymbol{\delta}(r) \in \mathcal{R}$, is the representation in our semantics of the real number r. For instance if $M = x + x$ (so that $A = \rho$) then the semantics of $N\,[M/x] = M + M$ is $g(\mu) = \mu * \mu$ and the semantics $\nu \in \mathcal{R}$ of $R = \mathtt{sample}(x, N, x + x)$ is

$$\int \boldsymbol{\delta}(2r)\mu(dr) = \beta_*(\mu)$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is defined by $\beta(r) = 2r$. In Section 9.1, we will understand that this sampling operation is simply a `let` construct, exactly as in the discrete PCS setting of [ET19]. See Example 9.9 for a more developed explanation.

In [EPT18b, Ehr20] the cones were accordingly equipped with a *measurability structure* defined in reference to a collection of basic measurable spaces (such a collection can be simply $\{\mathbb{R}\}$, what we assume in this introduction for simplicity). Given a cone P equipped with such a measurability structure $\mathcal{M}$ it is then possible to define a class of bounded[2] functions $\mathbb{R} \to P$ that we call the *measurable paths* of P. And then a (linear or stable) function $P \to Q$ is *measurable* from $(P, \mathcal{M})$ to $(Q, \mathcal{N})$ if its pre-composition with each $\mathcal{M}$-measurable path of P gives a $\mathcal{N}$-measurable path of Q. Equipped with their measurability paths, these measurable cones (more precisely, their unit balls) can be considered as QBSs, and the condition above of measurable path preservation is exactly the same as the definition of a morphism of QBSs (however notions such as linearity, stability or analyticity, which are crucial for us, do not arise naturally in the framework of QBSs).

[2]With respect to the norm of P.

These measurable cones were sufficient in [EPT18b] to allow sampling over the type ρ in a probabilistic extension of PCF because all types in such a language can be written $\sigma_1 \Rightarrow \cdots \Rightarrow \sigma_n \Rightarrow \rho$ and hence integrability for paths valued in such a type boils down to the integrability of $\mathcal{R}$-valued paths (with additional parameters in $\sigma_1, \dots, \sigma_n$) which is possible by our measurability assumptions. But if we want to interpret a call-by-value (or even call-by-push-value) language then we face the problem of integrating functions valued in more general cones such as for instance $!\mathcal{R}$ (in the sense of LL, $\mathcal{R}$ being the cone of finite measures on $\mathbb{R}$). So we must deal with cones where measurable paths can be integrated. Fortunately it turns out that, thanks to the properties of the measurability structure $\mathcal{M}$ of a cone P, it is easy to define the integral of a P-valued path $\gamma : \mathbb{R} \to P$ wrt. a finite measure μ on $\mathbb{R}$: it is an $x \in P$ such that, for each measurability test m on P, the real number $m(x)$ is equal to the standard Lebesgue integral $\int m(\gamma(r))\mu(dr)$ which is well defined and belongs to $\mathbb{R}_{\geq 0}$ since $m \circ \gamma$ is measurable and bounded, and μ is finite. And when such an x exists it is unique by our assumptions that the measurability tests associated with a cone separate it. So we can define a cone to be integrable if such integrals always exist, whatever be the choices of γ and μ.

In that way we are able to define a category of *integrable cones* and *linear and integrable maps*, that is, linear and measurable maps of cones which moreover commute with all integrals, a property which can be understood as a strong form of linearity. Such linear maps will sometimes be called integrable. It is rather easy to prove that this locally small category is complete, has a cogenerator and is well-powered so that we know by the special adjoint functor theorem that each continuous functor from this category to any other locally small category has a left adjoint. This allows first to equip our category with a tensor product: given two integrable cones B, C (we keep the measurability structures implicit), we can form the integrable cone $B \multimap C$ whose elements are the linear integrable maps from B to C, addition is defined pointwise and the norm is defined by $\|f\| = \sup_{\|x\| \leq 1} \|f(x)\|$. Then the functor $B \multimap \_$ is easily seen to preserve all limits and hence has a left adjoint $\_ \otimes B$. And we can prove that one defines in that way a tensor product $\_ \otimes \_$ which makes our category symmetric monoidal closed[3].

There is a faithful functor from the category of measurable spaces and sub-probability kernels to the category of measurable cones which maps a measurable space X to the cone $\mathsf{FMeas}(X)$ of finite non-negative measures on X. As already explained in [Geo21] (in a slightly different context) the integral preservation property that we enforce on linear morphisms on cones has the major benefit of making this functor not only faithful but also full.

Nonlinear functions: stability and analyticity. In a second part of the paper we define two cartesian closed categories of integrable cones and non-linear morphisms which are Scott continuous and measurable. We also develop the associated notions of exponential comonad (in the sense of the semantics of LL, see for instance [Mel09]) applying the special adjoint functor theorem to the continuous inclusion functor from the category of integrable cones and integrable linear functions to the non-linear category.

- In the first case the non-linear morphisms between integrable cones are the *measurable and stable functions* that were introduced in [EPT18b]. These morphisms are Scott continuous functions satisfying a "total monotonicity" condition, which is an iterated form

[3] In [Ehr20] we used the fact that PCSs are dense in cones to prove this result but this is actually not necessary, thanks to a slightly stronger assumption on the measurability structure of cones.

of monotonicity (plus preservation of measurable paths by post-composition of course). A peculiarity of this construction is that apparently no integral preservation condition is imposed on these morphisms[4].

- This fact can be considered as an issue for which we propose a solution by defining a notion of *analytic morphism* as the bounded limits of polynomial functions which are themselves described as finite sums of functions of shape $x \mapsto f(x, \dots, x)$ where f is an n-linear symmetric integrable and measurable function. These analytic functions are of course stable and measurable but not all stable and measurable functions are analytic because this latter notion is based on integrable linearity[5].

For each measurable space X, we show that for both exponential comonads $!\_$ described above, the integrable cone $\mathsf{FMeas}(X)$ has a canonical structure of coalgebra, which means that this cone can be considered as a *data-type* in the sense of [Kri90] or in the sense of the *positive formulas* of Polarized Linear Logic [Gir91, LR03, Ehr16]. It is very important to observe that this construction uses integration in a crucial way: as already explained above, the associated `let` operator can also be understood as a sampling construct, it is interpreted using this coalgebra structure which is defined using integration in the integrable cone $!\mathsf{FMeas}(X)$. Combined with the fact that the Kleisli categories of these comonads are cartesian closed and ω-cpo enriched, this means that integrable cones provide a semantics for a large number of functional programming languages with continuous data types and basic probability features.

Convex QBSs. Besides measurable cones, one major source of inspiration of this work is [Geo21], which introduces the notion of *convex QBS*, which are a particular class of algebras on the Giry-Panangaden monad of sub-probability measures in the category of QBSs. In other words, a convex QBS is a QBS equipped with an abstract, algebraic operation of "integration" from which all elementary operations of a cone can be derived. As in the present work, linear morphisms are required to commute with integration, *i.e.* to be morphisms of algebras on the sub-probability monad. The main differences with respect to the present setting are, first, that linear negation in convex QBSs is involutive (because they are defined as dual pairs) whereas we strongly conjecture that this is not true for integrable cones; and second, that measurability in convex QBSs is axiomatized in the QBS manner, by equipping each object with a collection of "measurable paths" from $\mathbb{R}$ to this object, satisfying sheaf-like conditions[6]. In integrable cones, following [EPT18b], measurability is axiomatized by means of a "measurability structure", *i.e.* a collection of "test functions" that map a real number and an element of the cone to a non-negative real number, measurably with respect to the first variable, and linearly and continuously with respect to the second. In turn, this measurability structure induces a class of measurable maps from $\mathbb{R}$ to the cone, turning the latter into a QBS: a map from $\mathbb{R}$ to the cone is measurable if and only if its composition with each test function is a measurable map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}_{\geq 0}$ (by composition, we mean that the second argument of the test function is replaced by the map, and the first argument is left alone). A map between integrable cones is measurable when it is a morphism of

[4]Notice that it is not possible to expect that non-linear morphisms will preserve integrals but one could expect that they satisfy a weakened version of this condition.

[5]An n-ary integrable multilinear function is a function with n-arguments which is linear and integrable in each parameter.

[6]In fact, these two differences are closely linked: negation in convex QBSs can be involutive precisely because their measurability is axiomatized in the QBS manner, without restrictions on the QBS-structure.

QBSs. This means that, from the point of view of measurability alone (*i.e.* if we forget the algebraic structure), integrable cones can be seen as a particular class of QBSs whose QBS structure can be defined as the "dual" of a set of test functions. This restriction has the pleasant consequence of making the theory of measurability and integration in cones quite easy, reducing it to standard Lebesgue integration by means of post-composition with tests.

Similarly defined integrals of functions ranging in topological vector spaces separated by their topological duals have been introduced by Pettis a long time ago [Pet38], and are also known as *weak integrals* or *Gelfand-Pettis integrals*. The transposition of this definition in our positive cone setting turns out to be quite suitable, thanks to its compatibility with categorical limits.

Contents

1. Preliminaries

1.1. **Notations.** In the whole paper, we say that a set is countable if it is finite or has the same cardinality as $\mathbb{N}$.

We use notations borrowed from the lambda-calculus to denote mathematical functions: if e is a mathematical expression for an element of B depending on a parameter $x \in A$, we use $\boldsymbol{\lambda} x \in A \cdot e$ for the corresponding function $A \to B$.

1.1.1. *Categorical notations borrowed from* LL. We also borrow notations from intuitionistic LL for denoting objects of our categories and construction on these objects. These notations are quite coherent although they somehow depart from the categorical traditions. In what follows, the word "linear" has to be understood in an intuitive way: as explained in the Introduction, our constructions are based on notions of linear morphisms which will be defined precisely later.

- We use $E \multimap F$ to denote a space of linear morphisms from E to F;
- we use $G \otimes E$ to denote the tensor product of G and E, such that a linear morphism from $G \otimes E$ to F is the same thing as linear morphism from G to $E \multimap F$;
- we use 1 for the unit of $\otimes$ (instead of the more traditional I);
- we use $\&$ (instead of the more traditional $\times$ that we use for denoting the standard cartesian product of sets) for the categorical product (aka. direct product) and $\top$ (instead of the more traditional 1) for the associated unit, which is the terminal object;
- we use $\oplus$ for the coproduct (aka. direct sum) and 0 for the associated unit which is the initial object;
- we use $!E$ for the linear logic exponential, which is not a symmetric tensor algebra but rather a symmetric tensor coalgebra.

Even if in our categories 0 and $\top$ are the same object (just as in the category of vector spaces), we prefer to keep distinct notations because we have in mind a refinement of our model where these objects are distinct, and we use the two notations depending on the context. Similarly, in some context where 1 is considered as a dualizing object, we denote it as $\perp$, again in accordance with the tradition of LL.

1.1.2. *Measure theory and other notations.* We use **Meas** for the category of measurable spaces and measurable functions.

If X and Y are measurable spaces, recall that a *kernel* from X to Y is a map $\kappa : X \times \sigma_Y \to \overline{\mathbb{R}_{\geq 0}}$ (where σ_Y denotes the σ-algebra of Y) such that:

- for all $x \in X$, the map $\boldsymbol{\lambda} U \cdot \kappa(x, U)$ is a measure on Y,
- for all $U \in \sigma_Y$, the map $\boldsymbol{\lambda} x \cdot \kappa(x, U)$ is measurable.

We write $\kappa : X \rightsquigarrow Y$ for "κ is a kernel from X to Y". We say that κ is *bounded* if the set $\{\kappa(x, Y) \mid x \in X\}$ has a finite upper bound.

If X is a measurable space, μ a non-negative measure on X and $f : X \to \mathbb{R}_{\geq 0}$ a non-negative measurable function, we use

$$\int f(r)\mu(dr)$$

for the integral, which belongs to $\overline{\mathbb{R}_{\geq 0}}$, rather than the more usual $\int f(r)d\mu(r)$. The reason of this choice is that it is much more convenient when the measure arises as the image of a kernel $\kappa : Y \rightsquigarrow X$ in which case we can use the non ambiguous notation $\int f(r)\kappa(s, dr)$. This notation is also intuitively compelling if we see dr as representing metaphorically an "infinitesimal" measurable subset of X.

If a is an element and $n \in \mathbb{N}$ we use $\overline{a}^n$ for the n-tuple $(a, \ldots, a)$.

We use $\mathbb{N}^+$ for $\mathbb{N} \setminus \{0\}$.

If $n \in \mathbb{N}$ we set $[n] = \{1, \ldots, n\}$.

If I is a set, we use $\mathcal{M}_{\mathrm{fin}}(I)$ for the set of all finite multisets of elements of I, which are the functions $m : I \to \mathbb{N}$ such that the set $\mathsf{supp}(m) = \{i \in I \mid m(i) \neq 0\}$ is finite.

1.2. **Categories.** The following is an easy consequence of the Yoneda lemma which gives a simple tool for proving that two functors are naturally isomorphic by checking that two associated indexed classes of homsets are in natural bijective correspondence.

Lemma 1.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories,* $F, G : \mathbf{C} \to \mathbf{D}$ *be functors and let* $\psi_{C,D} : \mathbf{D}(F(C), D) \to \mathbf{D}(G(C), D)$ *be a natural bijection. Then the family of morphisms* $\eta_C = \psi_{C,F(C)}(\mathsf{Id}_{F(C)}) \in \mathbf{D}(G(C), F(C))$ *is a natural isomorphism whose inverse is the family of morphisms* $\theta_C = {\psi_{C,G(C)}}^{-1}(\mathsf{Id}_{G(C)}) \in \mathbf{D}(F(C), G(C))$.

2. Cones

Cones are the basic objects of our model. They are algebraic structures with numerical features (the non-negative real half line acts on them) as well as domain theoretic features. The algebraic and numerical aspects will be essential to account for the probabilistic aspects of the model and the domain theoretic aspects will be crucial to give to our model a suitable computational expressive power, allowing to interpret arbitrary recursive definitions.

The purpose of the present section is to introduce this basic algebraic and numerical infrastructure and give its basic properties. Our definition of cones is borrowed without major modifications from [Sel04]. As explained in that paper, they are close to the domain theoretic treatment of positive cones developed in [Tix98], with the difference that Selinger's cones are equipped with a norm and that their order-theoretic completeness is deeply related to this norm.

The notion of positive cone itself is pervasive in functional analysis and it would be a very difficult task to describe its genealogy and many avatars in the literature. Our (and Selinger's) cones seem very similar to normal cones in Banach spaces, and it seems actually possible, given one or our cones P, to define an enveloping Banach space of which P is a normal positive cone. However, the linear morphisms that we consider between our cones are assumed to be continuous in a domain theoretic sense, and this seems to be a stronger property than continuity wrt. the topology induced by the norm (when the linear morphism is extended to the associated Banach space). This difference in the definition of morphisms seems to be a major drift wrt. the standard uses of cones in analysis.

Both Selinger and Tix assume that their cones are continuous (in the domain-theoretic sense) which makes it possible to prove a separation property similar to a Hahn-Banach theorem. This is an assumption that we cannot afford here because we will need our category of cones and linear maps to be complete and continuity does not seem to be preserved by equalizers in general. We will see that dropping this assumption is essentially harmless in the setting of this paper: our measurability structures of Section 3 will provide us the required separation properties.

Another difference between Selinger's cones and ours is that we do not assume order-theoretic completeness wrt. arbitrary norm-bounded directed sets, as it is usual in domain theory, but only wrt. norm-bounded ω-increasing sequences (or, equivalently, to countable directed sets). This assumption is sufficient for computing arbitrary fixpoints, see Section 9.2, and cannot be significantly strengthened because of our constant use of the monotone convergence theorem.

Measurability notions for cones will be necessary as well to deal with probabilities on arbitrary measurable spaces such as the real line; this will be done in Section 3.

2.1. **Basic definitions.** A *precone* is a $\mathbb{R}_{\geq 0}$-semimodule P which satisfies

(**Cancel**) $\forall x_1, x_2, x \in P\ x_1 + x = x_2 + x \Rightarrow x_1 = x_2$

(**Pos**) $\forall x_1, x_2 \in P\ x_1 + x_2 = 0 \Rightarrow x_1 = 0$

Given $x_1, x_2 \in P$, we stipulate that $x_1 \leq x_2$ if $\exists x \in P\ x_2 = x_1 + x$. By (**Cancel**) and (**Pos**) this defines a partial order relation on P: *the cone order* of P. Moreover when $x_1 \leq x_2$ there is exactly one $x \in P$ such that $x_2 = x_1 + x$, that we denote as $x_2 - x_1$. Notice that this subtraction between elements of P is only partially defined, and that it satisfies all the usual laws of subtraction.

A *cone* is a precone P equipped with a function $\|\_\|_P : P \to \mathbb{R}_{\geq 0}$ (or simply $\|\_\|$), called the *norm of* P, which satisfies the following properties.

(**Normh**) $\forall \lambda \in \mathbb{R}_{\geq 0} \forall x \in P\ \ \|\lambda x\| = \lambda \|x\|$

(**Normz**) $\forall x \in P\ \ \|x\| = 0 \Rightarrow x = 0$

(**Normt**) $\forall x_1, x_2 \in P\ \ \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

(**Normp**) $\forall x_1, x_2 \in P\ \ \|x_1\| \leq \|x_1 + x_2\|$ or, equivalently $\forall x_1, x_2 \in P\ x_1 \leq x_2 \Rightarrow \|x_1\| \leq \|x_2\|$.

Condition (**Normp**) expresses the positiveness of P and implies (**Pos**), but it is seems more sensible to require (**Pos**) at the beginning because of its purely algebraic nature, and because this allows to define the useful notion of precone.

(**Normc**) Each sequence $(x_n)_{n\in\mathbb{N}}$ of elements of P which is increasing[7] (for the cone order relation of P) and satisfies $\forall n \in \mathbb{N}\ \|x_n\| \leq 1$ has a lub $x = \sup_{n\in\mathbb{N}} x_n$ in P which satisfies $\|x\| \leq 1$.

A subset A of P is

- *bounded* if $\exists\lambda \in \mathbb{R}_{\geq 0} \forall x \in A\ \|x\| \leq \lambda$. We set $\mathcal{B}P = \{x \in P \mid \|x\| \leq 1\}$ and call this set the *unit ball* of P (*unit tip* might be more appropriate but seems less standard). With this notation, A is bounded iff $\exists\lambda \in \mathbb{R}_{\geq 0}\ A \subseteq \lambda\mathcal{B}P$.
- $\leq$-*bounded* if there is $y \in P$ such that $\forall x \in A\ x \leq y$. This implies that A is bounded (but the converse is not true).
- ω-*closed* if $\forall x_1, x_2 \in P\ (x_1 \leq x_2$ and $x_2 \in A) \Rightarrow x_1 \in A$ and for each bounded increasing sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A one has $\sup_{n\in\mathbb{N}} x_n \in A$.

Notice that P and $\mathcal{B}P$ are ω-closed subsets of P.

Definition 2.1. Let S be a set and P be a cone. A function $f : S \to P$ is *bounded* if $f(S)$ is bounded in P.

Definition 2.2. Let P and Q be cones, let $A \subseteq P$ be ω-closed and let $f : A \to Q$ be a function.

- f is *increasing* if $\forall x_1, x_2 \in A\ x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. Notice that if f is increasing and $(x_n)_{n\in\mathbb{N}}$ is a bounded and increasing sequence in A then the sequence $(f(x_n))_{n\in\mathbb{N}}$ is bounded by $\|f(\sup_{n\in\mathbb{N}} x_n)\|$ in Q, by (**Normp**) and monotonicity of f.
- f is ω-*continuous*, or simply continuous (no other notion of continuity will be considered in this paper), if f is monotonic and for each bounded increasing sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A, one has $f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n)$, that is $f(\sup_{n\in\mathbb{N}} x_n) \leq \sup_{n\in\mathbb{N}} f(x_n)$ since the converse holds by monotonicity of f.
- f is *linear* if $A = P$, $f(\lambda x) = \lambda f(x)$ and $f(x_1 + x_2) = f(x_1) + f(x_2)$, for all $\lambda \in \mathbb{R}_{\geq 0}$ and $x, x_1, x_2 \in P$. Notice that if f is linear then f is increasing because, given $x_1, x_2 \in P$, if $x_1 \leq x_2$ then $f(x_2 - x_1) + f(x_1) = f(x_2)$, and moreover we have $f(x_2 - x_1) = f(x_2) - f(x_1)$. One says that f is linear and continuous if it is linear and ω-continuous.
- If $f : P \to Q$ is linear, one says that f is *bounded* if its restriction to $\mathcal{B}P$ is a bounded function.

One major interest of this kind of continuity is the fact that separate continuity implies continuity (see Lemma 2.19), a property that usual topological continuity does not satisfy.

There are plenty of examples of cones:

Example 2.3. Let X be a measurable space. The space of all bounded measurable maps from X to $\mathbb{R}_{\geq 0}$ forms a cone: the operations are defined pointwise, and the norm is given by the supremum.

Example 2.4. Section 2.2 describes the cone of finite measures on a measurable space which provides one of the main motivations for this work. All the objects of the probabilistic coherence space model of LL can be seen as cones; the interested reader can have a look at the beginning of Section 10 to see more about them. Here are some instances of this particular class.

[7] A reader acquainted with domain-theory might expect here a stronger completeness requirement using arbitrary directed sets instead of ω-chains (or, equivalently, countable directed sets). It is absolutely crucial to use this restricted definition because we will often have to use the monotone convergence theorem to prove this property, and this theorem is valid only for countable families.

- The cone N whose elements are the $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} u_n < \infty$, with algebraic operations defined pointwise, and $\|u\| = \sum_{n\in\mathbb{N}} u_n$. This is also a special case of the cones of Section 2.2 where the measurable space is $\mathbb{N}$ with the discrete σ-algebra.
- The dual of N (in the sense of Definition 2.14) which can be described as the cone $N^{\perp}$ of bounded families $u \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ with norm defined by $\|u\| = \sup_{n\in\mathbb{N}} u_n$.
- The cone $P \Rightarrow 1$ where $P \in \{\mathsf{N}, \mathsf{N}^{\perp}\}$, whose elements are the families $t \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(\mathsf{N})}$ such that there is $\lambda \in \mathbb{R}_{\geq 0}$ such that
$$\forall u \in P \quad \|u\| \leq 1 \Rightarrow \sum_{m\in\mathcal{M}_{\mathrm{fin}}(\mathbb{N})} t_m u^m \leq \lambda$$
where $u^m = \prod_{n\in\mathbb{N}} u_n^{m(n)}$, with algebraic operations defined componentwise and norm defined by $\|t\| = \sup_{u\in\mathcal{B}P} \widehat{t}(u)$ where $\widehat{t}(u) = \sum_{m\in\mathcal{M}_{\mathrm{fin}}(\mathbb{N})} t_m u^m$. In both cases $P = \mathsf{N}$ and $P = \mathsf{N}^{\perp}$, $\widehat{t}$ can be seen as a bounded function $\mathcal{B}P \to \mathbb{R}_{\geq 0}$. The set of these families t equipped with that norm is easily seen to be a cone. An element of $\mathsf{N}^{\perp} \Rightarrow 1$ can be seen as a power series with infinitely many parameters, defining a function $\mathcal{B}\mathsf{N}^{\perp} = [0,1]^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$. An element of $\mathsf{N} \Rightarrow 1$ is an analytic function on the subprobability distributions on the natural numbers, we give an example of such a function. Given two such distributions u and v, we can define their convolution product $u * v \in \mathsf{N}$ by $(u * v)_n = \sum_{i=0}^{n} u_i v_{n-i}$ which is again a subprobability distribution (the push-forward of addition). If u (resp. v) is the probability distribution of a $\mathbb{N}$-valued random variable X (resp. Y) and X and Y are independent, then $u * v$ is the probability distribution associated with $X + Y$. Given a family $(a_n \in \mathbb{R}_{\geq 0})_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} a_n = 1$, a non trivial example of element of $\mathsf{N} \Rightarrow 1$ is t given by
$$\widehat{t}(u) = \sum_{n=0}^{\infty} a_n \overbrace{u * \cdots * u}^{n} .$$
whose coefficients are
$$t_m = \frac{\#m!}{m!}\, a_{\#m+1}$$
where $\#m$ is the number of elements of m (taking multiplicities into account) and $m! = \prod_{i\in\mathbb{N}} m(i)!$. Such power series are typical examples of the analytic functions that we will meet in Section 8 in the general setting of integrable cones.

Remark 2.5. We can already observe that our analytic functions will be defined in general only on the unit ball of their domain. The reason is that the analytic functions which interpret programs will in general be characterized by recursive equations. For instance, it is quite easy to define a probabilistic program (of type `unit` $\to$ `unit`) whose interpretation is a function $f : [0,1] \to [0,1]$ such that $f(u) = \frac{1}{2}u + \frac{1}{2}f(u)^2$, so that $f(u) = 1 - \sqrt{1-u}$: this is the only solution of the quadratic equation which gives a power series with nonnegative coefficients $(t_n)_{n\in\mathbb{N}}$ such that $f(u) = \sum_{n\in\mathbb{N}} t_n u^n$ for all $u \in [0,1]$, but the series diverges for $u > 1$.

Remark 2.6. As we have seen in the basic definitions and in the first examples, all the real numbers we consider are non-negative. A natural question is whether this restriction could be dropped and we argue that this issue is more tricky than it might seem at first sight. Consider an analytic function (in the sense described above) $f : [0,1]^2 \to [0,1]$, given by a family $t \in \mathbb{R}^2_{\geq 0}$, so that $f(u,v) = \sum_{n,p\in\mathbb{N}} t_{n,p} u^n v^p$. Then for each $u \in [0,1]$, the function $f_u : [0,1] \to [0,1]$ such that $f_u(v) = f(u,v)$ has a least fixpoint $g(u)$, and we shall see in Section 9.2 that the function g is itself analytic, that is $g(u) = \sum_{n\in\mathbb{N}} s_n u^n$ for some $(s_n \in \mathbb{R}_{\geq 0})_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} s_n \leq 1$. If we relax this positivity requirement, then our function f could be $f(u,v) = 1-(1-u)(1-v) = u+v-uv$ (mentioned in particular in [EHS04]). It is still true that f_u has a least fixpoint $g(u)$, but one checks easily that $g(0) = 0$ and $g(u) = 1$ if $u > 0$, which shows that g cannot be analytic, even with possibly negative coefficients.

2.2. **An archetypal example: the cone of finite measures.** Let X be a measurable space. The set $\underline{\mathsf{FMeas}(X)}$ of all *finite* (non-negative, real-valued) measures on X is naturally equipped with the structure of a cone:

- the algebraic operations of $\underline{\mathsf{FMeas}(X)}$ are defined pointwise (*e.g.* $(\mu_1+\mu_2)(U) = \mu_1(U) + \mu_2(U)$ for all $U \in \sigma_X$);
- the norm is given by $\|\mu\| = \mu(X)$ (this is the total variation norm of μ since μ is non-negative);
- observing that $\mu_1 \leq \mu_2$ means $\forall U \in \sigma_X\ \mu_1(U) \leq \mu_2(U)$, it is clear that each increasing sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{B}P$ has a least upper bound $\mu \in \mathcal{B}P$ which is computed pointwise: $\mu(U) = \sup_{n\in\mathbb{N}} \mu_n(U)$.

The set $\underline{\mathsf{FMeas}(X)}$ itself can be equipped with a σ-algebra, in the spirit of the Giry monad.

Let $\kappa : X \rightsquigarrow Y$ be a bounded kernel. Then the map $\widehat{\kappa} : \underline{\mathsf{FMeas}(X)} \rightsquigarrow \underline{\mathsf{FMeas}(Y)}$ defined by $\widehat{\kappa}(\mu)(V) = \int_{x\in X} \kappa(x,V)\,\mu(dx)$ is linear, continuous and measurable.

In fact, this map $\widehat{\kappa}$ has a stronger property: it preserves (S-)finite integrals. Namely, whenever μ is a finite measure on $\mathsf{FMeas}(X)$ (or more generally an S-finite[8] measure such that $\int_{\nu\in\mathsf{FMeas}(X)} \nu(X)\ \mu(d\nu) < \infty$), we have

$$\widehat{\kappa}\Big(\int_{\nu\in\mathsf{FMeas}(X)} \nu\,\mu(d\nu)\Big) = \int_{\nu\in\mathsf{FMeas}(X)} \widehat{\kappa}(\nu)\,\mu(d\nu)\,,$$

where the integrals are defined pointwise (recall that a measure on X is in particular a map from σ_X to $\overline{\mathbb{R}_{\geq 0}}$). Conversely, if a map $f : \underline{\mathsf{FMeas}(X)} \rightsquigarrow \underline{\mathsf{FMeas}(Y)}$ preserves S-finite integrals, then there exists a unique bounded kernel $\kappa : X \rightsquigarrow Y$ such that $f = \widehat{\kappa}$. It is given by $\kappa(x) = f(\boldsymbol{\delta}^X(x))$, where $\boldsymbol{\delta}^X(x)$ denotes the Dirac measure at x on X. If we think of S-finite measures as a generalization of formal linear combinations, then commutation with S-finite integrals is simply a generalization of linearity.

Preservation of S-finite integrals implies continuity and linearity, but the converse does not hold in general as illustrated in Remark 2.7.

Remark 2.7. Consider the map $\mathsf{cont} : \underline{\mathsf{FMeas}(\mathbb{R})} \to \underline{\mathsf{FMeas}(\mathbb{R})}$ defined by

$$\mathsf{cont}(\mu) = \frac{d\mu}{d\lambda}\,\lambda$$

where λ is the Lebesgue measure on $\mathbb{R}$ and $\frac{d\mu}{d\lambda}$ is the Radon-Nikodym derivative of μ with respect to λ. In other words, this map extracts the continuous part of the measures on $\mathbb{R}$.

[8] Recall that a measure is S-finite if it is a countable sum of finite measures. This is a weaker property than σ-finiteness.

This map is linear, ω-continuous and measurable (because the map $\mu \mapsto \frac{d\mu}{d\lambda}$ is measurable [Kal17, Theorem 1.28]). On the other hand,

$$\mathsf{cont}\Big(\int \boldsymbol{\delta}^{\mathbb{R}}(r)\lambda_{[0,1]}(dr)\Big) = \mathsf{cont}\left(\lambda_{[0,1]}\right) = \lambda_{[0,1]}\,,$$

(where $\lambda_{[0,1]}$ is the Lebesgue measure on $[0,1]$), while

$$\int \mathsf{cont}(\boldsymbol{\delta}^{\mathbb{R}}(r))\lambda_{[0,1]}(dr) = \int 0\lambda_{[0,1]}(dr) = 0\,.$$

Therefore, there exists no kernel $\kappa : \mathbb{R} \rightsquigarrow \mathbb{R}$ such that $\mathsf{cont} = \widehat{\kappa}$. As we shall see, avoiding this kind of situation is one of our main motivations for introducing integrability.

2.3. **Basic properties.** The following means that the notion of continuity we consider for linear maps behaves in an essentially algebraic way.

Lemma 2.8. *Let P and Q be cones and let $f : P \to Q$ be linear and continuous. If f is bijective then f^{-1} is linear and continuous.*

Proof. Linearity follows from the injectivity of f: let $y_1, y_2 \in Q$, $x_1 = f^{-1}(y_1 + y_2)$ and $x_2 = f^{-1}(y_1) + f^{-1}(y_2)$, we have $f(x_1) = y_1 + y_2$ and $f(x_2) = y_1 + y_2$ by linearity of f, hence $x_1 = x_2$. Scalar multiplication is dealt with similarly. Since f^{-1} is linear, it is increasing.

Let $(y_n \in \mathcal{B}Q)_{n=1}^{\infty}$ be an increasing sequence and let $y \in \mathcal{B}Q$ be its lub. The sequence $(f^{-1}(y_n) \in P)_{n=1}^{\infty}$ is increasing and upper bounded by $f^{-1}(y)$ and hence bounded in norm by $\left\|f^{-1}(y)\right\|_P$, so it has a lub $x \in P$ such that $x \leq f^{-1}(y)$. By continuity of f we have $f(x) = f(\sup_{n=1}^{\infty} f^{-1}(y_n)) = \sup_{n=1}^{\infty} y_n = y$ and hence $x = f^{-1}(y)$ which shows that f^{-1} is continuous. □

Using the notations of LL for the multiplicative constants, there is a cone $1 = \bot$ whose set of elements is $\mathbb{R}_{\geq 0}$ and $\|x\| = x$: the 1-dimensional cone. And using the notations of LL for the additive constants, there is also a cone $0 = \top$ whose only element is 0: the 0-dimensional cone.

Lemma 2.9. *Let P be a cone. Addition $P \times P \to P$ and scalar multiplication $1 \times P \to P$ are increasing and ω-continuous.*

Proof. See [Sel04]. □

The following lemma will be quite useful to prove that the difference between two linear and continuous functions is also linear and continuous, when it exists.

Lemma 2.10. *Let P and Q be cones, let $A \subseteq P$ be ω-closed and let $f, g : A \to Q$ be functions such that f is increasing, g is ω-continuous, $\forall x \in P\ f(x) \leq g(x)$ and the function $g - f = \boldsymbol{\lambda} x \in P \cdot (g(x) - f(x))$ is increasing. Then $g - f$ is ω-continuous.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded increasing sequence in A and let $x = \sup_{n\in\mathbb{N}} x_n$. For all $n \in \mathbb{N}$ we have $f(x_n) \leq f(x)$ and hence $g(x_n) \leq f(x) + g(x_n) - f(x_n)$. The sequence $(f(x) + g(x_n) - f(x_n))_{n\in\mathbb{N}}$ is increasing by our assumption that $g - f$ is increasing, and it is

$\leq$-bounded by $f(x) + g(x)$. We have

$$
\begin{aligned}
g(x) &= g(\sup_{n\in\mathbb{N}} x_n) \\
&= \sup_{n\in\mathbb{N}} g(x_n) \quad \text{since } g \text{ is } \omega\text{-continuous} \\
&\leq \sup_{n\in\mathbb{N}} (f(x) + g(x_n) - f(x_n)) \\
&= f(x) + \sup_{n\in\mathbb{N}} (g(x_n) - f(x_n)) \quad \text{by Lemma 2.9}
\end{aligned}
$$

and hence $g(x) - f(x) \leq \sup_{n\in\mathbb{N}}(g(x_n) - f(x_n))$. Since $g - f$ is increasing, if follows that $g - f$ is ω-continuous. □

Lemma 2.11. *If $f : P \to Q$ is linear then $f(\mathcal{B}P)$ is bounded. We set*

$$\|f\| = \sup_{x\in\mathcal{B}P} \|f(x)\| \in \mathbb{R}_{\geq 0}\,.$$

Proof. See [Sel04], we give the proof because it is short and interesting. If the lemma does not hold there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ \ \|x_n\| \leq 1$ and $\forall n \in \mathbb{N}\ \ \|f(x_n)\| \geq 4^n$. Then let $y_n = \sum_{k=1}^{n} \frac{1}{2^k} x_k \in P$, we have $\|y_n\| \leq \sum_{k=1}^{n} \frac{1}{2^k} \|x_n\| \leq 1$ and $(y_n)_{n\in\mathbb{N}}$ is an increasing sequence which therefore has a lub $y \in \mathcal{B}P$, and we have $\|f(y)\| \geq \|f(y_n)\| \geq \frac{1}{2^n} \|f(x_n)\| \geq 2^n$ by (**Normp**) and linearity of f. Since this holds for all $n \in \mathbb{N}$ we have a contradiction. □

Remark 2.12. We have obtained this property without even requiring f to be ω-continuous.

2.4. **The category of cones and linear and continuous maps.** Given cones P and Q the set $P \multimap Q$ of all linear and continuous maps from P to Q, equipped with obvious pointwise defined algebraic operations, is a precone. Notice that $f_1 \leq f_2$ simply means that $\forall x \in P\ f_1(x) \leq f_2(x)$ and then the difference is given by $(f_2 - f_1)(x) = f_2(x) - f_1(x)$: it suffices to check that this latter map is linear which is obvious, and that it is continuous which results from Lemma 2.10.

Lemma 2.13. *Equipped with the norm defined in Lemma 2.11, the precone $P \multimap Q$ is a cone.*

The proof is easy. By definition of $\|f\|$ and by (**Normh**) we have

$$\forall x \in P \quad \|f(x)\| \leq \|f\|\,\|x\|\,.$$

Definition 2.14. We set $P' = (P \multimap \bot)$. If $x \in P$ and $x' \in P'$ we write $\langle x, x'\rangle = x'(x) \in \mathbb{R}_{\geq 0}$.

Notice that, with these notations, $\|x'\| = \sup_{x\in\mathcal{B}P} \langle x, x'\rangle$.

Remark 2.15. The cone P' should be understood as an analog of the topological dual of a normed vector space; for instance one can define a linear and continuous morphism $\eta : P \to P''$ which corresponds to the usual embedding of a vector space into its bidual. However, in the case of cones, this morphism η is not necessarily injective: see the counter-example in Remark 2.16. As already mentioned we know from [Sel04] that an additional requirement on morphisms — namely, continuity, in the domain-theoretic sense — could guarantee the injectivity of this morphism; however continuity is too strong a requirement for what we are trying to accomplish. For this reason, the structures we define below (measurable, and later, integrable cones) will contain the axiom (**Mssep**) which states precisely that this morphism is injective.

Remark 2.16 (A cone whose dual is zero)**.** The following construction provides a non-trivial cone P whose dual P' contains only 0. This construction was suggested to us by one of the reviewers: many thanks to her/him. Consider the quotient set $P = P_0/\sim$, where P_0 is the set of all bounded measurable maps from $[0,1]$ to $\mathbb{R}_{\geq 0}$ (Example 2.3), and $\sim$ is the following equivalence relation: $f \sim g$ if and only if f and g differ only on a meager[9] subset of $[0,1]$.

The set P_0 inherits the structure of a cone from P, with $\|[f]\|$ defined as $\inf\{\|g\| \mid g \sim f\}$ (where $[f]$ denotes the equivalence class of f and $\|g\| = \sup_{x\in[0,1]} g(x)$). This is indeed a cone: the least obvious part is the fact that $\|[f]\| = 0$ implies $[f] = 0$. Indeed, assume $\|[f]\| = 0$. Then for all $n > 0$, there exists $f_n \sim f$ such that $\|f_n\| \leq \frac{1}{n}$. Thus, there exists a co-meager set X_n such that $f(x) \leq \frac{1}{n}$ for all $x \in X_n$. The intersection of all the X_n is itself co-meager, and therefore $f \sim 0$.

We shall prove that $P' = \{0\}$. Let $\alpha \in P'$. The map $f \mapsto \alpha([f])$ (from P_0 to $\mathbb{R}_{\geq 0}$) must commute with countable sums, therefore there must exist a finite measure μ on $[0,1]$ such that $\alpha([f]) = \int_{x\in[0,1]} f(x)\mu(dx)$ for all f. Moreover, we have that $\mu(Y) = 0$ for all meager sets Y. On the other hand, there exists a co-meager set X such that $\mu(X) = 0$. Indeed, let D be a dense countable subset of $[0,1]$ such that $\mu(D) = 0$ (such a set must exist because μ has at most countably many atoms). For all $n > 0$, there exists an open set X_n that contains D (and is therefore dense) and such that $\mu(X_n) \leq \frac{1}{n}$. The intersection X of all the X_n is co-meager, and its measure is 0. Therefore, $\mu([0,1]) = \mu(X) + \mu([0,1] \setminus X) = 0$, and thus $\alpha = 0$.

Definition 2.17. The category **Cones** has the cones as objects, and **Cones**(P, Q) is the set of all linear and continuous $f : P \to Q$ such that $\|f\| \leq 1$.

Theorem 2.18. *The category* **Cones** *has all small products. Given a family* $(P_i)_{i\in I}$ *of cones (with no cardinality restrictions on* I*), their categorical product* $(\&_{i\in I} P_i, (\mathsf{pr}_i)_{i\in I})$ *is defined as follows:*

- $\&_{i\in I} P_i$ *is the set of all* $\overrightarrow{x} = (x_i)_{i\in I} \in \prod_{i\in I} P_i$ *such that the family* $(\|x_i\|)_{i\in I}$ *is bounded, equipped with the obvious algebraic operations defined componentwise*
- *and* $\|\overrightarrow{x}\| = \sup_{i\in I} \|x_i\|$.

In particular, the terminal object (corresponding to the case where $I = \emptyset$*) is* $\top$.

The projections are the standard projections of the cartesian product in **Set**. *Given a family of morphisms* $(f_i \in \mathbf{Cones}(Q, P_i))_{i\in I}$*, the associated morphism* $f = \langle f_i\rangle_{i\in I} \in$ **Cones**$(Q, \&_{i\in I} P_i)$ *is characterized by* $f(y) = (f_i(y))_{i\in I}$.

The cone order of $\&_{i\in I} P_i$ *is the product of the cone orders of the* P_i*'s and the lubs of bounded sequences of elements of* $\&_{i\in I} P_i$ *are computed componentwise.*

See [Sel04]. There is a clear similarity with the ℓ^∞ construct of Banach spaces.

As announced, we can check now that separate continuity implies continuity.

Lemma 2.19. *Let* P*,* Q *and* R *be cones, let* $A \subseteq P$ *and* $B \subseteq Q$ *be* ω*-closed, so that* $A \times B$ *is an* ω*-closed subset of the product cone* $P \,\&\, Q$*, and let* $f : A \times B \to R$ *be separately*

[9]Recall that a subset of a topological space is meager if it is contained in a countable union of closed sets whose interiors are empty. Conversely, a subset is co-meager if it contains a countable intersection of dense open subsets.

ω-continuous (that is, for all $y \in B$ the function $\boldsymbol{\lambda} x \in A \cdot f(x,y)$ is ω-continuous and for all $x \in A$ the function $\boldsymbol{\lambda} y \in B \cdot f(x,y)$ is ω-continuous). Then f is ω-continuous.

Proof. Let $((x_n, y_n) \in A \times B)_{n\in\mathbb{N}}$ be an increasing bounded sequence in $A \times B$ so that $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded in A, and $(y_n)_{n\in\mathbb{N}}$ is increasing and bounded in B, and $\sup_{n\in\mathbb{N}}(x_n, y_n) = (\sup_{n\in\mathbb{N}} x_n, \sup_{n\in\mathbb{N}} y_n)$. We have

$$\begin{aligned} f(\sup_{n\in\mathbb{N}}(x_n, y_n)) &= \sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} f(x_n, y_k) \quad \text{by separate continuity} \\ &= \sup_{n\in\mathbb{N}} f(x_n, y_n) \end{aligned}$$

because f is increasing. □

Theorem 2.20. *The category* **Cones** *has all binary equalizers and therefore is complete. Moreover, if $(E, e \in \mathbf{Cones}(E,P))$ is the equalizer of $f, g \in \mathbf{Cones}(P,Q)$ then e reflects the order relation: if $x, y \in E$ satisfy $e(x) \leq_P e(y)$ then $x \leq_E y$.*

Proof. Let $f, g \in \mathbf{Cones}(P,Q)$. Let $E = \{x \in P \mid f(x) = g(x)\}$. We equip E with the algebraic operations of P which makes sense since f and g are linear: if $x_1, x_2 \in E$ then $f(x_1 + x_2) = f(x_1) + f(x_2) = g(x_1) + g(x_2) = g(x_1 + x_2)$ so that $x_1 + x_2 \in E$ and similarly for scalar multiplication. Next, for $x \in E$, we set $\|x\|_E = \|x\|_P$ which easily satisfies (**Normh**), (**Normz**), (**Normt**) and (**Normp**). Let $x_1, x_2 \in E$. It is obvious that $x_1 \leq_E x_2 \Rightarrow x_1 \leq_P x_2$. Conversely assume that $x_1 \leq_P x_2$ so that $x_2 - x_1$ exists in P, we have $f(x_2 - x_1) = f(x_2) - f(x_2)$ by linearity of f and similarly $g(x_2 - x_1) = g(x_2) - g(x_2)$ and hence $x_2 - x_1 \in E$ so that $x_1 \leq_E x_2$. We have proven that $x_1 \leq_E x_2 \Leftrightarrow x_1 \leq_P x_2$.

We prove that the norm of E satisfies (**Normc**), so let $(x_n)_{n\in\mathbb{N}}$ be a sequence which is increasing in E and such that $\forall n \in \mathbb{N}\ \|x_n\|_E \leq 1$. Then this sequence is increasing in P and satisfies $\forall n \in \mathbb{N}\ \|x_n\|_P \leq 1$ and hence it has a supremum $x \in P$ such that $\|x\|_P \leq 1$. Moreover by continuity of f and g we have $x \in E$. We have $\forall n \in \mathbb{N}\ x_n \leq_P x$ and hence $\forall n \in \mathbb{N}\ x_n \leq_E x$. Let $y \in E$ be such that $\forall n \in \mathbb{N}\ x_n \leq_E y$, we have $\forall n \in \mathbb{N}\ x_n \leq_P y$ and hence $x \leq_P y$ which implies $x \leq_E y$. This shows that x is the supremum of the x_n's in E, and since $\|x\|_E = \|x\|_P \leq 1$, we have proven (**Normc**) and hence E is a cone. Let $e : E \to P$ be the inclusion of E into P, it is clear that $e \in \mathbf{Cones}(E,P)$.

Finally, if $h \in \mathbf{Cones}(Q,P)$ satisfies $f\,h = g\,h$, we have that $h(u) \in E$ for all $u \in Q$ by definition of E. So $h = e\,h'$ where $h' : Q \to E$ is defined exactly as h (the only difference is the codomain), and we have $h' \in \mathbf{Cones}(Q,E)$ since the operations in E are defined as in P. The uniqueness of h' results from the injectivity of e. So (E,e) is the equalizer of f and g.

It follows that **Cones** is complete since it has also all small products. □

Lemma 2.21. *Let P and Q be cones and assume that $P \neq 0$. Let $f : P \to Q$ be linear and continuous, and bijective. Then $\|f\| \neq 0$ and $\left\|f^{-1}\right\| \geq \|f\|^{-1}$.*

Proof. By assumption there is $x \in P$ such that $x \neq 0$ and since f is bijective and $f(0) = 0$, we have that $Q \neq 0$. If follows that $\|f\| \neq 0$ and similarly $\left\|f^{-1}\right\| \neq 0$. Let $\varepsilon > 0$, we can find $x \in P$ such that $\|f\| \geq \frac{\|f(x)\|}{\|x\|} - \varepsilon$ hence $\frac{\|x\|}{\|f(x)\|} \geq \frac{1}{\|f\|+\varepsilon}$. Setting $y = f(x)$, we have $\frac{\left\|f^{-1}(y)\right\|}{\|y\|} \geq \frac{1}{\|f\|+\varepsilon}$ and hence $\left\|f^{-1}\right\| \geq \frac{1}{\|f\|+\varepsilon}$ and since this holds for all $\varepsilon > 0$ we get $\left\|f^{-1}\right\| \geq \|f\|^{-1}$. □

Proposition 2.22. *If $f \in \mathbf{Cones}(P,Q)$ is an iso and $P \neq 0$, then $\|f\| = 1$.*

Proof. By Lemma 2.21 we have $\|f^{-1}\| \geq \|f\|^{-1}$ and since f is a morphism $\|f\| \leq 1$ so $\|f^{-1}\| \geq 1$. Hence $\|f^{-1}\| = 1$ since f^{-1} is a morphism. Applying this to f^{-1} we get $\|f\| = 1$. □

The following technical lemma will be useful for proving that our category of integrable cones is well-powered in Theorem 4.18, a crucial property for being able to apply the special adjoint functor theorem.

Lemma 2.23. *Let P be a cone, S be a set and $f : P \to S$ be a bijective function. There is exactly one cone structure on S for which f becomes an iso in* **Cones**.

Proof. Given $s_1, s_2 \in S$, we set $s_1 + s_2 = f(f^{-1}(s_1) + f^{-1}(s_2))$ and similarly $\lambda s = f(\lambda f^{-1}(s))$ for $s \in S$ and $\lambda \in \mathbb{R}_{\geq 0}$. And we set $\|s\|_S = \|f^{-1}(s)\|_P$. It is straightforward that one defines a cone in that way, and that f is an iso. It is also obvious that this structure of cone one S is the only one such that f is an iso. □

3. Measurable cones

Let **Ar** be a *small*[10] full subcategory of **Meas** (the category of measurable spaces and measurable functions) which is closed under cartesian products and contains the terminal object 0 which is the one point measurable space (we use this notation because the one point measurable space is geometrically 0-dimensional). We also assume all the objects of **Ar** to be non-empty measurable spaces. We also use **Ar** for the set of all objects of the category **Ar**.

Remark 3.1. In most situations we could assume that **Ar** has all finite products $\mathbb{R}^n$ as objects, and measurable maps as morphisms. We could even assume that **Ar** has $\mathbb{R}$ as single object, or more precisely, two objects: $\mathbb{R}$ and 0 since all the $\mathbb{R}^n$ are isomorphic[11] to $\mathbb{R}$ in **Meas** for $n > 0$.

Definition 3.2. A *measurability structure* on a cone P is a family $\mathcal{M} = (\mathcal{M}_X)_{X \in \mathbf{Ar}}$ with $\mathcal{M}_X \subseteq (P')^X$ (where we recall that $P' = (P \multimap \bot)$ is the dual of the cone P) satisfying the four next conditions (**Msmeas**), (**Mscomp**), (**Mssep**) and (**Msnorm**). When $X = 0$ we consider $m \in \mathcal{M}_X$ as an element of P'.

(**Msmeas**) For each $m \in \mathcal{M}_X$ and $x \in \mathcal{B}P$, one has $\boldsymbol{\lambda} r \in X \cdot m(r, x) \in \mathbf{Meas}(X, [0,1])$ where $[0,1] \subseteq \mathbb{R}$ is equipped with its standard Borel σ-algebra. This implies in particular that if $r \in X$, then $\boldsymbol{\lambda} x \in P \cdot m(r, x) \in \mathbf{Cones}(P, 1)$.

(**Mscomp**) For each $m \in \mathcal{M}_X$ and $\varphi \in \mathbf{Ar}(Y, X)$ one has $\boldsymbol{\lambda}(s, x) \in (Y \times P) \cdot m(\varphi(s), x) = m \circ (\varphi \times P) \in \mathcal{M}_Y$.

In particular, since 0 is the terminal object of **Ar**, each element $m \in \mathcal{M}_0$ induces an element $\boldsymbol{\lambda}(r, x) \in (X \times P) \cdot m(x) \in \mathcal{M}_X$ and in this way we consider $\mathcal{M}_0$ as a subset of $\mathcal{M}_X$ for each $X \in \mathbf{Ar}$.

(**Mssep**) If $x_1, x_2 \in P$ satisfy $\forall m \in \mathcal{M}_0\ m(x_1) = m(x_2)$ then $x_1 = x_2$.

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[10]This assumption is crucial for making the use of the special adjoint functor theorem possible.

[11]Such isomorphisms involve however non canonical encoding methods so we prefer to avoid using this property explicitly.

(**Msnorm**) For all $x \in P$, one has

$$\|x\| = \sup\left\{\frac{m(x)}{\|m\|} \mid m \in \mathcal{M}_0 \text{ and } m \neq 0\right\}$$

or, equivalently, $\|x\| \leq \sup\{m(x)/\|m\| \mid m \in \mathcal{M}_0 \text{ and } m \neq 0\}$.

Indeed, for each $x' \in P' \setminus \{0\}$ and $x \in P$ one has $\|x\| \geq \frac{\langle x,x'\rangle}{\|x'\|}$.

Remark 3.3. The condition (**Msnorm**) can also be formulated as follows: for each $x \in P \setminus \{0\}$ and for each $\varepsilon > 0$ there exists $m \in \mathcal{M}_0 \setminus \{0\}$ such that $\|x\| \leq \frac{m(x)}{\|m\|} + \varepsilon$. The condition that $x \neq 0$ is required because we could possibly have $\mathcal{M}_0 = \{0\}$, but in that situation, by (**Mssep**) we must have $P = \{0\}$. The condition (**Msnorm**) was absent in [Ehr20] which made it much more difficult to prove that the category of cones and linear measurable cones is symmetric monoidal: we had to use a property of density of the category of PCSs. (**Msnorm**) makes the whole theory much better behaved.

Remark 3.4. We do not require $\mathcal{M}_0$ to be the whole unit ball of the dual P', but only a subset of it, sufficiently large for satisfying our requirements (**Mscomp**), (**Mssep**) and (**Msnorm**). As we will see in various constructions, $\mathcal{M}_0$ will often be a very small part of this unit ball.

Remark 3.5. Instead of (**Mssep**) we could also consider the following stronger separation condition: if for all $m \in \mathcal{M}_0$ one has $m(x) \leq m(y)$ then $x \leq y$. However this would complicate the definition of the measurability structures of the spaces of stable and measurable functions in Section 7.2 and of analytic functions in Section 8. This stronger separability does not seem to be necessary (at least for the purpose of what we do in this paper) but one should keep in mind that all our constructions could be performed within this restricted class.

Definition 3.6 (Measurable cone)**.** A *measurable cone* is a pair $C = (\underline{C}, \mathcal{M}^C)$ where $\underline{C}$ is a cone and $\mathcal{M}^C$ is a measurability structure on $\underline{C}$.

The main purpose of the measurability structure of a measurable cone is to equip the underlying cone with a structure of QBS by defining a collection of paths ranging in the cone.

Definition 3.7 (Measurable path)**.** Let $X \in \mathbf{Ar}$ and let C be a measurable cone. A *(measurable) path* of arity X is a function $\gamma : X \to \underline{C}$ which is bounded and such that, for each $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^C_Y$, the function $\boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma(r)) : Y \times X \to \mathbb{R}_{\geq 0}$ is measurable. We use $\underline{\mathsf{Path}(X, C)}$ for the set of measurable paths of arity X of the measurable cone C.

Remark 3.8. Measurable paths should be thought of as a generalization of finite kernels, and in the case where C is the measurable cone of finite measures on a measurable space Y, each bounded kernel from X to Y is a measurable path from X to C. One of the purposes of introducing integrals is to make the converse true.

Lemma 3.9. *Let $x \in \underline{C}$ and $\gamma = \boldsymbol{\lambda} r \in X \cdot x : X \to \underline{C}$ be the constant function. Then γ is a measurable path.*

This immediately results from the definitions.

Lemma 3.10. *Let $\gamma : X \to \underline{C}$ be a measurable path and let $\varphi \in \mathbf{Ar}(Y, X)$ for some $Y \in \mathbf{Ar}$. Then $\gamma \circ \varphi : Y \to \underline{C}$ is also a measurable path.*

Proof. Let $Y' \in \mathbf{Ar}$ and $m \in \mathcal{M}^C_{Y'}$, we have $\boldsymbol{\lambda}(s',s) \in Y' \times Y \cdot m(s', \gamma(\varphi(s))) = (\boldsymbol{\lambda}(s',r) \in Y' \times X \cdot m(s', \gamma(r))) \circ (Y' \times \varphi)$ which is measurable as the composition of two measurable maps. □

We turn the cone $1 = \perp$ into a measurable cone by defining $\mathcal{M}^1_X$ as the set of all functions mapping each element $r \in X$ to $\mathsf{Id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ for all $X \in \mathbf{Ar}$.

Proposition 3.11. *Let B be a measurable cone and let $x \in \underline{B}$. Then*

$$\|x\| = \sup_{x' \in \mathcal{B}\underline{B}'} \langle x, x' \rangle .$$

Proof. By definition of the norm in $\underline{B}'$ we have $\|x\| \geq \langle x, x' \rangle$ for all $x' \in \mathcal{B}\underline{B}'$. We can assume that $x \neq 0$ since otherwise the announced equation trivially holds. Let $\varepsilon > 0$ and $m \in \mathcal{M}^B_0 \setminus \{0\}$ be such that $\|x\| \leq \frac{m(x)}{\|m\|} + \varepsilon$. Let $x' = m/\|m\|$, we have $x' \in \mathcal{B}\underline{B}'$. Since $\|x\| \leq \langle x, x' \rangle + \varepsilon$ our contention is proven. □

Remark 3.12. One main purpose of the condition (**Msnorm**) is to get the above highly desirable property. We could have expected to get (**Mssep**) and (**Msnorm**) for free by means of a Hahn Banach theorem for cones as in [Sel04]. However, the counter-example of Remark 2.16, suggested to us by one of the reviewers of this paper, shows that such a separation property does not hold in our setting. The very nice Hahn Banach theorem proven in [Sel04] relies on the assumption that cones are continuous domains, an assumption that we cannot afford here because we need our cones to define a complete category in order to apply the special adjoint functor theorem which is our main tool for equipping **ICones** with a tensor product and with an exponential. Fortunately, we can take this Hahn Banach separation property as one of our axioms on the measurability tests, and proving that this property is preserved by all limits does not induce noticeable technical difficulties.

3.1. The category of measurable cones and linear, continuous and measurable maps.

We can now define our first main category of interest.

Definition 3.13. The category **MCones** has measurable cones as objects and an element of $\mathbf{MCones}(B, C)$ is an $f \in \mathbf{Cones}(\underline{B}, \underline{C})$ such that for each $X \in \mathbf{Ar}$ and each measurable path $\beta : X \to \underline{B}$ the function $f \circ \beta$ is a measurable path. Equivalently

$$\forall Y \in \mathbf{Ar}\ \forall m \in \mathcal{M}^C_Y \quad \boldsymbol{\lambda}(s,r) \in X \times Y \cdot m(s, f(\beta(r))) \text{ is measurable.}$$

Remark 3.14. An isomorphism $f \in \mathbf{MCones}(B, C)$ is a bijection $f : \underline{B} \to \underline{C}$ which is linear and continuous, satisfies $\forall x \in \underline{B}\ \ \|f(x)\|_C = \|x\|_B$ and, for each $X \in \mathbf{Ar}$ and each function $\beta : X \to \underline{B}$, one has $\beta \in \underline{\mathsf{Path}(X, B)} \Leftrightarrow f \circ \beta \in \underline{\mathsf{Path}(X, C)}$. This means that B and C can be isomorphic even if the measurability structures $\mathcal{M}^C$ and $\mathcal{M}^D$ are quite different: it suffices that they induce the same measurable paths.

Definition 3.15. Let B be a measurable cone and $\alpha \in \mathbb{R}$ with $\alpha > 0$. Then αB is the measurable cone which is defined exactly as B apart for the norm which is given by $\|x\|_{\underline{\alpha B}} = \alpha^{-1} \|x\|_{\underline{B}}$.

Notice that $\mathcal{B}(\underline{\alpha B}) = \alpha \mathcal{B}\underline{B} = \{x \in \underline{B} \mid \|x\|_{\underline{B}} \leq \alpha\}$.

3.2. **Examples: the measurable cones of measures and paths.** We introduce two important examples of measurable cones. The measurable cone of finite measures on an object of **Ar** will allow us to understand **Ar** as our category of basic data-types.

As noticed by one of the reviewers, it would not be strictly necessary to introduce the measurable cone of measurable paths since we will see that, in the setting of integrable cones, the cone of measurable paths can be described as an internal linear hom, see Theorem 6.1. Our motivations for presenting this construction are:

- it illustrates for the first time the reason why our tests have parameters in objects of **Ar**;
- it is a simple and natural construction, quite different from the cone of finite measures (and somehow dual to it), and completely independent from our integrability assumptions.

3.2.1. *The measurable cone of finite measures.* Let X be a measurable space (not necessarily in **Ar**). Recall that in Section 2.2 we defined the cone $\underline{\mathsf{FMeas}(X)}$ of all finite measures on X.

For all $Y \in \mathbf{Ar}$ and all $U \in \sigma_X$ we define $\widetilde{U} : Y \times \underline{\mathsf{FMeas}(X)} \to \mathbb{R}_{\geq 0}$ by $\widetilde{U}(s, \mu) = \mu(U)$. Then we define $\mathcal{M}_Y = \{\widetilde{U} \mid U \in \sigma_X\}$, and $\mathsf{FMeas}(Y) = (\underline{\mathsf{FMeas}(X)}, (\mathcal{M}_Y)_{Y \in \mathbf{Ar}})$ is clearly a measurable cone.

Remark 3.16. We could have taken another measurability structure as follows. For all $Y \in \mathbf{Ar}$ and all $W \in \sigma_{Y \times X}$ we define $\widetilde{W} : Y \times \underline{\mathsf{FMeas}(X)} \to \mathbb{R}_{\geq 0}$ by $\widetilde{W}(s, \mu) = \mu(\{r \mid (s, r) \in W\})$. Then we define $\mathcal{M}_Y = \{\widetilde{W} \mid W \in \sigma_{Y \times X}\}$. Then $\mathsf{FMeas}(X) = (\underline{\mathsf{FMeas}(X)}, (\mathcal{M}_Y)_{Y \in \mathbf{Ar}})$ is a measurable cone. As easily checked, these two measurability structures define exactly the same measurable paths on $\mathsf{FMeas}(X)$. This example shows that a given cone (namely $\underline{\mathsf{FMeas}(X)}$) can be given two distinct measurability structures. This is also an example of the situation mentioned in Remark 3.14: the measurability cones defined by these two measurability structures are isomorphic in the category **MCones**.

Notice that if $Y \in \mathbf{Ar}$ then a measurable path $\gamma : Y \to \underline{\mathsf{FMeas}(X)}$ is the same thing as a bounded kernel from Y to X.

Let $\varphi \in \mathbf{Meas}(X, Y)$ (remember that this means that φ is a measurable function $X \to Y$), then given $\mu \in \underline{\mathsf{FMeas}(X)}$ we can define $\nu = \varphi_*(\mu) \in \underline{\mathsf{FMeas}(Y)}$ by $\nu(V) = \mu(\varphi^{-1}(V))$ for each $V \in \sigma_Y$ (the push-forward of μ along φ).

Lemma 3.17. *We have $\varphi_* \in \mathbf{MCones}(\mathsf{FMeas}(X), \mathsf{FMeas}(Y))$. The operation* FMeas *on measurable cones extends to a functor* $\mathsf{FMeas} : \mathbf{Ar} \to \mathbf{MCones}$*, acting on morphisms by measure push-forward:* $\mathsf{FMeas}(\varphi) = \varphi_*$*.*

Proof. Linearity and continuity being obvious, as well as the fact that $\|\mathsf{FMeas}(f)\| \leq 1$, we only have to check measurability. Let $\kappa : Y' \to \underline{\mathsf{FMeas}(X)}$ be a measurable path. We must prove that $\kappa' = \mathsf{FMeas}(\varphi) \circ \kappa$ is a measurable path. Let $p \in \mathcal{M}_{Y''}^{\mathsf{FMeas}(Y)}$ for some $Y'' \in \mathbf{Ar}$, that is $p = \widetilde{V}$ for some $V \in \sigma_Y$. For $(s'', s') \in Y'' \times Y'$ we have $p(s'', \kappa'(s')) = \kappa'(s')(V) = \kappa(s')(\varphi^{-1}(V))$ and hence $\boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot p(s'', \kappa'(s'))$ is measurable because κ is a kernel and φ is measurable. Functoriality of FMeas is obvious. $\square$

3.2.2. *The measurable cone of paths.* Let C be an object of **MCones** and $X \in \mathbf{Ar}$. Let P be the set of all measurable paths $\gamma : X \to \underline{C}$. We turn P into a precone by defining the algebraic laws in the obvious pointwise manner. For instance let $\gamma_1, \gamma_2 \in P$, we define $\gamma = \gamma_1 + \gamma_2$ by $\gamma(r) = \gamma_1(r) + \gamma_2(r)$ which is bounded by (**Normt**). To check measurability, take $m \in \mathcal{M}^C_Y$, we have $\boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma(r)) = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma_1(r)) + \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma_2(r))$ (pointwise addition) by linearity of m in its second parameter, which is measurable in r by continuity of addition on $\mathbb{R}_{\geq 0}$.

Then we have $\gamma_1 \leq \gamma_2$ iff $\forall r \in X\ \gamma_1(r) \leq \gamma_2(r)$: it suffices to check that, when this latter condition holds, the map $\boldsymbol{\lambda} r \in X \cdot (\gamma_2(r) - \gamma_1(r))$ is a path which results from the continuity (and hence measurability) of subtraction of real numbers.

Given $\gamma \in P$ we set

$$\|\gamma\| = \sup_{r \in X} \|\gamma(r)\| \in \mathbb{R}_{\geq 0}$$

which is well defined by our assumption that γ is bounded. This satisfies all the required conditions for turning P into a cone, the only non obvious one being (**Normc**). So let $(\gamma_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements of P such that $\forall n \in \mathbb{N}\, \forall r \in X\ \ \|\gamma_n(r)\| \leq 1$. We define $\gamma : X \to P$ by $\gamma(r) = \sup_{n \in \mathbb{N}} \gamma_n(r) \in \mathcal{B}\underline{C}$ which is well defined since for each $r \in X$ the sequence $(\gamma_n(r))_{n \in \mathbb{N}}$ is increasing in $\mathcal{B}\underline{C}$. It suffices to check that γ satisfies the measurability condition, so let $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^C_Y$, we have by ω-continuity of m in its second argument

$$\boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma(r)) = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot \sup_{n \in \mathbb{N}} m(s, \gamma_n(r))$$

which is measurable by the monotone convergence theorem of measure theory (observing that $(\boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma_n(r)))_{n \in \mathbb{N}}$ is an increasing sequence of measurable functions $Y \times X \to [0,1]$).

Remark 3.18. Remember that it is precisely for being able to prove this kind of properties that we assume the unit balls of cones to be complete only for increasing chains and not for arbitrary directed sets.

So far we have equipped P (the set of measurable paths from X to C) with a structure of cone in the algebraic sense of Section 2. We equip now this cone with a measurability structure. This definition will illustrate, for the first time in this paper, the usefulness of the "additional" parameter of tests, spanning measurable spaces taken in **Ar**.

Let $Y \in \mathbf{Ar}$, $\varphi \in \mathbf{Ar}(Y, X)$ and $m \in \mathcal{M}^C_Y$, we define

$$\begin{aligned} \varphi \triangleright m : Y \times P &\to \mathbb{R}_{\geq 0} \\ (s, \gamma) &\mapsto m(s, \gamma(\varphi(s)))\,. \end{aligned}$$

Observe first that for each $s \in Y$, the map $\boldsymbol{\lambda}\gamma \in P \cdot (\varphi \triangleright m)(s, \gamma)$ is linear and continuous by linearity and continuity of m in its second argument. We check that the family $(\mathcal{M}_Y \subseteq P'^Y)_{Y \in \mathbf{Ar}}$ defined by $\mathcal{M}_Y = \{\varphi \triangleright m \mid \varphi \in \mathbf{Ar}(Y, X) \text{ and } m \in \mathcal{M}^C_Y\}$ is a measurability structure on P.

▶ (**Msmeas**). Let $p \in \mathcal{M}_Y$ and $\gamma \in P$, so that $p = \varphi \triangleright m$ for some $\varphi \in \mathbf{Ar}(Y, X)$ and $m \in \mathcal{M}^C_Y$, then let $\theta = \boldsymbol{\lambda} s \in Y \cdot p(s, \gamma) = \boldsymbol{\lambda} s \in Y \cdot m(s, \gamma(\varphi(s)))$. We know that $\psi = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \gamma(r))$ is measurable $Y \times X \to [0,1]$ and hence $\theta = \psi \circ \langle Y, \varphi \rangle$ is measurable $Y \to [0,1]$ since φ is measurable.

▶ (**Mscomp**). Let $p \in \mathcal{M}_Y$ and $\psi \in \mathbf{Ar}(Y', Y)$. We have $p = \varphi \triangleright m$ for some $\varphi \in \mathbf{Ar}(Y, X)$ and $m \in \mathcal{M}^C_Y$. Then we have $p \circ (\psi \times P) = (\varphi \circ \psi) \triangleright (m \circ (\psi \times \underline{C})) \in \mathcal{M}_{Y'}$.

▶ (**Mssep**). Let $\gamma_1, \gamma_2 \in P$ and assume that $\forall p \in \mathcal{M}_0\ p(\gamma_1) = p(\gamma_2)$. Let $r \in X$ that we consider as an element of $\mathbf{Ar}(0, X)$. Let $m \in \mathcal{M}^C_0$, by our assumption we have $(r \triangleright m)(\gamma_1) = (r \triangleright m)(\gamma_2)$, that is $m(\gamma_1(r)) = m(\gamma_2(r))$ and since this holds for all $m \in \mathcal{M}^C_0$ we have $\gamma_1(r) = \gamma_2(r)$ by (**Mssep**) in C.

▶ (**Msnorm**). Let $\gamma \in P \setminus \{0\}$ and $\varepsilon > 0$. We can find $r \in X$ such that $\gamma(r) \neq 0$ and $\|\gamma\| \leq \|\gamma(r)\| + \frac{\varepsilon}{2}$. By (**Msnorm**) holding in C we can find $m \in \mathcal{M}^C_0 \setminus \{0\}$ such that $\|\gamma(r)\| \leq \frac{m(\gamma(r))}{\|m\|} + \frac{\varepsilon}{2}$. Remember that $r \triangleright m \in \mathcal{M}_0$ and notice that $\|r \triangleright m\| = \sup\{m(\delta(r)) \mid \delta \in \mathcal{B}P\} = \|m\|$ by Lemma 3.9. So we have

$$\|\gamma\| \leq \|\gamma(r)\| + \frac{\varepsilon}{2} \leq \frac{(r \triangleright m)(\gamma)}{\|r \triangleright m\|} + \varepsilon$$

and hence $\|\gamma\| = \sup\{\frac{p(\gamma)}{\|p\|} \mid p \in \mathcal{M}_0 \text{ and } p \neq 0\}$ as required since $\mathcal{M}_0 = \{r \triangleright m \mid r \in X \text{ and } m \in \mathcal{M}^C_0\}$.

We use $\mathsf{Path}(X, C)$ for the measurable cone $(P, \mathcal{M})$ defined above. We end this section with the following lemma which will be useful when dealing with the tensor product of measurable cones, and in particular for proving Theorem 6.7.

Lemma 3.19. *Let B be a cone and $X, Y \in \mathbf{Ar}$. There is an iso*

$$\mathsf{fl}_{X,Y} \in \mathbf{MCones}(\mathsf{Path}(X, \mathsf{Path}(Y, B)), \mathsf{Path}(X \times Y, B))$$

which "flattens" $\eta \in \underline{\mathsf{Path}(X, \mathsf{Path}(Y, B))}$ into $\mathsf{fl}_{X,Y}(\eta) = \boldsymbol{\lambda}(r, s) \in X \times Y \cdot \eta(r)(s)$. As a consequence

$$\mathsf{fl}^{-1}_{Y,X}\, \mathsf{fl}_{X,Y} \in \mathbf{MCones}(\mathsf{Path}(X, \mathsf{Path}(Y, B)), \mathsf{Path}(Y, \mathsf{Path}(X, B)))\,,$$

the function which swaps the parameters of a path of paths, is an iso in **MCones***.*

Proof. Let $\eta \in \underline{\mathsf{Path}(X, \mathsf{Path}(Y, B))}$, we need first to prove that $\eta' = \mathsf{fl}(\eta) \in \underline{\mathsf{Path}(X \times Y, B)}$ so let $Y' \in \mathbf{Ar}$ and let $m \in \mathcal{M}^B_{Y'}$, we must prove that

$$\varphi = \boldsymbol{\lambda}(s', r, s) \in Y' \times X \times Y \cdot m(s', \eta'(r, s)) = \boldsymbol{\lambda}(s', r, s) \in Y' \times X \times Y \cdot m(s', \eta(r)(s))$$

is measurable. Let $m' = m \circ (\mathsf{pr}_1 \times \underline{B}) \in \mathcal{M}^B_{Y' \times Y}$ (that is $m'(s', s, x) = m(s', x)$) so that $\mathsf{pr}_2 \triangleright m' \in \mathcal{M}^{\mathsf{Path}(Y,B)}_{Y' \times Y}$, we know that $\boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot (\mathsf{pr}_2 \triangleright m')(s', s, \eta(r)) = \boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot m(s', \eta(r)(s))$ is measurable from which it follows that φ is measurable. Moreover it is clear that $\eta'(X \times Y) \subseteq \|\eta\|\, \mathcal{B}\underline{B}$ is bounded in $\underline{B}$ and hence $\eta' \in \underline{\mathsf{Path}(X \times Y, B)}$ as announced.

The linearity and ω-continuity of fl are clear so we check its measurability. Let $Y' \in \mathbf{Ar}$ and let $\eta \in \underline{\mathsf{Path}(Y', \mathsf{Path}(X, \mathsf{Path}(Y, B)))}$, we must prove that

$$\mathsf{fl} \circ \eta \in \underline{\mathsf{Path}(Y', \mathsf{Path}(X \times Y, B))}\,.$$

So let $Y'' \in \mathbf{Ar}$ and let $p \in \mathcal{M}^{\mathsf{Path}(X \times Y, B)}_{Y''}$. Let $\varphi' = \langle \varphi, \psi \rangle \in \mathbf{Ar}(Y'', X \times Y)$ and $m \in \mathcal{M}^B_{Y''}$ be such that $p = \varphi' \triangleright m$, we have that

$$\begin{aligned}
\varphi'' &= \boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot p(s'', \mathsf{fl}(\eta(s'))) \\
&= \boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot m(s'', \mathsf{fl}(\eta(s'))(\varphi(s''), \psi(s''))) \\
&= \boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot m(s'', \eta(s')(\varphi(s''))(\psi(s'')))
\end{aligned}$$

is measurable because

$$\varphi'' = \boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot (\varphi \triangleright (\psi \triangleright m))(s'', \eta(s'))$$

and by our assumption about η. Last notice that $\|\mathsf{fl}(\eta)\| = \|\eta\|$ which shows that $\mathsf{fl} \in \mathbf{MCones}(\mathsf{Path}(X, \mathsf{Path}(Y, B)), \mathsf{Path}(X \times Y, B))$.

As to the converse direction, given $\eta \in \underline{\mathsf{Path}(X \times Y, B)}$ let $\mathsf{fl}'(\eta) = \boldsymbol{\lambda} r \in X \cdot \boldsymbol{\lambda} s \in Y \cdot \eta(r, s)$, we must first prove that $\mathsf{fl}'(\eta) \in \underline{\mathsf{Path}(X, \mathsf{Path}(Y, B))}$, we just check measurability, boundedness being obvious. Let $p \in \mathcal{M}_{Y'}^{\mathsf{Path}(Y,B)}$ for some $Y' \in \mathbf{Ar}$. Let $\varphi \in \mathbf{Ar}(Y', Y)$ and $m \in \mathcal{M}_{Y'}^{B}$ be such that $p = \varphi \triangleright m$, we must prove that $\psi = \boldsymbol{\lambda}(s', r) \in Y' \times X \cdot p(s', \mathsf{fl}'(\eta)(r)) = \boldsymbol{\lambda}(s', r) \in Y' \times X \cdot m(s', \eta(r, \varphi(s')))$ is measurable. This follows from the fact that φ and $\boldsymbol{\lambda}(s', r, s) \in Y' \times X \times Y \cdot m(s', \eta(r, s))$ are measurable, the latter by our assumption about η.

Checking that fl' is a morphism in **MCones** follows exactly the same pattern as for fl, using the obvious bijection between $\mathcal{M}_{Y'}^{\mathsf{Path}(X,\mathsf{Path}(Y,B))}$ and $\mathcal{M}_{Y'}^{\mathsf{Path}(X \times Y,B)}$ induced by the fact that $\mathbf{Ar}$ is cartesian. Finally the observation that $\mathsf{fl}' = \mathsf{fl}^{-1}$ shows that fl is an iso in **MCones**. □

Remark 3.20. So a test on the space of C-valued and X-parameterized paths is provided by a test $m \in \mathcal{M}_Y^C$ — itself parameterized by a space $Y \in \mathbf{Ar}$ — and a "variable argument" which is a measurable function φ from Y to the space X. When φ is not a constant function, the value of $(\varphi \triangleright m)(s, \gamma) = m(s, \gamma(\varphi(s)))$ depends in general on s when γ is not a constant path, *even if the function* $m : Y \times \underline{C} \to \mathbb{R}_{\geq 0}$ *does not depend on its first argument.* This definition of tests in the cones of paths plays a crucial role in the proof of Lemma 3.19.

Imagine that we want to use a simpler notion of tests, with $\mathcal{M}^C \subseteq \underline{C}'$, that is, assume that our tests do not have the further parameter taken in a $Y \in \mathbf{Ar}$. The first difficulty we face consists in finding a suitable definition for $\mathcal{M}^{\mathsf{Path}(X,C)}$. The simplest option consists in taking all the $r \triangleright m$ where $r \in X$ and $m \in \mathcal{M}^C$, defined by $(r \triangleright m)(\beta) = m(\beta(r))$. This choice fulfills all the expected separation properties. With this definition, an element of $\underline{\mathsf{Path}(Y, \mathsf{Path}(X, C))}$ is the same thing as a bounded function $\gamma : Y \times X \to \underline{B}$ such that, for each $m \in \mathcal{M}^C$, the function $\boldsymbol{\lambda} s \in Y \cdot m(\beta(s, r)) : Y \to \mathbb{R}_{\geq 0}$ is measurable for all $r \in X$ and the function $\boldsymbol{\lambda} r \in X \cdot m(\beta(s, r)) : X \to \mathbb{R}_{\geq 0}$ is measurable for all $s \in Y$. Let us assume that $C = \mathsf{FMeas}(Z)$ for some $Z \in \mathbf{Ar}$, let $\varphi : Y \times X \to Z$ be a function, and let $\gamma : Y \times X \to \underline{C}$ be given by $\gamma(s, r) = \boldsymbol{\delta}^Z(\varphi(s, r))$. Saying that $\gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, C))}$ means that the function φ is separately measurable in both arguments, a condition which is strictly weaker than measurability on $Y \times X$ in general. On the other hand, with our definition of measurability tests for $\mathsf{Path}(X, C)$, Lemma 3.19 tells us that $\gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, C))}$ iff φ is measurable $Y \times X \to Z$ for the very simple reason that $Y \times X \in \mathbf{Ar}$. Notice that we have equipped P, the cone of measurable paths from X to C, with two different measurability structures: the original one made of all tests $\varphi \triangleright m$ where $\varphi \in \mathbf{Ar}(Y, X)$ and $m \in \mathcal{M}_Y^C$, and the simplified one, made of tests $\varphi \triangleright m$ where $\varphi \in \mathbf{Ar}(Y, X)$ is a *constant* function and $m \in \mathcal{M}_Y^C$. The two measurable cones obtained in that way are not isomorphic[12] in **MCones** since the associated measurable paths are distinct as we have seen. This complements Remarks 3.16 and 3.14.

We will meet a completely similar definition of tests for the space $B \multimap C$ of linear, continuous and integrable functions from B to C in Section 5.

[12] More precisely, the identity function between these two cones is not an isomorphism.

4. Integrable cones

We now introduce the main novelties of this paper, which are the definition of the integral of a measurable path wrt. a finite measure, the notion of integrable cone, and the notion of linear, continuous, measurable and integral preserving functions between integrable cones.

The following definition is quite similar to Definition 2.1 in [Pet38] of the integral of a function valued in a topological vector space. Our integrals are valued in cones instead of vector spaces.

Definition 4.1. Let B be a measurable cone, $X \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(X, B)}$ and $\mu \in \underline{\mathsf{FMeas}(X)}$. An *integral of* β *over* μ is an element x of $\underline{B}$ such that, for all $m \in \mathcal{M}_0^B$, one has

$$m(x) = \int m(\beta(r))\mu(dr)\,.$$

Notice indeed that $m \circ \beta : X \to \mathbb{R}_{\geq 0}$ is a bounded measurable function so that the integral above is well defined and belongs to $\mathbb{R}_{\geq 0}$ (remember that the measure μ is finite). Notice also that by (**Mssep**) if such an integral x exists, it is unique, so we can introduce a notation for it, we write

$$x = \int \beta(r)\mu(dr)\,.$$

When we want to stress the cone B where this integral is computed we denote it as $\int^B \beta(r)\mu(dr)$ and when we want to insist on the measurable space on which the integral is computed we write $\int_X \beta(r)\mu(dr)$ or $\int_{r\in X} \beta(r)\mu(dr)$.

Lemma 4.2. *If* $\beta \in \underline{\mathsf{Path}(X, B)}$ *is integrable over* $\mu \in \mathsf{FMeas}(X)$ *then*

$$\left\| \int_X \beta(r)\mu(dr) \right\|_B \leq \|\beta\|_{\mathsf{Path}(X,B)} \|\mu\|_{\mathsf{FMeas}(X)}\,.$$

Proof. Let $x = \int \beta(r)\mu(dr)$. If $x = 0$ there is nothing to prove so assume that $x \neq 0$. Let $\varepsilon > 0$ and let $m \in \mathcal{M}_0^B \setminus \{0\}$ be such that $\|x\| \leq \varepsilon + \frac{m(x)}{\|m\|}$, that is

$$\|x\| \leq \varepsilon + \frac{1}{\|m\|} \int m(\beta(r))\mu(dr)\,.$$

For each $r \in X$ we have $m(\beta(r)) \leq \|m\| \|\beta(r)\| \leq \|m\| \|\beta\|$. Our contention follows from $\|\mu\| = \mu(X) = \int \mu(dr)$. □

Definition 4.3. A measurable cone is *integrable* if, for all $X \in \mathbf{Ar}$, each $\beta \in \underline{\mathsf{Path}(X, B)}$ has an integral in $\underline{B}$ over each measure $\mu \in \underline{\mathsf{FMeas}(X)}$. When this is the case we use $\mathcal{I}_X^B$ for the uniquely defined function $\underline{\mathsf{Path}(X, B)} \times \underline{\mathsf{FMeas}(X)} \to \underline{B}$ such that $\mathcal{I}_X^B(\beta, \mu) = \int \beta(r)\mu(dr)$.

Remark 4.4. A very natural question is whether there are measurable cones which are not integrable. We strongly conjecture that such cones do exist but we have not yet tried to exhibit some.

The fundamental example of an integrable cone is the measurable cone of finite measures described in Section 3.2.1.

Theorem 4.5. *For each measurable space* X*, the measurable cone* $\mathsf{FMeas}(X)$ *is integrable.*

This is just a reformulation of the standard integration of a kernel.

Proof. Let $Y \in \mathbf{Ar}$, $\kappa \in \underline{\mathsf{Path}(Y, \mathsf{FMeas}(X))}$, which means that κ is a bounded kernel $Y \rightsquigarrow X$, and let $\nu \in \underline{\mathsf{FMeas}(Y)}$, which means that ν is a finite measure. We define $\mu : \sigma_X \to \mathbb{R}_{\geq 0}$ by

$$\forall U \in \sigma_X \quad \mu(U) = \int \kappa(s, U)\nu(ds) \in \mathbb{R}_{\geq 0}\,.$$

The fact that μ defined in that way is a finite measure is completely standard in measure theory and μ is the integral of κ by the very definition of $\mathcal{M}_0^{\mathsf{FMeas}(X)}$. □

In the sequel we assume that B is an integrable cone. We state and prove some basic expected properties of integration.

Lemma 4.6. *Let $\varphi : Y \times X \to \mathbb{R}_{\geq 0}$ be measurable and bounded and let $\kappa : Y \to \underline{\mathsf{FMeas}(X)}$ be a bounded kernel. Then the function $\boldsymbol{\lambda} s \in Y \cdot \int \varphi(s, r)\kappa(s, dr)$ is measurable.*

Proof. The property is obvious when φ is simple[13], and the result follows from the monotone convergence theorem by the fact that each $\mathbb{R}_{\geq 0}$-measurable function is the lub of a increasing sequence of simple functions. □

Lemma 4.7. *For each $X \in \mathbf{Ar}$, the map $\mathcal{I}_X^B$ is bilinear, continuous and measurable. This means that $\mathcal{I}_X^B : \underline{\mathsf{Path}(X, B)} \;\&\; \underline{\mathsf{FMeas}(X)} \to \underline{B}$ is continuous, separately linear in each of its two arguments and that for each $Y \in \mathbf{Ar}$, $\eta \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, B))}$ and $\kappa \in \underline{\mathsf{Path}(Y, \mathsf{FMeas}(X))}$, the function $\beta = \mathcal{I}_X^B \circ \langle \eta, \kappa \rangle : Y \to \underline{B}$ is a measurable path.*

Proof. Separate linearity in both argumets results from the linearity of integration and from (**Mssep**) satisfied by B, let us prove separate continuity (which implies continuity by Lemma 2.19). Let $(\beta_n)_{n \in \mathbb{N}}$ be an increasing sequence in $\mathcal{B}\underline{\mathsf{Path}(X, B)}$ and let $\mu \in \underline{\mathsf{FMeas}(X)}$. The sequence $(\mathcal{I}_X^B(\beta_n, \mu) \in \underline{B})_{n \in \mathbb{N}}$ is increasing by linearity of $\mathcal{I}_X^B$ and for all $n \in \mathbb{N}$ we have $\left\|\mathcal{I}_X^B(\beta_n, \mu)\right\| \leq \|\beta_n\| \, \|\mu\| \leq \|\mu\|$ so that $\sup_{n \in \mathbb{N}} \mathcal{I}_X^B(\beta_n, \mu) \in \underline{B}$ exists. Let $\beta = \sup_{n \in \mathbb{N}} \beta_n \in \mathcal{B}\underline{\mathsf{Path}(X, B)}$, that is $\forall r \in X \;\; \beta(r) = \sup_{n \in \mathbb{N}} \beta_n(r)$. Let $m \in \mathcal{M}_0^B$, since $(m \circ \beta_n)_{n \in \mathbb{N}}$ is an increasing sequence of measurable functions by linearity of m and since $m \circ \beta = \sup_{n \in \mathbb{N}} m \circ \beta_n$ (pointwise) by continuity of m, we have

$$\int m(\beta(r))\mu(dr) = \sup_{n \in \mathbb{N}} \int m(\beta_n(r))\mu(dr)$$

by the monotone convergence theorem. That is $m(\mathcal{I}_X^B(\beta, \mu)) = \sup_{n \in \mathbb{N}} m(\mathcal{I}_X^B(\beta_n, \mu)) = m(\sup_{n \in \mathbb{N}} \mathcal{I}_X^B(\beta_n, \mu))$ by continuity of m. By (**Mssep**) we get $\mathcal{I}_X^B(\beta, \mu) = \sup_{n \in \mathbb{N}} \mathcal{I}_X^B(\beta_n, \mu)$ as required.

Let $\beta \in \underline{\mathsf{Path}(X, B)}$ and let $(\mu_n \in \mathcal{B}\underline{\mathsf{FMeas}(X)})_{n \in \mathbb{N}}$ be an increasing sequence with lub μ. It is a standard fact that for each measurable and bounded $\varphi : X \to \mathbb{R}_{\geq 0}$ the sequence $(\int \varphi(r)\mu_n(dr))_{n \in \mathbb{N}}$ is increasing and has $\int \varphi(r)\mu(dr)$ as lub: this is due to the fact that $\int \varphi(r)\mu(dr) = \sup_{k \in \mathbb{N}} \int \varphi_k(r)\mu(dr)$ where $(\varphi_k \leq \varphi)_{k \in \mathbb{N}}$ is an increasing family of simple functions whose pointwise lub is φ, and to the fact that $\int \psi(r)\mu(dr) = \sup_{n \in \mathbb{N}} \int \psi(r)\mu_n(dr)$ holds trivially when ψ is simple. As above the sequence $(\mathcal{I}_X^B(\beta, \mu_n))_{n \in \mathbb{N}}$ is increasing with $\forall n \in \mathbb{N} \;\; \left\|\mathcal{I}_X^B(\beta, \mu_n)\right\| \leq \|\beta\| \, \|\mu\|$ and therefore has a lub $\sup_{n \in \mathbb{N}} \mathcal{I}_X^B(\beta, \mu_n) \in \underline{B}$. Let $m \in \mathcal{M}_0^B$,

[13] A $\mathbb{R}$-valued measurable function is simple iff it ranges in a finite subset of $\mathbb{R}$.

we have

$$\begin{aligned}m(\sup_{n\in\mathbb{N}}\mathcal{I}_X^B(\beta,\mu_n)) &= \sup_{n\in\mathbb{N}} m(\mathcal{I}_X^B(\beta,\mu_n))\\ &= \sup_{n\in\mathbb{N}}\int m(\beta(r))\mu_n(dr)\\ &= \int m(\beta(r))\mu(dr)\\ &= m(\mathcal{I}_X^B(\beta,\mu))\end{aligned}$$

and the announced continuity follows by (**Mssep**) in B.

Now we prove measurability, so let $Y\in\mathbf{Ar}$, $\eta\in\underline{\mathsf{Path}(Y,\mathsf{Path}(X,B))}$ and let $\kappa\in\underline{\mathsf{Path}(Y,\mathsf{FMeas}(X))}$, we prove that the function $\beta=\mathcal{I}_X^B\circ\langle\eta,\kappa\rangle : Y\to\underline{B}$ belongs to $\underline{\mathsf{Path}(Y,B)}$. The fact that $\beta(X)$ is bounded results from Lemma 4.2. Let $Y'\in\mathbf{Ar}$ and $m\in\mathcal{M}_{Y'}^B$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot m(s',\beta(s)) &= \boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot m(s',\mathcal{I}_X^B(\eta(s),\kappa(s)))\\ &= \boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot\int m(s',\eta(s,r))\kappa(s,dr)\end{aligned}$$

and this function is measurable by Lemma 4.6 and by our assumption about η. □

Lemma 4.8 (Change of variable)**.** *Let $X,Y\in\mathbf{Ar}$, $\beta\in\underline{\mathsf{Path}(X,B)}$, $\nu\in\underline{\mathsf{FMeas}(Y)}$ and $\varphi\in\mathbf{Ar}(Y,X)$. We have*

$$\int_{s\in Y}\beta(\varphi(s))\nu(ds)=\int_{r\in X}\beta(r)\varphi_*(\nu)(dr)\,.$$

In other words $\mathcal{I}_X^B$ is extranatural in X.

Proof. By the usual change of variable formula, through the use of measurability tests $m\in\mathcal{M}_0^B$ and (**Mssep**) for B. □

Lemma 4.9. *If B is an integrable cone and $\alpha\in\mathbb{R}$ is such that $\alpha>0$ then the measurable cone αB is integrable, and has the same integrals as B.*

We can define now the category which is at the core of the present study.

Definition 4.10. The category **ICones** has integrable cones as objects and an element of $\mathbf{ICones}(B,C)$ is an $f\in\mathbf{MCones}(B,C)$ such that, for all $X\in\mathbf{Ar}$ and all $\beta\in\underline{\mathsf{Path}(X,B)}$ and $\mu\in\underline{\mathsf{FMeas}(X)}$ one has

$$f\Big(\int\beta(r)\mu(dr)\Big)=\int f(\beta(r))\mu(dr)\,.$$

This property of f will be called *integral preservation* and when it holds we often simply say that f is *integrable*.

Notice that the right hand term of the above equation is well defined because $f\circ\beta\in\underline{\mathsf{Path}(X,C)}$ by our assumption on f. It is obvious that we define a category in that way.

Lemma 4.11. *The functor* $\mathsf{FMeas}:\mathbf{Ar}\to\mathbf{MCones}$ *introduced in Lemma 3.17 is a functor* $\mathbf{Ar}\to\mathbf{ICones}$.

Proof. Let $\varphi \in \mathbf{Ar}(X, Y)$ and $\kappa \in \underline{\mathsf{Path}(Y', \mathsf{FMeas}(X))}$ be a bounded kernel. Given $\mu' \in \underline{\mathsf{FMeas}(Y')}$ and $V \in \sigma_Y$ we have

$$\begin{aligned}\varphi_*\Big(\int \kappa(s')\mu'(ds')\Big)(V) &= \Big(\int \kappa(s')\mu'(ds')\Big)(\varphi^{-1}(V)) \\ &= \int \kappa(s', \varphi^{-1}(V))\mu'(ds') \quad \text{by def. of integration in } \mathsf{FMeas}(X) \\ &= \int \varphi_*(\kappa(s'))(V)\mu'(ds') \\ &= \Big(\int \varphi_*(\kappa(s'))\mu'(ds')\Big)(V) \quad \text{by def. of integration in } \mathsf{FMeas}(Y)\end{aligned}$$

so that φ_* preserves integrals. □

4.1. **Integrable cones as quasi-Borel spaces with additional structure.** In this section, we assume, as in Remark 3.1, that $\mathbf{Ar}$ has only two objects $\mathbb{R}$ and 0.

Then every integrable cone C can be seen as a QBS [VKS19] by letting M_C (which is by definition the set of all QBS-morphisms from $\mathbb{R}$ to C) be the set of all maps $\alpha : \mathbb{R} \to \underline{C}$ such that for all $m \in \mathcal{M}^C_{\mathbb{R}}$, the map $\boldsymbol{\lambda}(r, s) \cdot m(\alpha(r), s)$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is measurable.

There is a well-defined notion of S-finite measure (respectively: probability measure, sub-probability measure) on QBSs. The operation that maps a QBS Q to the set of all S-finite (respectively: probability, sub-probability) measures on Q defines a commutative strong monad on the category of QBSs [HKS+18, end of §2] (this is similar to the Giry monad on the category of measurable spaces and measurable maps). For each S-finite measure μ on an integrable cone C, there exists at most one element $y \in \underline{C}$ such that for all $m \in \mathcal{M}^C_0$, $m(y) = \int_{x \in \underline{C}} m(x)\, \mu(dx)$. If this element exists (which is always the case if μ is finite and has a bounded support), we will denote it by $\int_{x \in \underline{C}} x\, \mu(dx)$. One can check that for each integrable cone C, this construction makes the unit ball $\mathcal{B}C$ into an algebra over the monad of sub-probability measures on QBSs.

In this situation a map $f : \underline{C} \to \underline{B}$ between two integrable cones is a morphism in **ICones** if and only if:

- it is a morphism of QBSs,
- it preserves S-finite integrals: for each S-finite measure μ on C, if $\int_{x \in \underline{C}} x\, \mu(dx)$ exists (as an element of $\underline{C}$), then
$$\int_{x \in \underline{C}} f(x)\, \mu(dx) = f\Big(\int_{x \in \underline{C}} x\, \mu(dx)\Big),$$
- and it is non-expansive: for all $x \in \underline{C}$, $\|f(x)\| \leq \|x\|$.

In particular, for each morphism $f \in \mathbf{ICones}(B, C)$, the restriction of f to $\mathcal{B}\underline{B}$ is a morphism of algebras: this was one of our main guidelines in the design of integrable cones.

However, it is not clear whether or not each morphism of algebras from $\mathcal{B}\underline{B}$ to $\mathcal{B}\underline{C}$ is the restriction of some morphism in $\mathbf{ICones}(B, C)$ (which would make **ICones** a full subcategory of the category of algebras over the monad of sub-probability measures on QBSs). It is also not clear whether the construction of Section 9.2 (which will define a fixpoint operator on integrable cones) can be replicated in the category of algebras over the monad of sub-probability measures on QBSs. Indeed, we conjecture that in both cases the answer is no. On the other hand, integrable cones are not quite algebras over the monad of S-finite measures, because the would-be monad multiplication (namely, S-finite integration) is only partially defined. We do not know whether there exists a monad on (a full subcategory of) QBSs such that **ICones** is equivalent to the category of algebras over this monad.

4.2. The integrable cone of paths and a Fubini theorem for cones.

Theorem 4.12. *For each $X \in \mathbf{Ar}$ and each integrable cone B, the measurable cone $\mathsf{Path}(X, B)$ is integrable.*

Proof. Let $Y \in \mathbf{Ar}$, $\eta \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, B))}$ and $\nu \in \underline{\mathsf{FMeas}(Y)}$, we define $\beta : X \to \underline{B}$ by $\beta(r) = \int \eta(s)(r)\nu(ds)$, in other words the integral of a path of paths is defined pointwise. For each $r \in X$ we have

$$\begin{aligned}\|\beta(r)\| &= \left\|\int \eta(s)(r)\nu(ds)\right\| \\ &\leq \|\boldsymbol{\lambda} s \in Y \cdot \eta(s)(r)\|\,\|\nu\| \quad \text{by Lemma 4.2} \\ &\leq \|\eta\|\,\|\nu\|\end{aligned}$$

so β is a bounded function. This function is a measurable path by Lemma 4.7 so β belongs to $\underline{\mathsf{Path}(X, B)}$. Let $p \in \mathcal{M}_0^{\mathsf{Path}(X,B)}$ so that $p = r \triangleright m$ for some $r \in X$ and $m \in \mathcal{M}_0^B$, we have

$$\begin{aligned}p(\beta) &= m(\beta(r)) \\ &= m\Big(\int \eta(s)(r)\nu(ds)\Big) \\ &= \int m(\eta(s)(r))\nu(ds) \\ &= \int p(\eta(s))\nu(ds) \quad \text{by definition of } p\end{aligned}$$

and hence $\beta = \int \eta(s)\nu(ds)$. ☐

Theorem 4.13. *The operation* Path*, extended to morphisms by pre- and post-composition, is a functor $\mathbf{Ar}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$. In other words, given $f \in \mathbf{ICones}(B, C)$ and $\varphi \in \mathbf{Ar}(Y, X)$, we have*

$$\mathsf{Path}(\varphi, f) = \boldsymbol{\lambda}\beta \in \underline{\mathsf{Path}(X, B)} \cdot (f \circ \beta \circ \varphi) \in \mathbf{ICones}(\mathsf{Path}(X, B), \mathsf{Path}(Y, C))\,.$$

Proof. Functoriality is obvious. We check first measurability of $\mathsf{Path}(\varphi, f)$ so let $Y' \in \mathbf{Ar}$ and let $\eta \in \underline{\mathsf{Path}(Y', \mathsf{Path}(X, B))}$, we must check that $\mathsf{Path}(\varphi, f) \circ \eta \in \underline{\mathsf{Path}(Y', \mathsf{Path}(Y, C))}$. Let $Y'' \in \mathbf{Ar}$ and $p \in \mathcal{M}_{Y''}^{\mathsf{Path}(Y,C)}$, we check that $\psi = \boldsymbol{\lambda}(s'', s') \in Y'' \times Y' \cdot p(s'', \mathsf{Path}(\varphi, f)(\eta(s')))$ is measurable. So let $\rho \in \mathbf{Ar}(Y'', Y)$ and $m \in \mathcal{M}_{Y''}^C$ be such that $p = \rho \triangleright m$. Give $s'' \in Y''$ and $s' \in Y'$, we set

$$\begin{aligned}\psi(s'', s') &= m(s'', \mathsf{Path}(\varphi, f)(\eta(s'))(\rho(s''))) \\ &= m(s'', f(\eta(s')(\varphi(\rho(s''))))) \\ &= m(s'', f(\mathsf{fl}(\eta)(s', \varphi(\rho(s'')))))\end{aligned}$$

and the map ψ is measurable by Lemma 3.19 because $f \circ \mathsf{fl}(\eta) \in \underline{\mathsf{Path}(Y' \times X, C)}$ and $\varphi \circ \rho$ is measurable. We prove that $\mathsf{Path}(\varphi, f)$ preserves integrals. Let $Y' \in \mathbf{Ar}$, $\eta \in \underline{\mathsf{Path}(Y', \mathsf{Path}(X, B))}$, $\nu' \in \underline{\mathsf{FMeas}(Y')}$ and let $s \in Y$. We have

$$\begin{aligned}
\mathsf{Path}(\varphi, f)\Big(\int_{s' \in Y'}^{\mathsf{Path}(X,B)} \eta(s')\nu'(ds')\Big)(s) &= f\Big(\Big(\int_{s' \in Y'}^{\mathsf{Path}(X,B)} \eta(s')\nu'(ds')\Big)(\varphi(s))\Big) \\
&= f\Big(\int_{s' \in Y'}^{B} \eta(s')(\varphi(s))\nu'(ds')\Big) \quad \text{by def. of integration in } \mathsf{Path}(X, B) \\
&= \int_{s' \in Y'}^{C} f(\eta(s')(\varphi(s)))\nu'(ds') \quad \text{since } f \text{ preserves integrals} \\
&= \Big(\int_{s' \in Y'}^{\mathsf{Path}(Y,C)} \mathsf{Path}(\varphi, f)(\eta(s'))\nu'(ds')\Big)(s)
\end{aligned}$$

which proves our contention. □

Lemma 4.14. *The bijection $\mathsf{fl}_{X,Y}$ defined in Lemma 3.19, as well as its inverse, preserve integrals and hence*

$$\mathsf{fl}_{X,Y} \in \mathbf{ICones}(\mathsf{Path}(X, \mathsf{Path}(Y, B)), \mathsf{Path}(X \times Y, B))$$

is an iso in **ICones**.

Proof. Results straightforwardly from the "pointwise" definition of integration in the cones of paths. □

Theorem 4.15 (Fubini)**.** *Let $X, Y \in \mathbf{Ar}$, $\eta \in \underline{\mathsf{Path}(X, \mathsf{Path}(Y, B))}$, $\mu \in \underline{\mathsf{FMeas}(X)}$ and $\nu \in \underline{\mathsf{FMeas}(Y)}$. We have*

$$\int_Y \Big(\int_X \eta(r)\mu(dr)\Big)(s)\nu(ds) = \int_{X \times Y} \mathsf{fl}(\eta)(t)(\mu \times \nu)(dt)$$

Proof. Denoting by x_1 and x_2 these two elements of $\underline{B}$ it suffices to prove that for each $m \in \mathcal{M}_0^B$ one has $m(x_1) = m(x_2)$. Setting $\eta' = \mathsf{fl}(\eta)$ we have

$$x_1 = \int_Y \Big(\int_X \eta'(r, s)\mu(dr)\Big)\nu(ds) \qquad\qquad x_2 = \int_{X \times Y} \eta'(t)(\mu \times \nu)(dt)$$

and the equation follows by application of the usual Fubini theorem to the bounded non-negative measurable function $m \circ \eta'$ and to the finite measures μ and ν. Notice that in the expression of x_2 the variable t ranges over pairs. □

4.3. **The category of integrable cones.**

We start with proving that the category **ICones** of Definition 4.10 has all (projective) limits. This is not only a very pleasant property of the probabilistic model of LL that we are defining[14], it will play a crucial role in our definition of the tensor product and of the exponentials.

Theorem 4.16. *The category* **ICones** *is complete.*

[14]Which is not so common among models of LL; there is however a price to pay, it is very likely that the category **ICones** has no $*$-autonomous structure.

There is a faithful forgetful functor $\mathbf{ICones} \to \mathbf{Set}$ which maps each integrable cone C to $\underline{C}$, considered as a set, and each morphism to itself; we will see that this functor actually creates all the small limits in $\mathbf{ICones}$.

Proof. We prove first that $\mathbf{ICones}$ has all small products. We use implicitly Theorem 2.18 at several places. Let $(C_i)_{i\in I}$ be a collection of integrable cones and let $P = \&_{i\in I} \underline{C_i}$ which is the product of the $\underline{C_i}$'s in $\mathbf{Cones}$. Given $X \in \mathbf{Ar}$, $i \in I$ and $m \in \mathcal{M}_X^{C_i}$ we define $\mathsf{in}_i(m) : X \times P \to \mathbb{R}_{\geq 0}$ by $\mathsf{in}_i(m)(r, \overrightarrow{x}) = m(r, x_i)$. We set $\mathcal{M} = (\mathcal{M}_X)_{X\in\mathbf{Ar}}$ where $\mathcal{M}_X = \{\mathsf{in}_i(m) \mid i \in I \text{ and } m \in \mathcal{M}_X^{C_i}\}$. With the notations above, given $\overrightarrow{x} \in P$, the function $\boldsymbol{\lambda} r \in X \cdot \mathsf{in}_i(m)(r, \overrightarrow{x}) = \boldsymbol{\lambda} r \in X \cdot m(r, x_i)$ is measurable since $\mathcal{M}^{C_i}$ satisfies (**Msmeas**).

Let $\varphi \in \mathbf{Ar}(Y, X)$, we have $\mathsf{in}_i(m) \circ (\varphi \times P) = \mathsf{in}_i(m \circ (\varphi \times \underline{C_i})) \in \mathcal{M}_Y$ since $m \circ (\varphi \times \underline{C_i}) \in \mathcal{M}_Y^{C_i}$ by (**Mscomp**) in C_i.

Let $\overrightarrow{x(1)}, \overrightarrow{x(2)} \in P$ be such that $\forall p \in \mathcal{M}_0\ p(\overrightarrow{x(1)}) = p(\overrightarrow{x(2)})$. Then for each $i \in I$ we have $x(1)_i = x(2)_i$ by (**Mssep**) holding in C_i and hence $\overrightarrow{x(1)} = \overrightarrow{x(2)}$.

Let $\overrightarrow{x} \in P \setminus \{0\}$ and $\varepsilon > 0$. Since $\|\overrightarrow{x}\| = \sup_{i\in I} \|x_i\|$ there is $i \in I$ such that $\|\overrightarrow{x}\| \leq \|x_i\| + \varepsilon/2$ and $x_i \neq 0$. We can find $m \in \mathcal{M}_0^{C_i} \setminus \{0\}$ such that $\|x_i\| \leq m(x_i)/\|m\| + \varepsilon/2$. Let $p = \mathsf{in}_i(m) \in \mathcal{M}_0$, notice that $\|p\| = \|m_i\|$ since for each $x \in \mathcal{B}\underline{C_i}$ the family $\overrightarrow{y}$ defined by $y_i = x$ and $y_j = 0$ if $j \neq i$ satisfies $\overrightarrow{y} \in \mathcal{B}P$. So we have $\|\overrightarrow{x}\| \leq p(\overrightarrow{x})/\|p\| + \varepsilon$ which shows that $\mathcal{M}$ satisfies (**Msnorm**).

So the pair $(P, \mathcal{M})$ is a measurable cone $C = \&_{i\in I} C_i$, we prove that it is integrable. An element of $\underline{\mathsf{Path}(X, C)}$ is a family $(\gamma_i \in \underline{\mathsf{Path}(X, C_i)})_{i\in I}$ such that $(\|\gamma_i\|)_{i\in I}$ is bounded and, given $\mu \in \underline{\mathsf{FMeas}(X)}$, the family

$$\overrightarrow{x} = \Big(\int \gamma_i(r)\mu(dr)\Big)_{i\in I}$$

is in P by Lemma 4.2 and is the integral of γ over μ by definition of $\mathcal{M}^C$.

With the same notations as above, for each $i \in I$, the map $\mathsf{pr}_i \circ \gamma$ is a measurable path since, given $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}_Y^{C_i}$ one has $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot \mathsf{in}_i(m)(s, \gamma(r)) = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \mathsf{pr}_i(\gamma(r))$. The fact that $\mathsf{pr}_i \in \mathbf{ICones}(C, C_i)$ results from the definition of integration in C.

Let $(f_i \in \mathbf{ICones}(D, C_i))_{i\in I}$, then we know that $f = \langle f_i\rangle_{i\in I} \in \mathbf{Cones}(\underline{D}, \underline{C})$. Let $\delta \in \underline{\mathsf{Path}(X, D)}$, we prove that $f \circ \delta \in \underline{\mathsf{Path}(X, C)}$ so let $i \in I$ and $m \in \mathcal{M}_Y^{C_i}$. We have $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot \mathsf{in}_i(m)(s, f(\delta(r))) = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, f_i(\delta(r)))$ and this latter map is measurable for each $i \in I$ thus proving that $f \circ \delta$ is measurable. Using the same notations, let furthermore $\mu \in \underline{\mathsf{FMeas}(X)}$, we have

$$\begin{aligned} f\Big(\int \delta(r)\mu(dr)\Big) &= \Big(f_i\Big(\int \delta(r)\mu(dr)\Big)\Big)_{i\in I} \\ &= \Big(\int f_i(\delta(r))\mu(dr))\Big)_{i\in I} \quad \text{since each } f_i \text{ preserves integrals} \\ &= \int f(\delta(r))\mu(dr) \end{aligned}$$

which shows that $f \in \mathbf{ICones}(D, C)$ as required. This proves that $\mathbf{ICones}$ has all small products.

We prove now that **ICones** has equalizers, so let $f, g \in \mathbf{ICones}(C, D)$. Let $(P, e \in \mathbf{Cones}(P, \underline{C}))$ be the equalizer of f and g in **Cones**, see Theorem 2.20. Remember that if $x, y \in P$ satisfy $x \leq_{\underline{C}} y$ then $y - x \in P$.

We define $\mathcal{M}_X$ as the set of all $p : X \times P \to \mathbb{R}_{\geq 0}$ such that there is $m \in \mathcal{M}^C_X$ satisfying $\forall x \in P\, \forall r \in X\; p(r, x) = m(r, x)$. Then it is clear that $p \in (P')^X$ and we actually identify $\mathcal{M}_X$ with $\mathcal{M}^C_X$ although several elements of the latter can induce the same element of the former. We prove that $(\mathcal{M}_X)_{X \in \mathbf{Ar}}$ defines a measurability structure on P, the only non trivial property being (**Msnorm**). Let $x \in P \setminus \{0\}$ and $\varepsilon > 0$. Let $\varepsilon' > 0$ be such that $\varepsilon' \leq \varepsilon$ and $\varepsilon' < \|x\|$ (remember that we have assumed that $x \neq 0$ and hence $\|x\| > 0$). Applying (**Msnorm**) in C we can find $m \in \mathcal{M}^C_0 \setminus \{0\}$ such that $\|x\| = \|x\|_C$ satisfies $\|x\| \leq m(x)/\|m\|^{\underline{C}} + \varepsilon'$ where we have added the superscript to $\|m\|$ to insist on the fact that it is computed in $\underline{C}$, that is $\|m\|^{\underline{C}} = \sup_{y \in \mathcal{B}\underline{C}} m(y)$. By our assumption that $\varepsilon' < \|x\|$ we must have $m(x) \neq 0$. By definition of $\|\_\|_P$ we have $\mathcal{B}P = P \cap \mathcal{B}\underline{C} \subseteq \mathcal{B}\underline{C}$ and hence $\|m\|^P = \sup_{y \in \mathcal{B}P} m(y) \leq \sup_{y \in \mathcal{B}\underline{C}} m(y) = \|m\|^{\underline{C}}$ (and $\|m\|^P \neq 0$ since $m(x) \neq 0$ and $x \in P$) and hence

$$\|x\|_P = \|x\|_C \leq \frac{m(x)}{\|m\|^{\underline{C}}} + \varepsilon' \leq \frac{m(x)}{\|m\|^P} + \varepsilon$$

since $\|m\|^P \leq \|m\|^{\underline{C}}$ and $\varepsilon' \leq \varepsilon$, and since this holds for all $\varepsilon > 0$, it follows that P satisfies (**Msnorm**).

So we have defined a measurable cone $E = (P, \mathcal{M})$, we check that it is integrable. Let $X \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(X, E)}$ and $\mu \in \underline{\mathsf{FMeas}(X)}$, we have

$$f\Big(\int \beta(r)\mu(dr)\Big) = \int f(\beta(r))\mu(dr) = \int g(\beta(r))\mu(dr) = g\Big(\int \beta(r)\mu(dr)\Big)$$

since β ranges in $\underline{E} = P$ and f and g preserve integrals. Hence $\int \beta(r)\mu(dr) \in \underline{E}$ and this element of $\underline{E}$ is the integral of β over μ by definition of $\mathcal{M}^E$.

We check now that (E, e) is the equalizer of f, g in **ICones**. The inclusion $e \in \mathbf{Cones}(\underline{E}, \underline{C})$ is measurable $E \to C$ by definition of the measurability structure of E which is essentially the same as that of C and preserves integrals because the integral in E is defined as in C.

We already know that $f\, e = g\, e$. Let H be an integrable cone and $h \in \mathbf{ICones}(H, C)$ be such that $f\, h = g\, h$. Let h_0 be the unique element of $\mathbf{Cones}(\underline{H}, \underline{E})$ such that $h = e\, h_0$. Let $X \in \mathbf{Ar}$ and $\gamma \in \underline{\mathsf{Path}(X, H)}$ be a measurable path of H. Let $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^E_Y$ so that actually $m \in \mathcal{M}^{\underline{C}}_Y$. We have $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, h_0(\gamma(r))) = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, h(\gamma(r)))$ which is measurable since h is. With the same notation, taking also μ in $\underline{\mathsf{FMeas}(X)}$, we have

$$\begin{aligned} h_0\Big(\int^H \gamma(r)\mu(dr)\Big) &= h\Big(\int^H \gamma(r)\mu(dr)\Big) \quad \text{by definition of } h_0 \\ &= \int^C h(\gamma(r))\mu(dr) \quad \text{since } h \text{ preserves integrals} \\ &= \int^E h_0(\gamma(r))\mu(dr) \end{aligned}$$

and hence $h_0 \in \mathbf{ICones}(H, E)$. Since $h = e\, h_0$ and is unique with this property in **Cones**, it has the same properties in **ICones**.

This shows that **ICones** has all small limits. □

Lemma 4.17. *Let C be an integrable cone, S be a set, and let $f : \underline{C} \to S$ be a bijection. There is an integrable cone structure on S such that f is an iso in* **ICones***.*

This structure is not unique *a priori* (other choices for the measurability structure are possible in general), but this is not an issue for the use that we will make of this lemma.

Proof. By Lemma 2.23 we can equip S with a cone structure such that $f \in \mathbf{Cones}(\underline{C}, S)$ (we use S for the cone obtained by equipping S with this structure), and hence f is an iso in **Cones** since f is a bijection. Let $X \in \mathbf{Ar}$ and let $m \in \mathcal{M}^C_X$. We define $f_*(m) : X \times S \to \mathbb{R}_{\geq 0}$ by $f_*(m)(r, z) = m(r, f^{-1}(z))$. We set $\mathcal{M}_X = \{f_*(m) \mid m \in \mathcal{M}^C_X\}$. In view of the definition of the algebraic structure and of the norm of S, it is clear that $(\mathcal{M}_X)_{X\in\mathbf{Ar}}$ is a measurability structure on S, we still use S for denoting this measurable cone and we observe that f is an iso from C to S in **MCones**. It is also easy to check that $\mathsf{Path}(X, S) = \{f \circ \gamma \mid \gamma \in \mathsf{Path}(X, C)\}$. This cone S is integrable: given $\mu \in \mathsf{FMeas}(X)$ and $\gamma \in \mathsf{Path}(X, S)$, we have $f^{-1} \circ \gamma \in \mathsf{Path}(X, C)$ and hence the integral $x = \int_C f^{-1}(\gamma(r))\mu(dr) \in \underline{C}$ exists. Then for each $p = f_*(m) \in \mathcal{M}^S_0$ where $m \in \mathcal{M}^C_0$, we have $p(f(x)) = m(x) = \int_{\mathbb{R}_{\geq 0}} m(f^{-1}(\gamma(r)))\mu(dr) = \int_{\mathbb{R}_{\geq 0}} p(\gamma(r))\mu(dr)$ which shows that $\int_S \gamma(r)\mu(dr)$ exists and is $f(x)$. It follows also trivially that f preserves integrals. □

Theorem 4.18. *In the category* **ICones** *the object* 1 *is a coseparator and a separator*[15] *and* **ICones** *is well-powered.*

Proof. Let $f \neq g \in \mathbf{ICones}(C, D)$ and let $x \in \underline{C}$ be such that $f(x) \neq g(x)$. By (**Mssep**) there is $m \in \mathcal{M}^C_0$ such that $m(f(x)) \neq m(g(x))$ and since $m \in \mathbf{ICones}(C, 1)$ (using the definition of integrals) this shows that 1 is a coseparator.

Given $x \in \underline{C}$ we check that the function $\widehat{x} : \mathbb{R}_{\geq 0} \to \underline{C}$ defined by $\widehat{x}(\lambda) = \lambda x$ belongs to $\mathbf{ICones}(1, C)$. It is clearly linear and continuous. Let $\beta \in \underline{\mathsf{Path}(X, 1)}$ for some $X \in \mathbf{Ar}$. This simply means that β is a measurable and bounded function $X \to \mathbb{R}_{\geq 0}$, we must check that $\widehat{x} \circ \beta \in \underline{\mathsf{Path}(X, C)}$ so let $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^C_Y$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \widehat{x}(\beta(r))) &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \beta(r)x)\\ &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot \beta(r)m(s, x)\end{aligned}$$

which is measurable by measurability of multiplication. With the same notations and using moreover some $\mu \in \underline{\mathsf{FMeas}(X)}$ we must prove that

$$\widehat{x}\Big(\int^1 \beta(r)\mu(dr)\Big) = \int^C \widehat{x}(\beta(r))\mu(dr)\,.$$

[15]In the literature one also finds the words *generator* and *cogenerator* for such objects.

Remember that the second member of this equation is well defined since we have shown that $\widehat{x}\circ\beta \in \underline{\mathsf{Path}(X,C)}$. To check the equation, let $m\in\mathcal{M}_0^C$, we have

$$\begin{aligned} m\Big(\widehat{x}\Big(\int^1 \beta(r)\mu(dr)\Big)\Big) &= m\Big(\Big(\int^1 \beta(r)\mu(dr)\Big)x\Big)\\ &= \Big(\int^1 \beta(r)\mu(dr)\Big)m(x)\\ &= \int^1 \beta(r)m(x)\mu(dr)\\ &= \int^1 m(\widehat{x}(\beta(r)))\mu(dr)\\ &= m\Big(\int^C \widehat{x}(\beta(r))\mu(dr)\Big)\,. \end{aligned}$$

So $\widehat{x}\in\mathbf{ICones}(1,C)$ as contended. Since $\widehat{x}(1)=x$ this shows that 1 is a separator.

Let D be a subobject of C; more precisely let $h\in\mathbf{ICones}(D,C)$ be a mono. This implies that h is injective because 1 is a separator. Let $S=h(\underline{D})\subseteq\underline{C}$. By Lemma 4.17, there is an integrable cone D' such that $\underline{D'}=S$ (as sets) and f, the corestriction of h to S, is an iso in **ICones** from D to D'. Moreover the inclusion e of S into the set $\underline{C}$ satisfies $e=h\circ f^{-1}$ and hence $e\in\mathbf{ICones}(D',C)$.

We have proven that, in the slice category $\mathbf{ICones}/C$, each subobject (D,h) of C is isomorphic to a subobject (D',h') of C such that h' is an inclusion (that is $\forall y\in\underline{D'}\ h'(y)=y$). Notice finally that that the class of subobjects (D',h') of C such that h' is an inclusion is a set because **Ar** is a set[16]. Consider indeed a subset S of $\underline{C}$. The class of all structures of measurable cones D' whose underlying set is S is contained in

$$\begin{aligned} &S^{S\times S} \quad \text{(contains all possible additions)}\\ &\times S^{\mathbb{R}_{\geq 0}\times S} \quad \text{(contains all possible scalar multiplications)}\\ &\times \mathbb{R}_{\geq 0}^{S} \quad \text{(contains all possible norms)}\\ &\times \prod_{X\in\mathbf{Ar}} \mathcal{P}\left(\mathbb{R}_{\geq 0}^{X\times S}\right) \quad \text{(contains all possible measurability structures)} \end{aligned}$$

which is a set $\mathcal{F}(S)$ *because* **Ar** *is small*. Now the class of all subobjects (D',h') of C such that h' is an inclusion is contained in $\{(S,F)\mid S\subseteq\underline{C}\text{ and }F\in\mathcal{F}(S)\}$, which is a set. This shows that the class of subobjects of C is essentially small, that is **ICones** is well-powered. □

Theorem 4.19. *If $\mathcal{C}$ is a locally small category and $R:\mathbf{ICones}\to\mathcal{C}$ is a functor which preserves all limits, then R has a left adjoint.*

Proof. Apply the special adjoint functor theorem. □

Remark 4.20. This implies in particular that the forgetful functor $\mathbf{ICones}\to\mathbf{MCones}$ (which obviously preserves all limits) has a left adjoint, meaning that each measurable cone can be “completed with integrals”.

[16] It is only here that we use this assumption but it is essential.

4.4. Colimits and coproducts.

Theorem 4.21. *The category* **ICones** *has all small colimits.*

Proof. Let I be a small category, we use $\mathbf{ICones}^I$ for the category whose objects are the functors $I \to \mathbf{ICones}$ and the morphisms are the natural transformations, which is locally small since I is small. Then we have a "diagonal" functor $\Delta : \mathbf{ICones} \to \mathbf{ICones}^I$ which maps each object of **ICones** to the corresponding constant functor and each morphism to the identity natural transformation. It is easily checked that Δ preserves all limits and hence it has a left adjoint by Theorem 4.19. By definition of an adjunction, this functor maps each functor $I \to \mathbf{ICones}$ to its colimit which shows that **ICones** is cocomplete. □

This theorem does not give any insight on the structure of these colimits[17], so it is reasonable to have at least a closer look at coproducts.

Coproducts of cones. Let I be a set, without any restrictions on its cardinality for the time being. Let $(P_i)_{i\in I}$ be a family of cones. Let P be the set of all families $\overrightarrow{x} = (x_i)_{i\in I} \in \prod_{i\in I} P_i$ such that $\sum_{i\in I} \|x_i\| < \infty$. Notice that for such a family $\overrightarrow{x}$, the set $\{i \in I \mid x_i \neq 0\}$ is countable. We turn P into a cone by defining the operations componentwise and by setting $\|\overrightarrow{x}\| = \sum_{i\in I} \|x_i\|$. The induced cone order relation on P is the pointwise order and ω-completeness is easily proven (by commutations of lubs with sums in $\overline{\mathbb{R}_{\geq 0}}$). In **Cones**, this cone P is the coproduct of the P_i's with obvious injections $\mathsf{in}_i \in \mathbf{Cones}(P_i, P)$ mapping x to the family $\overrightarrow{x}$ such that $x_i = x$ and $x_j = 0$ for $j \neq i$. Given a family $(f_i \in \mathbf{Cones}(P_i, Q))_{i\in I}$ the unique map $[f_i]_{i\in I} \in \mathbf{Cones}(P, Q)$ such that $\forall j \in I\ \ [f_i]_{i\in I}\ \mathsf{in}_j = f_j$ is given by

$$[f_i]_{i\in I}\,(\overrightarrow{x}) = \sum_{i\in I} f_i(x_i)\,.$$

This sum converges because for each finite $J \subseteq I$ one has

$$\left\|\sum_{i\in J} f_i(x_i)\right\| \leq \sum_{i\in J} \|f_i(x_i)\| \leq \sum_{i\in J} \|x_i\| = \|x\|$$

and this map $[f_i]_{i\in I}$ is easily seen to be linear and continuous. We use $\oplus_{i\in I} P_i$ for the cone P defined in that way.

Lemma 4.22. *For each cone Q the cones $(\oplus_{i\in I} P_i) \multimap Q$ and $\&_{i\in I}\,(P_i \multimap Q)$ are isomorphic in* **Cones***.*

Proof. The fact that $\oplus_{i\in I} P_i$ is the coproduct of the P_i's means that the function

$$\mathcal{B}((\bigoplus_{i\in I} P_i) \multimap Q) \to \mathcal{B}(\mathop{\&}_{i\in I}(P_i \multimap Q))$$

which maps f to $(f\,\mathsf{in}_i)_{i\in I}$ is a bijection. It is linear and continuous by linearity and continuity of composition of morphisms. So this bijection is an isomorphism. □

In particular $(\oplus_{i\in I} P_i)' \simeq \&_{i\in I} P'_i$. Given $\overrightarrow{x'} \in \&_{i\in I} P'_i$ the associated linear and continuous form $\mathsf{fun}(\overrightarrow{x'})$ on $\oplus_{i\in I} P_i$ is given by

$$\mathsf{fun}(\overrightarrow{x'})(\overrightarrow{x}) = \langle \overrightarrow{x}, \overrightarrow{x'} \rangle = \sum_{i\in I} \langle x_i, x'_i \rangle \leq \|\overrightarrow{x}\| \left\|\overrightarrow{x'}\right\|\,.$$

[17] In particular it would be interesting to have a more explicit description of coequalizers.

We use these observations in the sequel.

Coproduct of measurable cones. Let $(C_i)_{i\in I}$ be a family of measurable cones. Let $P = \oplus_{i\in I} \underline{C_i}$. Let $\mathcal{M} = (\mathcal{M}_X)_{X\in\mathbf{Ar}}$ where $\mathcal{M}_X$ is the set of all $p \in (P')^X$ such that there is a family of coefficients $(\lambda_i \in \mathbb{R}_{\geq 0})_{i\in I}$ with $\boldsymbol{\lambda} r \in X \cdot \lambda_i p(r)_i \in \mathcal{M}^{C_i}_X$, identifying P' with $\&_{i\in I} \underline{C_i}'$ as explained above. In other words $p \in \mathcal{M}_X$ means that there are families $\overrightarrow{m} = (m_i \in \mathcal{M}^{C_i}_X)_{i\in I}$ and $\overrightarrow{\lambda} = (\lambda_i \in \mathbb{R}_{\geq 0})_{i\in I}$ such that, for all $r \in X$, the family $(\lambda_i \|m_i(r)\|)_{i\in I}$ is bounded by 1, and we have $p(r) = \mathsf{fun}(\overrightarrow{\lambda}\overrightarrow{m}(r))$ (where $\overrightarrow{\lambda}\overrightarrow{x} = (\lambda_i x_i)_{i\in I}$). Remember indeed from (**Msmeas**) that, for each measurable cone C, it is assumed that each $m \in \mathcal{M}^C_X$ satisfies that $m(r,x) \in [0,1]$ for all $r \in X$ and $x \in \mathcal{B}\underline{C}$.

Remark 4.23. These coefficients $\lambda_i \in \mathbb{R}_{\geq 0}$ are necessary because in the definition of measurable cones, we make very weak assumptions about the sets of measurability tests, in particular we do not assume that they are closed under multiplication by nonnegative coefficients ≤ 1. Such assumptions — and stronger ones, for instance, as suggested by one of the reviewers, we could require these sets of tests to be cones with operations defined pointwise — would be quite meaningful, but would require to check additional conditions in the proofs, for artificial reasons. The sets of measurability tests of a cone C should be understood as a kind of "predual" of the cone of paths $\mathsf{FMeas}(X, C)$, in the sense that the criterion for a bounded map $X \to \underline{C}$ to belong to this cone is the measurability of the (suitably defined) composition of this map with all measurability tests.

We prove that $\mathcal{M}$ is a measurability structure on P. Given $p \in \mathcal{M}_X$ and $\overrightarrow{x} \in P$ the map $\boldsymbol{\lambda} r \in X \cdot p(r)(\overrightarrow{x})$ is measurable by the monotone convergence theorem because the set $\{i \in I \mid x_i \neq 0\}$ is countable so the condition (**Msmeas**) holds. The conditions (**Mscomp**) and (**Mssep**) obviously hold, let us check (**Msnorm**). Let $\overrightarrow{x} \in P \setminus \{0\}$ and let $\varepsilon > 0$. Let $J = \{i \in I \mid x_i \neq 0\}$ which is countable and let $(i(n))_{n\in\mathbb{N}}$ be an enumeration of this set (assuming that it is infinite; the case where it is finite is simpler). For each $n \in \mathbb{N}$ let $m_n \in \mathcal{M}^{C_{i(n)}}_X$ be such that $m_n \neq 0$ and

$$\|x_{i(n)}\|_{C_{i(n)}} \leq \frac{m_n(x_{i(n)})}{\|m_n\|} + \frac{\varepsilon}{2^{n+1}}\,.$$

Let $p \in \mathcal{M}_0$ be given by $p(\overrightarrow{y}) = \sum_{n\in\mathbb{N}} \frac{m_n(y_{i(n)})}{\|m_n\|}$. We have $p(y) \leq \sum_{n\in\mathbb{N}} \|y_{i(n)}\| \leq 1$ and hence $0 < \|p\| \leq 1$. So we have

$$\|\overrightarrow{x}\| = \sum_{n\in\mathbb{N}} \|x_{i(n)}\| \leq \sum_{n\in\mathbb{N}} \frac{m_n(x_{i(n)})}{\|m_n\|} + \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^{n+1}} = p(\overrightarrow{x}) + \varepsilon \leq \frac{p(\overrightarrow{x})}{\|p\|} + \varepsilon$$

proving our contention. We have shown that $C = (P, \mathcal{M})$ is a measurable cone that we denote as $\oplus_{i\in I} C_i$.

Theorem 4.24. *For each $j \in I$ one has $(\mathsf{in}_j \in \mathbf{MCones}(C_j, \oplus_{i\in I} C_i))_{i\in I}$.*

If I is countable then $(\oplus_{i\in I} C_i, (\mathsf{in}_i)_{i\in I})$ is the coproduct of the C_i's in **MCones**. *If moreover the C_i's are integrable then so is $\oplus_{i\in I} C_i$, the in_i's preserve integrals and $(\oplus_{i\in I} C_i, (\mathsf{in}_i)_{i\in I}$ is the coproduct of the C_i's in* **ICones**.

Proof. The measurability of the in_i's is easy to prove.

We assume that I is countable. Let $(f_i \in \mathbf{MCones}(C_i, D))_{i\in I}$, we have already defined $f = [f_i]_{i\in I} \in \mathbf{Cones}(\oplus_{i\in I} \underline{C_i}, \underline{D})$ and we must prove that this function is measurable. Let

$X \in \mathbf{Ar}$ and $\gamma \in \underline{\mathsf{Path}(X, \oplus_{i\in I} C_i)}$, we prove that $f \circ \gamma \in \underline{\mathsf{Path}(X, D)}$ so let $Y \in \mathbf{Ar}$ and $q \in \mathcal{M}_Y^D$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s,r) \in Y \times X \cdot q(s, f(\gamma(r))) &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot q\big(s, \sum_{i\in I} f_i(\gamma(r)_i)\big) \\ &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot \sum_{i\in I} q(s, f_i(\gamma(r)_i))\end{aligned}$$

which is measurable by the monotone convergence theorem since I is countable.

Assume moreover that the C_i's are integrable and let $\mu \in \underline{\mathsf{FMeas}(X)}$. For each $i \in I$ we have $\boldsymbol{\lambda} r \in X \cdot \gamma(r)_i \in \underline{\mathsf{Path}(X, C_i)}$ because, for each $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}_Y^{C_i}$ we know that $\boldsymbol{\lambda}(s,r) \in Y \times X \cdot p(s, \gamma(r))$ is measurable, where $p = \mathsf{fun}(\overrightarrow{m})$ with $m_j = m$ if $i = j$ and $m_j = 0$ otherwise[18]. Therefore we can define $\overrightarrow{x} \in \prod_{i\in I} \underline{C_i}$ by $x_i = \int \gamma(r)_i \mu(dr)$.

Given $p = \mathsf{fun}(\overrightarrow{\lambda}\,\overrightarrow{m}) \in \mathcal{M}_0^C$, the map $p \circ \gamma : X \to \mathbb{R}_{\geq 0}$ is bounded and measurable, and we have

$$\begin{aligned}\int p(\gamma(r))\mu(dr) &= \int \big(\sum_{i\in I} \lambda_i m_i(\gamma(r)_i)\big)\mu(dr) \\ &= \sum_{i\in I} \int \lambda_i m_i(\gamma(r)_i)\mu(dr) \quad \text{by the monotone convergence theorem,} \\ &\qquad\qquad \text{since } I \text{ is countable} \\ &= \sum_{i\in I} \lambda_i m_i\Big(\int \gamma(r)_i \mu(dr)\Big) \quad \text{by definition of integrals} \\ &= p(\overrightarrow{x})\,.\end{aligned}$$

By (**Msnorm**) holding in C as shown above, this proves that $\|\overrightarrow{x}\| < \infty$ and hence $\overrightarrow{x} \in \underline{C}$ and the computation above shows also that $\overrightarrow{x} = \int \gamma(r)\mu(dr)$ and hence the cone C is integrable. The proof that it is the coproduct of the C_i's in **ICones** is routine. □

Even if I is not countable we know that $(C_i)_{i\in I}$ has a coproduct in **ICones** by Theorem 4.21, but we don't know yet how to describe it concretely.

5. Internal linear hom and the tensor product

The main goal of this section is to define a tensor product of two integrable cones, to prove that this operation is functorial and that **ICones** can be equipped with a structure of symmetric monoidal category (SMC) which is closed (SMCC).

Remark 5.1. Of course we first tried to define the tensor product concretely as one usually does in algebra, using some quotient. However the complicated interaction between the algebraic and the order theoretic properties of cones made the resulting description ineffective for proving basic properties expected from a tensor product. Now that we know that the tensor product exists for abstract reasons, and has the required structures and properties,

[18] It is harmless to assume that $0 \in \mathcal{M}_X^B$ for each measurable cone B and $X \in \mathbf{Ar}$: if B is a measurable cone and $C = (\underline{B}, \mathcal{M})$ where $\mathcal{M}_X = \mathcal{M}_X^B \cup \{0\}$ then C is a measurable cone and B and C are isomorphic in **MCones**, see Remark 3.14. And similarly in the category of integrable cones.

the quest for a reasonably simple concrete description can be undertaken with a more relaxed mind.

We define first the integrable cone $C \multimap D$ of linear, continuous, measurable and integrable morphisms $C \to D$. There are two good reasons for doing so.

- The definition of this object is easy and natural.
- Building this object will be necessary for proving that the SMC we define is closed.

Moreover, it is easy to prove that the associated internal hom functor $C \multimap \_ : \mathbf{ICones} \to \mathbf{ICones}$ preserves all limits. Then, thanks to Theorem 4.19 this functor has a left adjoint $\_ \otimes C : \mathbf{ICones} \to \mathbf{ICones}$, and this operation is also functorial wrt. C because $\_ \multimap \_$ is a functor $\mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ (thanks to a standard result in category theory), and we have a natural bijection of sets $\mathbf{ICones}(B \otimes C, D) \to \mathbf{ICones}(B, C \multimap D)$.

Last we prove that this natural bijection is actually an isomorphism $(B \otimes C \multimap D) \to (B \multimap (C \multimap D))$ in **ICones** and we show how to derive the SMC structure of **ICones** from this property.

We are convinced that this method is exactly the one described axiomatically in [EK66]. However we do not prove explicitly that **ICones** is closed in the sense of that paper and do not apply explicitly its results, first for the sake of self-containedness and also because, due to the concrete features of our category (basically: our morphisms are functions) the direct approach remains tractable.

Another approach, equivalent but conceptually more elegant, would have been to describe first **ICones** as a multicategory, introducing from the beginning a notion of multilinear morphism on **ICones** in a completely standard way. Then the tensor product would have been defined by a familiar universal property wrt. bilinear morphisms.

5.1. **The cone of linear morphisms.** Let C and D be objects of **ICones** and let P be the set of all $f : \underline{C} \to \underline{D}$ such that, for some $\varepsilon > 0$, one has $\varepsilon f \in \mathbf{ICones}(C, D)$, equipped with the same algebraic structure as $\underline{C} \multimap \underline{D}$ (see Lemma 2.13). This makes sense since the algebraic laws of the cone $\underline{C} \multimap \underline{D}$ preserve measurability and since integration is linear. Moreover given an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of measurable and integral preserving elements of $\underline{C} \multimap \underline{D}$ such that $\|f_n\| \leq 1$ (remember that $\|f\| = \sup_{x \in \mathcal{B}\underline{C}} \|f(x)\|$), the linear and continuous map $f = \sup_{n\in\mathbb{N}} f_n$ is measurable and preserves integrals by the monotone convergence theorem, as we show now.

Let $\gamma \in \overline{\mathsf{Path}(X, C)}$ be a measurable path and let $m \in \mathcal{M}_Y^D$ for some $Y \in \mathbf{Ar}$. The function $\varphi = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, f(\gamma(r))) : Y \times X \to [0, 1]$ satisfies $\varphi(s, r) = \sup_{n\in\mathbb{N}} \varphi_n$ where $(\varphi_n = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, f_n(\gamma(r)))_{n\in\mathbb{N}}$ is an increasing sequence of measurable functions by measurability of the f_n's and linearity and continuity of m in its second parameter, so φ is measurable which shows that f is measurable. Next, with the same notations and taking moreover some $\mu \in \underline{\mathsf{FMeas}(X)}$ we have, for each $m \in \mathcal{M}^D_0$,

$$\begin{aligned}
m\Big(f\Big(\int^C \gamma(r)\mu(dr)\Big)\Big) &= m\Big(\sup_{n\in\mathbb{N}} f_n\Big(\int^C \gamma(r)\mu(dr)\Big)\Big)\\
&= m\Big(\sup_{n\in\mathbb{N}} \int^D f_n(\gamma(r))\mu(dr)\Big)\\
&= \sup_{n\in\mathbb{N}} m\Big(\int^D f_n(\gamma(r))\mu(dr)\Big) \quad \text{by cont. of } m\\
&= \sup_{n\in\mathbb{N}} \int^1 m(f_n(\gamma(r)))\mu(dr) \quad \text{by def. of integration in } D\\
&= \int^1 \sup_{n\in\mathbb{N}} m(f_n(\gamma(r)))\mu(dr) \quad \text{by the monotone conv. th.}\\
&= \int^1 m(f(\gamma(r)))\mu(dr) \quad \text{by continuity of } m\\
&= m\Big(\int^D f(\gamma(r))\mu(dr)\Big) \quad \text{by def. of integration in } D
\end{aligned}$$

and hence f preserves integrals by (**Mssep**).

Given $\gamma \in \underline{\mathsf{Path}(X, C)}$ and $m \in \mathcal{M}^D_X$ we define $\gamma \triangleright m = \boldsymbol{\lambda}(r, f) \in X \times P \cdot m(r, f(\gamma(r)))$: $X \times P \to \mathbb{R}_{\geq 0}$. For each $r \in X$ the function $l = (\gamma \triangleright m)(r, \_) : \underline{C \multimap D} \to \mathbb{R}_{\geq 0}$ is linear and continuous by linearity and continuity of m in its second argument. We define $\mathcal{M}_X = \{\gamma \triangleright m \mid \gamma \in \underline{\mathsf{Path}(X, C)} \text{ and } m \in \mathcal{M}^D_X\}$.

Remark 5.2. We use the same notations for measurability tests on the cone of linear, continuous and integrable morphisms as for the cone of measurable paths in Section 3.2.2 for two reasons. The first one is that these tests are defined in a very similar way, the second one is Theorem 6.1 which allows to see measurable paths as linear, continuous and integrable maps.

We check that the family $(\mathcal{M}_X)$ is a measurability structure on the cone $P = (\underline{C} \multimap \underline{D})$.

► (**Msmeas**) Let $f \in \mathcal{B}P$, $\gamma \in \underline{\mathsf{Path}(X, C)}$ and $m \in \mathcal{M}^D_X$, then the map $\varphi = \boldsymbol{\lambda} r \in X \cdot m(r, f(\gamma(r)))$ is measurable because $f \circ \gamma \in \underline{\mathsf{Path}(X, D)}$ by measurability of f and hence $\boldsymbol{\lambda}(s, r) \in X \times X \cdot m(s, f(\gamma(r))) : X \times X \to [0, 1]$ is measurable from which follows the measurability of φ. The fact that φ ranges in $[0, 1]$ results from the assumption that $\|f\| \leq 1$.

► (**Mscomp**) Let $\gamma \in \underline{\mathsf{Path}(X, C)}$ and $m \in \mathcal{M}^D_X$, and let $\varphi \in \mathbf{Ar}(Y, X)$ for some $Y \in \mathbf{Ar}$. Then we have $(\gamma \triangleright m) \circ (\varphi \times P) = \boldsymbol{\lambda}(s, f) \in Y \times P \cdot m(\varphi(s), f(\gamma(\varphi(s))))) = (\gamma \circ \varphi) \triangleright (m \circ (\varphi \times \underline{D}))$ and since $\gamma \circ \varphi \in \underline{\mathsf{Path}(Y, C)}$ by Lemma 3.10 and $m \circ (\varphi \times \underline{D}) \in \mathcal{M}^D_Y$ by property (**Mscomp**) satisfied in D, we have $(\gamma \triangleright m) \circ (\varphi \times P) \in \mathcal{M}_Y$.

► (**Mssep**) Let $f_1, f_2 \in P$ and assume that for all $x \in \underline{C}$ and $m \in \mathcal{M}^D_0$ one has $(x \triangleright m)(f_1) = (x \triangleright m)(f_2)$, that is $m(f_1(x)) = m(f_2(x))$. By (**Mssep**) in D we have $f_1(x) = f_2(x)$, and since this holds for all $x \in \underline{C}$ we have $f_1 = f_2$.

► (**Msnorm**) Let $f \in P \setminus \{0\}$. Let $\varepsilon > 0$, we can assume without loss of generality that $\varepsilon < 2\|f\|$. By definition of $\|f\|$ there is $x \in \mathcal{B}\underline{C}$ such that $\|f\| \leq \|f(x)\| + \varepsilon/3$ and hence

$\|f\| < \|f(x)\| + \varepsilon/2$. This implies in particular that $f(x) \neq 0$ by our assumption that $\varepsilon < 2\|f\|$. By (**Msnorm**) in D there is $m \in \mathcal{M}_0^D \setminus \{0\}$ such that

$$\|f(x)\| \leq m(f(x))/\|m\| + \min(\varepsilon/2, \|f\| - \varepsilon/2)\,.$$

If $m(f(x)) = 0$ we have $\|f(x)\| \leq \|f\| - \varepsilon/2$ which is not possible since $\|f\| < \|f(x)\| + \varepsilon/2$, so $(x \triangleright m)(f) = m(f(x)) \neq 0$. This implies in particular that $\|x \triangleright m\| \neq 0$. We have $\|x \triangleright m\| = \sup_{g\in\mathcal{B}P} m(g(x)) \leq \|m\| \sup_{g\in\mathcal{B}P} \|g(x)\| \leq \|m\|$ since $x \in \mathcal{B}\underline{C}$, and hence

$$\begin{aligned}\|f\| \leq \|f(x)\| + \varepsilon/2 &\leq \frac{m(f(x))}{\|m\|} + \varepsilon = \frac{(x \triangleright m)(f)}{\|m\|} + \varepsilon \\ &\leq \frac{(x \triangleright m)(f)}{\|x \triangleright m\|} + \varepsilon \quad \text{since } 0 < \|x \triangleright m\| \leq \|m\|\,.\end{aligned}$$

So we have defined a measurable cone that we denote as $C \multimap D$.

We will need a technical lemma whose intuitive meaning is interesting *per se*: a linear morphisms valued in a cone of paths is the same thing as a path valued in the measurable cone of linear morphisms just defined. This lemma will be essential for proving that the measurable cone $C \multimap D$ is integrable.

Lemma 5.3. *There is an argument swapping isomorphism*

$$\mathsf{sw} \in \mathbf{MCones}(C \multimap \mathsf{Path}(X, D), \mathsf{Path}(X, C \multimap D))$$

which maps f to $\boldsymbol{\lambda} r \in X \cdot \boldsymbol{\lambda} x \in \underline{C} \cdot f(x)(r)$.

This iso preserves integrals, but this property is not required for what follows.

Proof. Let $f \in \underline{C \multimap \mathsf{Path}(X, D)}$. If $r \in X$, the map $g = \boldsymbol{\lambda} x \in \underline{C} \cdot f(x)(r) : \underline{C} \to \underline{D}$ is linear and continuous because f is, and the algebraic operations and the lubs are computed pointwise in $\underline{\mathsf{Path}(X, D)}$, we prove that g is measurable. Let $Y, Y' \in \mathbf{Ar}$, $\gamma \in \underline{\mathsf{Path}(Y, C)}$ and $m \in \mathcal{M}_{Y'}^D$, we set $\varphi = \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot m(s', g(\gamma(s))) = \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot m(s', f(\gamma(s))(r))$. Notice that, identifying r with the constant r-valued measurable function $Y' \to X$, we have $r \triangleright m \in \mathcal{M}_{Y'}^{\mathsf{Path}(X,D)}$ and $\varphi = \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot (r \triangleright m)(s', f(\gamma(s)))$ which is measurable because $f \circ \gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, D))}$ since $f \in \underline{C \multimap \mathsf{Path}(X, D)}$. This shows that g is measurable, we prove that g preserves integrals so let moreover $\nu \in \underline{\mathsf{FMeas}(Y)}$, we have

$$\begin{aligned}g\Big(\int_{s\in Y}^{C} \gamma(s)\nu(ds)\Big) &= f\Big(\int_{s\in Y}^{C} \gamma(s)\nu(ds)\Big)(r) \\ &= \Big(\int_{s\in Y}^{\mathsf{Path}(X,D)} f(\gamma(s))\nu(ds)\Big)(r) \quad \text{since } f \text{ preserves integrals.} \\ &= \int_{s\in Y}^{D} f(\gamma(s))(r)\nu(ds) \\ &= \int_{s\in Y}^{D} g(\gamma(s))\nu(ds)\,.\end{aligned}$$

since integrals in $\mathsf{Path}(X, D)$ are computed pointwise. This shows that $g = \mathsf{sw}(f)(r) \in \underline{C \multimap D}$ for all $r \in X$.

We prove next that $\eta = \mathsf{sw}(f)$ belongs to $\underline{\mathsf{Path}(X, C \multimap D)}$ so let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{C\multimap D}$. Let $\gamma \in \underline{\mathsf{Path}(Y, C)}$ and $m \in \mathcal{M}_Y^D$ be such that $p = \gamma \triangleright m$. The function

$\varphi = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot p(s,\eta(r))$ satisfies

$$\begin{aligned}\varphi &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s,\eta(r)(\gamma(s)))\\ &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s,f(\gamma(s))(r))\,.\end{aligned}$$

We know that $\delta = f \circ \gamma \circ \mathsf{pr}_1 \in \underline{\mathsf{Path}(Y \times X, \mathsf{Path}(X,D))}$ because $\gamma \circ \mathsf{pr}_1 \in \underline{\mathsf{Path}(Y \times X, C)}$ and $f \in \mathbf{MCones}(C, \mathsf{Path}(X,D))$. Let $m' \in \mathcal{M}^{D}_{Y\times X}$ be defined by $m'(s,r,y) = m(s,y)$, we have $\mathsf{pr}_2 \triangleright m' \in \mathcal{M}^{\mathsf{Path}(X,D)}_{Y\times X}$ and hence $\varphi' = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot (\mathsf{pr}_2 \triangleright m')(s,r,\delta(s,r))$ is measurable. But

$$\varphi'(s,r) = m'(s,r,\delta(s,r)(\mathsf{pr}_2(s,r))) = m(s,f(\gamma(s))(r)) = \varphi(s,r)$$

so that φ is measurable, this shows that $\mathsf{sw}(f) \in \underline{\mathsf{Path}(X, C \multimap D)}$. The linearity and continuity of sw are obvious (the algebraic operations and lubs are defined pointwise) as well as the fact that $\|\mathsf{sw}\| \leq 1$. Its measurability relies on the obvious bijection between $\mathcal{M}^{C\multimap\mathsf{Path}(X,D)}_{Y'}$ and $\mathcal{M}^{\mathsf{Path}(X,C\multimap D)}_{Y'}$ which maps $\gamma \triangleright (\varphi \triangleright m)$ to $\varphi \triangleright (\gamma \triangleright m)$ for all $Y' \in \mathbf{Ar}$ (with $\gamma \in \underline{\mathsf{Path}(Y',C)}$, $\varphi \in \mathbf{Ar}(Y',X)$ and $m \in \mathcal{M}^{D}_{Y'}$). We have proven that sw is a morphism in $\mathbf{MCones}$.

Conversely given $\eta \in \underline{\mathsf{Path}(X, C \multimap D)}$ we define $f = \mathsf{sw}'(\eta) = \boldsymbol{\lambda}x \in \underline{C} \cdot \boldsymbol{\lambda}r \in X \cdot \eta(r)(x)$ and prove first that $f \in \underline{C \multimap \mathsf{Path}(X,D)}$. Let $x \in \underline{C}$ and $\delta = f(x) : X \to \underline{D}$. If $r \in X$ we have $\eta(r) \leq \|\eta\|$ and hence $\|\delta(r)\| = \|\eta(r)(x)\| \leq \|\eta\|\,\|x\|$ which shows that the function δ is bounded. Let $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^{D}_{Y}$, we set $\varphi = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s,\delta(r))$. For $s \in Y$ and $r \in X$, we have

$$\varphi(s,r) = m(s,\delta(r)) = m(s,\eta(r)(x)) = (x \triangleright m)(s,\eta(r))$$

where we identify x with the path $\gamma \in \underline{\mathsf{Path}(Y,C)}$ such that $\gamma(s) = x$, so that $x \triangleright m \in \mathcal{M}^{C\multimap D}_{Y}$. It follows that φ is measurable and hence $\delta \in \underline{\mathsf{Path}(X,D)}$.

Linearity of f is obvious and continuity results from the fact that lubs in $\underline{\mathsf{Path}(X,D)}$ are computed pointwise. Let $\gamma \in \underline{\mathsf{Path}(Y,C)}$ for some $Y \in \mathbf{Ar}$, we must prove next that $f \circ \gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X,D))}$. Let $Y' \in \mathbf{Ar}$ and $p \in \mathcal{M}^{\mathsf{Path}(X,D)}_{Y'}$, we must prove that $\psi = \boldsymbol{\lambda}(s',s) \in Y' \times Y \cdot p(s', f(\gamma(s)))$ is measurable. Let $\varphi \in \mathbf{Ar}(Y',X)$ and $m \in \mathcal{M}^{D}_{Y'}$ be such that $p = \varphi \triangleright m$. For $s' \in Y'$ and $s \in Y$ we have

$$\begin{aligned}\psi(s',s) &= m(s', f(\gamma(s))(\varphi(s')))\\ &= m(s',\eta(\varphi(s'))(\gamma(s)))\\ &= ((\gamma \circ \mathsf{pr}_2) \triangleright (m \circ (\mathsf{pr}_1 \times \underline{D})))(s',s,\eta\circ\varphi)\end{aligned}$$

and hence ψ is measurable since $\eta \circ \varphi$ is a measurable path. This shows that f is measurable, we prove last that f preserves integrals. So let $\gamma \in \underline{\mathsf{Path}(Y,C)}$ and $\mu \in \underline{\mathsf{FMeas}(Y)}$. Given

$r \in X$ we have

$$\begin{aligned}
f\Big(\int_{s\in Y}^{C}\gamma(s)\mu(ds)\Big)(r) &= \eta(r)\Big(\int_{s\in Y}^{C}\gamma(s)\mu(ds)\Big)\\
&= \int_{s\in Y}^{D}\eta(r)(\gamma(s))\mu(ds) \quad \text{since } \eta(r) \in \underline{C \multimap D}\\
&= \int_{s\in Y}^{D} f(\gamma(s))(r)\mu(ds) \quad \text{by definition of } f\\
&= \Big(\int_{s\in Y}^{\mathsf{Path}(X,D)} f(\gamma(s))\mu(ds)\Big)(r)
\end{aligned}$$

since integrals in $\mathsf{Path}(X, D)$ are computed pointwise.

The proof that sw' is a morphism in **MCones** follows the same pattern as for sw. □

Lemma 5.4. *The measurable cone $C \multimap D$ is integrable.*

Proof. Let $X \in \mathbf{Ar}$, $\eta \in \mathsf{Path}(X, C \multimap D)$ and $\mu \in \underline{\mathsf{FMeas}(X)}$. Let

$$f = \boldsymbol{\lambda} x \in \underline{C} \cdot \int^{D} \eta(r)(x)\mu(dr) = \boldsymbol{\lambda} x \in \underline{C} \cdot \int^{D} \mathsf{sw}(\eta)(x)(r)\mu(dr)\,.$$

This function is well defined since for each $x \in \underline{C}$ one has $\mathsf{sw}(\eta)(x) \in \underline{\mathsf{Path}(X, D)}$ by Lemma 5.3 so that the integral $\int \mathsf{sw}(\eta)(x)(r)\mu(dr) \in \underline{D}$ is well defined. The fact that $f : \underline{C} \to \underline{D}$ is linear and continuous results from the linearity of integration and from the monotone convergence theorem. Let us check that f is measurable so let $Y \in \mathbf{Ar}$ and let $\gamma \in \underline{\mathsf{Path}(Y, C)}$, we must prove that

$$\boldsymbol{\lambda} s \in Y \cdot \int^{D} \mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr) \in \underline{\mathsf{Path}(Y, D)}$$

so let $Y' \in \mathbf{Ar}$ and $m \in \mathcal{M}^{D}_{Y'}$, we must check that the function

$$\begin{aligned}
\psi &= \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot m\Big(s', \int^{D} \mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr)\Big)\\
&= \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot \int^{\mathbb{R}_{\geq 0}} m(s', \mathsf{sw}(\eta)(\gamma(s))(r))\mu(dr)
\end{aligned}$$

is measurable. We know that the function $\boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot m(s', \mathsf{sw}(\eta)(\gamma(s))(r))$ is measurable and bounded because $\mathsf{sw}(\eta) \circ \gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, D))}$ by Lemma 5.3 and we get the announced measurability by Lemma 4.7 (in the special case where κ is the kernel constantly equal to μ). Next we prove that f preserves integrals, so let moreover

$\nu \in \underline{\mathsf{FMeas}(Y)}$, we have

$$\begin{aligned}
f\Big(\int_{s\in Y}^{C}\gamma(s)\nu(ds)\Big) &= \int_{r\in X}^{D}\eta(r)\Big(\int_{s\in Y}^{C}\gamma(s)\nu(ds)\Big)\mu(dr)\\
&= \int_{r\in X}^{D}\Big(\int_{s\in Y}^{D}\eta(r)(\gamma(s))\nu(ds)\Big)\mu(dr)\quad\text{since } \eta(r)\in\underline{C\multimap D}\\
&= \int_{r\in X}^{D}\Big(\int_{s\in Y}^{D}\mathsf{sw}(\eta)(\gamma(s))(r)\nu(ds)\Big)\mu(dr)\\
&= \int_{s\in Y}^{D}\Big(\int_{r\in X}^{D}\mathsf{sw}(\eta)(\gamma(s))(r)\mu(dr)\Big)\nu(ds)\\
&\qquad\text{by Th. 4.15 (Fubini), since } \mathsf{sw}(\eta)\circ\gamma\in\underline{\mathsf{Path}(Y,\mathsf{Path}(X,D))}\\
&= \int_{s\in Y}^{D} f(\gamma(s))\nu(ds)\,.
\end{aligned}$$

This completes the proof that $f \in \underline{C \multimap D}$ as contended.

Let $p \in \mathcal{M}_0^{C\multimap D}$. Let $x \in \underline{C}$ and $m \in \mathcal{M}_0^D$ be such that $p = x \triangleright m$, we have

$$\begin{aligned}
p(f) &= m\Big(\int^{D}\eta(r)(x)\mu(dr)\Big)\\
&= \int m(\eta(r)(x))\mu(dr)\\
&= \int p(\eta(r))\mu(dr)\,,
\end{aligned}$$

so η is integrable over μ, and $\int^{C\multimap D}\eta(r)\mu(dr) = f$. □

This is the right place to insert a lemma very similar to Lemma 5.3 which will be useful for exhibiting the symmetry of our tensor product.

Lemma 5.5. *There is an argument swapping natural isomorphism*

$$\mathsf{sw} \in \mathbf{ICones}(B_1 \multimap (B_2 \multimap C), B_2 \multimap (B_1 \multimap C))$$

which maps f to $\boldsymbol{\lambda} x_1 \in \underline{B_1}\cdot\boldsymbol{\lambda}_2 \in \underline{B_2}\cdot f(x_1)(x_2)$.

Proof. One checks that

$$\mathsf{sw} \in \mathbf{MCones}(B_1 \multimap (B_2 \multimap C), B_2 \multimap (B_1 \multimap C))$$

as in the proof of 5.3, and this morphism preserves integrals because integrals are computed pointwise in $D \multimap E$ for all integrable cones D and E and by the Fubini theorem because all the considered measures are finite. □

5.2. **Bilinear maps.** After the linear function space $C \multimap D$ that we have just defined, the next concept deeply related to the tensor product is of course the concept of bilinear map that we introduce now.

Definition 5.6. Let C_1, C_2, D be integrable cones, we define formally

$$C_1, C_2 \multimap D = C_1 \multimap (C_2 \multimap D)$$

and call this integrable cone *the cone of integrable bilinear and continuous maps $C_1, C_2 \to D$.*

Indeed, thanks to Lemma 2.19, an element of $\underline{C_1, C_2 \multimap D}$ can be seen as a function $f : \underline{C_1} \mathbin{\&} \underline{C_2} \to \underline{D}$ which is separately linear and ω-continuous. Measurability of f is expressed equivalently by saying that given $(X_i \in \mathbf{Ar})_{i=1,2}$ and $(\gamma_i \in \underline{\mathsf{Path}(X_i, C_i)})_{i=1,2}$ the map $\boldsymbol{\lambda}(r_1, r_2) \in X_1 \times X_2 \cdot f(\gamma_1(r_1), \gamma_2(r_2)) : X_1 \times X_2 \to \underline{D}$ is a measurable path or that, given $X \in \mathbf{Ar}$ and $(\gamma_i \in \underline{\mathsf{Path}(X, C_i)})_{i=1,2}$, the map $\boldsymbol{\lambda} r \in X \cdot f(\gamma_1(r), \gamma_2(r))$ is a measurable path. Preservation of integrals means that, given moreover $(\mu_i \in \underline{\mathsf{FMeas}(X_i)})_{i=1,2}$, we have

$$f\Big(\int^{C_1} \gamma_1(r_1)\mu_1(dr_1), \int^{C_2} \gamma_2(r_2)\mu_2(dr_2)\Big) = \iint^{D} f(\gamma_1(r_1), \gamma_2(r_2))\mu_1(dr_1)\mu_2(dr_2)$$

where we can use the double integral symbol by Theorem 4.15.

Continuing to spell out the definition above of the integrable cone $C_1, C_2 \multimap D$, we see that, given $X \in \mathbf{Ar}$, an element of $\mathcal{M}_X^{C_1, C_2 \multimap D}$ is a

$$\gamma_1, \gamma_2 \triangleright m = \boldsymbol{\lambda}(r, f) \in X \times \underline{(C_1, C_2 \multimap D)} \cdot m(r, f(\gamma_1(r), \gamma_2(r))$$

where $(\gamma_i \in \underline{\mathsf{Path}(X, C_i)})_{i=1,2}$ and $m \in \mathcal{M}_X^D$. Last the integral of a measurable path $\eta \in \underline{\mathsf{Path}(X, (C_1, C_2 \multimap D))}$ over $\mu \in \underline{\mathsf{FMeas}(X)}$ is characterized by

$$\Big(\int \eta(r)\mu(dr)\Big)(x_1, x_2) = \int \eta(r)(x_1, x_2)\mu(dr)\,.$$

5.3. **The linear hom functor.** In order to define the tensor product as a left adjoint, we need to consider the operation $\multimap$ as an operation on morphisms of **ICones**, not only on objects. We define this operation and prove that it preserves all limits in its second argument.

Definition 5.7. Let $g \in \mathbf{ICones}(D_1, D_2)$ and $h \in \mathbf{ICones}(C_2, C_1)$. The function $h \multimap g : \underline{C_1 \multimap D_1} \to \underline{C_2 \multimap D_2}$ is defined by $(h \multimap g)(f) = g\, f\, h$.

Proposition 5.8. *If $g \in \mathbf{ICones}(D_1, D_2)$ and $h \in \mathbf{ICones}(C_2, C_1)$ then $h \multimap g \in \mathbf{ICones}(C_1 \multimap D_1, C_2 \multimap D_2)$.*

Proof. The linearity and continuity of $h \multimap g$ result from the same properties satisfied by g and h. The fact that $\|h \multimap g\| \leq 1$ results from the fact that $\|g\|, \|h\| \leq 1$, so let us check that $h \multimap g$ is measurable. Let $\eta_1 \in \underline{\mathsf{Path}(X, C_1 \multimap D_1)}$ for some $X \in \mathbf{Ar}$. We must prove that $(h \multimap g) \circ \eta_1 \in \underline{\mathsf{Path}(X, C_2 \multimap D_2)}$ so let $p \in \mathcal{M}_Y^{C_2 \multimap D_2}$ for some $Y \in \mathbf{Ar}$, we must prove that

$$\varphi = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, (h \multimap g)(\eta_1(r)))$$

is measurable. Let $\gamma \in \underline{\mathsf{Path}(Y, C_2)}$ and $m \in \mathcal{M}_Y^{D_2}$ be such that $p = \gamma \triangleright m$. For $s \in Y$ and $r \in X$ we have

$$\varphi(s, r) = m(s, g(\eta_1(r)(h(\gamma(s)))) = m(s, g(\delta_1(s)(r))) = m(s, g(\mathsf{fl}(\delta_1)(s, r)))$$

where $\delta_1 = \mathsf{sw}^{-1}(\eta_1) \circ h \circ \gamma \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, D_1))}$ by Lemma 5.3 and hence $g \circ \mathsf{fl}(\delta_1) \in \underline{\mathsf{Path}(Y \times X, D_2)}$ by Lemma 3.19 so that φ is measurable. We need last to prove that $h \multimap g$

preserves integrals so let moreover $\mu \in \underline{\mathsf{FMeas}(X)}$, we have

$$\begin{aligned}
(h \multimap g)\Big(\int^{C_1 \multimap D_1} \eta_1(r)\mu(dr)\Big) &= \boldsymbol{\lambda} x \in \underline{C_2} \cdot g\Big(\Big(\int^{C_1 \multimap D_1} \eta_1(r)\mu(dr)\Big)(h(x))\Big) \\
&= \boldsymbol{\lambda} x \in \underline{C_2} \cdot g\Big(\int^{D_1} \eta_1(r)(h(x))\mu(dr)\Big) \\
&= \boldsymbol{\lambda} x \in \underline{C_2} \cdot \int^{D_2} g\Big(\eta_1(r)(h(x))\Big)\mu(dr) \\
&= \boldsymbol{\lambda} x \in \underline{C_2} \cdot \int^{D_2} (h \multimap g)(\eta_1(r))(x)\mu(dr) \\
&= \int^{C_2 \multimap D_2} (h \multimap g)(\eta_1(r))\mu(dr)\,.
\end{aligned}$$

□

So we have defined a functor $\_ \multimap \_ : \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$. We identify $1 \multimap \_$ with the identity functor: we make no distinction between $x \in \underline{C}$ and the function $\widehat{x} \in \underline{1 \multimap C}$ (this notation is introduced in the proof of Theorem 4.18).

Theorem 5.9. *For each integrable cone C, the functor $C \multimap \_$ has a left adjoint.*

Proof. By Theorem 4.19 it suffices to prove that $C \multimap \_$ preserves all limits.

▶ Products. Let $(D_i)_{i \in I}$ be a family of measurable cones and let $D = \&_{i \in I} D_i$ as described in the proof of Theorem 4.16. We have a morphism

$$k = \langle C \multimap \mathsf{pr}_i \rangle_{i \in I} \in \mathbf{ICones}(C \multimap D, \mathop{\&}_{i \in I} (C \multimap D_i))$$

and we must prove that k is an iso. It is clearly injective, to prove surjectivity, let $\overrightarrow{f} = (f_i \in \underline{C \multimap D_i})_{i \in I} \in \underline{\&_{i \in I} (C \multimap D_i)}$ so that $(\|f_i\|)_{i \in I}$ is bounded in $\mathbb{R}_{\geq 0}$ and hence for each $x \in \underline{C}$ the family $(\|f_i(x)\|)_{i \in I}$ is bounded. So we can define a function $f : \underline{C} \to \underline{D}$ by $f(x) = (f_i(x))_{i \in I}$. This function is clearly linear and continuous. To prove measurability, take $\gamma \in \underline{\mathsf{Path}(X, C)}$ for some $X \in \mathbf{Ar}$ and $p \in \mathcal{M}^D_Y$ for some $Y \in \mathbf{Ar}$. This means that $p = \mathsf{in}_i(m)$ for some $i \in I$ and $m \in \mathcal{M}^{D_i}_Y$. Then $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, f(\gamma(r))) = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, f_i(\gamma(r)))$ is measurable because f_i is. Last let moreover $\mu \in \underline{\mathsf{FMeas}(X)}$, we have $f(\int^C \gamma(r)\mu(dr)) = \int^D f(\gamma(r))\mu(dr)$ by definition of f and of integration in D. This shows that $f \in \underline{C \multimap D}$ and hence that k is a bijection since $f_i = \mathsf{pr}_i\, f$ for each $i \in I$ and hence $\overrightarrow{f} = k(f)$.

We prove that $k^{-1} \in \mathbf{ICones}(\&_{i \in I} (C \multimap D_i), C \multimap D)$. Linearity and continuity follow from the fact that all operations are defined componentwise in $\underline{\&_{i \in I}} (C \multimap D_i)$. Next, given $\overrightarrow{f} \in \mathcal{B}(\underline{\&_{i \in I}} (C \multimap D_i))$, we have

$$\begin{aligned}
\Big\|k^{-1}(\overrightarrow{f})\Big\| &= \sup_{x \in \mathcal{B}\underline{C}} \Big\|k^{-1}(\overrightarrow{f})(x)\Big\| \\
&= \sup_{x \in \mathcal{B}\underline{C}} \sup_{i \in I} \|f_i(x)\| \\
&= \sup_{i \in I} \sup_{x \in \mathcal{B}\underline{C}} \|f_i(x)\| \\
&= \sup_{i \in I} \|f_i\| \leq 1\,.
\end{aligned}$$

Next we prove that k^{-1} is measurable so let $\eta \in \underline{\mathsf{Path}(X, \&_{i\in I}(C \multimap D_i))}$ for some $X \in \mathbf{Ar}$, we must prove that $\eta' = k^{-1} \circ \eta \in \underline{\mathsf{Path}(X, C \multimap D)}$. Notice that for all $r \in X$ we can write $\eta(r) = (\eta_i(r))_{i\in I}$ where $\eta_i = \mathsf{pr}_i \circ \eta \in \underline{\mathsf{Path}(X, C \multimap D_i)}$ for each $i \in I$. Let $Y \in \mathbf{Ar}$, $\gamma \in \underline{\mathsf{Path}(Y, C)}$ and $p \in \mathcal{M}_Y^D$, so that $p = \mathsf{in}_i(m)$ for some $i \in I$ and $m \in \mathcal{M}_Y^{D_i}$. We have $\boldsymbol{\lambda}(s,r) \in Y \times X \cdot (\gamma \triangleright p)(s, \eta'(r)) = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m_i(s, \eta_i(r))$ which is measurable since each η_i is a measurable path. Last let moreover $\mu \in \underline{\mathsf{FMeas}(X)}$, we must prove that $g_1 = k^{-1}(\int^{\&_{i\in I}(C\multimap D_i)} \eta(r)\mu(dr))$ and $g_2 = \int^{C \multimap D} k^{-1}(\eta(r))\mu(dr)$ are the same function. Let $x \in \underline{C}$, we have $g_1(x) = (\int^{D_i} \eta_i(r)(x)\mu(dr))_{i\in I} = g_2(x)$. This ends the proof that k is an iso in $\mathbf{ICones}$ and hence that $C \multimap \_$ preserves all products.

▶ Equalizers. Let $f, g \in \mathbf{ICones}(D_1, D_2)$ and let (E, e) be the corresponding equalizer in $\mathbf{ICones}$, as described in the proof of Theorem 4.16. Then we have $(C \multimap f)\ (C \multimap e) = (C \multimap g)\ (C \multimap e)$ by functoriality of $C \multimap \_$ and it will be sufficient to prove that $(C \multimap E, C \multimap e)$ has the universal property of an equalizer. Let H be an integrable cone and $h \in \mathbf{ICones}(H, C \multimap D_1)$ be such that $(C \multimap f)\ h = (C \multimap g)\ h$. Identifying h with its "uncurried" version $h' \in \underline{H, C \multimap D_1}$, the integrable bilinear and continuous map (see Section 5.2) given by $h'(z, x) = h(z)(x)$, we have $f \circ h' = g \circ h'$. In other words h' ranges in $\underline{E} \subseteq \underline{D_1}$, allowing to define $h'_0 \in \underline{H, C \multimap E}$ which is the same function as h' and is bilinear continuous and integrable by definition of E (which inherits the norm, the measurability and integrability structure of C). We use h_0 for the corresponding element of $\mathbf{ICones}(H, C \multimap E)$, so that $h = (C \multimap e)h_0$. The fact that h_0 is unique with this property results from the fact that $C \multimap e$ is a mono (it is actually the inclusion of $\underline{C \multimap E}$ into $\underline{C \multimap D_1}$ resulting from the inclusion e of $\underline{E}$ into $\underline{D_1}$). This shows that $(C \multimap E, C \multimap e)$ is the equalizer of $C \multimap f$ and $C \multimap g$ and ends the proof that $C \multimap \_$ preserves all limits. □

Lemma 5.10. *Let $X \in \mathbf{Ar}$ and let B, C, D be measurable cones. Let f be an element of $\mathbf{ICones}(B, \mathsf{Path}(X, C \multimap D))$. Then $f' = \boldsymbol{\lambda}(y, r, x) \in \underline{C} \times X \times \underline{B} \cdot f(x, r, y)$ belongs to $\mathbf{ICones}(C, \mathsf{Path}(X, B \multimap D))$.*

Proof. This results from the following sequence of isos in $\mathbf{ICones}$:

$$
\begin{aligned}
&B \multimap \mathsf{Path}(X, C \multimap D) \\
&\qquad \simeq B \multimap (C \multimap \mathsf{Path}(X, D)) \quad \text{by Lemma 5.3 and functoriality of } C \multimap \_ \\
&\qquad = B, C \multimap \mathsf{Path}(X, D) \\
&\qquad \simeq C, B \multimap \mathsf{Path}(X, D) \quad \text{by Section 5.2.}
\end{aligned}
$$
□

5.4. **The tensor product of integrable cones.** Let C be an integrable cone. We denote by $\_ \otimes C$ the left adjoint of the functor $C \multimap \_$, see Theorem 5.9. Because $\_ \multimap \_$ is a functor $\mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ (see Section 5.3), we know by the adjunction with a parameter theorem ([Mac71], Chapter IV, Section 7, Theorem 3), that the so defined operation[19] $\otimes$ can uniquely be extended in a bifunctor $\otimes : \mathbf{ICones}^2 \to \mathbf{ICones}$ in such a way that the bijection

$$\Phi_{B,C,D} : \mathbf{ICones}(B \otimes C, D) \to \mathbf{ICones}(B, C \multimap D)$$

[19]According to one of the reviewers of this paper, this tensor product can be understood as the adaptation to the setting of integrable cones of the standard projective tensor product of locally convex spaces.

given by the adjunction for each C is natural in B, C, D. We define

$$\tau_{B,C} = \Phi_{B,C,B\otimes C}(\mathsf{Id}_{B\otimes C}) \in \mathbf{ICones}(B, C \multimap B \otimes C) = \mathcal{B}(\underline{B, C \multimap B \otimes C})$$

and, for $x \in \underline{B}$ and $y \in \underline{C}$ we use the notation $x \otimes y = \tau_{B,C}(x, y)$. By naturality of Φ we have that, for each $f \in \underline{B \otimes C \multimap D}$,

$$\Phi_{B,C,D}(f) = f \circ \tau_{B,C}. \tag{5.1}$$

The next lemma is the key observation for proving that the above bijection is a cone isomorphism.

Lemma 5.11. *Let $X \in \mathbf{Ar}$ and B, C be integrable cones. Let $\eta : X \to \underline{B \otimes C \multimap 1}$ be a function. One has $\eta \in \underline{\mathsf{Path}(X, B \otimes C \multimap 1)}$ as soon as*

- *$\eta(X) \subseteq \underline{B \otimes C \multimap 1}$ is bounded*
- *and for all $Y \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(Y, B)}$ and $\gamma \in \underline{\mathsf{Path}(Y, C)}$, the function $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot \eta(r)(\beta(s) \otimes \gamma(s)) : Y \times X \to \mathbb{R}_{\geq 0}$ is measurable.*

Proof. Let $\eta' : \underline{B} \times \underline{C} \times X \to \mathbb{R}_{\geq 0}$ be defined by $\eta'(x, y, r) = \eta(r)(x \otimes y)$. We have $\eta' \in \mathbf{ICones}(B, C \multimap \mathsf{Path}(X, 1))$ by our assumptions. Let us check that η' indeed preserves integrals. Let $Y, Z \in \mathbf{Ar}$, $\mu \in \mathsf{FMeas}(Y)$, $\nu \in \mathsf{FMeas}(Z)$, $\beta \in \mathsf{Path}(Y, B)$, $\gamma \in \mathsf{Path}(Z, C)$, and $m = \mathcal{M}_0^{\mathsf{Path}(X,1)}$ (*i.e.* there is $r \in X$ such that $m(\xi) = \xi(r)$ for all $\xi \in \mathsf{Path}(X, 1)$). Then

$$\begin{aligned}
m\Big(\eta'\Big(\int \beta(s)\mu(ds), \int \gamma(t)\nu(dt)\Big)\Big) &= \eta'\Big(\int \beta(s)\mu(ds), \int \gamma(t)\nu(dt)\Big)(r) \\
&= \eta(r)\Big(\Big(\int \beta(s)\mu(ds)\Big) \otimes \Big(\int \gamma(t)\nu(dt)\Big)\Big) \\
&= \iint \eta(r)(\beta(s) \otimes \gamma(t))\mu(ds)\nu(dt) \quad \text{since } \tau \text{ preserves integrals} \\
&= \iint m(\eta'(\beta(s), \gamma(t)))\mu(ds)\nu(dt) \\
&= m\Big(\iint \eta'(\beta(s), \gamma(t))\mu(ds)\nu(dt)\Big).
\end{aligned}$$

Let

$$\eta'' = \Phi^{-1}_{B,C,\mathsf{Path}(X,1)}(\eta') \in \mathbf{ICones}(B \otimes C, \mathsf{Path}(X, 1)) = \mathbf{ICones}(B \otimes C, \mathsf{Path}(X, 1 \multimap 1))$$

up to a trivial **ICones** iso and so by Lemma 5.10 there is a $h \in \mathbf{ICones}(1, \mathsf{Path}(X, B \otimes C \multimap 1))$ such that $h(1)(r)(z) = \eta''(z)(r)$ for all $z \in \underline{B \otimes C}$ and $r \in X$. So we have $h(1)(r)(x \otimes y) = \eta(r)(x \otimes y)$ and hence $\eta(r) = h(1)(r)$ since both are elements of $\underline{B \otimes C \multimap 1}$. Since this holds for all $r \in X$ we have proven that $\eta = h(1)$ and hence $\eta \in \mathsf{Path}(X, B \otimes C \multimap 1)$ as contended. ☐

Now we can prove the main property of our tensor product which will allow us to prove that it has a structure of monoidal product on **ICones**.

Theorem 5.12. *For each integrable cones B, C, D, the function $\Phi_{B,C,D}$ is an isomorphism of integrable cones from $B \otimes C \multimap D$ to $B \multimap (C \multimap D) = (B, C \multimap D)$.*

Proof. By linearity and ω-continuity of composition on the left, the function $\Phi_{B,C,D}$ — characterized by (5.1) — is a linear and continuous map $\Phi_{B,C,D} : \underline{B \otimes C \multimap D} \to \underline{B, C \multimap D}$ which satisfies $\|\Phi_{B,C,D}(f)\| \leq \|f\|$ for all $f \in \underline{B \otimes C \multimap D}$. This latter property is due to the fact that if $f \in \underline{B \otimes C \multimap D}$ satisfies $\|f\| \leq 1$ then $f \in \mathbf{ICones}(B \otimes C, D)$ and hence $\Phi_{B,C,D}(f) \in \mathbf{ICones}(B, C \multimap D)$ so that $\|\Phi_{B,C,D}(f)\| \leq 1$ and hence for an arbitrary $f \in \underline{B \otimes C \multimap D}$ such that $f \neq 0$ we have $\|(1/\|f\|)f\| \leq 1$ and hence $\|\Phi_{B,C,D}((1/\|f\|)f)\| \leq 1$ which is exactly our contention, which trivially also holds when $f = 0$.

Let us prove that $\Phi_{B,C,D}$ is measurable, so let $X \in \mathbf{Ar}$ and $\eta \in \underline{\mathsf{Path}(X, B \otimes C \multimap D)}$, we must prove that $\Phi_{B,C,D} \circ \eta \in \underline{\mathsf{Path}(X, (B, C \multimap D))}$. So let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{B,C\multimap D}$, which means that $p = \beta, \gamma \triangleright m$ for some $\beta \in \underline{\mathsf{Path}(Y, B)}$, $\gamma \in \underline{\mathsf{Path}(Y, C)}$ and $m \in \mathcal{M}_Y^D$. We have

$$\begin{aligned}\boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, \Phi_{B,C,D}(\eta(r))) &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, \eta(r) \circ \tau_{B,C}) \\ &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \eta(r)(\beta(s) \otimes \gamma(s)))\end{aligned}$$

which is measurable because $\beta \otimes \gamma \in \underline{\mathsf{Path}(Y, B \otimes C)}$ (defining $\beta \otimes \gamma$ by $(\beta \otimes \gamma)(s) = \beta(s) \otimes \gamma(s)$) by measurability of τ, and by our assumption that η is a measurable path. Altogether we have proven that

$$\Phi_{B,C,D} \in \mathbf{MCones}((B \otimes C \multimap D), (B, C \multimap D))$$

and we prove now that this morphism preserves integrals, so let moreover $\mu \in \underline{\mathsf{FMeas}(X)}$, we have

$$\begin{aligned}\Phi_{B,C,D}\Big(\int \eta(r)\mu(dr)\Big) &= \boldsymbol{\lambda}(x, y) \in \underline{B} \times \underline{C} \cdot \Big(\int \eta(r)\mu(dr)\Big)(x \otimes y) \\ &= \boldsymbol{\lambda}(x, y) \in \underline{B} \times \underline{C} \cdot \Big(\int \eta(r)(x \otimes y)\mu(dr)\Big) \\ &= \boldsymbol{\lambda}(x, y) \in \underline{B} \times \underline{C} \cdot \Big(\int \Phi_{B,C,D}(\eta(r))(x, y)\mu(dr)\Big) \\ &= \int \Phi_{B,C,D}(\eta(r))\mu(dr)\,.\end{aligned}$$

This shows that $\Phi_{B,C,D} \in \mathbf{ICones}((B \otimes C \multimap D), (B, C \multimap D))$ and we show now that this morphism is an iso.

We know that this function is bijective, let us use $\Psi_{B,C,D}$ for its inverse, which is linear and continuous by Lemma 2.8. Since $\Psi_{B,C,D} : \mathbf{ICones}(B, C \multimap D) \to \mathbf{ICones}(B \otimes C, D)$, we have $\|\Psi_{B,C,D}(g)\| \leq \|g\|$ for all $g \in \underline{B, C \multimap D}$, using also the linearity of $\Psi_{B,C,D}$. We prove that $\Psi_{B,C,D}$ is measurable. Let $X \in \mathbf{Ar}$ and $\eta \in \underline{\mathsf{Path}(X, (B, C \multimap D))}$, we must prove that $\Psi_{B,C,D} \circ \eta \in \underline{\mathsf{Path}(X, (B \otimes C \multimap D))}$. Without loss of generality we assume that $\|\eta\| \leq 1$. Let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{B\otimes C\multimap D}$, we must check that $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, \Psi_{B,C,D}(\eta(r)))$ is measurable. There is $\theta \in \mathsf{Path}(Y, B \otimes C)$ and $m \in \mathcal{M}_Y^D$ such that $p = \theta \triangleright m$, and we must check that $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \Psi_{B,C,D}(\eta(r))(\theta(s)))$ is measurable. For this, since $\mathbf{Ar}$ is cartesian, it suffices to prove that

$$\boldsymbol{\lambda}(s, s', r) \in Y \times Y \times X \cdot m(s, \Psi_{B,C,D}(\eta(r))(\theta(s')))$$

is measurable. Since $\theta \in \underline{\mathsf{Path}(Y, B \otimes C)}$ it suffices to prove that

$$\eta' = \boldsymbol{\lambda}(s, r, z) \in Y \times X \times \underline{B \otimes C} \cdot m(s, \Psi_{B,C,D}(\eta(r))(z)) \in \underline{\mathsf{Path}(Y \times X, B \otimes C \multimap 1)}$$

and to this end we apply Lemma 5.11. The boundedness assumption is satisfied because $\|\eta\| \leq 1$ and hence $\|\Psi_{B,C,D}(\eta(r))\| \leq 1$ for each $r \in X$. So let $Y' \in \mathbf{Ar}$, $\beta \in \mathsf{Path}(Y', B)$

and $\gamma \in \mathsf{Path}(Y', C)$. We have

$$\begin{aligned}\boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot \eta'(s, r)(\beta(s') \otimes \gamma(s'))\\ = \boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot m(s, \Psi_{B,C,D}(\eta(r))(\beta(s') \otimes \gamma(s')))\\ = \boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot m(s, \eta(r)(\beta(s'), \gamma(s')))\end{aligned}$$

which is measurable by our assumption about η. Last we must prove that $\Psi_{B,C,D}$ preserves integrals. Using the same path η let furthermore $\mu \in \underline{\mathsf{FMeas}(X)}$, we must prove that

$$g_1 = \Psi_{B,C,D}\Big(\int \eta(r)\mu(dr)\Big) \in \underline{B \otimes C \multimap D}$$

$$\text{and } g_2 = \int \Psi_{B,C,D}(\eta(r))\mu(dr) \in \underline{B \otimes C \multimap D}$$

are equal. Since $\Phi_{B,C,D}$ preserves integrals we have $\Phi_{B,C,D}(g_1) = \Phi_{B,C,D}(g_2)$ and the required property follows from the injectivity of $\Phi_{B,C,D}$. □

Theorem 5.13. *For each $x \in \underline{B}$ and $y \in \underline{C}$ we have $\|x \otimes y\| = \|x\|\,\|y\|$.*

Proof. Since $\tau_{B,C} \in \mathbf{ICones}(B, C \multimap B \otimes C)$ we have $\|x \otimes y\| \leq \|x\|\,\|y\|$, we just have to prove the converse. If $x = 0$ or $y = 0$ our contention trivially holds so we can assume without loss of generality that $\|x\| = \|y\| = 1$ and let $\varepsilon > 0$. By Proposition 3.11 there is $x' \in \mathcal{B}\underline{B}'$ and $y' \in \mathcal{B}\underline{C}'$ such that $\langle x, x'\rangle \geq 1 - \varepsilon/2$ and $\langle y, y'\rangle \geq 1 - \varepsilon/2$. Let $g : \underline{B} \times \underline{C} \to \mathbb{R}_{\geq 0}$ be defined by $g(x_0, y_0) = \langle x_0, x'\rangle\langle y_0, y'\rangle$. Then $g \in \underline{B, C \multimap 1}$ and moreover $\|g\| \leq 1$. Let $z' = \Phi^{-1}_{B,C,1}(g) \in \mathcal{B}(\underline{B \otimes C})'$, we have

$$\begin{aligned}\|x \otimes y\| &\geq \langle x \otimes y, z'\rangle\\ &= \langle x, x'\rangle\langle y, y'\rangle\\ &\geq \big(1 - \frac{\varepsilon}{2}\big)^2 > 1 - \varepsilon\end{aligned}$$

so that $\|x \otimes y\| \geq 1$. □

5.5. **The symmetric monoidal structure of ICones.** We want now to exploit Theorem 5.12 to show that the category **ICones** can be endowed with a symmetric monoidal structure whose monoidal functor is our tensor product $\otimes$.

One very convenient tool for proving the associated coherence diagrams will be Proposition 5.14 which uses binary trees given by the following syntax: $*$ is a tree (a leaf) and if t_1 and t_2 are trees then $\langle t_1, t_2\rangle$ is a tree. We use $\mathcal{T}_n$ for the set of trees which have n leaves (for $n \in \mathbb{N}^+$).

These trees are used to specify arbitrary "tensor expressions" as follows. If $n \in \mathbb{N}^+$, $\overrightarrow{B} = (B_i)_{i=1}^n$ is a sequence of objects of **ICones** and $t \in \mathcal{T}_n$, we define an object $t^\otimes(\overrightarrow{B})$ of **ICones** by a straightforward induction, for instance $\langle *, \langle *, *\rangle\rangle^\otimes(B_1, B_2, B_3) = B_1 \otimes (B_1 \otimes B_2)$. In the same way, given $\overrightarrow{x} = (x_i \in \underline{B_i})_{i=1}^n$ one defines $t^\otimes(\overrightarrow{x}) \in \underline{t^\otimes(\overrightarrow{B})}$, for instance $\langle *, \langle *, *\rangle\rangle^\otimes(x_1, x_2, x_3) = x_1 \otimes (x_1 \otimes x_2)$.

Proposition 5.14. *Let $n \in \mathbb{N}^+$, $B_1, \dots, B_n, C$ be integrable cones and $t \in \mathcal{T}_n$. Let $f, g \in \mathbf{ICones}(t^\otimes(\overrightarrow{B}), C)$. If, for all $(x_i \in \underline{B_i})_{i=1}^n$ one has $f(t^\otimes(\overrightarrow{x})) = g(t^\otimes(\overrightarrow{x}))$, then $f = g$.*

Proof. By induction on the structure of t. The base case $t = *$ being trivial, assume that $t = \langle t_1, t_2 \rangle$ with $(t_i \in \mathcal{T}_{n_i})_{i=1,2}$ and $n_1 + n_2 = n$ (notice that $n_1, n_2 < n$) so that we can write $\overrightarrow{B} = (B^1_1, \dots, B^1_{n_1}, B^2_1, \dots, B^2_{n_2}) = (\overrightarrow{B^1}, \overrightarrow{B^2})$ and we have $t^{\otimes}(\overrightarrow{B}) = D_1 \otimes D_2$ where $(D_i = t_i^{\otimes}(\overrightarrow{B^i}))_{i=1,2}$. With these notations, we have $f, g \in \mathbf{ICones}(D_1 \otimes D_2, C)$ and so it suffices to prove that $\Phi_{D_1,D_2,C}(f) = \Phi_{D_1,D_2,C}(g) \in \mathbf{ICones}(D_1, D_2 \multimap C)$. By inductive hypothesis, it suffices to prove that for all $\overrightarrow{x^1} = (x^1_i \in \underline{B^1_i})_{i=1}^{n_1}$, one has $\Phi_{D_1,D_2,C}(f)(t_1^{\otimes}(\overrightarrow{x^1})) = \Phi_{D_1,D_2,C}(g)(t_1^{\otimes}(\overrightarrow{x^1})) \in \mathbf{ICones}(B_2, C)$ (the fact that both morphisms have norm ≤ 1 is true but not essential), and so, by inductive hypothesis again, it suffices to prove that for all $\overrightarrow{x^2} = (x^2_i \in \underline{B^2_i})_{i=1}^{n_2}$, one has $\Phi_{D_1,D_2,C}(f)(t_1^{\otimes}(\overrightarrow{x^1}))(t_2^{\otimes}(\overrightarrow{x^2})) = \Phi_{D_1,D_2,C}(g)(t_1^{\otimes}(\overrightarrow{x^1}))(t_2^{\otimes}(\overrightarrow{x^2})) \in \underline{C}$ which results from $\Phi_{D_1,D_2,C}(f)(t_1^{\otimes}(\overrightarrow{x^1}))(t_2^{\otimes}(\overrightarrow{x^2})) = f(t_1^{\otimes}(\overrightarrow{x^1}) \otimes t_2^{\otimes}(\overrightarrow{x^2})) = f(t^{\otimes}(\overrightarrow{x}))$ and similarly for g, and from our assumption about f and g. □

Theorem 5.15. *The category* **ICones***, equipped with the bifunctor* $\otimes$ *and unit* 1 *has a structure of symmetric monoidal category, and this SMC is closed.*

Proof. Let us deal first with the associator. We have two natural bijections

$$\begin{array}{c} \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, C) \\ \big\downarrow \Phi_{B_1 \otimes B_2, B_3, C} \\ \mathbf{ICones}(B_1 \otimes B_2, B_3 \multimap C) \\ \big\downarrow \Phi_{B_1, B_2, B_3 \multimap C} \\ \mathbf{ICones}(B_1, B_2 \multimap (B_3 \multimap C)) \end{array}$$

and — notice that here we use Theorem 5.12 in a crucial way —

$$\begin{array}{c} \mathbf{ICones}(B_1 \otimes (B_2 \otimes B_3), C) \\ \big\downarrow \Phi_{B_1, B_2 \otimes B_3, C} \\ \mathbf{ICones}(B_1, B_2 \otimes B_3 \multimap C) \\ \big\downarrow \mathbf{ICones}(B_1, \Phi_{B_2,B_3,C}) \\ \mathbf{ICones}(B_1, B_2 \multimap (B_3 \multimap C)) \end{array}$$

that we call respectively $\Psi_{B_1,B_2,B_3,C}$ and $\Psi'_{B_1,B_2,B_3,C}$ so that $(\Psi_{B_1,B_2,B_3,C})^{-1} \circ \Psi'_{B_1,B_2,B_3,C}$ is a natural bijection $\mathbf{ICones}(B_1 \otimes (B_2 \otimes B_3), C) \to \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, C)$ and hence, setting $C = B_1 \otimes (B_2 \otimes B_3)$, we know by Lemma 1.1 that

$$\alpha_{B_1,B_2,B_3} = (\Psi_{B_1,B_2,B_3,C})^{-1}(\Psi'_{B_1,B_2,B_3,C}(\mathsf{Id}_C)) \in \mathbf{ICones}((B_1 \otimes B_2) \otimes B_3, B_1 \otimes (B_2 \otimes B_3))$$

is a natural iso. Moreover the definition of the natural iso Φ implies that for all $x_1 \in \underline{B_1}$, $x_2 \in \underline{B_2}$ and $x_3 \in \underline{B_3}$, one has

$$\alpha_{B_1,B_2,B_3}((x_1 \otimes x_2) \otimes x_3) = x_1 \otimes (x_2 \otimes x_3) \tag{5.2}$$

Indeed $\Psi'_{B_1,B_2,B_3,B_1 \otimes (B_2 \otimes B_3)}(\mathsf{Id}_C)$ is

$$f = \boldsymbol{\lambda} x_1 \in \underline{B_1} \cdot \boldsymbol{\lambda} x_2 \in \underline{B_2} \cdot \boldsymbol{\lambda} x_3 \in \underline{B_3} \cdot x_1 \otimes (x_2 \otimes x_3)$$

and α_{B_1,B_2,B_3} must satisfy $\Psi_{B_1,B_2,B_3,C}(\alpha_{B_1,B_2,B_3}) = f$ which is exactly Equation (5.2).

Similarly one defines natural isos $\lambda_B \in \mathbf{ICones}(1 \otimes B, B)$ (using the obvious natural bijection $\mathbf{ICones}(1, B \multimap C) \to \mathbf{ICones}(B, C)$), $\rho_B \in \mathbf{ICones}(B \otimes 1, C)$ (using the obvious natural iso $(1 \multimap C) \to C$ in $\mathbf{ICones}$) and $\gamma_{B_1,B_2} \in \mathbf{ICones}(B_1 \otimes B_2, B_2 \otimes B_1)$) (using the natural iso of Lemma 5.5). These isos satisfy the following equations

$$\forall x \in \underline{B}, \forall u \in \mathbb{R}_{\geq 0} \quad \lambda_B(u \otimes x) = ux = \rho_B(x \otimes u) \tag{5.3}$$

$$\forall x_1 \in \underline{B_1}, \forall x_2 \in \underline{B_2} \quad \gamma_{B_1,B_2}(x_1 \otimes x_2) = x_2 \otimes x_1 \,. \tag{5.4}$$

The required coherence diagrams are easily proven using Equations (5.2), (5.3) and (5.4) combined with Proposition 5.14. In that way, we have endowed **ICones** with an SMC structure whose monoidal product is our tensor product $\otimes$. The natural isomorphism Φ tells us moreover that this SMC is closed. □

6. Categorical properties of integration

From now on all the cones we consider are integrable cones, unless otherwise specified. We use letters B, C, D and E, possibly with subscripts, to denote such cones.

In Lemma 4.11 we have defined the functor $\mathsf{FMeas} : \mathbf{Ar} \to \mathbf{ICones}$ which maps each $X \in \mathbf{Ar}$ to the integrable cone $\mathsf{FMeas}(X)$ of finite non-negative measures on X and acts on measurable functions by the standard push-forward operation, $\mathsf{FMeas}(\varphi) = \varphi_*$.

Notice that for each $X \in \mathbf{Ar}$ we have a specific element $\boldsymbol{\delta}^X \in \underline{\mathsf{Path}(X, \mathsf{FMeas}(X))}$ such that $\boldsymbol{\delta}^X(r)$ is the Dirac mass at $r \in X$, the measure defined by

$$\boldsymbol{\delta}^X(r)(U) = \begin{cases} 1 & \text{if } r \in U \\ 0 & \text{otherwise.} \end{cases}$$

The boundedness of $\boldsymbol{\delta}^X$ is obvious and its measurability results from the observation that if $m = \widetilde{U}$ (for some $U \in \sigma_X$) we have $m \circ \boldsymbol{\delta}^X = \chi_U$ (the characteristic function of U) which is measurable.

Theorem 6.1. *For each $X \in \mathbf{Ar}$ and integrable cone B, one has*

$$\mathcal{I}_X^B \in \mathbf{ICones}(\mathsf{Path}(X, B), \mathsf{FMeas}(X) \multimap B)$$

and $\mathcal{I}_X^B$ (this notation is introduced in Definition 4.3) is an isomorphism which is natural in X and in B (between functors $\mathbf{Ar}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$).

This means that $\mathcal{I}_X^B$ is bilinear continuous and measurable, and preserves integrals on both sides, and that, considered as a linear morphism acting on $\mathsf{Path}(X, B)$, it is an iso in **ICones**.

Proof. For the first statement we just have to prove preservation of integrals in both arguments since bilinearity, continuity and measurability have already been proven in Lemma 4.7. So let $Y \in \mathbf{Ar}$, $\nu \in \underline{\mathsf{FMeas}(Y)}$, $\eta \in \underline{\mathsf{Path}(Y, \mathsf{Path}(X, B))}$ and $\mu \in \underline{\mathsf{FMeas}(X)}$, we have

$$\begin{aligned}\mathcal{I}_X^B\Big(\int_{s\in Y}^{\mathsf{Path}(X,B)}\eta(s)\nu(ds),\mu\Big) &= \int_{r\in X}^{B}\Big(\int_{s\in Y}^{\mathsf{Path}(X,B)}\eta(s)\nu(ds)\Big)(r)\mu(dr)\\ &= \int_{r\in X}^{B}\Big(\int_{s\in Y}^{B}\eta(s)(r)\nu(ds)\Big)\mu(dr)\\ &= \int_{s\in Y}^{B}\Big(\int_{r\in X}^{B}\eta(s)(r)\mu(dr)\Big)\nu(ds) \quad \text{by Theorem 4.15}\\ &= \int \mathcal{I}_X^B(\eta(s),\mu)\nu(ds)\,.\end{aligned}$$

Next let $\beta\in\underline{\mathsf{Path}(X,B)}$ and $\kappa\in\underline{\mathsf{Path}(Y,\mathsf{FMeas}(X))}$, we have

$$\mathcal{I}_X^B\Big(\beta,\int_{s\in Y}^{\mathsf{FMeas}(X)}\kappa(s)\nu(ds)\Big) = \int_{r\in X}^{B}\beta(r)\Big(\int_{s\in Y}^{\mathsf{FMeas}(X)}\kappa(s)\nu(ds)\Big)(dr)$$

where one should remember that the value of the integral $\int\kappa(s)\nu(ds)$ is the finite measure on X which maps $U\in\sigma_X$ to $\int\kappa(s,U)\nu(ds)\in\mathbb{R}_{\geq 0}$. We claim that $x_1=x_2$ where

$$x_1=\int_{r\in X}^{B}\beta(r)\Big(\int_{s\in Y}^{\mathsf{FMeas}(X)}\kappa(s)\nu(ds)\Big)(dr) \qquad\qquad x_2=\int_{s\in Y}^{B}\Big(\int_{r\in X}^{B}\beta(r)\kappa(s,dr)\Big)\nu(ds)\,.$$

Upon applying to both members an element of $\mathcal{M}_0^B$ and using (**Mssep**) for B we can assume that $B=1$. By the monotone convergence theorem and the fact that each measurable function is the lub of a monotone sequence of simple measurable functions, we can assume that β is simple, and by linearity of integrals we can assume that $\beta=\chi_U$ for some $U\in\sigma_X$. Then we have $x_1=\int\kappa(s,U)\nu(ds)=x_2$.

Now we define a function $\mathcal{K}_X^B:\underline{\mathsf{FMeas}(X)\multimap B}\to\underline{\mathsf{Path}(X,B)}$ by setting

$$\mathcal{K}_X^B(f)=f\circ\boldsymbol{\delta}^X\,,$$

which belongs indeed to $\underline{\mathsf{Path}(X,B)}$ because $\boldsymbol{\delta}^X$ is a bounded measurable path. Linearity and continuity of $\mathcal{K}_X^B$ result from linearity and continuity of composition. We prove measurability so let $Y\in\mathbf{Ar}$ and $\eta\in\underline{\mathsf{Path}(Y,\mathsf{FMeas}(X)\multimap B)}$, we contend that $\mathcal{K}_X^B\circ\eta\in\underline{\mathsf{Path}(Y,\mathsf{Path}(X,B))}$. So let $Y'\in\mathbf{Ar}$ and let $p\in\mathcal{M}_{Y'}^{\mathsf{Path}(X,B)}$, we must prove that

$$\psi=\boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot p(s',\mathcal{K}_X^B(\eta(s)))$$

is measurable. Let $\varphi\in\mathbf{Ar}(Y',X)$ and $m\in\mathcal{M}_{Y'}^B$ be such that $p=\varphi\triangleright m$, we have

$$\begin{aligned}\psi &= \boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot m(s',\mathcal{K}_X^B(\eta(s))(\varphi(s')))\\ &= \boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot m(s',\eta(s)(\boldsymbol{\delta}^X(\varphi(s'))))\\ &= \boldsymbol{\lambda}(s',s)\in Y'\times Y\cdot (((\boldsymbol{\delta}^X\circ\varphi)\triangleright m)(s',\eta(s))\end{aligned}$$

which is measurable since $(\boldsymbol{\delta}^X\circ\varphi)\triangleright m\in\mathcal{M}_{Y'}^{\mathsf{FMeas}(X)\multimap B}$ and $\eta\in\underline{\mathsf{Path}(Y,\mathsf{FMeas}(X)\multimap B)}$.

We prove that $\mathcal{K}_X^B$ preserves integrals so let moreover $\nu \in \underline{\mathsf{FMeas}(Y)}$, we have

$$\begin{aligned}\mathcal{K}_X^B\Big(\int_{s\in Y}^{\mathsf{FMeas}(X)\multimap B}\eta(s)\nu(ds)\Big) &= \boldsymbol{\lambda} r\in X\cdot\int_{s\in Y}^{B}\eta(s)(\boldsymbol{\delta}^X(r))\nu(ds)\\ &= \boldsymbol{\lambda} r\in X\cdot\int_{s\in Y}^{B}\mathcal{K}_X^B(\eta(s))(r)\nu(ds)\\ &= \int_{s\in Y}^{\mathsf{Path}(X,B)}\mathcal{K}_X^B(\eta(s))\nu(ds)\end{aligned}$$

so that $\mathcal{K}_X^B \in \mathbf{ICones}(\mathsf{FMeas}(X)\multimap B, \mathsf{Path}(X,B))$.

Let $f \in \underline{\mathsf{FMeas}(X)\multimap B}$, we have

$$\begin{aligned}\mathcal{I}_X^B(\mathcal{K}_X^B(f)) &= \boldsymbol{\lambda}\mu\in\underline{\mathsf{FMeas}(X)}\cdot\int_{r\in X}^{B}f(\boldsymbol{\delta}^X(r))\mu(dr)\\ &= \boldsymbol{\lambda}\mu\in\underline{\mathsf{FMeas}(X)}\cdot f\Big(\int_{r\in X}^{\mathsf{FMeas}(X)}\boldsymbol{\delta}^X(r)\mu(dr)\Big)\quad\text{since } f \text{ preserves integrals}\\ &= \boldsymbol{\lambda}\mu\in\underline{\mathsf{FMeas}(X)}\cdot f(\mu) = f\end{aligned}$$

and let $\beta \in \underline{\mathsf{Path}(X,B)}$, we have

$$\mathcal{K}_X^B(\mathcal{I}_{r\in X}^B(\beta)) = \boldsymbol{\lambda} r\in X\cdot\Big(\int_{r'\in X}^{B}\beta(r')\boldsymbol{\delta}^X(r,dr')\Big) = \beta\,.$$

Checking naturality is routine. □

Theorem 6.2. *Let $X \in \mathbf{Ar}$, B be an object of $\mathbf{ICones}$ and $f_1, f_2 \in \mathbf{ICones}(\mathsf{FMeas}(X), B)$. If, for all $r \in X$, one has $f_1(\boldsymbol{\delta}^X(r)) = f_2(\boldsymbol{\delta}^X(r))$ then $f_1 = f_2$.*

Proof. This results from Theorem 6.1. □

Remark 6.3. In other words the Dirac measures $\boldsymbol{\delta}^X(r)$ are "dense" in the integrable cone $\mathsf{FMeas}(X)$, in the sense that two **ICones** morphisms which take the same values on Dirac measures are equal. This property is one of the main benefits of integrability of cones and it does not hold in **MCones** as shown in Remark 2.7.

It is easy to check that for each $X \in \mathbf{Ar}$ the functor $\mathsf{Path}(X, \_) : \mathbf{ICones} \to \mathbf{ICones}$ preserves all limits. It follows by Theorem 4.19 that it has a left adjoint. We provide an explicit description of this adjoint. We define the functor $\mathsf{FMeas}^{\otimes} : \mathbf{Ar}\times\mathbf{ICones} \to \mathbf{ICones}$ by $\mathsf{FMeas}^{\otimes}(X,B) = \mathsf{FMeas}(X)\otimes B$ and similarly for morphisms.

Theorem 6.4. *For each $X \in \mathbf{Ar}$ we have $\mathsf{FMeas}^{\otimes}(X,\_) \dashv \mathsf{Path}(X,\_)$*

Proof. We have the following sequence of natural bijections:

$$\begin{aligned}&\mathbf{ICones}(\mathsf{FMeas}(X)\otimes B, C)\\ &\qquad\simeq \mathbf{ICones}(B\otimes\mathsf{FMeas}(X), C)\quad\text{by symmetry of }\otimes\text{, Th. 5.15}\\ &\qquad\simeq \mathbf{ICones}(B, \mathsf{FMeas}(X)\multimap C)\quad\text{since }\mathbf{ICones}\text{ is an SMCC, Th. 5.15}\\ &\qquad\simeq \mathbf{ICones}(B, \mathsf{Path}(X,C))\quad\text{by Theorem 6.1.}\end{aligned}$$

□

6.1. **The category of substochastic kernels as a full subcategory of ICones.** If $X, Y \in \mathbf{Ar}$, a substochastic kernel from X to Y is an element of $\mathbf{Skern}(X,Y) = \underline{\mathcal{B}\mathsf{Path}(X, \mathsf{FMeas}(Y))}$: this is an equivalent characterization of this standard measure theory and probability notion. Then **Skern** is the category whose objects are those or **Ar** and:

- the identity at X is $\boldsymbol{\delta}^X \in \mathbf{Skern}(X,X)$
- and given $\kappa_1 \in \mathbf{Skern}(X_1, X_2)$ and $\kappa_2 \in \mathbf{Skern}(X_2, X_3)$, their composite $\kappa = \kappa_2\, \kappa_1$ is given by

$$\kappa(r_1)(U_3) = \int_{r_2 \in X_2}^1 \kappa_2(r_2, U_3)\kappa_1(r_1, dr_2)$$

  for $U_3 \in \sigma_{X_3}$, that is $\kappa(r_1) = \mathcal{I}_{X_2}^{\mathsf{FMeas}(X_3)}(\kappa_2)(\kappa_1(r_1))$: this formula is a continuous generalization of the product of substochastic matrices.

As is well known the category of measurable spaces and substochastic kernels can be presented as the Kleisli category of the Giry monad (or more precisely, of the Panangaden monad since we consider substochastic kernels instead of stochastic kernels), but this point of view does not apply to our case because the set of objects of our small category **Ar** has no reason to be stable under the action of the Panangaden monad.

If $\kappa \in \mathbf{Skern}(X,Y)$, we set $\mathsf{Klin}(\kappa) = \mathcal{I}_X^{\mathsf{FMeas}(Y)}(\kappa) \in \mathbf{ICones}(\mathsf{FMeas}(X), \mathsf{FMeas}(Y))$ defining a functor $\mathsf{Klin} : \mathbf{Skern} \to \mathbf{ICones}$ which maps $X \in \mathbf{Ar}$ to $\mathsf{FMeas}(X)$. Remember that we use FMeas for the functor $\mathbf{Ar} \to \mathbf{ICones}$ defined on morphisms by $\mathsf{FMeas}(\varphi) = \varphi_* = \mathsf{Klin}(\boldsymbol{\delta}^Y \circ \varphi) \in \mathbf{ICones}(\mathsf{FMeas}(X), \mathsf{FMeas}(Y))$ for $\varphi \in \mathbf{Ar}(X,Y)$.

Theorem 6.5. *The functor* $\mathsf{Klin} : \mathbf{Skern} \to \mathbf{ICones}$ *is full and faithful.*

Proof. By Theorem 6.1. □

Remark 6.6. So we can consider the category of measurable spaces (at least those sorted out by **Ar**) and *substochastic kernels* as a full subcategory of **ICones** and again, this is a major consequence of the assumption that linear morphisms must preserve integrals. This has to be compared with QBSs which form a cartesian closed category which contain the category of measurable spaces and *measurable functions* (or a full subcategory thereof such as our **Ar**) as a full subcategory through the Yoneda embedding.

In Section 9.1.1 we will see that, under very reasonable assumptions about its objects, **Ar** arises as a full subcategory of $\mathbf{ICones}^!$, the category of !-coalgebras for an exponential comonad ! based on stable and measurable functions, or on analytical morphisms.

Theorem 6.7. *There is an iso in* $\mathbf{ICones}(\mathsf{FMeas}(0), 1)$ *and, given* $X, Y \in \mathbf{Ar}$*, there is an iso in*

$$\mathbf{ICones}(\mathsf{FMeas}(X \times Y), \mathsf{FMeas}(X) \otimes \mathsf{FMeas}(Y))$$

which is natural in X *and* Y *on the category* **Ar***. These isos turn* FMeas *into a strong monoidal functor* $(\mathbf{Ar}, \times) \to (\mathbf{ICones}, \otimes)$.

Proof. Remember first that FMeas is a functor $\mathbf{Ar} \to \mathbf{ICones}$ which acts on morphisms by push-forward (see Lemma 4.11).

The first statement is obvious since a finite measure on a one element space is the same thing as an element of $\mathbb{R}_{\geq 0}$ whose norm is its value, and in that case our definition of measurability and integrals coincide with the usual ones.

Given an object B of **ICones** we have the following sequence of natural bijections between functors $\mathbf{Ar}^{\mathsf{op}} \times \mathbf{Ar}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{Set}$

$$\begin{aligned}
\mathbf{ICones}(\mathsf{FMeas}(X \times Y), B) &\simeq \mathcal{B}\underline{\mathsf{FMeas}(X \times Y) \multimap B} \\
&\simeq \mathcal{B}\underline{\mathsf{Path}(X \times Y, B)} \quad \text{by Theorem 6.1} \\
&\simeq \mathcal{B}\underline{\mathsf{Path}(X, (\mathsf{Path}(Y, B)))} \quad \text{by Lemma 3.19} \\
&\simeq \mathcal{B}\underline{\mathsf{FMeas}(X) \multimap (\mathsf{FMeas}(Y) \multimap B)} \quad \text{by Theorem 6.1} \\
&\simeq \mathbf{ICones}(\mathsf{FMeas}(X), (\mathsf{FMeas}(Y) \multimap B)) \\
&\simeq \mathbf{ICones}(\mathsf{FMeas}(X) \otimes \mathsf{FMeas}(Y), B)
\end{aligned}$$

because **ICones** is an SMCC and so by Lemma 1.1 we have a natural transformation

$$\psi_{X,Y} \in \mathbf{ICones}(\mathsf{FMeas}(X) \otimes \mathsf{FMeas}(Y), \mathsf{FMeas}(X \times Y))$$

between functors $\mathbf{Ar} \times \mathbf{Ar} \to \mathbf{ICones}$ and this natural transformation is completely characterized by $\psi_{X,Y}(\mu \otimes \nu) = \mu \times \nu$ by the definition of the iso fl used in Lemma 3.19. Using this characterization as well as Proposition 5.14, it is easy to prove that this natural isomorphism (together with its 0-ary version) turns FMeas into a monoidal functor from the cartesian category **Ar** to the monoidal category **ICones**. The proof uses also the fact that $(\varphi_1 \times \varphi_2)_*(\mu_1 \times \mu_2) = (\varphi_1)_*(\mu_1) \times (\psi_2)_*(\mu_2)$ for $\mu_i \in \underline{\mathsf{FMeas}X_i}$ and $\varphi_i \in \mathbf{Ar}(X_i, Y_i)$ for $i = 1, 2$. □

Remark 6.8. This is yet another highly desirable property of the tensor product which results from the preservation of integrals by linear morphisms. It means that an element π of $\mathsf{FMeas}(X) \otimes \mathsf{FMeas}(Y)$ whose norm is 1 can be considered as the joint probability distribution of two (not necessarily independent) random variables valued in X and Y respectively. The case where $\pi = \mu \otimes \nu$ corresponds to the situation where the random variables are independent, of associated distributions μ and ν.

7. Stable and measurable functions

We start studying the non-linear maps between integrable cones and we will consider actually two kinds of non-linear morphisms:

- the stable and measurable morphisms in the present section
- and the analytic ones in Section 8.

The first ones were introduced in [EPT18b] in a weaker setting (no general notion of integration was considered in that paper). The second ones are naturally derived from the monoidal structure of **ICones** and from the ω-completeness of cones. The two notions are deeply connected: analytic morphisms are in particular stable, and some properties proven in Section 7 will be useful in Section 8. We will also see in Remark 9.6 that there are stable functions which are not analytic, the fundamental reason for that being that the definition of stability does not refer to integrals. In the *discrete probability* setting of probabilistic coherence spaces (see Section 10), it has been proved that the two notions are equivalent, see [Cru18].

Stable morphisms satisfy a strong form of monotonicity, which, in ordinary real analysis, can be expressed in terms of derivatives: $\forall n \in \mathbb{N}\ f^{(n)}(x) \geq 0$ (absolute monotonicity). Because we are working in the setting of cones, we use iterated differences instead of derivatives, following an idea first introduced by Bernstein in 1914.

The definition of these iterated differences uses a notion of *local cone* introduced in [EPT18b] and that we recall now. This construction could be rephrased in terms of summability structure [Ehr23], but this is not necessary for our purpose here.

7.1. **The local cone.** One major feature of stable functions[20] from an integrable cone B to an integrable cone C is that, contrarily to linear morphisms, they will be defined, in general, only on the closed "unit ball" $\mathcal{B}\underline{B}$ of B, see Remark 2.5. A typical example is the already mentioned function $f : \mathcal{B}\underline{1} = [0,1] \to \underline{1}$ given by $f(x) = 1 - \sqrt{1-x}$, see Remark 2.5. To deal with such a function $\mathcal{B}\underline{B} \to \underline{C}$ and its strong monotonicity properties at a given $x \in \mathcal{B}\underline{B}$, it will be important to be able to consider the set U of all $u \in \underline{B}$ such that $x + u \in \mathcal{B}\underline{B}$, or rather of all elements of $\mathcal{B}\underline{B}$ of shape λu for such an u and for $\lambda \in \mathbb{R}_{\geq 0}$: we will see that U can itself be considered as a cone, the *local cone* of B at x.

Let B be an integrable cone and $x \in \mathcal{B}\underline{B}$. Let

$$P = \{u \in \underline{B} \mid \exists \varepsilon > 0 \ x + \varepsilon u \in \mathcal{B}\underline{B}\} .$$

Then P is a precone whose 0 element is the 0 of $\underline{B}$. Indeed if $u_1, u_2 \in P$ there is $\varepsilon > 0$ such that $(x + \varepsilon u_i \in \mathcal{B}\underline{B})_{i=1,2}$ and hence

$$x + \frac{\varepsilon}{2}(u_1 + u_2) = \frac{1}{2}(x + \varepsilon u_1) + \frac{1}{2}(x + \varepsilon u_2) \in \mathcal{B}\underline{B}$$

so that $u_1 + u_2 \in P$. The fact that $u \in P \Rightarrow \forall \lambda \in \mathbb{R}_{\geq 0} \ \lambda u \in P$ is clear. The condition (**Cancel**) and (**Pos**) result easily from the fact that they hold in $\underline{B}$.

We can equip P with a map (sometimes called a gauge) $N : P \to \mathbb{R}_{\geq 0}$ defined by

$$N(u) = (\sup\{\lambda > 0 \mid x + \lambda u \in \mathcal{B}\underline{B}\})^{-1} = \inf\{\lambda^{-1} \mid \lambda > 0 \text{ and } x + \lambda u \in \mathcal{B}\underline{B}\}$$

Lemma 7.1. *The function N is a norm on P and, equipped with this norm, P is a cone whose 0 element and algebraic operations are those of $\underline{B}$.*

Proof. Assume that $N(u) = 0$, this means that $\forall \lambda \in \mathbb{R}_{\geq 0} \ \|x + \lambda u\| \leq 1$ and hence $\forall \lambda \in \mathbb{R}_{\geq 0} \ \lambda \|u\| \leq 1$ so that $u = 0$. Let $u_1, u_2 \in P$ and let $\varepsilon > 0$. We can find $\lambda_1, \lambda_2 > 0$ such that $x + \lambda_i u_i \in \mathcal{B}\underline{B}$ and $\lambda_i^{-1} \leq N(u_i) + \varepsilon/2$ for $i = 1, 2$. We have

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}(x + \lambda_2 u_2) + \frac{\lambda_2}{\lambda_1 + \lambda_2}(x + \lambda_1 u_1) \in \mathcal{B}\underline{B}$$

so that

$$x + \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}(u_1 + u_2) \in \mathcal{B}\underline{B}$$

so that $N(u_1 + u_2) \leq (\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2})^{-1} = \lambda_1^{-1} + \lambda_2^{-1} \leq N(u_1) + N(u_2) + \varepsilon$. Since this holds for all $\varepsilon > 0$ we have $N(u_1 + u_2) \leq N(u_1) + N(u_2)$. The property (**Normp**) is easy, let us prove (**Normc**). Observe first that, for $u, v \in P$, we have $u \leq_P v$ iff $u \leq_B v$. Let $(u_n)_{n \in \mathbb{N}}$ be an increasing sequence in P such that $\forall n \in \mathbb{N} \ N(u_n) \leq 1$. Then we have $\forall n \in \mathbb{N} \ x + u_n \in \mathcal{B}\underline{B}$ and hence the sequence $(x + u_n)_{n \in \mathbb{N}}$ is an increasing sequence in $\mathcal{B}\underline{B}$ and so it has a lub in $\mathcal{B}\underline{B}$ which coincides with $x + u$ where u is the lub of $(u_n)_{n \in \mathbb{N}}$ in $\underline{B}$. Thus we have $u \in P$ and $N(u) \leq 1$. Last observe that u is *a fortiori* the lub of the u_n's in P. □

[20] This will be also the case of analytic morphisms in Section 8.

We use now the standard notation $\|\_\|$ for that norm. Notice that

$$\mathcal{B}P = \{u \in \underline{B} \mid x + u \in \mathcal{B}\underline{B}\}$$

and also that $\|u\|_B \leq \|u\|$, for all $u \in P$.

For each $X \in \mathbf{Ar}$ we define $\mathcal{M}_X$ as the set of all test functions $\boldsymbol{\lambda} r \in X \cdot \boldsymbol{\lambda} u \in P \cdot m(r,u)$ for the elements m of $\mathcal{M}_X^B$ (such an element of $\mathcal{M}_X$ will still be denoted by m even if two different elements of $\mathcal{M}_X^B$ possibly induce the same test). The fact that $\forall r \in X,\, u \in \mathcal{B}P\ m(r,u) \leq 1$ whenever $m \in \mathcal{M}_X$ results from $\mathcal{B}P \subseteq \mathcal{B}\underline{B}$.

Then it is straightforward to check that $(P, (\mathcal{M}_X)_{X \in \mathbf{Ar}})$ is a measurable cone, that we denote as B_x and call *the local cone of* B *at* x and it is also clear that this measurable cone is integrable, the integral of a path in B_x being defined exactly as in B.

It is important to keep in mind the meaning of the norm in this local cone, which is most usefully described as follows.

Lemma 7.2. *Let* $x \in \mathcal{B}\underline{B}$ *and* $u \in \underline{B_x} \setminus \{0\}$. *Then we have* $x + \|u\|_{B_x}^{-1} u \in \mathcal{B}\underline{B}$ *and* $x + \lambda u \notin \mathcal{B}\underline{B}$ *for all* $\lambda > \|u\|_{\underline{B_x}}^{-1}$.

Proof. By definition of the norm in B_x and by the ω-closedness of $\mathcal{B}\underline{B}$. □

Example 7.3. Let $B = 1$ so that $\underline{1} = \mathbb{R}_{\geq 0}$ and $\mathcal{B}\underline{1} = [0,1]$. If $x \in [0,1]$ we have two cases: if $x = 1$ then $B_x = 0$ and if $x < 1$ we have $\underline{B_x} = \mathbb{R}_{\geq 0}$. If $u \in \underline{B_x} \setminus \{0\}$ (which implies $x < 1$) then the largest $\lambda > 0$ such that $x + \lambda u \in [0,1]$ is $\frac{1-x}{u}$ and hence $\|u\|_{B_x} = \frac{u}{1-x} = \frac{1}{1-x}\|u\|_B$ so the local cone B_x can be seen as an homothetic image of B by a factor $\frac{1}{1-x}$ which goes to ∞ when x gets closer to the border 1 of the unit ball $[0,1]$. This very simple example gives an intuition of what happens in general, with the difference that B_x has no reason to be always homothetic to B, think for instance of the case where $B = 1 \,\&\, 1$ and $x = (1,0)$: then B_x is isomorphic to 1.

7.2. The integrable cone of stable and measurable functions.

Given $n \in \mathbb{N}$, we use $\mathcal{P}^-(n)$ (resp. $\mathcal{P}^+(n)$) for the set of all subsets I of $\{1,\dots,n\}$ such that $n - \#I$ is odd (resp. even). In particular $\{1,\dots,n\} \in \mathcal{P}^+(n)$.

Lemma 7.4. *Let* $n \in \mathbb{N}$, $j \in \{1,\dots,n+1\}$ *and* $\varepsilon \in \{-,+\}$. *Given* $I \in \mathcal{P}^\varepsilon(n)$, *the set*

$$\mathsf{inj}_j(I) = \{i \in I \mid i < j\} \cup \{j\} \cup \{i+1 \mid i \in I \text{ and } i \geq j\}$$

belongs to $\mathcal{P}^\varepsilon(n+1)$ *and* inj_j *defines a bijection between* $\mathcal{P}^\varepsilon(n)$ *and the set of all* $J \subseteq \{1,\dots,n+1\}$ *such that* $J \in \mathcal{P}^\varepsilon(n+1)$ *and* $j \in J$.

This is obvious.

Definition 7.5. Let P and Q be cones, a function $f : \mathcal{B}P \to Q$ is *totally monotonic* if for each $n \in \mathbb{N}$ and each $x, u_1, \dots, u_n \in P$ such that $x + \sum_{i=1}^n u_i \in \mathcal{B}P$ one has

$$\sum_{I \in \mathcal{P}^-(n)} f(x + \sum_{i \in I} u_i) \leq \sum_{I \in \mathcal{P}^+(n)} f(x + \sum_{i \in I} u_i)\,. \tag{7.1}$$

For $n = 1$ this condition means that f is increasing. For $n = 2$ we have $\mathcal{P}^-(2) = \{\{1\},\{2\}\}$ and $\mathcal{P}^+(2) = \{\{1,2\},\emptyset\}$, so Condition (7.1) means

$$f(x + u_1) + f(x + u_2) \leq f(x + u_1 + u_2) + f(x)$$

that is, assuming that f is increasing,

$$f(x+u_1)-f(x)\leq f(x+u_1+u_2)-f(x+u_2)$$

in other words, the map $x\mapsto f(x+u_1)-f(x)$ is increasing (where it is defined). For $n=3$ we have $\mathcal{P}^-(3)=\{\{1,2\},\{2,3\},\{1,3\},\emptyset\}$ and $\mathcal{P}^+(3)=\{\{1,2,3\},\{1\},\{2\},\{3\}\}$, so Condition (7.1) means:

$$\begin{aligned}f(x+u_1+u_2)+f(x+u_2+u_3)+f(x+u_1+u_3)+f(x)\\ \leq f(x+u_1+u_2+u_3)+f(x+u_1)+f(x+u_2)+f(x+u_3)\,.\end{aligned}$$

Remark 7.6. This kind of definition appears in many places in the literature in real analysis, differential equations, Laplace transforms *etc.* The corresponding conditions, first considered by Hausdorff, are then usually expressed in terms of derivatives: for instance a function f from some open interval I of $\mathbb{R}$ to $\mathbb{R}$ is *absolutely monotonic* (resp. *completely monotonic*) if it is C^∞ and satisfies $f^{(n)}(x)\geq 0$ (resp. $(-1)^n f^{(n)}(x)\geq 0$) for all $x\in I$ and $n\in\mathbb{N}$. Bernstein introduces in [Ber14], in the case of real functions of one real parameter, iterated differences allowing to characterize absolutely monotonic functions — which in turn can be shown to be analytic — by a condition which is equivalent to (7.1). We use the expression "totally monotonic" for this extension to cones of Bernstein's definition to avoid confusion with "completely monotonic" and "absolutely monotonic".

This definition arose in denotational semantics during the work of the first author reported in [EPT18b], when the authors of that paper tried to build a *cartesian closed category* whose objects are Selinger's cones, ordered by the cone order ($x\leq y$ if there is a z such that $x+z=y$). A careful analysis of these constraints (which are actually quite strong) leads unavoidably to the conclusion that the morphisms of the sought CCC should be totally monotonic. We will see in Section 7.4 how total monotonicity leads indeed to cartesian closedness.

Definition 7.7. Let P and Q be cones. A function $f:\mathcal{B}P\to Q$ is *stable* if f is totally monotonic, bounded and ω-continuous (see Definition 2.2).

Remark 7.8. This terminology is motivated by the fact that stable functions (in the sense of Berry [Ber78]) on Girard's coherence spaces [Gir86] can be characterized by a property completely similar to total monotonicity. We have the intuition that our stable morphisms on cones are a quantitative analog of the notion of stable function introduced in "qualitative" denotational semantics.

Example 7.9. Let $B=\mathsf{FMeas}(\{0,1\})$ so that $\underline{B}=\mathbb{R}^2_{\geq 0}$ with the norm given by $\|x\|=x_0+x_1$. Then for each $\overrightarrow{a}\in\mathbb{R}_{\geq 0}^{\mathbb{N}\times\mathbb{N}}$ which satisfies $\forall t\in[0,1]\quad\sum_{p,q\in\mathbb{N}}a_{p,q}t^p(1-t)^q\leq 1$, the function $f:\mathcal{B}\underline{B}\to\mathbb{R}_{\geq 0}$ defined by $f(x)=\sum_{p,q\in\mathbb{N}}a_{p,q}x_0^p x_1^q$ is totally monotonic. An example of such a function is $f(x)=\sum_{n=1}^{\infty}2^n x_0^n x_1^n$ since for $x\in\mathcal{B}\underline{B}$ one has $x_0x_1\leq\frac{1}{4}$. One might think that f is convex since all its iterated partial derivatives are ≥ 0 everywhere, but this is not true since for instance $f(0,1)=f(1,0)=0$ whereas $f(\frac{1}{2},\frac{1}{2})=1$. In general, total monotonicity of functions defined by power series with one or more parameters as in this example correspond to the fact that all the partial derivatives are everywhere ≥ 0, which is related simply to convexity only in the one parameter case.

Definition 7.10. Let C,D be measurable cones. A stable function $f:\mathcal{B}\underline{C}\to\underline{D}$ is measurable if for each $X\in\mathbf{Ar}$ and $\gamma\in\mathcal{B}\underline{\mathsf{Path}(X,C)}$ one has $f\circ\gamma\in\underline{\mathsf{Path}(X,D)}$.

Lemma 7.11. *The set of stable and measurable functions $C \to D$, equipped with algebraic operations defined pointwise, is a precone.*

This is straightforward, we use P for this precone. We need first to understand the order on stable functions induced by the addition operation of P. As usual, this order relation is simply denoted as $\leq$ of $\leq_P$.

Lemma 7.12. *Let $f, g \in P$. One has $f \leq g$ iff for each $n \in \mathbb{N}$ and each $x, u_1, \ldots, u_n \in \mathcal{B}\underline{C}$ such that $x + \sum_{i=1}^{n} u_i \in \mathcal{B}\underline{C}$ one has*

$$\sum_{I \in \mathcal{P}^-(n)} g(x + \sum_{i \in I} u_i) + \sum_{I \in \mathcal{P}^+(n)} f(x + \sum_{i \in I} u_i) \leq \sum_{I \in \mathcal{P}^+(n)} g(x + \sum_{i \in I} u_i) + \sum_{I \in \mathcal{P}^-(n)} f(x + \sum_{i \in I} u_i)\,.$$

Proof. Assume first that $f \leq g$ and let $h = g - f$. Notice that since addition is defined pointwise in P we have $h(x) = g(x) - f(x)$ for all $x \in \mathcal{B}\underline{C}$, and by definition of the order relation of P, this function h is stable. Given $n \in \mathbb{N}$ and $x, u_1, \ldots, u_n \in \mathcal{B}\underline{C}$ such that $x + \sum_{i=1}^{n} u_i \in \mathcal{B}\underline{C}$ we have

$$\sum_{I \in \mathcal{P}^-(n)} h(x + \sum_{i \in I} u_i) \leq \sum_{I \in \mathcal{P}^+(n)} h(x + \sum_{i \in I} u_i)$$

and the announced inequality follows. Conversely if the property expressed in the lemma holds we have in particular $\forall x \in \mathcal{B}P\ f(x) \leq g(x)$ (take $n = 0$) and so we can define a function $h : \mathcal{B}\underline{C} \to \underline{D}$ by $h(x) = g(x) - f(x)$ and this function is totally monotonic. This function is ω-continuous by Lemma 2.10 and is measurable because subtraction is measurable on $\mathbb{R}^2$. ◻

Remark 7.13. Notice that if $f \leq g$ (still for the cone order relation of P, characterized by Lemma 7.12) then $f(x) \leq g(x)$ for all $x \in \mathcal{B}\underline{B}$, but the converse is not true. As an example take $f(x) = x$ and $g(x) = 1$, defining two stable functions (for $B = C = 1$) which do not satisfy $f \leq_P g$ but are such that $f(x) \leq g(x)$ holds for all $x \in [0, 1]$. It is natural to call this order relation on stable functions the *stable order* in reference to [Ber78, Gir86] where the stable order behaves in a very similar way, and admits a similar characterization, in terms of differences.

We equip this precone P with the norm $\|f\| = \sup_{x \in \mathcal{B}\underline{C}} \|f(x)\|$ which is well defined by our assumptions that stable functions are bounded.

Lemma 7.14. *Let $(f_n \in \mathcal{B}P)_{n \in \mathbb{N}}$ be an increasing sequence (for the stable order). Then $f : \mathcal{B}\underline{C} \to \underline{D}$ defined by $f(x) = \sup_{n \in \mathbb{N}} f_n(x)$ is bounded, totally monotonic, ω-continuous and measurable, that is, $f \in P$. This map f is the lub of the f_n's in P.*

Proof sketch. Total monotonicity follows from ω-continuity of addition, ω-continuity is straightforward and measurability results from the monotone convergence theorem. The fact that f is the lub of the f_n's results from the fact that it is defined as a pointwise lub. ◻

So P is a cone that we equip with a measurability structure $\mathcal{M}$ defined as in $C \multimap D$: a $p \in \mathcal{M}_X$ is a function $p = \gamma \triangleright m$ where $\gamma \in \underline{\mathsf{Path}(X, C)}$ and $m \in \mathcal{M}_Y^D$, given by

$$\gamma \triangleright m = \boldsymbol{\lambda}(r, f) \in X \times P \cdot m(r, f(\gamma(r)))\,.$$

Then we check that $\mathcal{M}$ satisfies the required conditions exactly as we did for $C \multimap D$ in Section 5.1. We have defined a measurable cone $C \Rightarrow_{\mathsf{s}} D$ that we prove now to be integrable.

Let $X \in \mathbf{Ar}$ and $\eta \in \underline{\mathsf{Path}(X, C \Rightarrow_{\mathsf{s}} D)}$, and let $\mu \in \underline{\mathsf{FMeas}(X)}$. We define a function $f : \mathcal{B}\underline{C} \to \underline{D}$ by

$$f(x) = \int_{r \in X} \eta(r)(x)\mu(dr)\,.$$

This integral is well defined because, for each $x \in \mathcal{B}\underline{B}$, the function $\boldsymbol{\lambda} r \in X \cdot \eta(r)(x)$ is measurable and bounded since η is a measurable path. The function f is totally monotonic by bilinearity of integration, ω-continuous by the monotone convergence theorem, we check that it is measurable. Let $Y \in \mathbf{Ar}$ and $\gamma \in \underline{\mathsf{Path}(Y, C)}$, we have

$$f \circ \gamma = \boldsymbol{\lambda} s \in Y \cdot \int_{r \in X} \eta(r)(\gamma(s))\mu(dr)$$

and we must prove that $f \circ \gamma \in \underline{\mathsf{Path}(Y, D)}$, so let $Y' \in \mathbf{Ar}$ and $m \in \mathcal{M}^{D}_{Y'}$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot m(s', (f \circ \gamma)(s)) &= \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot m\Big(s', \int_{r \in X} \eta(r)(\gamma(s))\mu(dr)\Big)\\ &= \boldsymbol{\lambda}(s', s) \in Y' \times Y \cdot \int_{r \in X} m(s', \eta(r)(\gamma(s)))\mu(dr)\end{aligned}$$

which is measurable because $\boldsymbol{\lambda}(s', s, r) \in Y' \times Y \times X \cdot m(s', \eta(r)(\gamma(s)))$ is measurable by our assumption about η and by Lemma 4.6. This shows that $f \in \underline{C \Rightarrow_{\mathsf{s}} D}$. Let $p \in \mathcal{M}^{C \Rightarrow D}_{0}$ so that $p = x \triangleright m$ for some $x \in \underline{C}$ and $m \in \mathcal{M}^{D}_{0}$, we have

$$\begin{aligned}p(f) &= m\Big(\int_{r \in X} \eta(r)(x)\mu(dr)\Big)\\ &= \int_{r \in X} m(\eta(r)(x))\mu(dr)\\ &= \int_{r \in X} p(\eta(r))\mu(dr)\end{aligned}$$

so that f is the integral of η over μ, this shows that $C \Rightarrow_{\mathsf{s}} D$ is an integrable cone.

7.3. **Finite differences.** We introduce a natural difference operator on totally monotonic functions which provides an inductive characterization of total monotonicity that we will use to prove two basic properties of totally monotonic functions, Lemmas 7.26 and 7.27, which will show useful to establish a property which is not completely obvious: the composition of two stable functions is stable. In the setting of complete and absolute monotonicity in real analysis, the corresponding property can be obtained by means of the Faà di Bruno formula (higher derivative of composite of functions) as mentioned in [LN83], Section 7. Our notion of total monotonicity being defined in terms of iterated differences, we need a specific reasoning.

Given $\overrightarrow{u} \in \underline{B}^n$ such that $\sum_{i=1}^{n} u_i \in \mathcal{B}\underline{B}$ we use $B_{\overrightarrow{u}}$ for the local cone $B_{\sum_{i=1}^{n} u_i}$ (see Section 7.1).

Let B, C be cones, $f : \mathcal{B}\underline{B} \to \underline{C}$ be a function, $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \underline{B}$ such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$ we define

$$\begin{aligned}\Delta^\varepsilon f(\overrightarrow{u}) : \mathcal{B}\underline{B_{\overrightarrow{u}}} &\to \underline{C} \\ x &\mapsto \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x + \sum_{i \in I} u_i)\end{aligned}$$

for $\varepsilon \in \{-, +\}$ and if f is totally monotonic we set

$$\Delta f(\overrightarrow{u}) = \Delta^+ f(\overrightarrow{u}) - \Delta^- f(\overrightarrow{u}) : \mathcal{B}\underline{B_{\overrightarrow{u}}} \to \underline{C}\,,$$

the difference being computed pointwise. Notice that for $n = 0$ (so that $\overrightarrow{u} = (\,)$) we have $\Delta f((\,)) = f$ since $\mathcal{P}^+(0) = \{\emptyset\}$ and $\mathcal{P}^-(0) = \emptyset$.

Definition 7.15. Let $f \in \mathcal{B}\underline{B} \to \underline{C}$ be a function and let $n \in \mathbb{N}$. We say that f is *n-increasing from B to C* if

- $n = 0$ and f is increasing
- or $n > 0$, f is increasing and, for all $u \in \mathcal{B}\underline{B}$ the function $\Delta f(u) : \mathcal{B}\underline{B_u} \to \underline{C}$ (which maps x to $f(x + u) - f(x)$) is $(n-1)$-increasing from B_u to C.

Lemma 7.16. *Let $f \in \mathcal{B}\underline{B} \to \underline{C}$ be a function which is n-increasing for all $n \in \mathbb{N}$. Then for all $u \in \mathcal{B}\underline{B}$, the function $\Delta f(u) : \mathcal{B}\underline{B_u} \to \underline{C}$ is n-increasing for all $n \in \mathbb{N}$.*

Proof. Immediate consequence of the definition of n-increasing functions. ☐

Lemma 7.17. *Let $f : \underline{B} \to \underline{C}$ be totally monotonic. For $u, u_1, \dots, u_n \in \underline{B}$ and $x \in \mathcal{B}\underline{B_{u,\overrightarrow{u}}}$, one has $\Delta^\varepsilon f(u, \overrightarrow{u})(x) = \Delta^\varepsilon f(\overrightarrow{u})(x + u) + \Delta^{-\varepsilon} f(\overrightarrow{u})(x)$ for $\varepsilon \in \{+, -\}$. Moreover*

$$\Delta f(\overrightarrow{u})(x) \le \Delta f(\overrightarrow{u})(x + u) \quad \text{and} \quad \Delta f(u, \overrightarrow{u}) = \Delta(\Delta f(\overrightarrow{u}))(u)\,.$$

Proof. Let $\overrightarrow{v} = (u, \overrightarrow{u})$, of length $n + 1$. We have

$$\Delta^\varepsilon f(\overrightarrow{v})(x) = \sum_{I \in \mathcal{P}^\varepsilon(n+1)} f(x + \sum_{i \in I} v_i) = \sum_{\substack{I \in \mathcal{P}^\varepsilon(n+1) \\ 1 \in I}} f(x + \sum_{i \in I} v_i) + \sum_{\substack{I \in \mathcal{P}^\varepsilon(n+1) \\ 1 \notin I}} f(x + \sum_{i \in I} v_i)\,.$$

Now observe that

$$\sum_{\substack{I \in \mathcal{P}^\varepsilon(n+1) \\ 1 \notin I}} f(x + \sum_{i \in I} v_i) = \sum_{I \in \mathcal{P}^{-\varepsilon}(n)} f(x + \sum_{i \in I} u_i) = \Delta^{-\varepsilon} f(\overrightarrow{u})(x)$$

by definition of $\overrightarrow{v}$ and, using Lemma 7.4, observe also that

$$\sum_{\substack{I \in \mathcal{P}^\varepsilon(n+1) \\ 1 \in I}} f(x + \sum_{i \in I} v_i) = \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x + u + \sum_{i \in I} u_i) = \Delta^\varepsilon f(\overrightarrow{u})(x + u)\,.$$

So we have $\Delta^\varepsilon f(u, \overrightarrow{u})(x) = \Delta^\varepsilon f(\overrightarrow{u})(x + u) + \Delta^{-\varepsilon} f(\overrightarrow{u})(x)$. Since f is totally monotonic, we have

$$\Delta^- f(u, \overrightarrow{u})(x) \le \Delta^+ f(u, \overrightarrow{u})(x)$$

and hence

$$\Delta^- f(\overrightarrow{u})(x + u) - \Delta^- f(\overrightarrow{u})(x) \le \Delta^+ f(\overrightarrow{u})(x + u) - \Delta^+ f(\overrightarrow{u})(x)$$

both subtractions being defined since f is increasing. Therefore $\Delta f(\overrightarrow{u})(x) \leq \Delta f(\overrightarrow{u})(x+u)$. Moreover

$$\begin{aligned}\Delta f(u,\overrightarrow{u})(x) &= \Delta^+ f(u,\overrightarrow{u})(x) - \Delta^- f(u,\overrightarrow{u})(x)\\ &= (\Delta^+ f(\overrightarrow{u})(x+u) + \Delta^- f(\overrightarrow{u})(x)) - (\Delta^- f(\overrightarrow{u})(x+u) + \Delta^+ f(\overrightarrow{u})(x))\\ &= (\Delta^+ f(\overrightarrow{u})(x+u) - \Delta^- f(\overrightarrow{u})(x+u)) - (\Delta^+ f(\overrightarrow{u})(x) - \Delta^- f(\overrightarrow{u})(x))\\ &= \Delta f(\overrightarrow{u})(x+u) - \Delta f(\overrightarrow{u})(x)\\ &= \Delta(\Delta f(\overrightarrow{u}))(u)(x)\,. \end{aligned}$$

□

Lemma 7.18. *If a function $f \in \mathcal{B}\underline{B} \to \underline{C}$ is totally monotonic, then for each $u \in \mathcal{B}\underline{B}$, the function $\Delta f(u) : \mathcal{B}\underline{B_u} \to \underline{C}$ is totally monotonic.*

Proof. Let $\overrightarrow{u} \in \mathcal{B}\underline{B_u}$, notice that $(B_u)_{\overrightarrow{u}} = B_{u,\overrightarrow{u}}$. Let $x \in \mathcal{B}\underline{B_{u,\overrightarrow{u}}}$. We have

$$\Delta^\varepsilon(\Delta f(u))(\overrightarrow{u})(x) = \Delta^\varepsilon f(\overrightarrow{u})(x+u) - \Delta^\varepsilon f(\overrightarrow{u})(x)$$

where the subtraction makes sense because f is increasing. By our assumption on f we have

$$\Delta^- f(u,\overrightarrow{u})(x) \leq \Delta^+ f(u,\overrightarrow{u})(x)$$

that is

$$\Delta^- f(\overrightarrow{u})(x+u) + \Delta^+ f(\overrightarrow{u})(x) \leq \Delta^+ f(\overrightarrow{u})(x+u) + \Delta^- f(\overrightarrow{u})(x)$$

by Lemma 7.17, and hence

$$\Delta^-(\Delta f(u))(\overrightarrow{u})(x) \leq \Delta^+(\Delta f(u))(\overrightarrow{u})(x)$$

□

Theorem 7.19. *A function $f \in \mathcal{B}\underline{B} \to \underline{C}$ is totally monotonic iff it is n-increasing for all $n \in \mathbb{N}$.*

Proof. Remember that f is totally monotonic iff for all $n \in \mathbb{N}$, all $\overrightarrow{u} \in \underline{B}^n$ and $x \in \underline{B_{\overrightarrow{u}}}$ we have $\Delta^- f(\overrightarrow{u})(x) \leq \Delta^+ f(\overrightarrow{u})(x)$.

We prove first by induction on $k \in \mathbb{N}$ that for all $f \in \mathcal{B}\underline{B} \to \underline{C}$, if f is totally monotonic then f is k-increasing.

For $k = 0$, we have to prove that f is increasing, which results from total monotonicity applied with $n = 1$.

For $k > 0$ we have to prove that f is increasing (which results from total monotonicity applied with $n = 1$) and that for all $u \in \mathcal{B}\underline{B}$ the function $\Delta f(u) : \mathcal{B}\underline{B_u} \to \underline{C}$ is $(k-1)$-increasing, for which, by inductive hypothesis, it suffices to prove that $\Delta f(u)$ is totally monotonic, and this property results from Lemma 7.18.

Conversely, we prove by induction on $n \in \mathbb{N}$ that, for each $f \in \mathcal{B}\underline{B} \to \underline{C}$, if f is k-increasing for all $k \in \mathbb{N}$ then for all $\overrightarrow{u} \in \mathcal{B}\underline{B}^n$ and $x \in \mathcal{B}\underline{B_{\overrightarrow{u}}}$, one has $\Delta^- f(\overrightarrow{u})(x) \leq \Delta^+ f(\overrightarrow{u})(x)$. For $n = 0$ there is nothing to prove so assume that $n > 0$. Let $(u,\overrightarrow{u}) \in \mathcal{B}\underline{B}^n$ and let $x \in \mathcal{B}\underline{B_{u,\overrightarrow{u}}}$. Notice that $\overrightarrow{u} \in \mathcal{B}\underline{B}^{n-1}$ and $x \in \mathcal{B}\underline{B_{\overrightarrow{u}}}$. Since $\Delta f(u)$ is k-increasing for all $k \in \mathbb{N}$ by Lemma 7.16, we know by applying the inductive hypothesis to $\Delta f(u)$ that

$$\Delta^-(\Delta f(u))(\overrightarrow{u})(x) \leq \Delta^+(\Delta f(u))(\overrightarrow{u})(x)$$

that is

$$\Delta^- f(\overrightarrow{u})(x+u) - \Delta^- f(\overrightarrow{u})(x) \leq \Delta^+ f(\overrightarrow{u})(x+u) - \Delta^+ f(\overrightarrow{u})(x)$$

which implies

$$\Delta^- f(\overrightarrow{u})(x+u) + \Delta^+ f(\overrightarrow{u})(x) \leq \Delta^+ f(\overrightarrow{u})(x+u) + \Delta^- f(\overrightarrow{u})(x)$$

that is $\Delta^- f(u,\overrightarrow{u})(x) \leq \Delta^+ f(u,\overrightarrow{u})(x)$ by Lemma 7.17, as expected. □

Lemma 7.20. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotonic and $\overrightarrow{u} \in \underline{B}^n$ be such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$. Then the functions $\Delta^+ f(\overrightarrow{u}), \Delta^- f(\overrightarrow{u}), \Delta f(\overrightarrow{u}) : \mathcal{B}\underline{B_{\overrightarrow{u}}} \to \underline{C}$ are totally monotonic.*

Proof. The total monotonicity of $\Delta^\varepsilon f(\overrightarrow{u})$ results from the easy observation that if $g : \mathcal{B}\underline{B} \to \mathcal{B}\underline{C}$ and $u \in \mathcal{B}\underline{B}$ then the map $g_u : \mathcal{B}\underline{B_u} \to \underline{C}$ defined by $g_u(x) = g(x+u)$ is also totally monotonic, and each sum of totally monotonic functions is totally monotonic.

The total monotonicity of $\Delta f(\overrightarrow{u})$ results from Theorem 7.19. □

Lemma 7.21. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotonic. Then for each $\overrightarrow{u} \in \underline{B}^n$ such that $\sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$ and $x \in \underline{B_{\overrightarrow{u}}}$ we have $\Delta f(\overrightarrow{u})(x) \leq f(x + \sum_{i=1}^n u_i)$.*

Proof. By induction on n. The base case $n = 0$ is trivial since then $\Delta f(\overrightarrow{u})(x) = f(x)$. For the inductive case, let $(u,\overrightarrow{u}) \in \underline{B}^{n+1}$ with $u + \sum_{i=1}^n u_i \in \mathcal{B}B$ and $x \in \underline{B_{u,\overrightarrow{u}}}$, that is $x + u \in \underline{B_{\overrightarrow{u}}}$. We have $\Delta f(u,\overrightarrow{u})(x) = \Delta f(\overrightarrow{u})(x+u) - \Delta f(\overrightarrow{u})(x) \leq \Delta f(\overrightarrow{u})(x+u) \leq f(x + u + \sum_{i=1}^n u_i)$ by inductive hypothesis. □

Lemma 7.22. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be a totally monotonic function. Let $n \in \mathbb{N}$, $u, v \in \mathcal{B}\underline{B}$ and $\overrightarrow{u} \in \mathcal{B}\underline{B}^n$, and assume that $u + v + \sum_{i=1}^n u_i \in \mathcal{B}\underline{B}$. Then for each $x \in \mathcal{B}\underline{B_{\overrightarrow{u}}}$ we have*

$$\Delta f(\overrightarrow{u})(x+u) = \Delta f(\overrightarrow{u})(x) + \Delta f(u,\overrightarrow{u})(x)$$
$$\Delta f(u+v,\overrightarrow{u})(x) = \Delta f(u,\overrightarrow{u})(x) + \Delta f(v,\overrightarrow{u})(x+u)\,.$$

Proof. The first equation results from $\Delta f(u,\overrightarrow{u}) = \Delta(\Delta f(\overrightarrow{u}))(u)$, see Lemma 7.18. For the second equation take $x \in \mathcal{B}\underline{B_{u+v,\overrightarrow{u}}}$. Let n be the length of $\overrightarrow{u}$. Setting $\overrightarrow{v} = (u,\overrightarrow{u})$ and $\overrightarrow{w} = (v,\overrightarrow{u})$ (both of length $n+1$), we have

$$\begin{aligned}
\Delta f(\overrightarrow{v})(x) + \Delta f(\overrightarrow{w})(x+u) &= \sum_{I\in\mathcal{P}^+(n+1)} f(x + \sum_{i\in I} v_i) - \sum_{I\in\mathcal{P}^-(n+1)} f(x + \sum_{i\in I} v_i) \\
&\quad + \sum_{I\in\mathcal{P}^+(n+1)} f(x + u + \sum_{i\in I} w_i) - \sum_{I\in\mathcal{P}^-(n+1)} f(x + u + \sum_{i\in I} w_i) \\
&= \sum_{I\in\mathcal{P}^-(n)} f(x + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^+(n)} f(x + u + \sum_{i\in I} u_i) \\
&\quad - (\sum_{I\in\mathcal{P}^+(n)} f(x + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^-(n)} f(x + u + \sum_{i\in I} u_i)) \\
&\quad + \sum_{I\in\mathcal{P}^-(n)} f(x + u + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^+(n)} f(x + u + v + \sum_{i\in I} u_i) \\
&\quad - (\sum_{I\in\mathcal{P}^+(n)} f(x + u + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^-(n)} f(x + u + v + \sum_{i\in I} u_i)) \\
&= \sum_{I\in\mathcal{P}^+(n)} f(x + u + v + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^-(n)} f(x + \sum_{i\in I} u_i) \\
&\quad - (\sum_{I\in\mathcal{P}^+(n)} f(x + \sum_{i\in I} u_i) + \sum_{I\in\mathcal{P}^-(n)} f(x + u + v + \sum_{i\in I} u_i))\,.
\end{aligned}$$

□

Lemma 7.23. *Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be totally monotonic. Let $n \in \mathbb{N}$, $u \in \underline{B}$ and $\overrightarrow{u}, \overrightarrow{v} \in \underline{B}^n$, and assume that $u + \sum_{i=1}^{n}(u_i + v_i) \in \mathcal{B}\underline{B}$. Then for each $x \in \mathcal{B}\underline{B_{u,\overrightarrow{u},\overrightarrow{v}}}$ we have*

$$\begin{aligned}\Delta f(\overrightarrow{u} + \overrightarrow{v})(x + u) = \Delta f(\overrightarrow{u})(x) &+ \Delta f(u, \overrightarrow{u} + \overrightarrow{v})(x)\\ &+ \Delta f(v_1, u_2 + v_2, \dots, u_n + v_n)(x + u_1)\\ &+ \Delta f(u_1, v_2, u_3 + v_3, \dots, u_n + v_n)(x + u_2) + \cdots\\ &+ \Delta f(u_1, \dots, u_{n-1}, v_n)(x + u_n)\,.\end{aligned}$$

Proof. Simple computations using Lemma 7.22. □

Let $\mathsf{S}^n B$ be the cone defined by $\underline{\mathsf{S}^n B} = \underline{B}^{n+1}$ with operations defined pointwise and norm defined by $\|(x, \overrightarrow{u})\|_{\mathsf{S}^n B} = \|x + \sum_{i=1}^{n} u_i\|_B$. It is easy to check that one actually defines a cone in that way.

Remark 7.24. This cone is not the $(n+1)$-fold coproduct of $\underline{B}$ with itself. Take indeed $B = 1 \,\&\, 1$ and $n = 1$, then $\|((1, 0), (0, 1)\|_{\mathsf{S}^1 B} = \|(1, 1)\|_{\underline{B}} = 1$ whereas $\|((1, 0), (0, 1)\|_{\underline{B} \oplus \underline{B}} = 2$. Neither is it the $(n + 1)$-fold product of $\underline{B}$; it is actually (isomorphic to) $\overbrace{1 \,\&\, \cdots \,\&\, 1}^{n+1} \multimap B$. This construct is at the origin of coherent differentiation [Ehr23].

Lemma 7.25. *If $f : \mathcal{B}\underline{B} \to \underline{C}$ is totally monotonic, the map $(x, \overrightarrow{u}) \to \Delta f(\overrightarrow{u})(x)$ is increasing $\mathcal{B}\mathsf{S}^n B \to \underline{C}$.*

Proof. Follows easily from Theorem 7.19. □

Now we can state and prove the main lemma which allows to prove that totally monotonic functions are closed under composition. Remember that even in the setting of completely and absolutely monotonic functions (where derivatives instead of differences are used), the corresponding result is not completely trivial as it requires the use of the Faà di Bruno formula.

Lemma 7.26. *Let $n \in \mathbb{N}$, $f, h_1, \dots, h_n : \mathcal{B}\underline{B} \to \underline{C}$ and $g : \mathcal{B}\underline{C} \to \underline{D}$ be totally monotonic functions such that $\forall x \in \mathcal{B}\underline{B}\ f(x) + \sum_{i=1}^{n} h_i(x) \in \mathcal{B}\underline{C}$. Then the function $k : \mathcal{B}\underline{B} \to \underline{D}$ defined by $k(x) = \Delta g(h_1(x), \dots, h_n(x))(f(x))$ is totally monotonic.*

Proof. With the notations and conventions of the statement, we prove by induction on p that, for all $p \in \mathbb{N}$, for all $n \in \mathbb{N}$, for all $f, h_1, \dots, h_n, g$ which are totally monotonic and satisfy $\forall x \in \mathcal{B}\underline{B}\ f(x) + \sum_{i=1}^{n} h_i(x) \in \mathcal{B}\underline{C}$, the function k is p-increasing.

For $p = 0$, the property results from Lemma 7.25.

We assume the property for p and prove it for $p + 1$. Let $u \in \mathcal{B}\underline{B}$ we have to prove that the function $\Delta k(u)$ is p-increasing from B_u to D. Let $x \in \mathcal{B}\underline{B_u}$, we have

$$\begin{aligned}\Delta k(u)(x) &= \Delta g(h_1(x + u), \dots, h_n(x + u))(f(x + u)) - \Delta g(h_1(x), \dots, h_n(x))(f(x))\\ &= \Delta g(h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x) + \Delta f(u)(x))\\ &\quad - \Delta g(h_1(x), \dots, h_n(x))(f(x)) \quad \text{by definition of } \Delta h_i(u)\\ &= \Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x))\\ &\quad + \Delta g(\Delta h_1(u)(x), h_2(x) + \Delta h_2(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x) + h_1(x))\\ &\quad + \Delta g(h_1(x), \Delta h_2(u)(x), h_3(x) + \Delta h_3(u)(x), \dots,\\ &\qquad\qquad h_n(x) + \Delta h_n(u)(x))(f(x) + h_2(x))\\ &\quad + \cdots + \Delta g(h_1(x), \dots, h_{n-1}(x), \Delta h_n(u)(x))(f(x) + h_n(x))\end{aligned}$$

by Lemma 7.23, observing that the first term of the sum which appears in Lemma 7.23 is annihilated by the subtraction above.

We can apply the inductive hypothesis to each of the terms of this sum. Let us consider for instance the first of these expressions:

$$\Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_n(x) + \Delta h_n(u)(x))(f(x))\,.$$

We know that the functions $h'_1, \dots, h'_{n+1}$ defined by $h'_1(x) = \Delta f(u)(x)$, $h'_2(x) = h_1(x) + \Delta h_1(u)(x) = h_1(x+u)$,..., $h'_{n+1}(x) = h_n(x) + \Delta h_n(u)(x) = h_n(x+u)$ are totally monotonic from $\underline{B_u}$ to $\underline{C}$: this results from Lemma 7.20. Moreover we have

$$\forall x \in \mathcal{B}\underline{B}\ f(x) + \sum_{i=1}^{n+1} h'_i(x) = f(x+u) + \sum_{i=1}^{n} h_i(x+u) \in \mathcal{B}\underline{C}\,.$$

Therefore the inductive hypothesis applies and we know that the function

$$x \mapsto \Delta g(\Delta f(u)(x), h_1(x) + \Delta h_1(u)(x), \dots, h_p(x) + \Delta h_p(u)(x))(f(x))$$

is p-increasing. The same reasoning applies to all terms and hence the function $\Delta k(u)$ is p-increasing from $\mathcal{B}\underline{B_u}$ to $\underline{C}$, as contended. □

Lemma 7.27. *Let $f : \underline{B} \times \mathcal{B}\underline{C} \to \underline{D}$ be linear in its first argument and totally monotonic in its second argument. Then, when restricted to $\mathcal{B}\underline{B} \times \mathcal{B}\underline{C}$, the function f is totally monotonic.*

Proof. Let $n \in \mathbb{N}$, $(x, y), (u_1, v_1), \dots, (u_n, v_n) \in \underline{B} \times \underline{C}$ be such that $(x, y) + \sum_{i=1}^{n}(u_i, v_i) \in \mathcal{B}\underline{B} \times \mathcal{B}\underline{C}$. For $\varepsilon \in \{+, -\}$, we have

$$\begin{aligned}
\Delta^\varepsilon f((u_1, v_1), \dots, (u_n, v_n))(x, y) &= \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x + \sum_{i\in I} u_i, y + \sum_{i \in I} v_i) \\
&= \sum_{I \in \mathcal{P}^\varepsilon(n)} f(x, y + \sum_{i\in I} v_i) + \sum_{I \in \mathcal{P}^\varepsilon(n)} \sum_{j \in I} f(u_j, y + \sum_{i\in I} v_i)
\end{aligned}$$

by linearity of f in its first argument. By total monotonicity of f in its second argument we have

$$\sum_{I \in \mathcal{P}^+(n)} f(x, y + \sum_{i\in I} v_i) \geq \sum_{I \in \mathcal{P}^-(n)} f(x, y + \sum_{i\in I} v_i)\,.$$

Next, assuming that $n > 0$, we have

$$\begin{aligned}
\sum_{I \in \mathcal{P}^\varepsilon(n)} \sum_{j \in I} f(u_j, y + \sum_{i\in I} v_i) &= \sum_{j=1}^{n} \sum_{\substack{I \in \mathcal{P}^\varepsilon(n) \\ j \in I}} f(u_j, y + \sum_{i\in I} v_i) \\
&= \sum_{j=1}^{n} \sum_{I \in \mathcal{P}^\varepsilon(n-1)} f(u_j, y + \sum_{i \in \mathsf{inj}_j(I)} v_i) \text{ by Lemma 7.4} \\
&= \sum_{j=1}^{n} \sum_{I \in \mathcal{P}^\varepsilon(n-1)} f(u_j, y + v_j + \sum_{i\in I} v(j)_i)
\end{aligned}$$

where $(v(j)_i)_{i=1}^{n-1}$ is defined by

$$v(j)_i = \begin{cases} v_i & \text{if } i < j \\ v_{i+1} & \text{if } i \geq j\,. \end{cases}$$

By our assumption that f is totally monotonic in its second argument we have, for each $j = 1, \ldots, n$,

$$\sum_{I \in \mathcal{P}^+(n-1)} f(u_j, y + v_j + \sum_{i \in I} v(j)_i) \geq \sum_{I \in \mathcal{P}^-(n-1)} f(u_j, y + v_j + \sum_{i \in I} v(j)_i)$$

from which it follows that

$$\sum_{I \in \mathcal{P}^+(n)} \sum_{j \in I} f(u_j, y + \sum_{i \in I} v_i) \geq \sum_{I \in \mathcal{P}^-(n)} \sum_{j \in I} f(u_j, y + \sum_{i \in I} v_i)$$

and hence

$$\Delta^+ f((u_1, v_1), \ldots, (u_n, v_n))(x, y) \geq \Delta^- f((u_1, v_1), \ldots, (u_n, v_n))(x, y)$$

for $n > 0$. This inequation also holds trivially for $n = 0$. □

An immediate consequence of this lemma is the following observation which will be useful in Section 8.

Lemma 7.28. *Let $f : \prod_{i=1}^n \underline{B_i} \to \underline{C}$ be linear in each of its n arguments. Then the restriction of f to $\prod_{i=1}^n \mathcal{B}\underline{B_i}$ is totally monotonic $\mathcal{B}\underline{\&_{i=1}^n B_i} \to \underline{C}$.*

Proof. Simple induction on n using Lemma 7.27. □

Remark 7.29. In this statement, and other similar ones, we restrict f to the unit ball not for deep reason but only because the notion of totally monotonic function has been defined on unit balls, see Definition 7.5.

7.4. The cartesian closed category of integrable cones and stable and measurable functions.

A quite remarkable property of total monotonicity is that it gives rise to cartesian *closed* categories, as we will see in this section. We do not know if this phenomenon has been observed before [EPT18b].

Let $\mathbf{SCones}(B, C)$ be the set of all stable and measurable functions from B to C whose norm is ≤ 1.

Theorem 7.30. *If $f \in \mathbf{SCones}(B, C)$ and $g \in \mathbf{SCones}(C, D)$ then $g \circ f \in \mathbf{SCones}(B, D)$.*

Proof. The only non-obvious fact is that $g \circ f$ is totally monotonic, which is obtained by Lemma 7.26 (applied with $n = 0$). □

So we have defined a category **SCones** whose objects are the integrable cones, and the morphisms are the stable and measurable functions.

Lemma 7.31. $\mathbf{ICones}(B, C) \subseteq \mathbf{SCones}(B, C)$*.*

Proof. Indeed linearity clearly implies total monotonicity. □

So we have a functor $\mathsf{Der^s} : \mathbf{ICones} \to \mathbf{SCones}$ which acts as the identity on objects and morphisms. We can consider this functor as a forgetful functor since it forgets linearity, whence its name: in LL the purpose of the *dereliction* rules allows to forget the linearity of morphisms. The functor $\mathsf{Der^s}$ is obviously faithful but of course not full: see Examples 2.4 and 7.9 which provide nonlinear totally monotonic functions.

Theorem 7.32. *The category* **SCones** *has all products and is cartesian closed.*

Proof. If $(B_i)_{i\in I}$ is a family of integrable cones, we have already defined $B = \&_{i\in I} B_i$ which is the categorical product of the B_i's in **ICones** (when equipped with the projections $\mathsf{pr}_i \in \mathbf{ICones}(B, B_i)$). So $\mathsf{Der}^{\mathsf{s}}(\mathsf{pr}_i) \in \mathbf{SCones}(B, B_i)$ for each $i \in I$. Let $(f_i \in \mathbf{SCones}(C, B_i))_{i\in I}$, we define $f : \mathcal{B}\underline{C} \to \mathcal{B}\underline{B}$ by $f(x) = (f_i(x))_{i\in I}$ which is well defined by our assumption that $\forall i \in I\ \ \|f_i\| \leq 1$. Then f is easily seen to be stable because all the operations of B, as well as its cone order relation, are defined componentwise. Measurability of f is proven as in the proof of Theorem 4.16. This shows that B is the categorical product of the B_i's in **SCones**.

Let B and C be integrable cones. We have defined in Section 7.2 the integrable cone $B \Rightarrow_{\mathsf{s}} C$ of stable and measurable functions $B \to C$, we show that it is the internal hom of B and C in **SCones**. We define $\mathsf{Ev} : \mathcal{B}\underline{((B \Rightarrow_{\mathsf{s}} C) \,\&\, B)} \to \underline{C}$ by $\mathsf{Ev}(f, x) = f(x)$. The total monotonicity of Ev results from Lemma 7.27. We have

$$\mathcal{B}\underline{((B \Rightarrow_{\mathsf{s}} C) \,\&\, B)} = \mathcal{B}\underline{(B \Rightarrow_{\mathsf{s}} C)} \times \mathcal{B}\underline{B}$$

by definition of the norm in the categorical product. It follows that $\|\mathsf{Ev}\| \leq 1$. We prove that Ev is measurable so let $X \in \mathbf{Ar}$ and $\delta \in \mathcal{B}\underline{\mathsf{Path}(X, ((B \Rightarrow_{\mathsf{s}} C) \,\&\, B))}$ which means that $\delta = \langle \eta, \beta \rangle$ with $\eta \in \mathcal{B}\underline{\mathsf{Path}(X, B \Rightarrow_{\mathsf{s}} C)}$ and $\beta \in \mathcal{B}\underline{\mathsf{Path}(X, B)}$, we must prove that $\mathsf{Ev} \circ \delta \in \underline{\mathsf{Path}(X, C)}$ so let $Y \in \mathbf{Ar}$ and $m \in \mathcal{M}^C_Y$. We must prove that

$$\varphi = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \mathsf{Ev}(\delta(r))) = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \eta(r)(\beta(r)))$$

is measurable. We have $p = (\beta \circ \mathsf{pr}_2) \triangleright (m \circ \mathsf{pr}_1) \in \mathcal{M}^{B \Rightarrow_{\mathsf{s}} C}_{Y\times X}$ and by our assumption about η we know that

$$\boldsymbol{\lambda}(r, s, r') \in X \times Y \times X \cdot p(s, r, \eta(r')) = \boldsymbol{\lambda}(r, s, r') \in X \times Y \times X \cdot m(s, \eta(r')(\beta(r)))$$

is measurable and hence so is φ and we have shown that $\mathsf{Ev} \in \mathbf{SCones}((B \Rightarrow_{\mathsf{s}} C) \,\&\, B, C)$.

We prove that $(B \Rightarrow_{\mathsf{s}} C, \mathsf{Ev})$ is the internal hom of B, C in the cartesian category **SCones**. So let $f \in \mathbf{SCones}(D \,\&\, B, C)$. For each given $z \in \mathcal{B}\underline{D}$ we see easily that $g = \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(z, x) \in \mathcal{B}\underline{B \Rightarrow_{\mathsf{s}} C}$, it remains to check that $g \in \mathbf{SCones}(D, B \Rightarrow_{\mathsf{s}} C)$. Let us first check that g is totally monotonic so let $n \in \mathbb{N}$ and $z, w_1, \dots, w_n \in \underline{D}$ be such that $z + \sum_{i=1}^n w_i \in \mathcal{B}\underline{D}$. We must prove that

$$h^- = \sum_{I\in\mathcal{P}^-(n)} g(z + \sum_{i\in I} w_i) \leq \sum_{I\in\mathcal{P}^+(n)} g(z + \sum_{i\in I} w_i) = h^+$$

in $\underline{B \Rightarrow_{\mathsf{s}} C}$. We use the characterization of the cone order in that cone given by Lemma 7.12. So let $k \in \mathbb{N}$ and let $x, u_1, \dots, u_k \in \underline{B}$ be such that $x + \sum_{j=1}^k u_j \in \mathcal{B}B$. We must prove that

$$\begin{aligned} y^- = \sum_{J\in\mathcal{P}^-(k)} h^+(x + \sum_{i\in J} u_j) + \sum_{J\in\mathcal{P}^+(k)} h^-(x + \sum_{i\in J} u_j) \\ \leq \sum_{J\in\mathcal{P}^+(k)} h^+(x + \sum_{i\in J} u_j) + \sum_{J\in\mathcal{P}^-(k)} h^-(x + \sum_{i\in J} u_j) = y^+ \end{aligned}$$

in $\underline{C}$. We have

$$y^- = \sum_{\substack{I\in\mathcal{P}^+(n)\\ J\in\mathcal{P}^-(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j) + \sum_{\substack{I\in\mathcal{P}^-(n)\\ J\in\mathcal{P}^+(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j)$$

$$y^+ = \sum_{\substack{I\in\mathcal{P}^+(n)\\ J\in\mathcal{P}^+(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j) + \sum_{\substack{I\in\mathcal{P}^-(n)\\ J\in\mathcal{P}^-(k)}} g(z+\sum_{i\in I} w_i, x+\sum_{i\in J} u_j)$$

Notice that $(\mathcal{P}^+(n)\times\mathcal{P}^-(k))\cap(\mathcal{P}^-(n)\times\mathcal{P}^+(k))=\emptyset$ and that there is a bijection

$$\begin{aligned}(\mathcal{P}^+(n)\times\mathcal{P}^-(k))\cup(\mathcal{P}^-(n)\times\mathcal{P}^+(k)) &\to \mathcal{P}^-(n+k)\\ (I,J) &\mapsto I\cup(J+n)\end{aligned}$$

and similarly that $(\mathcal{P}^+(n)\times\mathcal{P}^+(k))\cap(\mathcal{P}^-(n)\times\mathcal{P}^-(k))=\emptyset$ and that there is a bijection

$$\begin{aligned}(\mathcal{P}^+(n)\times\mathcal{P}^+(k))\cup(\mathcal{P}^-(n)\times\mathcal{P}^-(k)) &\to \mathcal{P}^+(n+k)\\ (I,J) &\mapsto I\cup(J+n)\,.\end{aligned}$$

We define a sequence $(w'_l,u'_l)_{l=1}^{n+k}$ of elements of $\underline{D}\times\underline{B}$ as follows:

$$(w'_l,u'_l)=\begin{cases}(w_l,0) & \text{if } l\in\{1,\dots,n\}\\ (0,u_{l-n}) & \text{if } l\in\{n+1,\dots,n+k\}\,.\end{cases}$$

so that $(z,x)+\sum_{l=1}^{n+k}(w'_l,u'_l)\in\mathcal{B}\underline{D}\times\mathcal{B}\underline{B}$. With these notations, we have

$$y^- = \sum_{K\in\mathcal{P}^-(n+k)} g((z,x)+\sum_{l\in K}(w'_l,u'_l))$$

$$y^+ = \sum_{K\in\mathcal{P}^+(n+k)} g((z,x)+\sum_{l\in K}(w'_l,u'_l))$$

and hence $y^-\le y^+$ since g is totally monotonic.

The ω-continuity of g results from Lemma 7.14. We prove that g is measurable so let $X\in\mathbf{Ar}$ and $\delta\in\mathcal{B}\underline{\mathsf{Path}(X,D)}$, we must prove that $g\circ\delta\in\underline{\mathsf{Path}(X,B\Rightarrow_{\mathsf{s}}C)}$ so let $Y\in\mathbf{Ar}$ and $p\in\mathcal{M}_Y^{B\Rightarrow_{\mathsf{s}}C}$. Let $\beta\in\mathcal{B}\underline{\mathsf{Path}(Y,B)}$ and $m\in\mathcal{M}_Y^C$ be such that $p=\beta\triangleright m$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s,r)\in Y\times X\cdot p(s,g(\delta(r))) &= \boldsymbol{\lambda}(s,r)\in Y\times X\cdot m(s,g(\delta(r))(\beta(s)))\\ &= \boldsymbol{\lambda}(s,r)\in Y\times X\cdot m(s,f(\delta(r),\beta(s)))\end{aligned}$$

and this map is measurable because $\langle\delta\circ\mathsf{pr}_2,\beta\circ\mathsf{pr}_1\rangle\in\mathcal{B}\underline{\mathsf{Path}(Y\times X,D\ \&\ B)}$ and by measurability of f. □

Remark 7.33. Note that contrarily to **ICones** it is very likely that the category **SCones** does not have all equalizers and therefore is not complete. For instance we have f,g : **SCones**$(1,1)$ given by $f(x)=x$ and $g(x)=x^2$, and the set of all $x\in\mathcal{B}\underline{1}=[0,1]$ such that $f(x)=g(x)$ is $\{0,1\}$ which does not look like a cone. It would be interesting to understand if the set of solutions of such an equation could be considered as some kind of manifold, with a local structure of integrable cone, as in differential geometry. The same observation applies to the category **ACones** studied in Section 8.

So we have a functor $\_ \Rightarrow_{\mathsf{s}} \_ : \mathbf{SCones}^{\mathsf{op}} \times \mathbf{SCones} \to \mathbf{SCones}$ mapping (B,C) to $B \Rightarrow_{\mathsf{s}} C$ and $f \in \mathbf{SCones}(B',B), g \in \mathbf{SCones}(C,C'))$ to $f \Rightarrow_{\mathsf{s}} g \in \mathbf{SCones}(B \Rightarrow_{\mathsf{s}} C, B' \Rightarrow_{\mathsf{s}} C')$ which is given by $(f \Rightarrow_{\mathsf{s}} g)(h) = g \circ h \circ f$. Observe that if $g \in \mathbf{ICones}(C,C')$ we have $f \Rightarrow_{\mathsf{s}} g \in \mathbf{ICones}(B \Rightarrow_{\mathsf{s}} C, B' \Rightarrow_{\mathsf{s}} C')$ so that in the sequel we consider only $\_ \Rightarrow_{\mathsf{s}} \_$ as a functor $\mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ (using implicitly a pre-composition with $\mathsf{Der}^{\mathsf{s}}$).

Theorem 7.34. *The functor* $\mathsf{Der}^{\mathsf{s}} : \mathbf{ICones} \to \mathbf{SCones}$ *preserves all limits.*

Proof. By Theorem 4.19 it suffices to prove that $\mathsf{Der}^{\mathsf{s}}$ preserves all products and all equalizers. Since products are defined in the same way in both categories, the first property is obvious, let us check the second one.

Let B, C be objects of **ICones** and $f, g \in \mathbf{ICones}(B,C)$, we have already defined the equalizer $(E, e \in \mathbf{ICones}(E,B))$ of f, g in the proof of Theorem 4.16. We just have to check that (E,e) is the equalizer of f,g in **SCones** as well. So let H be an integrable cone and $h \in \mathbf{SCones}(H,B)$ be such that $f \circ h = g \circ h$. This simply means that $h(\mathcal{B}\underline{H}) \subseteq \mathcal{B}\underline{E}$ from which it follows that $h \in \mathbf{SCones}(H,E)$ because the cone order relation of E is the restriction of that of B to $\underline{E} \subseteq \underline{B}$ (and similarly for the measurability structure). Let us call h' this version of h ranging in $\mathcal{B}\underline{E}$ instead of $\mathcal{B}\underline{B}$, so that $h = e \circ h'$. It is obvious that h' is the only morphism in **SCones** having this property. □

The study of the exponential induced by this cartesian closed structure on the category **ICones** is developed in Section 9.

8. Analytic and integrable functions on cones

Our goal now is to associate with **ICones** another cartesian closed category based on a notion of morphisms which are analytic in the sense that they are limits of polynomial functions. These analytic functions are actually stable and measurable, but their definition is based on a notion of multilinear maps[21] in **ICones** which preserve integrals so that analytic functions have an implicit "integral preservation" property[22] that general stable and measurable functions don't have.

8.1. The cone of multilinear and symmetric functions.

The basic ingredient for defining our analytic functions is the notion of multilinear morphisms. More precisely, they will allow first to define homogeneous polynomial functions (obtained by applying an n-linear function to "diagonal" tuples $(x,\dots,x)$), and then analytic functions as converging sums thereof.

Definition 8.1. Let $B_1,\dots,B_n,C$ be integrable cones. A function $f : \prod_{i=1}^{n} \underline{B_i} \to \underline{C}$ is said to be *multilinear and continuous* if it is linear and continuous, separately, with respect to each of its n arguments.

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[21]As in [KT18] but without the support of complex analysis.

[22]A property that we don't really know yet how to express simply and directly in terms of the functions; of course it is not plain integral preservation which cannot be expected from non-linear maps. We know that it is a property of the analytic functions themselves because the symmetric multilinear functions of their Taylor expansion at 0 are associated with analytic functions in a unambiguous way by means of standard polarization formulas as we shall see.

Observe that when this holds, f is bounded (use for instance Lemma 2.11 and monoidal closedness in a proof by induction on n). One says that f is *symmetric* if $B_1 = \cdots = B_n = B$ and, for all $\sigma \in \mathfrak{S}_n$ (the group of permutations on $\{1, \ldots, n\}$), one has

$$\forall x_1, \ldots, x_n \in \underline{B} \quad f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\,.$$

One says that f is *measurable* if for all $X \in \mathbf{Ar}$ and $(\beta_i \in \underline{\mathsf{Path}(X, B_i)})_{i=1}^n$, one has $f \circ \langle \beta_1, \ldots, \beta_n \rangle \in \underline{\mathsf{Path}(X, C)}$. Lastly, one says that f is *integrable* (or that it *preserves integrals*) if it is separately integrable with respect to each of its arguments.

The multilinear, continuous, symmetric, measurable and integrable functions $\underline{B}^n \to \underline{C}$ are easily seen to form a cone $\mathbf{Sym}_n(B, C)$ with operations defined pointwise and norm defined by

$$\|f\| = \sup\{\|f(x_1, \ldots, x_n)\| \mid x_1, \ldots, x_n \in \mathcal{B}\underline{B}\}\,.$$

We equip this cone with a measurability structure $(\mathcal{M}_X^{\mathbf{Sym}_n(B,C)})_{X \in \mathbf{Ar}}$ where $\mathcal{M}_X^{\mathbf{Sym}_n(B,C)}$ is the set of all $p = \overrightarrow{\beta} \triangleright m$ where $\overrightarrow{\beta} = (\beta_i \in \underline{\mathsf{Path}(X, B)})_{i=1}^n$ and $m \in \mathcal{M}_X^C$, given by $p(f) = \boldsymbol{\lambda} r \in X \cdot m(r, f(\beta_1(r), \ldots, \beta_n(r)))$. The order relation in this cone is the pointwise order as easily checked.

Notice last that this cone is integrable. Let indeed $X \in \mathbf{Ar}$, $\mu \in \underline{\mathsf{FMeas}(X)}$ and $\eta \in \underline{\mathsf{Path}(X, \mathbf{Sym}_n(B, C))}$ and let us define a function $f : \underline{B}^n \to \underline{C}$ by

$$f(x_1, \ldots, x_n) = \int \eta(r)(x_1, \ldots, x_n)\mu(dr)$$

then it is easy to check as usual that f is well defined, $f \in \underline{\mathbf{Sym}_n(B, C)}$ and that $f = \int \eta(r)\mu(dr)$.

Remark 8.2. This integrable cone is a "subcone" of the integrable cone $B \otimes \cdots \otimes B \multimap C$, but we have not developed the notion of subcone in the present paper.

Remark 8.3. We could define $\mathbf{Sym}_n(B, C)$ more abstractly as follows: first, generalizing Definition 5.6, we define the integrable cone of n-linear morphisms $(B_1, \ldots, B_n \multimap C) = (B_1 \multimap \cdots \multimap B_n \multimap C)$, then we observe that, when $B_1 = \cdots = B_n = B$, for each permutation $\sigma \in \mathfrak{S}_n$ there is an automorphism on that cone which acts by permutation of the arguments of n-linear functions, and we define $\mathbf{Sym}_n(B, C)$ as the equalizer of all these automorphisms using the completeness of **ICones**. Of course we would have obtained the same object of **ICones** (possibly up to an isomorphism), but the explicit description above will be useful.

8.2. **The cone of homogeneous polynomial functions.** As announced, this is the next basic concept in the definition of analytic functions.

Definition 8.4. An n-homogeneous polynomial function from B to C is a function $f : \underline{B} \to \underline{C}$ such that there exists $h \in \underline{\mathbf{Sym}_n(B, C)}$ satisfying

$$\forall x \in \underline{B} \quad f(x) = h(x, \ldots, x)\,.$$

Then h is called a *linearization* of f. We use $\underline{\mathbf{Hpol}_n(B, C)}$ for the set of n-homogeneous polynomial functions from B to C and set $\mathsf{M}_n(h) = f$.

Notice that we would define exactly the same class of functions without requiring h to be symmetric. We make this choice only to reduce the number of notions at hand.

Lemma 8.5. *If $f \in \underline{\mathbf{Hpol}}_n(B,C)$ then the restriction of f to $\mathcal{B}\underline{B}$ is totally monotonic.*

Proof. Let h be a linearization of f. By Lemma 7.28 we know that the restriction of h to $\mathcal{B}\underline{B}^n \to \underline{C}$ is totally monotonic, and since the diagonal map $d : \underline{B} \to \underline{B}^n$ is linear of norm ≤ 1, the map $f = h \circ d$ is totally monotonic. □

Lemma 8.6. *An n-homogeneous polynomial function f has exactly one linearization $\mathsf{L}_n f$. Moreover*

$$\|\mathsf{L}_n f\| \leq \frac{n^n}{n!} \|f\|$$

where $\|f\| = \sup_{x \in \mathcal{B}\underline{B}} \|f(x)\|$.

Notice that we also have $\|f\| \leq \|\mathsf{L}_n f\|$ so that we could interpret this lemma as expressing that the norms of $\underline{\mathbf{Sym}_n(B,C)}$ and $\underline{\mathbf{Hpol}}_n(B,C)$ are equivalent and that these two cones are isomorphic in a weak sense (remember that in **ICones**, isomorphisms must have norm ≤ 1).

Proof. Let $f : \underline{B} \to \underline{C}$ be an n-homogeneous polynomial and let $h \in \underline{\mathbf{Sym}_n(B,C)}$ be a linearization of f. By Lemma 8.5 we can define a function $h' : \underline{B}^n \to \underline{C}$ by

$$h'(x_1,\dots,x_n) = \frac{1}{n!}\Big(\sum_{I \in \mathcal{P}^+(n)} f(\sum_{i \in I} x_i) - \sum_{I \in \mathcal{P}^-(n)} f(\sum_{i \in I} x_i)\Big) = \frac{1}{n!}\Delta f(x_1,\dots,x_n)(0)$$

and the usual proof of the polarization theorem shows that necessarily $h' = h$. So $\mathsf{L}_n f = h$ is completely determined by f, proving our contention.

Next, given $x_1,\dots,x_n \in \mathcal{B}\underline{B}$, we have

$$\begin{aligned}
\|h(x_1,\dots,x_n)\| &\leq \frac{1}{n!}\left\|f(\sum_{i=1}^n x_i)\right\| \quad \text{by Lemma 7.21}\\
&= \frac{n^n}{n!}\left\|h(\frac{1}{n}\sum_{i=1}^n x_i,\dots,\frac{1}{n}\sum_{i=1}^n x_i)\right\|\\
&= \frac{n^n}{n!}\left\|f(\frac{1}{n}\sum_{i=1}^n x_i)\right\|\\
&\leq \frac{n^n}{n!}\|f\|
\end{aligned}$$

since $\left\|\frac{1}{n}\sum_{i=1}^n x_i\right\| \leq 1$. □

The set $\underline{\mathbf{Hpol}}_n(B,C)$ is canonically a precone. Indeed if $f, g \in \underline{\mathbf{Hpol}}_n(B,C)$ then $f+g$ (defined pointwise) belongs to $\underline{\mathbf{Hpol}}_n(B,C)$ because clearly $f + g = \mathsf{M}_n(\mathsf{L}_n f + \mathsf{L}_n g)$ and we know that $\mathsf{L}_n f + \mathsf{L}_n g \in \underline{\mathbf{Sym}_n(B,C)}$. Multiplication by a scalar in $\mathbb{R}_{\geq 0}$ is dealt with similarly. Notice that this reasoning also shows that the maps $\mathsf{L}_n : \underline{\mathbf{Hpol}}_n(B,C) \to \underline{\mathbf{Sym}_n(B,C)}$ and $\mathsf{M}_n : \underline{\mathbf{Sym}_n(B,C)} \to \underline{\mathbf{Hpol}}_n(B,C)$ are linear.

We define $\|\text{-}\|_{\underline{\mathbf{Hpol}}_n(B,C)}$ as usual by $\|f\|_{\underline{\mathbf{Hpol}}_n(B,C)} = \sup_{x \in \mathcal{B}\underline{B}} \|f(x)\|_C$ so that clearly $\|f\|_{\underline{\mathbf{Hpol}}_n(B,C)} \leq \|\mathsf{L}_n f\|_{\mathbf{Sym}_n(B,C)}$. Equipped with this norm, $\underline{\mathbf{Hpol}}_n(B,C)$ is a cone: the only non obvious property is completeness, so let $(f_k \in \mathcal{B}\underline{\mathbf{Hpol}}_n(B,C))_{k=1}^\infty$ be an increasing sequence and let $f : \underline{B} \to \underline{C}$ be the pointwise lub of this sequence which is well defined since for each $x \in \underline{B}$ we have

$$\|f_k(x)\| \leq \|\mathsf{L}_n f_k\| \|x\|^n \leq \frac{n^n}{n!} \|f_k\| \|x\|^n \leq \frac{n^n}{n!} \|x\|^n \ .$$

The sequence $(\mathsf{L}_n f_k)_{k=1}^\infty$ is increasing in $\frac{n^n}{n!}\mathcal{B}\underline{\mathbf{Sym}_n(B,C)}$ and has therefore a lub $h \in \mathcal{B}\underline{\mathbf{Sym}_n(B,C)}$ and remember that this lub is defined pointwise on $\underline{B}^n$. It follows that $f = \mathsf{M}_n(h) \in \underline{\mathbf{Hpol}}_n(B,C)$ and that we have

$$\forall x \in \underline{B} \quad f(x) = \sup_k f_k(x)\,.$$

Since $\forall k\ f_k \leq f$ by monotonicity of M_n it follows that $f = \sup_k f_k$. Last observe that $\|f\| \leq 1$ which ends the proof of ω-completeness of $\mathcal{B}\underline{\mathbf{Hpol}}_n(B,C)$.

So we have shown that $\underline{\mathbf{Hpol}}_n(B,C)$ is a cone, and also that the linear maps L_n and M_n are ω-continuous.

Remark 8.7. It is important to notice that, contrarily to $\mathbf{Sym}_n(B,C)$, the cone order relation of $\underline{\mathbf{Hpol}}_n(B,C)$ is not the pointwise order. As an example take $n = 2$, $B = 1 \mathbin{\&} 1$, $C = 1$, and consider $f, g \in \underline{\mathbf{Hpol}}_n(B,C)$ given by $f(x,y) = 2xy$ and $g(x,y) = x^2 + y^2$. Then

$$\begin{aligned}\mathsf{L}_2 f((x_1,y_1),(x_2,y_2)) &= \frac{1}{2}(f(x_1+x_2,y_1+y_2) - f(x_1,y_1) - f(x_2,y_2))) \\ &= x_1y_2 + x_2y_1\end{aligned}$$

and similarly $\mathsf{L}_2 g((x_1,y_1),(x_2,y_2)) = x_1x_2 + y_1y_2$ and therefore we do not have $\mathsf{L}_2 f \leq \mathsf{L}_2 g$ (we have $\mathsf{L}_2 f((1,0),(0,1)) = 1$ and $\mathsf{L}_2 g((1,0),(0,1)) = 0$) whereas $\forall (x,y) \in \underline{1 \mathbin{\&} 1} = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\ f(x,y) \leq g(x,y)$.

Given $X \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(X,B)}$ and $m \in \mathcal{M}_X^C$ we define $\beta \triangleright m : X \times \underline{\mathbf{Hpol}}_n(B,C) \to \mathbb{R}_{\geq 0}$ as usual by $(\beta \triangleright m)(r,f) = m(r, f(\beta(r)))$ for all $f \in \underline{\mathbf{Hpol}}_n(B,C)$. Notice that

$$\boldsymbol{\lambda} r \in X \cdot m(r, f(\beta(r))) = \boldsymbol{\lambda} r \in X \cdot m(r, \mathsf{L}_n f(\overline{\beta(r)}^n))$$

and this function is measurable because $\mathsf{L}_n f \in \underline{\mathbf{Sym}_n(B,C)}$, which implies that $\mathsf{L}_n f \circ \langle \beta, \dots, \beta\rangle \in \underline{\mathsf{Path}(X,C)}$. Then it is easily checked that setting $\mathcal{M}_X = \{\beta \triangleright m \mid \beta \in \underline{\mathsf{Path}(X,B)}$ and $m \in \mathcal{M}_X^C\}$ we define a measurability structure on $\underline{\mathbf{Hpol}}_n(B,C)$ so that $E = (\underline{\mathbf{Hpol}}_n(B,C), (\mathcal{M}_X)_{X\in\mathbf{Ar}})$ is a measurable cone that we denote as $\mathbf{Hpol}_n(B,C)$.

Lemma 8.8. $\mathsf{L}_n \in \mathbf{MCones}(\underline{\mathbf{Hpol}}_n(B,C), \frac{n^n}{n!}\mathbf{Sym}_n(B,C))$.

Proof. In view of what we know about L_n, it suffices to prove that $\mathsf{L}_n : \underline{\mathbf{Hpol}_n(B,C)} \to \underline{\mathbf{Sym}_n(B,C)}$ is measurable so let $X \in \mathbf{Ar}$ and $\eta \in \underline{\mathsf{Path}(X, \mathbf{Hpol}_n(B,C))}$, we must check that $\mathsf{L}_n \circ \eta \in \underline{\mathsf{Path}(X, \mathbf{Sym}_n(B,C))}$. Let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{\mathbf{Sym}_n(B,C)}$. We have $p = \overrightarrow{\beta} \triangleright m$ where $\overrightarrow{\beta} \in \underline{\mathsf{Path}(Y,B)}^n$ and $m \in \mathcal{M}_Y^B$ so that

$$\boldsymbol{\lambda}(s,r) \in Y \times X \cdot p(s, \mathsf{L}_n(\eta(r))) = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \mathsf{L}_n(\eta(r))(\overrightarrow{\beta}(s)))$$

which, coming back to the characterization of L_n explicitly provided in the proof of Lemma 8.6, is measurable by measurability of addition and subtraction in $\mathbb{R}$ and linearity of $m(s, \_)$. $\square$

Theorem 8.9. $\mathbf{Hpol}_n(B,C)$ *is an integrable cone and*

$$\mathsf{L}_n \in \mathbf{ICones}(\underline{\mathbf{Hpol}}_n(B,C), \frac{n^n}{n!}\mathbf{Sym}_n(B,C))\,.$$

Proof. We prove integrability of $E = \mathbf{Hpol}_n(B,C)$ so let $X \in \mathbf{Ar}$, $\eta \in \underline{\mathsf{Path}(X,E)}$ and $\mu \in \underline{\mathsf{FMeas}(X)}$, we define $f : \underline{B} \to \underline{C}$ by $f(x) = \int \eta(r)(x)\mu(dr)$ using the fact that C is an integrable cone. Since L_n is measurable we can define $h : \underline{B}^n \to \underline{C}$ by

$$h(\overrightarrow{x}) = \int \mathsf{L}_n(\eta(r))(\overrightarrow{x})\mu(dr)\,.$$

which is clearly symmetric. It is n-linear, ω-continuous, measurable by Lemma 4.7, and integrable by the first statement of Theorem 6.1. So we have $h \in \underline{\mathbf{Sym}_n(B,C)}$ and $h(x,\dots,x) = \int \eta(r)(x)\mu(dx) = f(x)$ for all $x \in \underline{B}$ which proves that $f \in \underline{\mathbf{Hpol}_n(B,C)}$. Last let $p \in \mathcal{M}_0^{\mathbf{Hpol}_n(B,C)}$ so that $p = x \triangleright m$ for some $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$. We have $p(f) = m(f(x)) = m(\int \eta(r)(x)\mu(dx)) = \int m(\eta(r)(x))\mu(dr)$ by definition of an integral in C. So $p(f) = \int p(\eta(r))\mu(dr)$ by definition of p, which shows that f is the integral of η in $\mathbf{Hpol}_n(B,C)$ and hence that this measurable cone is also integrable.

The integrability of L_n results from its definition and from the fact that integrals commute with finite sums and differences. ☐

8.3. **The cone of analytic functions.** We can finally define and study our analytic functions.

Definition 8.10. A function $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic if it is bounded, and there is a sequence $(f_n \in \underline{\mathbf{Hpol}}_n(B,C))_{n\in\mathbb{N}}$ such that

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n=0}^{\infty} f_n(x)\,. \tag{8.1}$$

Such a sequence $(f_n)_{n\in\mathbb{N}}$ is called a *homogeneous polynomial decomposition* of f.

Notice that the precise meaning of (8.1) is that, for all $x \in \mathcal{B}\underline{B}$, the increasing sequence $(\sum_{n=0}^{N} f_n(x))_{N\in\mathbb{N}}$ is bounded in $\underline{C}$ (in the sense of the norm) and has $f(x)$ as lub.

Lemma 8.11. *If $f : \mathcal{B}\underline{B} \to \underline{C}$ is analytic, then f has exactly one homogeneous polynomial decomposition.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a homogeneous polynomial decomposition of an analytic f that we can assume without loss of generality to range in $\mathcal{B}\underline{C}$ since f is bounded. Let $x' \in \underline{B'}$ and $x \in \mathcal{B}\underline{B}$. Let

$$\begin{aligned}\varphi : [0,1] &\to \mathbb{R}_{\geq 0}\\ t &\mapsto x'(f(tx))\,.\end{aligned}$$

We have $\varphi(t) = \sum_{n=0}^{\infty} x'(f_n(x))t^n$ by linearity and continuity of x' and hence

$$\forall n \in \mathbb{N} \quad x'(f_n(x)) = \frac{1}{n!}\varphi^{(n)}(0) = \frac{1}{n!}\frac{d^n}{dt^n}x'(f(tx))\mid_{t=0}$$

so that if $(g_n)_{n\in\mathbb{N}}$ is another homogeneous polynomial decomposition of f we have $x'(f_n(x)) = x'(g_n(x))$ for all x, n and x'. Since this holds in particular for all $x' = m \in \mathcal{M}_0^B$ our claim is proven by (**Mssep**). ☐

If $f : \mathcal{B}\underline{B} \to C$ is analytic, we use $\mathsf{P}_n(f)$ for the nth component of its unique homogeneous polynomial decomposition and we set $\mathsf{D}_0^{(n)} = n!(\mathsf{L}_n \circ \mathsf{P}_n)$ so that $\mathsf{D}_0^{(n)} f \in \underline{\mathbf{Sym}_n(B,C)}$ and we have

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{x}^n)$$

which can be understood as the Taylor expansion of f, motivating the notation: $\mathsf{D}_0^{(n)} f$ can be understood as the nth derivative of f at 0, which is an n-linear symmetric function. As usual we say that f is measurable if, for all $X \in \mathbf{Ar}$ and $\beta \in \underline{\mathsf{Path}(X,B)}$, one has $f \circ \beta \in \underline{\mathsf{Path}(X,C)}$.

We define now a cone of analytic and measurable functions $\mathcal{B}\underline{B} \to \underline{C}$ so let P be the set of these functions. We define the algebraic operations on P pointwise: if $f, g \in P$ then $f + g \in P$ since $(f+g)(x) = f(x) + g(x) = \sum_{n=0}^{\infty} (\mathsf{P}_n f(x) + \mathsf{P}_n g(x))$ by continuity of addition. Notice that if $f, g \in P$ then

$$f \leq g \Leftrightarrow \forall n \in \mathbb{N} \quad \mathsf{D}_0^{(n)} f \leq \mathsf{D}_0^{(n)} g\,.$$

since $f \leq g$ means that $\forall x \in \mathcal{B}\underline{B}\ f(x) \leq g(x)$ and $\boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot (g(x) - f(x)) \in P$. Each map $\mathsf{D}_0^{(n)} : P \to \underline{\mathbf{Sym}_n(B,C)}$ is linear by Lemmas 8.6 and 8.11.

We set as usual

$$\|f\| = \sup\{\|f(x)\| \mid x \in \mathcal{B}\underline{B}\}$$

and define in that way a cone. Let indeed $(f^k)_{k\in\mathbb{N}}$ be an increasing sequence in $\mathcal{B}P$. For each $k, n \in \mathbb{N}$ we have $\left\|\mathsf{D}_0^{(n)} f^k\right\| \leq n^n$ by Lemma 8.8 and the sequence $(\mathsf{D}_0^{(n)} f^k)_{k\in\mathbb{N}}$ is increasing and hence has a lub $h_n \in \underline{\mathbf{Sym}_n(B,C)}$ and we have

$$\forall x_1, \dots, x_n \in \underline{B} \quad h_n(x_1, \dots, x_n) = \sup_{k\in\mathbb{N}} \mathsf{D}_0^{(n)} f^k(x_1, \dots, x_n)\,.$$

In particular we can define the homogeneous polynomial map $f_n = \mathsf{M}_n(\frac{1}{n!} h_n)$, which means

$$f_n(x) = \frac{1}{n!} h_n(\overline{x}^n) = \sup_{k\in\mathbb{N}} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n)\,.$$

Let $f : \mathcal{B}\underline{B} \to \underline{C}$ be defined by $f(x) = \sup_{k\in\mathbb{N}} f^k(x)$, we have

$$\begin{aligned} f(x) &= \sup_{k\in\mathbb{N}} f^k(x) \\ &= \sup_{k\in\mathbb{N}} \sum_{n=0} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n) \\ &= \sum_{n=0} \sup_{k\in\mathbb{N}} \frac{1}{n!} \mathsf{D}_0^{(n)} f^k(\overline{x}^n) \\ &= \sum_{n=0}^{\infty} f_n(x) \end{aligned}$$

which shows that f is analytic and is the lub $(f^k)_{k\in\mathbb{N}}$ in $\mathcal{B}P$ since f is clearly measurable (as usual by the monotone convergence theorem).

Then we define a family $\mathcal{M} = (\mathcal{M}_X)_{X\in\mathbf{Ar}}$ of sets of measurability tests by stipulating that $p \in \mathcal{M}_X$ if $p = \beta \triangleright m$ where $\beta \in \mathcal{B}\underline{\mathsf{Path}(X,B)}$ and $m \in \mathcal{M}_X^C$, and, if $f \in P$ and $r \in X$

then $p(r,f) = m(r, f(\beta(r)))$. It is easily checked that $(P, \mathcal{M})$ is a measurable cone, that we denote as $B \Rightarrow_{\mathsf{a}} C$.

We check that $B \Rightarrow_{\mathsf{a}} C$ is integrable so let $\eta \in \underline{\mathsf{Path}(X, B \Rightarrow_{\mathsf{a}} C)}$ for some $X \in \mathbf{Ar}$ and let $\mu \in \underline{\mathsf{FMeas}(X)}$. We define a function $f : \mathcal{B}\underline{B} \to \underline{C}$ by

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \int_X^C \eta(r)(x)\mu(dr)\,.$$

This function is well defined because for each given $x \in \mathcal{B}\underline{B}$ one has $\boldsymbol{\lambda} r \in X \cdot \eta(r)(x) \in \underline{\mathsf{Path}(X, C)}$. For each $r \in X$ we can write

$$\eta(r)(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)$$

and hence

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_X^C \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\int_X^{\mathbf{Sym}_n(B,C)} \mathsf{D}_0^{(n)}(\eta(r))\mu(dr) \Big)(\overline{x}^n)$$

by definition of integrals in the integrable cone $\mathbf{Sym}_n(B, C)$, and hence $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$. Let $p = (x \triangleright m) \in \mathcal{M}_0^{B \Rightarrow_{\mathsf{a}} C}$ where $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$, we have

$$\begin{aligned}
p(f) &= m(f(x)) \\
&= m\Big(\sum_{n=0}^{\infty} \frac{1}{n!} \int \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) \Big) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} m\Big(\int \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)\mu(dr) \Big) \quad \text{by lin. and cont. of } m \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int m(\mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n))\mu(dr) \quad \text{by def. of integrals in } C \\
&= \int \Big(\sum_{n=0}^{\infty} \frac{1}{n!} m(\mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n)) \Big)\mu(dr) \quad \text{by the monotone convergence th.} \\
&= \int m\Big(\sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)}(\eta(r))(\overline{x}^n) \Big)\mu(dr) \\
&= \int m(\eta(r)(x))\mu(dr) = \int p(\eta(r))\mu(dr)
\end{aligned}$$

which shows that $f = \int \eta(r)\mu(dr)$, and hence the measurable cone $B \Rightarrow_{\mathsf{a}} C$ is integrable.

Lemma 8.12. *For each $n \in \mathbb{N}$, the function $\mathsf{P}_n : \underline{B \Rightarrow_{\mathsf{a}} C} \to \underline{\mathbf{Hpol}_n(B, C)}$ is linear, continuous, measurable, integrable and has norm ≤ 1.*

Proof. Linearity and continuity result straightforwardly from the fact that the homogeneous polynomial decomposition $(f_n = \mathsf{P}_n(f))_{n \in \mathbb{N}}$ of f is uniquely determined by its defining property:

$$\forall x \in \mathcal{B}\underline{B} \quad f(x) = \sum_{n \in \mathbb{N}} f_n(x)\,.$$

Let $\eta \in \underline{\mathsf{Path}(X, B \Rightarrow_{\mathsf{a}} C)}$, we must check next that $\mathsf{P}_n \circ \eta \in \underline{\mathsf{Path}(X, \mathbf{Hpol}_n(B,C))}$ so let $Y \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(Y, B)}$ and $m \in \mathcal{M}_Y^C$, we must prove that

$$\begin{aligned}\theta &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot (\beta \triangleright m)(s, \mathsf{P}_n(\eta(r))) \\ &= \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, \mathsf{P}_n(\eta(r))(\beta(s)))\end{aligned}$$

is measurable $Y \times X \to \mathbb{R}_{\geq 0}$. This results from the fact that

$$\theta(s,r) = \frac{1}{n!}\frac{d^n}{dt^n} m(s, \eta(r, t\beta(s))) \mid_{t=0}$$

and from the measurability and smoothness wrt. t of the map $(s,r,t) \mapsto m(s, \eta(r, t\varphi(s)))$. Indeed the following is standard: if Z is a measurable space then if a function $Z \times [0,1) \to \mathbb{R}_{\geq 0}$ is measurable, and smooth in its second argument, then so is its derivative wrt. its second argument.

Last we check integrability of P_n so let moreover $\mu \in \mathsf{FMeas}(X)$, and let $p \in \mathcal{M}_0^{\mathbf{Hpol}_n(B,C)}$ so that $p = x \triangleright m$ for some $x \in \underline{B}$ and $m \in \mathcal{M}_0^C$, we have

$$\begin{aligned}p\Big(\mathsf{P}_n\Big(\int_Y^{\mathbf{Hpol}_n(B,C)} \eta(s)\mu(ds)\Big)\Big) &= \frac{1}{n!}\frac{d^n}{dt^n} m\Big(\int_Y^C \eta(s,tx)\mu(ds)\Big) \mid_{t=0} \\ &= \frac{1}{n!}\Big(\frac{d^n}{dt^n}\int_Y m(\eta(s,tx))\mu(ds)\Big) \mid_{t=0} \\ &= \frac{1}{n!}\int_Y \frac{d^n}{dt^n} m(\eta(s,tx)) \mid_{t=0} \mu(ds) \\ &= \int_Y p(\mathsf{P}_n(\eta(s)))\mu(ds)\end{aligned}$$

by standard properties of integration. The fact that $\|\mathsf{P}_n\| \leq 1$ results from the obvious fact that $\mathsf{P}_n f(x) \leq f(x)$ for all $x \in \mathcal{B}\underline{B}$. ☐

Theorem 8.13. *For all $n \in \mathbb{N}$ we have $\mathsf{D}_0^{(n)} \in \mathbf{ICones}(B \Rightarrow_{\mathsf{a}} C, n^n \mathbf{Sym}_n(B,C))$.*

Proof. Remember that $\mathsf{D}_0^{(n)} = n!(\mathsf{L}_n \circ \mathsf{P}_n)$ and apply Theorem 8.9 and Lemma 8.12. ☐

Theorem 8.14. *Each analytic function is stable and measurable.*

Proof. Immediate consequence of the definition of analytic functions and of Lemma 8.5. ☐

Remark 8.15. The converse is not true, as shown by Remark 9.6. Indeed since the stable and measurable function cont introduced in Remark 2.7 is actually linear, if cont were analytic we would have $\mathsf{D}_0^{(1)}\mathsf{cont} = \mathsf{cont}$ and $\mathsf{D}_0^{(n)}\mathsf{cont} = 0$ if $n \neq 1$ which is not possible since cont does not preserve integrals.

8.4. **The category of integrable cones and analytic functions.** Our goal in this section is to show that integrable cones, together with analytic functions as morphisms, form a category which is cartesian closed.

8.4.1. *Composing analytic functions.* We start with a special case of composition that we can think of as the restriction of an analytic function to a *local cone* in the sense of Section 7.1.

Theorem 8.16. *Let $x \in \mathcal{B}\underline{B}$ and let $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$. Then the function $g : \mathcal{B}\underline{B_x} \to \underline{C}$ defined by $g(u) = f(x+u)$ is analytic, that is $g \in \underline{B_x \Rightarrow_{\mathsf{a}} C}$.*

Proof. Given $u \in \mathcal{B}\underline{B_x}$ we have

$$\begin{aligned}
g(u) &= f(x+u) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{x+u}^n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{k!(n-k)!} \mathsf{D}_0^{(n)} f(\overline{u}^{n-k}, \overline{x}^k) \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k)
\end{aligned}$$

so it suffices to show that for each $l \in \mathbb{N}$ the function $g_l : \mathcal{B}\underline{B_x} \to \underline{C}$ defined by

$$g_l(u) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overline{u}^{l}, \overline{x}^k)$$

satisfies $g_l(u) = \varphi_l(\overline{u}^l)$ for some $\varphi_l \in \mathbf{Sym}_l(B_x, C)$. We show that we can set

$$\varphi_l(\overrightarrow{u}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overrightarrow{u}, \overline{x}^k)$$

for all $\overrightarrow{u} = (u_1, \dots, u_l) \in \underline{B_x}^l$. So let $\overrightarrow{u} = (u_1, \dots, u_l) \in \underline{B_x}^l$ and let $\lambda \geq \max_{i=1}^{l} \|u_i\|_{\underline{B_x}}$ be such that $\lambda > 0$ so that for each i we have $\frac{1}{\lambda} u_i \in \mathcal{B}\underline{B_x}$.

For $N \in \mathbb{N}$ let $\varphi_l^N(\overrightarrow{u}) = \sum_{k=0}^{N} \frac{1}{k!} \mathsf{D}_0^{(l+k)} f(\overrightarrow{u}, \overline{x}^k)$ so that $\varphi_l^N \in \underline{\mathbf{Sym}_l(B_x, C)}$; actually we even have $\varphi_l^N \in \underline{\mathbf{Sym}_l(B, C)}$. Observe that, setting $u = \frac{1}{l\lambda} \sum_{i=1}^{l} u_i \in \mathcal{B}\underline{B_x}$ we have $u_i \leq l\lambda u$ for each $i = 1, \dots, l$ so that

$$\begin{aligned}
\varphi_l^N(\overrightarrow{u}) &\leq \varphi_l^N(\overline{l\lambda u}^l) = (l\lambda)^l \varphi_l^N(\overline{u}^l) \\
&\leq (l\lambda)^l g(u) = (l\lambda)^l f(x+u)
\end{aligned}$$

so that $\left\|\varphi_l^N(\overrightarrow{u})\right\|_C \leq (l\lambda)^l \|f\|$ and since neither l nor λ depend on N the sequence $(\varphi_l^N(\overrightarrow{u}))_{N \in \mathbb{N}}$ is increasing in $(l\lambda)^l \mathcal{B}\underline{C}$, it has a lub which is $\varphi_l(\overrightarrow{u})$ which is therefore well-defined and belongs to $(l\lambda)^l \mathcal{B}\underline{C}$. The fact that the map $\varphi_l : \underline{B_x} \to \underline{C}$ defined in that way is l-linear symmetric and ω-continuous results from the ω-continuity of addition, scalar multiplication and from the basic properties of lubs. The measurability and integrability of φ_l result as usual from the monotone convergence theorem. □

Let $f \in \mathcal{B}\underline{B \Rightarrow_{\mathsf{a}} C}$ and $g \in \underline{C \Rightarrow_{\mathsf{a}} D}$, since $g(\mathcal{B}\underline{B}) \subseteq \mathcal{B}\underline{C}$, the function $g \circ f : \mathcal{B}\underline{B} \to \underline{D}$ is well defined and bounded. We assume first that $f(0) = 0$ so that the first term of the Taylor expansion of f vanishes and we have

$$g(f(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} g\Big(\overline{\sum_{k=1}^{\infty} \frac{1}{k!} \mathsf{D}_0^{(k)} f(\overline{x}^k)}^{\,n}\Big)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma:[n]\to\mathbb{N}^+} \frac{n!}{\sigma!} \mathsf{D}_0^{(n)} g(\mathsf{D}_0^{(\sigma(1))} f(\overline{x}^{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} f(\overline{x}^{\sigma(n)}))$$

by multilinearity and continuity of the $\mathsf{D}_0^{(n)} f$'s, with the notation $\sigma! = \prod_{i=1}^{n} \sigma(i)!$. If $n, l \in \mathbb{N}$ we define $\mathsf{L}(n, l)$ as the set of all $\sigma : [n] = \{1, \dots, n\} \to \mathbb{N}^+$ such that $\sum_{i=1}^{n} \sigma(i) = l$. This set is finite and empty as soon as $n > l$ (it is for obtaining this effect that we have assumed that $f(0) = 0$). We have

$$g(f(x)) = \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{n=0}^{l} \sum_{\sigma\in\mathsf{L}(n,l)} \frac{l!}{\sigma!} \mathsf{D}_0^{(n)} f(\mathsf{D}_0^{(\sigma(1))} g(\overline{x}^{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} g(\overline{x}^{\sigma(n)}))\,.$$

For each $l \in \mathbb{N}$, the function

$$h_l : \underline{B}^l \to \underline{D}$$

$$(x_1, \dots, x_l) \mapsto \sum_{n=0}^{l} \sum_{\sigma\in\mathsf{L}(n,l)} \frac{l!}{\sigma!} \mathsf{D}_0^{(l)} f(\mathsf{D}_0^{(\sigma(1))} g(x_1, \dots, x_{\sigma(1)}), \dots, \mathsf{D}_0^{(\sigma(n))} g(x_{l-\sigma(n)+1}, \dots, x_l))$$

is l-linear, measurable and integrable as a finite sum of such functions, however it is not necessarily symmetric (for instance, for $l = 4$, this sum contains the expression $\frac{4!}{(2!)^2} \mathsf{D}_0^{(2)} f(\mathsf{D}_0^{(2)} g(x_1, x_2), \mathsf{D}_0^{(2)} g(x_3, x_4)))$, but not $\frac{4!}{(2!)^2} \mathsf{D}_0^{(2)} f(\mathsf{D}_0^{(2)} g(x_1, x_3), \mathsf{D}_0^{(2)} g(x_2, x_4)))$, so we set

$$k_l(\overrightarrow{x}) = \frac{1}{l!} \sum_{\theta\in\mathfrak{S}_l} h_l(x_{\theta(1)}, \dots, x_{\theta(l)})$$

and k_l is again a finite sum of l-linear, measurable and integrable functions and hence obviously belongs to $\underline{\mathbf{Sym}_l(B, D)}$, and we have

$$g(f(x)) = \sum_{l=0}^{\infty} \frac{1}{l!} k_l(\overline{x}^l)$$

for each $x \in \mathcal{B}\underline{B}$ which proves that $g \circ f$ is analytic since this function is obviously bounded.

Now we don't assume anymore that $f(0) = 0$, and we define an obviously analytic function $f_0 \in \mathcal{B}\underline{B} \to \mathcal{B}\underline{C_{f(0)}}$ by $f_0(x) = f(x) - f(0)$. By Lemma 8.16 the function $g_0 : \mathcal{B}\underline{C_{f(0)}} \to \underline{D}$ given by $g_0(v) = g(f(0) + v)$ is analytic and hence $g \circ f = g_0 \circ f_0$ is analytic since $f_0(0) = 0$. The measurability of $g \circ f$ is obvious so $g \circ f \in \underline{B \Rightarrow_{\mathsf{a}} D}$.

This shows that we have defined a category **ACones** whose objects are the integrable cones and where a morphism from B to C is a $f \in \underline{B \Rightarrow_{\mathsf{a}} C}$ such that $\|f\| \leq 1$. We aim now at proving that this category is cartesian closed.

Lemma 8.17. *For all measurable cones B, C we have* $\mathbf{ICones}(B, C) \subseteq \mathbf{ACones}(B, C)$*.*

This is obvious and shows that there is a forgetful faithful functor $\mathsf{Der}^{\mathsf{a}} : \mathbf{ICones} \to \mathbf{ACones}$ which acts as the identity on objects and morphisms.

Proposition 8.18. *The category* $\mathbf{ACones}$ *has all (small) products.*

Proof. We already know that each family $(B_i)_{i\in I}$ of integrable cones has a product $B = \&_{i\in I} B_i$ in $\mathbf{ICones}$ with projections $(\mathsf{pr}_i \in \mathbf{ICones}(B, B_i))_{i\in I}$. We show that B is also the product of the family $(B_i)_{i\in I}$ with projections $(\mathsf{Der}^{\mathsf{a}}(\mathsf{pr}_i))_{i\in I}$ in $\mathbf{ACones}$. Remember that an element of $\underline{B}$ is a family $(x_i \in \underline{B_i})_{i\in I}$ such that the family $(\|x_i\|_{B_i})_{i\in I}$ is bounded in $\mathbb{R}_{\geq 0}$.

So let $(f_i \in \mathbf{ACones}(C, B_i))_{i\in I}$, it suffices to prove that the function $f : \mathcal{B}\underline{C} \to \underline{B}$ given by $f(y) = (f_i(y))_{i\in I}$ belongs to $\mathbf{ACones}(C, B)$. The fact that $\forall y \in \mathcal{B}\underline{C}\ f(y) \in \mathcal{B}\underline{B}$ results from the definition of the norm of B and from the fact that $\forall i \in I\ \ \|f_i\| \leq 1$. The measurability of f results trivially from its definition and from the definition of $\mathcal{M}^B$. We know that $f_i(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathsf{D}_0^{(n)} f_i(\overline{y}^n)$. For each $n \in \mathbb{N}$ the map $\varphi_n : \underline{C}^n \to \underline{B}$ defined by $\varphi_n(\overrightarrow{y}) = (\mathsf{D}_0^{(n)} f_i(\overrightarrow{y}))_{i\in I}$ belongs to $\underline{\mathbf{Sym}_n(C, B)}$ since we know that $\left\|\mathsf{D}_0^{(n)} f\right\| \leq n^n$ by Theorem 8.13. It follows that f is analytic since $f(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\varphi_n(\overline{y}^n)$. □

Theorem 8.19. *The category* $\mathbf{ACones}$ *is cartesian closed.*

Proof. We already know that $B \Rightarrow_{\mathsf{a}} C$ is an integrable cone and we have an obvious function

$$\begin{aligned} \mathsf{Ev} : \underline{(B \Rightarrow_{\mathsf{a}} C) \mathbin{\&} B} = (\underline{B \Rightarrow_{\mathsf{a}} C}) \times \underline{B} &\to \underline{C} \\ (f, x) &\mapsto f(x) \end{aligned}$$

which satisfies $\|\mathsf{Ev}\| \leq 1$, we show that it is measurable. Let $\theta \in \underline{\mathsf{Path}(X, (B \Rightarrow_{\mathsf{a}} C) \mathbin{\&} B)}$ for some $X \in \mathbf{Ar}$, so that $\theta = \langle \eta, \beta \rangle$ where $\eta \in \underline{\mathsf{Path}(X, B \Rightarrow_{\mathsf{a}} C)}$ and $\beta \in \underline{\mathsf{Path}(X, B)}$, we must prove that $\mathsf{Ev} \circ \langle \eta, \beta \rangle \in \underline{\mathsf{Path}(X, C)}$ so let $m \in \mathcal{M}_Y^C$ for some $Y \in \mathbf{Ar}$, we must prove that the function $\varphi = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot m(s, \eta(r)(\beta(r))) : Y \times X \to \mathbb{R}_{\geq 0}$ is measurable. We build $p = (\beta \circ \mathsf{pr}_1) \triangleright (m \circ \mathsf{pr}_2) \in \mathcal{M}_{X\times Y}^{B\Rightarrow_{\mathsf{a}} C}$ and since $\eta \in \underline{\mathsf{Path}(d, B \Rightarrow_{\mathsf{a}} C)}$ the map

$$\begin{aligned} \psi &= \boldsymbol{\lambda}(r_1, s, r_2) \in X \times Y \times X \cdot p(r_1, s, \eta(r_2)) \\ &= \boldsymbol{\lambda}(r_1, s, r_2) \in X \times Y \times X \cdot m(s, \eta(r_2)(\beta(r_1))) \end{aligned}$$

is measurable which shows that $\varphi = \boldsymbol{\lambda}(s, r) \in Y \times X \cdot \psi(r, s, r)$ is measurable.

We prove that Ev is analytic. We have

$$\begin{aligned} \mathsf{Ev}(f, x) &= f(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{x}^n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \varphi_n(\overline{(f, x)}^{n+1}) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (n+1) \varphi_n(\overline{(f, x)}^{n+1}) \end{aligned}$$

where $\varphi_n : \underline{((B \Rightarrow_{\mathsf{a}} C) \,\&\, B)}^{n+1} \to \underline{C}$ is given by

$$\varphi_n((f_1,x_1),\dots,(f_{n+1},x_{n+1})) = \frac{1}{n+1}\sum_{i=1}^{n+1} \mathsf{D}_0^{(n)} f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})$$

and therefore belongs to $\underline{\mathbf{Sym}_{n+1}(B,C)}$; the measurability of φ_n follows from that of $\mathsf{D}_0^{(n)} f$. If follows that Ev is analytic, with

$$\mathsf{D}_0^{(0)}\,\mathsf{Ev}() = 0$$
$$\mathsf{D}_0^{(n+1)}\,\mathsf{Ev}((f_1,x_1),\dots,(f_{n+1},x_{n+1})) = \sum_{i=1}^{n+1} \mathsf{D}_0^{(n)} f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})\,.$$

Now we deal with the Curry transpose of analytic functions. So let D be an integrable cone and let $f \in \mathbf{ACones}(D \,\&\, B, C)$. Given $z \in \mathcal{B}\underline{D}$ let $f_z : \mathcal{B}\underline{B} \to \underline{C}$ be given by $f_z(x) = f(z,x)$. We know that $f_z \in \underline{B \Rightarrow_{\mathsf{a}} C}$ by Theorem 8.16 applied at $(z,0) \in \mathcal{B}\underline{D \,\&\, B}$ and by precomposing the obtained "local" analytic function $g : (D \,\&\, B)_{(z,0)} \to C$ defined by $g(w,y) = f(z+w,y)$ with the obviously analytic function $x \mapsto (0,x)$: this composition of functions coincides with f_z.

We are left with proving that the function $g : \mathcal{B}\underline{D} \to \underline{B \Rightarrow_{\mathsf{a}} C}$ defined by $g(z) = f_z$ belongs to $\mathbf{ACones}(D, B \Rightarrow_{\mathsf{a}} C)$. It is obvious that $\|g\| \le 1$ so let us check that g is measurable. Let $\delta \in \underline{\mathsf{Path}(X,D)}$, we must prove that $g \circ \delta \in \underline{\mathsf{Path}(X, B \Rightarrow_{\mathsf{a}} C)}$ so let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{B \Rightarrow_{\mathsf{a}} C}$, we must prove that the function $\varphi = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot p(s, g(\delta(r))) : Y \times X \to \mathbb{R}_{\ge 0}$ is measurable. We have $p = \beta \triangleright m$ where $\beta \in \underline{\mathsf{Path}(Y,B)}$ and $m \in \mathcal{M}_Y^D$ so that $\varphi = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, g(\delta(r))(\beta(s))) = \boldsymbol{\lambda}(s,r) \in Y \times X \cdot m(s, f(\delta(r),\beta(s)))$ is measurable because f is measurable and $\boldsymbol{\lambda}(r,s) \in Y \times X \cdot (\delta(r),\beta(s)) \in \underline{\mathsf{Path}(Y \times X, D \,\&\, B)}$. We are left with proving that g is analytic. For $z \in \mathcal{B}\underline{D}$ we have

$$\begin{aligned}
g(z) &= \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(z,x)\\
&= \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{D}_0^{(n)} f(\overline{(z,x)}^{n})\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(n)} f(\overline{(z,0)+(0,x)}^{n})\\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \sum_{k=0}^{n} \binom{n}{k} \mathsf{D}_0^{(n)} f(\overline{(z,0)}^{k}, \overline{(0,x)}^{n-k})\\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(n)} f(\overline{(z,0)}^{k}, \overline{(0,x)}^{n-k})\\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{\infty} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f(\overline{(z,0)}^{k}, \overline{(0,x)}^{l}) = \sum_{k=0}^{\infty} \frac{1}{k!} h_k(\overline{z}^{k})
\end{aligned}$$

where

$$h_k(z_1,\dots,z_k) = \sum_{l=0}^{\infty} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f((z_1,0),\dots,(z_k,0),\overline{(0,x)}^{l})$$

is well defined for all $z_1, \dots, z_k \in \underline{D}$. Indeed, as usual it suffices to take some $\lambda > 0$ such that $\lambda \geq \|z_i\|_D$ for $i = 1, \dots, k$ and observe that for all $N \in \mathbb{N}$ one has, setting $z = \sum_{i=1}^{k} z_i$ so that $\frac{1}{k\lambda} z \in \mathcal{B}\underline{D}$,

$$\begin{aligned}\sum_{l=0}^{N} \frac{1}{l!} \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot \mathsf{D}_0^{(k+l)} f((z_1, 0), \dots, (z_k, 0), \overline{(0, x)}^l) &\leq h_k(\overline{z}^k) \\ &= (k\lambda)^k h_k(\frac{1}{k\lambda} \overline{z}^k) \\ &\leq k!(k\lambda)^k g(\frac{1}{k\lambda} z)\,.\end{aligned}$$

The map h_k is multilinear by ω-continuity of the algebraic operations in each cone, it is obviously symmetric by the symmetry of the $\mathsf{D}_0^{(n)} f$. Its ω-continuity follows from that of the $\mathsf{D}_0^{(n)} f$ and from commutations of lubs. Last, measurability and integrability follow as usual from the monotone convergence theorem. So we have $h_k \in \mathbf{Sym}_n(D, B \Rightarrow_{\mathsf{a}} C)$ and this shows that g is analytic.

To prove that **ACones** is cartesian closed it suffices to prove that g is the unique morphism in $\mathbf{ACones}(D, B \Rightarrow_{\mathsf{a}} C)$ such that

$$\mathsf{Ev} \circ (g \mathbin{\&} \mathsf{Id}_B) = f$$

which results straightforwardly from the fact that Ev is defined exactly as in **Set**. □

Theorem 8.20. *The functor* $\mathsf{Der}^{\mathsf{a}} : \mathbf{ICones} \to \mathbf{ACones}$ *preserves all limits.*

Proof. Preservation of categorical products resulting easily from the construction of products in **ACones** (Proposition 8.18) and in **ICones** (Theorem 4.16), let us deal with equalizers. So let $f, g \in \mathbf{ICones}(B, C)$ and let (E, e) be their equalizer in **ICones**: remember that $\underline{E} = \{x \in \underline{B} \mid f(x) = g(x)\}$ and that $e \in \mathbf{ICones}(E, B)$ is the obvious injection. Let $h \in \mathbf{ACones}(D, B)$ be such that $f \circ h = g \circ h$. This means that $h(\mathcal{B}\underline{D}) \subseteq \mathcal{B}\underline{E}$. We know that

$$h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} h_n(z)$$

where $h_n \in \mathbf{Hpol}_n(D, B)$ is fully characterized by

$$\forall x' \in \underline{B'} \quad x'(h_n(z)) = \frac{d^n}{dt^n} x'(h(tz)) \mid_{t=0}\,, \tag{8.2}$$

see the proof of Lemma 8.11. We contend that $f \circ h_n = g \circ h_n$ so let $z \in \mathcal{B}\underline{D}$ and let $p \in \mathcal{M}_0^C$, we have

$$p(f(h_n(z))) = \frac{d^n}{dt^n} p(f(h(tz)))) \mid_{t=0}\,,$$

by Equation (8.2) applied with $x' = p\, f \in \underline{B'}$ and hence $p(f(h_n(z))) = p(g(h_n(z)))$ which proves our contention by (**Mssep**). This shows that $h_n(\mathcal{B}\underline{D}) \subseteq \mathcal{B}\underline{E}$. Therefore since the operator L_n is defined in terms of addition, subtraction and multiplication by non-negative real numbers we have $\mathsf{L}_n h_n \in \underline{\mathbf{Sym}_n(B, E)}$ – measurability and integrability follow from the fact that measurability tests and integrals in E are defined as in B. Finally this shows that $\mathsf{D}_0^{(n)} h \in \underline{\mathbf{Sym}_n(B, E)}$ and hence $h \in \mathbf{ACones}(D, E)$ which shows that (E, e) is the equalizer of f, g in **ACones**. □

9. The linear-non-linear adjunction, in the stable and analytic cases

From now on we use $\mathcal{C}$ to denote one of the two cartesian closed categories **SCones** and **ACones**, which are both locally small. In both cases we use Der to denote the functor $\mathbf{ICones} \to \mathcal{C}$ (which was denoted by $\mathsf{Der}^{\mathsf{s}}$ when $\mathcal{C} = \mathbf{SCones}$ and by $\mathsf{Der}^{\mathsf{a}}$ when $\mathcal{C} = \mathbf{ACones}$).

Remember that Der preserves all limits, see Theorems 7.34 and 8.20.
For that reason the two categories **ICones** and $\mathcal{C}$ can be related by a linear-non-linear adjunction in the sense of [Mel09], and hence form a categorical model of Intuitionistic LL. We describe directly the associated Seely category.

Let $\mathsf{E} : \mathcal{C} \to \mathbf{ICones}$ be the left adjoint of Der, which exists by Theorem 4.19, and let us introduce the notation $\Theta_{B,C} : \mathbf{ICones}(\mathsf{E}B, C) \to \mathcal{C}(B, \mathsf{Der}\, C) = \mathcal{C}(B, C)$ for the associated natural bijection (remember that $\mathsf{Der}\, C = C$).

Remark 9.1. Just as for the tensor product (see Remark 5.1), we have no concrete description of the E functor for the time being.

We use $(!, \mathsf{der}, \mathsf{dig})$ for the induced comonad on **ICones** whose Kleisli category is (equivalent to) $\mathcal{C}$ since

$$\begin{aligned}\mathbf{ICones}_!(B, C) &= \mathbf{ICones}(!B, C)\\ &= \mathbf{ICones}(\mathsf{E}\ \mathsf{Der}\, B, C)\\ &\simeq \mathcal{C}(\mathsf{Der}\, B, \mathsf{Der}\, C)\\ &= \mathcal{C}(B, C)\,.\end{aligned}$$

Notice that actually $!B = \mathsf{E}B$ and similarly for morphisms. The notation der for the counit of this comonad comes from the dereliction rule of LL, and the notation dig for the comultiplication comes from the LL digging derived rule.

Let $\mathsf{nl}_B = \Theta_{B,\mathsf{E}B}(\mathsf{Id}_{\mathsf{E}B}) \in \mathcal{C}(B, !B)$ be the unit of the adjunction, which is the "universal nonlinear map" on B in the sense that for each integrable cone C and each $f \in \mathcal{C}(B, C)$ one has $f = \varphi \circ \mathsf{nl}_B$ for a unique $\varphi \in \mathbf{ICones}(!B, C)$, namely $\varphi = (\Theta_{B,C})^{-1}(f)$ (dropping the Der symbol since this functor acts as the identity on objects and morphisms considered as functions). So that for $h \in \mathbf{ICones}(!B, C)$, one has

$$\Theta_{B,C}(h) = h \circ \mathsf{nl}_B\,.$$

For each $x \in \mathcal{B}\underline{B}$ we set $x^! = \mathsf{nl}_B(x) \in \mathcal{B}\underline{\mathsf{E}B}$ so that, for $f \in \mathcal{C}(B, C)$ we have $f(x) = (\Theta_{B,C})^{-1}(f)(x^!)$.

The next lemma is similar to Proposition 5.14.

Lemma 9.2. *Let $n \geq 1$, let $B_1, \dots, B_n, C$ be objects of* **ICones** *and f and g be elements of* $\mathbf{ICones}(!B_1 \otimes \cdots \otimes !B_n, C)$ *such that $f({x_1}^! \otimes \cdots \otimes {x_n}^!) = g({x_1}^! \otimes \cdots \otimes {x_n}^!)$ for all $(x_i \in \mathcal{B}\underline{B_i})_{i=1}^n$. Then $f = g$.*

Proof. By induction on n. For $n = 1$ this comes from the fact that $f(x) = (\Theta_{B_1,C})^{-1} f(x^!)$ for all $x \in \mathcal{B}\underline{B}$ so that our assumption entails $f = g$.

For $n > 1$, we have $\mathsf{cur}(f), \mathsf{cur}(g) \in \mathbf{ICones}(!B_1, !B_2 \otimes \cdots \otimes !B_n \multimap C)$. For each $x_1 \in \mathcal{B}\underline{B_1}$, the two functions $\mathsf{cur}(f)({x_1}^!), \mathsf{cur}(g)({x_1}^!) \in \mathbf{ICones}(!B_2 \otimes \cdots \otimes !B_n, C)$ satisfy

$$\forall x_2 \in \mathcal{B}\underline{B_2}, \dots, x_n \in \mathcal{B}\underline{B_n} \quad \mathsf{cur}(f)({x_1}^!)({x_2}^! \otimes \cdots \otimes {x_n}^!) = \mathsf{cur}(g)({x_1}^!)({x_2}^! \otimes \cdots \otimes {x_n}^!)$$

and hence $\mathsf{cur}(f)({x_1}^!) = \mathsf{cur}(g)({x_1}^!)$ by inductive hypothesis. As in the base case we get $f = g$. ☐

The counit $\mathsf{der}_B \in \mathbf{ICones}(!B, B)$ of the comonad $!\_$ is also the counit of the adjunction. It satisfies therefore

$$\forall x \in \mathcal{B}\underline{B} \quad \mathsf{der}_B(x^!) = x\,.$$

The comultiplication $\mathsf{dig}_B \in \mathbf{ICones}(!B, !!B) = \mathbf{ICones}(\mathsf{E}\,\mathsf{Der}\,B, \mathsf{E}\,\mathsf{Der}\,\mathsf{E}\,\mathsf{Der}\,B)$ is defined by $\mathsf{dig}_B = \mathsf{E}(\mathsf{nl}_B)$ so that we have

$$\forall x \in \mathcal{B}\underline{B} \quad \mathsf{dig}_B(x^!) = x^{!!}\,.$$

since by naturality of nl_B we have $\mathsf{Der}(\mathsf{E}(\mathsf{nl}_B)) \circ \mathsf{nl}_B = \mathsf{nl}_{\mathsf{Der}(\mathsf{E}B)} \circ \mathsf{nl}_B$ in $\mathcal{C}$.

Lemma 9.3. *Let $f \in \mathbf{ICones}(B, C)$ and $x \in \mathcal{B}\underline{B}$. We have $(!f)(x^!) = f(x)^!$.*

Proof. We have $(!f)(x^!) = (\mathsf{E}(\mathsf{Der}\,f))(x^!) = ((\mathsf{Der}(\mathsf{E}(\mathsf{Der}\,f))) \circ \mathsf{nl}_{\mathsf{Der}\,B})(x)$ where the composition is taken in $\mathcal{C}$. By naturality we get $(!f)(x^!) = (\mathsf{nl}_{\mathsf{Der}\,C} \circ \mathsf{Der}\,f)(x) = f(x)^!$. □

Consider the two functors $L, R : \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones}^{\mathsf{op}} \times \mathbf{ICones} \to \mathbf{ICones}$ defined on objects by $L(B, C, D) = (B \Rightarrow (C \multimap D))$ and $R(B, C, D) = (C \multimap (B \Rightarrow D))$, and similarly on morphisms.

Lemma 9.4. *Let B, C, D be integrable cones. There is an isomorphism in* **ICones** *from $L(B, C, D) = (B \Rightarrow (C \multimap D))$ to $R(B, C, D) = (C \multimap (B \Rightarrow D))$ which is natural in B, C and D.*

Proof sketch. This needs a separate proof in each case $\mathcal{C} = \mathbf{SCones}$ and $\mathcal{C} = \mathbf{ACones}$, which follows a pattern that we have seen many times. The natural isomorphism maps $f \in \underline{B \Rightarrow (C \multimap D)}$ to $\boldsymbol{\lambda} y \in \underline{C} \cdot \boldsymbol{\lambda} x \in \mathcal{B}\underline{B} \cdot f(x, y)$. □

Then we have

$$\begin{aligned}
\mathbf{ICones}(!(B_1 \mathbin{\&} B_2), C) &\simeq \mathcal{C}(B_1 \mathbin{\&} B_2, C) \\
&\simeq \mathcal{C}(B_1, B_2 \Rightarrow C) \\
&\simeq \mathbf{ICones}(!B_1, B_2 \Rightarrow C) \\
&\simeq \mathbf{ICones}(1, !B_1 \multimap (B_2 \Rightarrow C)) \\
&\simeq \mathbf{ICones}(1, B_2 \Rightarrow (!B_1 \multimap C)) \text{ by Lemma 9.4} \\
&\simeq \mathcal{C}(\top, B_2 \Rightarrow (!B_1 \multimap C)) \\
&\simeq \mathcal{C}(B_2, (!B_1 \multimap C)) \\
&\simeq \mathbf{ICones}(!B_2, (!B_1 \multimap C)) \\
&\simeq \mathbf{ICones}(!B_2 \otimes !B_1, C) \\
&\simeq \mathbf{ICones}(!B_1 \otimes !B_2, C)
\end{aligned}$$

by a sequence of natural bijections and hence by Lemma 1.1 we have a natural isomorphism $\mathsf{m}^2_{B_1,B_2}$ in

$$\mathbf{ICones}(!B_1 \otimes !B_2, !(B_1 \mathbin{\&} B_2))$$

which satisfies

$$\mathsf{m}^2_{B_1,B_2}({x_1}^! \otimes {x_2}^!) = \langle x_1, x_2 \rangle^!\,.$$

Notice that this equation fully characterizes $\mathsf{m}^2_{B_1,B_2}$ by Lemma 9.2.

Similarly we define an iso $\mathsf{m}^0 \in \mathbf{ICones}(1, !\top)$ which is such that $\mathsf{m}^0(t) = t\,0^!$ for all $t \in \mathbb{R}_{\geq 0}$. Then one can prove using Lemma 1.1 again that $!$ is a strong monoidal comonad.

Theorem 9.5. *Equipped with the strong monoidal comonad* !, *the category* **ICones** *is a Seely category in the sense of* [Mel09].

Proof. Using Lemma 9.2 it is easy to prove the remaining properties, which regard $!\_$ and its associated morphisms. As an example let us prove that the following diagram commutes.

$$\begin{array}{ccc} !B_1 \otimes !B_2 & \xrightarrow{\mathsf{m}^2_{B_1,B_2}} & !(B_1 \mathbin{\&} B_2) \\ & & \downarrow{\scriptstyle \mathsf{dig}_{B_1 \& B_2}} \\ {\scriptstyle \mathsf{dig}_{B_1} \otimes \mathsf{dig}_{B_2}} \downarrow & & !!(B_1 \mathbin{\&} B_2) \\ & & \downarrow{\scriptstyle !\langle !\mathsf{pr}_1, !\mathsf{pr}_2\rangle} \\ !!B_1 \otimes !!B_2 & \xrightarrow{\mathsf{m}^2_{!B_1,!B_2}} & !(!B_1 \mathbin{\&} !B_2) \end{array}$$

Given $(x_i \in \mathcal{B}\underline{B_i})_{i=1,2}$, we have

$$\begin{aligned} \mathsf{m}^2_{!B_1,!B_2}((\mathsf{dig}_{B_1} \otimes \mathsf{dig}_{B_2})(x_1{}^! \otimes x_2{}^!)) &= \mathsf{m}^2_{!B_1,!B_2}(x_1{}^{!!} \otimes x_2{}^{!!}) \\ &= \langle x_1{}^!, x_2{}^! \rangle^! \end{aligned}$$

and

$$\begin{aligned} !\langle !\mathsf{pr}_1, !\mathsf{pr}_2\rangle(\mathsf{dig}_{B_1 \& B_2}(\mathsf{m}^2_{B_1,B_2}(x_1{}^! \otimes x_2{}^!))) &= !\langle !\mathsf{pr}_1, !\mathsf{pr}_2\rangle(\mathsf{dig}_{B_1 \& B_2}(\langle x_1, x_2\rangle^!)) \\ &= !\langle !\mathsf{pr}_1, !\mathsf{pr}_2\rangle(\langle x_1, x_2\rangle^{!!}) \\ &= (\langle !\mathsf{pr}_1, !\mathsf{pr}_2\rangle(\langle x_1, x_2\rangle^!))^! \\ &= \langle (!\mathsf{pr}_1)(\langle x_1, x_2\rangle^!), (!\mathsf{pr}_2)(\langle x_1, x_2\rangle^!)\rangle^! \\ &= \langle x_1{}^!, x_2{}^! \rangle^! \end{aligned}$$

□

Remark 9.6. Assume that $\mathcal{C} = \mathbf{SCones}$ and that, as in Remark 3.1 and Section 4.1, **Ar** is the category whose only objects are $\mathbb{R}$ and 0 (the one element measurable space), and all measurable functions as morphisms. Then the underlying set of $\underline{\mathsf{FMeas}(\mathbb{R})}$ is the set of all finite measures on $\mathbb{R}$.

Consider, as in Remark 2.7, the map $\mathsf{cont} : \underline{\mathsf{FMeas}(\mathbb{R})} \to \underline{\mathsf{FMeas}(\mathbb{R})}$ which extracts the continuous part of each measure on $\mathbb{R}$. In addition to being linear and ω-continuous, this map is also measurable (because the map $\mu \mapsto \frac{d\mu}{d\lambda}$ is measurable [Kal17, Theorem 1.28]). However, as noted in Remark 2.7, this map does not commute with integrals and is therefore not a morphism in **ICones**.

Nevertheless, cont is a morphism in **SCones**, so there exists $f \in \mathbf{ICones}(!\mathsf{FMeas}(\mathbb{R}), \mathsf{FMeas}(\mathbb{R}))$ such that $\mathsf{cont}(\mu) = f(\mu^!)$. It would be interesting to understand how this function f works to get some insight on the internal structure of the stable exponential, which is defined in a rather implicit way (by the special adjoint functor theorem).

9.1. The coalgebra structure of $\mathsf{FMeas}(X)$.

In this section we derive additional consequences of the integrability condition on cones and linear morphisms. First, we show that for each $X \in \mathbf{Ar}$, the integrable cone $\mathsf{FMeas}(X)$ has a structure of !-coalgebra. The definition of this coalgebra structure strongly uses the fact that $!\mathsf{FMeas}(X)$ is an integrable cone, that is, all measurable paths valued in that cone have an integral wrt. each subprobability measure on the measurable space (belonging to $\mathbf{Ar}$) where it is defined. This is typically a property which was not available in [EPT18b].

Let $X \in \mathbf{Ar}$. In Section 6 we defined the Dirac path $\boldsymbol{\delta}^X \in \mathcal{B}\underline{\mathsf{Path}(X, \mathsf{FMeas}(X))}$ which maps $r \in X$ to $\boldsymbol{\delta}^X(r)$, the Dirac measure at r. Since morphisms in $\mathcal{C}$ are measurable we have

$$\mathsf{nl}_{\mathsf{FMeas}(X)} \circ \boldsymbol{\delta}^X \in \mathcal{B}\underline{\mathsf{Path}(X, !\mathsf{FMeas}(X))}$$

and we define

$$\mathsf{h}_X = \mathcal{I}_X^{!\mathsf{FMeas}(X)}(\mathsf{nl}_{\mathsf{FMeas}(X)} \circ \boldsymbol{\delta}^X) \in \mathbf{ICones}(\mathsf{FMeas}(X), !\mathsf{FMeas}(X))$$

using Theorem 6.1. In other words h_X is defined by

$$\mathsf{h}_X(\mu) = \int_{r \in X} \boldsymbol{\delta}^X(r)^! \mu(dr)$$

and satisfies $\mathsf{h}_X(\boldsymbol{\delta}^X(r)) = \boldsymbol{\delta}^X(r)^!$.

Theorem 9.7. *Equipped with* h_X*, the object* $\mathsf{FMeas}(X)$ *of* $\mathbf{ICones}$ *is a coalgebra of the comonad* $!\_$*. Moreover for each* $\varphi \in \mathbf{Ar}(X, Y)$*, we have*

$$\mathsf{FMeas}(\varphi) = \varphi_* \in \mathbf{ICones}^!(\mathsf{FMeas}(X), \mathsf{FMeas}(Y))$$

so that FMeas *is a functor* $\mathbf{Ar} \to \mathbf{ICones}^!$.

Proof. We must first prove: $\mathsf{der}_{\mathsf{FMeas}(X)}\, \mathsf{h}_X = \mathsf{Id}_{\mathsf{FMeas}(X)} \in \mathbf{ICones}(\mathsf{FMeas}(X), \mathsf{FMeas}(X))$. By Theorem 6.2 this results from the fact that for all $r \in X$ one has $(\mathsf{der}_{\mathsf{FMeas}(X)}\, \mathsf{h}_X)(\boldsymbol{\delta}^X(r)) = \mathsf{der}_{\mathsf{FMeas}(X)}(\boldsymbol{\delta}^X(r)^!) = \boldsymbol{\delta}^X(r)$.

Next we must prove that $f_1 = f_2 \in \mathbf{ICones}(\mathsf{FMeas}(X), !!(\mathsf{FMeas}(X)))$ where

$$f_1 = \mathsf{dig}_{\mathsf{FMeas}(X)}\, \mathsf{h}_X \quad \text{and} \quad f_2 = !\mathsf{h}_X\, \mathsf{h}_X\,.$$

Let $r \in X$, we have

$$f_1(\boldsymbol{\delta}^X(r)) = \mathsf{dig}_{!\mathsf{FMeas}(X)}(\boldsymbol{\delta}^X(r)^!) = \boldsymbol{\delta}^X(r)^{!!}$$
$$f_2(\boldsymbol{\delta}^X(r)) = !\mathsf{h}_X(\boldsymbol{\delta}^X(r)^!) = (\mathsf{h}_X(\boldsymbol{\delta}^X(r)))^!$$

by Lemma 9.3. And hence $f_2(\boldsymbol{\delta}^X(r)) = \boldsymbol{\delta}^X(r)^{!!}$ so that $f_1 = f_2$ by Theorem 6.2.

Let now $\varphi \in \mathbf{Ar}(X, Y)$, we must prove that $f_1 = f_2$ where $f_1 = \mathsf{h}_Y\, \varphi_*$ and $f_2 = !(\varphi_*)\, \mathsf{h}_X$. Let $r \in X$, we have $f_1(\boldsymbol{\delta}^X(r)) = \mathsf{h}_Y(\boldsymbol{\delta}^Y(\varphi(r)) = \boldsymbol{\delta}^Y(\varphi(r))^!$ and $f_2(\boldsymbol{\delta}^X(r)) = !(\varphi_*)(\boldsymbol{\delta}^X(r)^!) = (\varphi_*(\boldsymbol{\delta}^X(r)))^! = \boldsymbol{\delta}^Y(\varphi(r))^!$ by Lemma 9.3 and so $f_1 = f_2$ by Theorem 6.2. This proves the second part of the theorem. □

Remark 9.8. One of the main goals in introducing integrable cones was precisely to get this additional structure for each cone $\mathsf{FMeas}(X)$ (notably because this structure is required in order to interpret call-by-value languages). It means more specifically that for each $f \in \mathcal{C}(\mathsf{FMeas}(X), B) = \mathbf{ICones}(!\mathsf{FMeas}(X), B)$ we can define $g = f\, \mathsf{h}_X \in \mathbf{ICones}(\mathsf{FMeas}(X), B)$ such that

$$\forall \mu \in \underline{\mathsf{FMeas}(X)} \quad g(\mu) = \int_{r \in X} f(\boldsymbol{\delta}^X(r)) \mu(dr)$$

which is a "linearization" of f allowing to interpret the sampling operation of probabilistic programming languages: one samples a $r \in X$ (a real number if $X = \mathbb{R}$) according to the distribution μ and feeds the program f with the value r represented as the Dirac measure $\boldsymbol{\delta}^X(r)$. This Dirac measure represents, in our semantics, the real number r considered as a value. This observation strongly supports the idea of taking the objects or $\mathbf{Ar}$ (such as the real line, or the set of natural numbers) as our basic data-types and the measurable functions $\varphi \in \mathbf{Ar}(X,Y)$ as the basic functions of our programming language, through the functor $\mathsf{FMeas} : \mathbf{Ar} \to \mathbf{ICones}$: remember that $\mathsf{FMeas}(\varphi)(\boldsymbol{\delta}^X(r)) = \varphi_*(\boldsymbol{\delta}^X(r)) = \boldsymbol{\delta}^X(\varphi(r))$.

From the viewpoint of LL this means that each $X \in \mathbf{Ar}$ can be seen as a positive type, that is, a type equipped with structural rules allowing to erase and duplicate its values, see for instance [Gir91, LR03, ET19]. Another way to understand h_X is to see it as a *storage operator* in the sense of [Kri94], that is, $\mathsf{FMeas}(X)$ is a *data-type*. This idea will be confirmed in Section 9.1.1 where we will see that FMeas is a full and faithful functor from $\mathbf{Ar}$ to $\mathbf{ICones}^!$.

Example 9.9. A typical probabilistic programming language that we can interpret in the model $\mathbf{ICones}$ (we assume that $\mathbb{R} \in \mathbf{Ar}$) is the probabilistic version of PCF presented in [EPT18b] (to which we refer for more details and examples), which features continuous data-types and can be extended in various ways. Such a language could feature a type ρ of real numbers, a constant unif such that $\Gamma \vdash \mathsf{unif} : \rho$ corresponding to the uniform probability distribution on the interval $[0,1]$ *etc.* All the types of this language, which are given by the following grammar

$$\sigma, \tau \cdots := \rho \mid \sigma \Rightarrow \tau \mid \ \cdots$$

are interpreted as objects of $\mathbf{ICones}$: $[\![\rho]\!] = \mathsf{FMeas}(\mathbb{R})$, $[\![\sigma \Rightarrow \tau]\!] = [\![\sigma]\!] \Rightarrow [\![\tau]\!]$, *etc.* A term M such that $\Gamma \vdash M : \tau$ where $\Gamma = (x_1 : \sigma_1, \dots, x_k : \sigma_k)$ is a typing context, will then be interpreted as a stable and measurable morphism, or as an analytic morphism $[\![M]\!]_\Gamma \in \mathbf{ICones}_!([\![\sigma_1]\!] \,\&\, \cdots \,\&\, [\![\sigma_k]\!], [\![\tau]\!])$. So if $\Gamma, x : \rho \vdash M : \sigma$ has a free variable of type ρ and $\Gamma \vdash N : \rho$, we should consider that N represents a (sub)probability distribution $[\![N]\!]_\Gamma$ on $\mathbb{R}$ and we may want to sample a real number r along this distribution and feed M with the resulting real value that M will use *as many times as it wants*: this value will be represented as the Dirac measure $\boldsymbol{\delta}^{\mathbb{R}}(r)$. In our language, the corresponding construct is a simple let which allows to deal with the ground type ρ in a call-by-value way[23]:

$$\Gamma \vdash \mathsf{let}(x, N, M) : \tau$$

and the semantics of this construct is

$$[\![\mathsf{let}(x,N,M)]\!]_\Gamma = \int_{r\in\mathbb{R}}^{[\![\tau]\!]} [\![M]\!]_\Gamma(\boldsymbol{\delta}^{\mathbb{R}}(r))[\![N]\!]_\Gamma(dr)$$

which is well defined since $[\![M]\!]_\Gamma$ is stable and measurable (or analytic), the function $\boldsymbol{\lambda} r \in \mathbb{R} \cdot \boldsymbol{\delta}^{\mathbb{R}}(r)$ belongs to $\mathcal{B}\underline{\mathsf{Path}(\mathbb{R}, \mathsf{FMeas}(\mathbb{R}))}$ and the cone $[\![\tau]\!]$ is integrable. The constant unif is interpreted as the probability measure on $\mathbb{R}$ which maps a measurable set U to the Lebesgue measure of $U \cap [0,1]$. For each $r \in \mathbb{R}$, the language has a constant $\underline{r}$ of type ρ and $[\![\underline{r}]\!]_\Gamma = \boldsymbol{\delta}^{\mathbb{R}}(r) \in \mathcal{B}\underline{\mathsf{FMeas}(\mathbb{R})}$ (in each context Γ). Our language will also have constructs $\mathsf{log}(M)$, $\mathsf{sqrt}(M)$ *etc.* corresponding to the usual functions which are all measurable, and typed for instance by

$$\frac{\Gamma \vdash M : \rho}{\Gamma \vdash \mathsf{log}(M) : \rho}$$

[23]This idea was already central in [EPT14, EPT18a, ET19], in the discrete setting of probabilistic coherence spaces.

with semantics given by push-forward: $[\![\mathsf{log}(M)]\!]_\Gamma = \log_*([\![M]\!]_\Gamma)$. So for instance we can define a closed term N such that $\vdash N : \rho$ by

$$N = \mathsf{let}(x, \mathsf{unif}, \mathsf{let}(y, \mathsf{unif}, \mathsf{mult}(\mathsf{sqrt}(\mathsf{mult}(\underline{-2.}, \mathsf{log}(x))), \mathsf{cos}(\mathsf{mult}(\underline{6.28\cdots}, y)))))$$

and then $[\![N]\!]$ is the normal distribution $\mathcal{N}(0,1)$ defined by the Box Muller method, and we can define a term N' with two free variable x and s for $\mathcal{N}(x,s)$ as

$$N' = \mathsf{let}(y, N, \mathsf{plus}(\mathsf{mult}(s, y), x)))$$

such that $x : \rho, s : \rho \vdash N' : \rho$. Then $[\![N'[\underline{4.2}/x, \underline{0.7}/s]]\!] = [\![N']\!]_{x:\rho,s:\rho}(\boldsymbol{\delta}^{\mathbb{R}}(4.2), \boldsymbol{\delta}^{\mathbb{R}}(0.7)) \in \underline{\mathsf{FMeas}(\mathbb{R})}$ is the measure $\mathcal{N}(4.2, 0.7)$.

9.1.1. *The category of measurable functions as a full subcategory of the Eilenberg Moore category.* So we have extended the operation FMeas on the measurable spaces of $\mathbf{Ar}$ into a functor $\mathbf{Ar} \to \mathbf{ICones}^!$ which acts on morphisms by push-forward. This functor is clearly faithful, we prove that, under very reasonable assumptions about $\mathbf{Ar}$, it is also full, which is quite a remarkable fact: the Eilenberg-Moore category of ! contains $\mathbf{Ar}$ as a full subcategory. Again, integration is an essential ingredient in the proof of this result.

A Polish space is a complete metric space which has a countable dense subset.

We will need two lemmas which are folklore in measure theory.

Lemma 9.10. *Let X be a Polish space, equipped with its standard Borel σ-algebra σ_X. Let μ be a probability measure on X and assume that $\forall U \in \sigma_X\ \mu(U) \in \{0,1\}$. Then μ is a Dirac measure.*

Proof. Given $r \in X$ and $\varepsilon \geq 0$ we use $B(r,\varepsilon) \subseteq X$ for the closed ball of radius ε centered at r. Let D be a countable dense subset of X. Let $F \subseteq X$ be closed and such that $\mu(F) = 1$ and let $\varepsilon > 0$, we have $F \subseteq \bigcup_{r\in D\cap F} B(r,\varepsilon)$ and hence $1 = \mu(F) \leq \sum_{r\in D\cap F} \mu(B(r,\varepsilon))$ and hence $\exists r \in D \cap F\ \mu(B(r,\varepsilon)) = 1$. We define a sequence $(r_n)_{n\in\mathbb{N}}$ of elements of D such that $\forall n \in \mathbb{N}\ \mu(B(r_n, 2^{-n})) = 1$ as follows. We obtain r_0 by applying the property above with $F = X$ and $\varepsilon = 1$. We get r_{n+1} by applying the property above with $F = B(r_n, 2^{-n})$ and $\varepsilon = 2^{-(n+1)}$. Then the sequence $(r_n)_{n\in\mathbb{N}}$ is Cauchy and has therefore a limit r and we have $\{r\} = \bigcap_{n\in\mathbb{N}} B(r_n, 2^{-n})$ so that $\mu(\{r\}) = \inf_{n\in\mathbb{N}} \mu(B(r_n, 2^{-n})) = 1$ since μ is a measure. It follows that $\mu(U) = 0$ for each measurable U such that $r \notin U$ since we must have $\mu(\{r\} \cup U) = 1$ and hence $\mu = \boldsymbol{\delta}^X(r)$. □

Lemma 9.11. *If X and Y are measurable spaces such that the σ-algebra of Y contains all singletons (this is true in particular if Y is a Polish space), and if κ is a kernel from X to Y such that for all $r \in X$ the measure $\kappa(r)$ is a Dirac measure on Y, then there is a uniquely defined measurable function $\varphi : X \to Y$ such that $\kappa = \boldsymbol{\delta}^Y \circ \varphi$.*

This is obvious.

Theorem 9.12. *Let $X, Y \in \mathbf{Ar}$ be such that Y is a Polish space and let f be a morphism from $\mathsf{FMeas}(X)$ to $\mathsf{FMeas}(Y)$ in $\mathbf{ICones}$. Then f is a coalgebra morphism from $(\mathsf{FMeas}(X), \mathsf{h}_X)$ to $(\mathsf{FMeas}(Y), \mathsf{h}_Y)$ iff there is a $\varphi \in \mathbf{Ar}(X,Y)$ such that $f = \mathsf{FMeas}(\varphi) = \varphi_*$.*

As a consequence, if we assume that X is a Polish space for all $X \in \mathbf{Ar}$, then $\mathbf{Ar}$ is a full subcategory of the Eilenberg Moore category of the comonad $!\_$ through the FMeas functor.

Most measurable spaces which appear in probability theory are Polish spaces: discrete spaces, the real line, countable products and measurable subspaces of Polish spaces (and hence the Cantor Space and the Baire Space, the Hilbert Cube *etc.*) are Polish spaces. So the restriction to Polish spaces is not a serious one.

Proof. Saying that f is a coalgebra morphism means that the following diagram commutes in **ICones**:

$$\begin{array}{ccc} \mathsf{FMeas}(X) & \xrightarrow{f} & \mathsf{FMeas}(Y) \\ {\scriptstyle \mathsf{h}_X}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{h}_Y} \\ !\mathsf{FMeas}(X) & \xrightarrow{!f} & !\mathsf{FMeas}(Y) \end{array}$$

which, by Theorem 6.2, is equivalent to

$$(f(\boldsymbol{\delta}^X(r)))^! = !f(\mathsf{h}_X(\boldsymbol{\delta}^X(r)) = \mathsf{h}_Y(f(\boldsymbol{\delta}^X(r)) = \int_{s\in Y}^{!\mathsf{FMeas}(Y)} \boldsymbol{\delta}^Y(s)^! f(\boldsymbol{\delta}^X(r))(ds) \tag{9.1}$$

for all $r \in X$, and this equation trivially holds if $f = \varphi_*$. Assume conversely that f satisfies (9.1). Let V be a measurable subset of Y and let $g \in \mathcal{C}(\mathsf{FMeas}(Y), 1)$ be defined by $g(\nu) = \nu(V)^2$ and let $g_0 = (\Theta_{\mathsf{FMeas}(Y),1})^{-1}(g) \in \mathbf{ICones}(!\mathsf{FMeas}(Y), 1)$ which is characterized by $\forall \nu \in \underline{\mathsf{FMeas}(Y)}\ g(\nu) = g_0(\nu^!)$. We have $g_0(f(\boldsymbol{\delta}^X(r))^!) = g(f(\boldsymbol{\delta}^X(r))) = f(\boldsymbol{\delta}^X(r))(V)^2$ and, since g_0 preserves integrals,

$$\begin{aligned} f(\boldsymbol{\delta}^X(r))(V)^2 &= g_0\Big(\int_{s\in Y}^{!\mathsf{FMeas}(Y)} \boldsymbol{\delta}^Y(s)^! f(\boldsymbol{\delta}^X(r))(ds)\Big) \quad \text{by Equation (9.1)} \\ &= \int_{s\in Y} g_0(\boldsymbol{\delta}^Y(s)^!) f(\boldsymbol{\delta}^X(r))(ds) \\ &= \int_{s\in Y} \boldsymbol{\delta}^Y(s)(V)^2 f(\boldsymbol{\delta}^X(r))(ds) \\ &= \int_{s\in Y} \boldsymbol{\delta}^Y(s)(V) f(\boldsymbol{\delta}^X(r))(ds) \\ &= f(\boldsymbol{\delta}^X(r))(V) \end{aligned}$$

so we have $f(\boldsymbol{\delta}^X(r))(V) \in \{0,1\}$ for all $V \in \sigma_Y$. Let $g \in \mathcal{C}(\mathsf{FMeas}(Y), 1)$ be defined now by $g(\nu) = 1$ and let $g_0 = (\Theta_{\mathsf{FMeas}(Y),1})^{-1}(g) \in \mathbf{ICones}(!\mathsf{FMeas}(Y), 1)$, we have $g_0(f(\boldsymbol{\delta}^X(r))^!) = g(f(\boldsymbol{\delta}^X(r))) = 1$ and, since g_0 preserves integrals,

$$\begin{aligned} 1 = g_0\Big(\int_{s\in Y}^{!\mathsf{FMeas}(Y)} \boldsymbol{\delta}^Y(s)^! f(\boldsymbol{\delta}^X(r))(ds)\Big) &= \int_{s\in Y} g_0(\boldsymbol{\delta}^Y(s)^!) f(\boldsymbol{\delta}^X(r))(ds) \\ &= \int_{s\in Y} f(\boldsymbol{\delta}^X(r))(ds) \\ &= f(\boldsymbol{\delta}^X(r))(Y) \end{aligned}$$

and hence the measure $f(\boldsymbol{\delta}^X(r))$ is a Dirac measure by Lemma 9.10 and it follows that $f = \mathsf{FMeas}(\varphi) = \varphi_*$ for a uniquely determined $\varphi \in \mathbf{Ar}(X, Y)$ by Lemma 9.11. □

9.2. **Fixpoint operators in the cartesian closed category.** Remember that $\mathcal{C}$ is a CCC having the following property:

> The objects of $\mathcal{C}$ are integrable cones. In particular, for each object B of $\mathcal{C}$, the set $\mathcal{B}\underline{B}$ has a structure of ω-cpo by the condition (**Normc**), with 0 as least element. And each $f \in \mathcal{C}(B, B)$ is in particular an increasing and ω-continuous function $\mathcal{B}\underline{B} \to \mathcal{B}\underline{B}$ and therefore has a least fixpoint which is $\sup_{n=0}^{\infty} f^n(0) \in \mathcal{B}\underline{B}$.

It is completely standard to apply this property to the map $\mathcal{Z} \in \mathcal{C}((B \Rightarrow B) \Rightarrow B, (B \Rightarrow B) \Rightarrow B)$ given by

$$\mathcal{Z}(F)(f) = f(F(f))$$

which is well-defined and belongs to $\mathcal{C}((B \Rightarrow B) \Rightarrow B, (B \Rightarrow B) \Rightarrow B)$ by cartesian closedness of $\mathcal{C}$. The least fixpoint $\mathcal{Y}$ of $\mathcal{Z}$ is an element of $\mathcal{C}(B \Rightarrow B, B)$ which is easily seen to satisfy

$$\mathcal{Y}(f) = \sup_{n=0}^{\infty} f^n(0)$$

and is therefore a least fixpoint operator that we have proven here to be a morphism in $\mathcal{C}$, that is, a stable and measurable or an analytic map depending on the considered category $\mathcal{C}$. This morphism $\mathcal{Y}$ is the key ingredient to interpret recursively defined functional programs in the CCC $\mathcal{C}$.

Example 9.13. The function $f : [0, 1] \to [0, 1]$ given by $f(x) = \frac{1}{2} + \frac{1}{4}x^2$ belongs to $\mathcal{C}(1, 1)$ so it has a least fixpoint $x \in [0, 1]$ which must satisfy $x^2 - 4x + 2 = 0$ and is therefore $2 - \sqrt{2}$. Of course, when $a \in \mathbb{R}_{\geq 0}$ and $a > 0$, the function $x \mapsto x + a$ from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ has no fixpoint, but this is not a contradiction because it does not restrict to a function $[0, 1] \to [0, 1]$.

10. Probabilistic coherence spaces as integrable cones

So far we have seen several ways of building integrable cones: as spaces of measures, or of paths, as products and tensor products, as spaces of analytic maps *etc.* As announced in Example 2.4 we describe here another source of integrable cones: the probabilistic coherence spaces. Intuitively, they form a model of LL based on *discrete* but not necessarily finite probabilities. So their definition does not require measure theory.

We use $\overline{\mathbb{R}_{\geq 0}}$ for the completed real half-line, that is $\overline{\mathbb{R}_{\geq 0}} = \mathbb{R} \cup \{\infty\}$, considered as a semi-ring with multiplication satisfying $0\,\infty = 0$, which is the only possible choice since we want multiplication to be ω-continuous.

Let I be a set. If $i \in I$ we use $\mathsf{e}(i)$ for the element of $(\mathbb{R}_{\geq 0})^I$ such that $\mathsf{e}(i)_j = \delta_{i,j}$.

If $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ we define $\mathcal{P}^\perp \subseteq \overline{\mathbb{R}_{\geq 0}}^I$ by

$$\mathcal{P}^\perp = \{x' \in (\mathbb{R}_{\geq 0})^I \mid \forall x \in \mathcal{P} \ \sum_{i \in I} x_i x'_i \leq 1\}$$

and we use the notation $\langle x, x' \rangle = \sum_{i \in I} x_i x'_i$. As usual we have $\mathcal{P} \subseteq \mathcal{Q} \Rightarrow \mathcal{Q}^\perp \Rightarrow \mathcal{P}^\perp$ and $\mathcal{P} \subseteq \mathcal{P}^{\perp\perp}$, and as a consequence $\mathcal{P}^\perp = \mathcal{P}^{\perp\perp\perp}$. In other words, it is equivalent to say that $\mathcal{P} = \mathcal{P}^{\perp\perp}$ or to say that $\mathcal{P} = \mathcal{Q}^\perp$ for some $\mathcal{Q}$.

Theorem 10.1. *Let $\mathcal{P} \subseteq \overline{\mathbb{R}_{\geq 0}}^I$, one has $\mathcal{P} = \mathcal{P}^{\perp\perp}$ iff the following conditions hold*

- *$\mathcal{P}$ is convex (that is, if $x, y \in \mathcal{P}$ and $\lambda \in [0, 1]$ then $\lambda x + (1 - \lambda)y \in \mathcal{P}$)*
- *$\mathcal{P}$ is down-closed for the product order*

- *and, for each sequence $(x(n))_{n\in\mathbb{N}}$ of element of $\mathcal{P}$ which is increasing for the pointwise order, the pointwise lub $\in \overline{\mathbb{R}_{\geq 0}}^I$ of this sequence belongs to $\mathcal{P}$.*

A proof is outlined in [Gir04] and a complete proof can be found in [Ehr22].

Definition 10.2. A probabilistic coherence space (PCS) is a pair $\mathcal{X} = (|\mathcal{X}|, \mathsf{P}\mathcal{X})$ where $|\mathcal{X}|$ is a set which is at most countable[24] and $\mathsf{P}\mathcal{X} \subseteq (\mathbb{R}_{\geq 0})^{|\mathcal{X}|}$ satisfies

- $\mathsf{P}\mathcal{X} = \mathsf{P}\mathcal{X}^{\perp\perp}$
- for all $a \in |\mathcal{X}|$ there is $x \in \mathsf{P}\mathcal{X}$ such that $x_a > 0$
- and for all $a \in |\mathcal{X}|$ the set $\{x_a \mid x \in \mathsf{P}\mathcal{X}\} \subseteq \mathbb{R}_{\geq 0}$ is bounded.

The 2nd and 3rd conditions are required to keep the coefficients finite and are dual of each other.

Given two sets I, J, a vector $u \in \overline{\mathbb{R}_{\geq 0}}^I$ and a matrix $w \in \overline{\mathbb{R}_{\geq 0}}^{I\times J}$, we define $w \cdot u \in \overline{\mathbb{R}_{\geq 0}}^J$ by

$$w \cdot u = \Big(\sum_{i\in I} w_{i,j} u_i\Big)_{j\in J}$$

and then, given $v \in \overline{\mathbb{R}_{\geq 0}}^J$, observe that

$$\langle w \cdot u, v\rangle = \langle w, u \mathbin{\overline{\otimes}} v\rangle = \sum_{i\in I, j\in J} w_{i,j} u_i v_j \in \overline{\mathbb{R}_{\geq 0}}$$

where $u \mathbin{\overline{\otimes}} v \in \overline{\mathbb{R}_{\geq 0}}^{I\times J}$ is defined by $(u \mathbin{\overline{\otimes}} v)_{i,j} = u_i v_j$ (we use this notation here to avoid confusions with the tensor operations we have introduced for integrable cones).

Given $w_1 \in \overline{\mathbb{R}_{\geq 0}}^{I_1\times I_2}$ and $w_2 \in \overline{\mathbb{R}_{\geq 0}}^{I_2\times I_3}$, one defines $w_2\, w_1 \in \overline{\mathbb{R}_{\geq 0}}^{I_1\times I_3}$ (product of matrices written in reversed order) by

$$(w_2\, w_1)_{i_1,i_3} = \sum_{i_2\in I_2} (w_1)_{i_1,i_2} (w_2)_{i_2,i_3}\,.$$

It is easily checked that, given PCSs $\mathcal{X}$ and $\mathcal{Y}$, one defines a PCS $\mathcal{X} \multimap \mathcal{Y}$ by $|\mathcal{X} \multimap \mathcal{Y}| = |\mathcal{X}| \times |\mathcal{Y}|$ and

$$\mathsf{P}(\mathcal{X} \multimap \mathcal{Y}) = \{t \in (\mathbb{R}_{\geq 0})^{|\mathcal{X}\multimap\mathcal{Y}|} \mid \forall x \in \mathsf{P}\mathcal{X}\ t \cdot x \in \mathsf{P}\mathcal{Y}\}\,.$$

Indeed one can check that

$$\mathsf{P}(\mathcal{X} \multimap \mathcal{Y}) = \{x \mathbin{\overline{\otimes}} y' \mid x \in \mathsf{P}\mathcal{X} \text{ and } y' \in \mathsf{P}\mathcal{X}^{\perp}\}^{\perp}\,.$$

Then given $s \in \mathsf{P}(\mathcal{X} \multimap \mathcal{Y})$ and $t \in \mathsf{P}(\mathcal{Y} \multimap \mathcal{Z})$ one has

$$t\, s \in \mathsf{P}(\mathcal{X} \multimap \mathcal{Z})$$

and the diagonal matrix $\mathsf{Id} = (\delta_{a,a'})_{(a,a')\in|\mathcal{X}\multimap\mathcal{X}|}$ belongs to $\mathsf{P}(\mathcal{X} \multimap \mathcal{X})$. This defines the category **Pcoh** of probabilistic coherence spaces.

Let $t \in \mathbf{Pcoh}(\mathcal{X}, \mathcal{Y})$. We use $\mathsf{fun}(t) : \mathsf{P}\mathcal{X} \to \mathsf{P}\mathcal{Y}$ for the function defined by $\mathsf{fun}(t)(x) = t \cdot x$.

The orthogonal (or linear negation) $\mathcal{X}^{\perp}$ of a PCS $\mathcal{X}$ is defined by $|\mathcal{X}^{\perp}| = |\mathcal{X}|$ and $\mathsf{P}(\mathcal{X}^{\perp}) = (\mathsf{P}\mathcal{X})^{\perp}$ so that $\mathcal{X}^{\perp\perp} = \mathcal{X}$. We use $\perp$ for the PCS such that $|\perp| = \{*\}$ and $\mathsf{P}\perp = [0, 1]$. Setting $1 = \perp^{\perp}$ we have obviously $1 = \perp$ and $\mathcal{X}^{\perp}$ is trivially isomorphic to

[24] This countability assumption is crucial in the present setting, again because of our use of the monotone convergence theorem.

$\mathcal{X} \multimap \perp$. Under this iso, the function $\mathsf{fun}(x') : \mathsf{P}\mathcal{X} \to [0,1]$ associated with $x' \in \mathsf{P}\mathcal{X}^\perp$ is given by $\mathsf{fun}(x')(x) = \langle x, x' \rangle$.

This linear negation is a functor $\mathbf{Pcoh}^{\mathsf{op}} \to \mathbf{Pcoh}$, mapping $t \in \mathbf{Pcoh}(\mathcal{X}, \mathcal{Y})$ to it transpose $t^\perp$ defined by $(t^\perp)_{b,a} = t_{a,b}$.

Each PCS $\mathcal{X}$ induces a measurable cone $\mathsf{ic}(\mathcal{X})$ defined by $\underline{\mathsf{ic}(\mathcal{X})} = \{\lambda x \mid x \in \mathsf{P}\mathcal{X} \text{ and } \lambda \in \mathbb{R}_{\geq 0}\}$ with algebraic operations defined in the obvious pointwise way. Notice that if $t \in \mathbf{Pcoh}(\mathcal{X}, \mathcal{Y})$ we can extend $\mathsf{fun}(t)$ to a function $\underline{\mathsf{ic}(\mathcal{X})} \to \underline{\mathsf{ic}(\mathcal{Y})}$ by setting $\mathsf{fun}(t)(x) = \lambda^{-1}\,\mathsf{fun}(t)(\lambda x)$ for each $\lambda > 1$ such that $\lambda x \in \mathsf{P}\mathcal{X}$ (the function does not depend on the choice of λ).

The norm of this cone is defined by

$$\|x\|_{\mathsf{ic}(\mathcal{X})} = \sup_{x' \in \mathcal{B}\underline{\mathsf{ic}(\mathcal{X})'}} \langle x, x' \rangle = \inf\{\lambda > 0 \mid x \in \lambda \mathsf{P}\mathcal{X}\}$$

so that $\mathcal{B}\underline{\mathsf{ic}(\mathcal{X})} = \mathsf{P}\mathcal{X}$. The measurability structure of $\mathsf{ic}(\mathcal{X})$ is given by $\mathcal{M}_0^{\mathsf{ic}(\mathcal{X})} = \{\mathsf{fun}(x') \mid x' \in \mathsf{P}\mathcal{X}^\perp\}$ and $\mathcal{M}_X^{\mathsf{ic}(\mathcal{X})}$ is the set of all constant functions from $X \in \mathbf{Ar}$ to $\mathcal{M}_0^{\mathsf{ic}(\mathcal{X})}$.

Lemma 10.3. *Let $\mathcal{X}$ be a PCS and let $\mathcal{P} \subseteq (\mathbb{R}_{\geq 0})^{|\mathcal{X}|}$ be such that $\mathsf{P}\mathcal{X} = \mathcal{P}^\perp$. Then*

$$\|x\|_{\mathsf{ic}(\mathcal{X})} = \sup_{x' \in \mathcal{P}} \langle x, x' \rangle\,.$$

The proof is easy. Notice that for each $a \in |\mathcal{X}|$ one has $\mathsf{e}(a) \in \underline{\mathsf{ic}(\mathcal{X})}$ by the second condition in the definition of a PCS, and that $\|\mathsf{e}(a)\|$ is not necessarily equal to 1.

Lemma 10.4. *Let $\mathcal{X}$ be a PCS and $X \in \mathbf{Ar}$. A function $\beta : X \to \underline{\mathsf{ic}(\mathcal{X})}$ is a measurable path of the measurable cone $\mathsf{ic}(\mathcal{X})$ iff $\beta(X)$ is bounded and, for all $a \in |\mathcal{X}|$, the function $\boldsymbol{\lambda} r \in X \cdot \beta(r)_a : X \to \mathbb{R}_{\geq 0}$ is measurable.*

Proof. The $\Rightarrow$ direction results from the observation that, for each $a \in |\mathcal{X}|$ there is a $\lambda > 0$ such that $\lambda \mathsf{e}(a) \in \mathsf{P}\mathcal{X}^\perp$. For the $\Leftarrow$ direction let $\beta : X \to \mathsf{P}\mathcal{X}$ be such that the function $\boldsymbol{\lambda} r \in X \cdot \beta(r)_a$ is measurable for all $a \in |\mathcal{X}|$. Let $x' \in \mathsf{P}\mathcal{X}^\perp$, we must prove that $\varphi = \boldsymbol{\lambda} r \in X \cdot \langle \beta(r), x' \rangle$ is measurable. Since $|\mathcal{X}|$ is countable, this results from the monotone convergence theorem and from the fact that

$$\varphi(r) = \sum_{a \in |\mathcal{X}|} \beta(r)_a x'_a\,. \qquad \square$$

Theorem 10.5. *For each PCS $\mathcal{X}$ the measurable cone $\mathsf{ic}(\mathcal{X})$ is integrable.*

Proof. Let $\beta : X \to \mathsf{P}\mathcal{X}$ be a measurable path (by Lemma 10.4 this is equivalent to saying that $\beta_a = \boldsymbol{\lambda} r \in X \cdot \beta(r)_a$ is measurable $X \to \mathbb{R}_{\geq 0}$ for all $a \in |\mathcal{X}|$ since $\forall r \in X \;\; \|\beta(r)\| \leq 1$). Let $\mu \in \underline{\mathsf{FMeas}(X)}$. We define $x \in (\mathbb{R}_{\geq 0})^{|\mathcal{X}|}$ by

$$x_a = \int \beta_a(r) \mu(dr)$$

which is a well defined element of $\mathbb{R}_{\geq 0}$ since the function β_a is bounded by definition of a PCS. Let $x' \in \mathsf{P}\mathcal{X}^\perp$, we have, applying the monotone convergence theorem,

$$\begin{aligned}\langle x, x'\rangle &= \sum_{a\in|\mathcal{X}|}\Big(\int \beta_a(r)\mu(dr)\Big)x'_a \\ &= \sum_{a\in|\mathcal{X}|}\int \big(\beta_a(r))x'_a\big)\,\mu(dr) \\ &= \int \langle\beta(r), x'\rangle\mu(dr) \leq \|\mu\|\end{aligned}$$

so if $\lambda > 0$ is such that $\lambda\|\mu\| \leq 1$ we get $\langle\lambda x, x'\rangle \leq 1$ for all $x' \in \mathsf{P}\mathcal{X}^\perp$ so that $x \in \underline{\mathsf{ic}(\mathcal{X})}$. The equation $\langle x, x'\rangle = \int\langle\beta(r), x'\rangle\mu(dr)$ which holds for all $x' \in \mathsf{P}\mathcal{X}^\perp$ shows that x is the integral of β over μ by definition of $\mathcal{M}^{\mathsf{ic}(\mathcal{X})}$. □

Theorem 10.6. *If* $t \in \mathsf{P}(\mathcal{X} \multimap \mathcal{Y})$ *then* $\mathsf{fun}(t) \in \mathbf{ICones}(\mathsf{ic}(\mathcal{X}), \mathsf{ic}(\mathcal{Y}))$ *and extended to morphisms in that way, the operation* ic *is a full and faithful functor* $\mathbf{Pcoh} \to \mathbf{ICones}$*.*

Proof. The fact that $\mathsf{fun}(t)$ is linear and continuous is easy (the proof can be found in [DE11] for instance). Measurability and integral preservation of $\mathsf{fun}(t)$ boil down again to the monotone convergence theorem. Faithfulness results from the fact that t is completely determined by the action of $\mathsf{fun}(t)$ on the elements $\mathsf{e}(a)$ of $\underline{\mathsf{ic}(\mathcal{X})}$ (for all $a \in |\mathcal{X}|$; remember that indeed $\forall a \in |\mathcal{X}|\ \mathsf{e}(a) \in \underline{\mathsf{ic}(\mathcal{X})}$). Last let $f \in \mathbf{ICones}(\mathsf{ic}(\mathcal{X}), \mathsf{ic}(\mathcal{Y}))$. We define $t \in (\mathbb{R}_{\geq 0})^{|\mathcal{X} \multimap \mathcal{Y}|}$ by $t_{a,b} = f(\mathsf{e}(a))_b$. Given $x \in \mathsf{P}\mathcal{X}$ and $y' \in \mathsf{P}\mathcal{Y}^\perp$ we have

$$\begin{aligned}\langle t\cdot x, y'\rangle &= \sum_{a\in|\mathcal{X}|, b\in|\mathcal{Y}|} t_{a,b}x_a y'_b \\ &= \sum_{a\in|\mathcal{X}|, b\in|\mathcal{Y}|} f(\mathsf{e}(a))_b x_a y'_b \\ &= \sum_{b\in|\mathcal{Y}|} f(x)_b y'_b \quad \text{by linearity and continuity of } f \\ &= \langle f(x), y'\rangle \leq 1\end{aligned}$$

since $\|f\| \leq 1$, which shows that $t\cdot x \in \mathsf{P}\mathcal{Y}$ and hence $t \in \mathbf{Pcoh}(\mathcal{X}, \mathcal{Y})$. The equation $\langle t\cdot x, y'\rangle = \langle f(x), y'\rangle$ for all $y' \in \mathsf{P}\mathcal{Y}^\perp$ also shows that $t\cdot x = f(x)$ and hence the functor ic is full. □

We use $\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}$ for the tensor product operation in $\mathbf{Pcoh}$, that is $|\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}| = |\mathcal{X}| \times |\mathcal{Y}|$ and $\mathsf{P}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}) = \{x \mathbin{\overline{\otimes}} y \mid x \in \mathsf{P}\mathcal{X} \text{ and } y \in \mathsf{P}\mathcal{Y}\}^{\perp\perp} = \big(\mathcal{X} \multimap \mathcal{Y}^\perp\big)^\perp$.

Theorem 10.7. *If* $\mathcal{X}, \mathcal{Y}$ *are PCSs then* fun *is an iso from the integrable cone* $\mathsf{ic}(\mathcal{X} \multimap \mathcal{Y})$ *to the integrable cone* $\mathsf{ic}(\mathcal{X}) \multimap \mathsf{ic}(\mathcal{Y})$ *in* $\mathbf{ICones}$*, and this iso is natural in* $\mathcal{X}$ *and* $\mathcal{Y}$*.*

Proof sketch. We know by Theorem 10.6 that fun is an iso of cones. We need to prove that fun and fun^{-1} are measurable and that fun preserve integrals (then fun^{-1} also preserves integrals by injectivity of fun).

Let $X \in \mathbf{Ar}$ and $\eta \in \underline{\mathsf{Path}(X, \mathsf{ic}(\mathcal{X} \multimap \mathcal{Y}))}$, we show that

$$\mathsf{fun} \circ \eta \in \underline{\mathsf{Path}(X, \mathsf{ic}(\mathcal{X}) \multimap \mathsf{ic}(\mathcal{Y}))}$$

so let $Y \in \mathbf{Ar}$, $\beta \in \underline{\mathsf{Path}(Y, \mathsf{ic}(\mathcal{X}))}$ and $m \in \mathcal{M}_Y^{\mathsf{ic}(\mathcal{Y})}$ meaning that $m = \mathsf{fun}(y')$ for some $y' \in \mathsf{P}\mathcal{Y}^\perp$, we have, for all $(s, r) \in Y \times X$,

$$\begin{aligned}(\beta \triangleright m)(s, \mathsf{fun}(\eta(r))) &= \langle \mathsf{fun}(\eta(r))(\beta(s)), y' \rangle \\ &= \langle \eta(r) \cdot \beta(s), y' \rangle \\ &= \sum_{(a,b) \in |\mathcal{X}| \times |\mathcal{Y}|} \eta(r)_{a,b} \beta(s)_a y'_b\end{aligned}$$

and the function $\boldsymbol{\lambda}(s, r) \in Y \times X \cdot (\beta \triangleright m)(s, \mathsf{fun}(\eta(r)))$ is measurable as a countable sum of measurable functions. Conversely let now $\eta \in \underline{\mathsf{Path}(X, \mathsf{ic}(\mathcal{X}) \multimap \mathsf{ic}(\mathcal{Y}))}$, we must prove that $\mathsf{fun}^{-1} \circ \eta \in \underline{\mathsf{Path}(X, \mathsf{ic}(\mathcal{X} \multimap \mathcal{Y}))}$ so let $Y \in \mathbf{Ar}$ and $p \in \mathcal{M}_Y^{\mathsf{ic}(\mathcal{X} \multimap \mathcal{Y})}$, that is $p = \mathsf{fun}(z)$ for some $z \in \mathsf{P}(\mathcal{X} \multimap \mathcal{Y})^\perp = \mathsf{P}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}^\perp)$, we have

$$\begin{aligned}\boldsymbol{\lambda}(s, r) \in Y \times X \cdot p(s, \mathsf{fun}^{-1}(\eta(r))) &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot \langle z, \mathsf{fun}^{-1}(\eta(r)) \rangle \\ &= \boldsymbol{\lambda}(s, r) \in Y \times X \cdot \sum_{(a,b) \in |\mathcal{X}| \times |\mathcal{Y}|} z_{a,b} \eta(r)(\mathsf{e}(a))_b\end{aligned}$$

which is measurable as a countable sum of measurable functions since we know that for all a, b the function $\boldsymbol{\lambda} r \in X \cdot \eta(\mathsf{e}(a))_b$ is measurable by our assumption that η is a measurable path.

The fact that fun preserves integrals results from the pointwise definition of integration in $\mathsf{ic}(\mathcal{X}) \multimap \mathsf{ic}(\mathcal{Y})$. □

Theorem 10.8. *There is a natural isomorphism* $\varphi_{\mathcal{X},\mathcal{Y}} \in \mathbf{ICones}(\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y}), \mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}))$.

Proof sketch. The map $\boldsymbol{\lambda}(x, y) \in \underline{\mathsf{ic}(\mathcal{X})} \times \underline{\mathsf{ic}(\mathcal{Y})} \cdot x \mathbin{\overline{\otimes}} y$ is easily seen to be bilinear, ω-continuous, measurable and separately integrable so that we have an associated $\varphi_{\mathcal{X},\mathcal{Y}} \in \mathbf{ICones}(\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y}), \mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}))$ characterized by $\varphi_{\mathcal{X},\mathcal{Y}}(x \otimes y) = x \mathbin{\overline{\otimes}} y$. We define now $\psi_{\mathcal{X},\mathcal{Y}} : \mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}) \to \mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})$. First, given $(a, b) \in |\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}|$ we set $\psi_{\mathcal{X},\mathcal{Y}}(\mathsf{e}(a, b)) = \mathsf{e}(a) \otimes \mathsf{e}(b)$. Next given $z \in \underline{\mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})}$ such that $\mathsf{supp}(z) = \{(a, b) \in |\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y}| \mid z_{(a,b)} \neq 0\}$ is finite we set $\psi_{\mathcal{X},\mathcal{Y}}(z) = \sum_{(a,b) \in |\mathcal{X} \overline{\otimes} \mathcal{Y}|} z_{(a,b)} \mathsf{e}(a) \otimes \mathsf{e}(b)$ which is a well defined finite sum in the cone $\underline{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})}$. We contend that

$$\|\psi_{\mathcal{X},\mathcal{Y}}(z)\|_{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})} \leq \|z\|_{\mathsf{ic}(\mathcal{X} \overline{\otimes} \mathcal{Y})}$$

so let $\varepsilon > 0$ and assume without loss of generality that $\|z\| \leq 1$. By Proposition 3.11 there is $g \in \mathcal{B}\underline{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y}) \multimap \perp}$ such that $\|\psi_{\mathcal{X},\mathcal{Y}}(z)\|_{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})} \leq g(\psi_{\mathcal{X},\mathcal{Y}}(z)) + \varepsilon$. Let $h \in \mathcal{B}\underline{(\mathsf{ic}(\mathcal{X}) \multimap (\mathsf{ic}(\mathcal{Y}) \multimap \perp))}$ be the the bilinear morphism associated to g by the iso of Theorem 5.12. We have

$$\begin{aligned}g(\psi_{\mathcal{X},\mathcal{Y}}(z)) &= \sum_{(a,b) \in |\mathcal{X} \otimes \mathcal{Y}|} z_{(a,b)} g(\mathsf{e}(a) \otimes \mathsf{e}(b)) \\ &= \sum_{(a,b) \in |\mathcal{X} \otimes \mathcal{Y}|} z_{(a,b)} h(\mathsf{e}(a), \mathsf{e}(b)) \leq 1\end{aligned}$$

because $(h(\mathsf{e}(a), \mathsf{e}(b)))_{(a,b) \in |\mathcal{X} \otimes \mathcal{Y}|} \in \mathsf{P}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})^\perp$ by Theorem 10.7 and by our assumption that $\|z\| \leq 1$. So we have $\|\psi_{\mathcal{X},\mathcal{Y}}(z)\|_{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})} \leq 1 + \varepsilon$ and since this holds for all $\varepsilon > 0$ our contention is proven. Now let z be any element of $\underline{\mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})}$ and assume again that

$\|z\| \leq 1$. Let $(I_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite sets such that $\bigcup I_n = |\mathcal{X}| \times |\mathcal{Y}|$ and let $z(n) \in \underline{\mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})}$ be defined by

$$z(n)_{(a,b)} = \begin{cases} z_{(a,b)} & \text{if } (a,b) \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

so that the sequence $(z(n))_{n\in\mathbb{N}}$ is increasing and has z as lub in $\underline{\mathsf{ic}(\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})}$. The sequence $(\psi_{\mathcal{X},\mathcal{Y}}(z(n)))_{n\in\mathbb{N}}$ is increasing and all its elements have norm ≤ 1 since each $z(n)$ has finite support and norm ≤ 1 and hence it has a lub in $\underline{\mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y})}$. It is easy to check that this lub does not depend on the choice of the I_n's, so we can set $\psi_{\mathcal{X},\mathcal{Y}}(z) = \sup_{n\in\mathbb{N}} \psi_{\mathcal{X},\mathcal{Y}}(z(n))$ so that actually

$$\psi_{\mathcal{X},\mathcal{Y}}(z) = \sum_{(a,b)\in|\mathcal{X}|\times|\mathcal{Y}|} z_{a,b}\mathsf{e}(a) \otimes \mathsf{e}(b)\,.$$

The proof that $\psi_{\mathcal{X},\mathcal{Y}} \in \mathbf{ICones}(\mathsf{ic}((\mathcal{X} \mathbin{\overline{\otimes}} \mathcal{Y})), \mathsf{ic}(\mathcal{X}) \otimes \mathsf{ic}(\mathcal{Y}))$ follows the standard pattern and it is obvious that it is the inverse of $\varphi_{\mathcal{X},\mathcal{Y}}$. □

10.1. **More constructions.** We outline very briefly the additive and exponential constructions on PCSs. The categorical product $\mathcal{X} = \mathop{\&}_{i\in I} \mathcal{X}_i$ of a family $(\mathcal{X}_i)_{i\in I}$ of PCSs can be described by $|\mathcal{X}| = \bigcup_{i\in I} \{i\} \times |\mathcal{X}_i|$ and $x \in (\mathbb{R}_{\geq 0})^{|\mathcal{X}|}$ belongs to $\mathsf{P}\mathcal{X}$ if $\forall i \in I$ $\mathsf{pr}_i \cdot x \in \mathsf{P}\mathcal{X}_i$ where $\mathsf{pr}_i \in (\mathbb{R}_{\geq 0})^{|\mathcal{X}|\times|\mathcal{X}_i|}$ is given by $(\mathsf{pr}_i)_{(j,a),b} = \delta_{i,j}\delta_{a,b}$, so that $\mathsf{pr}_i \in \mathsf{P}(\mathcal{X}, \mathcal{X}_i)$ for each $i \in I$. Then it is easy to check that $(\mathcal{X}, (\mathsf{pr}_i)_{i\in I})$ is the categorical product of the family $(\mathcal{X}_i)_{i\in I}$ in $\mathbf{Pcoh}$ and that there is a natural isomorphism from $\mathop{\&}_{i\in I} \mathsf{ic}(\mathcal{X}_i)$ to $\mathsf{ic}(\mathcal{X})$.

For the coproduct $\mathcal{Y} = \oplus_{i\in I} \mathcal{X}_i$ we can take $\mathcal{Y} = (\mathop{\&}_{i\in I} \mathcal{X}_i^{\perp})^{\perp}$ so that $|\mathcal{Y}| = |\mathcal{X}|$, and $\mathsf{P}\mathcal{Y} = \{x \in \mathsf{P}\mathcal{X} \mid \sum_{i\in I} \|\mathsf{pr}_i \cdot x\| \leq 1\}$, equipped with injections $\mathsf{in}_i = \mathsf{pr}_i^{\perp} \in \mathbf{Pcoh}(\mathcal{X}_i, \mathcal{Y})$. So for instance the coproduct $1 \oplus 1$ has $\{0, 1\}$ as web, and $\mathsf{P}(1 \oplus 1) = \{u \in \mathbb{R}^2_{\geq 0} \mid u_0 + u_1 \leq 1\}$.

For the exponential, we introduce the following notations. Given $u \in (\mathbb{R}_{\geq 0})^I$ we define $u^! \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(I)}$ by $u^!_m = u^m = \prod_{i\in I} u_i^{m(i)}$. Given a PCS $\mathcal{X}$ we define a PCS $!\mathcal{X}$ by $|!\mathcal{X}| = \mathcal{M}_{\mathrm{fin}}(|\mathcal{X}|)$ and $\mathsf{P}(!\mathcal{X}) = \{x^! \mid x \in \mathsf{P}\mathcal{X}\}^{\perp\perp}$ so that $t \in \mathbf{Pcoh}(!\mathcal{X}, \mathcal{Y})$ means exactly that $t \in (\mathbb{R}_{\geq 0})^{\mathcal{M}_{\mathrm{fin}}(|\mathcal{X}|)\times|\mathcal{Y}|}$ and

$$\widehat{t}(x) = \Big(\sum_{m\in\mathcal{M}_{\mathrm{fin}}(|\mathcal{X}|)} t_{m,b} x^m\Big)_{b\in|\mathcal{Y}|} \in \mathsf{P}\mathcal{Y}$$

from which it is not hard to derive that the integrable cones $\mathsf{ic}(!\mathcal{X} \multimap \mathcal{Y})$ and $\mathsf{ic}(\mathcal{X}) \Rightarrow_{\mathsf{a}} \mathsf{ic}(\mathcal{Y})$ are isomorphic.

Remark 10.9. There is a morphism $\varphi_{\mathcal{X}} \in \mathbf{ICones}(!^{\mathsf{a}}\mathsf{ic}(\mathcal{X}), \mathsf{ic}(!\mathcal{X}))$ for all PCSs $\mathcal{X}$ such that $\varphi_{\mathcal{X}}(x^{!_{\mathsf{a}}}) = x^!$ for all $x \in \mathsf{P}\mathcal{X}$. This morphism is similar to the one of Theorem 10.8, we conjecture that it is an iso.

10.2. **Example: the Cantor Space as an equalizer of Pcoh morphisms.** Since **ICones** is a complete category, the equalizer of two parallel **Pcoh** morphisms is an integrable cone. As we shall see now, this cone needs not be a PCS which means that, contrarily to the larger category **ICones**, the category **Pcoh** is not complete. This example also shows that interesting “non discrete” cones arise as limits of diagrams in **Pcoh**.

Consider the PCS $\mathcal{S}$ such that $|\mathcal{S}| = \{0,1\}^{<\omega}$ is the set of finite sequences of 0's and 1's and where $x \in (\mathbb{R}_{\geq 0})^{|\mathcal{S}|}$ belongs to $\mathsf{P}\mathcal{S}$ if, for each $u \subseteq |\mathcal{S}|$ which is an antichain (meaning that if $s, s' \in u$ then $s \leq s' \Rightarrow s = s'$ where $\leq$ is the prefix order), one has $\sum_{s\in u} x_s \leq 1$. Since $\mathsf{P}\mathcal{S} = A^{\perp}$ where A is the set of all characteristic functions of antichains, $\mathcal{S}$ is a PCS (the second and third conditions of Definition 10.2 result from the observation that each singleton is an antichain). Notice that

$$\|x\|_{\mathcal{S}} = \sup_{x' \in A} \langle x, x' \rangle \tag{10.1}$$

by Lemma 10.3.

The PCS $\mathcal{S}$ is the “least solution” (in the sense explained in [DE11, ET19]) of the equation $\mathcal{S} = 1 \mathbin{\&} (\mathcal{S} \oplus \mathcal{S})$.

There is a morphism $\theta \in \mathbf{Pcoh}(\mathcal{S}, \mathcal{S})$ which is given by

$$\theta_{s,t} = \begin{cases} 1 & \text{if } s = ta \text{ for some } a \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

where we use simple juxtaposition for concatenation. Indeed given an antichain u and $x \in \mathsf{P}\mathcal{S}$ we have

$$\sum_{t \in u} (\theta \cdot x)_t = \sum_{s \in v} x_s \leq 1$$

where $v = \{sa \mid s \in u \text{ and } a \in \{0,1\}\}$ is an antichain since u is an antichain. Let C be the integrable cone which is the equalizer of θ and Id_S, considered as morphisms of **ICones** through the full and faithful functor ic.

Theorem 10.10. *The integrable cone C is isomorphic to $\mathsf{FMeas}(\mathcal{C})$ where $\mathcal{C}$ is the Cantor Space equipped with the Borel sets of its usual topology (the product topology of $\{0,1\}^{\omega}$ where $\{0,1\}$ has the discrete topology).*

Proof. We have

$$\underline{C} = \{x \in \mathsf{ic}(\mathcal{S}) \mid \theta \cdot x = x\},$$

that is, an element of $\underline{C}$ is an $x \in (\mathbb{R}_{\geq 0})^{|\mathcal{S}|}$ such that $x \in \mathsf{P}\mathcal{S}$ and

$$\forall s \in |\mathcal{S}| \quad x_s = x_{s0} + x_{s1}\,.$$

Given $s \in |\mathcal{S}|$ we set $\uparrow s = \{\alpha \in \mathcal{C} \mid s < \alpha\} \subseteq \mathcal{C}$, which is a clopen of $\mathcal{C}$. Let U be an open subset of $\mathcal{C}$, the set $\downarrow U$ of all $s \in |\mathcal{S}|$ which are minimal (for the prefix order) such that $\uparrow s \subseteq U$ is an antichain, and we have

$$U = \bigcup \{\uparrow s \mid s \in \downarrow U\} \tag{10.2}$$

by definition of the topology of $\mathcal{C}$. Given $x \in \underline{\mathsf{ic}(\mathcal{S})}$ we define a function $\mathsf{meas}(x) : \mathcal{O}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ on the open sets of $\mathcal{C}$ by

$$\mathsf{meas}(x)(U) = \sum_{s \in \downarrow U} x_s$$

and we have $\mathsf{meas}(x)(U) \leq \|x\|$ by our assumption that $x \in \underline{\mathsf{ic}(\mathcal{S})}$. This function $\mathsf{meas}(x)$ is additive (that is $\mathsf{meas}(x)(\bigcup_{i\in I} U_i) = \sum_{i\in I} \mathsf{meas}(x)(U_i)$ for each countable family $(U_i)_{i\in I}$ of pairwise disjoint open subsets of $\mathcal{C}$). And so $\mathsf{meas}(x)$ extends to a uniquely defined finite measure on the Borel sets of the Cantor Space, that is to an element of $\underline{\mathsf{FMeas}(\mathcal{C})}$. Notice that $\mathsf{meas} : \underline{\mathsf{ic}(\mathcal{S})} \to \mathsf{FMeas}(\mathcal{C})$ is linear and satisfies $\|\mathsf{meas}(x)\| = x_{\langle\rangle} \leq \|x\|$ where $\langle\rangle \in |\mathcal{S}|$ is the empty sequence.

Let $\mu \in \underline{\mathsf{FMeas}(\mathcal{C})}$, we define $\mathsf{rep}(\mu) \in (\mathbb{R}_{\geq 0})^{|\mathcal{S}|}$ by $\mathsf{rep}(\mu)_s = \mu(\uparrow s)$. Given an antichain $u \subseteq |\mathcal{S}|$ notice that the clopens $(\uparrow s)_{s\in u}$ are pairwise disjoint and that $U = \bigcup_{s\in u} \uparrow s$ is open and hence measurable, so, since μ is a measure, we have

$$\sum_{s\in u} \mathsf{rep}(\mu)_s = \sum_{s\in u} \mu(\uparrow s) = \mu\Big(\bigcup_{s\in u} \uparrow s\Big) = \mu(U) \leq \mu(\mathcal{C})\,.$$

Since this holds for each antichain u we have shown that $\mathsf{rep}(\mu) \in \mathsf{P}\mathcal{S}$. Notice that for each $s \in |S|$ we have $\uparrow s = \uparrow s0 \cup \uparrow s1$ and that this union is disjoint, so that $\mu(\uparrow s) = \mu(\uparrow s0) + \mu(\uparrow s1)$ since μ is a measure, that is $\mathsf{rep}(\mu) \in \underline{C}$. Notice also that the function rep is linear and satisfies $\mathsf{rep}(\mu) \leq \|\mu\|$ since for each antichain $u \subseteq |\mathcal{S}|$ one has $\sum_{s\in u} \mathsf{rep}(\mu)_s = \sum_{s\in u} \mu(\uparrow s) = \mu(\bigcup_{s\in u} \uparrow s) \leq \mu(\mathcal{C}) = \|\mu\|$.

We prove that the functions meas and rep are inverse of each other. Let first $x \in \underline{C}$, we have, for all $s \in |\mathcal{S}|$,

$$\mathsf{rep}(\mathsf{meas}(x))_s = \mathsf{meas}(x)(\uparrow s) = x_s$$

since $\downarrow\uparrow s = \{s\}$. Let now $\mu \in \mathsf{FMeas}(\mathcal{C})$ and let $U \in \mathcal{O}(\mathcal{C})$ we have

$$\mathsf{meas}(\mathsf{rep}(\mu))(U) = \sum_{s\in\downarrow U} \mathsf{rep}(\mu)_s = \sum_{s\in\downarrow U} \mu(\uparrow s) = \mu(U)$$

by Formula (10.2). It follows that $\mathsf{meas}(\mathsf{rep}(\mu)) = \mu$.

It follows that meas and rep define an order isomorphism between $\underline{\mathsf{FMeas}(\mathcal{C})}$ and $\underline{C}$ and therefore are ω-continuous (this uses also the fact that $\|\mathsf{meas}(x)\| = \|x\|$ since $\|x\| = \|\mathsf{rep}(\mathsf{meas}(x))\| \leq \|\mathsf{meas}(x)\|$ and similarly $\|\mathsf{rep}(\mu)\| = \|\mu\|$).

The fact that the map $\mathsf{meas} : \underline{\mathsf{ic}(\mathcal{S})} \to \underline{\mathsf{FMeas}(\mathcal{C})}$ is linear is measurable and integrable results as usual from the monotone convergence theorem. So we have $\mathsf{meas} \in \mathbf{ICones}(\mathsf{ic}(\mathcal{S}), \mathsf{FMeas}(\mathcal{C}))$ and hence by restriction $\mathsf{meas} \in \mathbf{ICones}(\underline{C}, \mathsf{FMeas}(\mathcal{C}))$ since $\|\mathsf{meas}(x)\| = \mathsf{meas}(x)(\mathcal{C}) \leq \|x\|$ for all $x \in \underline{C}$. Again, checking that rep is measurable and integrable is routine; as an example let us prove the last property so let $X \in \mathbf{Ar}$ and let $\kappa \in \underline{\mathsf{Path}(X, \mathsf{FMeas}(\mathcal{C}))}$. Let $m \in \mathcal{M}_0^C$, that is $m = \mathsf{fun}(x')$ for some $x' \in \mathsf{P}\mathcal{S}^\perp$. We have

$$\begin{aligned}
m\Big(\int^C \mathsf{rep}(\kappa(r))\mu(dr)\Big) &= \sum_{s\in|\mathcal{S}|} x'_s\Big(\int^{\mathsf{ic}(S)} \mathsf{rep}(\kappa(r))\mu(dr)\Big)_s \\
&= \sum_{s\in|\mathcal{S}|} x'_s \int \mathsf{rep}(\kappa(r))_s\mu(dr) \\
&= \int \big(\sum_{s\in|\mathcal{S}|} x'_s\kappa(r)(\uparrow s)\big)\mu(dr) \\
&= \int m(\mathsf{rep}(\kappa(r)))\mu(dr)\,.
\end{aligned}$$

By Formula (10.1) we have $\|\mathsf{rep}\| \leq 1$ and hence $\mathsf{rep} \in \mathbf{ICones}(\mathsf{FMeas}(\mathcal{C}), C)$. □

Conclusion

Elaborating on earlier work by the first author (together with Michele Pagani and Christine Tasson) on a denotational semantics based on measurable cones and by the second author on a notion of convex QBS where integration is the fundamental algebraic operation [Geo21], we have developed a theory of integration for measurable cones, introducing the category of *integrable cones* and of linear morphisms preserving integrals. We have shown that this category is a model of Intuitionistic LL featuring two exponential comonads; for defining the tensor product and the exponentials we have used the special adjoint functor theorem which avoids providing explicit combinatorial constructions of these objects.

The construction is parameterized by a small full subcategory **Ar** of the category of measurable spaces and measurable functions. The model obtained in that way has many pleasant properties.

- It contains the category of probabilistic coherence spaces as a full subcategory.
- It contains the category whose objects are those of **Ar** and whose morphisms are the substochastic kernels as a full subcategory.
- For both exponentials, the associated Eilenberg Moore category contains **Ar** as a full subcategory, if we assume that all the objects of **Ar** are standard Borel spaces which btw. is a very natural and harmless requirement.

The two latter properties strongly rely on the fact that the morphisms of the underlying linear category preserve integrals. The last one means that **Ar** can be considered as a category of basic data-types (the objects of **Ar**) and basic operations on them (the morphisms of **Ar**).

In future work we will explain how this model can be used for interpreting call-by-value or even call-by-push-value probabilistic functional programming languages with continuous data-types (interpreted as the aforementioned coalgebras) as well as recursive types.

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