DECIDABILITY OF ONE-CLOCK WEIGHTED TIMED GAMES WITH ARBITRARY WEIGHTS

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ABSTRACT. Weighted Timed Games (WTG for short) are the most widely used model to describe controller synthesis problems involving real-time issues. Unfortunately, they are notoriously difficult, and undecidable in general. As a consequence, one-clock WTGs have attracted a lot of attention, especially because they are known to be decidable when only non-negative weights are allowed. However, when arbitrary weights are considered, despite several recent works, their decidability status was still unknown. In this paper, we solve this problem positively and show that the value function can be computed in exponential time (if weights are encoded in unary).

1. INTRODUCTION

The task of designing programs is becoming more and more involved. Developing formal methods to ensure their correctness is thus an important challenge. Programs sensitive to real-time allow one to measure time elapsing in order to take decisions. The design of such programs is a notoriously difficult problem because timing issues may be intricate, and a posteriori debugging such issues is hard. The model of timed automata [AD94] has been widely adopted as a natural and convenient setting to describe real-time systems. This model extends finite-state automata with finitely many real-valued variables, called clocks, and transitions can check clocks against lower/upper bounds and reset some clocks.

Model-checking aims at verifying whether a real-time system modelled as a timed automaton satisfies some desirable property. Instead of verifying a system, one can try to synthesise one automatically. A successful approach, widely studied during the last decade, is one of the two-player games. In this context, a player represents the *controller*, and an antagonistic player represents the *environment*. Being able to identify a winning strategy

of the controller, i.e. a recipe on how to react to uncontrollable actions of the environment, consists in the synthesis of a system that is guaranteed to be correct by construction.

In the realm of real-time systems, timed automata have been extended to timed games [AM99] by partitioning locations between the two players. In a turn-based fashion, the player that must play proposes a delay and a transition. The controller aims at satisfying some ω-regular objective however the environment player behaves. Deciding the winner in such turn-based timed games has been shown to be EXPTIME-complete [JT07], and a symbolic algorithm allowing tool development has been proposed [BCD$^+$07].

In numerous application domains, in addition to real-time, other quantitative aspects have to be taken into account. For instance, one could aim at minimising the energy used by the system. To address this quantitative generalisation, weighted (aka priced) timed games (WTG for short) have been introduced [BCFL04, BFH$^+$01]. Locations and transitions are equipped with integer weights, allowing one to define the accumulated weight associated with a play. In this context, one focuses on a simple, yet natural, reachability objective: given some target location, the controller, that we now call Min, aims at ensuring that it will be reached while minimising the accumulated weight. The environment, that we now call Max, has the opposite objective: avoid the target location or, if not possible, maximise the accumulated weight. This allows one to define the value of the game as the minimal weight Min can guarantee. The associated decision problem asks whether this value is less than or equal to some given threshold.

In the earliest studies of this problem, some semi-decision procedures have been proposed to approximate this value for WTGs with non-negative weights [ABM04, BCFL04]. In addition, a subclass of strictly non-Zeno cost WTGs for which their algorithm terminates has been identified in [BCFL04]. This approximation is motivated by the undecidability of the problem, first shown in [BBR05]. This restriction has recently been lifted to WTGs with arbitrary weights in [BMR17].

An orthogonal research direction to recover decidability is to reduce the number of clocks and more precisely to focus on *one-clock WTGs*. Though restricted, a single clock is often sufficient for modelling purposes. When only non-negative weights are considered, decidability has been proven in [BLMR06] and later improved in [Rut11, HIM13] to obtain exponential time algorithms. Despite several recent works, the decidability status of one-clock WTGs with arbitrary weights is still open. In the present paper, we show the decidability of the value problem for this class. More precisely, we prove that the value function can be computed in exponential time (if weights are encoded in unary and not in binary).

Before exposing our approach, let us briefly recap the existing results. Positive results obtained for one-clock WTGs with non-negative weights are based on a reduction to so-called *simple WTG*, where the underlying timed automata contain no guard, no reset, and the clock value along with the execution exactly spans the $[0, 1]$ interval. In simple WTG, it is possible to compute with various techniques inspired by the paradigm of value iteration, adapted by a computation of the whole value function starting at time 1 and going back in time until 0 [BLMR06, Rut11, HIM13], leading to an exponential-time algorithm. A PSPACE lower-bound is also known for related decision problems [FIJS20].

Recent works extend the positive results of simple WTGs to arbitrary weights [BGH$^+$15, BGH$^+$22], yielding decidability of *reset-acyclic* one-clock WTGs with arbitrary weights, with a *pseudo-polynomial time* complexity (that is polynomial if weights are encoded in unary). It is also explained how to extend the result to all WTGs where no cyclic play containing a reset may have a negative weight arbitrarily close to 0. Moreover, it is shown that Min needs

memory to play (almost-)optimally, in a very structured way: Min uses *switching strategies*, that are composed of two memoryless strategies, the second one being triggered after a given (pseudo-polynomial) number κ of steps.

The crucial ingredient to obtain decidability for non-negative weights or reset-acyclic weighted timed games is to limit the number of reset transitions taken along a play. This is no longer possible in presence of cycles of negative weights containing a reset. There, Min may need to iterate cycles for a number κ of times that depends on the desired precision ε on the value (to play ε-optimally, Min needs to cycle $O(1/\varepsilon)$ times, see Example 2.3). To rule out these annoying behaviours, we rely on three main ingredients:

- As there is a single clock, a cyclic path ending with a reset corresponds to a cycle of configurations. We define the *value* of such a cycle, that allows us to identify which player may benefit from iterating it.
- Using the classical region graph construction, we prove stronger properties on the value function (it is continuous on the closure of region intervals). This allows us to prove that Max has an optimal memoryless strategy that avoids cycles whose value is negative (Section 3).
- We introduce in Section 4 a partial unfolding of the game, so as to obtain an acyclic WTG, for which decidability is known. To do so, we rely on the existence of (almost-)optimal switching strategies for Min, allowing us to limit the depth of exploration. Also we keep track of cycles encountered and handle them according to their value. We transport the previous result on the existence of a "smart" optimal strategy for Max in the context of this unfolding in Section 5. This allows us to show that the unfolding has the same value as the original WTG in Section 6.

We finally wrap up the proof in Section 7. Along the way, we crucially need that the value function is obtained as a fixed point (indeed the greatest one) of an operator that was already used in many contributions before [ABM04, BCFL04, BMR17]. We formally show this statement in Section 8.

This article is an extended version of the conference article [MPR22], with respect to which we have incorporated the full proofs of the result (in particular Section 8 is entirely new), in a clarified way.

2. WEIGHTED TIMED GAMES

2.1. **Definitions.** We only consider weighted timed games with a single clock, denoted by x. The valuation ν of this clock is a non-negative real number, i.e. $\nu \in \mathbb{R}_{\geq 0}$. On such a clock, transitions of the timed games will be able to check some interval constraints, called *guards*, on the clock, i.e. intervals I of real values with closed or open bounds that are natural numbers (or $+\infty$). For every interval I having finite bounds a and b, we denote its closure by $\bar{I} = [a, b]$.

Definition 2.1. A *weighted timed game* (WTG for short) is a tuple $\mathcal{G} = \langle Q_{\mathsf{Min}}, Q_{\mathsf{Max}}, Q_t, Q_u, \Delta, \mathsf{wt}, \mathsf{wt}_t \rangle$ with

- $Q = Q_{\mathsf{Min}} \uplus Q_{\mathsf{Max}} \uplus Q_t$ a finite set of locations split between players Min and Max, and a set of target locations;
- $Q_u \subseteq Q_{\mathsf{Min}} \uplus Q_{\mathsf{Max}}$ a set of *urgent* locations where time cannot be delayed;

- Δ a finite set of transitions each of the form (q, I, R, w, q'), with q and q' two locations (with $q \notin Q_t$), I an interval, $w \in \mathbb{Z}$ the weight of the transition, and R being either $\{x\}$ when the clock must be reset, or $\emptyset$ when it does not;
- $\mathsf{wt} \colon Q \to \mathbb{Z}$ a weight function associating an integer weight with each location: for uniformisation of the notations, we extend this weight function to also associate with each transition the weight it contains, i.e. $\mathsf{wt}\big((q, I, R, w, q')\big) = w$;
- and $\mathsf{wt}_t \colon Q_t \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ a function mapping each target configuration to a final weight, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

We note that our definition is not usual. Indeed, the addition of final weights in WTGs is not standard, but we use it in the process of solving those games: in any case, it is possible to simply map a given target location to the weight 0, allowing us to recover the standard definitions of the literature. The presence of urgent locations is also unusual: in a timed automaton with several clocks, urgency can be modelled with an additional clock u that is reset just before entering the urgent location and with constraints $u \in [0, 0]$ on outgoing transitions. However, when limiting the number of clocks to one, we regain modelling capabilities by allowing for such urgent locations. The weight of an urgent location is never used and will thus not be given in drawings: instead, urgent locations will be displayed with u inside.

Given a WTG $\mathcal{G}$, its semantics, denoted by $[\![\mathcal{G}]\!]$, is defined in terms of a game on an infinite transition system whose vertices are *configurations* of $\mathcal{G}$, i.e. the set of pairs $(q, \nu) \in Q \times \mathbb{R}_{\geq 0}$. Configurations are split into players according to the location q, and a configuration (q, ν) is a target if $q \in Q_t$. To encode the delay spent in the current location before firing a certain transition, edges linking vertices will be labelled by elements of $\mathbb{R}_{\geq 0} \times \Delta$. Formally, for every delay $t \in \mathbb{R}_{\geq 0}$, transition $\delta = (q, I, R, w, q') \in \Delta$ and valuation ν, we add a labelled edge $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$ if

- $\nu + t \in I$;
- $\nu' = 0$ if $R = \{x\}$, and $\nu' = \nu + t$ otherwise;
- and $t = 0$ if $q \in Q_u$.

This edge is given a weight $t \times \mathsf{wt}(q) + \mathsf{wt}(\delta)$ taking into account discrete and continuous weights. Without loss of generality by applying classical techniques [BPDG98, Lemma 5], we suppose the absence of deadlocks except on target locations, i.e. for each location $q \in Q \backslash Q_t$ and valuation ν, there exist $t \in \mathbb{R}_{\geq 0}$ and $\delta = (q, I, R, w, q') \in \Delta$ such that $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$ and no transitions start from Q_t.

Paths and plays. We call *path* a finite or infinite sequence of consecutive transitions $q_0 \xrightarrow{\delta_0} q_1 \xrightarrow{\delta_1} \cdots$ where $\delta_0, \delta_1, \ldots \in \Delta$ and $q_0, q_1, \ldots \in Q$. We sometimes denote $\pi_1 \cdot \pi_2$ the concatenation of a finite path π_1 ending in location q and another path π_2 starting in location q. We call *play* a finite or infinite sequence of edges in the semantics of the game $(q_0, \nu_0) \xrightarrow{t_0, \delta_0} (q_1, \nu_1) \xrightarrow{t_1, \delta_1} (q_2, \nu_2) \cdots$. A play is said to *follow* a path if both use the same sequence of transitions. We let FPaths (resp. FPlays) be the set of all finite paths (resp. plays).

Given a finite path π or a finite play ρ, we let $|\pi|$ or $|\rho|$ its *length* which is its number of transitions (or edges), and $|\pi|_\delta$ or $|\rho|_\delta$ the number of occurrences of a given transition δ in π (or ρ). More generally, for a play ρ and a set A of transitions, we let $|\rho|_A$ be the number of occurrences of all transitions from A in ρ, i.e. $|\rho|_A = \sum_{\delta \in A} |\rho|_\delta$. We also let $\mathsf{last}(\pi)$
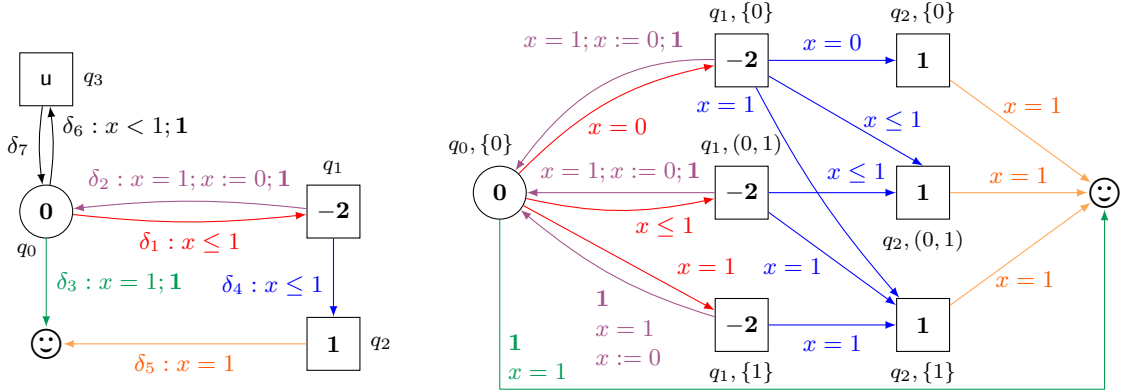
Figure 1: On the left, a WTG with a cyclic path of weight $[-1, 1]$ containing a reset. Its weights are depicted in bold font, and the missing ones are 0. Locations belonging to Min (resp. Max) are depicted by circles (resp. squares). Transitions that contain the reset of x are labelled with $x := 0$. The intervals of guards are described, as classically done in timed automata, via equality or inequality constraints on the unique clock x. The target location is ☺, whose final weight function is zero. Location q_3 is urgent. On the right, the restriction of its closure to locations q_0, q_1, q_2 and ☺.

and $\mathsf{last}(\rho)$ be the last location or configuration. Finally, $\mathsf{FPaths}_{\mathsf{Max}}$ (resp. $\mathsf{FPaths}_{\mathsf{Min}}$) and $\mathsf{FPlays}_{\mathsf{Max}}$ (resp. $\mathsf{FPlays}_{\mathsf{Min}}$) denote the subset of finite paths or plays whose last element belong to player Max (resp. Min).

A finite play $\rho = (q_0, \nu_0) \xrightarrow{t_0, \delta_0} (q_1, \nu_1) \cdots (q_k, \nu_k)$ can be associated with the cumulated weight of the edges it traverses:

$$\mathsf{wt}_\Sigma(\rho) = \sum_{i=0}^{k-1} \left(\mathsf{wt}(\ell_i) \times t_i + \mathsf{wt}(\delta_i) \right).$$

A *maximal play* ρ (either infinite or trapped in a deadlock that is necessarily a target configuration) is associated with a *payoff* $\mathsf{P}(\rho)$ as follows: the payoff of an infinite play (meaning that it never visits a target location) is $+\infty$, while the payoff of a finite play, thus ending in a target configuration (q, ν), is $\mathsf{wt}_\Sigma(\rho) + \mathsf{wt}_t(q, \nu)$. By [BFH$^+$01], the set of weights of plays following a given path is known to be an interval of values. Moreover, when all the guards along the path are closed intervals, this interval has closed bounds.

A *cyclic path* is a finite path that starts and ends in the same location. A *cyclic play* is a finite play that starts and ends in the same configuration: it necessarily follows a cyclic path, but the reverse might not be true since some non-cyclic plays can follow a cyclic path (if they do not end in the same valuation as the one in which they start).

Example 2.2. Plays that follow the cyclic path $\pi = q_0 \xrightarrow{\delta_1} q_1 \xrightarrow{\delta_2} q_0$ of the WTG depicted on the left in Figure 1 have weight between -1 (with the play $(q_0, 0) \xrightarrow{0, \delta_1} (q_1, 0) \xrightarrow{1, \delta_2} (q_0, 0)$) and 1 (with the play $(q_0, 0) \xrightarrow{1, \delta_1} (q_1, 1) \xrightarrow{0, \delta_2} (q_0, 0)$), so $\mathsf{wt}_\Sigma(\pi) = [-1, 1]$. Another cyclic path is $\pi' = q_0 \xrightarrow{\delta_6} q_3 \xrightarrow{\delta_7} q_0$ which goes via an urgent location. In particular, all plays that

follow this one are of the form $(q_0, \nu) \xrightarrow{t, \delta_6} (q_3, \nu + t) \xrightarrow{0, \delta_7} (q_0, \nu + t)$ with ν and $\nu + t$ less than 1: they all have weight 1.

Strategies and value. A *strategy* gives a set of choices to one of the players. A strategy of Min is a function $\sigma \colon \mathsf{FPlays}_{\mathsf{Min}} \to \mathbb{R}_{\geq 0} \times \Delta$ mapping each finite play ρ whose last configuration belongs to Min to a pair (t, δ) of delay and transition, such that the play ρ can be extended by an edge labelled with (t, δ). A play ρ is said to be *conforming* to a strategy σ if the choice made in ρ at each location of Min is the one prescribed by σ. Moreover, a finite path π is said to be conforming to a strategy σ if there exists a finite play following π that is conforming to σ. Similar definitions hold for strategies τ of Max. We let $\mathsf{Strat}_{\mathsf{Min}, \mathcal{G}}$ (resp., $\mathsf{Strat}_{\mathsf{Max}, \mathcal{G}}$) be the set of strategies of Min (resp., Max) in the game $\mathcal{G}$, or simply $\mathsf{Strat}_{\mathsf{Min}}$ and $\mathsf{Strat}_{\mathsf{Max}}$ if the game is clear from the context: we will always use letters σ and τ to differentiate from strategies of Min and Max.

A strategy is said to be *memoryless* if it only depends on the last configuration of the plays. More formally, Max's strategy τ is memoryless if for all plays ρ and ρ' such that $\mathsf{last}(\rho) = \mathsf{last}(\rho')$, we have $\tau(\rho) = \tau(\rho')$.

After both players have chosen their strategies σ and τ, each initial configuration (q, ν) gives rise to a unique maximal play that we denote by $\mathsf{Play}((q, \nu), \sigma, \tau)$. The *value* of the configuration (q, ν) is then obtained by letting players choose their strategies as they want, first Min and then Max, or vice versa since WTGs are known to be determined [BGH$^+$22]:

$$\mathsf{Val}_{\mathcal{G}}(q, \nu) = \sup_{\tau} \inf_{\sigma} \mathsf{P}(\mathsf{Play}((q, \nu), \sigma, \tau)) = \inf_{\sigma} \sup_{\tau} \mathsf{P}(\mathsf{Play}((q, \nu), \sigma, \tau)) \,.$$

The value of a strategy σ of Min (symmetric definitions can be given for strategies τ of Max) is defined as:

$$\mathsf{Val}_{\mathcal{G}}^{\sigma}(q, \nu) = \sup_{\tau} \mathsf{P}(\mathsf{Play}((q, \nu), \sigma, \tau)) \,.$$

Then, a strategy σ^* of Min is *optimal* if, for all initial configurations (q, ν),

$$\mathsf{Val}_{\mathcal{G}}^{\sigma^*}(q, \nu) \leq \mathsf{Val}_{\mathcal{G}}(q, \nu) \,.$$

Because of the infinite nature of the timed games, optimal strategies may not exist: for example, a player may want to let time elapse as much as possible, but with a delay $t < 1$ because of a strict guard, preventing them to obtain the optimal value. We will see in Example 3.1 that this situation can even happen when all guards contain only *closed* comparisons. We naturally extend the definition to *almost-optimal strategies*, taking into account small possible errors: we say that a strategy σ^* of Min is *ε-optimal* if, for all initial configurations (q, ν),

$$\mathsf{Val}_{\mathcal{G}}^{\sigma^*}(q, \nu) \leq \mathsf{Val}_{\mathcal{G}}(q, \nu) + \varepsilon \,.$$

Example 2.3. We have seen, in Example 2.2, that in q_0 (on the left in Figure 1), Min has no interest in following the cycle $q_0 \xrightarrow{\delta_6} q_3 \xrightarrow{\delta_7} q_0$ since all plays following it have weight 1. Jumping directly to the target location via δ_3 leads to a weight of 1. But Min can do better: from valuation 0, by jumping to q_1 after a delay of $t \leq 1$, it leaves a choice to Max to either jump to q_2 and the target leading to a total weight of $1 - t$, or to loop back in q_0 thus closing a cyclic play of weight $-2(1 - t) + 1 = 2t - 1$. If t is chosen too close to 1, the value of the cycle is greater than 1, and Max will benefit from it by increasing the total weight. If t is chosen smaller than $1/2$, the weight of the cycle is negative, and Max will prefer to go to the target to obtain a weight $1 - t$ close to 1, not very beneficial to Min. Thus, Min prefers

to play just above $1/2$, for example at $1/2 + \varepsilon$. In this case, Max will choose to go to the target with a total weight of $1/2 + \varepsilon$. The value of the game, in configuration $(q_0, 0)$, is thus $\mathsf{Val}_{\mathcal{G}}(q_0, 0) = 1/2$. Not only Min does not have an optimal strategy (but only ε-optimal ones, for every $\varepsilon > 0$), but needs memory to play ε-optimally, since Min cannot play *ad libitum* transition δ_2 with a delay $1/2 - \varepsilon$: in this case, Max would prefer staying in the cycle, thus avoiding the target. Thus, Min will play the transition δ_1 at least $1/4\varepsilon$ times so that the cumulated weight of all the cycles is below $-1/2$, in which case Min can safely use transition δ_1 still earning $1/2$ in total.

Clock bounding. Seminal works in WTGs [ABM04, BCFL04] have assumed that clocks are bounded. This is known to be without loss of generality for (weighted) timed automata [BFH+01, Theorem 2]: it suffices to replace transitions with unbounded delays with self-loop transitions periodically resetting the clock. We do not know if it is the case for the WTGs defined above since this technique cannot be directly applied. This would give too much power to player Max that would then be allowed to loop in a location (and thus avoid the target) where an unbounded delay could originally be taken before going to the target. In [BCFL04], since the WTGs are concurrent, this new power of Max is compensated by always giving Min a chance to move outside of such a situation. Trying to detect and avoid such situations in our turn-based case seems difficult in the presence of negative weights since the opportunities of Max crucially depend on the configurations of value $-\infty$ that Min could control afterwards: the problem of detecting such configurations (for all classes of WTGs) is undecidable [Bus19, Prop. 9.2], which is additional evidence to motivate the decision to focus only on bounded WTGs. We thus suppose from now on that the clock is bounded by a constant $M \in \mathbb{N}$, i.e. every transition of the WTG is equipped with the interval $[0, M]$.

Regions. In the following, we rely on the crucial notion of regions introduced in the seminal work on timed automata [AD94] to obtain a partition of the set of valuations $[0, M]$. To reduce the number of regions concerning the more usual one of [AD94] in the case of a single clock, we define regions by a construction inspired by Laroussinie, Markey, and Schnoebelen [LMS04]. Formally, we call regions of $\mathcal{G}$ the set

$$\mathsf{Reg}_{\mathcal{G}} = \{(M_i, M_{i+1}) \mid 0 \le i \le k-1\} \cup \{\{M_i\} \mid 0 \le i \le k\}$$

where $M_0 = 0 < M_1 < \cdots < M_k$ are all the endpoints of the intervals appearing in the guards of $\mathcal{G}$ (to which we add 0 if needed). As usual, if I is a region, then the time successor of valuations in I forms a finite union of regions, and the reset $I[x := 0] = \{0\}$ is also a region. A region I' is said to be a *time successor* of the region I if there exists $\nu \in I$, $\nu' \in I'$, and $t > 0$ such that $\nu' = \nu + t$.

Final weights. We also assume that the final weight functions satisfy a sufficient property ensuring that they can be encoded in finite space: we require final weight functions to be piecewise affine with a finite number of pieces and continuous on each region. More precisely, we assume that cutpoints (the value of the clock in-between two affine pieces) and coefficients are rational and given in binary.

We let W_{loc}, W_{tr} and W_{fin} be the maximum absolute value of weights of locations, transitions and final functions, i.e.

$$W_{\mathsf{loc}} = \max_{q \in Q} |\mathsf{wt}(q)| \qquad W_{\mathsf{tr}} = \max_{\delta \in \Delta} |\mathsf{wt}(\delta)| \qquad W_{\mathsf{fin}} = \sup_{\substack{q \in Q_t \\ \mathsf{wt}_t(q, \cdot) \notin \{+\infty, -\infty\}}} \sup_{\nu} |\mathsf{wt}_t(q, \nu)|$$

We also let W be the maximum of W_{loc}, W_{tr}, and W_{fin}.

2.2. **Fixpoint characterisation of the value.** The value function $\mathsf{Val}_{\mathcal{G}}\colon Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ of WTGs can sometimes be characterised as a fixpoint (and even the greatest fixpoint) of some operator $\mathcal{F}$ defined as follows: for all configurations (q, ν) and all mappings $X\colon Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$, we let:

$$\mathcal{F}(X)(q, \nu) = \begin{cases} \mathsf{wt}_t(q, \nu) & \text{if } q \in Q_t \\ \inf_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(q', \nu')\big) & \text{if } q \in Q_{\mathsf{Min}} \\ \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(q', \nu')\big) & \text{if } q \in Q_{\mathsf{Max}} \end{cases}$$

This operator is the basis of the decidability result for (many-clocks) WTGs with non-negative weights with some divergence conditions on the weight of cycles [BCFL04], since the value iteration algorithm that iterates the operator over an initial well-chosen function is supposed to converge (in finite time) towards the desired value. However the proof given in [BCFL04, Bou16] of the claim that $\mathsf{Val}_{\mathcal{G}}$ is indeed the greatest fixpoint of $\mathcal{F}$ contains flaws since they suppose that the limit of the iterates of $\mathcal{F}$ is a continuous function of $\mathbb{R}_{\geq 0}$ to prove that this limit is the value function. Since the limit of a sequence of continuous functions may not be continuous, this fact needs to be proven.

Fortunately, the necessary claim can be recovered in the case of such *divergent* WTGs (at least in the turn-based case that we consider in this article, and not necessarily in the concurrent case studied in [BCFL04]) even in presence of both negative and non-negative weights, as can be recovered from [BMR18].

Moreover, in the non-divergent case, with negative weights in WTGs, the continuity of the value function is indeed not guaranteed [BGH+22, Remark 3.3]. In particular, this implies that the proof (even if we somehow obtain the continuity of the limit) can not a priori be adapted to all WTGs with negative weights.

In our specific one-clock case, we are able to correct the proof of [BCFL04, Bou16]. As the proof is long and technical, and orthogonal to the rest of the paper, we defer it to Section 8. We obtain there the following result:

Theorem 2.4 . *The value function of all (one-clock) WTGs is the greatest fixpoint of the operator $\mathcal{F}$.*

2.3. **Closure.** A game $\mathcal{G}$ can be populated with the region information without loss of generality, building what is called the *region game* in [BMR17], the addition of the classical region automaton with information on the owner of locations inherited from $\mathcal{G}$. To solve one-clock WTGs without reset transitions in [BGH+22], authors do not use the usual region game. Indeed, their method is based on a construction that consists in not only enhancing the locations with regions (as the region game) but also closing all guards while preserving the value of the original game.

Definition 2.5. The *closure* of a WTG $\mathcal{G}$ is the WTG $\overline{\mathcal{G}} = \langle L_{\mathsf{Min}}, L_{\mathsf{Max}}, L_t, L_u, \overline{\Delta}, \overline{\mathsf{wt}}, \overline{\mathsf{wt}_t} \rangle$ where:

- $L = L_{\mathsf{Min}} \uplus L_{\mathsf{Max}} \uplus L_t$ with $L_{\mathsf{Min}} = Q_{\mathsf{Min}} \times \mathsf{Reg}_{\mathcal{G}}$, $L_{\mathsf{Max}} = Q_{\mathsf{Max}} \times \mathsf{Reg}_{\mathcal{G}}$, $L_t = Q_t \times \mathsf{Reg}_{\mathcal{G}}$, and $L_u = Q_u \times \mathsf{Reg}_{\mathcal{G}}$;

- for all $(q,I) \in L$, $\big((q,I), \overline{I_g \cap I''}, R, w, (q',I')\big) \in \overline{\Delta}$ if and only if there exist a transition $(q, I_g, R, w, q') \in \Delta$, and a region I'' such that $I_g \cap I'' \neq \emptyset$, the lower bound of I'' is a time successor of I, and I' is equal to I'' if $R = \emptyset$ and to $\{0\}$ otherwise: $\overline{I_g \cap I''}$ stands for the topological closure of the non-empty interval $I_g \cap I''$;
- for all (q,I), we have $\overline{\mathsf{wt}}(q,I) = \mathsf{wt}(q)$;
- for all $(q,I) \in L_t$, for $\nu \in I$, $\overline{\mathsf{wt}_t}((q,I),\nu) = \mathsf{wt}_t(q,\nu)$ and extend $\nu \mapsto \overline{\mathsf{wt}_t}((q,I),\nu)$ by continuity on $\overline{I}$, the closure of the interval I. We may also let $\overline{\mathsf{wt}_t}((q,I),\nu) = +\infty$ for all $\nu \notin \overline{I}$, even though we will never use this in the following.

An example of closure is given in Figure 1, which depicts the closure (right) of the WTG (left) restricted to locations q_0, q_1, q_2, and ☺ (we have seen that q_3 is anyway useless).

The semantic of the closure is obtained by concentrating on the following set of configurations which is an invariant of the closure (i.e. starting from such configuration fulfilling the invariant, we can only reach configurations fulfilling the invariant):

- configurations $((q, \{M_k\}), M_k)$;
- and configurations $((q, (M_k, M_{k+1})), \nu)$ with $\nu \in [M_k, M_{k+1}]$ (and not only in (M_k, M_{k+1}) as one might expect in the region game).

The closure of the guards allows players to mimic a move in $\mathcal{G}$ "arbitrarily close" to M_k (or M_{k+1}) in (M_k, M_{k+1}) to be simulated by jumping on M_k (or M_{k+1}) still staying in the region (M_k, M_{k+1}). In particular, it is shown in [BGH$^+$22] that we can transform an ε-optimal strategy of $\overline{\mathcal{G}}$ into an ε'-optimal strategy of $\mathcal{G}$ with $\varepsilon' < 2\varepsilon$ and vice-versa. Thus, the closure of a WTG preserves its value.

Lemma 2.6 [BGH$^+$22]. *For all WTGs $\mathcal{G}$, $(q,I) \in Q \times \mathsf{Reg}_{\mathcal{G}}$ and $\nu \in I$,*

$$\mathsf{Val}_{\mathcal{G}}(q,\nu) = \mathsf{Val}_{\overline{\mathcal{G}}}((q,I),\nu).$$

Moreover, the closure construction also makes the value function more manageable for our purpose. Indeed, as shown in [BGH$^+$22], the mapping $\nu \mapsto \mathsf{Val}_{\mathcal{G}}(\ell,\nu)$ is continuous over all regions, but there might be discontinuities at the borders of the regions. The closure construction clears this issue by softening the borders of each region independently: we show the continuity of the value function on each closed region (and not only on the regions) in the closure game by following a very similar sketch as the one of [BGH$^+$22, Theorem 3.2]. The completed proof is given in Appendix A.

Lemma 2.7. *For all WTGs $\mathcal{G}$ and $(q,I) \in Q \times \mathsf{Reg}_{\mathcal{G}}$, the mapping $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q,I),\nu)$ is continuous over $\overline{I}$.*

In [BGH$^+$22], it is also shown that the mapping $\nu \mapsto \mathsf{Val}_{\mathcal{G}}(\ell,\nu)$ is piecewise affine on each region where it is not infinite, that the total number of pieces (and thus of cutpoints in-between two such affine pieces) is pseudo-polynomial (i.e. polynomial in the number of locations and the biggest weight W), and that all cutpoints and the value associated to such a cutpoint are rational numbers. We will only use this result on *reset-acyclic* WTGs, i.e. that do not contain cyclic paths with a transition with a reset, which we formally cite here:

Theorem 2.8 [BGH$^+$22]. *If $\mathcal{G}$ is a reset-acyclic WTG, then for all locations q, the piecewise affine mapping $\nu \mapsto \mathsf{Val}_{\mathcal{G}}(q,\nu)$ is computable in time polynomial in $|Q|$ and W.*

In [BGH$^+$22], this result is extended to take into allow for cyclic paths containing reset transitions when the weight of all the plays following them is not arbitrarily close to 0 and negative.

Example 2.9. Notice that the game on the left in Figure 1 does not fulfil this hypothesis: indeed, the play $(q_0, 0) \xrightarrow{1/2-\varepsilon,\delta_1} (q_1, 1/2 - \varepsilon) \xrightarrow{1/2+\varepsilon,\delta_2} (q_0, 0)$ is a cyclic play that contains a transition with a reset, and of weight -2ε negative and arbitrarily close to 0.

2.4. **Contribution.** In this work, we use a different technique of [BGH$^+$22] to push the decidability frontier and prove that the value function is computable for all WTGs (in particular the one of Figure 1):

Theorem 2.10. *For all WTGs $\mathcal{G}$ and all locations q_i, the mapping $\nu \mapsto \mathsf{Val}_{\mathcal{G}}(q_i, \nu)$ is computable in time exponential in $|Q|$ and W.*

Remark 2.11. The complexities of Theorems 2.8 and 2.10 would be more traditionally considered as exponential and doubly-exponential if weights of the WTG were encoded in binary as usual. In this work, we thus count the complexities as if all weights were encoded in unary and thus consider W to be the bound of interest. For Theorem 2.8, the obtained bound is classically called *pseudo-polynomial* in the literature.

The rest of this article gives the proof of Theorem 2.10. We fix a WTG $\mathcal{G}$ and an initial location q_i. We let $\overline{\mathcal{G}} = \langle L_{\mathsf{Min}}, L_{\mathsf{Max}}, L_t, L_u, \overline{\Delta}, \overline{\mathsf{wt}}, \overline{\mathsf{wt}_t} \rangle$ be its closure. We first use Lemma 2.6, which allows us to deduce the result by computing the value functions $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q_i, I), \nu)$ for all regions I. Regions I over which $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q_i, I), \nu)$ is constantly equal to $+\infty$ or $-\infty$ are computable in polynomial time, as explained in [BGH$^+$22]. We therefore remove them from $\overline{\mathcal{G}}$ from now on. We now fix an initial region I_i and let $\ell_i = (q_i, I_i)$, and explain how to compute $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q_i, I_i), \nu)$ on the interval I_i.

As in the non-negative case [BLMR06], the objective is to limit the number of transitions with a reset taken into the plays while not modifying the value of the game. When all weights are non-negative, this is fairly easy to achieve since, intuitively speaking, Min has no interest in using any cycles containing such a transition (since it has non-negative weight and is thus non-beneficial for Min). The game can thus be transformed so that each transition with a reset is taken at most once. To obtain a smaller game, it is even possible to simply count the number of transitions with a reset taken so far in the play and stop the game (with a final weight $+\infty$) in case the counter goes above the number of such transitions in the game. The transformed game has a polynomial number of locations with respect to the original game and is reset-acyclic, which allows one to solve it by using Theorem 2.8, with a time complexity polynomial in $|Q|$ and W (instead of the exponential time complexity originally achieved in [BLMR06, Rut11] with respect to $|Q|$).

The situation is much more intricate in the presence of negative weights since negative cycles containing a transition with a reset can be beneficial for Min, as we have seen in Example 2.3. Notice that this is still true in the closure of the game, as can be checked on the right in Figure 1. Moreover, some cyclic paths may have both plays following it with a positive weight and plays following it with a negative weight, making it difficult to determine whether it is beneficial to Min (or not). To overcome this situation, we will consider the point of view of Max, benefiting from the determinacy of the WTG. We will show that, in the closure $\overline{\mathcal{G}}$, Max can play *optimally* with *memoryless* strategies while *avoiding negative cyclic plays*. This will simplify our further study since, by following this strategy, Max ensures that only non-negative cyclic plays will be encountered, which is not beneficial to Min. Therefore, as in [BLMR06], we will limit the firing of transitions with a reset to at most once. However,
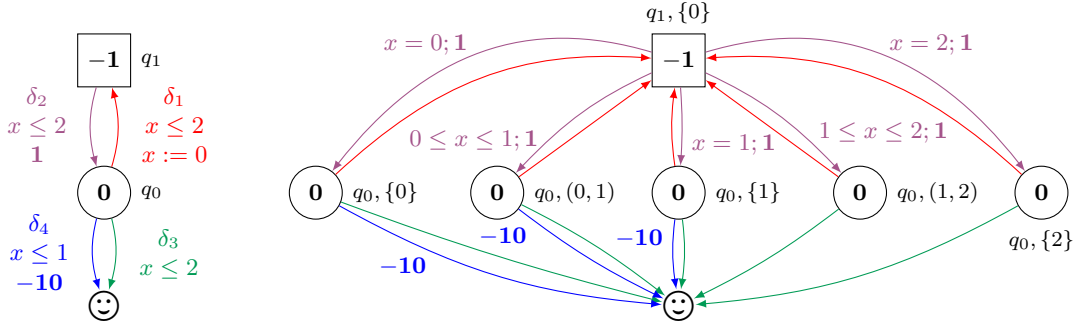
Figure 2: On the left, a WTG where Max needs memory to play ε-optimally. On the right, its closure where we merged several transitions by removing unnecessary guards.

we are not able to do it without blowing up exponentially the number of locations of the games. Instead, along the unfolding of the game, we need to record enough information in order to know, in case a cyclic path ending with a reset is closed, whether this cyclic path has a potential negative weight (in which case Max will indeed not follow it) or non-negative weight (in which case it is not beneficial for Min to close the cycle). Determining in which case we are will be made possible by introducing the notion of value of a cyclic path in Section 3. Then, Max has even an optimal strategy to avoid closing cyclic paths with a negative value (which is stronger than only avoiding creating negative cyclic plays). The unfolding, denoted $\mathcal{U}$, will be defined in Section 4. Section 5 shows that Max keeps its ability to play without falling in negative "cycles" in the unfolding. This allows us to show in Section 6 that the unfolding game has a value equal to the closure game. This allows us to wrap up the proof of Theorem 2.10 in Section 7.

3. How Max can control negative cycles

One of the main arguments of our proof is that, in the closure of a WTG $\overline{\mathcal{G}}$, Max can play *optimally* with *memoryless* strategies while *avoiding negative cyclic plays*. As already noticed in [BGH+22], this is not always true in all WTGs: Max may need memory to play ε-optimally without the possibility of avoiding some negative cyclic plays.

Example 3.1. In the WTG depicted on the left in Figure 2, we can see that $\mathsf{Val}(q_1, 0) = 0$, but Max does not have an optimal strategy, needs memory to play ε-optimally, and cannot avoid negative cyclic plays. Indeed, an optimal strategy for Max always chooses a delay greater than 1: if at some point, a strategy of Max chooses a delay less than or equal to 1, then Min can always choose δ_4, and the value of this strategy is -10. However, Max must choose a delay closer and closer to 1. Otherwise, we suppose that there exists $\beta > 0$ such that all delays chosen by the strategy are greater than $1 + \beta$, and Min has a family of strategies that stay longer and longer in the cycle with a weight at most $-\beta$. Thus, the value of this strategy will tend to $-\infty$. In particular, Max does not have an optimal strategy, and the ε-optimal strategy requires infinite memory to play with delays closer and closer to 1 (for instance, after the n-th round in the cycle, Max delays $\varepsilon/2^n$ time units, to sum up, all weights to a value at most $-\varepsilon$).

Such convergence phenomena needed by Max do not exist in $\overline{\mathcal{G}}$ since all guards are closed (this is not sufficient alone though) and by the regularity of Val given by Lemma 2.7.

Example 3.2. We consider the closure of the WTG depicted in Figure 2. The ε-optimal strategy (with memory) of Max in $\mathcal{G}$ translates into an optimal memoryless strategy in $\overline{\mathcal{G}}$: in $(q_1, \{0\})$, Max can delay 1 time unit and jump into the location $(q_0, (1, 2))$. Then, cyclic plays that Min can create have a zero weight and are thus not profitable for either player.

To generalise this explanation, we start by defining the value of cyclic paths ending with a reset in a given WTG. Intuitively, the value of this cyclic path is the weight that Min (or Max) can guarantee regardless of the delays chosen by Max (or Min) during this one.

Definition 3.3. Let $\mathcal{G}$ be a WTG. We define by induction the *value* $\mathsf{Val}^\nu_{\mathcal{G}}(\pi)$ of a finite path π in $\mathcal{G}$ from an initial valuation ν of the clock: if π has length 0 (i.e. if $\pi \in Q$), we let:

$$\mathsf{Val}^\nu_{\mathcal{G}}(\pi) = 0\,.$$

Otherwise, π can be written $q_0 \xrightarrow{\delta_0} \pi'$ (with π' starting in location q_1), and we let:

$$\mathsf{Val}^\nu_{\mathcal{G}}(\pi) = \begin{cases} \inf_{(q_0,\nu)\xrightarrow{t_0,\delta_0}(q_1,\nu') \text{ edge of } [\![\mathcal{G}]\!]} \left(t_0\,\mathsf{wt}(q_0) + \mathsf{wt}(\delta_0) + \mathsf{Val}^{\nu'}_{\mathcal{G}}(\pi')\right) & \text{if } q_0 \in L_{\mathsf{Min}} \\ \sup_{(q_0,\nu)\xrightarrow{t_0,\delta_0}(q_1,\nu') \text{ edge of } [\![\mathcal{G}]\!]} \left(t_0\,\mathsf{wt}(q_0) + \mathsf{wt}(\delta_0) + \mathsf{Val}^{\nu'}_{\mathcal{G}}(\pi')\right) & \text{if } q_0 \in L_{\mathsf{Max}} \end{cases}$$

Then, for a cyclic path π of $\mathcal{G}$ ending by a transition with a reset, we let $\mathsf{Val}_{\mathcal{G}}(\pi) = \mathsf{Val}^0_{\mathcal{G}}(\pi)$.

Example 3.4. Let $\pi = q_0 \xrightarrow{\delta_1} q_1 \xrightarrow{\delta_2} q_0$ be the cyclic path of the WTG $\mathcal{G}$ depicted on the left in Figure 1. To evaluate the value of π, Min only needs to choose a delay $t_1 \in [0, 1]$ when firing δ_1, while Max has no choice but to play a delay $1 - t_1$ when firing δ_2, generating a finite play ρ of weight $\mathsf{wt}_\Sigma(\rho) = 2t_1 - 1$. We deduce that $\mathsf{Val}_{\mathcal{G}}(\pi) = \inf_{t_1 \in [0,1]}(2\,t_1 - 1) = -1$ (when Min chooses $t_1 = 0$).

A cyclic path with a negative value ensures that Min can always guarantee to obtain a cyclic play that follows it with a negative weight, even when there are other cyclic plays (that follow it) with a non-negative weight. It is exactly those cycles that are problematic for Max since Min can benefit from them. We now show our key lemma: in the closure of a WTG, Max can play optimally and avoid cyclic paths of negative value.

Lemma 3.5. *In a closure WTG $\overline{\mathcal{G}}$, Max has a memoryless optimal strategy τ such that*
(1) *all cyclic plays conforming to τ have a non-negative weight;*
(2) *all cyclic paths ending by a reset conforming to τ have a non-negative value.*

Proof. We use Theorem 2.4 to define the memoryless strategy τ. Indeed, the identity $\mathsf{Val}_{\overline{\mathcal{G}}} = \mathcal{F}(\mathsf{Val}_{\overline{\mathcal{G}}})$, applied over configurations belonging to Max, suggests a choice of transition and delay to play almost optimally. As $\mathcal{F}$ computes a supremum on the set of possible (transitions and) delays, this does not directly lead to a specific choice: in general, this would give rise to ε-optimal strategies and not an optimal one. This is where we rely on the continuity of $\mathsf{Val}_{\overline{\mathcal{G}}}$ (Lemma 2.7) on each closure of region to deduce that this supremum is indeed a maximum. More precisely, for $\ell \in L_{\mathsf{Max}}$, we can write $\mathcal{F}(\mathsf{Val}_{\overline{\mathcal{G}}})(\ell, \nu)$ as:

$$\max_{\delta \in \overline{\Delta}} \sup_{t \text{ s.t. } (\ell,\nu)\xrightarrow{t,\delta}(\ell',\nu')} \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu')\right).$$

The guard of transition δ is the closure $\overline{I}$ of a region $I \in \mathsf{Reg}_{\mathcal{G}}$, therefore, t is in a closed interval J of values such that $\nu + t$ falls in $\overline{I}$. Notice that ν' is either 0 if δ contains a reset or is $\nu + t$: in both cases, this is a continuous function of t. Relying on the continuity of $\mathsf{Val}_{\overline{\mathcal{G}}}$, the mapping $t \in J \mapsto \overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu')$ is thus continuous over a compact set so that its supremum is indeed a maximum.

We thus let the memoryless strategy τ be such that, for all configurations (ℓ, ν), $\tau(\ell, \nu)$ is chosen arbitrarily in:

$$\underset{\delta \in \overline{\Delta}}{\operatorname{argmax}} \quad \underset{t \text{ s.t. } \ell,\nu \xrightarrow{t,\delta} \ell',\nu'}{\operatorname{argmax}} \quad \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \right) \tag{3.1}$$

This mapping τ is then extended into a memoryless strategy, defining it over finite plays by only considering the last configuration of the play. To conclude the proof, we show that τ is an optimal strategy that satisfies the two properties of the lemma.

We first show that τ is an optimal strategy by proving that $\mathsf{Val}_{\overline{\mathcal{G}}}^{\tau}(\ell, \nu) \geq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu)$ for all configurations (ℓ, ν). In particular, we show that for all plays ρ from (ℓ, ν) conforming to τ, we have $\mathsf{P}(\rho) \geq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu)$. We remark that if ρ does not reach L_t, then $\mathsf{P}(\rho) = +\infty$, and the inequality is satisfied. Now, we suppose that ρ reaches L_t, and we reason by induction on the length of ρ to show that $\mathsf{P}(\rho) \geq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu)$ for all plays ρ starting in a configuration (ℓ, ν) reaching L_t. If ρ has length 0, it starts directly in $\ell \in L_t$, and $\mathsf{P}(\rho) = \overline{\mathsf{wt}}_t(\ell, \nu) = \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu)$. Otherwise, $\rho = (\ell, \nu) \xrightarrow{t,\delta} \rho'$, with ρ' starting in a configuration (ℓ', ν'). In particular, by inductive hypothesis, we have:

$$\mathsf{P}(\rho) = \overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{P}(\rho') \geq \overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \,.$$

Now, if $\ell \in L_{\mathsf{Min}}$, then we conclude by using that $\mathsf{Val}_{\overline{\mathcal{G}}}$ is a fixed point of $\mathcal{F}$, i.e.

$$\mathsf{P}(\rho) \geq \inf_{(\ell,\nu) \xrightarrow{t,\delta} (\ell',\nu')} \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \right) = \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) \,.$$

Otherwise, we suppose that $\ell \in L_{\mathsf{Max}}$ and (t, δ) is defined by τ. Thus, by (3.1) and using again that $\mathsf{Val}_{\overline{\mathcal{G}}}$ is a fixed point of $\mathcal{F}$, we obtain that

$$\mathsf{P}(\rho) \geq \sup_{(\ell,\nu) \xrightarrow{t,\delta} (\ell',\nu')} \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell, I) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \right) = \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) \,.$$

Finally, we conclude the proof by showing that τ satisfies the two properties of the lemma.

(1) Let $\rho = (\ell_1, \nu_1) \xrightarrow{t_1,\delta_1} \cdots (\ell_k, \nu_k) \xrightarrow{t_k,\delta_k} (\ell_{k+1}, \nu_{k+1}) = (\ell_1, \nu_1)$ be a cyclic play conforming to τ. We show that $\mathsf{wt}_{\Sigma}(\rho) \geq 0$ by claiming that for all $i \in \{1, \ldots, k\}$,

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu_i) \leq \overline{\mathsf{wt}}(\delta_i) + t_i\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{i+1}, \nu_{i+1}) \tag{3.2}$$

Indeed by summing this inequality along ρ, we obtain:

$$\sum_{i=1}^{k} \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu_i) \leq \sum_{i=1}^{k} \left(\overline{\mathsf{wt}}(\delta_i) + t_i\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{i+1}, \nu_{i+1}) \right)$$

i.e., since ρ is a cyclic play,

$$\mathsf{wt}_{\Sigma}(\rho) = \sum_{i=1}^{k} \left(\overline{\mathsf{wt}}(\delta_i) + t_i\,\overline{\mathsf{wt}}(\ell_i) \right) \geq 0 \,.$$

To conclude this point, we show (3.2). For $i \in \{1, \ldots, k\}$, we distinguish two cases. First, we suppose that $\ell_i \in L_{\mathsf{Min}}$ and we conclude as $\mathsf{Val}_{\overline{\mathcal{G}}}$ is a fixed point of $\mathcal{F}$:

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu_i) = \inf_{(\ell_i, \nu_i) \xrightarrow{t, \delta} (\ell', \nu')} \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \right)$$
$$\leq \overline{\mathsf{wt}}(\delta_i) + t_i\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{i+1}, \nu_{i+1}) \,.$$

Otherwise, $\ell_i \in L_{\mathsf{Max}}$ then, as $\mathsf{Val}_{\overline{\mathcal{G}}}$ is a fixed point of $\mathcal{F}$ and by using (3.1), we have:

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu_i) = \sup_{(\ell_i, \nu_i) \xrightarrow{t, \delta} (\ell', \nu')} \left(\overline{\mathsf{wt}}(\delta) + t\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell', \nu') \right)$$
$$\leq \overline{\mathsf{wt}}(\delta_i) + t_i\,\overline{\mathsf{wt}}(\ell_i) + \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{i+1}, \nu_{i+1}) \,.$$

(2) Let $\pi = \ell_0 \xrightarrow{\delta_0} \ell_1 \cdots \ell_k \xrightarrow{\delta_k} \ell_0$ be a cyclic path conforming to τ such that δ_k contains a reset. By grouping all infimum/supremum together in the previous definition, we can see that $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi)$ can be rewritten as:

$$\inf_{\substack{\left(f_i \colon (t_0, \ldots, t_{i-1}) \mapsto t_i \right) \underset{\ell_i \in L_{\mathsf{Min}}}{0 \leq i \leq k}}} \sup_{\substack{\left(f_i \colon (t_0, \ldots, t_{i-1}) \mapsto t_i \right) \underset{\ell_i \in L_{\mathsf{Max}}}{0 \leq i \leq k}}} \mathsf{wt}_\Sigma(\rho)$$

where ρ is the finite play $(\ell_0, 0) \xrightarrow{t_0 = f_0, \delta_0} (\ell_1, \nu_1) \xrightarrow{t_1 = f_1(t_0), \delta_1} \cdots \xrightarrow{t_k = f_k(t_0, \ldots, t_{k-1}), \delta_k} (\ell_0, 0)$. Notice that the mapping f_i, chosen by the player owning location ℓ_i, describes the delay before taking the transition δ_i as a function of the previously chosen delays. In particular, for all $\varepsilon > 0$, there exists $\left(f_i \colon (t_0, \ldots, t_{i-1}) \mapsto t_i \right) \underset{\ell_i \in L_{\mathsf{Min}}}{0 \leq i \leq k}$ such that for all $\left(f_i \colon (t_0, \ldots, t_{i-1}) \mapsto t_i \right) \underset{\ell_i \in L_{\mathsf{Max}}}{0 \leq i \leq k}$:

$$\mathsf{wt}_\Sigma(\rho) \leq \mathsf{Val}_{\overline{\mathcal{G}}}(\pi) + \varepsilon$$

with ρ the finite play described above. Since π is conforming to τ, a particular choice of delays $\left(f_i \colon (t_0, \ldots, t_{i-1}) \mapsto t_i \right) \underset{\ell_i \in L_{\mathsf{Max}}}{0 \leq i \leq k}$ is given by τ itself. In this case, the latter finite play ρ is conforming to τ. By the previous item, we know that $\mathsf{wt}_\Sigma(\rho) \geq 0$, therefore:

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\pi) \geq -\varepsilon \,.$$

Since this holds for all $\varepsilon > 0$, we deduce that $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi) \geq 0$ as expected. $\qquad \square$

As a side note, it is tempting to strengthen Lemma 3.5.(2) so as to ensure that all plays following a cyclic path ending by a reset conforming to τ have a non-negative weight. Unfortunately, this does not hold, as shown in the following example.

Example 3.6. We consider the closure of the WTG depicted in Figure 3. Let $\pi = (q_0, \{0\}) \xrightarrow{\delta_1} (q_1, \{0\}) \xrightarrow{\delta_2} (q_0, \{0\})$. It is a cyclic path such that plays following it have a weight in $[-1, 1]$. To evaluate the value of π in $\overline{\mathcal{G}}$, Min and Max need to choose delays $t_1, t_2 \in [0, 1]$ when firing δ_1 and δ_2. We obtain a set of finite plays ρ parametrised by t_1 and t_2 of weight $\mathsf{wt}_\Sigma(\rho) = -t_1 + t_2$. We deduce that $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi) = \inf_{t_1} \sup_{t_2} (t_2 - t_1) = 0$ (when Min and Max choose $t_1 = t_2 = 1$). In particular, from the configuration $((q_0, \{0\}), 0)$, the cyclic path π is not interesting for Min since he only can guarantee the weight 0. Thus, he must play transition δ_4 after a delay of 1 unit of time to lead to a value of -1. To play optimally, Max must avoid the transition δ_3, i.e. all optimal strategies of Max play in the previous cyclic
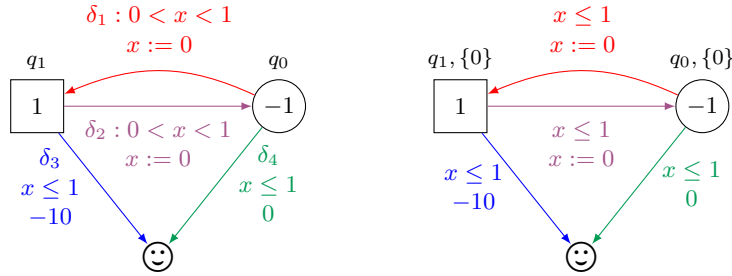
Figure 3: On the left, a WTG such that its closure on the right contains a cyclic path of value 0, but some cyclic paths of negative weight. Moreover, Max uses the cyclic path to play optimally.

path π that has a non-negative value but such that certain plays following it have a negative weight.

Finally, we note that Lemma 3.5 does not allow us to conclude on the decidability of the value problem since we use the unknown value $\mathsf{Val}_{\overline{\mathcal{G}}}$ to define the optimal strategy.

4. DEFINITION OF THE UNFOLDING

To compute $\mathsf{Val}_{\overline{\mathcal{G}}}$, we now define the partial *unfolding* of the WTG $\overline{\mathcal{G}}$ by allowing only one occurrence of each cyclic path (from $\overline{\mathcal{G}}$) ending by a reset. In particular, when a transition with a reset is taken for the first time, we go into a new copy of the WTG, from which, if this transition happens to be chosen one more time, we stop the play by jumping into a new target location. The final weight of this target location is determined by the value of the cyclic path (ending with a reset) that would have just been closed. If the cyclic path has a negative value, then we go in a leaf $\mathsf{t}_{<0}$ of final weight $-\infty$ since this is a desirable cycle for Min. Otherwise, we go in a leaf $\mathsf{t}_{\geq 0}$ of final weight big enough to be an undesirable behaviour for Min, i.e. $|L|(W_{\mathsf{tr}} + M\, W_{\mathsf{loc}}) + W_{\mathsf{fin}}$ (for technical reasons that will become clear later, we can not simply put a final weight $+\infty$).

A single transition with a reset can be part of two distinct cyclic paths, one of negative value and the other of non-negative value, as demonstrated in Example 4.1. Thus, knowing the last transition of the cycle is not enough to compute the value of the cyclic path. Instead, we need to record the whole cyclic path: copying the game (as done in the non-negative setting [BLMR06]) is not enough. Our unfolding needs to remember the path followed so far: their locations are thus finite paths of $\overline{\mathcal{G}}$.

Example 4.1. In Figure 4, we have depicted a WTG (left) and a portion of its closure (right), where δ_2' is contained in a cyclic path of negative value:

$$(q_0, \{0\}) \xrightarrow{\delta_3'} (q_2, \{0\}) \xrightarrow{\delta_4''} (q_0, \{1\}) \xrightarrow{\delta_1''} (q_1, \{0\}) \xrightarrow{\delta_2'} (q_0, \{0\})$$

and another cyclic path of non-negative (zero) value:

$$(q_0, \{0\}) \xrightarrow{\delta_1'} (q_1, \{0\}) \xrightarrow{\delta_2'} (q_0, \{0\}) \,.$$

In order to obtain a finite acyclic unfolding, we also need to stop cyclic paths without resets. To do so, we will rely on a property of reset-acyclic WTGs. For such WTGs, it can
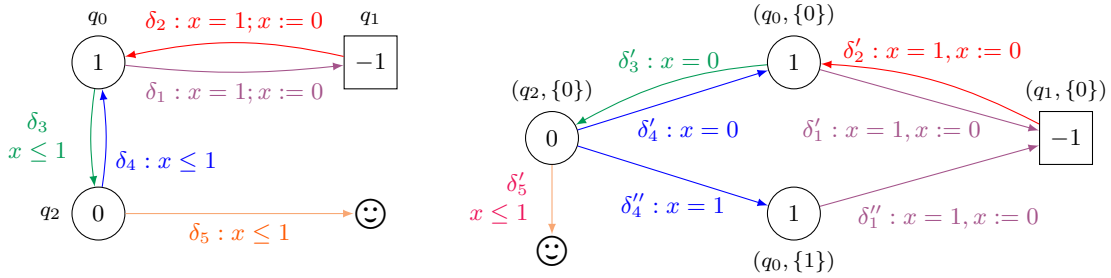
Figure 4: A WTG (left) and a portion of its closure (right) where δ_2' belongs to a cyclic path of non-negative value and another cyclic path of negative value.

be shown the existence of an ε-optimal strategy for Min with a particular shape [BGH$^+$22] defined as follows:

Definition 4.2 . A *switching* strategy σ is described by two memoryless strategies σ^1, and σ^2, as well as a switching threshold κ'. The strategy σ then consists in playing strategy σ^1 until either we reach a target location or the finite play has a length of at least κ', in which case we switch to strategy σ^2.

Intuitively, σ^1 aims at reaching a cyclic play with negative weight, while σ^2 is an attractor to the target. As a consequence, we can estimate the maximal number of steps needed by σ^2 to reach the target. Combining this with the switching threshold κ', we can deduce a threshold κ that upper bounds the number of steps under the switching strategy σ to reach the target. Moreover, we can explicitly give the pseudo-polynomial bound κ since it is given by the previous work of [BGH$^+$22]. From [BGH$^+$22, Lemma 3.9], we know that

$$\kappa' = O(|L| \times (W_{\mathsf{loc}} + |\sigma^1| \times W_{\mathsf{tr}}|L|) + |\sigma^1|)$$

where $|\sigma^1|$ is the size of this strategy, i.e. the number of cutpoints in $\mathsf{Val}_{\overline{\mathcal{G}}}$ (by [BGH$^+$22, Theorem 5.9]). Moreover, by [BGH$^+$22, Theorem 5.13], we have a bound over the number of cutpoints in $\mathsf{Val}_{\overline{\mathcal{G}}}$, i.e. $|\sigma^1| = O(W_{\mathsf{tr}}^4|L|^9)$. Thus, we deduce that the switching threshold κ' is approximated by

$$\kappa' = O\left(|L| \times \left[W_{\mathsf{loc}} + W_{\mathsf{tr}}^4|L|^9 \times W_{\mathsf{tr}}|L|\right] + W_{\mathsf{tr}}^4|L|^9\right) = O\left(|L|^{11}(W_{\mathsf{loc}} + W_{\mathsf{tr}}^5)\right) .$$

Then, we fix κ'' to be the number of turns taken by σ^2 to reach the target location, which is polynomial in the number of locations of the region automaton underlying the game, thus polynomial in the number of locations of the game (since there is only one-clock). Overall, this gives a definition for κ as:

$$\kappa = \kappa' + \kappa'' = O\left(|L|^{12}(W_{\mathsf{loc}} + W_{\mathsf{tr}}^5)\right)$$

that is polynomial in $|Q|$ (as $|L|$ is polynomial in $|Q|$) and in W. Thus, we obtain the following result.

Lemma 4.3 [BGH$^+$22]**.** *Let $\mathcal{G}$ be a reset-acyclic WTG. Min has an ε-optimal switching strategy σ such that all plays conforming to σ reach the target within κ steps. Moreover, κ is polynomial in $|Q|$ and W.*

Now, between two transitions with a reset, we obtain a reset-acyclic WTG. As a consequence, since Min can play almost optimally using a switching strategy, we can bound
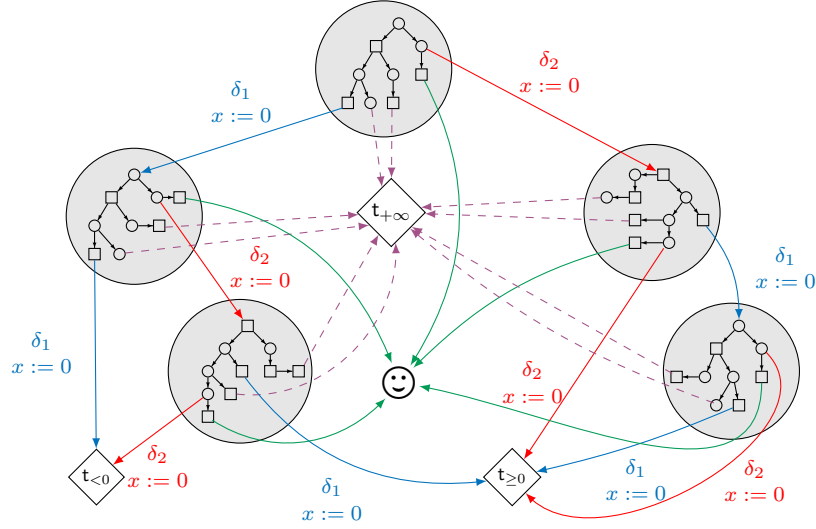
Figure 5: Scheme of the unfolding of a closure of a WTG.

the number of steps between two transitions with a reset by κ. This property allows us to avoid incorporating cycles in the unfolding: we cut the unfolding when the play becomes longer than κ since the last seen transition with a reset. In this case, we will jump into a new target location, $t_{+\infty}$, whose final weight is equal to $+\infty$ since it is an undesirable behaviour for Min.

The scheme of the unfolding is depicted in Figure 5 when the closure of a WTG contains two transitions with a reset, δ_1 and δ_2, each belonging to several cycles of different values (negative and non-negative). Inside each grey component, only transitions with no reset are unfolded for at most κ steps by only keeping, in the current location, the path followed so far. Transitions with a reset induce a change of components: these are in between the components. The second time they are visited, the value of the cycle it closes is computed, and we jump in $t_{<0}$ or $t_{\geq 0}$ depending on the sign of the value.

Definition 4.4. The *unfolding* of $\overline{\mathcal{G}}$ from the initial location ℓ_i is the (a priori infinite) WTG $\mathcal{U} = \langle L'_{\mathsf{Min}}, L'_{\mathsf{Max}}, L'_t, L'_u, \Delta', \mathsf{wt}', \mathsf{wt}'_t \rangle$ with $L'_{\mathsf{Min}} \subseteq \mathsf{FPaths}_{\mathsf{Min}}$, $L'_{\mathsf{Max}} \subseteq \mathsf{FPaths}_{\mathsf{Max}}$, $L'_t \subseteq L_t \cup \{t_{\geq 0}, t_{<0}, t_{+\infty}\}$ such that

- $L' = L'_{\mathsf{Min}} \uplus L'_{\mathsf{Max}} \uplus L'_t$ and Δ' are the smallest sets such that $\ell_i \in L'$ and for all $\pi \in L'_{\mathsf{Min}} \uplus L'_{\mathsf{Max}}$ and $\delta \in \Delta$, if $\textsc{Next}(\pi, \delta) = (\pi', \delta')$ then $\pi' \in L'$ and $\delta' \in \Delta'$ (where $\textsc{Next}$ is defined in Algorithm 1);
- $L'_u = \{\pi \in L' \mid \mathsf{last}(\pi) \in L_u\}$;
- for all $\pi \notin L'_t$, $\mathsf{wt}'(\pi) = \overline{\mathsf{wt}}(\mathsf{last}(\pi))$;
- for all $\pi \in L'_t$, for all ν,

$$\mathsf{wt}'_t(\pi, \nu) = \overline{\mathsf{wt}_t}(\pi, \nu) \quad \text{if } \pi \in L_t \qquad \mathsf{wt}'_t(t_{\geq 0}, \nu) = |L|(W_{\mathsf{tr}} + M\, W_{\mathsf{loc}}) + W_{\mathsf{fin}}$$
$$\mathsf{wt}'_t(t_{<0}, \nu) = -\infty \qquad\qquad\qquad\quad \mathsf{wt}'_t(t_{+\infty}, \nu) = +\infty \,.$$

As expected, the definition of $\textsc{Next}$ guarantees that a "new" target location is reached when the length between two resets is too long or when a transition with a reset appears

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Algorithm 1 Function Next that maps pairs $(\pi, \delta) \in \mathsf{FPaths}_{\overline{\mathcal{G}}} \times \overline{\Delta}$ to pairs (π', δ') composed of a finite path π' of $\overline{\mathcal{G}}$ (or $\mathsf{t}_{\geq 0}$, or $\mathsf{t}_{<0}$, or $\mathsf{t}_{+\infty}$) and a new transition δ' of the unfolding $\mathcal{U}$.

1: **function** Next$(\pi, \delta = (\ell_1, I, R, w, \ell_2))$: $\qquad\qquad\qquad\qquad\qquad$ ▷ $\quad$ last$(\pi) = \ell_1$
2: $\quad$ **if** $\ell_2 \in L_t$ **then** $\pi' := \ell_2$
3: $\quad$ **else if** $R = \{x\}$ **then**
4: $\quad\quad$ **if** $|\pi|_\delta = 0$ **then** $\pi' := \pi \xrightarrow{\delta} \ell_2$
5: $\quad\quad$ **else**$\{ \ \pi := \pi_1 \xrightarrow{\delta} \pi_2$
6: $\quad\quad\quad$ **if** $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi_2 \xrightarrow{\delta} \ell_2) \geq 0$ **then** $\pi' := \mathsf{t}_{\geq 0}$ **else** $\pi' := \mathsf{t}_{<0}\}$
7: $\quad$ **else**$\{ \ \pi := \pi_1 \cdot \pi_2$ where π_2 contains no reset and $|\pi_2|$ is maximal
8: $\quad\quad$ **if** $|\pi_2| = \kappa$ **then** $\pi' := \mathsf{t}_{+\infty}$ $\ \ $ **else** $\pi' := \pi \xrightarrow{\delta} \ell_2 \ \}$
9: $\quad$ $\delta' := (\pi, I, R, w, \pi')$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ▷ $\quad$ $\Delta\mathsf{proj}(\delta') := \delta$
10: $\quad$ **return** (π', δ')

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two times. Moreover, the length of the path in a location that is not a target, given by the application of Next, strictly increases. This allows us to show that $\mathcal{U}$ is a finite and acyclic WTG.

Lemma 4.5. *The WTG $\mathcal{U}$ is acyclic and has a finite set of locations of cardinality at most exponential in $|Q|$ and W.*

Proof. We start by proving that $\mathcal{U}$ is an acyclic WTG. The function Next never removes a transition from a path $\pi \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$ that is given as input. In particular, the function Next produces a transition from π to π' that is an extension of π (i.e. $|\pi'| > |\pi|$) or a target location. Thus, all paths in $\mathcal{U}$ are acyclic, i.e. $\mathcal{U}$ is acyclic.

Now, we prove that $\mathcal{U}$ is a finite WTG by proving that for all locations $\pi \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$, the length of the path is upper-bounded by $(|\Delta_R| + 1)\kappa$, where we let Δ_R be the subset of transitions with a reset. Locations of $\mathcal{U}$ are built by successive applications of Next. We show by induction (on the number of such applications) that every location $\pi \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$ can be decomposed as follows: $\pi = \pi'_0 \xrightarrow{\delta_0} \pi'_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_k} \pi'_k$ where transitions δ_i belong to Δ_R and are pairwise distinct, and where π'_i belong to $\mathsf{FPaths}_{\overline{\mathcal{G}}}$ and have length at most κ. As a direct consequence of this property, we have $k \leq |\Delta_R|$, and we easily deduce the expected bound on $|\pi|$, and thus the desired bound on the number of locations of $\mathcal{U}$ by the bound on κ of Lemma 4.3.

We now proceed to the induction on the number of applications of Next. As a base case, we have $\pi = \ell_i$, and the property trivially holds. Assume now that the property holds for π, with a decomposition $\pi = \pi'_0 \xrightarrow{\delta_0} \pi'_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_k} \pi'_k$. We fix some $\delta = (\ell_1, I, R, w, \ell_2) \in \Delta$ with last$(\pi) = \ell_1$, and we consider Next$(\pi, \delta) = (\pi', \delta')$ with $\pi' \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$. We distinguish cases according to the definition of Next. Observe that $\pi' \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$ excludes the case when Next sets π' in lines 2, 6, or 8 (when $|\pi'_k| = \kappa$). The following cases may occur:

- if Next sets π' in line 4, then we have $\delta \in \Delta_R$, $|\pi|_\delta = 0$, and $\pi' = \pi \xrightarrow{\delta} \ell_2$. Hence, a correct decomposition of π' is obtained by adding δ and an empty path π'_{k+1} to the ones of π.
- if Next sets π' in line 8, while $\pi' \neq \mathsf{t}_{+\infty}$ (by hypothesis), then $|\pi'_k| < \kappa$. A correct decomposition of π' is then obtained from the one of π by replacing π'_k with $\pi'_k \xrightarrow{\delta} \ell_2$. $\qquad \square$

5. How Max can control negative "cycles" in the unfolding

In this section, we show another good property of $\mathcal{U}$, mimicking the one of Lemma 3.5 that Max has an optimal memoryless strategy in $\overline{\mathcal{G}}$ avoiding cyclic plays with a negative weight. In the unfolding $\mathcal{U}$, there are no cyclic plays, but we will be able to obtain a similar result: Max can play optimally with a memoryless strategy while making sure that in-between two occurrences of the same transition with a reset (which would result in a cyclic play in the original game $\overline{\mathcal{G}}$), the play has a non-negative weight. More formally, we want to obtain the following lemma, that we show in the rest of the section:

Lemma 5.1. *In the WTG $\mathcal{U}$, Max has a memoryless optimal strategy τ such that if a finite play $\rho = \rho_1 \xrightarrow{t_1, \delta'_1} \rho_2 \xrightarrow{t_2, \delta'_2} (\mathsf{t}_{\geq 0}, 0)$ is conforming to τ with $\Delta\mathsf{proj}(\delta'_1) = \Delta\mathsf{proj}(\delta'_2) \in \Delta_R$ (i.e. the same transition with a reset in the original WTG $\overline{\mathcal{G}}$), then $\mathsf{wt}_\Sigma(\rho_2 \xrightarrow{t_2, \delta'_2} (\mathsf{t}_{\geq 0}, 0)) \geq 0$.*

Remark 5.2. The fact that $\mathcal{U}$ is acyclic is crucial in this result: we can not guarantee that the value of the path ending in $\mathsf{t}_{\geq 0}$ is non-negative if we would have defined $\mathcal{U}$ with grey components (in Figure 5) containing cyclic paths without a reset. Indeed, the values of cyclic paths are not preserved by concatenation. For instance, in the WTG $\mathcal{G}$ depicted on the left of Figure 4, we can see that $\mathsf{Val}_\mathcal{G}(q_0 \xrightarrow{\delta_1} q_1 \xrightarrow{\delta_2} q_0) = 0$ (Min and Max must delay 1 in each location), and $\mathsf{Val}_\mathcal{G}(q_0 \xrightarrow{\delta_3} q_2 \xrightarrow{\delta_4} q_0) = 0$. However, when we concatenate these two cyclic paths, we obtain the cycle $q_0 \xrightarrow{\delta_3} q_2 \xrightarrow{\delta_4} q_0 \xrightarrow{\delta_1} q_1 \xrightarrow{\delta_2} q_0)$ of value -1.

The proof of Lemma 5.1 essentially consists in applying the result of Lemma 3.5 in $\mathcal{U}$ to define a memoryless optimal strategy with the desired property. However, Lemma 3.5 holds only in the closure of WTGs (Example 3.1 gives a counter-example when the WTG is not a closure), i.e. a priori, the result holds only on the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$. Nevertheless, we notice that the guards of all transitions of $\mathcal{U}$ come from $\overline{\mathcal{G}}$ and the regions of $\mathcal{U}$ are thus the ones of $\overline{\mathcal{G}}$. Therefore, apart from target locations, only locations (π, I) in $\overline{\mathcal{U}}$ with π ending in a location of the form (ℓ, I) are reachable. Thus $\mathcal{U}$ and $\overline{\mathcal{U}}$ are the same WTG, and the result of Lemma 3.5 transfers to $\mathcal{U}$ as well.

The second argument of the proof is checking that $\mathcal{U}$ preserves the value of paths, i.e. the value in $\mathcal{U}$ of a finite path $\pi_\mathcal{U}$, $\mathsf{Val}^\nu_\mathcal{U}(\pi_\mathcal{U})$, is equal to the value in $\overline{\mathcal{G}}$ of its projection given by $\Delta\mathsf{proj}$. In particular, we define a new projection function $\Pi\mathsf{proj}$ as an extension over finite paths of $\Delta\mathsf{proj}$ such that for all finite paths in $\mathcal{U}$ with at least one transition, $\pi_\mathcal{U} = \pi_1 \xrightarrow{\delta'} \pi'_\mathcal{U}$ with $\pi_1 \in L' \setminus L'_t$, we let $\Pi\mathsf{proj}(\pi_\mathcal{U})$ be equal to:

$$\begin{cases} \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \ell_2 & \text{if } \pi'_\mathcal{U} \in L' \text{ and } \Delta\mathsf{proj}(\delta') = (\mathsf{last}(\pi_1), I, R, w, \ell_2) \\ \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \Pi\mathsf{proj}(\pi'_\mathcal{U}) & \text{otherwise .} \end{cases}$$

We note that, $\Pi\mathsf{proj}(\pi_\mathcal{U})$ is always a finite path in $\overline{\mathcal{G}}$ with the same length of $\pi_\mathcal{U}$ and it satisfies the following properties:

Lemma 5.3. *Let $\pi_\mathcal{U} \in \mathsf{FPaths}_\mathcal{U}$ be a path with at least one transition, then*
(1) for all valuations ν, $\mathsf{Val}^\nu_\mathcal{U}(\pi_\mathcal{U}) = \mathsf{Val}^\nu_{\overline{\mathcal{G}}}(\Pi\mathsf{proj}(\pi_\mathcal{U}))$;
(2) if $\mathsf{last}(\pi_\mathcal{U}) \notin L'_t$, then $\Pi\mathsf{proj}(\pi_\mathcal{U})$ is a suffix of $\mathsf{last}(\pi_\mathcal{U})$.

Proof. (1) We prove this property when the first location of $\pi_\mathcal{U}$ belongs to Min. The case where it belongs to Max is analogous when we replace the infimum by a supremum. We

reason by induction on the length of $\pi_{\mathcal{U}}$. First, we suppose that $\pi_{\mathcal{U}}$ contains exactly one transition, i.e. $\pi_{\mathcal{U}} = \pi_1 \xrightarrow{\delta'} \pi_2$ and :

$$\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \inf_t \left(t\,\mathsf{wt}'(\pi_1) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_2) \right)$$

where $\nu' = \nu + t$ if δ' does not contain a reset, or $\nu' = 0$ otherwise. Since the value of an empty path is null, we have $\mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_2) = 0 = \mathsf{Val}_{\mathcal{G}}^{\nu'}(\ell_2)$ where ℓ_2 is given by $\Delta\mathsf{proj}(\delta')$. Moreover, as $\mathcal{U}$ preserves the weight of transitions and locations, we obtain that

$$\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \inf_t \left(t\,\mathsf{wt}(\mathsf{last}(\pi_1)) + \mathsf{wt}(\Delta\mathsf{proj}(\delta')) + \mathsf{Val}_{\mathcal{G}}^{\nu'}(\ell_2) \right).$$

By definition of $\Pi\mathsf{proj}$, we remark that $\Pi\mathsf{proj}(\pi_{\mathcal{U}}) = \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \ell_2$. Thus, since $\mathcal{U}$ preserves transitions with a reset, we deduce that $\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \mathsf{Val}_{\mathcal{G}}^{\nu}(\Pi\mathsf{proj}(\pi_{\mathcal{U}}))$.

Now, we suppose that $\pi_{\mathcal{U}} = \pi_1 \xrightarrow{\delta'} \pi_{\mathcal{U}}'$ with $\pi_{\mathcal{U}}'$ a path in $\mathcal{U}$. Since $\mathcal{U}$ preserves the weight of transitions and locations, we have:

$$\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \inf_t \left(t\,\mathsf{wt}'(\pi_1) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_{\mathcal{U}}') \right)$$
$$= \inf_t \left(t\,\mathsf{wt}(\mathsf{last}(\pi_1)) + \mathsf{wt}(\Delta\mathsf{proj}(\delta')) + \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_{\mathcal{U}}') \right)$$

where $\nu' = \nu + t$ if δ' does not contain a reset, or $\nu' = 0$ otherwise. Now, the induction hypothesis applied to $\pi_{\mathcal{U}}'$ implies that

$$\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \inf_t \left(t\,\mathsf{wt}(\mathsf{last}(\pi_1)) + \mathsf{wt}(\Delta\mathsf{proj}(\delta')) + \mathsf{Val}_{\mathcal{G}}^{\nu'}(\Pi\mathsf{proj}(\pi_{\mathcal{U}}')) \right)$$

Finally, we obtain that $\mathsf{Val}_{\mathcal{U}}^{\nu}(\pi_{\mathcal{U}}) = \mathsf{Val}_{\mathcal{G}}^{\nu}(\Pi\mathsf{proj}(\pi_{\mathcal{U}}))$, since $\mathcal{U}$ preserves transitions with a reset and $\Pi\mathsf{proj}(\pi_{\mathcal{U}}) = \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \Pi\mathsf{proj}(\pi_{\mathcal{U}}')$.

(2) We reason by induction on the length of $\pi_{\mathcal{U}}$. First, we suppose that $\pi_{\mathcal{U}}$ contains only one transition, i.e. $\pi_{\mathcal{U}} = \pi_1 \xrightarrow{\delta'} \pi_2$ with $\pi_2 \in L'_{\mathsf{Min}} \cup L'_{\mathsf{Max}}$, and $\Pi\mathsf{proj}(\pi_{\mathcal{U}}) = \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \ell_2$ where ℓ_2 is given by $\Delta\mathsf{proj}(\delta')$. By definition of Next, since $\pi_2 \notin L'_t$, we note that $\pi_2 = \pi_1 \xrightarrow{\Delta\mathsf{proj}(\delta')} \ell_2$. Since $\mathsf{last}(\pi_1)$ is a suffix of π_1, it follows that $\Pi\mathsf{proj}(\pi_{\mathcal{U}})$ is a suffix of $\mathsf{last}(\pi_{\mathcal{U}}) = \pi_2$.

Otherwise, we suppose that $\pi_{\mathcal{U}} = \pi_1 \xrightarrow{\delta'} \pi_{\mathcal{U}}'$ with $\mathsf{last}(\pi_{\mathcal{U}}') \notin L'_t$, and $\Pi\mathsf{proj}(\pi_{\mathcal{U}}) = \mathsf{last}(\pi_1) \xrightarrow{\Delta\mathsf{proj}(\delta')} \Pi\mathsf{proj}(\pi_{\mathcal{U}}')$. By induction hypothesis, $\Pi\mathsf{proj}(\pi_{\mathcal{U}}')$ is a suffix of $\mathsf{last}(\pi_{\mathcal{U}}')$, i.e. there exists a finite path π of $\overline{\mathcal{G}}$ such that $\mathsf{last}(\pi_{\mathcal{U}}') = \pi \cdot \Pi\mathsf{proj}(\pi_{\mathcal{U}}')$. Now, we remark that $\pi = \pi_1$, since each application of Next (that does not reach a target location) adds exactly one transition in the path of the next location: δ' is a transition between π_1 and π_2 where $\pi_2 = \pi_1 \xrightarrow{\Delta\mathsf{proj}(\delta')} \ell_2$ is the first location of $\pi_{\mathcal{U}}'$. Finally, we obtain a suffix of $\mathsf{last}(\pi_{\mathcal{U}})$ since $\mathsf{last}(\pi_1)$ is a suffix of π_1. $\qquad\square$

Finally, we have the tools to finish the proof of Lemma 5.1. As explained before, we apply the result of Lemma 3.5 in $\mathcal{U}$ (since the closure of $\mathcal{U}$ describes the same WTG as $\mathcal{U}$) to obtain a memoryless optimal strategy τ for Max. It remains to show that if $\rho = \rho_1 \xrightarrow{t,\delta_1'} \rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)$ is conforming to τ with $\Delta\mathsf{proj}(\delta_1') = \Delta\mathsf{proj}(\delta_2')$ containing a reset, then $\mathsf{wt}_{\Sigma}(\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)) \geq 0$. Let $\pi_{\mathcal{U}}$ be the path of $\mathcal{U}$ followed by $\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)$.

We start by claiming that

$$\mathsf{P}(\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)) \geq \mathsf{Val}_{\mathcal{U}}(\pi_{\mathcal{U}}) + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \tag{5.1}$$

where this is the place where the fact that $\mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0)$ is not equal to $+\infty$ is crucial.

Equation (5.1) and Lemma 5.3.(1) (with $\nu = 0$) allow us to conclude as follows. First,

$$\mathsf{wt}_\Sigma(\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)) = \mathsf{P}(\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)) - \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \geq \mathsf{Val}_{\mathcal{U}}(\pi_{\mathcal{U}}) \geq \mathsf{Val}_{\overline{\mathcal{G}}}(\pi)$$

where $\pi = \Pi\mathsf{proj}(\pi_{\mathcal{U}})$. Then, since $\pi_{\mathcal{U}} = \pi_{\mathcal{U}}' \xrightarrow{\delta_2'} \mathsf{t}_{\geq 0}$ where ρ_2 follows $\pi_{\mathcal{U}}'$, we deduce that π' is a suffix of $\pi_{\mathcal{U}}'$ where $\pi = \pi' \xrightarrow{\delta} \ell$ (by Lemma 5.3.(2) applied to $\pi_{\mathcal{U}}'$). In particular, the definition of NEXT on π' and δ (as $\mathsf{t}_{\geq 0}$ is reached) guarantees that $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi) \geq 0$.

To conclude the proof, we need to show (5.1). We reason by induction on suffixes ρ' of $\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)$ showing that

$$\mathsf{P}(\rho') \geq \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_{\mathcal{U}}') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0)$$

where $\pi_{\mathcal{U}}'$ is the path followed by ρ', and ν' is the first valuation of ρ'. For the suffix $\rho' = (\mathsf{t}_{\geq 0}, 0)$, then

$$\mathsf{P}(\rho') = \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) = \mathsf{Val}_{\mathcal{U}}^0(\mathsf{t}_{\geq 0}) + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \,.$$

Otherwise, we suppose that $\rho' = (\pi, \nu') \xrightarrow{t,\delta'} \rho''$. In particular, we fix $\nu'' = \nu' + t$ the first valuation of ρ'' ($\nu'' \neq 0$ since ρ_2 does not contain a transition with a reset) and $\pi_{\mathcal{U}}' = \pi \xrightarrow{\delta'} \pi_{\mathcal{U}}''$ with ρ'' follows $\pi_{\mathcal{U}}''$. Moreover, we deduce that

$$\mathsf{P}(\rho') = t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{P}(\rho'')$$
$$\geq t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'+t}(\pi_{\mathcal{U}}'') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \qquad \text{(by induction hypothesis)} \,.$$

To conclude the induction case, we distinguish two cases.

- If $\pi \in L_{\mathsf{Min}}'$, then

$$\mathsf{P}(\rho') \geq \inf_{t \text{ s.t. } (\pi,\nu') \xrightarrow{t,\delta'} (\pi',\nu'+t)} \left(t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'+t}(\pi_{\mathcal{U}}'') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \right)$$
$$= \inf_{t \text{ s.t. } (\pi,\nu') \xrightarrow{t,\delta'} (\pi',\nu'+t)} \left(t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'+t}(\pi_{\mathcal{U}}'') \right) + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0)$$
$$= \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_{\mathcal{U}}') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \,.$$

- If $\pi \in L_{\mathsf{Max}}'$, since τ chooses δ', we can deduce that

$$\mathsf{P}(\rho') \geq \sup_{t \text{ s.t. } (\pi,\nu') \xrightarrow{t,\delta'} (\pi',\nu'+t)} \left(t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'+t}(\pi_{\mathcal{U}}'') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \right)$$
$$= \sup_{t \text{ s.t. } (\pi,\nu') \xrightarrow{t,\delta'} (\pi',\nu'+t)} \left(t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') + \mathsf{Val}_{\mathcal{U}}^{\nu'+t}(\pi_{\mathcal{U}}'') \right) + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0)$$
$$= \mathsf{Val}_{\mathcal{U}}^{\nu'}(\pi_{\mathcal{U}}') + \mathsf{wt}_t(\mathsf{t}_{\geq 0}, 0) \,.$$

Since it is obtained after transition δ_1' that resets the clock, the first valuation of $\rho_2 \xrightarrow{t_2,\delta_2'} (\mathsf{t}_{\geq 0}, 0)$ is 0. Thus, by induction, we obtain (5.1) as expected.
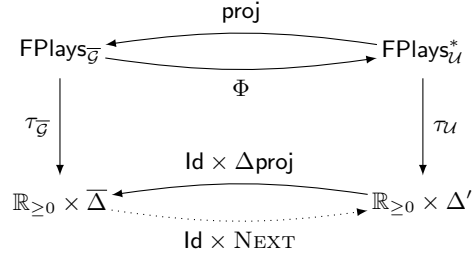
Figure 6: Scheme showing the links between the different objects defined for the proof of Theorem 6.1 where $\mathsf{FPlays}^*_{\mathcal{U}}$ is the set of finite plays of $\mathcal{U}$ avoiding target locations $\mathsf{t}_{\geq 0}$ and $\mathsf{t}_{< 0}$.

6. Value of the unfolding

The most difficult part of the proof of Theorem 2.10 is to show that the unfolding preserves the value from $\overline{\mathcal{G}}$. Remember that we have fixed an initial location $\ell_{\mathsf{i}} = (q_{\mathsf{i}}, I_{\mathsf{i}})$ to build $\mathcal{U}$.

Theorem 6.1. *For all $\nu \in I_{\mathsf{i}}$, $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu) = \mathsf{Val}_{\mathcal{U}}(\ell_{\mathsf{i}}, \nu)$.*

We prove Theorem 6.1 in this section, splitting the proof into two inequalities.

First inequality. We prove first that $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu) \leq \mathsf{Val}_{\mathcal{U}}(\ell_{\mathsf{i}}, \nu)$, which can be rewritten as:

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu) \leq \sup_{\tau_{\mathcal{U}}} \mathsf{Val}^{\tau_{\mathcal{U}}}_{\mathcal{U}}(\ell_{\mathsf{i}}, \nu) \,.$$

We must thus show that Max can guarantee to always do at least as good in $\mathcal{U}$ as in $\overline{\mathcal{G}}$. We thus fix an optimal strategy $\tau_{\overline{\mathcal{G}}}$ in $\overline{\mathcal{G}}$ obtained by Lemma 3.5: in particular, $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu) = \mathsf{Val}^{\tau_{\overline{\mathcal{G}}}}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu)$. We show the existence of a strategy $\tau_{\mathcal{U}}$ in $\mathcal{U}$ such that $\mathsf{Val}^{\tau_{\overline{\mathcal{G}}}}_{\overline{\mathcal{G}}}(\ell_{\mathsf{i}}, \nu) \leq \mathsf{Val}^{\tau_{\mathcal{U}}}_{\mathcal{U}}(\ell_{\mathsf{i}}, \nu)$, i.e. for all plays ρ conforming to $\tau_{\mathcal{U}}$, there exists a play conforming to $\tau_{\overline{\mathcal{G}}}$ with a weight at most the weight of ρ. As it is depicted in Figure 6, the strategy $\tau_{\mathcal{U}}$ is defined via a *projection* of plays of $\mathcal{U}$ in $\overline{\mathcal{G}}$: we use the mapping NEXT to send back transitions of $\overline{\Delta}$ to Δ'.

More formally, the projection operator proj projects finite plays of $\mathcal{U}$ starting in ℓ_{i} (since these are the only ones we need to take care of) to finite plays of $\overline{\mathcal{G}}$. For this reason, from now on, $\mathsf{FPlays}_{\mathcal{U}}$ and $\mathsf{FPlays}_{\overline{\mathcal{G}}}$ denote the subsets of plays that start in location ℓ_{i}. Moreover, we limit ourselves to projecting plays of $\mathcal{U}$ that do not reach the targets $\mathsf{t}_{< 0}$ and $\mathsf{t}_{\geq 0}$, since otherwise there is no canonical projection in $\overline{\mathcal{G}}$. We thus let $\mathsf{FPlays}^*_{\mathcal{U}}$ be all such finite plays of $\mathsf{FPlays}_{\mathcal{U}}$ that do not end in $\mathsf{t}_{< 0}$ or $\mathsf{t}_{\geq 0}$. The projection function $\mathsf{proj} \colon \mathsf{FPlays}^*_{\mathcal{U}} \to \mathsf{FPlays}_{\overline{\mathcal{G}}}$ is defined inductively on finite plays $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$ by letting $\mathsf{proj}(\rho)$ be

$$\begin{cases} (\ell_{\mathsf{i}}, \nu) & \text{if } \rho = (\ell_{\mathsf{i}}, \nu) \in L'; \\ \mathsf{proj}(\rho') \xrightarrow{t, \Delta\mathsf{proj}(\delta')} (\mathsf{last}(\pi), \nu) & \text{if } \rho = \rho' \xrightarrow{t, \delta'} (\pi, \nu); \\ \mathsf{proj}(\rho') \xrightarrow{t, \Delta\mathsf{proj}(\delta')} (\ell', \nu) & \text{if } \rho = \rho' \xrightarrow{t, \delta'} (\mathsf{t}_{+\infty}, \nu) \text{ and } \Delta\mathsf{proj}(\delta') = (\ell, I, R, w, \ell'). \end{cases}$$

It fulfils the following properties:

Lemma 6.2. *For all plays $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$,*
(1) *if $\mathsf{last}(\rho) = (\pi, \nu)$ with $\pi \neq \mathsf{t}_{+\infty}$, then $\mathsf{last}(\mathsf{proj}(\rho)) = (\mathsf{last}(\pi), \nu)$;*
(2) *$\mathsf{wt}_\Sigma(\rho) = \mathsf{wt}_\Sigma(\mathsf{proj}(\rho))$;*

(3) *if* $\mathsf{last}(\rho) = (\pi, \nu)$ *with* $\pi \notin L_t$, *then* $\mathsf{proj}(\rho)$ *follows* π.

Proof. (1) Since $\pi \neq \mathsf{t}_{+\infty}$, this is direct from a case analysis on the definition of proj.

(2) We reason by induction on the length of $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$. First, we suppose that $\rho = (\ell_{\mathsf{i}}, \nu)$, then we have $\mathsf{proj}(\rho) = \rho$ and $\mathsf{wt}_\Sigma(\rho) = 0 = \mathsf{wt}_\Sigma(\mathsf{proj}(\rho))$. Now, we suppose that $\rho = \rho' \xrightarrow{t,\delta'} (\pi, \nu)$, with $\rho' \in \mathsf{FPlays}^*_{\mathcal{U}}$ ending in location π' such that $\pi' \notin L'_t$. Then,

$$\mathsf{wt}_\Sigma(\rho) = \mathsf{wt}_\Sigma(\rho') + t\,\mathsf{wt}'(\pi') + \mathsf{wt}'(\delta')$$
$$= \mathsf{wt}_\Sigma(\rho') + t\,\overline{\mathsf{wt}}(\mathsf{last}(\pi')) + \overline{\mathsf{wt}}(\Delta\mathsf{proj}(\delta'))$$

since $\mathcal{U}$ preserves the weights of $\overline{\mathcal{G}}$, i.e. $\mathsf{wt}'(\pi') = \overline{\mathsf{wt}}(\mathsf{last}(\pi'))$, and $\mathsf{wt}'(\delta') = \overline{\mathsf{wt}}(\Delta\mathsf{proj}(\delta'))$. Moreover, the induction hypothesis applied to ρ' implies that

$$\mathsf{wt}_\Sigma(\rho) = \mathsf{wt}_\Sigma(\mathsf{proj}(\rho')) + t\,\overline{\mathsf{wt}}(\mathsf{last}(\pi')) + \overline{\mathsf{wt}}(\Delta\mathsf{proj}(\delta'))$$
$$= \mathsf{wt}_\Sigma(\mathsf{proj}(\rho')) + t\,\overline{\mathsf{wt}}(\mathsf{last}(\mathsf{proj}(\rho'))) + \overline{\mathsf{wt}}(\Delta\mathsf{proj}(\delta'))$$

since $\mathsf{last}(\pi') = \mathsf{last}(\mathsf{proj}(\rho'))$ by the first item (as $\pi' \neq \mathsf{t}_{+\infty}$). Finally, by the definition of $\mathsf{proj}(\rho)$, we conclude that $\mathsf{wt}_\Sigma(\rho) = \mathsf{wt}_\Sigma(\mathsf{proj}(\rho))$.

(3) We reason by induction on the length of $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$ that does not reach a target location. If $\rho = (\ell_{\mathsf{i}}, \nu)$, the property is trivial. Now, we suppose that $\rho = \rho' \xrightarrow{t,\delta'} (\pi, \nu)$, with $\rho' \in \mathsf{FPlays}^*_{\mathcal{U}}$ ending in a configuration (π', ν') such that $\pi' \notin L'_t$. In particular, we have $\mathsf{proj}(\rho) = \mathsf{proj}(\rho') \xrightarrow{t,\delta} (\mathsf{last}(\pi), \nu)$ with $\delta = \Delta\mathsf{proj}(\delta')$, and, by the induction hypothesis, $\mathsf{proj}(\rho')$ follows π'. Moreover, we have $\textsc{Next}(\pi', \delta) = (\pi, \delta')$ such that π must be obtained from π' on lines 4 or 8 of Algorithm 1, i.e. $\pi = \pi' \xrightarrow{\delta} \ell_2$ where ℓ_2 is given by δ. Thus, we deduce that $\mathsf{proj}(\rho)$ follows π. $\qquad\square$

Now, for all plays $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$ such that $\mathsf{last}(\rho) = (\pi, \nu)$ and $\pi \in L'_{\mathsf{Max}}$ (for plays not starting in ℓ_{i}, the decision over ρ is irrelevant), we define a strategy $\tau_{\mathcal{U}}$ for Max in $\mathcal{U}$ by

$$\tau_{\mathcal{U}}(\rho) = (t, \delta') \quad \text{if } \tau_{\overline{\mathcal{G}}}(\mathsf{proj}(\rho)) = (t, \delta) \text{ and } \textsc{Next}(\pi, \delta) = (\pi', \delta')$$

We note that this is a valid decision for Max: we apply the same delay (since delays chosen in $\tau_{\overline{\mathcal{G}}}$ and $\tau_{\mathcal{U}}$ are the identical) from the same configuration (as $\mathsf{last}(\mathsf{proj}(\rho)) = (\mathsf{last}(\pi), \nu)$, by Lemma 6.2.(1)), through the same guard (since guards of δ and δ' are identical). Thus, whether or not the location π is urgent (i.e. $\mathsf{last}(\pi)$ is urgent), the decision (t, δ') gives rise to an edge in $[\![\mathcal{U}]\!]$. Moreover, since the definition of $\tau_{\mathcal{U}}$ relies on the projection, it is of no surprise that:

Lemma 6.3. *Let* $\rho \in \mathsf{FPlays}^*_{\mathcal{U}}$ *be a play conforming to* $\tau_{\mathcal{U}}$. *Then,* $\mathsf{proj}(\rho)$ *is conforming to* $\tau_{\overline{\mathcal{G}}}$.

Proof. We reason by induction on the length of ρ. If $\rho = (\ell_{\mathsf{i}}, \nu)$, then $\mathsf{proj}(\rho) = (\ell_{\mathsf{i}}, \nu)$, and the property is trivial. Otherwise, we suppose that $\rho = \rho' \xrightarrow{t,\delta'} (\pi, \nu)$ and $\mathsf{proj}(\rho) = \mathsf{proj}(\rho') \xrightarrow{t,\delta} (\mathsf{last}(\pi), \nu)$ where $\delta = \Delta\mathsf{proj}(\delta')$. By the induction hypothesis, $\mathsf{proj}(\rho')$ is conforming to $\tau_{\overline{\mathcal{G}}}$. Letting $\mathsf{last}(\mathsf{proj}(\rho')) = (\ell', \nu')$, we conclude by distinguishing two cases. First, if $\ell' \in L_{\mathsf{Min}}$, we directly conclude that $\mathsf{proj}(\rho)$ is conforming to $\tau_{\overline{\mathcal{G}}}$ too. Otherwise, we suppose that $\ell' \in L_{\mathsf{Max}}$. Since ρ is conforming to $\tau_{\mathcal{U}}$ and ρ' also belongs to Max (by Lemma 6.2.(1)), we have $\tau_{\mathcal{U}}(\rho') = (t, \delta')$. In particular, by definition of $\tau_{\mathcal{U}}$, $\tau_{\overline{\mathcal{G}}}(\mathsf{proj}(\rho')) = (t, \Delta\mathsf{proj}(\delta')) = (t, \delta)$. Thus, $\rho_{\overline{\mathcal{G}}}$ is conforming to $\tau_{\overline{\mathcal{G}}}$. $\qquad\square$

Finally, we prove $\mathsf{Val}_{\overline{\mathcal{G}}}^{\tau_{\overline{\mathcal{G}}}}(\ell_i, \nu) \leq \mathsf{Val}_{\mathcal{U}}^{\tau_{\mathcal{U}}}(\ell_i, \nu)$ by showing that for all plays $\rho_{\mathcal{U}}$ from (ℓ_i, ν) conforming to $\tau_{\mathcal{U}}$, there exists a play $\rho_{\overline{\mathcal{G}}}$ from (ℓ_i, ν) conforming to $\tau_{\overline{\mathcal{G}}}$ such that $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{P}(\rho_{\mathcal{U}})$. If $\rho_{\mathcal{U}}$ does not reach a target location of $\mathcal{U}$ or reaches target $\mathsf{t}_{+\infty}$, then $\mathsf{P}(\rho_{\mathcal{U}}) = +\infty$, and for all plays $\rho_{\overline{\mathcal{G}}}$ conforming to $\tau_{\overline{\mathcal{G}}}$, we have $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq +\infty = \mathsf{P}(\rho_{\mathcal{U}})$. Now, we suppose that $\rho_{\mathcal{U}}$ reaches a target location different from $\mathsf{t}_{+\infty}$.

- If the target location reached by $\rho_{\mathcal{U}}$ is not in $\{\mathsf{t}_{\geq 0}, \mathsf{t}_{<0}\}$, then $\rho_{\mathcal{U}} \in \mathsf{FPlays}_{\mathcal{U}}^*$, and we can use the projector operator to let $\rho_{\overline{\mathcal{G}}} = \mathsf{proj}(\rho_{\mathcal{U}})$. It is conforming to $\tau_{\overline{\mathcal{G}}}$ (by Lemma 6.3). Moreover, by letting $\mathsf{last}(\rho_{\mathcal{U}}) = (\pi, \nu)$ (with $\pi \neq \mathsf{t}_{+\infty}$ by hypothesis), we have $\mathsf{wt}_t'(\pi, \nu) = \overline{\mathsf{wt}_t}(\mathsf{last}(\pi), \nu)$ since $\mathsf{last}(\rho_{\overline{\mathcal{G}}}) = (\mathsf{last}(\pi), \nu)$, by Lemma 6.2.(1). We conclude that $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) = \mathsf{P}(\rho_{\mathcal{U}})$, since proj preserves the weight (by Lemma 6.2.(2)).

- If the target location reached by $\rho_{\mathcal{U}}$ is $\mathsf{t}_{\geq 0}$, then we decompose $\rho_{\mathcal{U}}$ as $\rho_{\mathcal{U}} = \rho_{\mathcal{U}}^1 \xrightarrow{t, \delta'} (\mathsf{t}_{\geq 0}, \nu)$ with $\rho_{\mathcal{U}}^1 \in \mathsf{FPlays}_{\mathcal{U}}^*$ and $(\pi', \nu') = \mathsf{last}(\rho_{\mathcal{U}}^1)$. Since the value in $\overline{\mathcal{G}}$ is supposed to be finite (we removed configurations of value $+\infty$ or $-\infty$), Min can always guarantee to reach the target, i.e. there exists an (attractor) memoryless strategy $\sigma_{\overline{\mathcal{G}}}$ that guarantees to reach L_t. Now, let $\rho_{\overline{\mathcal{G}}} = \rho_{\overline{\mathcal{G}}}^1 \rho_{\overline{\mathcal{G}}}^2$ be such that $\rho_{\overline{\mathcal{G}}}^1 = \mathsf{proj}(\rho_{\mathcal{U}}^1) \xrightarrow{t, \delta} (\ell, \nu)$ with $\delta = \Delta\mathsf{proj}(\delta')$ and $\rho_{\overline{\mathcal{G}}}^2$ be the play from (ℓ, ν) conforming to $\tau_{\overline{\mathcal{G}}}$ and $\sigma_{\overline{\mathcal{G}}}$. To conclude this case, we prove that $\rho_{\overline{\mathcal{G}}}$ is conforming to $\tau_{\overline{\mathcal{G}}}$ and $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{P}(\rho_{\mathcal{U}})$.

  First, since $\mathsf{proj}(\rho_{\mathcal{U}}^1)$ is conforming to $\tau_{\overline{\mathcal{G}}}$ (by Lemma 6.3), then $\rho_{\overline{\mathcal{G}}}^1$ is conforming to $\tau_{\overline{\mathcal{G}}}$ if and only if its last move is. If $\pi' \in L'_{\mathsf{Min}}$, then $\mathsf{proj}(\rho_{\mathcal{U}}^1)$ belongs to Min (by Lemma 6.2.(1)) and $\rho_{\overline{\mathcal{G}}}^1$ is conforming to $\tau_{\overline{\mathcal{G}}}$. Otherwise, we suppose that $\pi' \in L'_{\mathsf{Max}}$, then $\tau_{\mathcal{U}}(\rho_{\mathcal{U}}^1) = (t, \delta')$ and $\mathrm{NEXT}(\pi', \delta) = (\mathsf{t}_{<0}, \delta')$. Thus, since $\mathsf{proj}(\rho_{\mathcal{U}}^1)$ belongs to Max (by Lemma 6.2.(1)) and by the construction of $\tau_{\mathcal{U}}$, we deduce that $\tau_{\overline{\mathcal{G}}}(\mathsf{proj}(\rho_{\mathcal{U}}^1)) = (t, \delta)$, i.e. $\rho_{\overline{\mathcal{G}}}^1$ is conforming to $\tau_{\overline{\mathcal{G}}}$. Finally, we conclude that $\rho_{\overline{\mathcal{G}}}$ is conforming to $\tau_{\overline{\mathcal{G}}}$ by the choice of $\rho_{\overline{\mathcal{G}}}^2$.

  Now, we prove that $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{P}(\rho_{\mathcal{U}})$. First, we remark that

  $$\mathsf{P}(\rho_{\overline{\mathcal{G}}}) = \mathsf{wt}_\Sigma(\rho_{\overline{\mathcal{G}}}^1) + \mathsf{P}(\rho_{\overline{\mathcal{G}}}^2) = \mathsf{wt}_\Sigma(\mathsf{proj}(\rho_{\mathcal{U}}^1)) + t\,\overline{\mathsf{wt}}(\mathsf{last}(\pi')) + \overline{\mathsf{wt}}(\delta) + \mathsf{P}(\rho_{\overline{\mathcal{G}}}^2)\,.$$

  In particular, since $\overline{\mathsf{wt}}(\mathsf{last}(\pi')) = \mathsf{wt}'(\pi')$ (by definition of $\mathcal{U}$) and also by using Lemma 6.2.(2), we obtain:

  $$\mathsf{P}(\rho_{\overline{\mathcal{G}}}) = \mathsf{wt}_\Sigma(\rho_{\mathcal{U}}^1) + t\,\mathsf{wt}'(\pi') + \mathsf{wt}'(\delta') + \mathsf{P}(\rho_{\overline{\mathcal{G}}}^2) = \mathsf{wt}_\Sigma(\rho_{\mathcal{U}}) + \mathsf{P}(\rho_{\overline{\mathcal{G}}}^2)\,.$$

  Moreover, the length of $\rho_{\overline{\mathcal{G}}}^2$ is bounded by $|L|$ (since it is conforming to an attractor, and since regions are already encoded in $\overline{\mathcal{G}}$) and each of its edges has a weight bounded in absolute values by $W_{\mathsf{tr}} + M\,W_{\mathsf{loc}}$. By adding its final weight, we obtain:

  $$\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{wt}_\Sigma(\rho_{\mathcal{U}}) + |L|(W_{\mathsf{tr}} + M\,W_{\mathsf{loc}}) + W_{\mathsf{fin}}\,.$$

  Now, we remark that $\rho_{\mathcal{U}}$ reaches $\mathsf{t}_{\geq 0}$, and its weight is thus:

  $$\mathsf{P}(\rho_{\mathcal{U}}) = \mathsf{wt}_\Sigma(\rho_{\mathcal{U}}) + |L|(W_{\mathsf{tr}} + M\,W_{\mathsf{loc}}) + W_{\mathsf{fin}}\,.$$

  Therefore, $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{P}(\rho_{\mathcal{U}})$.

- Finally, we prove that the case where the target location reached by $\rho_{\mathcal{U}}$ is $\mathsf{t}_{<0}$ is not possible. As before we decompose $\rho_{\mathcal{U}}$ as $\rho_{\mathcal{U}} = \rho_{\mathcal{U}}^1 \xrightarrow{t, \delta'} (\mathsf{t}_{\geq 0}, \nu)$ with $\rho_{\mathcal{U}}^1 \in \mathsf{FPlays}_{\mathcal{U}}^*$ and $(\pi', \nu') = \mathsf{last}(\rho_{\mathcal{U}}^1)$. We consider $\rho_{\overline{\mathcal{G}}}^1 = \mathsf{proj}(\rho_{\mathcal{U}}^1) \xrightarrow{t, \delta} (\ell, \nu)$ with $\delta = \Delta\mathsf{proj}(\delta')$ that is conforming to $\tau_{\overline{\mathcal{G}}}$ (by the same reasoning than the previous case) and we prove that $\rho_{\overline{\mathcal{G}}}^1$ finishes with a play that follows the cyclic path with negative value that contradicts

Lemma 3.5.(2). By definition of $\mathcal{U}$, we have $\textsc{Next}(\pi', \delta) = (\mathsf{t}_{<0}, \delta')$ with $|\pi'|_\delta > 0$, so by letting $\pi' = \pi_1 \xrightarrow{\delta} \pi_2$ with $|\pi_2|_\delta = 0$, we have $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi_2 \xrightarrow{\delta} \ell_2) < 0$ where ℓ_2 is given by δ. Moreover, since $\mathsf{proj}(\rho^1_{\mathcal{U}})$ follows π (by Lemma 6.2.(3)), $\rho^1_{\overline{\mathcal{G}}}$ follows $\pi \xrightarrow{\delta} \ell_2$ that contains a cyclic path $\pi_2 \xrightarrow{\delta} \ell_2$ with a negative value.

To conclude the proof, we have shown that for all plays $\rho_{\mathcal{U}}$ from (ℓ_i, ν) conforming to $\tau_{\mathcal{U}}$, we can build a play $\rho_{\overline{\mathcal{G}}}$ from (ℓ_i, ν) conforming to $\tau_{\overline{\mathcal{G}}}$ such that $\mathsf{P}(\rho_{\overline{\mathcal{G}}}) \leq \mathsf{P}(\rho_{\mathcal{U}})$. In particular,

$$
\begin{aligned}
\mathsf{Val}^{\tau_{\overline{\mathcal{G}}}}_{\overline{\mathcal{G}}}(\ell_i, \nu) &= \inf_{\tau_{\overline{\mathcal{G}}} \in \mathsf{Strat}_{\mathrm{Min}, \overline{\mathcal{G}}}} \mathsf{P}(\mathsf{Play}((\ell_i, \nu), \sigma_{\overline{\mathcal{G}}}, \tau_{\overline{\mathcal{G}}})) \\
&\leq \inf_{\tau_{\mathcal{U}} \in \mathsf{Strat}_{\mathrm{Min}, \mathcal{U}}} \mathsf{P}(\mathsf{Play}((\ell_i, \nu), \sigma_{\mathcal{U}}, \tau_{\mathcal{U}})) \\
&\leq \mathsf{Val}^{\tau_{\mathcal{U}}}_{\mathcal{U}}(\ell_i, \nu) \,.
\end{aligned}
$$

Second inequality. We then prove the reciprocal inequality $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu) \geq \mathsf{Val}_{\mathcal{U}}(\ell_i, \nu)$ that can be rewritten as:

$$
\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu) \geq \sup_{\tau_{\mathcal{U}}} \mathsf{Val}^{\tau_{\mathcal{U}}}_{\mathcal{U}}(\ell_i, \nu) \,.
$$

It thus amounts to showing that Max can guarantee to always do at least as good in $\overline{\mathcal{G}}$ as in $\mathcal{U}$. We thus fix the optimal strategy $\tau_{\mathcal{U}}$ in $\mathcal{U}$ given by Lemma 5.1, and show that $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu) \geq \mathsf{Val}^{\tau_{\mathcal{U}}}_{\mathcal{U}}(\ell_i, \nu)$.

To do so, we show that there exists a strategy $\tau_{\overline{\mathcal{G}}}$ in $\overline{\mathcal{G}}$ such that for a particular play ρ conforming to $\tau_{\overline{\mathcal{G}}}$, there exists a play conforming to $\tau_{\mathcal{U}}$ with a weight at most the weight of ρ. As depicted in Figure 6, the strategy $\tau_{\overline{\mathcal{G}}}$ is defined via a function Φ that maps plays of $\overline{\mathcal{G}}$ into plays of $\mathcal{U}$. Intuitively, this function removes all cyclic plays ending with a reset from plays in $\overline{\mathcal{G}}$. Formally, it is defined by induction on the length of the plays by letting $\Phi(\ell_i, \nu) = (\ell_i, \nu)$, and for all plays $\rho \in \mathsf{FPlays}_{\overline{\mathcal{G}}}$, letting $\rho' = \rho \xrightarrow{t, \delta} (\ell, \nu)$,

(1) if $\Phi(\rho)$ ends in $\mathsf{t}_{+\infty}$, we fix $\Phi(\rho') = \Phi(\rho)$;

(2) else, if δ contains a reset and $\Phi(\rho) = \rho_1 \xrightarrow{t', \delta'} \rho_2$ with $\Delta\mathsf{proj}(\delta') = \delta$, letting π the first location of ρ_2, we fix $\Phi(\rho') = \rho_1 \xrightarrow{t', \delta'} (\pi, 0)$;

(3) otherwise, letting $\textsc{Next}(\pi, \delta) = (\pi', \delta')$ with π the last location of $\Phi(\rho)$, we fix $\Phi(\rho') = \Phi(\rho) \xrightarrow{t, \delta'} (\pi', \nu)$.

This function satisfies the following properties:

Lemma 6.4. *For all plays $\rho \in \mathsf{FPlays}_{\overline{\mathcal{G}}}$, if $\mathsf{last}(\Phi(\rho)) = (\pi, \nu)$ with $\pi \neq \mathsf{t}_{+\infty}$, then we have $\pi \notin \{\mathsf{t}_{<0}, \mathsf{t}_{\geq 0}\}$ and*

$$
\mathsf{last}(\rho) = \begin{cases} (\mathsf{last}(\pi), \nu) & \text{if } \pi \notin L_t\,; \\ (\pi, \nu) & \text{otherwise}\,. \end{cases}
$$

Proof. We show the property by induction on the length of ρ. If $\rho = (\ell_i, \nu)$, then $\Phi(\rho) = \rho$ and the property holds. Otherwise, we let $\rho' = \rho \xrightarrow{t, \delta} (\ell, \nu)$, and we suppose that the property holds for ρ (since it does not end in L'_t) and we follow the definition of Φ.

(1) If $\Phi(\rho)$ ends in $\mathsf{t}_{+\infty}$, we have $\Phi(\rho') = \Phi(\rho)$ and this case is thus not possible (since $\Phi(\rho')$ is supposed to not end in $\mathsf{t}_{+\infty}$).

(2) Else, if δ contains a reset and $\Phi(\rho) = \rho_1 \xrightarrow{t',\delta'} \rho_2$ with $\Delta\mathsf{proj}(\delta') = \delta = (\ell, I, R, w, \ell')$ and $\mathsf{last}(\rho_1) = (\pi_1, \nu)$, we have $\Phi(\rho') = \rho_1 \xrightarrow{t',\delta'} (\pi', 0)$, by letting π' the first location of ρ_2. Moreover, we have $\text{NEXT}(\pi_1, \delta) = (\pi', \delta')$. Now, by definition of NEXT, if $\ell' \in L_t$, then $\pi' = \ell' \in L_t$. Thus, we conclude that $\mathsf{last}(\rho') = (\ell', 0) = (\mathsf{last}(\Phi(\rho')), 0)$ as expected. Otherwise, $\ell' \notin L_t$ and we have $\mathsf{last}(\Phi(\rho)) = (\pi, 0)$. We note that $\pi \notin \{\mathsf{t}_{<0}, \mathsf{t}_{\geq 0}\}$ since ρ_1 does not contain a transition δ'_1 such that $\Delta\mathsf{proj}(\delta'_1) = \delta$ (otherwise, in $\Phi(\rho)$, we would have already fired twice the transition δ with a reset, before trying to fire it a third time). Thus $\pi = \pi' \xrightarrow{\delta} \ell'$, and we conclude.

(3) Otherwise, $\Phi(\rho') = \Phi(\rho) \xrightarrow{t,\delta'} (\pi', \nu)$ if $\text{NEXT}(\pi, \delta) = (\pi', \delta')$ with π the last location of $\Phi(\rho)$. Once again, we are in a case where $\pi' = \pi \xrightarrow{\delta} \ell'$, by letting $\delta = (\ell, I, R, w, \ell')$. Thus, we conclude as before. $\square$

Now, we define $\tau_{\overline{\mathcal{G}}}$ such that its behaviour is the same as the one given by $\tau_{\mathcal{U}}$ after the application of Φ on the finite play, i.e. after the removal of all cyclic paths between the same transition with a reset. Formally, for all plays $\rho \in \mathsf{FPlays}_{\overline{\mathcal{G}}}$, we let $\tau_{\overline{\mathcal{G}}}(\rho)$ be defined as any valid move (t, δ) if $\Phi(\rho)$ ends in $\mathsf{t}_{+\infty}$, and otherwise,

$$\tau_{\overline{\mathcal{G}}}(\rho) = (t, \Delta\mathsf{proj}(\delta')) \qquad \text{if } \tau_{\mathcal{U}}(\Phi(\rho)) = (t, \delta')$$

This is a valid decision for Max. First, by Lemma 6.4, $\mathsf{last}(\rho) = (\mathsf{last}(\pi), \nu)$ when $\mathsf{last}(\Phi(\rho)) = (\pi, \nu)$. Moreover, delays chosen in $\tau_{\overline{\mathcal{G}}}$ and $\tau_{\mathcal{U}}$ are the same, and the guards of δ' and $\Delta\mathsf{proj}(\delta')$ are identical. Thus, whether or not the location π is urgent, the decision $(t, \Delta\mathsf{proj}(\delta'))$ gives rise to an edge in $[\![\overline{\mathcal{G}}]\!]$. Since the definition of $\tau_{\overline{\mathcal{G}}}$ relies on the operation Φ, it is again not surprising that:

Lemma 6.5. *Let $\rho \in \mathsf{FPlays}_{\overline{\mathcal{G}}}$ be a play conforming to $\tau_{\overline{\mathcal{G}}}$. Then $\Phi(\rho)$ is conforming to $\tau_{\mathcal{U}}$.*

Proof. We reason by induction on the length of ρ. If $\rho = (\ell_\mathsf{i}, \nu)$, then $\Phi(\rho) = (\ell_\mathsf{i}, \nu)$ and the property is trivial. Otherwise, we suppose that $\rho' = \rho \xrightarrow{t,\delta} (\ell, \nu)$. By the induction hypothesis, $\Phi(\rho)$ conforms to $\tau_{\mathcal{U}}$.

(1) If $\Phi(\rho)$ ends in $\mathsf{t}_{+\infty}$, we have $\Phi(\rho') = \Phi(\rho)$ that is conforming to $\tau_{\mathcal{U}}$.

(2) If δ contains a reset and $\Phi(\rho) = \rho_1 \xrightarrow{t',\delta'} \rho_2$ with $\Delta\mathsf{proj}(\delta') = \delta$, letting π be the first location of ρ_2, we have $\Phi(\rho') = \rho_1 \xrightarrow{t',\delta'} (\pi, 0)$. This is a prefix of $\Phi(\rho)$ that is conforming to $\tau_{\mathcal{U}}$. Thus, $\Phi(\rho')$ is conforming to $\tau_{\mathcal{U}}$ too.

(3) Otherwise, $\Phi(\rho') = \Phi(\rho) \xrightarrow{t,\delta'} (\pi', \nu)$ if $\text{NEXT}(\pi, \delta) = (\pi', \delta')$ with π the last location of $\Phi(\rho)$. If $\Phi(\rho)$ ends in a location of Min, since it is conforming to $\tau_{\mathcal{U}}$, so does $\Phi(\rho')$. Otherwise, $\tau_{\overline{\mathcal{G}}}(\rho) = (t, \delta)$ which implies that $\tau_{\mathcal{U}}(\Phi(\rho)) = (t, \delta'')$ with $\Delta\mathsf{proj}(\delta'') = \delta$, meaning that $\text{NEXT}(\pi, \delta) = (\pi', \delta'')$, i.e. $\delta'' = \delta'$: in this case too, $\Phi(\rho')$ is conforming to $\tau_{\mathcal{U}}$. $\square$

Finally, we prove that $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_\mathsf{i}, \nu) \geq \mathsf{Val}_{\mathcal{U}}^{\tau_{\mathcal{U}}}(\ell_\mathsf{i}, \nu)$. Notice that we do not aim at comparing $\mathsf{Val}_{\mathcal{U}}^{\tau_{\mathcal{U}}}(\ell_\mathsf{i}, \nu)$ with $\mathsf{Val}_{\overline{\mathcal{G}}}^{\tau_{\overline{\mathcal{G}}}}(\ell_\mathsf{i}, \nu)$ but instead directly with $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_\mathsf{i}, \nu)$. This is helpful here since we do not need to start with any play ρ conforming to $\tau_{\overline{\mathcal{G}}}$. Instead, we pick a special play, choosing well the strategy followed by Min. Indeed, we suppose that Min follows an ε-optimal (switching) strategy σ in $\overline{\mathcal{G}}$, as given in [BGH$^+$22]. As we explained before in Definition 4.4, in WTGs without resets, this ensures that in all plays $\rho_{\overline{\mathcal{G}}}$ conforming to σ, the target is reached fast enough (with a number of transitions bounded by κ). We can easily enrich the result

of [BGH$^+$22] to take into account resets. Indeed, as performed in [BGH$^+$22, Theorem 6.6] to show that all one-clock WTGs have an (a priori non-computable) value function that is piecewise affine with a finite number of cutpoints, we can replace each transition with a reset by a new transition jumping in a fresh target location of value given by the value function we aim at computing. From a strategy perspective, this means that in each component of our unfolding (in-between two transitions with a reset), Min follows a switching strategy. Notice that such strategies are a priori not knowing to be computable (since we cannot perform the transformation described above, using the value function), but we use only its existence in this proof.

We thus consider an ε-optimal strategy σ for Min in $\overline{\mathcal{G}}$ such that in all plays $\rho_{\overline{\mathcal{G}}}$ conforming to σ, in-between two transitions with a reset and after the last such transition, the number of transitions is bounded by κ. We now fix the special play ρ from (ℓ_i, ν) conforming to σ and $\tau_{\overline{\mathcal{G}}}$. It reaches a target since σ is ε-optimal and $\mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu) \neq +\infty$. We show that

$$\exists \rho_{\mathcal{U}} \in \mathsf{FPlays}_{\mathcal{U}} \text{ conforming to } \tau_{\mathcal{U}} \quad \mathsf{P}(\rho_{\mathcal{U}}) \leq \mathsf{P}(\rho) \tag{$\star$}$$

As a consequence, we obtain:

$$\mathsf{Val}_{\mathcal{U}}^{\tau_{\mathcal{U}}}(\ell_i, \nu) = \inf_{\sigma_{\mathcal{U}} \in \mathsf{Strat}_{\mathsf{Min},\mathcal{U}}} \mathsf{P}(\mathsf{Play}((\ell_i, \nu), \sigma_{\mathcal{U}}, \tau_{\mathcal{U}})) \leq \mathsf{P}(\rho_{\mathcal{U}}) \leq \mathsf{P}(\rho) \leq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have $\mathsf{Val}_{\mathcal{U}}^{\tau_{\mathcal{U}}}(\ell_i, \nu) \leq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell_i, \nu)$ as expected.

To show $(\star)$, we proceed by induction on the prefixes ρ' of ρ, proving that $(\star)$ holds or that $\Phi(\rho')$ does not end in $\mathsf{t}_{+\infty}$ and $\mathsf{wt}_\Sigma(\Phi(\rho')) \leq \mathsf{wt}_\Sigma(\rho')$. Indeed, at the end of the induction, we therefore obtain $(\star)$ or that $\Phi(\rho)$ does not end in $\mathsf{t}_{+\infty}$ and $\mathsf{wt}_\Sigma(\Phi(\rho)) \leq \mathsf{wt}_\Sigma(\rho)$. In the case where $(\star)$ does not hold, we fix $\rho_{\mathcal{U}} = \Phi(\rho)$ and $\mathsf{last}(\rho_{\mathcal{U}}) = (\pi, \nu)$. In particular, we have $\pi \in L_t$, and $\mathsf{last}(\rho) = (\pi, \nu)$: by Lemma 6.4, if $\pi \notin L_t$, then $\mathsf{last}(\rho) = (\mathsf{last}(\pi), \nu)$, with $\mathsf{last}(\pi) \notin L_t$ that contradicts the fact that ρ reaches the target. Therefore,

$$\mathsf{P}(\rho_{\mathcal{U}}) = \mathsf{P}(\Phi(\rho)) = \mathsf{wt}_\Sigma(\Phi(\rho)) + \mathsf{wt}_t'(\pi, \nu) \leq \mathsf{wt}_\Sigma(\rho) + \mathsf{wt}_t(\pi, \nu) = \mathsf{P}(\rho).$$

Since $\rho_{\mathcal{U}}$ is conforming to $\tau_{\mathcal{U}}$ (by Lemma 6.5), we obtain $(\star)$ here too.

Finally, we proceed to the proof by induction. First, we suppose that $\rho' = (\ell_i, \nu)$ and $\mathsf{wt}_\Sigma(\Phi(\rho')) = 0 = \mathsf{wt}_\Sigma(\rho')$. Otherwise, we suppose that $\rho' = \rho'' \xrightarrow{t,\delta} (\ell, \nu)$. By induction on ρ'', if $(\star)$ does not (already) hold, we know that $\Phi(\rho'')$ does not end in $\mathsf{t}_{+\infty}$ and $\mathsf{wt}_\Sigma(\Phi(\rho'')) \leq \mathsf{wt}_\Sigma(\rho'')$. We follow the three cases of the definition of $\Phi(\rho')$.

(1) We cannot have $\Phi(\rho'')$ ending in $\mathsf{t}_{+\infty}$ by hypothesis.

(2) Suppose now that δ contains a reset and $\Phi(\rho'') = \rho_1 \xrightarrow{t',\delta'} \rho_2$ with $\Delta\mathsf{proj}(\delta') = \delta$. Letting π the first location of ρ_2, we have $\Phi(\rho') = \rho_1 \xrightarrow{t',\delta'} (\pi, 0)$. Thus

$$\mathsf{wt}_\Sigma(\Phi(\rho')) = \mathsf{wt}_\Sigma(\Phi(\rho'')) - \mathsf{wt}_\Sigma(\rho_2) \leq \mathsf{wt}_\Sigma(\rho'') - \mathsf{wt}_\Sigma(\rho_2) \tag{6.1}$$

Let $(\pi', \nu') = \mathsf{last}(\rho_2)$, and $\rho_{\mathcal{U}} = \Phi(\rho'') \xrightarrow{t,\delta''} (\pi'', 0)$, with $\mathsf{NEXT}(\pi', \delta) = (\pi'', \delta'')$. Notice that $\rho_{\mathcal{U}}$ is conforming to $\tau_{\mathcal{U}}$, since $\Phi(\rho'')$ does and if π' belongs to Max, this follows directly from the definition of $\tau_{\overline{\mathcal{G}}}$ from $\tau_{\mathcal{U}}$ (since $\tau_{\overline{\mathcal{G}}}(\rho''_{\overline{\mathcal{G}}}) = (t, \delta)$, and $\Phi(\rho'') \notin \mathsf{t}_{+\infty}$). Moreover, it contains twice a transition with a reset coming from the same transition δ of $\overline{\mathcal{G}}$, therefore $\pi'' \in \{\mathsf{t}_{<0}, \mathsf{t}_{\geq 0}\}$. If $\pi'' = \mathsf{t}_{<0}$, $\mathsf{P}(\rho_{\mathcal{U}}) = -\infty$ and $(\star)$ holds. Otherwise, if $\pi'' = \mathsf{t}_{\geq 0}$, by Lemma 5.1 applied on $\rho_{\mathcal{U}}$, $\mathsf{wt}_\Sigma(\rho_2 \xrightarrow{t,\delta''} (\mathsf{t}_{\geq 0}, 0)) \geq 0$, i.e. $\mathsf{wt}_\Sigma((\pi', \nu') \xrightarrow{t,\delta''}$

$(\mathsf{t}_{\geq 0}, 0)) \geq -\mathsf{wt}_\Sigma(\rho_2)$. Combined with (6.1), we obtain that

$$
\begin{aligned}
\mathsf{wt}_\Sigma(\Phi(\rho')) &\leq \mathsf{wt}_\Sigma(\rho'') + \mathsf{wt}_\Sigma((\pi', \nu') \xrightarrow{t, \delta''} (\mathsf{t}_{\geq 0}, 0)) \\
&= \mathsf{wt}_\Sigma(\rho'') + t\,\mathsf{wt}'(\pi') + \mathsf{wt}'(\delta'') \\
&= \mathsf{wt}_\Sigma(\rho'') + t\,\overline{\mathsf{wt}}(\ell') + \overline{\mathsf{wt}}(\delta) = \mathsf{wt}_\Sigma(\rho')
\end{aligned}
$$

where we let ℓ' be the last location of ρ'', which is also the last location of π'.

(3) Otherwise, $\Phi(\rho') = \Phi(\rho'') \xrightarrow{t, \delta'} (\pi', \nu)$ if $\mathrm{NEXT}(\pi, \delta) = (\pi', \delta')$ with π the last location of $\Phi(\rho'')$. In this case,

$$
\begin{aligned}
\mathsf{wt}_\Sigma(\Phi(\rho')) &= \mathsf{wt}_\Sigma(\Phi(\rho'')) + t\,\mathsf{wt}'(\pi) + \mathsf{wt}'(\delta') \\
&\leq \mathsf{wt}_\Sigma(\rho'') + t\,\overline{\mathsf{wt}}(\ell') + \overline{\mathsf{wt}}(\delta) = \mathsf{wt}_\Sigma(\rho')
\end{aligned}
$$

where we let ℓ' e the last location of ρ''.

This ends the proof by induction.

7. Main decidability result

By using the unfolding, we are now able to conclude the proof of Theorem 2.10, i.e. to compute the value function of $\mathcal{G}$ in exponential time with respect to $|Q|$ and W.

Remember (by Lemma 2.6) that we only need to explain how to compute $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q_\mathsf{i}, I_\mathsf{i}), \nu)$ over I_i. By Theorem 6.1, this is equivalent to computing $\nu \mapsto \mathsf{Val}_{\mathcal{U}}((q_\mathsf{i}, I_\mathsf{i}), \nu)$ over I_i. We now explain why this is doable.

First, the definition of $\mathcal{U}$ is effective: we can compute it entirely, making use of Lemma 4.5 showing that it is a finite WTG. The only non-trivial part is the test of the sign of $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi_2 \xrightarrow{\delta} \ell_2)$ in line 4 of Algorithm 1 to determine in which target location we jump. Since $\pi_2 \xrightarrow{\delta} \ell_2$ is a finite path, we can apply Theorem 2.8 to compute the value of the corresponding game, which is exactly the value $\mathsf{Val}_{\overline{\mathcal{G}}}(\pi_2 \xrightarrow{\delta} \ell_2)$. The complexity of computing the value of a path is polynomial in the length of this path (that is exponential in $|Q|$ and W, by Lemma 4.5) and polynomial in $|Q|$ and W (notice that weights of $\overline{\mathcal{G}}$ are the same as the ones in $\mathcal{G}$): this is thus of complexity exponential in $|Q|$ and W. Since $\mathcal{U}$ has an exponential number of locations with respect to $|Q|$ and W, the total time required to compute $\mathcal{U}$ is also exponential with respect to $|Q|$ and W.

Lemma 4.5 ensures that $\mathcal{U}$ is acyclic, so we can apply Theorem 2.8 to compute the value mapping $\nu \mapsto \mathsf{Val}_{\mathcal{U}}((q_\mathsf{i}, I_\mathsf{i}), \nu)$ as a piecewise affine and continuous function. It requires a complexity polynomial in the number of locations of $\mathcal{U}$, and in W (since weights of $\mathcal{U}$ all come from $\mathcal{G}$). Knowing the previous bound on the number of locations of $\mathcal{U}$, this complexity translates into an exponential time complexity with respect to $|Q|$ and W, as announced.

8. The value function of (one-clock) WTGs is the greatest fixpoint of $\mathcal{F}$

We finally prove Theorem 2.4, i.e. that the value function of all (one-clock) WTGs is the greatest fixpoint of the operator $\mathcal{F}$. A natural way to prove this theorem would be to use the
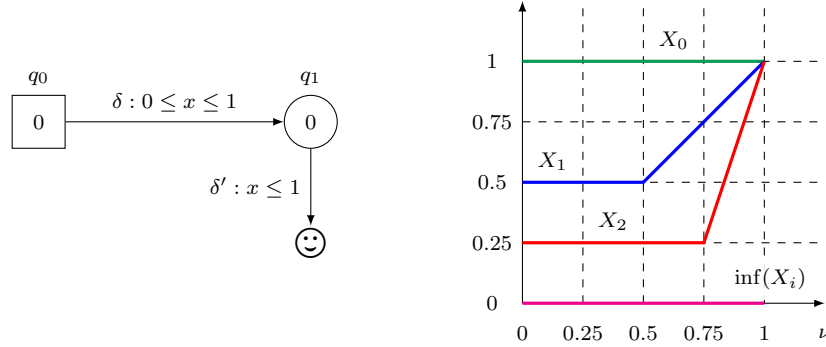
Figure 7: On the left, a WTG in which $\mathcal{F}$ is not Scott-continuous, for instance when we consider the non-increasing sequence of continuous functions on $(0,1)$ depicted on the right for all locations.

fixpoint theory and, more precisely, Kleene's theorem characterising the greatest fixpoint[1] as the limit of a sequence of iterates of $\mathcal{F}$ before showing that the limit is equal to the value of $\mathcal{G}$. To be applicable, this theorem requires $\mathcal{F}$ to be *Scott-continuous* over a complete partial order (CPO) [Win93, Chapter 8], i.e. monotonous and such that for all non-increasing sequence $(X_i)_{i \in \mathbb{N}}$ of elements of the CPO, $\inf_i \mathcal{F}(X^i) = \mathcal{F}(\inf_i X^i)$. Unfortunately, although the operator $\mathcal{F}$ can be shown to be monotonous (see Lemma 8.4.1), it is not necessarily Scott-continuous as demonstrated by the following example.

Example 8.1. We consider the WTG depicted on the left of Figure 7. We prove that $\mathcal{F}$ is not Scott-continuous by exhibiting a non-increasing sequence $(X_i)_i$ of functions (that are continuous over regions) such that $\inf_i X_i > \mathcal{F}(\inf_i X_i)$. The sequence is depicted in the right of Figure 7 and can be defined, for all configurations (q, ν) and all $i \in \mathbb{N}$, by:

$$X_i(q,\nu) = \begin{cases} \frac{1}{2^i} & \text{if } 0 < \nu \le \frac{2^i - 1}{2^i} \\ (2^i - 1)\nu + (2 - 2^i) & \text{if } \frac{2^i - 1}{2^i} < \nu < 1 \end{cases}$$

We consider the location q_0 of Max. For all $i \in \mathbb{N}$ and all valuations ν, we have:

$$\mathcal{F}(X_i)(q_0, \nu) = \sup_t (t\,\mathsf{wt}(q_0) + \mathsf{wt}(\delta) + X_i(q_1, \nu + t)) = \sup_t X_i(q_1, \nu + t) = 1$$

and thus $\inf_i \mathcal{F}(X_i)(q_0, \nu) = 1$. Moreover, since $\inf_i X_i$ is constant function whose value is 0 in the interval $(0,1)$, we deduce that $\mathcal{F}(\inf_i X_i)(q_0, \nu) = 0$ for all configurations (q_0, ν). Thus, we deduce that $\mathcal{F}(\inf_i X_i)(q_0, \nu) < \inf_i \mathcal{F}(X_i)(q_0, \nu)$.

We thus design a more pedestrian proof only using non-increasing sequences (V_i) defined by an iteration of the operator $\mathcal{F}$ (as in [Tar55]) that *uniformly* converge over each region, i.e. the restriction of the sequence to each region uniformly converges. In particular, we adapt and correct the sketch given in [Bou16] for concurrent hybrid games with only non-negative weights to the context of (one-clock) WTGs with negative weights. As in [Bou16], our proof is split into two parts:

---

[1] A careful reader will remark that Kleene's theorem characterises the least fixpoint for increasing sequences of elements in a complete partial order (CPO). Intuitively, we use this version with a reverse order (the CPO admits upper-bounds instead of lower-bounds). Formally, we can fit the hypothesis of Kleene's theorem by considering the operator $-\mathcal{F}$.

(1) In Section 8.2, we prove that the sequence V_i of iterates of $\mathcal{F}$ (used in the *value iteration-*
based algorithm of [BCFL04]) converges toward the greatest fixpoint of $\mathcal{F}$. In [Bou16],
it is proved that all non-increasing sequences of functions that uniformly converge over
each region are a fixpoint of the operator $\mathcal{F}$. The key argument of [Bou16] is to prove
the uniform convergence of the sequence V_i by using Dini's theorem. We show that this
is legal by showing that functions V_i are all k-Lipschitz-continuous for the same constant
k (which requires us to restrict to one-clock WTGs).
(2) In Section 8.3, we prove that the sequence V_i of iterates of $\mathcal{F}$ converges to the value
function. The key argument in [Bou16] is to remark that the mapping obtained after
i applications of $\mathcal{F}$ is a value function when we consider only plays of length i. We
formalise this intuition by inductively defining a strategy of Min that will increase the
length i of the plays such that its value is upper bounded by V_i. To do it, in Section 8.1,
we start by proving that a fixpoint of a restriction of $\mathcal{F}$ under a given strategy of Min is
the value of this strategy.

We now fix a (one-clock) WTG $\mathcal{G}$. We have supposed that the final weight functions
are continuous over each region. Without loss of generality, we may also suppose that final
weights $\mathsf{wt}_t(q, \nu)$ are different from $+\infty$, for all configurations (q, ν) with $q \in Q_t$. To do so,
it suffices to forbid the jump into a region I where the final weight function is constant equal
to $+\infty$, by modifying the guard on the incoming transitions.

8.1. **Restriction of $\mathcal{F}$ according to a strategy of Min.** Before to start the proof of
Theorem 2.4, we establish a partial result when we have fixed the strategy of Min. In
particular, we give a link between the value Val^σ of a strategy σ of Min and the restriction
$\mathcal{F}^\sigma$ of $\mathcal{F}$ according to this strategy, by replacing the infimum for locations of Min with the
choice given by σ. However, since we can not (and we do not want to) suppose that σ is
memoryless, we need to extend the functions that $\mathcal{F}^\sigma$ use. Formally, $\mathcal{F}^\sigma$ is a new operator
over functions $X \colon \mathsf{FPlays} \to \overline{\mathbb{R}}$ such that $\mathcal{F}^\sigma(X)(\rho)$ is equal to

$$
\begin{cases}
\mathsf{wt}_t(q, \nu) & \text{if } q \in Q_t \\
\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(\rho \xrightarrow{t,\delta} (q', \nu')) & \text{if } q \in Q_{\mathsf{Min}} \text{ and } \sigma(\rho) = (t, \delta) \\
\sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(\rho \xrightarrow{t,\delta} (q', \nu')) \right) & \text{if } q \in Q_{\mathsf{Max}}
\end{cases}
$$

where $\mathsf{last}(\rho) = (q, \nu)$.

Since the value function Val^σ has only been defined for configurations, we need to extend
it over all finite plays. To do that, we define the weight of a play given by two strategies (σ
and τ) from a given finite play ρ by the weight of the unique play ρ' conforming to σ and τ
from the last configuration of ρ (when σ and τ are initialised by ρ), i.e.

$$
\mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) = \mathsf{wt}(\rho') \,.
$$

Even if the weight of ρ is not taken into account in the weight of $\mathsf{Play}(\rho, \sigma, \tau)$, we observe
that $\mathsf{Play}(\mathsf{last}(\rho), \sigma, \tau)$ does not describe the same accumulated weight as $\mathsf{Play}(\rho, \sigma, \tau)$ (since
σ and τ may use some memory). We thus let, for all finite plays ρ,

$$
\mathsf{Val}^\sigma(\rho) = \sup_\tau \mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) \,.
$$

Lemma 8.2. Val^σ *is a fixpoint of* $\mathcal{F}^\sigma$.

Proof. Let ρ be a finite play and (q, ν) its last configuration. If $q \in Q_t$, then for all strategies τ of Max, $\mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) = \mathsf{wt}_t(q, \nu)$. Thus, we obtain that

$$\mathsf{Val}^\sigma(\rho) = \sup_\tau \mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) = \mathsf{wt}_t(q, \nu) = \mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho)$$

where the second equality follows by applying the supremum over strategies of Max.

Now, we suppose that $q \in Q_{\mathsf{Min}}$, and let $\sigma(\rho) = (t, \delta)$. Thus, for all strategies τ of Max, we obtain that

$$\mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) = \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau))\,.$$

In particular, by applying the supremum over strategies of Max, we obtain that

$$\mathsf{Val}^\sigma(\rho) = \sup_\tau \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau))\big)\,.$$

We note that the choice (t, δ) of σ depends only on ρ that is independent of the chosen strategy of Max. Thus, we deduce that

$$\mathsf{Val}^\sigma(\rho) = \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \sup_\tau \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau))$$

$$= \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q', \nu')) = \mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho)\,.$$

Finally, we suppose that $q \in Q_{\mathsf{Max}}$ and we reason by double inequalities. We begin by showing that $\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho) \leq \mathsf{Val}^\sigma(\rho)$. Let $\varepsilon > 0$, by the definition of $\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho)$, we obtain the existence of an edge $(q, \nu) \xrightarrow{t,\delta} (q', \nu')$ such that

$$\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho) \leq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q', \nu')) + \frac{\varepsilon}{2}\,.$$

Similarly, by the definition of Val^σ, there exists a strategy τ^* for Max such that

$$\mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q', \nu')) \leq \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau^*)) + \frac{\varepsilon}{2}\,.$$

In particular, by combining these two inequalities, we obtain:

$$\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho) \leq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau^*)) + \varepsilon\,.$$

We consider a new strategy τ for Max defined such that $\tau(\rho) = (t, \delta)$ and $\tau(\rho') = \tau^*(\rho')$, for all finite plays $\rho' \neq \rho$. In particular, since τ and τ^* make the same choice for all plays that extend $\rho \xrightarrow{t,\delta} (q', \nu')$, we obtain that $\mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau^*) = \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau)$. Thus, we deduce that

$$\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho) \leq \mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau)) + \varepsilon \leq \sup_{\tau'} \big(\mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau'))\big) + \varepsilon = \mathsf{Val}^\sigma(\rho) + \varepsilon\,.$$

Since this inequality holds for all $\varepsilon > 0$, it follows that $\mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho) \leq \mathsf{Val}^\sigma(\rho)$.

Conversely, we prove that $\mathsf{Val}^\sigma(\rho) \leq \mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho)$. Let $\varepsilon > 0$, and as for the previous inequality, there exists a strategy τ^* of Max such that

$$\mathsf{Val}^\sigma(\rho) - \varepsilon \leq \mathsf{P}(\mathsf{Play}(\rho, \sigma, \tau^*))\,.$$

In particular, by letting $(t, \delta) = \tau^*(\rho)$, we deduce that

$$\mathsf{Val}^\sigma(\rho) - \varepsilon \leq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{P}(\mathsf{Play}(\rho \xrightarrow{t,\delta} (q', \nu'), \sigma, \tau^*))$$

$$\leq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q', \nu'))$$

$$\leq \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q', \nu')) \right) = \mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho).$$

Finally, since this inequality holds for all $\varepsilon > 0$, we obtain that $\mathsf{Val}^\sigma(\rho) \leq \mathcal{F}^\sigma(\mathsf{Val}^\sigma)(\rho)$. $\quad\square$

8.2. **Iterates of $\mathcal{F}$ uniformly converge to the greatest fixpoint of $\mathcal{F}$.** We now prove the first result needed in the proof of Theorem 2.4. In particular, we consider the sequence $(V_i)_i$ of functions $Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ defined, for all $i \in \mathbb{N}$ and for all configurations (q, ν) by

$$V_i(q, \nu) = \begin{cases} +\infty & \text{if } i = 0 \text{ and } q \notin Q_t \\ \mathsf{wt}_t(q, \nu) & \text{if } i = 0 \text{ and } q \in Q_t \\ \mathcal{F}(V_{i-1})(q, \nu) & \text{otherwise.} \end{cases}$$

Proposition 8.3. $\inf_i V_i$ *is the greatest fixpoint of $\mathcal{F}$.*

This section is devoted to the proof of this proposition. In particular, our proof relies on the following technical results[2] providing sufficient condition on the limit of the sequence $(V_i)_i$ to be a fixpoint of $\mathcal{F}$.

Lemma 8.4. (1) *$\mathcal{F}$ is monotonous[3] over $Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$, where the partial order over $Q \times \mathbb{R}_{\geq 0}$ is the pointwise order over Q and the usual order over $\mathbb{R}_{\geq 0}$.*
(2) *For all $X : Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ and $a \geq 0$, $\mathcal{F}(X + a) \leq \mathcal{F}(X) + a$.*
(3) *For all non-increasing sequences $(X_i)_i$ of functions $X_i : Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ that uniformly converge over each region[4], $\inf_i \mathcal{F}(X_i) = \mathcal{F}(\inf_i X_i)$.*

Proof. (1) Let $X, X' : Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ be two functions such that $X \geq X'$ (i.e. $X(q, \nu) \geq X'(q, \nu)$ for all configurations (q, ν)), and let (q, ν) be a configuration. If $q \in Q_t$, then $\mathcal{F}(X)(q, \nu) = \mathsf{wt}_t(q, \nu) = \mathcal{F}(X')(q, \nu)$. Otherwise, since $X(q', \nu') \geq X'(q', \nu')$, for all edges $(q, \nu) \xrightarrow{t,\delta} (q', \nu')$, we have:

$$\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(q', \nu') \geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X'(q', \nu')$$

Finally, we apply the infimum (resp. supremum) over all edges in this inequality if $q \in Q_{\mathsf{Min}}$ (resp. $q \in Q_{\mathsf{Max}}$).
(2) Let (q, ν) be a configuration.
  • If $q \in Q_t$, then, since $a \geq 0$, we have:

$$\mathcal{F}(X + a)(q, \nu) = \mathsf{wt}_t(q, \nu) = \mathcal{F}(X)(q, \nu) \leq \mathcal{F}(X)(q, \nu) + a.$$

---

[2]These results hold for all WTGs and not only one-clock WTG.

[3]A function $f : X \to Y$ over partial orders X and Y is monotonous if for all $x \leq x'$ in X, we have $f(x) \leq f(x')$ in Y.

[4]A sequence of functions $(f_i)_i$ from partial orders X to $\overline{\mathbb{R}}$ uniformly converge over $A \subset X$ towards a function f if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ and $x \in A$, $|f_i(x) - f(x)| \leq \varepsilon$.

- If $q \in Q_{\mathsf{Min}}$, then, since a does not depend on edges, we have:

$$
\begin{aligned}
\mathcal{F}(X + a)(q, \nu) &= \inf_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(q', \nu') + a\big) \\
&= \inf_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + X(q', \nu')\big) + a \\
&= \mathcal{F}(X)(q, \nu) + a\,.
\end{aligned}
$$

- If $q \in Q_{\mathsf{Max}}$, then, for the same reason, we have $\mathcal{F}(X + a)(q, \nu) = \mathcal{F}(X)(q, \nu) + a$.

(3) Since $\mathcal{F}$ is monotonous (by item (1)), we remark that for all $j \in \mathbb{N}$, we have $\mathcal{F}(X_j) \geq \mathcal{F}(\inf_i X_i)$. In particular, as this inequality holds for all $j \in \mathbb{N}$, we obtain that

$$
\inf_i \mathcal{F}(X_i) \geq \mathcal{F}(\inf_i X_i)\,.
$$

Conversely, let $\varepsilon > 0$ and I be a region. Since $(X_i)_i$ uniformly converges over I to $\inf_i X_i$ (since the sequence (X_i) is non-increasing), there exists $j_I \in \mathbb{N}$ such that $X_{j_I} \leq \inf_i X_i + \varepsilon$ over I. Now, since there are only a finite number of regions, we fix $j = \max_I j_I$. Thus, since the sequence $(X_i)_i$ is non-increasing, for all regions I, $X_j \leq \inf_i X_i + \varepsilon$. Since $\mathcal{F}$ is monotonous and by item (2),

$$
\mathcal{F}(X_j) \leq \mathcal{F}\big(\inf_i X_i + \varepsilon\big) \leq \mathcal{F}\big(\inf_i X_i\big) + \varepsilon\,.
$$

In particular, we deduce that $\inf_i \mathcal{F}(X_i) \leq \mathcal{F}\big(\inf_i X_i\big) + \varepsilon$, for all $\varepsilon > 0$. $\qquad\square$

As a corollary of this result, we prove that $\inf_i V_i$ is a fixpoint of $\mathcal{F}$ by proving that it is a non-increasing sequence that uniformly converges. In particular, we observe that this sequence of functions is non-increasing since $\mathcal{F}$ is monotonous (by Lemma 8.4.(1)) and $V_0 \geq \mathcal{F}(V_0)$ (since $V_0(q, \nu) = +\infty$, or $V_0(q, \nu) = V_1(q, \nu) = \mathsf{wt}_t(q, \nu)$). In particular, it (simply) converges to $\inf_i V_i$. To prove that $(V_i)_i$ uniformly converges to $\inf_i V_i$, we will use Dini's theorem: a sequence of continuous functions that (simply) converges to a continuous function, uniformly converges. The main difficulty is to prove that $\inf_i V_i$ is continuous over regions. To do it, we note that if a sequence of k-Lipschitz-continuous functions (simply) converges, then its limit is a continuous function. In particular, we want to show that there exists $k \in \mathbb{R}_{\geq 0}$ such that, for all $i \in \mathbb{N}$, V_i is k-Lipschitz-continuous.

Definition 8.5. A function $f \colon \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ is *continuous* (respectively, *k-Lipschitz-continuous, for $k \in \mathbb{R}_{\geq 0}$) on regions* if for all regions I, the restriction of f over each region is a continuous (respectively, k-Lipschitz-continuous) function. A function $f \colon \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ is k-*Lipschitz-continuous on regions*, for $k \in \mathbb{R}_{\geq 0}$, if for all regions I and all valuations $\nu, \nu' \in I$, $|f(\nu) - f(\nu')| \leq \Lambda |\nu - \nu'|$.

A function $X \colon Q \times \mathbb{R}_{\geq 0} \to \overline{\mathbb{R}}$ is *continuous* (respectively, *k-Lipschitz-continuous) on regions*, if for all locations $q \in Q$, the restriction of X to q is continuous (respectively, k-Lipschitz-continuous).

We let Λ be the maximum absolute value of all weights of locations and of derivatives that appear in the piecewise-affine functions (the slopes of the affine pieces) of wt_t. Then, V_0 is trivially Λ-Lipschitz-continuous on regions. Indeed, being a Λ-Lipschitz-continuous function on regions when already being a continuous and piecewise affine function with finitely many pieces is equivalent to having its derivatives bounded by Λ in absolute value. In [Bus19, Lemma 10.10], it is shown in all WTGs, for all $i \in \mathbb{N}$, V_i is Λ_i-Lipschitz-continuous

on regions for a constant Λ_i that depends on i. We now refine the proof, in our one-clock setting, to show that the same constant Λ can be chosen for all i.

Lemma 8.6. *For all $i \in \mathbb{N}$, V_i is Λ-Lipschitz-continuous on regions.*

Proof. We rely on the knowledge that for all $i \in \mathbb{N}$, V_i is continuous and piecewise affine on each regions, with finitely many pieces, i.e. each V_i has a finite number of cutpoints[5].

We reason by induction on $i \in \mathbb{N}$ showing that the derivative of V_i is bounded by Λ in absolute values. The base case $i = 0$ is trivially satisfied as seen above. Let $i \in \mathbb{N}$ be such that V_i is continuous on regions and piecewise affine with finitely many pieces that have a derivative bounded by Λ. Let $q \in Q \setminus Q_t$ (otherwise, we conclude as for $i = 0$). By massaging the definition of $\mathcal{F}$, we have that

$$V_{i+1}(q, \nu) = \begin{cases} \min_\delta \inf_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu')\big) & \text{if } q \in Q_{\mathsf{Min}} \\ \max_\delta \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu')\big) & \text{if } q \in Q_{\mathsf{Max}} \end{cases}$$

For a fixed valuation ν, and once chosen the transition δ in the minimum or maximum, there are finitely many delays t to consider in the infimum or supremum: since V_i is piecewise affine, they are either delay 0 or all delays t such that $\nu + t$ are cutpoints ν^c of $V_i(q', \cdot)$. In particular, since there is only a finite number of such cutpoints, the function $\mathcal{F}(V_i)(q, \cdot)$ can be written as a finite nesting of min and max operations over affine terms, each corresponding to a choice of delay and a transition to take. There are several cases to define those terms, depending on the chosen transition δ and cutpoint ν^c. If the transition δ resets x:

- if a delay 0 is chosen, then the affine term is $V_i(q', 0) + \mathsf{wt}(\delta)$ that has derivative 0;
- otherwise, the affine term that it generates is of the form:

$$(\nu^c - \nu)\,\mathsf{wt}(q) + \mathsf{wt}(\delta) + V_i(q', 0)$$

whose derivative is bounded by W_{loc} in absolute value, and thus by Λ.

If the transition δ does not reset x:

- if a delay 0 is chosen, then the affine term is $\mathsf{wt}(\delta) + V_i(q', \nu)$, whose derivative is the same as in $V_i(q', \cdot)$ and thus bounded by Λ in absolute value;
- otherwise, the affine term that it generates is of the form:

$$(\nu^c - \nu)\,\mathsf{wt}(q) + \mathsf{wt}(\delta) + V_i(q', \nu^c)\,.$$

whose derivative is bounded by W_{loc} in absolute value, and thus by Λ. □

Now, we have tools to prove Proposition 8.3. First, we prove that $\inf_i V_i$ is a fixed point of $\mathcal{F}$, i.e. $\inf_i V_i = \mathcal{F}(\inf_i V_i)$. By Lemma 8.6, we know that for all $i \in \mathbb{N}$, V_i is Λ-Lipschitz-continuous over regions. Thus, we deduce that $(V_i)_i$ converges to a continuous function over regions, i.e. $\inf_i V_i(q)$ is continuous over regions, for all locations q. Now, by Dini's theorem, we deduce that $(V_i)_i$ uniformly converges over regions to $\inf_i V_i$. Finally, we apply Lemma 8.4.(3) to conclude that $\inf_i V_i = \inf_i \mathcal{F}(V_i) = \mathcal{F}(\inf_i V_i)$, and thus that $\inf_i V_i$ is a fixpoint of $\mathcal{F}$.

Finally, we prove that $\inf_i V_i$ is the greatest fixpoint V of $\mathcal{F}$. As V is the greatest fixpoint, we have $\inf_i V_i \leq V$. Conversely, we prove by induction on $i \in \mathbb{N}$ that $V \leq V_i$. If $i = 0$ and $q \notin Q_t$, then $V_0(q, \nu) = +\infty$ and $V(q, \nu) \leq V_0(q, \nu)$; otherwise, $q \in Q_t$ and $V_0(q, \nu) = \mathsf{wt}_t(q, \nu)$, while $V(q, \nu) = \mathsf{wt}_t(q, \nu)$ (since V is a fixpoint of $\mathcal{F}$). If $i \in \mathbb{N}$ is such

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[5]We recall that a cutpoint is the value of the clock in-between two affine pieces of the function.

that $V \leq V_i$, as $\mathcal{F}$ is monotonous, we have $\mathcal{F}(V) \leq \mathcal{F}(V_i)$. Thus, since V is a fixpoint of $\mathcal{F}$, we deduce that $V = \mathcal{F}(V) \leq \mathcal{F}(V_i) = V_{i+1}$ that concludes the proof of Proposition 8.3.

8.3. **The greatest fixpoint of $\mathcal{F}$ is equal to the value function.** To conclude the proof of Theorem 2.4, its remains to prove that $\inf_i V_i = \mathsf{Val}$. To do it, we adapt the proof given in [Bou16] to our context (turn-based games with negative and positive weights).

Proposition 8.7. $\inf_i V_i = \mathsf{Val}$

The main idea of this proof is the link between V_i and the value obtained when we consider only plays with at most i steps. We thus let W_i be the configurations from where Min can guarantee to reach a target location within i steps: this is a very classical sequence of configurations that is traditionally called *attractor*. Intuitively, for a configuration not in W_i, V_i is equal to $+\infty$ since Max can avoid the target in the i first steps. Formally, we define the sequence of $(W_i)_i$ by induction on $i \in \mathbb{N}$: $(q,\nu) \in W_0$ if $q \in Q_t$, and for all $i \in \mathbb{N}$, $(q,\nu) \in W_{i+1}$ if $(q,\nu) \in W_i$, or

(1) $q \in Q_{\mathsf{Min}}$, and there exists an edge $(q,\nu) \xrightarrow{t,\delta} (q',\nu')$ such that $(q',\nu') \in W_i$;
(2) $q \in Q_{\mathsf{Max}}$, and for all edges $(q,\nu) \xrightarrow{t,\delta} (q',\nu')$, we have $(q',\nu') \in W_i$.

The following lemma recalls the link between V_i and W_i:

Lemma 8.8. *Let $i \in \mathbb{N}$ and (q,ν) be a configuration. Then, $(q,\nu) \in W_i$ if and only if $V_i(q,\nu) < +\infty$.*

Proof. We prove the equivalence by induction on $i \in \mathbb{N}$. If $i = 0$, since the game has been modified so that final weight functions are finite, we conclude by definitions of W_0 and V_0. Now, we fix $i \in \mathbb{N}$ such that for all configurations (q',ν'), we have $(q',\nu') \in W_i$ if and only if $V_i(q',\nu') < +\infty$. Let (q,ν) be a configuration. First, we suppose that $q \in L_t$. In this case, $(q,\nu) \in W^{i+1}$ and $V_{i+1}(q,\nu) = \mathsf{wt}_t(q,\nu) < +\infty$ (by hypothesis).

Now, we suppose that $q \in Q_{\mathsf{Min}}$ and we have:
$$V_{i+1}(q,\nu) = \mathcal{F}(V_i)(q,\nu) = \inf_{(q,\nu)\xrightarrow{t,\delta}(q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q',\nu')\right)$$

In particular, $V_{i+1}(q,\nu) < +\infty$ if and only if there exists an edge $(q,\nu) \xrightarrow{t,\delta} (q',\nu')$ such that $V_i(q',\nu') < +\infty$. We deduce that $V_{i+1}(q,\nu) < +\infty$ if and only if there exists an edge $(q,\nu) \xrightarrow{t,\delta} (q',\nu')$ such that $(q',\nu') \in W_i$ (by applying the inductive hypothesis on (q',ν')). We conclude that $V_{i+1}(q,\nu) < +\infty$ if and only if $(q,\nu) \in W^{i+1}$, by item (1) of the definition of W_{i+1}.

Finally, we suppose that $q \in Q_{\mathsf{Max}}$ and we have:
$$V_{i+1}(q,\nu) = \mathcal{F}(V_i)(q,\nu) = \sup_{(q,\nu)\xrightarrow{t,\delta}(q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q',\nu')\right)$$

In particular, by inductive hypothesis, $V_{i+1}(q,\nu) < +\infty$ if and only if for all edges $(q,\nu) \xrightarrow{t,\delta} (q',\nu')$, we have $(q',\nu') \in W_i$. Thus, by item (2) of the definition of W_{i+1}, we obtain that $V_{i+1}(q,\nu) < +\infty$ if and only if $(q,\nu) \in W_{i+1}$. $\square$

To prove that the value iteration converges to the value function, we relate configurations in W_i with some particular strategies of Min. Given a configuration (q,ν), we fix $\mathsf{Strat}_i(q,\nu)$ to be the set of strategies of Min such that all plays from (q,ν) conforming to it reach the

target *in at most i steps*. More precisely, we require that for all plays starting from (q, ν) and conforming to a strategy of $\mathsf{Strat}_i(q, \nu)$, the jth configuration of the play belongs to W_{i-j}: in particular, the first configuration, (q, ν) must be in W_i, and the last one in W_0 (i.e. with a location being a target).

For all $\varepsilon > 0$, we inductively define a sequence of strategies $(\sigma_i^\varepsilon)_i$ whose i-th strategy will be shown to belong to $\mathsf{Strat}_i(q, \nu)$ if $(q, \nu) \in W_i$, and ε-optimal according to V_i. In particular, we prove that an almost-optimal strategy can be defined by choosing almost-optimal edges along the play. Intuitively, the i-th strategy chooses the first move as the best edge according to V_i, and then follows the $(i-1)$-th strategy (applying in the suffix of the play except the first choice).

Formally, we let σ_0^ε be any fixed strategy of Min. For $i \in \mathbb{N}$, relying on $\sigma_i^{\varepsilon/2}$, we inductively define σ_{i+1}^ε according to the length of all finite plays ending in a location of Min. If the play contains only one configuration, we fix $\sigma_{i+1}^\varepsilon(q, \nu)$ be any decision (t, δ) such that $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$ and $\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu') \leq V_{i+1}(q, \nu) + \varepsilon/2$ (that exists by definition of $V_{i+1}(q, \nu)$ as an infimum). Otherwise, the play can be decomposed as $(q, \nu) \xrightarrow{t, \delta} \rho$ with (q', ν') the first configuration of ρ, and we let:

$$\sigma_{i+1}^\varepsilon((q, \nu) \xrightarrow{t, \delta} \rho) = \sigma_i^{\varepsilon/2}(\rho)\,.$$

Lemma 8.9. *For all $i \in \mathbb{N}$, $\varepsilon > 0$ and $(q, \nu) \in W_i$,*
$$\sigma_i^\varepsilon \in \mathsf{Strat}_i(q, \nu) \quad \text{and} \quad V_i(q, \nu) + \varepsilon \geq \mathsf{Val}^{\sigma_i^\varepsilon}(q, \nu)\,.$$

Proof. We reason by induction on $i \in \mathbb{N}$. If $i = 0$, since $(q, \nu) \in W_0$, we have $q \in Q_t$ and thus any strategy (and thus the fixed strategy σ_0^ε) is in $\mathsf{Strat}_0(q, \nu)$, and $V_0(q, \nu) = \mathsf{wt}_t(q, \nu) = \mathsf{Val}^{\sigma_0^\varepsilon}(q, \nu)$.

Now, consider $i \in \mathbb{N}$ such that for all configurations $(q, \nu) \in W_i$, $\sigma_i^{\varepsilon/2} \in \mathsf{Strat}_i(q, \nu)$, and $V_i(q, \nu) + \varepsilon/2 \geq \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q, \nu)$. We show that σ_{i+1}^ε satisfies the properties for a given configuration $(q, \nu) \in W_{i+1}$. If $q \in Q_t$, we conclude as in the case $i = 0$ (since $q \in W_0$). Otherwise, we show $\sigma_{i+1}^\varepsilon \in \mathsf{Strat}_{i+1}(q, \nu)$ by contradiction. We thus suppose that there exists a finite play ρ' of length $i + 1$ conforming to σ_{i+1}^ε that does not reach Q_t. It can be decomposed as $(q, \nu) \xrightarrow{t, \delta} \rho$ where $\sigma_{i+1}^\varepsilon(q, \nu) = (t, \delta)$ and ρ is conforming to $\sigma_i^{\varepsilon/2}$. We show that $(q', \nu') \in W_i$ where (q', ν') is the first configuration of ρ.

- If $q \in Q_{\mathsf{Max}}$, then we conclude that $(q', \nu') \in W_i$ by item (2) of definition of W_{i+1}: all edges from (q, ν) reach a configuration in W_i.

- If $q \in Q_{\mathsf{Min}}$, then, by item (1) of definition of W_{i+1}, there exists an edge $(q, \nu) \xrightarrow{t', \delta'} (q'', \nu'')$ such that $(q'', \nu'') \in W_i$, i.e. $V_i(q'', \nu'') < +\infty$ (by Lemma 8.8). The choice of $\sigma_{i+1}^\varepsilon(q, \nu)$ is taken along all possible edges from (q, ν), at most ε away of the infimum. Thus, it chooses an edge $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$ such that $V_i(q', \nu') < +\infty$, i.e. $(q', \nu') \in W_i$ (by Lemma 8.8).

By induction hypothesis applied to $(q', \nu') \in W_i$, $\sigma_i^{\varepsilon/2} \in \mathsf{Strat}_i(q', \nu')$, and thus ρ reaches Q_t within i steps which contradicts the hypothesis.

We then prove that $V_{i+1}(q, \nu) + \varepsilon \geq \mathsf{Val}^{\sigma_{i+1}^\varepsilon}(q, \nu)$. By definition of σ_{i+1}^ε with $\sigma_i^{\varepsilon/2}$, we remark that, for all finite plays $(q, \nu) \xrightarrow{t, \delta} \rho$ of length at least one, we have $\sigma_{i+1}^\varepsilon((q, \nu) \xrightarrow{t, \delta} \rho) = \sigma_i^{\varepsilon/2}(\rho)$. In particular, the weight of all plays from $(q, \nu) \xrightarrow{t, \delta} \rho$ and conforming to σ_{i+1}^ε is equal to the weight of the play from ρ and conforming to $\sigma_i^{\varepsilon/2}$ under the same strategy of

Max, i.e. for all strategies of Max, τ, we have:

$$\mathsf{wt}(\mathsf{Play}((q,\nu) \xrightarrow{t,\delta} (q',\nu'), \sigma_{i+1}^\varepsilon, \tau)) = \mathsf{wt}(\mathsf{Play}((q',\nu'), \sigma_i^{\varepsilon/2}, \tau)).$$

Thus, by applying the supremum over strategies of Max, we deduce that

$$\mathsf{Val}^{\sigma_{i+1}^\varepsilon}((q,\nu) \xrightarrow{t,\delta} (q',\nu')) = \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q',\nu') \tag{8.1}$$

- If $q \in Q_{\mathsf{Max}}$, then we have:

$$V_{i+1}(q,\nu) = \mathcal{F}(V_i)(q,\nu) = \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q',\nu')\big)$$

Since $(q,\nu) \in W_{i+1}$, then we have $(q',\nu') \in W_i$ (by item (2)). Moreover, by induction hypothesis applying on $V_i(q',\nu')$, we deduce that $V_i(q',\nu') + \varepsilon/2 \geq \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q',\nu')$. Thus, for all (t,δ), we have:

$$V_{i+1}(q,\nu) \geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q',\nu')$$

$$\geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q',\nu') - \varepsilon/2$$

$$\geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma_{i+1}^\varepsilon}((q,\nu) \xrightarrow{t,\delta} (q',\nu')) - \varepsilon/2 \qquad \text{(by (8.1))}$$

Finally, since this inequality holds for all edges from (q,ν), we deduce that

$$V_{i+1}(q,\nu) \geq \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma_{i+1}^\varepsilon}((q,\nu) \xrightarrow{t,\delta} (q',\nu'))\big) - \varepsilon/2$$

$$\geq \mathsf{Val}^{\sigma_{i+1}^\varepsilon}(q,\nu) - \varepsilon \qquad \text{(by Lemma 8.2).}$$

- If $q \in Q_{\mathsf{Min}}$, then, by definition of σ_{i+1}^ε, and letting $\sigma_{i+1}^\varepsilon(q,\nu) = (t,\delta)$:

$$V_{i+1}(q,\nu) \geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q',\nu') - \varepsilon/2$$

Now, since $(q',\nu') \in W_i$ (as explain before to show that $\sigma_{i+1}^\varepsilon \in \mathsf{Strat}_{i+1}(q,\nu)$), by induction hypothesis, $V_i(q',\nu') + \varepsilon/2 \geq \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q',\nu')$. Thus, we deduce that

$$V_{i+1}(q,\nu) \geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma_i^{\varepsilon/2}}(q',\nu') - \varepsilon$$

$$\geq \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma_{i+1}^\varepsilon}((q,\nu) \xrightarrow{t,\delta} (q',\nu')) - \varepsilon \qquad \text{(by (8.1))}$$

$$\geq \mathsf{Val}^{\sigma_{i+1}^\varepsilon}(q,\nu) - \varepsilon \qquad \text{(by Lemma 8.2).} \qquad \square$$

As a corollary, we obtain:

Lemma 8.10. *For all $i \in \mathbb{N}$, and $(q,\nu) \in W_i$, $V_i(q,\nu) = \inf_{\sigma \in \mathsf{Strat}_i(q,\nu)} \mathsf{Val}^\sigma(q,\nu)$.*

Proof. We reason by induction on $i \in \mathbb{N}$. If $i = 0$, since $q \in Q_t$ for all strategies $\sigma \in \mathsf{Strat}_0(q,\nu)$, $V_0(q,\nu) = \mathsf{wt}_t(q,\nu) = \mathsf{Val}^\sigma(q,\nu)$.

For $i \in \mathbb{N}$ such that the property holds, let $(q,\nu) \in W_{i+1}$. If $q \in Q_t$, we have $(q,\nu) \in W_0$ and we conclude as in the case $i = 0$. Otherwise, Lemma 8.9 directly implies that

$$V_{i+1}(q,\nu) + \varepsilon \geq \mathsf{Val}^{\sigma_{i+1}^\varepsilon}(q,\nu) \geq \inf_{\sigma \in \mathsf{Strat}_{i+1}(q,\nu)} \mathsf{Val}^\sigma(q,\nu)$$

and $V_{i+1}(q,\nu) \geq \inf_{\sigma \in \mathsf{Strat}_{i+1}(q,\nu)} \mathsf{Val}^\sigma(q,\nu)$ since the inequality holds for all $\varepsilon > 0$.

Conversely, we show that $V_{i+1}(q,\nu) \leq \inf_{\sigma \in \mathsf{Strat}_{i+1}(q,\nu)} \mathsf{Val}^\sigma(q,\nu)$ by proving that for all $\sigma \in \mathsf{Strat}_{i+1}(q,\nu)$, we have $V_{i+1}(q,\nu) \leq \mathsf{Val}^\sigma(q,\nu)$. Let $\sigma \in \mathsf{Strat}_{i+1}(q,\nu)$.

- If $q \in Q_{\mathsf{Min}}$, then we let $(t, \delta) = \sigma(q, \nu)$ with $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$, so that $(q', \nu') \in W_i$. By induction hypothesis, we have $V_i(q', \nu') = \inf_{\sigma' \in \mathsf{Strat}_i(q', \nu')} \mathsf{Val}^\sigma(q', \nu')$. Consider the strategy σ' obtained from σ by adding as a first move the edge $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$. Formally, it is defined by:

$$\sigma_{q', \nu'}(\rho) = \begin{cases} \sigma\left((q, \nu) \xrightarrow{t, \delta} \rho\right) & \text{if } \rho \text{ starts in } (q', \nu'); \\ \sigma(\rho) & \text{otherwise.} \end{cases}$$

Given a play ρ' conforming to σ' starting from (q', ν'), we remark that $(q, \nu) \xrightarrow{t, \delta} \rho'$ is conforming to σ. In particular, we obtain that

$$\mathsf{Play}((q, \nu), \sigma', \tau) = \mathsf{Play}((q, \nu) \xrightarrow{t, \delta} (q', \nu'), \sigma, \tau) \qquad (8.2)$$

Thus, from (8.2), we deduce that $\sigma' \in \mathsf{Strat}_i(q', \nu')$. Thus, $V_i(q', \nu') \le \mathsf{Val}^{\sigma'}(q', \nu')$ and we obtain that

$$\begin{aligned} V_{i+1}(q, \nu) &= \mathcal{F}(V_i)(q, \nu) \\ &\le \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu') \\ &\le \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^{\sigma'}(q', \nu') \,. \end{aligned}$$

Moreover, by (8.2), we also obtain that, for all strategies τ of Max,

$$\mathsf{P}(\mathsf{Play}((q, \nu), \sigma', \tau)) = \mathsf{P}(\mathsf{Play}((q, \nu) \xrightarrow{t, \delta} (q', \nu'), \sigma, \tau)) \,.$$

In particular, we deduce that $\mathsf{Val}^{\sigma'}(q', \nu') = \mathsf{Val}^\sigma((q, \nu) \xrightarrow{t, \delta} (q', \nu'))$, and we can rewrite the previous inequality as:

$$\begin{aligned} V_{i+1}(q, \nu) &\le \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma((q, \nu) \xrightarrow{t, \delta} (q', \nu')) \\ &\le \mathsf{Val}^\sigma(q, \nu) \qquad \text{(by Lemma 8.2).} \end{aligned}$$

- If $q \in Q_{\mathsf{Max}}$, then, by Lemma 8.2, we have:

$$\mathsf{Val}^\sigma(q, \nu) = \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + \mathsf{Val}^\sigma((q, \nu) \xrightarrow{t, \delta} (q', \nu')) \right)$$

Letting $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$ be an edge from (q, ν), since $\sigma \in \mathsf{Strat}_{i+1}(q, \nu)$, we have $(q', \nu') \in W_i$, and thus by induction hypothesis, $V_i(q', \nu') \le \inf_{\sigma' \in \mathsf{Strat}_i(q', \nu')} \mathsf{Val}^\sigma(q', \nu')$. By considering the same strategy σ' as the one defined in the case of Min, we obtain that

$$V_i(q', \nu') \le \mathsf{Val}^{\sigma'}(q', \nu') = \mathsf{Val}^\sigma((q, \nu) \xrightarrow{t, \delta} (q', \nu'))$$

Thus, we deduce that

$$\mathsf{Val}^\sigma(q, \nu) \ge \mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu')$$

Since this holds for all edges $(q, \nu) \xrightarrow{t, \delta} (q', \nu')$, we deduce that

$$\mathsf{Val}^\sigma(q, \nu) \ge \sup_{(q,\nu) \xrightarrow{t,\delta} (q',\nu')} \left(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q) + V_i(q', \nu') \right) = V_{i+1}(q, \nu) \,. \qquad \square$$

Finally, we have tools to prove Proposition 8.7. In particular, we fix W be the set of configurations from where Min can ensure to reach Q_t (without restriction on the number of steps), that is the limit of $(W_i)_i$: $W = \bigcup_i W_i$. By classical results [FBBD$^+$23, Theorem 103] on the attractor computation in timed games, we know that there exists a finite $N \in \mathbb{N}$ such that $W = \bigcup_{i=0}^{N} W_i$. Now, by letting $V = \inf_i V_i$, we can finally prove that $V = \mathsf{Val}$.

We reason by double inequalities and we start by proving that $V \geq \mathsf{Val} = \inf_\sigma \mathsf{Val}^\sigma$. If $(q, \nu) \notin W$, we have for all $i \in \mathbb{N}$, $V_i(q, \nu) = +\infty$ (by Lemma 8.8), and thus $V(q, \nu) = +\infty$. Otherwise, $(q, \nu) \in W_N$. Let $\varepsilon > 0$. Since $(V_i)_i$ uniformly converges to its limit, there exists $k \geq N$ such that $V_k(q, \nu) \leq V(q, \nu) + \varepsilon$. By using Lemma 8.10, $\inf_{\sigma \in \mathsf{Strat}_k(q,\nu)} \mathsf{Val}^\sigma(q, \nu) \leq V(q, \nu) + \varepsilon$. By considering the infimum over all strategies, and since this holds for all ε, we get $\mathsf{Val} = \inf_\sigma \mathsf{Val}^\sigma \leq V$.

Conversely, we prove that $V \leq \mathsf{Val} = \inf_\sigma \mathsf{Val}^\sigma$. By contradiction, we suppose that there exists a strategy σ of Min and an initial configuration (q, ν) such that $V(q, \nu) > \mathsf{Val}^\sigma(q, \nu)$. Since then $\mathsf{Val}^\sigma(q, \nu) < +\infty$, all plays conforming to σ reach a target location. We (inductively) build a play ρ from (q, ν) conforming to σ such that at each step we guarantee that $\mathsf{last}(\rho) = (q', \nu')$ satisfies $q' \notin Q_t$, and $V(q', \nu') > \mathsf{Val}^\sigma(\rho)$. In particular, this implies that ρ is an infinite play that never reaches a target, and we get a contradiction.

Now, to finish the proof, we provide the construction of a such ρ. First, we suppose that $\rho = (q, \nu)$. To initiate the inductive construction of ρ, since $V(q, \nu) > \mathsf{Val}^\sigma(q, \nu)$, we deduce that $q \notin Q_t$ (otherwise $V(q, \nu) = \mathsf{wt}_t(q, \nu) = \mathsf{Val}^\sigma(q, \nu)$ by Lemma 8.2).

Then, we suppose that ρ is a play from (q, ν) conforming to σ such that $V(q', \nu') > \mathsf{Val}^\sigma(\rho)$ where $\mathsf{last}(\rho) = (q', \nu')$ and $q' \notin Q_t$. We define a new step for ρ as follows.

- If $q' \in Q_{\mathsf{Min}}$, then we extend ρ by $\rho' = \rho \xrightarrow{t,\delta} (q'', \nu'')$, by letting $\sigma(\rho) = (t, \delta)$. Since ρ is conforming to σ, then ρ' is also conforming to σ. By induction hypothesis and Lemma 8.2,

$$V(q', \nu') > \mathsf{Val}^\sigma(\rho) = \mathsf{wt}(\delta) + t\,\mathsf{wt}(q') + \mathsf{Val}^\sigma(\rho')$$

  Since V is a fixpoint of $\mathcal{F}$ by Proposition 8.3, with $V(q', \nu')$ being thus equal to an infimum over all possible edges, we obtain

$$V(q'', \nu'') \geq V(q', \nu') - \mathsf{wt}(\delta) - t\,\mathsf{wt}(q') > \mathsf{Val}^\sigma(\rho')$$

- If $q' \in Q_{\mathsf{Max}}$, then we prove that there exists an edge $(q', \nu') \xrightarrow{t,\delta} (q'', \nu'')$ such that $V(q'', \nu'') > \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q'', \nu''))$, and we define the new step of ρ with this edge (the resulting play is conforming to σ). To do that, we reason by contradiction, and we suppose that for all edges $(q', \nu') \xrightarrow{t,\delta} (q'', \nu'')$, we have $V(q'', \nu'') \leq \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q'', \nu''))$. In particular, we obtain a contradiction since:

$$V(q', \nu') > \mathsf{Val}^\sigma(\rho) \qquad \text{(by induction hypothesis)}$$

$$= \sup_{(q',\nu') \xrightarrow{t,\delta} (q'',\nu'')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q') + \mathsf{Val}^\sigma(\rho \xrightarrow{t,\delta} (q'', \nu''))\big) \qquad \text{(by Lemma 8.2)}$$

$$\geq \sup_{(q',\nu') \xrightarrow{t,\delta} (q'',\nu'')} \big(\mathsf{wt}(\delta) + t\,\mathsf{wt}(q') + V(q'', \nu'')\big) \qquad \text{(by monotonicity of $\mathcal{F}$)}$$

$$> V(q', \nu') \qquad \text{(since V is a fixpoint of $\mathcal{F}$ by Proposition 8.3).}$$

This concludes the proof that $\lim_i V_i = \mathsf{Val}$, and thus of Theorem 2.4.

9. Conclusion

We solve one-clock WTGs with arbitrary weights, an open problem for several years. We strongly rely on the determinacy of the game, taking the point of view of Max, instead of the one of Min as was done in previous work with only non-negative weights. We also use technical ingredients such as the closure of a game, switching strategies for Min, and acyclic unfoldings.

Regarding the complexity, our algorithm runs in exponential time (with weights encoded in unary), which does not match the known PSPACE lower bound with weights in unary [FIJS20]. Observe that this lower bound only uses non-negative weights. This complexity gap deserves further study.

To compute the value function with a PSPACE algorithm, a promising idea from a reviewer of this article consists in using the first-order theory over the reals with a fixed number of quantifier alternations where the satisfiability of a formula can be checked in PSPACE [BPR06, remark 13.10]. The idea is to encode the greatest fixpoint of $\mathcal{F}$ (that is the value of the game, by Theorem 2.4) in this logic. Indeed, since the value function of a one-clock WTG is piecewise affine with a pseudo-polynomial number of cutpoints (according to [BGH+22]), we can write such a formula by using a variable for each cutpoint and slope, and then expressing with inequalities and equalities that, for each cutpoint or line segment, the current valuation is at least as good as what can be obtained by either waiting until a later cutpoint, or jumping through a transition.

Our work also opens three research directions. First, as we unfold the game into a finite tree, it would be interesting to develop a symbolic approach that shares computation between subtrees in order to obtain a more efficient algorithm. Second, playing stochastically in WTGs with shortest path objectives has been recently studied in [MPR21]. One could study an extension of one-clock WTGs with stochastic transitions. In this context, Min aims at minimizing the expectation of the accumulated weight. Third, the analysis of cycles that we have done in the setting of one-clock WTGs can be an inspiration to identify new decidable classes of WTGs with arbitrarily many clocks.

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Appendix A. Continuity of the value function on closure of regions

Lemma 2.7. *For all WTGs $\mathcal{G}$ and $(q, I) \in Q \times \mathsf{Reg}_{\mathcal{G}}$, the mapping $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q, I), \nu)$ is continuous over $\overline{I}$.*

The main ingredient of our proof is, given a strategy σ of Min in $\overline{\mathcal{G}}$, a location $\ell = (q, I)$ of $\overline{\mathcal{G}}$, and valuations $\nu, \nu' \in \overline{I}$ (and not only $\nu, \nu' \in I$ as in the proof of [BGH$^+$22, Theorem 3.2]), to show how to build a strategy σ' in $\overline{\mathcal{G}}$ and a length-preserving function g that maps plays of $\overline{\mathcal{G}}$ starting in (ℓ, ν') and conforming to σ' to plays of $\overline{\mathcal{G}}$ starting in (ℓ, ν) conforming to σ with similar behaviour and weight. More precisely, we define σ' and g by induction on the length k of the finite play that is given as an argument and relies on the following set of induction hypotheses:

Induction hypothesis: There exist a strategy σ', only defined on plays of length at most $k - 1$ starting in (ℓ, ν'), and a function g mapping plays of length k starting in (ℓ, ν') conforming to σ' to plays of length k starting in (ℓ, ν) conforming to σ such that for all plays $\rho' = (\ell_0 = \ell, \nu'_0 = \nu') \xrightarrow{t'_0, \delta'_0} \cdots \xrightarrow{t'_{k-1}, \delta'_{k-1}} (\ell_k, \nu'_k)$ conforming to σ', letting $(\ell_0, \nu_0 = \nu) \xrightarrow{t_0, \delta_0} \cdots \xrightarrow{t_{k-1}, \delta_{k-1}} (\ell_k, \nu_k)$ the play $g(\rho')$, we have:

(1) $|\nu_k - \nu'_k| \leq |\nu - \nu'|$;
(2) $\mathsf{wt}_\Sigma(\rho') \leq \mathsf{wt}_\Sigma(g(\rho')) + W_{\mathsf{loc}}(|\nu - \nu'| - |\nu_k - \nu'_k|)$.

We note that no property is required on the strategy σ' for finite plays that do not start in (ℓ, ν'). Moreover, by the invariants of $\overline{\mathcal{G}}$, we have that for every $i \in \{0, \ldots, k\}$, ν_i and ν'_i belong to the interval $\overline{I_i}$ such that $\ell_i = (q_i, I_i)$.

Let us explain how this result would imply the desired result before going through the induction itself, i.e. why $\nu \mapsto \mathsf{Val}_{\overline{\mathcal{G}}}((q, I), \nu)$ is continuous over $\overline{I}$. We remark first that the result directly implies that if the value of the game is finite for some valuation ν in $\overline{I}$, then it is finite for all other valuation ν' in $\overline{I}$. Indeed, a finite value of the game in (ℓ, ν) implies that there exists a strategy σ such that every play starting in (ℓ, ν) and conforming to it reaches a target location in a final valuation such that the final weight function applying in this last configuration is finite. Moreover, denoting σ' the strategy obtained from σ thanks to the above result, any play ρ' starting in (ℓ, ν') and conforming to σ' reaches a target location (since $g(\rho')$ does as a play conforming to σ). Moreover, its final weight function is finite as the final valuation of ρ', and $g(\rho')$ sit in the same region and, by hypothesis, a final weight function is either always finite or always infinite within a region.

Now, assuming the value of the game is finite over $\overline{I}$ and we show that the value is continuous over $\overline{I}$. To do it, we show that, for all $\nu \in \overline{I}$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\nu' \in \overline{I}$ with $|\nu - \nu'| \leq \delta$, we have $|\mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) - \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu')| \leq \varepsilon$. To this end, we can show that:

$$|\mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) - \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu')| \leq (W_{\mathsf{loc}} + K)|\nu - \nu'| \qquad (A.1)$$

where K is the greatest absolute value of the slopes appearing in the piecewise affine functions within wt_t. Indeed, assume that this inequality holds, and consider $\nu \in \overline{I}$ and a positive real number ε. Then, we let $\delta = \frac{\varepsilon}{W_{\mathsf{loc}} + K}$, and we consider a valuation ν' such that $|\nu - \nu'| \leq \delta$. In this case, (A.1) becomes:

$$|\mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) - \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu')| \leq (W_{\mathsf{loc}} + K)|\nu - \nu'| \leq (W_{\mathsf{loc}} + K)\frac{\varepsilon}{W_{\mathsf{loc}} + K} \leq \varepsilon .$$
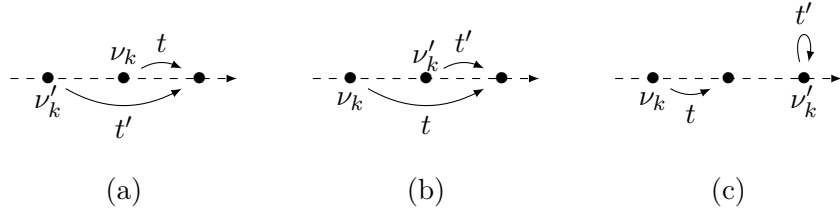
Thus, proving (A.1) is sufficient to establish continuity.

Figure 8: The definition of t' when (a) $\nu'_k \leq \nu_k$; (b) $\nu_k < \nu'_k < \nu_k + t$; (c) $\nu_k < \nu_k + t < \nu'_k$.

On the other hand, (A.1) is equivalent to:

$$\mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) \leq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu') + (W_{\mathsf{loc}} + K)|\nu - \nu'| \quad \text{and} \quad \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu') \leq \mathsf{Val}_{\overline{\mathcal{G}}}(\ell, \nu) + (W_{\mathsf{loc}} + K)|\nu - \nu'|.$$

As those two last equations are symmetric with respect to ν and ν', we only have to show either of them. We thus focus on the latter, which, by using the upper value, can be reformulated as: for all strategies σ of Min, there exists a strategy σ' such that

$$\mathsf{Val}_{\overline{\mathcal{G}}}^{\sigma'}(\ell, \nu') \leq \mathsf{Val}_{\overline{\mathcal{G}}}^{\sigma}(\ell, \nu) + (W_{\mathsf{loc}} + K)|\nu - \nu'|.$$

We note that this last equation is equivalent to saying that there exists a function g mapping plays ρ' from (ℓ, ν') conforming to σ' to plays from (ℓ, ν) conforming to σ such that, for all such ρ' the final valuations of ρ' and $g(\rho')$ differ by at most $|\nu - \nu'|$ and

$$\mathsf{wt}_\Sigma(\rho') \leq \mathsf{wt}_\Sigma(g(\rho')) + W_{\mathsf{loc}}|\nu - \nu'|$$

which is exactly what we claimed induction achieves since $-|\nu_k - \nu'_k| \leq 0$. Thus, to conclude this proof, let us now define σ' and g, by induction on the length k of ρ'.

Base case $k = 0$: In this case, σ' does not have to be defined since there are no plays of length -1. Moreover, in that case, $\rho' = (\ell, \nu')$ and $g(\rho') = (\ell, \nu)$, in which case both properties are trivial.

Inductive case: Let us suppose now that the construction is done for a given $k \geq 0$ and perform it for $k + 1$. We start with the construction of σ'. To that extent, we consider a play $\rho' = (\ell_0 = \ell, \nu'_0 = \nu') \xrightarrow{t'_0, \delta'_0} \cdots \xrightarrow{t'_{k-1}, \delta'_{k-1}} (\ell_k, \nu'_k)$ conforming to σ' (provided by induction hypothesis) such that $\ell_k \in L_{\mathsf{Min}}$. Let (t, δ) be the choice of delay and transition made by σ on $g(\rho')$, i.e. $\sigma(g(\rho')) = (t, \delta)$. Then, we define $\sigma'(\rho') = (t', \delta)$ where $t' = \max(0, \nu_k + t - \nu'_k)$. The delay t' respects the guard of transition δ, as can be seen from Figure 8. Indeed, either $\nu_k + t = \nu'_k + t'$ (cases (a) and (b) in Figure 8) or $\nu_k \leq \nu_k + t \leq \nu'_k$ (case (c) in Figure 8 where $t' = 0$), in which case ν'_k is in the same closure of region as $\nu_k + t$ since ν_k and ν'_k are in the same closure of region by induction hypothesis: we conclude by noticing that the guard of δ is closed.

Let us now build the mapping g. Let $\rho' = (\ell_0 = \ell, \nu'_0 = \nu') \xrightarrow{t'_0, \delta'_0} \cdots \xrightarrow{t'_k, \delta'_k} (\ell_{k+1}, \nu'_{k+1})$ be a play conforming to σ' and let $\tilde{\rho}' = (\ell_0, \nu'_0) \xrightarrow{t'_0, \delta'_0} \cdots \xrightarrow{t'_{k-1}, \delta'_{k-1}} (\ell_k, \nu'_k)$ its prefix of length k. Using the construction of g over plays of length k by induction, the play $g(\tilde{\rho}') = (\ell_0, \nu_0 = \nu) \xrightarrow{t_0, \delta_0} \cdots \xrightarrow{t_{k-1}, \delta_{k-1}} (\ell_k, \nu_k)$ satisfies properties (1) and (2). Then:

- if $\ell_k \in L_{\mathsf{Min}}$ and $\sigma(g(\tilde{\rho}')) = (t, \delta)$, then $g(\rho') = g(\tilde{\rho}') \xrightarrow{t, \delta} (\ell_{k+1}, \nu_{k+1})$ is obtained by applying those choices on $g(\tilde{\rho}')$. By the construction of σ', we moreover have $\delta'_k = \delta$;

- if $\ell_k \in L_{\mathsf{Max}}$, the last valuation ν_{k+1} of $g(\rho')$ is rather obtained by choosing action (t, δ'_k) verifying $t = \max(0, \nu'_k + t'_k - \nu_k)$. We note that transition δ'_k is allowed since both $\nu_k + t$ and $\nu'_k + t'_k$ are in the same closure of region (for similar reasons as above).

Moreover, by induction hypothesis $\tilde{\rho}'$ and $g(\tilde{\rho}')$ have the same length.

Now, to prove (1), we notice that we always have either

$$\nu_k + t = \nu'_k + t'_k \quad \text{or} \quad \nu_k \leq \nu_k + t \leq \nu'_k = \nu'_k + t'_k \quad \text{or} \quad \nu'_k \leq \nu'_k + t \leq \nu_k = \nu_k + t \,.$$

In all of these possibilities, we have $|(\nu_k + t) - (\nu'_k + t'_k)| \leq |\nu_k - \nu'_k|$.

We finally check property (2). Either ℓ_k belongs to Min or to Max, using the induction hypothesis, we have:

$$\begin{aligned}
\mathsf{wt}_\Sigma(\rho') &= \mathsf{wt}_\Sigma(\tilde{\rho}') + \mathsf{wt}(\delta'_k) + t'_k\,\mathsf{wt}(\ell_k) \\
&\leq \mathsf{wt}_\Sigma(g(\tilde{\rho}')) + W_{\mathsf{loc}}(|\nu - \nu'| - |\nu_k - \nu'_k|) + \mathsf{wt}(\delta'_k) + t'_k\,\mathsf{wt}(\ell_k) \\
&= \mathsf{wt}_\Sigma(g(\rho')) + (t'_k - t)\,\mathsf{wt}(\ell_k) + W_{\mathsf{loc}}(|\nu - \nu'| - |\nu_k - \nu'_k|) \,.
\end{aligned}$$

To conclude, let us claim that

$$|t'_k - t| \leq |\nu_k - \nu'_k| - |\nu'_{k+1} - \nu_{k+1}| \tag{A.2}$$

Thus, since $|\mathsf{wt}(\ell_k)| \leq W_{\mathsf{loc}}$, we conclude that

$$\mathsf{wt}_\Sigma(\rho') \leq \mathsf{wt}_\Sigma(g(\rho')) + W_{\mathsf{loc}}(|\nu - \nu'| - |\nu_{k+1} - \nu'_{k+1}|)$$

which concludes the induction.

To conclude the proof, we prove (A.2). First, we suppose that δ'_k does not contain a reset. In particular, we have $t'_k = \nu'_{k+1} - \nu'_k$ and $t = \nu_{k+1} - \nu_k$, thus $|t'_k - t| = |\nu'_{k+1} - \nu'_k - (\nu_{k+1} - \nu_k)|$. Then, two cases are possible: either $t'_k = \max(0, \nu_k + t - \nu'_k)$ or $t = \max(0, \nu'_k + t'_k - \nu_k)$. So we have three different possibilities:

- if $t'_k + \nu'_k = t + \nu_k$, then $\nu'_{k+1} = \nu_{k+1}$, thus

$$|t'_k - t| = |\nu_k - \nu'_k| = |\nu_k - \nu'_k| - |\nu'_{k+1} - \nu_{k+1}| \,;$$

- if $t = 0$, then $\nu_k = \nu_{k+1} \geq \nu'_{k+1} \geq \nu'_k$, thus

$$|t'_k - t| = \nu'_{k+1} - \nu'_k = (\nu_k - \nu'_k) - (\nu_{k+1} - \nu'_{k+1}) = |\nu_k - \nu'_k| - |\nu'_{k+1} - \nu_{k+1}| \,;$$

- if $t'_k = 0$, then $\nu'_k = \nu'_{k+1} \geq \nu_{k+1} \geq \nu_k$, thus

$$|t'_k - t| = \nu_{k+1} - \nu_k = (\nu'_k - \nu_k) - (\nu'_{k+1} - \nu_{k+1}) = |\nu_k - \nu'_k| - |\nu'_{k+1} - \nu_{k+1}| \,.$$

Otherwise, δ'_k contains a reset, then $\nu'_{k+1} = \nu_{k+1} = 0$. If $t'_k = \nu_k + t - \nu'_k$, we have that $|t'_k - t| = |\nu_k - \nu'_k|$. Otherwise, $t'_k = 0$ and $t \leq \nu'_k - \nu_k$. In all cases, we have proved (A.2).