


QUANTITATIVE EQUALITY IN SUBSTRUCTURAL LOGIC VIA LIPSCHITZ DOCTRINES

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ABSTRACT. Substructural logics naturally support a quantitative interpretation of formulas, as they are seen as consumable resources. Distances are the quantitative counterpart of equivalence relations: they measure how much two objects are similar, rather than just saying whether they are equivalent or not. Hence, they provide the natural choice for modelling equality in a substructural setting. In this paper, we develop this idea, using the categorical language of Lawvere’s doctrines. We work in a minimal fragment of Linear Logic enriched by graded modalities, which are needed to write a resource sensitive substitution rule for equality, enabling its quantitative interpretation as a distance. We introduce both a deductive calculus and the notion of Lipschitz doctrine to give it a sound and complete categorical semantics. The study of 2-categorical properties of Lipschitz doctrines provides us with a universal construction, which generates examples based for instance on metric spaces and quantitative realisability. Finally, we show how to smoothly extend our results to richer substructural logics, up to full Linear Logic with quantifiers.

1. INTRODUCTION

Equality is probably the most elementary and non-trivial predicate one can consider in logic. It enables a very basic task, that is, reasoning about the identity of expressions and thus of the objects they denote. Equality in First Order Logic is nowadays fairly well understood from both a syntactic and a semantic perspective. On the syntactic side, equality can be described as a binary predicate $t = u$, relating two terms t and u , which has to satisfy a couple of properties: every term has to be equal to itself (*reflexivity*) and, whenever we know that t and u are equal and a formula ϕ holds on t , then it holds on u as well (*substitutivity*). In other words, the former establishes that some trivial identities always hold, while the latter allows one to *transport* a property ϕ from t to u when they are equal; indeed, substitutivity is also known as the transport property of equality. More formally, the equality predicate has to satisfy the following entailments:

$$\top \vdash t = t \qquad \phi[t/x] \ \& \ t = u \vdash \phi[u/x]$$

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where \top is the true predicate, $\&$ is the conjunction and $\phi[t/x]$ denotes the substitution of the term t for the variable x in the formula ϕ .

A simple semantic counterpart of the above description of equality is given by *equivalence relations*. Indeed, considering a set X of objects/individuals one is interested in, a property/formula on X can be seen as a function $\alpha : X \rightarrow \{0, 1\}$, which says, for each object x in X , whether or not it has the property α . Then, equality can be safely modelled by an equivalence relation on X , namely, a binary function $\rho : X \times X \rightarrow \{0, 1\}$ enjoying the usual reflexivity, symmetry and transitivity properties, provided that formulas are modelled by those α compatible with ρ . That is, if x and y are equivalent according to ρ (i.e. $\rho(x, y) = 1$), then either both satisfy α or both do not (i.e. $\alpha(x) = \alpha(y)$).

The understanding of equality becomes much less clear when moving to *substructural* logics, such as, (fragments of) Linear Logic [Gir87]. The core idea of these logics is to see formulas as consumable resources in a proof, which therefore cannot be freely deleted or duplicated. To achieve this, some structural rules of usual deductive calculi are removed, specifically weakening and contraction rules, which are precisely those enabling deletion and duplication of formulas. As a consequence, usual true \top and conjunction $\&$ are replaced by $\mathbf{1}$ and \otimes , respectively, which behaves just as monoid-like operations (in particular, \otimes is not idempotent and $\mathbf{1}$ is not a top element).

Even though the model sketched above cannot work in general for the substructural case, since it obviously validates both weakening and contraction, from a semantic perspective, we can still have a quite natural understanding of equality, by taking a *quantitative* point of view. Intuitively, a quantitative semantics measures how much a property holds rather than just saying whether it holds or not. As a paradigmatic example, let us model a quantitative property on a set X by a function $\alpha : X \rightarrow [0, \infty]$ taking values in the set of extended non-negative real numbers (usually considered with the reversed order). Intuitively, the value $\alpha(x) = \epsilon$ means that x has the property α up to an error ϵ , hence α surely holds on x when $\alpha(x) = 0$. Following this intuitive reading of quantitative properties, conjunction is given by the pointwise extension of addition of real numbers, which, being non-idempotent, is capable of properly propagate errors. In this setting, the natural generalisation of equivalence relations are *distances*, that is, binary functions $\rho : X \times X \rightarrow [0, \infty]$ satisfying a quantitative version of reflexivity, symmetry and transitivity:

$$0 \geq \rho(x, x) \quad \rho(x, y) \geq \rho(y, x) \quad \rho(x, y) + \rho(y, z) \geq \rho(x, z)$$

Note that the quantitative transitivity property is nothing but the usual triangular inequality.

Now the question is: how can we introduce equality in substructural logics so that this quantitative semantics is supported? A first naive attempt could be to take the linear version of equality rules in First Order Logic, introducing a binary predicate $t = u$ which satisfies the linear reflexivity and substitutivity properties written below:

$$\mathbf{1} \vdash t = t \quad \phi[t/x] \otimes t = u \vdash \phi[u/x]$$

These entailments are precisely the usual ones where standard connectives \top and $\&$ are replaced by their linear versions $\mathbf{1}$ and \otimes . However smooth, this approach raises the unexpected issue that the equality predicate can be used an arbitrary number of times. Indeed, as already observed in [Doz96], the following entailments are provable from the two properties above:

$$t = u \vdash \mathbf{1} \quad t = u \vdash (t = u) \otimes (t = u)$$

The former says that equality can be deleted, while the latter that it can be duplicated. In other words, this shows that weakening and contraction rules, which in general are banned from substructural logics, are instead admissible for equality. This fact is undesirable per se in a substructural setting, as it is against its goal of controlling the use of formulas, but it is even worse from the semantic perspective, as the quantitative interpretation of equality is broken. Indeed, given a distance $\rho : X \times X \rightarrow [0, \infty]$ interpreting equality, the second entailment above forces it to satisfy the inequality $\rho(x, y) \geq \rho(x, y) + \rho(x, y)$, which is true only when $\rho(x, y)$ is either 0 or ∞ . This means that ρ is, de facto, an equivalence relation.

In this paper we design a novel approach to equality in substructural logic, which is capable of preserve its natural quantitative interpretation as a (non-trivial) distance. On the one hand, this treatment of equality, enabling more control on its usage, is more in the spirit of substructural logics and, on the other, it could also shed a new light on the connection between such logics and *quantitative reasoning*, defining a set of formal tools supporting it. Indeed, quantitative methods are increasingly used in many different domains, such as differential privacy [RP10, BKOB13, XCL14, BGHP16], denotational semantics [BM94, dAGH⁺17], program/behavioural metrics [DGJP04, vBW05, CGPX14, Gav18, DGY19, DG22], and rewriting [GF22]. With respect to usual techniques, quantitative methods better cope with the imprecision arising when one reasons about the behaviour of complex software systems also interacting with physical processes. The main formal system supporting these techniques is Quantitative Equational Logic, introduced and developed by Mardare et al. [MPP16, MPP17, MPP21, Adá22, BMPP18]. This is a logical system with all structural rules, specifically designed to reason about real-valued metrics, using ad-hoc symbols representing approximations of such metrics. Nevertheless, quantitative methods are also naturally connected to substructural logics, however, such a connection is still less explored, hence, in this paper, we investigate it systematically, developing both syntax and (categorical) semantics of a (minimal) substructural logic with *quantitative equality*.

Our analysis starts by observing that, in analogy with what happens in the specific case of program metrics [CD17, Gav18], the usual substitutive property is not resource sensitive as it does not take into account resources needed to perform the substitution: the equality $t = u$ has to be substitutive *for any* formula ϕ , while the “cost” of substituting t by u may be different for different formulas, since, for instance, it may depend on the number of occurrences of t . Hence, we need to rephrase the substitutive property to explicitly consider such resources. To this end, we work in (a fragment of) Linear Logic enriched with *graded modalities* [BP15]. Roughly, one adds to the logic a family of unary connectives $!_r$, where r is taken from a structure of resources, which make explicit “how much” a formula can be used.

Using graded modalities, we can rephrase the substitutive property as follows:

$$\phi[t/x] \otimes !_r(t = u) \vdash \phi[u/x]$$

where r depends on ϕ (in a way that will be described in the paper) and encodes the amount of resources needed to derive the substitution. In this way, the equality predicate cannot be duplicated, hence it admits a non-trivial quantitative interpretation.

The main mathematical tool we will use throughout the paper are *Lawvere’s doctrines* [Law69], which provide an elegant categorical/algebraic understanding of logic. The key feature of this approach is that theories are written as functors, named doctrines, whose domain category models contexts and terms, and the functor maps each object to a poset that models formulas on that object ordered by logical entailment. In this framework,

logical phenomena can be explained algebraically, abstracting away from details and focusing only on essential features. For instance, replicability of standard equality in substructural logics has a neat algebraic explanation: such an equality is defined by a left adjoint, as pioneered by Lawvere [Law69, Law70], and, as we will show, predicates defined in this way are always replicable. This shows that a quantitative equality *cannot be given by a left adjoint*, however, thanks to the language of doctrines, we manage to compare in a rigorous way quantitative equality with the standard one, proving they share other fundamental structural properties. More precisely, as proved in [EPR20], standard equality in non-linear doctrines has a coalgebraic nature and quantitative equality, generalising the standard one, is coalgebraic as well.

The rest of the paper is organised as follows. Section 2 recalls known facts on equality in doctrines and presents their extension to the linear setting, discussing the non-quantitative nature of such an equality. Section 3 presents our framework for quantitative equality in a (graded) substructural logic. In Section 3.1 we first define R -graded doctrines, which are doctrines modelling the $(\otimes, \mathbf{1})$ -fragment of Linear Logic enriched by R -graded modalities, where R is an ordered semiring of resources, and introduce R -Lipschitz doctrines, namely, R -graded doctrines with quantitative equality. Then, in Section 3.2, we present a core deductive calculus for quantitative equality, based on that in [BP15], describing its sound and complete categorical semantics in R -Lipschitz doctrines in Section 3.3. Section 4 compares our approach to Quantitative Equational Theories (QETs) [MPP16, MPP17], providing some examples of theories in our calculus, which cannot be expressed as QETs. Section 5 analyses 2-categorical properties of R -Lipschitz doctrines. In Section 5.1 we show the coalgebraic nature of R -Lipschitz doctrines by proving they arise as coalgebras for a 2-comonad on the 2-category of R -graded doctrines. This provides us with a universal construction yielding an R -Lipschitz doctrine from an R -graded one, and we use it to generate semantics for the calculus. In Section 5.2 we relate quantitative equality with the usual one defined by left adjoints, formally proving that the former indeed refines the latter. Then, in Section 6 we extend previous results to richer fragments of Linear Logic up to full LL with quantifiers. We conclude in Section 7 discussing related and future work.

This paper is the journal version of [DP22]. Here, we give more background on doctrines (Section 2.1) and more details on standard equality in the substructural setting (Section 2.2). Moreover, we provide a categorical comparison between our quantitative equality and the usual one by left adjoints (Section 5.2). Finally, we also present more examples of theories (Section 4) and report all proofs of our results.

2. PRELIMINARIES ON (LINEAR) DOCTRINES

Doctrines provide a simple categorical framework to study several kinds of logics. In this section, we first recall basic notions about doctrines for standard (*i.e.* non-linear) logics, then we introduce the class of doctrines modelling the fragment of linear logic we will be concerned with.

2.1. Doctrines in a nutshell. Denote by \mathcal{Pos} the category of posets and monotone functions. A left adjoint to a monotone function $g : K \rightarrow H$ is a monotone function $f : H \rightarrow K$ such that for every x in K and y in H , both $y \leq gf(y)$ and $fg(x) \leq x$ hold. Equivalently $y \leq g(x)$ if and only if $f(y) \leq x$.

A *doctrine* is a pair $\langle \mathcal{C}, P \rangle$ where \mathcal{C} is a category with finite products and $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is a functor. The category \mathcal{C} is named the *base* of the doctrine. For X an object in \mathcal{C} the poset $P(X)$ is called *fibre over X* . For $f : X \rightarrow Y$ an arrow in \mathcal{C} , the monotone function $P_f : P(Y) \rightarrow P(X)$ is called *reindexing along f* . We will often refer to a doctrine $\langle \mathcal{C}, P \rangle$ using only the functor $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$.

A doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is *primary* if all fibres have finite meets and these are preserved by reindexing. The top element of $P(A)$ will be denoted as \top_A , while binary meets by \wedge_A . We shall drop the subscripts when these are clear.

There is a large variety of examples of primary doctrines (see [Jac01, vO08, Pit00]), in this paper we will exemplify definitions considering the following.

- Examples 2.1.** (1) Let L be a many-sorted signature with sorts σ, τ, \dots , function symbols f, g, \dots and predicate symbols p, q, \dots . Let \mathcal{T} be a theory in the $(\&, \top)$ -fragment of First Order Logic over L . The category \mathbf{Ctx}_L has contexts $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$ as objects and lists of terms $\langle t_1, \dots, t_k \rangle : \vec{\sigma} \rightarrow \langle y_1 : \tau_1, \dots, y_k : \tau_k \rangle$ as arrows, where each t_i has sort τ_i in the context $\vec{\sigma}$. Binary products are given by context concatenation, up to renaming of variables. As detailed in [MR13b, Example 2.2] the *syntactic doctrine* $\mathit{Prp}_{\mathcal{T}}$ based on \mathbf{Ctx}_L maps each context $\vec{\sigma}$ to the poset reflection of the collection of well-formed formulas over L in the $(\&, \top)$ -fragment with free variables in $\vec{\sigma}$ preordered by the entailment in \mathcal{T} . That is, $[\phi] \leq [\psi]$ iff $\phi \vdash_{\vec{\sigma}} \psi$ is provable in \mathcal{T} (In other words, the poset $\mathit{Prp}_{\mathcal{T}}(\vec{\sigma})$ is the Lindenbaum-Tarski algebra of well-formed formulas over L in the $(\&, \top)$ -fragment with free variables in $\vec{\sigma}$). The doctrine $\mathit{Prp}_{\mathcal{T}}$ is primary where conjunctions give finite meets.
- (2) Suppose H is a poset, the functor $\mathcal{P}_H : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ sends a set A to the set of functions H^A ordered pointwise. For a function $f : X \rightarrow A$ and α in $\mathcal{P}_H(A)$ the function $\mathcal{P}_H(f)(\alpha)$ is the composite αf in $\mathcal{P}_H(X)$. The doctrine \mathcal{P}_H is primary when H is a meet-semilattice. For $H = \{0, 1\}$ the doctrine \mathcal{P}_H is the contravariant powerset functor that will be denoted simply by \mathcal{P} .
- (3) Suppose that $A = (|A|, \cdot, K, S)$ is a (partial) combinatory algebra as in [vO08]. The doctrine $\mathcal{R}_A : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ maps each set X to the poset reflection of the preorder $\mathcal{P}(|A|)^X$ where for $\alpha, \beta : X \rightarrow \mathcal{P}(|A|)$, $\alpha \leq \beta$ whenever there is a in $|A|$ such that, for every x in X and b in $\alpha(x)$, the application $a.b$ is defined and belongs to $\beta(x)$. One says that a realises $\alpha \leq \beta$. The action of \mathcal{R}_A on functions is given by pre-composition. \mathcal{R}_A is primary [vO08].

Doctrines are the objects of the 2-category \mathbf{Dtn} where 1-arrows $\langle F, f \rangle : \langle \mathcal{C}, P \rangle \rightarrow \langle \mathcal{D}, Q \rangle$ consist of a finite product preserving functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $f : P \rightarrow QF^{\text{op}}$ as in the diagram

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow F^{\text{op}} & \searrow P & \\
 \mathcal{D}^{\text{op}} & & \mathbf{Pos} \\
 & \downarrow f \cdot & \nearrow Q \\
 & &
 \end{array}$$

the 2-arrows $\theta : \langle F, f \rangle \Rightarrow \langle F', f' \rangle$ are natural transformations $\theta : F \rightarrow F'$ such that $f_X \leq_{FX} Q_{\theta_X} \circ f'_X$ for every object X in \mathcal{C} , i.e.

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \begin{array}{c} \downarrow \\ \theta^{\text{op}} \\ \downarrow \end{array} & \begin{array}{c} \xrightarrow{P} \\ (F')^{\text{op}} \quad f \cdot (\leq) \cdot f' \\ \downarrow \\ Q \end{array} & \text{Pos} \\
 \mathcal{D}^{\text{op}} & &
 \end{array}$$

The composition of two consecutive 1-arrows $\langle F, f \rangle$ and $\langle G, g \rangle$ is given by $\langle G, g \rangle \circ \langle F, f \rangle = \langle GF, (gF^{\text{op}})f \rangle$, while composition of 2-arrows is that of natural transformations.

Primary doctrines are the objects of the 2-full 2-subcategory \mathbf{PD} of \mathbf{Dtn} , where a 1-arrow from P to Q is a 1-arrow $\langle F, f \rangle$ in \mathbf{Dtn} such that each component of f preserves finite meets.

Remark 2.2. The 2-categorical structure of doctrines has a relevant logical meaning: objects correspond to theories, 1-arrows to models/translations between theories and 2-arrows to homomorphisms of models/translations. Hence, the hom-category of two doctrines P and Q corresponds to the category of models of P into Q .

Under this interpretation, each 2-category of doctrines corresponds to a class of theories with their models. For instance, the 2-category \mathbf{PD} corresponds to theories in the (\wedge, \top) -fragment of First Order Logic. Thus, given a theory \mathcal{T} in this fragment over a signature L as in Example 2.1(1), a categorical semantics of \mathcal{T} into a primary doctrine P is an object of $\mathbf{PD}(Prp_{\mathcal{T}}, P)$.

A categorical description of equality in terms of left adjoints goes back to Lawvere [Law70]. More recently, Maietti and Rosolini [MR13a] identify primary doctrines as the essential framework where equality can be studied. Primary doctrines with equality are called elementary [MR13a] and are defined as follows.

Definition 2.3. A primary doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is *elementary* if for every A in \mathcal{C} , there is an element δ_A in $P(A \times A)$ such that

- (1) $\top_A \leq P_{\Delta_A}(\delta_A)$
- (2) for all X in \mathcal{C} and α in $P(X \times A)$ it holds that $P_{\langle \pi_1, \pi_2 \rangle}(\alpha) \wedge P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha)$

Examples 2.4. (1) Consider a theory \mathcal{T} over the $(\&, \top, =)$ -fragment of First Order Logic.

Then, the syntactic doctrine $Prp_{\mathcal{T}}$, defined as in Example 2.1(1), is elementary. Indeed, for every context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$ one can take as $\delta_{\vec{\sigma}}$ the formula $x_1 = y_1 \& \dots \& x_n = y_n$. over the context $\vec{\sigma} \times \vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n, y_1 : \sigma_1, \dots, y_n : \sigma_n \rangle$; conditions (1) and (2) in Definition 2.3 require the provability of entailments of the form

$$\top \vdash x_1 = x_1 \& \dots \& x_n = x_n \quad \phi(\vec{z}, x_1, \dots, x_n) \& x_1 = y_1 \& \dots \& x_n = y_n \vdash \phi(\vec{z}, y_1, \dots, y_n)$$

which easily follows from reflexivity and substitutivity of $x = y$.

- (2) The primary doctrine \mathcal{P}_H of Example 2.1(2) is elementary when H has a bottom element. For a set X , the equality predicate $\delta_X \in H^{X \times X}$ is the Kronecker delta mapping (x, x') to \top if $x = x'$ and to \perp otherwise.
- (3) The primary doctrine \mathcal{R}_A of Example 2.1(3) is elementary where for a set X the function δ_X maps (x, x') to $|A|$ if $x = x'$ and to \emptyset otherwise [vO08].

Elementary doctrines are the objects of the 2-category \mathbf{ED} where a 1-arrow $\langle F, f \rangle : P \rightarrow Q$ is a 1-arrow in \mathbf{PD} such that f preserves equality predicates. The 2-arrows are those of \mathbf{PD} .

As studied in [EPR20], equality is coalgebraic in the sense that the obvious inclusion of \mathbf{ED} into \mathbf{PD} has a right 2-adjoint

$$\mathbf{ED} \xleftarrow{\quad \top \quad} \mathbf{PD} \quad (2.1)$$

and \mathbf{ED} is isomorphic to the 2-category of coalgebras of the induced 2-comonad on \mathbf{PD} .

Remark 2.5. Similarly to what we have already observed in Remark 2.2, given a theory \mathcal{T} in the $(\&, \top, =)$ -fragment of First Order Logic, a categorical semantics of \mathcal{T} in an elementary doctrine P is an object of $\mathbf{ED}(Prp_{\mathcal{T}}, P)$. Therefore the adjoint situation in Diagram 2.1 allows one to build semantics for theories with equality, from semantics of theories without equality.

A primary doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathcal{P}os$ is a *first order doctrine* if P factors through the category of Heyting algebras and their homomorphisms and for every projection $\pi : A \times B \rightarrow B$ in \mathcal{C} the map $P(\pi)$ has a left adjoint \mathcal{A}_{π} and a right adjoint \mathcal{V}_{π} and these adjoints are natural in B , that is for every square

$$\begin{array}{ccc} A \times X & \xrightarrow{\pi} & X \\ \text{id}_A \times f \downarrow & & \downarrow f \\ A \times B & \xrightarrow{\pi} & B \end{array}$$

it holds that $P_f \mathcal{A}_{\pi} = \mathcal{A}_{\pi} P_{\text{id}_A \times f}$ and $P_f \mathcal{V}_{\pi} = \mathcal{V}_{\pi} P_{\text{id}_A \times f}$. The naturality of left and right adjoints is often called Beck-Chevalley condition. A primary doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathcal{P}os$ is a *hyperdoctrine* if it is first order and elementary.

Examples 2.6. (1) The syntactic doctrine $Prp_{\mathcal{T}}$ over a theory \mathcal{T} in full First Order Logic is first order (recall also Example 2.1(1)). For every context $\vec{\sigma}$ the logical connectives between formulas with free variables in $\vec{\sigma}$ provide the operations of the Heyting algebra. For a formula α over $\vec{\sigma} \times \vec{\tau} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n, y_1 : \tau_1, \dots, y_k : \tau_k \rangle$ the left adjoint along $\pi : \vec{\sigma} \times \vec{\tau} \rightarrow \vec{\tau}$ is the formula $\mathcal{A}_{\pi} \alpha$ with free variables over $\vec{\tau}$ defined as

$$\exists_{x_1 : \sigma_1} \dots \exists_{x_n : \sigma_n} \alpha$$

The right adjoint $\mathcal{V}_{\pi} \alpha$ is similar where the universal quantification is used in place of the existential one. The syntactic doctrine $Prp_{\mathcal{T}}$ over a theory \mathcal{T} (see also Example 2.4(1)) in full First Order Logic with equality is a hyperdoctrine.

- (2) The primary doctrine \mathcal{P}_H of Example 2.1(2) is an hyperdoctrine when H is a complete Heyting algebra, i.e. it has arbitrary suprema that distribute over meets [Pit00, vO08].
- (3) The primary doctrine \mathcal{R}_A of Example 2.1(3) is an hyperdoctrine [vO08].

Denote by \mathbf{FOD} the 2-full 2-subcategory of \mathbf{PD} on first order doctrines and those 1-arrows of \mathbf{PD} that additionally commute with all connectives and quantifiers. Similarly, \mathbf{HD} denotes the 2-full 2-subcategory of \mathbf{ED} on hyperdoctrines¹. The adjoint situation depicted in Diagram 2.1 restricts to the inclusion of \mathbf{FOD} into \mathbf{PD} and co-restricts to the inclusion of \mathbf{HD} into \mathbf{ED} , leading to the following diagram

¹One can see \mathbf{HD} as the pullback of $\mathbf{ED} \rightarrow \mathbf{PD}$ along $\mathbf{FOD} \rightarrow \mathbf{PD}$.

$$\begin{array}{c}
[\text{Ax}] \frac{}{\phi \vdash \phi} \qquad [\text{Cut}] \frac{\Gamma \vdash \phi \quad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi} \\
[\text{1L}] \frac{\Gamma \vdash \phi}{\Gamma, \mathbf{1} \vdash \phi} \qquad [\text{1R}] \frac{}{\vdash \mathbf{1}} \qquad [\otimes\text{L}] \frac{\Gamma, \psi, \theta \vdash \phi}{\Gamma, \psi \otimes \theta \vdash \phi} \qquad [\otimes\text{R}] \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}
\end{array}$$

Figure 1: Rules for $(\otimes, \mathbf{1})$ -fragment of Linear Logic

$$\begin{array}{ccc}
\mathbf{ED} & \xleftrightarrow{\top} & \mathbf{PD} \\
\uparrow & & \uparrow \\
\mathbf{HD} & \xleftrightarrow{\top} & \mathbf{FOD}
\end{array} \tag{2.2}$$

Combining the observations made in Remark 2.2 and in Remark 2.5 it not surprising that, given a theory \mathcal{T} in full First Order Logic over a signature L , a categorical semantics of \mathcal{T} in a first order doctrine P is an object of $\mathbf{FOD}(Prp_{\mathcal{T}}, P)$; and if \mathcal{T} is with equality and P is a hyperdoctrine (and not only a first order doctrine) models of \mathcal{T} in P are objects $\mathbf{HD}(Prp_{\mathcal{T}}, P)$.

2.2. Primary and elementary linear doctrines. Primary doctrines, that models the $(\&, \top)$ -fragment of First Order Logic, are the minimal setting where one can formulate rules of equality (cf. Definition 2.3). We shall call their linear counterparts primary linear doctrines. They model the $(\otimes, \mathbf{1})$ -fragment of Linear Logic and provide the minimal structure needed to describe equality in a substructural setting. The following definition is carved out from Seely's definition of linear hyperdoctrines [See89].

Definition 2.7. A *primary linear doctrine* is a triple $\langle P, *, \kappa \rangle$ where $P : \mathcal{C}^{\text{op}} \rightarrow \mathcal{Pos}$ is a doctrine and $*$: $P \times P \dot{\rightarrow} P$ and $\kappa : 1_{\mathcal{C}} \dot{\rightarrow} P$ are natural transformations such that, for all objects X in \mathcal{C} , $\langle P(X), *_X, \kappa_X \rangle$ is a commutative monoid.

In other words, a primary linear doctrine is a doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathcal{Pos}$ that factors through the category of ordered commutative monoids. More explicitly, for each object X in \mathcal{C} , we have a monotone function $*_X : P(X) \times P(X) \rightarrow P(X)$ and an element $\kappa_X \in P(X)$ such that $*_X$ is associative and commutative and κ_X is the neutral element for $*_X$ and, for $f : X \rightarrow Y$ in \mathcal{C} and α, β in $P(Y)$, we have $P_f(\alpha) *_X P_f(\beta) = P_f(\alpha *_Y \beta)$ and $\kappa_X = P_f(\kappa_Y)^2$. We shall often omit subscripts from $*$ and κ when these are obvious.

Primary linear doctrines are the objects of the 2-category \mathbf{LD} where a 1-arrow from $\langle P, *_P, \kappa^P \rangle$ to $\langle Q, *_Q, \kappa^Q \rangle$ is a 1-arrow $\langle F, f \rangle : P \rightarrow Q$ in \mathbf{Dtn} such that f is a natural monoid homomorphism, and 2-arrows are those of \mathbf{Dtn} .

Examples 2.8. (1) For a theory \mathcal{T} in the $(\otimes, \mathbf{1})$ -fragment of Linear Logic i.e. in the fragment described by the rules in Figure 1 (see [Gir87]) the syntactic doctrine $Prp_{\mathcal{T}}$ as in Example 2.1(1) is primary linear.

(2) If $\langle H, *, \kappa \rangle$ is an ordered commutative monoid, then $\langle \mathcal{P}_H, *, \kappa \rangle$ is primary linear, where \mathcal{P}_H is as in Example 2.1(2) and $*$ and κ are defined pointwise.

²Concisely, a primary linear doctrine is a monoid in $[\mathcal{C}^{\text{op}}, \mathcal{Pos}]$.

- (3) A BCI algebra $A = \langle |A|, \cdot, B, C, I \rangle$ can be seen as a linear version of a combinatory algebra where linear combinators B, C, I are used in place of combinators K, S (a precise definition is in [AHS02]). Analogously to Example 2.1(3) we define the realisability doctrine $\mathcal{R}_A : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ as the functor that maps each set X to the poset reflection of the preorder $\mathcal{P}(|A|)^X$ where for $\alpha, \beta \in \mathcal{P}(|A|)^X$, it is $\alpha \leq \beta$ whenever there is $a \in |A|$ such that, for every $x \in X$ and $b \in \alpha(x)$, the application $a.b$ is defined and belongs to $\beta(x)$. The action of \mathcal{R}_A on functions is by pre-composition. Each $\mathcal{R}_A(X)$ is a poset as I and B realise reflexivity and transitivity of \leq , respectively. Given $\alpha, \beta \in \mathcal{P}(|A|)^X$, define $\alpha * \beta = x \mapsto \{\mathbf{P}ab \in |A| \mid a \in \alpha(x) \text{ and } b \in \beta(x)\}$, where \mathbf{P} is the BCI pairing [Hos07], and $\kappa_X = x \mapsto \{I\}$. Then, $\langle \mathcal{R}_A, *, \kappa \rangle$ is a primary linear doctrine.
- (4) Let $\mathbb{W} = \langle W, \leq, \circ, \epsilon \rangle$ be an ordered commutative monoid (a.k.a. monoidal Kripke frame). Adapting from [DG22] the notion of \mathbb{W} -relation, we define a doctrine $K_{\mathbb{W}} : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ as follows: for a set X , $K_{\mathbb{W}}(X)$ is the collection of those subsets U of $X \times W$ such that if $\langle x, w \rangle \in U$ and $w \leq w'$ then $\langle x, w' \rangle \in U$ ordered by set inclusion, and for a function $f : X \rightarrow Y$, $K_{\mathbb{W}}(f)$ maps $U \in K_{\mathbb{W}}(Y)$ to $(f \times \text{id}_W)^{-1}(U)$. This is actually a primary linear doctrine, where, for every set X , $\kappa_X = \{\langle x, w \rangle \mid \epsilon \leq w\}$ and $U *_X V = \{\langle x, w \rangle \mid \exists w_1, w_2. w_1 \circ w_2 \leq w, \langle x, w_1 \rangle \in U, \langle x, w_2 \rangle \in V\}$.

Recall that elementary doctrines can be understood as those primary doctrine endowed with equality predicates. The following definition introduces those primary linear doctrines that are elementary as a direct linearisation of Definition 2.3 (i.e. replacing of \wedge, \top by $*, \kappa$).

Definition 2.9. A primary linear doctrine $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ is *elementary* if for every A in \mathcal{C} , there is an element δ_A in $P(A \times A)$ such that

- (1) $\kappa_A \leq P_{\Delta_A}(\delta_A)$
- (2) for all X in \mathcal{C} and α in $P(X \times A)$ it holds that $P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha)$

Lawvere first noticed that equality is a left adjoint [Law70]. This turns out to be true also in the linear setting.

Proposition 2.10. Let $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ be a primary linear doctrine, the following are equivalent

- (1) P is elementary
- (2) For every A there is δ_A in $P(A \times A)$ such that for every X and every α in $P(X \times A)$ the assignment

$$P(X \times A) \xrightarrow{\mathbb{F}_{\text{id}_X \times \Delta_A}} P(X \times A \times A)$$

$$\alpha \longmapsto P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * P_{\langle \pi_2, \pi_3 \rangle}(\delta_A)$$

defines a left adjoint to $P_{\text{id}_X \times \Delta_A}$.

Proof. (1) \Rightarrow (2). Suppose $\mathbb{F}_{\text{id}_X \times \Delta_A}(\alpha) \leq \beta$ and apply $P_{\text{id}_X \times \Delta_A}$ to both side of the inequality to get $\alpha \leq P_{\text{id}_X \times \Delta_A}(\beta)$. Conversely suppose $\alpha \leq P_{\text{id}_X \times \Delta_A}(\beta)$. Consider β as an element over $(X \times A) \times A$ and let p_1, p_2 and p_3 denote projections from $(X \times A) \times A$ and π_1, π_2 and π_3 denote projections from $X \times A \times A$. It holds $P_{\langle p_1, p_2 \rangle}(\beta) * P_{\langle p_2, p_3 \rangle}(\delta_A) \leq P_{\langle p_1, p_3 \rangle}(\beta)$. Evaluate both side of the inequality under $P_{\langle \pi_1, \pi_2, \pi_2, \pi_3 \rangle}$ to get

$$P_{\langle \pi_1, \pi_2 \rangle} P_{\text{id}_X \times \Delta_A}(\beta) * P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq \beta$$

and hence the claim.

(2) \Rightarrow (1). Note that $\delta_A = \mathcal{E}_{\Delta_A}(\kappa_A)$, so $\kappa_A \leq P_{\Delta_A}(\delta_A)$. Given α in $P(X \times A)$, from $\alpha = P_{\langle \text{id}_X \times \Delta_A \rangle} P_{\langle \pi_1, \pi_3 \rangle}(\alpha)$ follows $\mathcal{E}_{\text{id}_X \times \Delta_A}(\alpha) = P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha)$. \square

The characterisation 2.10 will be helpful to prove some basic properties of equality predicates. Equality predicates in elementary doctrines (i.e. in the non-linear setting) are proved to be equivalence relations. Their linear counterpart are distances³, which can be defined in an abstract purely algebraic way in any primary linear doctrine $\langle P, *, \kappa \rangle$ as follows: a *P-distance* on an object A is an element $\rho \in P(A \times A)$ such that

$$\begin{aligned} \kappa &\leq P_{\Delta_A}(\rho) && \text{(reflexivity)} \\ \rho &\leq P_{\langle \pi_2, \pi_1 \rangle}(\rho) && \text{(symmetry)} \\ P_{\langle \pi_1, \pi_2 \rangle}(\rho) * P_{\langle \pi_2, \pi_3 \rangle}(\rho) &\leq P_{\langle \pi_1, \pi_3 \rangle}(\rho) && \text{(transitivity)} \end{aligned}$$

- Examples 2.11.** (1) For the syntactic doctrine $Prp_{\mathcal{T}}$ as in Example 2.8(1), a $Prp_{\mathcal{T}}$ -distance over a context $\vec{\sigma}$ is a formula $\rho(\vec{x}, \vec{y})$ in the context $\vec{\sigma} \times \vec{\sigma}$ such that the following entailments are derivable: $\mathbf{1} \vdash \rho(\vec{x}, \vec{x})$ and $\rho(\vec{x}, \vec{y}) \vdash \rho(\vec{y}, \vec{x})$ and $\rho(\vec{x}, \vec{y}) \otimes \rho(\vec{y}, \vec{z}) \vdash \rho(\vec{x}, \vec{z})$. (2) For the primary linear doctrine $\langle \mathcal{P}_{[0, \infty]}, +, 0 \rangle$ as in Example 2.8(2) (where $[0, \infty]$ is the Lawvere quantale [Law73]) a $\mathcal{P}_{[0, \infty]}$ -distance ρ over a set X is a metric $\rho : X \times X \rightarrow [0, \infty]$, i.e. $0 \geq \rho(x, x)$ and $\rho(x, y) \geq \rho(y, x)$ and $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$. (3) For the realisability doctrine \mathcal{R}_A as in Example 2.8(3), a \mathcal{R}_A -distance ρ over a set X is function $\rho : X \times X \rightarrow \mathcal{P}(|A|)$ such that there are three elements in $|A|$ realising reflexivity, symmetry, transitivity of ρ . (4) For the doctrine K_W as in Example 2.8(4), a K_W -distance over a set X is a ternary relation $\rho \subseteq X \times X \times W$ such that, for every x, y, z in X and every w, w_1, w_2 in W , we have $\epsilon \leq w$ implies $\langle x, x, w \rangle \in \rho$ and $\langle x, y, w \rangle \in \rho$ implies $\langle y, x, w \rangle \in \rho$ and $\langle x, y, w_1 \rangle \in \rho$ and $\langle y, z, w_2 \rangle \in \rho$ implies $\langle x, z, w_1 \circ w_2 \rangle \in \rho$.

Proposition 2.12. *Let $\langle P, *, \kappa \rangle$ be an elementary primary linear doctrine. For every object A in the base of P , the equality predicate δ_A is a P -distance.*

Proof. The first point of Definition 2.9 establishes reflexivity of δ_A . The second point gives transitivity when X is A and α is δ_A . To show symmetry, note that $\langle \pi_2, \pi_1 \rangle \circ \Delta_A = \Delta_A$, hence, from reflexivity we deduce $\kappa_A \leq P_{\Delta_A}(\delta_A) = P_{\Delta_A}(P_{\langle \pi_2, \pi_1 \rangle}(\delta_A))$ and from the adjunction $\mathcal{E}_{\Delta_A} \dashv P_{\Delta_A}$ and Proposition 2.10 we conclude $\delta_A = \mathcal{E}_{\Delta_A}(\kappa_A) \leq P_{\langle \pi_2, \pi_1 \rangle}(\delta_A)$. \square

As already mentioned in the introduction, the direct rephrasing of rules for equality from First Order Logic to Linear Logic leads to a replicable equality predicate. This was already observed by Dozen [Doz96] and it can be quickly seen in the following tree

$$\frac{\frac{\frac{}{\vdash t = t} \quad \frac{}{\vdash t = t}}{\vdash t = t \otimes t = t} \quad \frac{}{t = u \vdash t = u}}{\vdash t = t \otimes t = t \quad t = u \vdash t = u}}{t = u \vdash t = u \otimes t = u}$$

by reflexivity of $=$, we have $\vdash t = t$, then, introducing \otimes on the right, we get $\vdash t = t \otimes t = t$; by the assumption $t = u \vdash t = u$, then, since $t = t \otimes t = t$ is equal to $(t = x \otimes t = x)[t/x]$, using substitutivity we conclude $t = u \vdash t = u \otimes t = u$.

³Binary products in base \mathcal{C} of P play a crucial role in defining what a distance is, as one need diagonals and, more generally, tuples of arrows to formulate reflexivity and transitivity. This is clear from the syntactic point of view, as one can see from Example 2.11(1), where to write conditions for reflexivity and transitivity one uses formulas where the same variables occurs multiple times.

Doctrines allows to prove this drawback in more generality. Given a primary linear doctrine $\langle P, *, \kappa \rangle$, an element α in $P(X)$ is called *affine* if $\alpha \leq \kappa$ and *replicable* if $\alpha \leq \alpha * \alpha$. Note that on affine and replicable elements the monoidal operation $*$ behaves as a meet operation, hence, they can be deleted and duplicated.

- Examples 2.13.** (1) For the syntactic doctrine $Prp_{\mathcal{T}}$ over a theory \mathcal{T} in the $(\otimes, \mathbf{1})$ -fragment of Linear Logic, an element α in $Prp_{\mathcal{T}}(\vec{\sigma})$ (i.e. is a formula with free variables over the context $\vec{\sigma}$) is affine when weakening $(\alpha \vdash \mathbf{1})$ is derivable in \mathcal{T} , while α is replicable if contraction $(\alpha \vdash \alpha \otimes \alpha)$ is derivable in \mathcal{T} .
- (2) For the primary linear doctrine $\langle \mathcal{P}_{[0, \infty]}, +, 0 \rangle$ as in Example 2.11(2) every element $\alpha : A \rightarrow [0, \infty]$ is affine as 0 is also a top element, while α is replicable if and only if its values are either 0 or ∞ .

Proposition 2.14. *Let $\langle P, *, \kappa \rangle$ be primary linear and $f : X \rightarrow Y$ an arrow of the base. If P_f has a left adjoint \mathcal{A}_f , then $\mathcal{A}_f(\kappa_X)$ is affine and replicable.*

Proof. We prove affineness by noting that $\mathcal{A}_f(\kappa_X) = \mathcal{A}_f P_f(\kappa_Y) \leq \kappa_Y$. Replicability follows from $\kappa_X = \kappa_X * \kappa_X \leq P_f(\mathcal{A}_f \kappa_X) * P_f(\mathcal{A}_f \kappa_X) = P_f(\mathcal{A}_f(\kappa_X) * \mathcal{A}_f(\kappa_X))$ and the adjunction between \mathcal{A}_f and P_f . \square

By Proposition 2.10 it is $\delta_A = \mathcal{A}_{\Delta_A}(\kappa_A)$. Affineness and replicability follows by Proposition 2.14.

This shows that the standard approach based on left adjoints cannot support a quantitative notion of equality, which then needs to be defined in a different way.

3. QUANTITATIVE EQUALITY VIA GRADED MODALITIES

Graded modalities indexed by a semiring R are used in many linear calculi and type theories to enable resource sensitive reasoning [RP10, BGMZ14, GKO⁺16, OLI19, AB20, DG22, MIO21]. We introduce graded modalities in the framework of primary linear doctrines to formulate a quantitative variant of elementary doctrines that we call Lipschitz doctrines. We motivate the terminology after Corollary 3.9. Then, we define a core deductive calculus for quantitative equality and its categorical semantics in Lipschitz doctrines, proving it is sound and complete.

3.1. Graded and Lipschitz doctrines. A *resource semiring* is a semiring in the category \mathcal{Pos} , that is, a tuple $R = \langle |R|, \preceq, +, \cdot, 0, 1 \rangle$, where $\langle |R|, \preceq \rangle$ is a partially ordered set and $+$ and \cdot are monotone binary operations on it such that

- $\langle |R|, +, 0 \rangle$ is a commutative monoid and $\langle |R|, \cdot, 1 \rangle$ is a monoid,
- for all $r, s, q \in |R|$ we have $r \cdot (s + q) = r \cdot s + r \cdot q$, $(s + q) \cdot r = s \cdot r + q \cdot r$, $r \cdot 0 = 0$ and $0 \cdot r = 0$,
- $+$ and \cdot are monotone w.r.t. \preceq .

To define R -graded modalities, we adapt the approach in [DR21], following the definition of graded linear exponential comonads [BGMZ14, GKO⁺16, Kat18].

Definition 3.1. An R -graded (linear exponential) modality on a primary linear doctrine $\langle P, *, \kappa \rangle$ is an $|R|$ -indexed family of natural transformations $\flat_r : P \dashrightarrow P$ satisfying the axioms below for all objects X , α, β in $P(X)$ and r, s in $|R|$:

- (1) $\kappa \leq \flat_r \kappa$ (lax-monoidality-1)
- (2) $\flat_r \alpha * \flat_r \beta \leq \flat_r(\alpha * \beta)$ (lax-monoidality-2)

- (3) $b_0\alpha \leq \kappa$ (weakening)
- (4) $b_{r+s}\alpha \leq b_r\alpha * b_s\alpha$ (contraction)
- (5) $b_1\alpha \leq \alpha$ (counit)
- (6) $b_{r \cdot s}\alpha \leq b_r(b_s\alpha)$ (comultiplication)
- (7) $b_r\alpha \leq b_s\beta$ if $s \preceq r$ (contravariance)

An R -graded (primary linear) doctrine is a primary linear doctrine together with an R -graded modality on it.

Naturality ensures that b_r is monotone and stable under reindexing, for all r . making the graded modality a graded linear exponential comonad on each fibre.

We collect below some examples adapted from [BP15, OLI19, DG22].

- Examples 3.2.** (1) The singleton set $\{\infty\}$ with the trivial order and operations is a resource semiring. An $\{\infty\}$ -graded modality on P satisfies the following inequalities fibre-wise: $b_\infty\alpha \leq \kappa$, $b_\infty\alpha \leq b_\infty\alpha * b_\infty\alpha$, $b_\infty\alpha \leq \alpha$ and $b_\infty\alpha \leq b_\infty b_\infty\alpha$. Hence, b_∞ models the usual bang modality of Linear Logic [Gir87]. Syntactic doctrines on the $(\otimes, \mathbf{1}, !)$ -fragment of Linear Logic are $\{\infty\}$ -graded.
- (2) Let \mathbb{N} abbreviate the semiring $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ of natural numbers and \mathbb{N}_∞ its extension by a new point ∞ with $\infty + x = x + \infty = \infty \cdot x = x \cdot \infty = \infty$. Denote by \mathbb{N}^\equiv and \mathbb{N}_∞^\equiv the semirings \mathbb{N} and \mathbb{N}_∞ ordered by the equality relation. If b is an \mathbb{N}^\equiv -graded modality on P , since $n = 1 + \dots + 1$ (n times), contraction and counit imply $b_n\alpha \leq \alpha * \dots * \alpha$ (n times). Hence, $b_n\alpha$ provides exactly n copies of α . Syntactically, this corresponds to the *exact usage* modality $!_n$ of Bounded Linear Logic [GSS92]. Indeed, syntactic doctrines built out of it are \mathbb{N}^\equiv -graded. If we consider \mathbb{N}_∞^\equiv , we get the additional modality b_∞ which behaves as in Item 1, modelling unrestricted usage.
- (3) Let $\mathbb{R}_{\geq 0} = \langle [0, \infty), \leq, +, \cdot, 0, 1 \rangle$ be the semiring of non-negative real numbers, $[0, \infty]$ be the Lawvere quantale [Law73] and $\langle \mathcal{P}_{[0, \infty]}, +, 0 \rangle$ be a doctrine as in Example 2.8(2). For $\alpha \in \mathcal{P}_{[0, \infty]}(X)$ and $r \in [0, \infty)$, the assignment $\alpha \mapsto \mathbf{m}_r\alpha$, with $\mathbf{m}_r\alpha(x) = r \cdot \alpha(x)$ is an $\mathbb{R}_{\geq 0}$ -graded modality on $\mathcal{P}_{[0, \infty]}$. Here $r \cdot \infty = \infty$ when $r \neq 0$ and $0 \cdot \infty = 0$.
- (4) Consider an ordered R -Linear Combinatory Algebra (R -LCA) [Atk18], that is, a BCI algebra A (cf. Example 2.8(3)) with a function $!_r : |A| \rightarrow |A|$ for $r \in |R|$ and elements $K, W_{r,s}, D, d_{r,s}, F_r$ and $O_{r,s}$, with $r \preceq s$, of $|A|$ satisfying

$$\begin{array}{ll}
 K.x.!_0y = x & W_{r,s}.x.!_{r+s}y = x.!_ry.!_sy \\
 D.!_1x = x & d_{r,s}.!_{r \cdot s}x = !_r!_sx \\
 F_r.!_rx.!_ry = !_r(x.y) & O_{r,s}.!_sx = !_rx
 \end{array}$$

The realisability doctrine \mathcal{R}_A (cf. Example 2.8(3)) is R -graded: for $\alpha \in \mathcal{P}(|A|)^X$ and $r \in |R|$, define $b_r\alpha = x \mapsto \{!_ra \mid a \in \alpha(x)\}$.

- (5) Let R be a resource semiring and $\mathbb{W} = \langle W, \leq, \circ, \epsilon \rangle$ a monoidal Kripke frame as in Example 2.8(4). Recall from [DG22] that a lax action of R on \mathbb{W} is a monotone map $a : |R| \times W \rightarrow W$ such that

$$\begin{array}{ll}
 a_X(r, \epsilon) \leq \epsilon & a_X(r, w_1 \circ w_2) \leq a_X(r, w_1) \circ a_X(r, w_2) \\
 \epsilon \leq a(0, w) & a(r, w) \circ a(s, w) \leq a(r + s, w) \\
 w \leq a(1, w) & a(r, a(s, w)) \leq a(r \cdot s, w)
 \end{array}$$

This gives rise to the R -graded modality \bar{a} on the primary linear doctrine $K_{\mathbb{W}}$ of Example 2.8(4) whose component at X maps $U \in K_{\mathbb{W}}(X)$ to $\bar{a}_r U = \{\langle x, w \rangle \mid \exists v. a(r, v) \leq w, \langle x, v \rangle \in U\}$.

R -graded doctrines are the objects of the 2-category \mathbf{LD}_R where a 1-arrow from $\langle P, \flat \rangle$ to $\langle Q, \sharp \rangle$ is a 1-arrow $\langle F, f \rangle : P \rightarrow Q$ in \mathbf{LD} preserving modalities, that is, for all $r \in |R|$ and object X in the base of P , it holds $\sharp_r F_X \circ f_X = f_X \circ \flat_r X$; while 2-arrows, compositions and identities are those of \mathbf{LD} .

Proposition 3.3 establishes the inclusion $\mathbf{PD} \hookrightarrow \mathbf{LD}_R$.

Proposition 3.3. *Let $\langle P, *, \kappa \rangle$ be a primary linear doctrine and $\flat_r : P \dot{\rightarrow} P$ be the identity for all $r \in |R|$. Then, P is a primary doctrine iff \flat is an R -graded modality on it.*

Proof. Let X be an object in the base category of P and α an element in $P(X)$. The left-to-right implication is easy to check: the non-trivial axiom to verify is weakening, indeed, we have $\flat_0 \alpha = \alpha \leq \kappa$, as κ is the top element of $P(X)$. For the converse we have to show that κ is the top element of $P(X)$ and $\alpha \leq \alpha * \alpha$ for all $\alpha \in P(X)$. By weakening $\alpha = \flat_0 \alpha \leq \kappa$ and contraction gives $\alpha = \flat_{r+r} \alpha \leq \flat_r \alpha * \flat_r \alpha = \alpha * \alpha$. \square

A quantitative equality for an R -graded doctrine $\langle P, \flat \rangle$ is then defined to be a P -distance satisfying an *R -graded substitutivite property* as detailed below:

Definition 3.4. An *R -Lipschitz doctrine* is a triple $\langle P, \flat, \mathbf{d} \rangle$ where $\langle P, \flat \rangle$ is an R -graded doctrine and, for each object A in the base, \mathbf{d}_A is a P -distance on A such that, for all objects A and X and α in $P(X \times A)$

(a) there is an r in $|R|$ such that

$$P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * \flat_r P_{\langle \pi_2, \pi_3 \rangle}(\mathbf{d}_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha);$$

(b) $\mathbf{d}_{A \times X} = P_{\langle \pi_1, \pi_3 \rangle}(\mathbf{d}_A) * P_{\langle \pi_2, \pi_4 \rangle}(\mathbf{d}_X)$;

(c) $\mathbf{d}_{A \times X} \leq P_{\langle \pi_1, \pi_3 \rangle}(\mathbf{d}_A)$ and $\mathbf{d}_{A \times X} \leq P_{\langle \pi_2, \pi_4 \rangle}(\mathbf{d}_X)$;

(d) $\mathbf{d}_1 = \kappa$.

The key difference between elementary and R -Lipschitz doctrines is the substitutive property, which, taking advantage of graded modalities, in the latter is stated in a resource sensitive way. Indeed, to prove a substitution, we need to have *enough equality resources*.

Since symmetry and transitivity are no longer derivable from substitutivity, for Lipschitz doctrines we assumed \mathbf{d}_A to be a P -distance (and not only a reflexive relation), as these are natural properties for equality.

The product $A \times B$ mimics the concatenation of the context A with the context B , hence axioms (b), (c) and (d) encode the *independence of contexts*. It allows the independent use of equalities in a product.

In Section 5.1 we will describe a construction producing a Lipschitz doctrine out of any graded doctrine, which will provide us with several examples. For the moment we consider only the following one:

Examples 3.5. Denote by \mathcal{Met}_L the category of metric spaces (whose metrics take values in $[0, \infty]$) and Lipschitz continuous functions. For a metric space A , denote by $|A|$ the underlying set and by \mathbf{d}_A the metric of A . Note that \mathcal{Met}_L has finite products where $|A \times B| = |A| \times |B|$ and $\mathbf{d}_{A \times B}(x, y, x', y') = \mathbf{d}_A(x, x') + \mathbf{d}_B(y, y')$. Endow the Lawvere quantale $[0, \infty]$ with the Euclidean metric and let $Lip(A)$ be the set of Lipschitz continuous functions from a metric space A to $[0, \infty]$. Each $Lip(A)$ is an ordered commutative monoid where the order and the operations are defined pointwise. If $f : A \rightarrow B$ and $\alpha : B \rightarrow [0, \infty]$ are Lipschitz continuous, denote by $Lip(f)(\alpha)$ the composition $\alpha f : A \rightarrow [0, \infty]$. Then $Lip : \mathcal{Met}_L^{\text{op}} \rightarrow \mathcal{Pos}$ is a primary linear doctrine. Note that $\mathbf{d}_A \in Lip(A \times A)$ as the inequality

$d_A(x, x') + d_A(y, y') \geq |d_A(x', y') - d_A(x, y)|$ holds. Endow Lip with the $\mathbb{R}_{\geq 0}$ -graded modality \mathbf{m} which acts as in Example 3.2(3), i.e. $\mathbf{m}_r \alpha(x) = r \cdot \alpha(x)$, then $\langle Lip, \mathbf{m}, \mathbf{d} \rangle$ is a Lipschitz doctrine: indeed every $\alpha \in Lip(X \times A)$ is Lipschitz continuous, hence there is $r \in [0, \infty)$ such that $r \cdot d_{X \times A}(x, a, x', a') \geq |\alpha(x', a') - \alpha(x, a)|$ that, by reflexivity of d_X , implies $r \cdot d_A(a, a') \geq |\alpha(x, a') - \alpha(x, a)|$ and this is equivalent to $\alpha(x, a) + r \cdot d_A(a, a') \geq \alpha(x, a')$.

Axiom (a), describing substitution, can be equivalently rephrased as follows.

Proposition 3.6. *Let $\langle P, \flat \rangle$ be an R -graded doctrine and \mathbf{d} a family of P -distances. The following are equivalent:*

- (1) $\langle P, \flat, \mathbf{d} \rangle$ is R -Lipschitz;
- (2) $\langle P, \flat, \mathbf{d} \rangle$ satisfies axioms of Definition 3.4 where (a) is replaced by the following:
 - (a') for all objects A and α in $P(A)$, there is $r \in |R|$ such that

$$P_{\pi_1}(\alpha) * \flat_r \mathbf{d}_A \leq P_{\pi_2}(\alpha).$$

Proof. 1) \Rightarrow 2) take as X the terminal object of \mathcal{C} to get (a'). 2) \Rightarrow 1) Take α in $P(X \times A)$, then $P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * \flat_r \mathbf{d}_{X \times A} \leq P_{\langle \pi_3, \pi_4 \rangle}(\alpha)$. By (b) one has $P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * \flat_r P_{\langle \pi_2, \pi_4 \rangle} \mathbf{d}_A \leq P_{\langle \pi_3, \pi_4 \rangle}(\alpha)$. Reindexing along $\langle \pi_1, \pi_2, \pi_1, \pi_3 \rangle : X \times A \times A \rightarrow X \times A \times X \times A$ completes the proof. \square

Axioms (c) and (d) of Definition 3.4 are equivalent to affineness of equality.

Proposition 3.7. *Let $\langle P, \flat \rangle$ be an R -graded doctrine and \mathbf{d} be a family of P -distances. The following are equivalent:*

- (1) $\langle P, \flat, \mathbf{d} \rangle$ is R -Lipschitz;
- (2) \mathbf{d} satisfies the following axioms:
 - (a'') axioms (a) and (b) of Definition 3.4 hold,
 - (b'') \mathbf{d}_A is affine.

Proof. 1) \Rightarrow 2) follows from $\mathbf{d}_{A \times 1} = P_{\langle \pi_1, \pi_3 \rangle}(\mathbf{d}_A)$ and $\mathbf{d}_{A \times 1} \leq \mathbf{d}_1 = \kappa$. 2) \Rightarrow 1) is immediate. \square

The quantitative nature of this notion of equality can be noticed also from its relationship with reindexing, described by the following proposition.

Proposition 3.8. *Let $\langle P, \flat, \mathbf{d} \rangle$ be an R -Lipschitz doctrine. For any $f : A_1 \times \dots \times A_n \rightarrow B$ in the base category of P , there are $r_1, \dots, r_n \in |R|$ such that*

$$\flat_{r_1} P_{\langle \pi_1, \pi_{n+1} \rangle}(\mathbf{d}_{A_1}) * \dots * \flat_{r_n} P_{\langle \pi_n, \pi_{2n} \rangle}(\mathbf{d}_{A_n}) \leq P_{f \times f}(\mathbf{d}_B)$$

Proof. For all $i \in 1..n$, let $\vec{\pi}_i = \langle \pi_i, \dots, \pi_n \rangle$. By Definition 3.4((a)) we know that, for all $i \in 1..n$, there is $r_i \in |R|$ such that

$$P_{\langle \vec{\pi}_1^{i-1}, \vec{\pi}_i^n, \vec{\pi}_{n+1}^{i-1}, \vec{\pi}_i^n \rangle} (P_{f \times f}(\mathbf{d}_B)) * \flat_{r_i} (P_{\langle \pi_i, \pi_{n+i} \rangle}(\mathbf{d}_{A_i})) \leq P_{\langle \vec{\pi}_1^i, \vec{\pi}_{i+1}^n, \vec{\pi}_{n+1}^{i-1}, \vec{\pi}_{i+1}^n \rangle} (P_{f \times f}(\mathbf{d}_B))$$

holds in $A_1 \times \dots \times A_n \times A_1 \times \dots \times A_n$, where $\vec{\pi}_k^h$, with $1 \leq k \leq h+1 \leq 2n+1$, denotes the sequence of projections $\pi_k, \pi_{k+1}, \dots, \pi_{h-1}, \pi_h$. Combining these inequalities, using monotonicity of $*$ and transitivity of \leq , we get

$$P_{\langle \pi_1, \dots, \pi_n, \pi_1, \dots, \pi_n \rangle} (P_{f \times f}(\mathbf{d}_B)) * \flat_{r_1} (P_{\langle \pi_1, \pi_{n+1} \rangle}(\mathbf{d}_{A_1})) * \dots * \flat_{r_n} (P_{\langle \pi_n, \pi_{2n} \rangle}(\mathbf{d}_{A_n})) \leq P_{f \times f}(\mathbf{d}_B)$$

Note that $P_{\langle \pi_1, \dots, \pi_n, \pi_1, \dots, \pi_n \rangle} (P_{f \times f}(\mathbf{d}_B)) = P_{f \circ \langle \pi_1, \dots, \pi_n \rangle} (P_{\Delta_B}(\mathbf{d}_B))$. Then, the thesis follows because $\kappa \leq P_{f \circ \langle \pi_1, \dots, \pi_n \rangle} (P_{\Delta_B}(\mathbf{d}_B))$ holds by reflexivity of \mathbf{d}_B . \square

Corollary 3.9. *Let $\langle P, \flat, \mathbf{d} \rangle$ be an R -Lipschitz doctrine. For any $f : A \rightarrow B$ in the base category of P , there exists $r \in |R|$ such that $\flat_r \mathbf{d}_A \leq P_{f \times f}(\mathbf{d}_B)$.*

$$\begin{array}{c}
\text{[W]} \frac{\Gamma \vdash \phi}{\Gamma, !_0 \psi \vdash \phi} \qquad \text{[C]} \frac{\Gamma, !_r \psi, !_s \psi \vdash \phi}{\Gamma, !_{r+s} \psi \vdash \phi} \qquad \text{[DER]} \frac{\Gamma, \psi \vdash \phi}{\Gamma, !_1 \psi \vdash \phi} \\
\text{[PRO]} \frac{!_{s_1} \psi_1, \dots, !_{s_n} \psi_n \vdash \phi}{!_{r \cdot s_1} \psi_1, \dots, !_{r \cdot s_n} \psi_n \vdash !_r \phi} \qquad \text{[DECR]} \frac{\Gamma \vdash !_s \phi}{\Gamma \vdash !_r \phi} \quad r \preceq s
\end{array}$$

Figure 2: Rules for graded bang modality

Corollary 3.9 describes a condition similar to the Lipschitz condition on maps between metric spaces and motivates the name in Definition 3.4.

Finally, also R -Lipschitz doctrines can be organised into a 2-category \mathbf{LLD}_R , whose objects are R -Lipschitz doctrines, a 1-arrow from $\langle P, b, \mathbf{d} \rangle$ to $\langle Q, b', \mathbf{d}' \rangle$ is 1-arrow $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle Q, b' \rangle$ preserving the distance, that is, such that, for each object A in the base category of P , it holds $\mathbf{d}'_{FA} = f_{A \times A}(\mathbf{d}_A)$, and a 2-arrow from $\langle F, f \rangle$ to $\langle G, g \rangle$ is a 2-arrow $\theta : \langle F, f \rangle \Rightarrow \langle G, g \rangle$ in \mathbf{LD}_R .

3.2. A deductive calculus for quantitative equality. So far we have seen that theories \mathcal{T} over a signature L generate syntactic doctrines $Prp_{\mathcal{T}}$. This section describes a calculus whose associated syntactic doctrine is R -Lipschitz.

Assume $R = \langle |R|, \preceq, +, \cdot, 0, 1 \rangle$ is a resource semiring and L a first order signature. A judgment has the form $\Gamma \vdash_{\vec{\sigma}} \phi$ where $\Gamma = \phi_1, \dots, \phi_k$ is a finite multiset and $\phi_1, \dots, \phi_k, \phi$ are formulas over L in the context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$. We will often omit the context $\vec{\sigma}$ to improve readability.

We first introduce PLL_R , a calculus extending the $(\otimes, \mathbf{1})$ -fragment of Linear Logic by a family of modalities $!_r$, for $r \in |R|$. The front ‘P’ is for primary as its associated syntactic doctrine is a primary linear doctrine (endowed with an R -graded modality).

Rules of PLL_R are a subset of the calculus presented in [BP15], in Figure 2 we report only rules for modalities as structural rules and those for \otimes and $\mathbf{1}$ can be found in Figure 1. The first four rules are the graded variant of standard rules for the bang modality of Linear Logic. Rules [W] and [C] encode graded structural rules: by weakening we can add formulas marked as not used (labelled by 0) and contraction tracks the usage by addition. Rule [DER] tells that hypotheses with grade 1 can be treated as linear hypotheses and rule [PRO] introduces $!_r$, scaling by r the grades of the hypotheses. Rule [DECR] allows to approximate the usage of a formula following (contravariantly) the order of the semiring.

To extend PLL_R by a quantitative equality, we need to add a resource sensitive substitution rule. Hence, we need a way to compute for each formula ϕ and variable x the cost (represented as an element of the semiring) of substituting x in ϕ . To this end, we first enrich the notion of signature.

An R -graded signature is a first order signature L where symbols have an R -graded arity, that is, an assignment $|-|$ as the following:

$$|f| = \langle r_1, \sigma_1 \rangle, \dots, \langle r_n, \sigma_n \rangle \rightarrow \tau \qquad |p| = \langle r_1, \sigma_1 \rangle, \dots, \langle r_n, \sigma_n \rangle$$

where f and p are a function and a predicate symbol, respectively, and, σ_i is the sort of the i -th argument, while $r_i \in |R|$ says how much the cost of substituting a variable in the i -th position is amplified. We write $|f|_i$ (resp. $|p|_i$) in place of r_i in the assignments above.

$$\begin{array}{c}
\text{[R]} \frac{}{\vdash t \equiv_{\tau} t} \quad \text{[S]} \frac{\Gamma \vdash t \equiv_{\tau} u}{\Gamma \vdash u \equiv_{\tau} t} \quad \text{[T]} \frac{\Gamma \vdash t \equiv_{\tau} u \quad \Delta \vdash u \equiv_{\tau} v}{\Gamma, \Delta \vdash t \equiv_{\tau} v} \quad \text{[W-EQ]} \frac{\Gamma \vdash \phi}{\Gamma, t \equiv_{\tau} u \vdash \phi} \\
\text{[SUBST]} \frac{\Gamma \vdash \phi[t/x] \quad \Delta \vdash !_{\text{gr}(\phi, x)} t \equiv_{\tau} u}{\Gamma, \Delta \vdash \phi[u/x]}
\end{array}$$

Figure 3: Rules for graded equality.

Denote by V and Trm the sets of variables and terms inductively constructed from symbols in L in the usual way (using the standard arity obtained from the graded one by erasing resources). Resources in the graded arity determines a function $\text{gr} : Trm \times V \rightarrow |R|$ as follows:

- $\text{gr}(z, x) = 0$ if $x \neq z$
- $\text{gr}(x, x) = 1$
- $\text{gr}(f(t_1, \dots, t_n), x) = |f|_1 \cdot \text{gr}(t_1, x) + \dots + |f|_n \cdot \text{gr}(t_n, x)$

Intuitively, $\text{gr}(t, x)$ represents the cost of substituting the variable x inside the term t . Note that $\text{gr}(t, x)$ depends on the number of occurrences of x in t . For instance, if $f_0, f_1(x), \dots, f_n(x, x, \dots, x)$ are terms where $|f|_i = 1$ for all $i \in 0..n$, then $\text{gr}(f_i(x, \dots, x), x) = 1 + \dots + 1$ (i times). In particular, for constants, *i.e.* function symbols f with no arguments, we always have $\text{gr}(f, x) = 0$.

Denote by Wff the set of well-formed formulas constructed from symbols in L , the equality symbol \equiv and using \otimes , $\mathbf{1}$ and $!_r$ as connectives. We extend the function gr to a function $\text{gr} : Wff \times V \rightarrow |R|$ mapping a formula ϕ and a variable x to the amount of resources needed to substitute x in ϕ :

- $\text{gr}(p(t_1, \dots, t_n), x) = |p|_1 \cdot \text{gr}(t_1, x) + \dots + |p|_n \cdot \text{gr}(t_n, x)$,
- $\text{gr}(t \equiv_{\tau} u, x) = \text{gr}(t, x) + \text{gr}(u, x)$,
- $\text{gr}(\phi \otimes \psi, x) = \text{gr}(\phi, x) + \text{gr}(\psi, x)$,
- $\text{gr}(\mathbf{1}, x) = 0$,
- $\text{gr}(!_r \phi, x) = r \cdot \text{gr}(\phi, x)$.

Similarly to terms, the cost for substituting the variable x in a formula ϕ depends on the number of occurrences of x in ϕ . For instance, we have $\text{gr}(p(x) \otimes p(y), x) = |p|_1$, while $\text{gr}(p(x) \otimes p(x), x) = |p|_1 + |p|_1$. Again, for a predicate symbol p with no arguments, we have $\text{gr}(p, x) = 0$.

The calculus PLL_R extended by rules in Figure 3 will be called LPLL_R , where the first ‘L’ is for Lipschitz.

We have rules for reflexivity, symmetry and transitivity of \equiv_{σ} and a rule for weakening. Note the substitution rule: a substitution is derivable only if *enough equality resources* are available. These are determined by the function gr .

Remark 3.10. If $|f| = \langle |f|_1, \sigma_1 \rangle \dots \langle |f|_n, \sigma_n \rangle \rightarrow \tau$ is the arity of the function symbol f , then one can prove the entailment

$$!_{|f|_1}(x_1 \equiv_{\sigma_1} y_1), \dots, !_{|f|_n}(x_n \equiv_{\sigma_n} y_n) \vdash f(\vec{x}) \equiv_{\tau} f(\vec{y})$$

where $\vec{x} = x_1, \dots, x_n$ and $\vec{y} = y_1, \dots, y_n$. Interpreting the equality as a distance means that the application of the function f amplifies the distance between x_i and y_i by a factor $|f|_i$. Similarly, if p is a predicate symbol of arity $|p| = \langle |p|_1, \sigma_1 \rangle \dots \langle |p|_n, \sigma_n \rangle$, one can derive the

entailment

$$p(\vec{x}), !_{|p|_1}(x_1 \multimap_{\sigma_1} y_1), \dots, !_{|p|_n}(x_n \multimap_{\sigma_n} y_n) \vdash p(\vec{y})$$

meaning that one has to amplify equality between x_i and y_i by $|p|_i$ to derive $p(\vec{y})$ from $p(\vec{x})$.

Proposition 3.11. *Let \mathcal{T} be a theory in $LPLL_R$ over the R -graded signature L . The syntactic doctrine $Prp_{\mathcal{T}} : \mathcal{C}\mathcal{X}t_L^{\text{op}} \rightarrow \mathbf{Pos}$ is an R -Lipschitz doctrine with R -graded modality given by $!$ and a family of distances \mathbf{d}^{\equiv} inductively defined by*

$$\mathbf{d}_{\langle \rangle}^{\equiv} = \mathbf{1} \quad \mathbf{d}_{\langle \vec{\sigma}, x_{n+1} : \sigma_{n+1} \rangle}^{\equiv} = \mathbf{d}_{\vec{\sigma}}^{\equiv} \otimes (x_{n+1} \multimap_{\sigma_{n+1}} x'_{n+1})$$

Proof. To check that $Prp_{\mathcal{T}}$ is R -Lipschitz we first need to prove that it is R -graded where \otimes and $\mathbf{1}$ give the primary linear structure and $!$ the R -graded modality. The former is known (cf. Example 2.8(1)), while to prove the latter we need to show that rules in Figure 2 suffice to derive that $!$ satisfies axioms listed in Definition 3.1. Naturality of $!_r$ holds as it commutes with substitution, while its monotonicity easily follows by rules [DER] and [PRO]. Moreover, weakening, contraction, counit, comultiplication and controvariance follows immediately from rules [W], [C], [DER], [PRO] and [DECR]. To show lax-monoidality-2, we first derive $!_1\alpha, !_1\beta \vdash \alpha \otimes \beta$ from $\alpha, \beta \vdash \alpha \otimes \beta$, using rule [DER] twice. Then, by rule [PRO], we get $!_r\alpha, !_r\beta \vdash !_r(\alpha \otimes \beta)$. Finally, then introducing \otimes on the left, we get $!_r\alpha \otimes !_r\beta \vdash !_r(\alpha \otimes \beta)$, as needed. The proof of lax-monoidality-1 is similar: from $\vdash \mathbf{1}$ we derive $\vdash !_r\mathbf{1}$ using [PRO] and we conclude introducing $\mathbf{1}$ on the left.

To prove that $Prp_{\mathcal{T}}$ is R -Lipschitz, we have to show that rules in Figure 3 imply conditions in Definition 3.4. Rule [W-EQ] ensures that $\mathbf{d}_{\vec{\sigma}}^{\equiv}$ is affine for every context $\vec{\sigma}$, hence, by Proposition 3.7, it suffices to check only axioms (a) and (b) of Definition 3.4. Since products in $\mathcal{C}\mathcal{X}t_L$ are given by context concatenation, axiom (b) is straightforward, by definition of \mathbf{d}^{\equiv} . Finally, by Proposition 3.6, it remains to show that, for every context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$ and every formula α in $Prp_{\mathcal{T}}(\vec{\sigma})$, there is r in $|R|$ such that the entailment

$$\alpha(x_1, \dots, x_n) \otimes !_r \mathbf{d}_{\vec{\sigma}}^{\equiv} \vdash \alpha(y_1, \dots, y_n)$$

in the context $\langle x_1 : \sigma_1, \dots, x_n : \sigma_n, y_1 : \sigma_1, \dots, y_n : \sigma_n \rangle$ is provable in \mathcal{T} . To this end, first of all, note that, using rule [SUBST], for every Γ and $i \in 1..n$, from $\Gamma \vdash \alpha(y_1, \dots, y_{i-1}, x_i, \dots, x_n)$ and $!_{r_i}x_i \multimap_{\sigma_i} y_n \vdash !_{r_i}x_i \multimap_{\sigma_i} y_i$, we can derive $\Gamma, !_{r_i}x_i \multimap_{\sigma_i} y_i \vdash \alpha(y_1, \dots, y_i, x_{i+1}, \dots, x_n)$, where $r_i = \mathbf{gr}(\alpha(y_1, \dots, y_{i-1}, x_i, \dots, x_n), x_i)$. Therefore, starting from $\alpha(x_1, \dots, x_n) \vdash \alpha(x_1, \dots, x_n)$ and iteratively using this fact, we can prove

$$\alpha(x_1, \dots, x_n), !_{r_1}x_1 \multimap_{\sigma_1} y_1, \dots, !_{r_n}x_n \multimap_{\sigma_n} y_n \vdash \alpha(y_1, \dots, y_n)$$

Now, observe that, using rule [W-EQ], we have $\mathbf{d}_{\vec{\sigma}}^{\equiv} \vdash x_i \multimap_{\sigma_i} y_i$, for all $i \in 1..n$, and, by monotonicity of $!_s$, we get $!_s \mathbf{d}_{\vec{\sigma}}^{\equiv} \vdash !_s x_i \multimap_{\sigma_i} y_i$, for all $s \in |R|$ and $i \in 1..n$. Hence, by cut, we derive

$$\alpha(x_1, \dots, x_n), !_{r_1} \mathbf{d}_{\vec{\sigma}}^{\equiv}, \dots, !_{r_n} \mathbf{d}_{\vec{\sigma}}^{\equiv} \vdash \alpha(y_1, \dots, y_n)$$

Let us set $r = r_1 + \dots + r_n$. Then, by iteratively applying rule [C] we get $\alpha(x_1, \dots, x_n), !_r \mathbf{d}_{\vec{\sigma}}^{\equiv} \vdash \alpha(y_1, \dots, y_n)$, and so the thesis follows by introducing \otimes on the left. \square

3.3. Semantics in R -Lipschitz doctrines. The interpretation of a theory in a doctrine is standard [Pit00, Jac01]: it maps contexts and terms to objects and arrows of the base and formulas to elements of the fibres, respecting the entailments. The interpretation of $LPLL_R$ in an R -Lipschitz doctrine has to be defined with the additional requirement that gr agrees with the structure of the doctrine.

Let L be an R -graded signature. An R -graded interpretation of L into a R -Lipschitz doctrine $\langle P, \flat, \mathbf{d} \rangle$ assigns to every sort σ an object $\llbracket \sigma \rrbracket$ in the base of P , to every function symbol f of arity $\langle |f|_1, \sigma_1 \rangle \dots \langle |f|_n, \sigma_n \rangle \rightarrow \tau$ an arrow $\llbracket f \rrbracket : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow \llbracket \tau \rrbracket$ in the base of P such that

$$\prod_{i=1}^n \flat_{|f|_i} P_{\langle \pi_i, \pi_{n+i} \rangle}(\mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\llbracket f \rrbracket \times \llbracket f \rrbracket}(\mathbf{d}_{\llbracket \tau \rrbracket}) \quad (3.1)$$

and to every predicate symbol p of arity $\langle |p|_1, \sigma_1 \rangle \dots \langle |p|_n, \sigma_n \rangle$ an element $\llbracket p \rrbracket$ in $P(\llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket)$ such that

$$P_{\text{pr}_1}(\llbracket p \rrbracket) * \prod_{i=1}^n \flat_{|p|_i} P_{\langle \pi_i, \pi_{n+i} \rangle}(\mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\text{pr}_2}(\llbracket p \rrbracket) \quad (3.2)$$

where $\text{pr}_1 = \langle \pi_1, \dots, \pi_n \rangle$ and $\text{pr}_2 = \langle \pi_{n+1}, \dots, \pi_{2n} \rangle$.

Intuitively, we can pick as interpretation of a function symbol f an arrow $\llbracket f \rrbracket$ for which the cost of substituting the i -th argument is $|f|_i$, and similarly for predicate symbols. In other words, the graded arity of f determines valid Lipschitz constants for its interpretation $\llbracket f \rrbracket$.

Example 3.12. We can interpret an $\mathbb{R}_{\geq 0}$ -graded signature L in the Lipschitz doctrine $Lip : \text{Met}_L^{\text{op}} \rightarrow \text{Pos}$ presented in Example 3.5. The interpretation maps sorts to metric spaces, a function symbol f of arity $\langle |f|_1, \sigma_1 \rangle \dots \langle |f|_n, \sigma_n \rangle \rightarrow \tau$ to a Lipschitz function $\llbracket f \rrbracket : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow \llbracket \tau \rrbracket$ whose Lipschitz constant is $|f|_1 + \dots + |f|_n$ and a predicate symbol p of arity $\langle |p|_1, \sigma_1 \rangle \dots \langle |p|_n, \sigma_n \rangle$ to a Lipschitz function $\llbracket p \rrbracket : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow [0, \infty]$ whose Lipschitz constant is $|p|_1 + \dots + |p|_n$.

An R -graded interpretation of L induces an interpretation of contexts and terms in the base of P and an interpretation of formulas in the fibres of P . More precisely, a context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$ is interpreted by the product $\llbracket \vec{\sigma} \rrbracket_{\text{Trm}} = \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket$, a term t of sort τ in $\vec{\sigma}$ by an arrow $\llbracket t \rrbracket_{\text{Trm}} : \llbracket \vec{\sigma} \rrbracket_{\text{Trm}} \rightarrow \llbracket \tau \rrbracket$ and a formula ϕ in $\vec{\sigma}$ by an element $\llbracket \phi \rrbracket_{\text{Wff}} \in P(\llbracket \vec{\sigma} \rrbracket_{\text{Trm}})$, as follows:

$$\begin{aligned} \llbracket x_i \rrbracket_{\text{Trm}} &= \pi_i & \llbracket \mathbf{1} \rrbracket_{\text{Wff}} &= \kappa \\ \llbracket f(t_1, \dots, t_n) \rrbracket_{\text{Trm}} &= \llbracket f \rrbracket \circ \langle \llbracket t_1 \rrbracket_{\text{Trm}}, \dots, \llbracket t_n \rrbracket_{\text{Trm}} \rangle & \llbracket \phi \otimes \psi \rrbracket_{\text{Wff}} &= \llbracket \phi \rrbracket_{\text{Wff}} * \llbracket \psi \rrbracket_{\text{Wff}} \\ \llbracket p(t_1, \dots, t_n) \rrbracket_{\text{Wff}} &= P_{\langle \llbracket t_1 \rrbracket_{\text{Trm}}, \dots, \llbracket t_n \rrbracket_{\text{Trm}} \rangle}(\llbracket p \rrbracket) & \llbracket !_r \phi \rrbracket_{\text{Wff}} &= \flat_r \llbracket \phi \rrbracket_{\text{Wff}} \\ \llbracket t =_{\tau} u \rrbracket_{\text{Wff}} &= P_{\langle \llbracket t \rrbracket_{\text{Trm}}, \llbracket u \rrbracket_{\text{Trm}} \rangle}(\mathbf{d}_{\llbracket \tau \rrbracket}) \end{aligned}$$

An R -graded interpretation of a theory \mathcal{T} of $LPLL_R$ over L , is an R -graded interpretation $\llbracket - \rrbracket$ of L such that $\llbracket - \rrbracket_{\text{Wff}}$ respects entailments in \mathcal{T} .

We will often omit subscripts Trm and Wff from $\llbracket - \rrbracket$ when these are clear. If Γ is a list of formulas ϕ_1, \dots, ϕ_n then $\llbracket \Gamma \rrbracket$ abbreviates $\llbracket \phi_1 \rrbracket * \dots * \llbracket \phi_n \rrbracket$.

The following theorem states soundness for the semantics of $LPLL_R$.

Theorem 3.13. *Let $\llbracket - \rrbracket$ be an R -graded interpretation of a theory \mathcal{T} in $LPLL_R$ over L into an R -Lipschitz $\langle P, \flat, \mathbf{d} \rangle$. Then, $\langle \llbracket - \rrbracket_{\text{Trm}}, \llbracket - \rrbracket_{\text{Wff}} \rangle : \langle \text{Prp}_{\mathcal{T}}, !, \mathbf{d}^{\#} \rangle \rightarrow \langle P, \flat, \mathbf{d} \rangle$ is a 1-arrow in \mathbf{LLD}_R .*

The proof is carried out by the usual induction on rules [Pit00, Jac01], relying on the following lemma:

Lemma 3.14. *For every R -graded interpretation $\llbracket - \rrbracket$ of L into $\langle P, \flat, \mathbf{d} \rangle$ we have:*

(1) *for any term $t : \tau$ in the context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$*

$$\prod_{i=1}^n \flat_{\mathbf{gr}(t, x_i)} P_{\langle \pi_i, \pi_{n+i} \rangle}(\mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\llbracket t \rrbracket \times \llbracket \tau \rrbracket}(\mathbf{d}_{\llbracket \tau \rrbracket})$$

(2) *for any formula ϕ in the context $\vec{\sigma} = \langle x_1 : \sigma_1, \dots, x_n : \sigma_n \rangle$*

$$P_{\text{pr}_1}(\llbracket \phi \rrbracket) * \prod_{i=1}^n \flat_{\mathbf{gr}(\phi, x_i)} P_{\langle \pi_i, \pi_{n+i} \rangle}(\mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\text{pr}_2}(\llbracket \phi \rrbracket)$$

where $\text{pr}_1 = \langle \pi_1, \dots, \pi_n \rangle$ and $\text{pr}_2 = \langle \pi_{n+1}, \dots, \pi_{2n} \rangle$.

Proof. 1) is proved by induction on $t : \sigma$. If $t : \sigma$ is a variable in $\vec{\sigma}$, i.e. $t : \sigma$ is $x_p : \sigma_p$ for $1 \leq p \leq n$, then $\llbracket t \rrbracket$ is the projection $\pi_p : \llbracket \sigma_1 \rrbracket \times \dots \times \llbracket \sigma_n \rrbracket \rightarrow \llbracket \sigma_p \rrbracket$ and $\mathbf{gr}(t, x_j) = 1$ if $p = j$ and is 0 if $p \neq j$, so $\flat_{\mathbf{gr}(t, x_j)} \mathbf{d}_{\llbracket \sigma_j \rrbracket} \leq \kappa$ (for $j \neq p$ by weakening of \flat) and $\flat_{\mathbf{gr}(t, x_p)} \mathbf{d}_{\llbracket \sigma_p \rrbracket} \leq \mathbf{d}_{\llbracket \sigma_p \rrbracket}$ (by counit of \flat), so

$$\prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle}(\flat_{\mathbf{gr}(t, x_i)} \mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\langle \pi_p, \pi_{n+p} \rangle}(\flat_{\mathbf{gr}(t, x_p)} \mathbf{d}_{\llbracket \sigma_i \rrbracket})$$

whence the claim as $\langle \pi_p, \pi_{n+p} \rangle = \pi_p \times \pi_p = \llbracket t \rrbracket \times \llbracket t \rrbracket$.

Suppose $t : \tau$ is of the form $f(t_1, \dots, t_m)$ where f is a function symbol of arity $\tau_1, \dots, \tau_m \rightarrow \tau$, and each t_k is a term of type τ_k in the context $\sigma_1, \dots, \sigma_n$, satisfying the inductive hypothesis, i.e.

$$\prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle}(\flat_{\mathbf{gr}(t_k, x_i)} \mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\llbracket t_k \rrbracket \times \llbracket \tau_k \rrbracket}(\mathbf{d}_{\llbracket \tau_k \rrbracket})$$

Multiply both sides by $\flat_{|f|_k}$ and use comultiplication of \flat to get

$$\prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle}(\flat_{|f|_k} \cdot \mathbf{gr}(t_k, x_i) \mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\llbracket t_k \rrbracket \times \llbracket \tau_k \rrbracket}(\flat_{|f|_k} \mathbf{d}_{\llbracket \tau_k \rrbracket})$$

From $\llbracket t_k \rrbracket \times \llbracket \tau_k \rrbracket = \langle \pi_k, \pi_{m+k} \rangle \circ (\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket \times \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket)$ one has

$$P_{\llbracket t_k \rrbracket \times \llbracket \tau_k \rrbracket}(\flat_{|f|_k} \mathbf{d}_{\llbracket \tau_k \rrbracket}) = P_{\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket \times \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket} P_{\langle \pi_k, \pi_{m+k} \rangle}(\flat_{|f|_k} \mathbf{d}_{\llbracket \tau_k \rrbracket})$$

The way in which the interpretation of an R -graded signature is defined ensures the satisfaction of inequality (3.1) for f .

$$\prod_{k=1}^m P_{\langle \pi_k, \pi_{m+k} \rangle}(\flat_{|f|_k} \mathbf{d}_{\llbracket \tau_k \rrbracket}) \leq P_{\llbracket f \rrbracket \times \llbracket \tau \rrbracket}(\mathbf{d}_{\llbracket \tau \rrbracket})$$

Since $\llbracket t \rrbracket = \llbracket f(t_1, \dots, t_m) \rrbracket = \llbracket f \rrbracket \circ \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket$, evaluating both side of the inequality along $\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket \times \langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rrbracket$ one has

$$\prod_{k=1}^m P_{\llbracket t_k \rrbracket \times \llbracket \tau_k \rrbracket}(\flat_{|f|_k} \mathbf{d}_{\llbracket \tau_k \rrbracket}) \leq P_{\llbracket t \rrbracket \times \llbracket \tau \rrbracket}(\mathbf{d}_{\llbracket \tau \rrbracket})$$

whence

$$\prod_{k=1}^m \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{|f|_k \cdot \text{gr}(t_k, x_i)} d_{\llbracket \sigma_i \rrbracket}) \leq P_{\llbracket t \rrbracket \times \llbracket t \rrbracket} (d_{\llbracket \tau \rrbracket})$$

finally recall that $\text{gr}(t, x_i) = |f|_1 \cdot \text{gr}(t_1, x_i) + \dots + |f|_m \cdot \text{gr}(t_m, x_i)$. Contraction of b completes the proof.

2) If ϕ is $p(t_1, \dots, t_m)$ where p predicate symbol of L of arity τ_1, \dots, τ_m and t_i is a term of L of arity $\sigma_1, \dots, \sigma_n \rightarrow \tau_i$, then $\text{gr}(p, x_i) = |p|_1 \cdot \text{gr}(t_1, x_i) + \dots + |p|_n \cdot \text{gr}(t_n, x_i)$. Condition displayed in (3.2) together with contraction of b gives

$$P_{\text{pr}_1}(\llbracket p \rrbracket) * \prod_{i=1}^n b_{\text{gr}(\phi, x_i)} P_{\langle \pi_i, \pi_{n+i} \rangle} (d_{\llbracket \sigma_i \rrbracket}) \leq P_{\text{pr}_2}(\llbracket p \rrbracket)$$

where $\text{pr}_1 = \langle \pi_1, \dots, \pi_m \rangle$ and $\text{pr}_2 = \langle \pi_{m+1}, \dots, \pi_{2m} \rangle$.

The reindexing of both sides along $\langle \llbracket t_1 \rrbracket, \dots, \llbracket t_m \rrbracket \rangle$ yields that the right hand side, that becomes $P_{\langle \pi_{n+1}, \dots, \pi_{2n} \rangle}(\llbracket \phi \rrbracket)$, is greater or equal to the left hand side, that becomes

$$P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \phi \rrbracket) * \prod_{k=1}^m P_{\langle \pi_k, \pi_{n+k} \rangle} (b_{\text{gr}(p, x_k)} P_{\llbracket t_k \rrbracket \times \llbracket t_k \rrbracket} d_{\llbracket \tau_k \rrbracket})$$

By point 1) of this proposition, the displayed formula is greater than or equal to

$$P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \phi \rrbracket) * \prod_{k=1}^m \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(p, x_k) \cdot \text{gr}(t_k, x_i)} d_{\llbracket \sigma_i \rrbracket})$$

recall that $\text{gr}(\phi, x_i) = |p|_1 \cdot \text{gr}(t_1, x_i) + \dots + |p|_m \cdot \text{gr}(t_m, x_i)$, so contraction of b leads to the claim.

If ϕ is $t =_{\tau} u$ where tu are terms of arity $\sigma_1, \dots, \sigma_n \rightarrow \tau$, then $\llbracket \phi \rrbracket = P_{\llbracket t \rrbracket, \llbracket u \rrbracket} (d_{\llbracket \tau \rrbracket})$ and $\text{gr}(\phi, x_i) = \text{gr}(t, x_i) + \text{gr}(u, x_i)$, so by of b and conditions on t and u it holds that

$$\begin{aligned} & P_{\langle \pi_1, \dots, \pi_n \rangle} P_{\llbracket t \rrbracket, \llbracket u \rrbracket} (d_{\llbracket \tau \rrbracket}) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(\phi, x_i)} d_{\llbracket \sigma_i \rrbracket}) \\ & \leq P_{\langle \pi_1, \dots, \pi_n \rangle} P_{\llbracket t \rrbracket, \llbracket u \rrbracket} (d_{\llbracket \tau \rrbracket}) * P_{\llbracket t \rrbracket \times \llbracket t \rrbracket} (d_{\llbracket \tau \rrbracket}) * P_{\llbracket u \rrbracket \times \llbracket u \rrbracket} (d_{\llbracket \tau \rrbracket}) \\ & \leq P_{\langle \pi_{n+1}, \dots, \pi_{2n} \rangle} P_{\llbracket t \rrbracket, \llbracket u \rrbracket} (d_{\llbracket \tau \rrbracket}) \end{aligned}$$

by transitivity of $d_{\llbracket \tau \rrbracket}$.

If ϕ is $\psi \otimes \theta$ where both ψ and θ are well formed formulas in the context $\vec{\sigma}$ satisfying the inductive hypothesis, then $\llbracket \phi \rrbracket = \llbracket \psi \rrbracket * \llbracket \theta \rrbracket$ and $\text{gr}(\phi, x_i) = \text{gr}(\psi, x_i) + \text{gr}(\theta, x_i)$. By contraction of b , it follows

$$\begin{aligned} & P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \psi \rrbracket * \llbracket \theta \rrbracket) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(\psi, x_i) + \text{gr}(\theta, x_i)} d_{\llbracket \sigma_i \rrbracket}) \\ & \leq P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \psi \rrbracket * \llbracket \theta \rrbracket) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(\psi, x_i)} d_{\llbracket \sigma_i \rrbracket} * b_{\text{gr}(\theta, x_i)} d_{\llbracket \sigma_i \rrbracket}) \\ & = P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \psi \rrbracket) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(\psi, x_i)} d_{\llbracket \sigma_i \rrbracket}) * P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \theta \rrbracket) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle} (b_{\text{gr}(\theta, x_i)} d_{\llbracket \sigma_i \rrbracket}) \end{aligned}$$

By hypothesis on ψ the first formulas in the square brackets is less than or equal to $P_{\langle \pi_{n+1}, \dots, \pi_{2n} \rangle}(\llbracket \psi \rrbracket)$ and the same for θ , whence the claim.

If ϕ is $\mathbf{1}$ then $\text{gr}(\phi, x_i) = 0$. Then $\llbracket \mathbf{1} \rrbracket = \kappa$ and the claim is trivial by weakening to \flat .

If ϕ is $!_r \psi$ for some well formed formula ψ in the context $\vec{\sigma}$, then $\llbracket \phi \rrbracket = \flat_r \llbracket \psi \rrbracket$ and $\text{gr}(\phi, x_i) = r \cdot \text{gr}(\psi, x_i)$. By the hypothesis on ψ it is

$$P_{\langle \pi_1, \dots, \pi_n \rangle}(\llbracket \psi \rrbracket) * \prod_{i=1}^n P_{\langle \pi_i, \pi_{n+i} \rangle}(\flat_{\text{gr}(\psi, x_i)} \mathbf{d}_{\llbracket \sigma_i \rrbracket}) \leq P_{\langle \pi_{n+1}, \dots, \pi_{2n} \rangle}(\llbracket \psi \rrbracket)$$

It suffices to multiply both sides by \flat_r and use comultiplication of \flat to prove the claim. \square

We can now sketch the proof of Theorem 3.13.

Proof of Theorem 3.13. The only non-trivial part of the theorem is to prove that every component of $\llbracket - \rrbracket_{Wff}$ is monotone. As we said before, this is done by induction on rules of $LPLL_R$. We check only cases for [PRO] and [SUBST], leaving the other to the reader.

[pro]: Suppose $\llbracket !_{s_1} \psi_1 \rrbracket * \dots * \llbracket !_{s_n} \psi_n \rrbracket \leq \llbracket \phi \rrbracket$. By definition of $\llbracket - \rrbracket_{Wff}$, the left hand side of the inequality is equal to $\flat_{s_1} \llbracket \psi_1 \rrbracket * \dots * \flat_{s_n} \llbracket \psi_n \rrbracket$. Then, by monotonicity of \flat_r , we get $\flat_r(\flat_{s_1} \llbracket \psi_1 \rrbracket * \dots * \flat_{s_n} \llbracket \psi_n \rrbracket) \leq \flat_r \llbracket \phi \rrbracket$. By lax-monoidality-2 and comultiplication we get $\flat_{r \cdot s_1} \llbracket \psi_1 \rrbracket * \dots * \flat_{r \cdot s_n} \llbracket \psi_n \rrbracket \leq \flat_r \llbracket \phi \rrbracket$. Finally, again by definition of $\llbracket - \rrbracket_{Wff}$, we get the thesis.

[subst]: Consider a context $\vec{\sigma}$, a sort τ , terms t, u of sort τ in the context $\vec{\sigma}$ and a formula ϕ in the context $\vec{\sigma}, x : \tau$. Suppose that $\llbracket \Gamma \rrbracket \leq \llbracket \phi[t/x] \rrbracket$ and $\llbracket \Delta \rrbracket \leq \llbracket !_{\text{gr}(\phi, x)} t \multimap_{\tau} u \rrbracket$. By definition we have $\llbracket \phi[t/x] \rrbracket = P_{\langle \text{id}_{\llbracket \vec{\sigma} \rrbracket}, \llbracket t \rrbracket \rangle}(\llbracket \phi \rrbracket)$ and $\llbracket !_{\text{gr}(\phi, x)} t \multimap_{\tau} u \rrbracket = P_{\langle \llbracket t \rrbracket, \llbracket u \rrbracket \rangle}(\flat_{\text{gr}(\phi, x)} \mathbf{d}_{\llbracket \tau \rrbracket})$. Combining these inequalities, we get

$$\llbracket \Gamma, \Delta \rrbracket = \llbracket \Gamma \rrbracket * \llbracket \Delta \rrbracket \leq P_{\langle \text{id}_{\llbracket \vec{\sigma} \rrbracket}, \llbracket t \rrbracket \rangle}(\llbracket \phi \rrbracket) * P_{\langle \llbracket t \rrbracket, \llbracket u \rrbracket \rangle}(\flat_{\llbracket \tau \rrbracket})$$

Using Item (2) of Lemma 3.14 and reflexivity of equality predicates, the following inequality holds in the poset $P(\llbracket \vec{\sigma} \rrbracket \times \llbracket \tau \rrbracket \times \llbracket \tau \rrbracket)$: $P_{\langle \pi_1, \pi_2 \rangle}(\llbracket \phi \rrbracket) * \flat_{\text{gr}(\phi, x)} P_{\langle \pi_2, \pi_3 \rangle}(\mathbf{d}_{\llbracket \tau \rrbracket}) \leq P_{\langle \pi_1, \pi_3 \rangle}(\llbracket \phi \rrbracket)$, and, reindexing along $\langle \text{id}_{\llbracket \vec{\sigma} \rrbracket}, \llbracket t \rrbracket, \llbracket u \rrbracket \rangle : \llbracket \vec{\sigma} \rrbracket \rightarrow \llbracket \vec{\sigma} \rrbracket \times \llbracket \tau \rrbracket \times \llbracket \tau \rrbracket$, we obtain

$$P_{\langle \text{id}_{\llbracket \vec{\sigma} \rrbracket}, \llbracket t \rrbracket \rangle}(\llbracket \phi \rrbracket) * \flat_{\text{gr}(\phi, x)} P_{\langle \llbracket t \rrbracket, \llbracket u \rrbracket \rangle}(\mathbf{d}_{\llbracket \tau \rrbracket}) \leq P_{\langle \text{id}_{\llbracket \vec{\sigma} \rrbracket}, \llbracket u \rrbracket \rangle}(\llbracket \phi \rrbracket)$$

Putting together these inequalities and by definition of $\llbracket - \rrbracket_{Wff}$ we get the thesis. \square

Remark 3.15. Recall from Example 3.12 that an $\mathbb{R}_{\geq 0}$ -graded signature can be interpreted into the Lipschitz doctrine $Lip : \mathcal{Met}_L^{\text{op}} \rightarrow \mathcal{Pos}$. The equality predicate \multimap_{τ} is interpreted as the metric $\mathbf{d}_{\llbracket \tau \rrbracket}$ of the space $\llbracket \tau \rrbracket$. This implies that replicability of equality is not derivable in $LPLL_R$, as it would imply $\mathbf{d}_{\llbracket \tau \rrbracket}(x, y) \geq \mathbf{d}_{\llbracket \tau \rrbracket}(x, y) + \mathbf{d}_{\llbracket \tau \rrbracket}(x, y)$ which needs not to be true for a generic metric space $\llbracket \tau \rrbracket$.

The following proposition states completeness of the semantics of $LPLL_R$ in the sense of [Pit00].

Proposition 3.16. *Let \mathcal{T} be a theory in $LPLL_R$ over a signature L and ϕ, ψ formulas over L . If, for all R -Lipschitz doctrines $\langle P, \flat, \mathbf{d} \rangle$ and for all interpretations $\llbracket - \rrbracket$ of \mathcal{T} in $\langle P, \flat, \mathbf{d} \rangle$, we have $\llbracket \phi \rrbracket_{Wff} \leq \llbracket \psi \rrbracket_{Wff}$, then $\phi \vdash \psi$ is provable in \mathcal{T} .*

The proof of this result is straightforward, as we can take the trivial interpretation in the syntactic doctrine $Prp_{\mathcal{T}}$ as in Proposition 3.11, which is R -Lipschitz

Remark 3.17. One could prove a stronger completeness result, showing an *internal language theorem*. This consists in proving the equivalence between the 2-category of R -Lipschitz doctrines and a suitable 2-category of $LPLL_R$ theories. As it is well-known, the key lemma

of this kind of results shows that from any doctrine one can extract a theory, where, loosely speaking, sorts and function symbols are objects and arrows in the base, while predicate symbols and axioms are elements and entailments in the fibres. One could do this for R -Lipschitz doctrines, where the difficult part is to determine grades in the R -graded arities. As noticed in Remark 3.10, graded arities provide a choice of Lipschitz constants for the corresponding symbols, but from Proposition 3.8 we only know that in a R -Lipschitz doctrine such Lipschitz constants exist. Hence, in order to obtain graded arities, we need the axiom of choice. A rigorous proof is left for future work.

4. EXAMPLES FROM QUANTITATIVE EQUATIONAL THEORIES AND BEYOND

Mardare et al. [MPP16, MPP17] introduced the notion of *quantitative equational theory* (QET) as a formal tool to describe and reason about quantitative algebras, that is, algebras whose carrier is a metric space and whose operations are non-expansive maps. In this section, we show how the calculus $LPLL_R$ and its semantics can be used to reason about quantitative algebras, comparing it with QETs and obtaining at the same time a range of examples of $LPLL_R$ theories. In the literature [MPP16, MPP17, MPP21, Adá22, BMPP18], QETs are studied in great detail using the language of monads and proving many completeness results. Here, we just focus on examples, illustrating how $LPLL_R$ works, leaving for future work a formal and systematic comparison between $LPLL_R$ and QETs.

We first recall the key features of a QET. Syntactically, terms are built out from a single-sorted signature Ω of possibly infinitary function symbols, we write $f : I \in \Omega$, where I is a set, when f is a function symbol in Ω whose arity is I . In order to deal with distances, the key idea is to explicitly handle quantities, by working with labelled equations $t =_\epsilon u$, where ϵ is a non-negative rational number. Such an equation states that the distance between t and u is bounded by ϵ . A sequent has the shape $\Gamma \vdash t =_\epsilon u$, where Γ is a possibly infinite set of equations, and the calculus is defined by the following rules, where $f : I \in \Omega$ and σ is a substitution.

$$\begin{array}{ll}
\text{(Ref)} \quad \vdash t =_0 t & \text{(NExp)} \quad \{t_i =_\epsilon u_i \mid i \in I\} \vdash f((t_i)_{i \in I}) =_\epsilon f((u_i)_{i \in I}) \\
\text{(Sym)} \quad t =_\epsilon u \vdash u =_\epsilon t & \text{(Subst)} \quad \Gamma \vdash t =_\epsilon u \text{ implies } \sigma(\Gamma) \vdash \sigma(t) =_\epsilon \sigma(u) \\
\text{(Triang)} \quad \{t =_\epsilon u, u =_{\epsilon'} v\} \vdash t =_{\epsilon+\epsilon'} v & \text{(Assum)} \quad \text{if } t =_\epsilon u \in \Gamma, \text{ then } \Gamma \vdash t =_\epsilon u \\
\text{(Arch)} \quad \{t =_{\epsilon'} u \mid \epsilon' > \epsilon\} \vdash t =_\epsilon u & \text{(Cut)} \quad \text{if } \Gamma \vdash \psi \text{ for all } \psi \in \Theta \text{ and } \Theta \vdash t =_\epsilon u, \\
& \text{then } \Gamma \vdash t =_\epsilon u
\end{array}$$

A first crucial difference between a QET and $LPLL_R$ is that the underlying logic of the former is not linear, as contraction and weakening are derivable: the hypotheses Γ form a set, the rule **(Cut)** is in additive form and the rule **(Assum)** disregards some hypotheses. As a consequence, each formula in a QET has to be interpreted in a standard non-quantitative way, *e.g.* as a subset, while in $LPLL_R$ every formula admits a direct quantitative interpretation *e.g.*, as $[0, \infty]$ -valued function. Therefore, to reason quantitatively, in a QET one has to deal with an infinite family of predicates (*i.e.* labelled equations) representing all possible approximations of a metric, which requires an explicit handling of quantities. On the other hand, in $LPLL_R$ one has a single equality predicate which is directly interpreted as a $[0, \infty]$ -valued metric. Working with explicit quantities makes it necessary to add specific rules to manage them (such as **(Arch)**) and to allow sequents with infinitely many hypotheses. Note also that infinitary sequents are needed to deal with infinitary function symbols, which

QET	LPLL _R
(S0) $\vdash x + 0 =_0 x$	$\vdash x + 0 = x$
(S1) $\vdash x + x =_0 x$	$\vdash x + x = x$
(S2) $\vdash x + y =_0 y + x$	$\vdash x + y = y + x$
(S3) $\vdash (x + y) + z =_0 x + (y + z)$	$\vdash (x + y) + z = x + (y + z)$
(S4) $\{x =_\epsilon x', y =_{\epsilon'} y'\} \vdash x + y =_{\epsilon \vee \epsilon'} x' + y'$	

Figure 4: Axioms for quantitative semigroups as a QET and as an LPLL_R theory.

are allowed in QETs, while LPLL_R only deals with finitary ones to keep sequents finitary as well.

Another important difference is that in QETs function symbols are forced to be non-expansive by the rule **(NExp)**, while in LPLL_R they can be arbitrary Lipschitz functions. Since non-expansive maps are Lipschitz with constant equal to 1, LPLL_R can treat them, but, as we will see, it can go beyond this limitation, dealing with quantitative algebras where operations are arbitrary Lipschitz functions.

Remaining rules are instead similar: rules **(Ref)**, **(Sym)** and **(Triang)** above formalise the axioms of a distance and correspond to rules [R], [S] and [T] of LPLL_R, while rules **(Subst)** and **(Cut)** correspond to monotonicity of substitution and the cut rule of LPLL_R.

We now focus on examples. Examples in [MPP16] can be rephrased as LPLL_R theories where R is the semiring $\mathbb{R}_{\geq 0}$ of non-negative real numbers, in such a way that their intended semantics induces a sound interpretation of the corresponding LPLL_R theory in the doctrine of metric spaces and Lipschitz maps (cf. Example 3.5). We will shorten notation for functions symbols in a single-sorted graded signature: if f is a function symbol with $|f| = \langle r_1, \star \rangle \dots \langle r_n, \star \rangle \rightarrow \star$, we write $f : \langle r_1, \dots, r_n \rangle$.

We start by considering the theory **QS**₀ of quantitative semilattices with zero. To write **QS**₀ as a QET, one takes a signature whose function symbols are $+ : 2$ and $0 : 0$, while, to write **QS**₀ as an LPLL_R theory, one takes a $\mathbb{R}_{\geq 0}$ -graded signature with one sort, no predicate symbols and whose function symbols are $+ : \langle 1, 1 \rangle$ and $0 : \langle \rangle$, meaning that $+$ is non-expansive in both arguments. In Figure 4, we list on the left the axioms of **QS**₀ given in [MPP16] and on the right the corresponding ones in LPLL_R. The axiom **(S0)** says that the distance between $x + 0$ and x is less than 0. This information in LPLL_R is given forcing $\mathbf{1} \vdash x + 0 = x$, thanks to the fact that $\mathbf{1}$ is interpreted as the function constantly equal to 0. Axioms **(S1)**, **(S2)** and **(S3)** are translated similarly. Axiom **(S4)** in the QET refines the non-expansiveness of $+$ and it is needed to properly handle labels of equations. Since in LPLL_R there are no labels, this axiom is not needed, as non-expansiveness of $+$ follows from rules of LPLL_R and the graded arity in the considered signature. Note that the axioms in LPLL_R are the usual ones of the theory of semilattices: $+$ is associative, commutative and idempotent and 0 is the neutral element.

A semantics for **QS**₀ as QET is given in [MPP16] as follows. Let $\langle M, d \rangle$ be a metric space and consider the metric space $\langle C_d, H_d \rangle$ where C_d the set of compact subsets of M with respect to d and $H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$, with $d(m, X) = \inf_{x \in X} d(m, x)$ for $m \in M$ and $X \subseteq M$, is the Hausdorff distance. Then, 0 is interpreted as the empty set and $+$ as the function of type $\langle C_d, H_d \rangle \times \langle C_d, H_d \rangle \rightarrow \langle C_d, H_d \rangle$ mapping two compact sets to their union. This assignment gives rise to a semantics also of **QS**₀ seen as an LPLL_R theory in the $\mathbb{R}_{\geq 0}$ -Lipschitz doctrine $Lip : \mathcal{Met}_L^{\text{op}} \rightarrow \mathcal{Pos}$ (cf. Example 3.5). We interpret

QET	LPLL _R
(B1) $\vdash x +_1 y =_0 x$	$\vdash x +_1 y = x$
(B2) $\vdash x +_e x =_0 x$	$\vdash x +_e x = x$
(SC) $\vdash x +_e y =_0 y +_{1-e} x$	$\vdash x +_e y = y +_{1-e} x$
(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)$	$\vdash (x +_e y) +_{e'} z = x +_{ee'} (y +_{\frac{e'-ee'}{1-ee'}} z)$
(IB _p) $\{x =_{\epsilon_1} y, x' =_{\epsilon_2} y'\} \vdash x +_e y =_{\delta} x' +_e y'$ $(\delta \geq (e\epsilon_1^p + (1-e)\epsilon_2^p)^{1/p})$	

Figure 5: Axioms for interpolative barycentric algebras as a QET and as an LPLL_R theory.

a context with n variables as the product of n copies of $\langle C_d, H_d \rangle$ and a term with n free variables as a Lipschitz function from $\langle C_d, H_d \rangle^n$ to $\langle C_d, H_d \rangle$.

Other examples discussed in [MPP16] deal with metrics over probability distributions. The considered theories are defined over a signature whose function symbols are $+_e : 2$ for all $e \in [0, 1]$, which can be thought of as probabilistic choice operators with probability e , that is, intuitively, $x +_e y$ is x with probability e and y with probability $1 - e$. The QET **IBA_p** of p -Interpolative Barycentric Algebras, with $p \geq 1$, can be rephrased in LPLL_R using a single-sorted $\mathbb{R}_{\geq 0}$ -graded signature with no predicate symbols and whose function symbols are $+_e : \langle e^{1/p}, (1-e)^{1/p} \rangle$ for all $e \in [0, 1]$, meaning that $+_e$ scales the distance of the first argument by a factor $e^{1/p}$ and that of the second argument by a factor $(1-e)^{1/p}$. We report the axioms in Figure 5. The axioms are translated as for the theory **QS₀**. The last one has no counterpart in LPLL_R as it essentially states that $+_e$ is Lipschitz with coefficients $e^{1/p}$ and $(1-e)^{1/p}$,⁴ and this in LPLL_R follows from the fact that $+_e$ has gated arity $\langle e^{1/p}, (1-e)^{1/p} \rangle$ and so we can derive the entailment $!_{e^{1/p}} x = y, !_{(1-e)^{1/p}} x' = y' \vdash x +_e y = x' +_e y'$.

Mardare et al. [MPP16] define a semantics of this theory taking as domain of the interpretation the set $\Delta[M]$ of Borel probability measures over a metric space $\langle M, d \rangle$ with the p -Wasserstein distance W_d^p defined by

$$W_d^p(\mu, \nu) = \sup \left\{ \left| \int f^p d\mu - \int f^p d\nu \right|^{1/p} \mid f : M \rightarrow [0, \infty) \text{ is non-expansive} \right\}$$

and interpreting $+_e$ as the convex combination of two distributions, that is, the function mapping a pair of distributions $\langle \mu, \nu \rangle$ to $e\mu + (1-e)\nu$. This choice, induces an interpretation of the corresponding LPLL_R theory in $Lip : \mathcal{Met}_L^{\text{op}} \rightarrow \mathcal{Pos}$, as it is easy to see that the convex combination operation has Lipschitz constants $e^{1/p}$ and $(1-e)^{1/p}$ with respect to W_d^p .

A specialised version of the theory **IBA_p** is the theory **BA** of Left-Invariant Barycentric Algebras, which is obtained by replacing the axiom **(IB_p)** by the axiom **(LI)** stating $\vdash x +_e z =_{\epsilon} y +_e z$ where $\epsilon \geq e$. This axiom, together with the others, basically requires that Lipschitz constants of $+_e$ are e and $1 - e$, but it also forces the distance to be smaller than 1. To rephrase this theory in LPLL_R, we need to consider a single-sorted $\mathbb{R}_{\geq 0}$ -graded signature where function symbols are $+_e : \langle e, 1 - e \rangle$, for all $e \in [0, 1]$, and having a constant predicate symbol $\mathbf{u} : \langle \rangle$ modelling the real number 1. This matches the intuition that in our setting

⁴To check this we rely on the inequality $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$, which holds for non-negative real numbers a, b .

real numbers are “truth values”, hence they can be represented syntactically by constant predicate symbols. Finally, we have to add to the theory above the axiom $u \vdash x = y$, stating that the distance is smaller than 1. Note that, using the rules of LPLL_R , we can prove $!_\epsilon u \vdash x = x +_\epsilon z = y +_\epsilon z$, for all $\epsilon \geq e$, which corresponds exactly to the axiom **(LI)**.

Following again [MPP16], a semantics of the theory **BA** in the Lipschitz doctrine $D_{\langle \mathcal{P}_{[0,\infty]}, \mathbf{m} \rangle}$ can be constructed by taking as interpretation of the sort the set $\Delta[S, \Sigma]$ of probability measures over a measurable space $\langle S, \Sigma \rangle$, with the total variation distance $T(\mu, \nu) = \sup_{E \in \Sigma} |\mu(E) - \nu(E)|$, and interpreting $+_\epsilon$ again as the convex combination of distributions.

4.1. Beyond QETs. As already noticed, in QETs function symbols must be interpreted by non-expansive maps due to the rule **(NExp)**. The calculus LPLL_R , instead, can express theories where function symbols have to be interpreted by arbitrary Lipschitz maps. As examples of such situation, we briefly describe the theories **R-MVS**, of metric vector spaces over the field \mathbb{R} of real numbers, and **R-GC**, of R -graded combinators over a given resource semiring R , which, therefore, cannot be expressed as QETs.

The single-sorted $\mathbb{R}_{\geq 0}$ -graded signature of **R-MVS** has no predicate symbols and consists of function symbols $+ : \langle 1, 1 \rangle$, $- : \langle 1 \rangle$ and $0 : \langle \rangle$ for the group structure and $\lambda_a : \langle |a| \rangle$ for scalar multiplication by a , for all $a \in \mathbb{R}$. Note that the scalar multiplication λ_a is required to scale the distance by a factor $|a|$, hence in general it is not non-expansive; indeed, the entailment $!_{|a|} x = y \vdash \lambda_a(x) = \lambda_a(y)$ is provable. Below we report the axioms:

$$\begin{array}{ll}
 \text{(G1)} \vdash (x + y) + z = x + (y + z) & \text{(L1)} \vdash \lambda_a(x + y) = \lambda_a(x) + \lambda_a(y) \\
 \text{(G2)} \vdash x + y = y + x & \text{(L2)} \vdash \lambda_{a+b}(x) = \lambda_a(x) + \lambda_b(x) \\
 \text{(G3)} \vdash x + 0 = x & \text{(L3)} \vdash \lambda_{ab}(x) = \lambda_a(\lambda_b(x)) \\
 \text{(G4)} \vdash x + (-x) = 0 & \text{(L4)} \vdash \lambda_1(x) = x
 \end{array}$$

These axioms are actually the standard ones for vector spaces, the only difference is that the underlying calculus is substructural, hence equality can be interpreted as a distance. Any real vector space V with a norm $\|-\|$ gives a semantics for this theory: the distance $d(u, v) = \|u - v\|$ makes the group operations non-expansive and the scalar multiplication by a a Lipschitz function with constant $|a|$ as $\|au - av\| = |a| \cdot \|u - v\|$.

Another example where function symbols are not necessarily non-expansive is the theory **R-GC** of R -graded combinators an R -Linear Combinatory Algebra [Atk18] (see also Example 3.2(4)) over a given resource semiring R . The single-sorted R -graded signature of **R-GC** has no predicate symbols, function symbols $\cdot : \langle 1, 1 \rangle$ and $\mathbf{b}_r : \langle r \rangle$,⁵ for all $r \in |R|$, constants $B, C, I, K, W_{r,s}, D, d_{r,s}, F_r$, for all $r, s \in |R|$, and $O_{r,s}$, for all $r, s \in |R|$ with $r \preceq s$. The symbol \mathbf{b}_r has grade r as, intuitively, the expression $\mathbf{b}_r t$ provides “ r copies of t ”, hence, the cost of a substitution inside \mathbf{b}_r should be amplified by r . Indeed, as a consequence of the rules of LPLL_R the entailment $!_r t = u \vdash \mathbf{b}_r t = \mathbf{b}_r u$ is derivable. The axioms of the theory

⁵We use \mathbf{b}_r instead of the more standard $!_r$ to avoid confusion with the modality of the logical calculus.

are the following:

$$\begin{array}{ll}
(\mathbf{GC1}) \vdash ((B \cdot x) \cdot y) \cdot z = x \cdot (y \cdot z) & (\mathbf{GC2}) \vdash ((C \cdot x) \cdot y) \cdot z = (x \cdot z) \cdot y \\
(\mathbf{GC3}) \vdash I \cdot x = x & (\mathbf{GC4}) \vdash (K \cdot x) \cdot \mathbf{b}_0 y = x \\
(\mathbf{GC5}) \vdash (W_{r,s} \cdot x) \cdot \mathbf{b}_{r+s} = (x \cdot \mathbf{b}_r y) \cdot \mathbf{b}_s y & (\mathbf{GC6}) \vdash D \cdot \mathbf{b}_1 x = x \\
(\mathbf{GC7}) \vdash d_{r,s} \cdot \mathbf{b}_{r,s} x = \mathbf{b}_r(\mathbf{b}_s x) & (\mathbf{GC8}) \vdash (F_r \cdot \mathbf{b}_r x) \cdot \mathbf{b}_r y = \mathbf{b}_r(x \cdot y) \\
(\mathbf{GC9}) \vdash O_{r,s} \cdot \mathbf{b}_s x = \mathbf{b}_r x
\end{array}$$

When R is the semiring $\mathbb{R}_{\geq 0}$ of non-negative real numbers, models of $\mathbb{R}_{\geq 0}$ -**GC** in *Lip* provide a metric variant of (total) $\mathbb{R}_{\geq 0}$ -Linear Combinatory Algebras [Atk18] (see also Example 3.2(4)). Hence, equality is interpreted as a metric on combinators, which could be regarded as a program distance, and, for every $r \in [0, \infty)$, \mathbf{b}_r is interpreted as a Lipschitz map with constant r .

5. THE 2-CATEGORY OF LIPSCHITZ DOCTRINES

As we have seen in the previous section, R -Lipschitz doctrines are the objects of the 2-category \mathbf{LLD}_R . In this section we study its properties, relating it with with other 2-categories of doctrines. More in detail, we first show that R -Lipschitz doctrines arise as coalgebras for a 2-comonad on R -graded ones. This result smoothly generalises what happens in the non-linear setting and provides us with a universal construction of R -Lipschitz doctrines from R -graded ones. Then, we study how our quantitative equality relates to the standard one given by left adjoints, introducing elementary R -graded doctrines and comparing them with R -Lipschitz ones.

5.1. R -Lipschitz doctrines as coalgebras. A key property of standard equality in the non-linear setting, recognised by [Pas15, EPR20], is that it arises as a coalgebra structure over a primary doctrine, that is, a doctrine modelling conjunctions. More precisely, consider Diagram 2.1 from Section 2

$$\mathbf{ED} \xleftarrow{\top} \mathbf{PD}$$

It shows an adjoint situation between \mathbf{PD} and \mathbf{ED} , *i.e.* the 2-categories of primary doctrines and that of elementary ones that is, primary doctrines with equality. That adjoint situation is comonadic. This fact not only reveals the coalgebraic nature of equality, but provides a universal construction yielding elementary doctrines from primary ones.

The goal of this subsection is to generalise this fact to our linear setting. That is, we present a universal construction producing a Lipschitz doctrine out of any R -graded one. To this end, we define a 2-functor $R_{\text{Lip}} : \mathbf{LD}_R \rightarrow \mathbf{LLD}_R$ that is the right 2-adjoint of the obvious forgetful 2-functor $U_{\text{Lip}} : \mathbf{LLD}_R \rightarrow \mathbf{LD}_R$ and prove that R -Lipschitz doctrines are the coalgebras of the 2-comonad induced by such a 2-adjunction. On one hand, this result provides us with a universal construction of R -Lipschitz doctrines, which gives us a tool to produce semantics for the calculus LPLL_R . On the other hand, it shows that our quantitative notion of equality, although not falling within Lawvere's definition by left adjoints (Proposition 2.14), generalises it and preserves its coalgebraic nature.

To construct R_{Lip} , for any R -graded doctrine $\langle P, \flat \rangle$, we build an R -Lipschitz doctrine $D_{\langle P, \flat \rangle} : \mathcal{L}_{\langle P, \flat \rangle}^{\text{OP}} \rightarrow \mathcal{Pos}$, called *Lipschitz completion* of $\langle P, \flat \rangle$. Let \mathcal{C} be the base of P . The category $\mathcal{L}_{\langle P, \flat \rangle}$ is defined as follows:

- objects are pairs $\langle A, \rho \rangle$ where A is an object in \mathcal{C} and ρ is an affine P -distance on A ;
- an arrow from $\langle A, \rho \rangle$ to $\langle B, \sigma \rangle$ is an arrow $f : A \rightarrow B$ in \mathcal{C} such that $b_r \rho \leq P_{f \times f}(\sigma)$, for some $r \in |R|$;
- composition and identities are those of \mathcal{C} .

Given $f : \langle A, \rho \rangle \rightarrow \langle B, \sigma \rangle$ we say that r tracks f if r is such that $b_r \rho \leq P_{f \times f}(\sigma)$. So $f : A \rightarrow B$ underlies an arrow $f : \langle A, \rho \rangle \rightarrow \langle B, \sigma \rangle$ if there is r that tracks f .

Intuitively, objects of $\mathcal{L}_{\langle P, b \rangle}$ can be regarded as metric spaces inside P and its arrows as abstract Lipschitz functions between such spaces. It is easy to see that $\mathcal{L}_{\langle P, b \rangle}$ is a category: identities are well-defined as, for every $\langle A, \rho \rangle$, the identity id_A is tracked by 1 thanks to the counit axiom of b ; also composition is well defined as, if r tracks f and s tracks g , then $s \cdot r$ tracks $g \circ f$ by comultiplication and naturality of b_s

$$b_{s \cdot r} \rho \leq b_s b_r \rho \leq b_s P_{f \times f}(\sigma) = P_{f \times f}(b_s \sigma) \leq P_{f \times f}(P_{g \times g}(\tau))$$

Given elements $\rho \in P(A \times A)$ and $\sigma \in P(B \times B)$, we denote by $\rho \boxtimes \sigma$ the element $P_{\langle \pi_1, \pi_3 \rangle}(\rho) * P_{\langle \pi_2, \pi_4 \rangle}(\sigma)$ in $P(A \times B \times A \times B)$.

The proof of the following proposition is straightforward.

Proposition 5.1. *Let $\langle P, *, \kappa \rangle$ be a primary linear doctrine. If ρ and σ are P -distances, then $\rho \boxtimes \sigma$ is a P -distance. If ρ and σ are affine, then $\rho \boxtimes \sigma$ is affine.*

Relying on this property, we can prove the following result.

Proposition 5.2. *The category $\mathcal{L}_{\langle P, b \rangle}$ has finite products.*

Proof. A terminal object is $\langle 1, \kappa_{1 \times 1} \rangle$, where 1 is a terminal object in \mathcal{C} . Indeed, for any object $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$, the unique map $t_A : A \rightarrow 1$ in \mathcal{C} , induces a map from $\langle A, \rho \rangle$ to $\langle 1, \kappa_{1 \times 1} \rangle$ since by weakening it holds $b_0 \rho \leq \kappa_{A \times A} = P_{t_A \times t_A}(\kappa_{1 \times 1})$. Given $\langle A, \rho \rangle$ and $\langle B, \sigma \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$, the pair $\langle A \times B, \rho \boxtimes \sigma \rangle$ is an object in $\mathcal{L}_{\langle P, b \rangle}$ by Proposition 5.1, as both ρ and σ are affine P -distances.

Since both ρ and σ are affine, projections $\pi_A : A \times B \rightarrow A$ and $\pi_B : A \times B \rightarrow B$ induce maps from $\langle A \times B, \rho \boxtimes \sigma \rangle$ to $\langle A, \rho \rangle$ and $\langle B, \sigma \rangle$, respectively; indeed, using affineness and counit, we have $b_1(\rho \boxtimes \sigma) \leq b_1 P_{\langle \pi_1, \pi_3 \rangle}(\rho) \leq P_{\langle \pi_1, \pi_3 \rangle}(\rho) = P_{\pi_A \times \pi_A}(\rho)$ and similarly for π_B . Finally, to check the universal property, let $\langle C, \tau \rangle$ be an object in $\mathcal{L}_{\langle P, b \rangle}$ and $f : \langle C, \tau \rangle \rightarrow \langle A, \rho \rangle$ and $g : \langle C, \tau \rangle \rightarrow \langle B, \sigma \rangle$ arrows in $\mathcal{L}_{\langle P, b \rangle}$. It suffices to show that the arrow $\langle f, b \rangle : C \rightarrow A \times B$ in \mathcal{C} induced by f and g underlines an arrow from $\langle C, \tau \rangle$ to $\langle A \times B, \rho \boxtimes \sigma \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$. By hypothesis, we know that there are $r, s \in |R|$ such that $b_r \tau \leq P_{f \times f}(\rho)$ and $b_s \tau \leq P_{g \times g}(\sigma)$, hence, using contraction, we get

$$b_{r+s} \tau \leq b_r \tau * b_s \tau \leq P_{f \times f}(\rho) * P_{g \times g}(\sigma)$$

whence the claim. \square

We now focus on the fibres. Given an affine P -distance ρ on an object A in \mathcal{C} , an element α in $P(A)$ and r in $|R|$, we write r tracks α when $P_{\pi_1}(\alpha) * b_r \rho \leq P_{\pi_2}(\alpha)$. The suborder of $P(A)$ of R -graded descent data is:

$$\mathcal{Des}_\rho(A) = \{\alpha \in P(A) \mid r \text{ tracks } \alpha \text{ for some } r \in |R|\}$$

Consider $f : \langle A, \rho \rangle \rightarrow \langle B, \sigma \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$ and $\beta \in \mathcal{Des}_\sigma(B)$, then, r tracks f and s tracks β for some $r, s \in |R|$. By comultiplication and naturality of b_s , we get

$$b_{s \cdot r} \rho \leq b_s b_r \rho \leq b_s P_{f \times f}(\sigma) = P_{f \times f}(b_s \sigma)$$

Thus, since $\pi_i \circ (f \times f) = f \circ \pi_1$, for $i = 1, 2$, we get

$$P_{\pi_1}(P_f(\beta)) * b_{s \cdot r} \rho \leq P_{f \times f}(P_{\pi_1}(\beta) * b_s \sigma) \leq P_{\pi_2}(P_f(\beta))$$

showing that $s \cdot r$ tracks $P_f(\beta)$. In other words P_f applies $\mathcal{D}es_\sigma(B)$ to $\mathcal{D}es_\rho(A)$ and the assignments

$$D_{\langle P, b \rangle} \langle A, \rho \rangle := \mathcal{D}es_\rho(A) \quad D_{\langle P, b \rangle}(f) := P_f$$

determine a functor $D_{\langle P, b \rangle} : \mathcal{L}_{\langle P, b \rangle}^{\text{op}} \rightarrow \mathcal{P}os$.

Proposition 5.3. $D_{\langle P, b \rangle}$ is a R -Lipschitz doctrine.

Proof. We have $\kappa \in \mathcal{D}es_\rho$, as $\kappa * b_0 \rho \leq \kappa$, by weakening. For $\alpha, \beta \in \mathcal{D}es_\rho$ it holds $P_{\pi_1}(\alpha) * b_r \rho \leq P_{\pi_2}(\alpha)$ and $P_{\pi_1}(\beta) * b_s \rho \leq P_{\pi_2}(\beta)$, for some $r, s \in |R|$. Then, by contraction and commutativity of $*$

$$\begin{aligned} P_{\pi_1}(\alpha * \beta) * b_{r+s} \rho &\leq P_{\pi_1}(\alpha) * b_r \rho * P_{\pi_1}(\beta) * b_s \rho \\ &\leq P_{\pi_2}(\alpha) * P_{\pi_2}(\beta) = P_{\pi_2}(\alpha * \beta) \end{aligned}$$

For $\alpha \in \mathcal{D}es_\rho$ and $r \in |R|$ it holds $P_{\pi_1}(\alpha) * b_s \rho \leq P_{\pi_2}(\alpha)$, for some $s \in |R|$. Comultiplication, lax-monoidality and naturality of b_r lead to

$$\begin{aligned} P_{\pi_1}(b_r \alpha) * b_{r \cdot s} \rho &\leq b_r P_{\pi_1}(\alpha) * b_r b_s \rho \\ &\leq b_r (P_{\pi_1}(\alpha) * b_s \rho) \\ &\leq b_r P_{\pi_2}(\alpha) = P_{\pi_2}(b_r \alpha) \end{aligned}$$

In the end, to show that $D_{\langle P, b \rangle}$ is R -Lipschitz, we set $d_{\langle A, \rho \rangle} := \rho$, for each object $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$. First of all, note that $d_{\langle A, \rho \rangle} \in D_{\langle P, b \rangle}(\langle \langle A, \rho \rangle \times \langle A, \rho \rangle \rangle) = D_{\langle P, b \rangle} \langle A \times A, \rho \boxtimes \rho \rangle$ since, by symmetry and transitivity of ρ and counit, we have $P_{\langle \pi_1, \pi_2 \rangle}(\rho) * b_1(\rho \boxtimes \rho) \leq P_{\langle \pi_3, \pi_4 \rangle}(\rho)$. Then, the thesis follows by Proposition 3.7. \square

The construction we have just described extends to a 2-functor $\mathbf{R}_{\text{Lip}} : \mathbf{LD}_R \rightarrow \mathbf{LLD}_R$ as discussed below. Let $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle Q, b' \rangle$ be a 1-arrow in \mathbf{LD}_R and let $h : \langle A, \rho \rangle \rightarrow \langle B, \sigma \rangle$ be an arrow in $\mathcal{L}_{\langle P, b \rangle}$. The following assignments define a functor $\overline{F} : \mathcal{L}_{\langle P, b \rangle} \rightarrow \mathcal{L}_{\langle Q, b' \rangle}$:

$$\overline{F} \langle A, \rho \rangle := \langle FA, f_{A \times A}(\rho) \rangle \quad \overline{F} h := Fh$$

Then, \mathbf{R}_{Lip} is defined on arrows as follows:

$$\mathbf{R}_{\text{Lip}} \langle F, f \rangle := \langle \overline{F}, f \rangle \quad \mathbf{R}_{\text{Lip}}(\theta) := \theta$$

where $\theta : \langle F, f \rangle \Rightarrow \langle G, g \rangle$ is a 2-arrow in \mathbf{LD}_R .

Proposition 5.4. *The assignments above define a 2-functor*

$$\mathbf{R}_{\text{Lip}} : \mathbf{LD}_R \rightarrow \mathbf{LLD}_R$$

Proof. To prove the thesis, we just have to show that \mathbf{R}_{Lip} is well-defined, then algebraic identities follow immediately. Let $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle Q, b' \rangle$ be a 1-arrow in \mathbf{LD}_R . The fact that $\overline{F} : \mathcal{L}_{\langle P, b \rangle} \rightarrow \mathcal{L}_{\langle Q, b' \rangle}$ is a well-defined product-preserving functor follows from the following two facts.

- Let ρ be an affine P -distance on A , then $f_{A \times A}(\rho)$ is an affine Q -distance on FA . This is immediate as f is a natural transformation preserving the monoidal structure.

- Let $h : \langle A, \rho \rangle \rightarrow \langle B, \sigma \rangle$ be an arrow in $\mathcal{L}_{\langle P, \flat \rangle}$, then $Fh : \langle FA, f_{A \times A}(\rho) \rangle \rightarrow \langle FB, f_{B \times B}(\sigma) \rangle$ is an arrow in $\mathcal{L}_{\langle Q, \flat' \rangle}$. By definition of arrows in $\mathcal{L}_{\langle P, \flat \rangle}$, we know that $\flat_r \rho \leq P_h(\sigma)$ holds for some $r \in |R|$. Since $\langle F, f \rangle$ is a 1-arrow in \mathbf{LD}_R , we know that $\flat'_{rFA \times FA} \circ f_{A \times A} \leq f_{A \times A} \circ \flat_{rA \times A}$, hence, applying $f_{A \times A}$ we get

$$\flat'_r f_{A \times A}(\rho) \leq f_{A \times A}(\flat_r \rho) \leq f_{A \times A}(P_h(\sigma)) = Q_{Fh}(f_{B \times B}(\sigma))$$

To check that f_A applies $D_{\langle P, \flat \rangle} \langle A, \rho \rangle = \mathbf{Des}_\rho$ into $D_{\langle Q, \flat' \rangle} \langle FA, f_{A \times A}(\rho) \rangle = \mathbf{Des}_{f_{A \times A}(\rho)}$, it is enough to note that, since, for all $\alpha \in \mathbf{Des}_\rho$, we have $P_{\pi_1}(\alpha) *^P \flat_r \rho \leq P_{\pi_2}(\alpha)$, for some $r \in |R|$, we

$$\begin{aligned} Q_{F\pi_1}(f_A(\alpha)) *^Q \flat'_r f_{A \times A}(\rho) &\leq Q_{f\pi_1}(f_A(\alpha)) *^Q f_{A \times A}(\flat_r \rho) \\ &= f_{A \times A}(P_{\pi_1}(\alpha) *^P \flat_r \rho) \\ &\leq f_{A \times A} P_{\pi_2}(\alpha) \\ &= Q_{F\pi_2}(f_A(\alpha)) \end{aligned}$$

Since the R -graded structure of $D_{\langle P, \flat \rangle}$ and $D_{\langle Q, \flat' \rangle}$ are, respectively, that of $\langle P, \flat \rangle$ and $\langle Q, \flat' \rangle$, f preserves the structure by hypothesis, hence $\langle \overline{F}, f \rangle$ is a well-defined 1-arrow in \mathbf{LLD}_R .

Let $\theta : \langle F, f \rangle \Rightarrow \langle G, g \rangle$ be a 2-arrow in \mathbf{LD}_R . The fact that $\mathbf{R}_{\text{Lip}}(\theta) = \theta$ is a well-defined 2-arrow in \mathbf{LLD}_R follows from the following observation: let $\langle A, \rho \rangle$ be an object in $\mathcal{L}_{\langle P, \flat \rangle}$, then $\theta_A : \langle FA, f_{A \times A}(\rho) \rangle \rightarrow \langle GA, g_{A \times A}(\rho) \rangle$ is an arrow in $\mathcal{L}_{\langle Q, \flat' \rangle}$. This is immediate as, by definition of 2-arrow and counit, we get $\flat'_1 f_{A \times A}(\rho) \leq f_{A \times A}(\rho) \leq Q_{\theta_{A \times A}}(g_{A \times A}(\rho)) = Q_{\theta_A \times \theta_A}(g_{A \times A}(\rho))$. \square

For each R -graded doctrine $\langle P, \flat \rangle$ and each R -Lipschitz doctrine $\langle Q, \sharp, \mathbf{d} \rangle$ there are a 1-arrow $\langle U^P, u^P \rangle$ in \mathbf{LD}_R and a 1-arrow $\langle E^Q, e^Q \rangle$ in \mathbf{LLD}_R as depicted below:

$$\begin{array}{ccc} \mathcal{L}_{\langle P, \flat \rangle}^{\text{op}} & \xrightarrow{D_{\langle P, \flat \rangle}} & \mathcal{P}os \\ \downarrow (U^P)^{\text{op}} & \downarrow u^P & \nearrow P \\ \mathcal{C}\emptyset & & \end{array} \qquad \begin{array}{ccc} \mathcal{D}^{\text{op}} & \xrightarrow{Q} & \mathcal{P}os \\ \downarrow (E^Q)^{\text{op}} & \downarrow e^Q & \nearrow D_{\langle Q, \sharp \rangle} \\ \mathcal{L}_{\langle Q, \sharp \rangle}^{\text{op}} & & \end{array}$$

On the left, U^P forgets distances, mapping $\langle A, \rho \rangle$ to its underlying object A and being the identity on arrows, u^P is the obvious natural inclusion of $D_{\langle P, \flat \rangle} \langle A, \rho \rangle$ into $P(A)$. On the right, E^Q maps an object A to $\langle A, \mathbf{d}_A \rangle$ and is the identity on arrows and e^Q is the identity on the fibres. Note that E^P is well-defined on objects as $\langle Q, \sharp, \mathbf{d} \rangle$ is R -Lipschitz, hence \mathbf{d}_A is an affine Q -distance on A , and on arrows by Corollary 3.9; moreover, it preserves finite products by Items (b),(d) of Definition 3.4.

These families of 1-arrows will give rise, respectively, to the counit and the unit of the 2-adjunction $\mathbf{U}_{\text{Lip}} \dashv \mathbf{R}_{\text{Lip}}$ which is the core of the next theorem and also the main result of this subsection.

Theorem 5.5. *The 2-functor $\mathbf{U}_{\text{Lip}} : \mathbf{LLD}_R \rightarrow \mathbf{LD}_R$ is 2-comonadic:*

- (1) \mathbf{R}_{Lip} is the right 2-adjoint of \mathbf{U}_{Lip} and
- (2) \mathbf{LLD}_R is isomorphic to the 2-category of coalgebras for the 2-comonad $\mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}}$.

Proof. First we prove that $\mathbf{U}_{\text{Lip}} \dashv \mathbf{R}_{\text{Lip}}$. Consider the following 2-natural transformations:

$$\epsilon_{\langle P, \flat \rangle} := \langle U^P, u^P \rangle \qquad \eta_{\langle Q, \sharp \rangle} := \langle E^Q, e^Q \rangle$$

for each object $\langle P, b \rangle$ in \mathbf{LD}_R and $\langle Q, \sharp, e \rangle$ in \mathbf{LLD}_R . We have that $\epsilon : \mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}} \rightarrow \mathbf{Id}_{\mathbf{LD}_R}$ is a 2-natural transformation because, for any 1-arrow $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle P', b' \rangle$ in \mathbf{LD}_R , it holds $U^{P'} \circ \overline{F} = F \circ U^P$, by definition of \mathbf{R}_{Lip} . On the other hand, $\eta : \mathbf{Id}_{\mathbf{LLD}_R} \rightarrow \mathbf{R}_{\text{Lip}} \circ \mathbf{U}_{\text{Lip}}$ is a 2-natural transformation because each 1-arrow $\langle F, f \rangle : \langle Q, \sharp, d \rangle \rightarrow \langle Q', \sharp', d' \rangle$ in \mathbf{LLD}_R preserves distances, hence $E^{Q'} F A = \langle F A, d'_{FA} \rangle = \langle F A, f_{A \times A}(d_A) \rangle = \overline{F} E^Q A$, for any object A in the base category of Q . The two triangular identities hold as the following diagrams (in \mathbf{LD}_R) commute for any R -graded doctrine $\langle P, b \rangle$ and R -Lipschitz doctrine $\langle Q, \sharp, e \rangle$:

$$\begin{array}{ccc}
 \langle D_{\langle P, b \rangle}, b \rangle & \xrightarrow{\langle E^{D_{\langle P, b \rangle}}, e^{D_{\langle P, b \rangle}} \rangle} & \langle D_{\langle D_{\langle P, b \rangle}, b \rangle}, b \rangle \\
 \searrow \langle \text{Id}, \text{id} \rangle & & \downarrow \langle U^{D_{\langle P, b \rangle}}, u^{D_{\langle P, b \rangle}} \rangle \\
 & & \langle D_{\langle P, b \rangle}, b \rangle
 \end{array}
 \qquad
 \begin{array}{ccc}
 \langle Q, \sharp \rangle & \xrightarrow{\langle E^Q, e^Q \rangle} & \langle D_{\langle Q, \sharp \rangle}, \sharp \rangle \\
 \searrow \langle \text{Id}, \text{id} \rangle & & \downarrow \langle U^Q, u^Q \rangle \\
 & & \langle Q, \sharp \rangle
 \end{array}$$

Let \mathbf{T} be the 2-comonad on \mathbf{LD}_R induced by the adjunction. In order to prove that \mathbf{LLD}_R is isomorphic to the 2-category $\mathbf{LLD}_R^{\mathbf{T}}$ of coalgebras for \mathbf{T} , note that there is a comparison 2-functor $\mathbf{K} : \mathbf{LLD}_R \rightarrow \mathbf{LD}_R^{\mathbf{T}}$ mapping an R -Lipschitz doctrine $\langle Q, \sharp, e \rangle$ to the coalgebra $\langle E^Q, e^Q \rangle : \langle Q, \sharp \rangle \rightarrow \langle D_{\langle Q, \sharp \rangle}, \sharp \rangle$ and being the identity on arrows. On the other hand, given a coalgebra $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle D_{\langle P, b \rangle}, b \rangle$ on an R -graded doctrine $\langle P, b \rangle$, since $\langle U^P, u^P \rangle \circ \langle F, f \rangle = \langle \text{Id}, \text{id} \rangle$, we have $f = \text{id}$ and $FA = \langle A, \rho \rangle$ for any object A in the base category of P and $P(A) = \mathcal{D}es_{\rho}(A)$. Hence $\langle P, b, d \rangle$ is R -Lipschitz with $d_A = \rho$. On arrows the 2-functor is the identity, because being a coalgebra morphism is exactly the same as preserving distances. Such a 2-functor is the inverse of \mathbf{K} . \square

This result shows that R -Lipschitz doctrines can be seen as coalgebras for the 2-comonad $\mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}}$ on \mathbf{LD}_R , that is, pairs consisting of an R -graded doctrine $\langle P, b \rangle$ and a 1-arrow $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle D_{\langle P, b \rangle}, b \rangle$ in \mathbf{LD}_R . This means that R -Lipschitz doctrines are *structures* over R -graded one. The next theorem such structures in a very precise way.

Proposition 5.6. *Let $\langle F, f \rangle : \langle P, b \rangle \rightarrow \langle D_{\langle P, b \rangle}, b \rangle$ be a $\mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}}$ -coalgebra structure over $\langle P, b \rangle$. Then, $\langle F, f \rangle$ is a left adjoint of $\langle U^P, u^P \rangle$ in the 2-category \mathbf{LD}_R .*

Proof. Since $\langle F, f \rangle$ is a coalgebra, we have that $\langle U^P, u^P \rangle \circ \langle F, f \rangle = \langle \text{Id}, \text{id} \rangle$, hence we take the identity to be the unit of the adjunction. Now, given an object $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$, we have that $F U^P \langle A, \rho \rangle = \langle A, d_A \rangle$, where $d_A \in P(A \times A)$ is a P -distance satisfying the R -graded substitutivity condition because $\langle P, b, \delta \rangle$ is R -Lipschitz by Theorem 5.5. Therefore, since $\rho \in P(A \times A)$, we get that $P_{\langle \pi_1, \pi_2 \rangle}(\rho) * b_r P_{\langle \pi_2, \pi_3 \rangle}(d_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\rho)$, for some $r \in |R|$. Then, reindexing along $\langle \pi_1, \pi_1, \pi_2 \rangle : A \times A \rightarrow A \times A \times A$ and using reflexivity of ρ , we get $b_r d_A \leq \rho$. This proves that the identity on A gives rise to an arrow $\text{id}_A : \langle A, d_A \rangle \rightarrow \langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, b \rangle}$, which will be the components of the counit of the adjunction. Finally, the triangular laws of adjunctions trivially holds as both the unit and the counit have the identity as underlying arrow, therefore we get the thesis. \square

In other words, Proposition 5.6 proves that the 2-comonad $\mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}}$ is a *KZ comonad* [Koc95]. This express in categorical terms the fact that being R -Lipschitz is a *property* rather than a *structure*. Indeed, Proposition 5.6 shows that every $\mathbf{U}_{\text{Lip}} \circ \mathbf{R}_{\text{Lip}}$ -coalgebra structure on a R -graded doctrine $\langle P, b \rangle$ must be left adjoint of the counit $\langle U^P, u^P \rangle$ of the comonad, hence they are all isomorphic thanks to general properties of adjunctions in

2-categories. Equivalently, there is at most one $U_{\text{Lip}} \circ R_{\text{Lip}}$ -coalgebra structure on $\langle P, b \rangle$ up to isomorphism.

Comparing with the non-linear case, by Proposition 3.3, we know that \mathbf{PD} is a 2-subcategory of \mathbf{LD}_R and also that \mathbf{ED} is 2-subcategory of \mathbf{LLD}_R , since axioms of R -Lipschitz doctrines (cf. Definition 3.4) trivially holds for elementary ones when the modality is the identity as in Proposition 3.3. Then, it is easy to check that, by restricting $U_{\text{Lip}} \dashv R_{\text{Lip}}$ to \mathbf{PD} and \mathbf{ED} , we retrieve the 2-adjunction presented in [EPR20]. This is essentially because in a primary doctrine, viewed as a R -graded one, distances are actually equivalence relations and Lipschitz arrows just preserves such relations. This leads us to the following commutative diagram:

$$\begin{array}{ccc}
 \mathbf{LD}_R & \xrightarrow{R_{\text{Lip}}} & \mathbf{LLD}_R \\
 \uparrow & \dashv & \uparrow \\
 \mathbf{PD} & \xrightarrow{\quad} & \mathbf{ED}
 \end{array} \tag{5.1}$$

The 2-functor $R_{\text{Lip}} : \mathbf{LD}_R \rightarrow \mathbf{LLD}_R$ can be used to construct examples of R -Lipschitz doctrines, providing semantics for the calculus LPLL_R , obtained in the same way as in Example 3.12.

- Examples 5.7.** (1) The Lipschitz doctrine $\langle \text{Lip}, \mathbf{m}, \mathbf{d} \rangle$ of Example 3.5 is the Lipschitz completion of the $\mathbb{R}_{\geq 0}$ -graded doctrine $\langle \mathcal{P}_{[0, \infty]}, \mathbf{m} \rangle$ of Example 3.2(3).
- (2) The Lipschitz completion of the R -graded doctrine $\langle K_{\mathbb{W}}, \bar{a} \rangle$ of Example 3.2(5) (where \mathbb{W} is a monoidal Kripke frame and a a lax axioms of R on \mathbb{W}) is the Lipschitz doctrine $D_{\langle K_{\mathbb{W}}, \bar{a} \rangle} : \mathcal{L}_{\langle K_{\mathbb{W}}, \bar{a} \rangle}^{\text{op}} \rightarrow \mathcal{P}\text{os}$. Objects of the category $\mathcal{L}_{\langle K_{\mathbb{W}}, \bar{a} \rangle}$ are pairs $\langle X, \rho \rangle$ where the ternary relation $\rho \subseteq X \times X \times W$ is an affine $K_{\mathbb{W}}$ -distance in the sense of Example 2.11(4). An arrow $f : \langle X, \rho \rangle \rightarrow \langle Y, \sigma \rangle$ is a function $f : X \rightarrow Y$ for which there is $r \in |R|$ such that, for all $x, x' \in X$ and $w \in W$, $\langle x, x', w \rangle \in \rho$ implies $\langle f(x), f(x'), a(r, w) \rangle \in \sigma$. An element in $D_{\langle K_{\mathbb{W}}, \bar{a} \rangle} \langle X, \rho \rangle$ is $U \subseteq X \times W$ for which there is $r \in |R|$ such that, for all $x, x' \in X$ and $w_1, w_2 \in W$, $\langle x, w_1 \rangle \in U$ and $\langle x, x', w_2 \rangle \in \rho$ implies $\langle x', w_1 \circ a(r, w_2) \rangle \in U$.
- (3) The Lipschitz completion of the R -graded realizability doctrine $\langle \mathcal{R}_A, b \rangle$ of Example 3.2(4) (where A be an ordered R -LCA) is the R -Lipschitz doctrine $D_{\langle \mathcal{R}_A, b \rangle} : \mathcal{L}_{\langle \mathcal{R}_A, b \rangle}^{\text{op}} \rightarrow \mathcal{P}\text{os}$. Objects of $\mathcal{L}_{\langle \mathcal{R}_A, b \rangle}$ are pairs $\langle X, \rho \rangle$ such that $\rho : X \times X \rightarrow \mathcal{P}(|A|)$ is an affine \mathcal{R}_A -distance as in Example 2.11(1). An arrow $f : \langle X, \rho \rangle \rightarrow \langle Y, \sigma \rangle$ of $\mathcal{L}_{\langle \mathcal{R}_A, b \rangle}$ is a function such that there $r \in |R|$ and there is a realizer in $|A|$ mapping, for every $x, x' \in X$, elements of $!_r \rho(x, x')$ into elements of $\sigma(f(x), f(x'))$. An element α in $D_{\langle \mathcal{R}_A, b \rangle} \langle X, \rho \rangle$ is a function $\alpha : X \rightarrow \mathcal{P}(|A|)$ such that there $r \in |R|$ and a realizer in $|A|$ mapping, for every $x, x' \in X$, elements of $\{\mathbf{P}ab \in |A| \mid a \in \alpha(x), b \in !_r \rho(x, x')\}$ into elements of $\alpha(x')$.

The fact that R_{Lip} is a right 2-adjoint allows us to characterise in a precise way model of LPLL_R theories in R -Lipschitz doctrines obtained as Lipschitz completions of an R -graded one. Indeed, if \mathcal{T} is a theory in LPLL_R and $\langle P, b \rangle$ is an R -graded doctrine, the 2-adjunction $U_{\text{Lip}} \dashv R_{\text{Lip}}$ gives us the following isomorphism between hom-categories:

$$\mathbf{LD}_R(\langle \text{Prp}_{\mathcal{T}}, ! \rangle, \langle P, b \rangle) \simeq \mathbf{LLD}_R(\langle \text{Prp}_{\mathcal{T}}, !, \mathbf{d}^{\#} \rangle, \langle D_{\langle P, b \rangle}, b, \mathbf{d} \rangle)$$

which says that models (and their homomorphisms) into an R -graded doctrine $\langle P, b \rangle$, that is, ignoring equality, are the same as models into the Lipschitz completion of $\langle P, b \rangle$ that preserve equality.

5.2. Relating notions of equality. In this section we establish a relation between the quantitative equality introduced in Section 3 and the traditional Lawvere's notion of equality defined by left adjoints.

First of all, we cast Lawvere's equality to R -graded doctrines. This is quite easy since R -graded doctrines are in particular primary linear doctrines, hence one can always consider those that are elementary according to Definition 2.9. More explicitly, this means that, a R -graded doctrine $\langle P, \flat \rangle$ is *elementary* if for every object A in the base of P there is an element $\delta_A \in P(A \times A)$, which is reflexive and substitutive.

We have already seen in Proposition 2.14 that such an equality predicate is replicable. In the R -graded setting, this non-linearity is even more evident as detailed by the next proposition.

Given an R -graded doctrine $\langle P, \flat \rangle$, an element α in $P(A)$ is said to be \flat -intuitionistic if $\alpha \leq \flat_r \alpha$ for all $r \in |R|$. Intuitively, this means that α can provide an arbitrary amount of copies of itself, hence it can be used in an unrestricted way. Note that a \flat -intuitionistic element is, in particular, affine and replicable. Indeed, if α is \flat -intuitionistic, we have $\alpha \leq \flat_0 \alpha \leq \kappa$ (by weakening) and $\alpha \leq \flat_{1+1} \alpha \leq \flat_1 \alpha * \flat_1 \alpha \leq \alpha * \alpha$ (by contraction and counit). Notice also that, by lax monoidality of \flat , we have that κ is \flat -intuitionistic and, if α and β are \flat -intuitionistic, so is $\alpha * \beta$. Then, Proposition 2.14 is generalised as follows.

Proposition 5.8. *Let $\langle P, \flat \rangle$ be an R -graded doctrine and $f : X \rightarrow Y$ an arrow of the base. If P_f has a left adjoint \mathcal{A}_f , then $\mathcal{A}_f(\kappa_X)$ is \flat -intuitionistic.*

Proof. We have $\kappa_X \leq \flat_r \kappa_X$ by lax monoidality of \flat_r and $\kappa_X \leq P_f(\mathcal{A}_f(\kappa_X))$ by the adjunction $\mathcal{A}_f \dashv P_f$. This implies $\kappa_X \leq \flat_r P_f(\mathcal{A}_f(\kappa_X)) = P_f(\flat_r \mathcal{A}_f(\kappa_X))$, by naturality and monotonicity of \flat_r . Then we get $\mathcal{A}_f(\kappa_X) \leq \flat_r \mathcal{A}_f(\kappa_X)$, again by the adjunction $\mathcal{A}_f \dashv P_f$, as needed. \square

Relying on Proposition 5.8 implies that equality predicates in an elementary R -graded doctrines are \flat -intuitionistic.

At this point, the natural question is the following: how does this (standard) notion of equality relate to our quantitative equality? To answer this question in a precise way, first of all we observe that also elementary R -graded doctrines can be organised in a 2-category, and then we compare it with the 2-category of R -Lipschitz doctrines.

Elementary R -graded doctrines are the objects of the 2-category \mathbf{ELD}_R where a 1-arrow from $\langle P, \flat \rangle$ to $\langle Q, \sharp \rangle$ is a 1-arrow $\langle F, f \rangle : \langle P, \flat \rangle \rightarrow \langle Q, \sharp \rangle$ in \mathbf{LD}_R such that $\mathcal{A}_{\text{id}_{F_X} \times \Delta_{F_A}} \circ f_{X \times A} = f_{X \times A \times A} \circ \mathcal{A}_{\text{id}_X \times \Delta_A}$, for all objects X and A in the base of P . A 2-arrow from $\langle F, f \rangle$ to $\langle G, g \rangle$ is a 2-arrow $\theta : \langle F, f \rangle \Rightarrow \langle G, g \rangle$ in \mathbf{LD}_R . Compositions and identities are those of \mathbf{LD}_R .

First of all, we observe that an elementary R -graded doctrine $\langle P, \flat \rangle$ is also R -Lipschitz. Indeed, given an object A in the base of P , the equality predicate δ_A is an affine P -distance, by Proposition 2.12 and Proposition 5.8, and it satisfies the graded substitutive property since we have

$$P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * \flat_1 P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * P_{\langle \pi_2, \pi_3 \rangle}(\delta_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha)$$

by counit and substitutivity of δ_A (cf. Definition 2.9). This observation immediately provides us with a 2-functor (actually an inclusion) $\mathbf{I}_{\text{EL}} : \mathbf{ELD}_R \hookrightarrow \mathbf{LLD}_R$.

In the rest of the section we will show that the 2-functor \mathbf{I}_{EL} is 2-comonadic. We start by constructing a candidate to be its right 2-adjoint. Consider an R -Lipschitz doctrine $\langle P, \flat, \mathbf{d} \rangle$ where \mathcal{C} is the base of P . We denote by \mathcal{C}_\flat the full subcategory of \mathcal{C} on those objects A such that \mathbf{d}_A is \flat -intuitionistic. This category inherits finite products from \mathcal{C} . Indeed, the

terminal object of \mathcal{C} is in $\mathcal{C}_!$, as $\mathbf{d}_1 = \kappa$ is \flat -intuitionistic, and given objects A and B in $\mathcal{C}_!$, the product $A \times B$ is in $\mathcal{C}_!$ as well, since $\mathbf{d}_{A \times B} = \mathbf{d}_A \boxtimes \mathbf{d}_B$ is \flat -intuitionistic as both \mathbf{d}_A and \mathbf{d}_B are. Denote by $P_!$ the restriction of P to $\mathcal{C}_!$; then $\langle P_!, \flat \rangle$ is a R -graded doctrine.

Proposition 5.9. *$\langle P_!, \flat \rangle$ is an elementary R -graded doctrine.*

Proof. We just have to prove that $\langle P_!, \flat \rangle$ is elementary according to Definition 2.9. Since $\langle P, \flat, \mathbf{d} \rangle$ is R -Lipschitz, we know that, for every object A in $\mathcal{C}_!$, $\mathbf{d}_A \in P(A \times A) = P_!(A \times A)$ and \mathbf{d}_A is reflexive as it is a P -distance. The substitutivity property follows since \mathbf{d}_A is \flat -intuitionistic by definition of $\mathcal{C}_!$ and by the R -substitutivity property of R -Lipschitz doctrines as detailed below. Indeed, we have that, for every $\alpha \in P_!(X \times A) = P(X \times A)$, there is $r \in |R|$, such that

$$P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * P_{\langle \pi_2, \pi_3 \rangle}(\mathbf{d}_A) \leq P_{\langle \pi_1, \pi_2 \rangle}(\alpha) * {}_{b_r}P_{\langle \pi_2, \pi_3 \rangle}(\mathbf{d}_A) \leq P_{\langle \pi_1, \pi_3 \rangle}(\alpha) \quad \square$$

Indeed, if $\langle F, f \rangle : \langle P, \flat, \mathbf{d} \rangle \rightarrow \langle Q, \sharp, \mathbf{e} \rangle$ is a 1-arrow in \mathbf{LLD}_R and \mathbf{d}_A is \flat -intuitionistic, then \mathbf{e}_{FA} is \sharp -intuitionistic, as $\mathbf{e}_{FA} = f_{A \times A}(\mathbf{d}_A) \leq f_{A \times A}({}_{b_r}\mathbf{d}_A) = \sharp_r f_{A \times A}(\mathbf{d}_A) = \sharp_r \mathbf{e}_{FA}$.

Thanks to this observation, we can easily extend the construction above to a 2-functor $\mathbf{R}_{\text{EL}} : \mathbf{LLD}_R \rightarrow \mathbf{ELD}_R$, which just restricts 1-arrows and 2-arrows to the subcategories of shape $\mathcal{C}_!$ introduced above. Finally, we get the following theorem.

Theorem 5.10. *The 2-functor $\mathbf{I}_{\text{EL}} : \mathbf{ELD}_R \rightarrow \mathbf{LLD}_R$ is 2-comonadic:*

- (1) \mathbf{R}_{EL} is the right 2-adjoint of \mathbf{I}_{EL} and
- (2) \mathbf{ELD}_R is isomorphic to the 2-category of coalgebras for the KZ 2-comonad $\mathbf{I}_{\text{EL}} \circ \mathbf{R}_{\text{EL}}$.

Proof. (Sketched) Since $\mathbf{R}_{\text{EL}} \circ \mathbf{I}_{\text{EL}}$ is the identity, the unit is the identity as well, while the counit is $\langle I^P, i^P \rangle : \langle P_!, \flat, \mathbf{d} \rangle \rightarrow \langle P, \flat, \mathbf{d} \rangle$, where I^P is the inclusion of $\mathcal{C}_!$ into \mathcal{C} and i^P is the identity. Then, the two triangular identities trivially hold, proving the first item. Towards a proof of the second one, let $\langle F, f \rangle$ be a coalgebra on $\langle P, \flat, \mathbf{d} \rangle$, then the condition $\langle I^P, i^P \rangle \circ \langle F, f \rangle = \langle \text{Id}, \text{id} \rangle$ leads to $\langle F, f \rangle = \langle I^P, i^P \rangle = \langle \text{Id}, \text{id} \rangle$, since I^P is an inclusion and i^P is the identity. This in particular proves that the 2-comonad $\mathbf{I}_{\text{EL}} \circ \mathbf{R}_{\text{EL}}$ is KZ. Moreover, this implies that, for every object A in the base of P , \mathbf{d}_A is \flat -intuitionistic, hence, $\langle P, \flat, \mathbf{d} \rangle$ already is in \mathbf{ELD}_R . The converse inclusion is straightforward. \square

Composing the 2-adjunctions in Theorem 5.5 and Theorem 5.10 we get a 2-adjunction between \mathbf{LD}_R and \mathbf{ELD}_R . Although in general 2-comonadic 2-adjunctions do not compose, this 2-adjunction turns out to be 2-comonadic and as it can be observed following arguments similar to those already used in Theorem 5.5 and Theorem 5.10

$$\begin{array}{ccccc}
 \mathbf{LD}_R & \xrightleftharpoons[\top]{\mathbf{R}_{\text{Lip}}} & \mathbf{LLD}_R & \xrightleftharpoons[\top]{\mathbf{R}_{\text{EL}}} & \mathbf{ELD}_R \\
 \uparrow & & \swarrow & & \uparrow \\
 \mathbf{PD} & \xrightleftharpoons[\top]{} & & & \mathbf{ED}
 \end{array}$$

The square on the left is Diagram 5.1. The right triangle is commutative. This is due to the fact that the composition of \mathbf{R}_{EL} with the inclusion of \mathbf{ELD}_R into \mathbf{LLD}_R is the identity on \mathbf{ELD}_R .

6. RICHER FRAGMENTS OF LINEAR LOGIC

The calculus LPLL_R , introduced in Section 3.2, is the minimal core calculus for quantitative equality, as it only considers connectives necessary to deal with it, namely, \otimes , $\mathbf{1}$ and $!_r$. This section extends quantitative equality to larger fragments of Linear Logic, introducing other connective in LPLL_R .

Each new connective comes with usual rules, hence we do not report them. On the other hand, we have to extend the definition of the function gr to new connectives, in order to determine resources they require to derive substitutions.

Since constants do not depend on variables, there is nothing to substitute, hence the cost to derive a substitution is zero. Thus we set

$$\text{gr}(\mathbf{0}, x) = \text{gr}(\top, x) = 0$$

A substitution in a formula built out of binary multiplicative connectives costs the sum of the resources that the two subformulas need to derive the substitution, because both of them can be subsequently used. Thus we set

$$\text{gr}(\phi \multimap \psi, x) = \text{gr}(\phi, x) + \text{gr}(\psi, x)$$

A substitution in a formula built out of quantifiers costs the same resources as the subformula, only if the substituted and the quantified variable differs, otherwise the cost is zero. Thus we set

$$\text{gr}(\forall z.\phi, x) = \text{gr}(\exists z.\phi, x) = \begin{cases} \text{gr}(\phi, x) & \text{if } x \neq z \\ 0 & \text{if } x = z \end{cases}$$

This is because, since z is bounded in $\forall z.\phi$ and $\exists z.\phi$, substitutions propagate to subformulas only if the substituted variable is different from z , otherwise $\forall z.\phi$ and $\exists z.\phi$ behave as constants.

The extension of LPLL_R by additive connectives requires additional structure on R . A straightforward sufficient condition is that R has binary suprema denoted by \vee . Then, the cost of a substitution in a formula built out of binary additive connectives is the supremum of the resources that the two subformulas need to derive a substitution, because only one of them can be subsequently used. Thus we set

$$\text{gr}(\phi \& \psi, x) = \text{gr}(\phi \oplus \psi, x) = \text{gr}(\phi, x) \vee \text{gr}(\psi, x)$$

When all the connectives above are considered we abbreviate the resulting calculus by ILL_R .

A way to obtain classical calculi is by adding a constant \perp which is dualising, namely, it satisfies the entailment $(\phi \multimap \perp) \multimap \perp \vdash \phi$. Being a constant, \perp is such that $\text{gr}(\perp, x) = 0$. The classical version of ILL_R is denoted by LL_R .

It is straightforward to define R -graded doctrines that correspond to the various fragments of (classical or intuitionistic) Linear Logic. Here we consider only the following.

Definition 6.1. A *multiplicative (linear) doctrine* is a primary linear doctrine $\langle P, *, \kappa \rangle$ such that

- for every A and every α, β, γ in $P(A)$ there is $\alpha \multimap \beta$ in $P(A)$ such that $\gamma \leq \alpha \multimap \beta$ if and only if $\gamma * \alpha \leq \beta$
- for every $f : X \rightarrow A$ and every α, β in $P(A)$, it holds $P_f(\alpha \multimap \beta) = P_f(\alpha) \multimap P_f(\beta)$;

A *multiplicative-additive (linear) doctrine* is a multiplicative linear doctrine $\langle P, *, \kappa \rangle$ such that $P : \mathcal{C}^{\text{op}} \rightarrow \mathcal{Pos}$ factors through the category of lattices.

A multiplicative linear doctrine P has *quantifiers* if for every projection $\pi : A \times B \rightarrow B$ the map P_π has a left adjoint \mathcal{E}_π and a right adjoint \mathcal{V}_π and these are natural in A (this naturality condition is sometimes called Beck-Chevalley condition).

A multiplicative linear doctrine is *classical* if for each object A there is \perp_A in $P(A)$ such that $(\alpha \multimap \perp_A) \multimap \perp_A = \alpha$, which is preserved by reindexing.

A *first order linear doctrine* is a multiplicative-additive linear doctrine with quantifiers.

The syntactic doctrine associated with the $(\otimes, \mathbf{1}, \multimap)$ -fragment of ILL is a multiplicative linear doctrine; the one associated with the $(\otimes, \mathbf{1}, \multimap, \oplus, \mathbf{0}, \&, \top)$ -fragment of ILL is a multiplicative-additive linear doctrine; the one associated with full first order ILL is a first order linear doctrine. Similarly, syntactic doctrines associated with classical variants of the above fragments provides classical versions of the corresponding doctrines.

The following propositions show that the additional structure of R -graded doctrines modelling larger fragments of LL is inherited by their Lipschitz completion.

Proposition 6.2. *Suppose $\langle P, \flat \rangle$ is an R -graded doctrine and consider its Lipschitz completion $D_{\langle P, \flat \rangle} : \mathcal{L}_{\langle P, \flat \rangle}^{\text{op}} \rightarrow \mathcal{Pos}$.*

- If P is multiplicative, so is $D_{\langle P, \flat \rangle}$;
- if P is multiplicative and classical, so is $D_{\langle P, \flat \rangle}$;
- if P is multiplicative and has quantifier, so is $D_{\langle P, \flat \rangle}$.

Proof. Suppose P is multiplicative. Take $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, \flat \rangle}$ and α, β in $\mathcal{Des}_\rho(A)$, hence there are r, s in $|R|$ such that $P_{\pi_1}(\alpha) * \flat_r \rho \leq P_{\pi_2}(\alpha)$ and $P_{\pi_1}(\beta) * \flat_s \rho \leq P_{\pi_2}(\beta)$. Then, using contraction of \flat and symmetry of ρ , we have $P_{\pi_2}(\alpha) * P_{\pi_1}(\alpha \multimap \beta) * \flat_{r+s} \rho \leq P_{\pi_2}(\alpha) * P_{\pi_1}(\alpha \multimap \beta) * \flat_r \rho * \flat_s \rho \leq P_{\pi_1}(\alpha) * P_{\pi_1}(\alpha \multimap \beta) * \flat_s \rho \leq P_{\pi_1}(\beta) * \flat_s \rho \leq P_{\pi_2}(\beta)$, showing that $P_{\pi_1}(\alpha \multimap \beta) * \flat_{r+s} \rho \leq P_{\pi_2}(\alpha \multimap \beta)$. This makes $\alpha \multimap \beta$ an element of \mathcal{Des}_ρ .

Suppose P is multiplicative and classical and take $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, \flat \rangle}$. By weakening of \flat , it holds $P_{\pi_1}(\perp_A) * \flat_0 \rho \leq P_{\pi_1}(\perp_A) * \kappa = P_{\pi_1}(\perp_A) = \perp_{A \times A} = P_{\pi_2}(\perp_A)$ so \perp_A is in $\mathcal{Des}_\rho(A)$.

Suppose P is multiplicative with quantifiers and take γ in $\mathcal{Des}_{\rho \boxtimes \sigma}(A \times B)$. By naturality $P_{\pi_1}(\mathcal{E}_{\pi_1} \alpha) * \flat_r \sigma = \mathcal{E}_{\langle \pi_1, \pi_2 \rangle}(P_{\langle \pi_1, \pi_3 \rangle}(\alpha)) * \flat_r \sigma \leq \mathcal{E}_{\langle \pi_1, \pi_2 \rangle}(P_{\langle \pi_1, \pi_3 \rangle}(\alpha) * \flat_r P_{\langle \pi_1, \pi_2 \rangle}(\sigma)) \leq \mathcal{E}_{\langle \pi_1, \pi_2 \rangle} P_{\langle \pi_2, \pi_3 \rangle} \alpha = P_{\pi_2} \mathcal{E}_{\pi_1} \alpha$ for some r in R . So descent data are closed under left adjoints along projections. Similarly, for right adjoints we have $P_{\pi_1}(\mathcal{V}_{\pi_1} \alpha) * \flat_r \sigma = \mathcal{V}_{\langle \pi_1, \pi_2 \rangle}(P_{\langle \pi_1, \pi_3 \rangle}(\alpha)) * \flat_r \sigma \leq \mathcal{V}_{\langle \pi_1, \pi_2 \rangle}(P_{\langle \pi_1, \pi_3 \rangle}(\alpha) * \flat_r P_{\langle \pi_1, \pi_2 \rangle}(\sigma)) \leq \mathcal{V}_{\langle \pi_1, \pi_2 \rangle} P_{\langle \pi_2, \pi_3 \rangle} \alpha = P_{\pi_2} \mathcal{V}_{\pi_1} \alpha$ for some r in R . \square

Proposition 6.3. *Let $\langle P, \flat \rangle$ be an R -graded doctrine where R has finite suprema and consider its Lipschitz completion $D_{\langle P, \flat \rangle} : \mathcal{L}_{\langle P, \flat \rangle}^{\text{op}} \rightarrow \mathcal{Pos}$.*

- if P is multiplicative-additive, so is $D_{\langle P, \flat \rangle}$;
- if P is first order linear, so is $D_{\langle P, \flat \rangle}$.

Proof. Suppose P is multiplicative-additive. Take $\langle A, \rho \rangle$ in $\mathcal{L}_{\langle P, \flat \rangle}$ and α, β in $\mathcal{Des}_\rho(A)$, hence there are r, s in $|R|$ such that $P_{\pi_1}(\alpha) * \flat_r \rho \leq P_{\pi_2}(\alpha)$ and $P_{\pi_1}(\beta) * \flat_s \rho \leq P_{\pi_2}(\beta)$. Then, $P_{\pi_1}(\alpha \wedge \beta) * \flat_{r \vee s} \rho \leq (P_{\pi_1}(\alpha) * \flat_{r \vee s} \rho) \wedge (P_{\pi_1}(\beta) * \flat_{r \vee s} \rho)$. Since $\flat_{r \vee s} \rho \leq \flat_r \rho$ and $\flat_{r \vee s} \rho \leq \flat_s \rho$, it follows $P_{\pi_1}(\alpha \wedge \beta) * \flat_{r \vee s} \rho \leq (P_{\pi_1}(\alpha) * \flat_r \rho) \wedge (P_{\pi_1}(\beta) * \flat_s \rho) \leq P_{\pi_2}(\alpha) \wedge P_{\pi_2}(\beta)$, which proved $\mathcal{Des}_\rho(A)$ is closed under binary meets. By a similar argument proves the closure under binary joins. Top and bottom elements belong to $\mathcal{Des}_\rho(A)$ as one can prove using the same arguments as for \perp . The second item follows by previous item and Proposition 6.2. \square

Corollary 6.4. *Let $\langle P, \flat \rangle$ be an R -graded doctrine where R has finite suprema. If P is classical and first order linear, so is the Lipschitz completion $D_{\langle P, \flat \rangle} : \mathcal{L}_{\langle P, \flat \rangle}^{\text{op}} \rightarrow \mathcal{Pos}$.*

Theorem 5.5 shows that the notion of quantitative equality given in this paper is coalgebraic, in the sense that Lipschitz doctrines are the coalgebras of a comonad over the category of graded doctrines. This generalizes a known situation that holds in the non-linear case, where elementary doctrines are the coalgebras of a comonad over the category of primary doctrines. This is summarised by the Diagram 5.1 recalled below on the left.

$$\begin{array}{ccc}
 \mathbf{LD}_R & \xrightarrow{\mathbf{R}_{\text{Lip}}} & \mathbf{LLD}_R \\
 \uparrow & \xleftarrow{\top} & \uparrow \\
 \mathbf{PD} & \xrightarrow{\top} & \mathbf{ED}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbf{ED} & \xleftarrow{\top} & \mathbf{PD} \\
 \uparrow & & \uparrow \\
 \mathbf{HD} & \xleftarrow{\top} & \mathbf{FOD}
 \end{array}
 \quad (6.1)$$

In the non-linear case the coalgebraic nature of equality behaves well with respect to other connectives or quantifiers that the doctrine may have. In particular if one consider the 2-category \mathbf{FOD} of first order doctrines (i.e. those doctrines modelling full first order logic) and its subcategory \mathbf{HD} on hyperdoctrines (i.e. those first order doctrines that are also elementary) one shows that \mathbf{HD} is the category of coalgebras of a comonad on \mathbf{FOD} . Diagram 2.2 (rewritten above on the right) shows the adjoint situation that determines the mentioned comonad.

Gluing the two diagram together suggests the existence of a cube of 2-categories of doctrines. We devote the rest of the section to complete such a cube.

Denote by \mathbf{FLD}_R the 2-full 2-subcategory of \mathbf{LD}_R on first order R -graded doctrines and 1-arrows that preserve the first order linear structure. An R -quantitative hyperdoctrine is a first order R -Lipschitz. The 2-full 2-subcategory of \mathbf{LLD}_R on R -quantitative hyperdoctrines is \mathbf{QHD}_R . Proposition 6.3 shows that (if R has finite suprema) the 2-adjunction between \mathbf{LD}_R and \mathbf{LLD}_R restricts to a 2-adjunction between \mathbf{FLD}_R and \mathbf{QHD}_R . Then, we can extend the diagrams 6.1 to the first order case, obtaining the following:

$$\begin{array}{ccccc}
 & & \mathbf{ED} & \xleftarrow{\mathbf{R}_{\text{Lip}}} & \mathbf{PD} \\
 & & \uparrow & \xleftarrow{\top} & \uparrow \\
 \mathbf{LLD}_R & \xleftarrow{\mathbf{R}_{\text{Lip}}} & \mathbf{LD}_R & \xleftarrow{\top} & \mathbf{PD} \\
 \uparrow & \xleftarrow{\top} & \uparrow & \xleftarrow{\mathbf{R}_{\text{Lip}}} & \uparrow \\
 & & \mathbf{HD} & \xleftarrow{\mathbf{R}_{\text{Lip}}} & \mathbf{FOD} \\
 & & \uparrow & \xleftarrow{\top} & \uparrow \\
 \mathbf{QHD}_R & \xleftarrow{\mathbf{R}_{\text{Lip}}} & \mathbf{FLD}_R & \xleftarrow{\top} & \mathbf{FOD}
 \end{array}$$

The embedding of \mathbf{FOD} into \mathbf{FLD}_R follows from Proposition 3.3, as doctrines in \mathbf{FOD} are in particular primary and the rest of the structure does not interact with the modality. The same applies to the embedding of \mathbf{HD} into \mathbf{QHD}_R , but here one has also to notice that, since doctrines in \mathbf{HD} are in particular elementary, they trivially satisfy the axioms of Lipschitz doctrines (cf. Definition 3.4).

Example 6.5. The $\mathbb{R}_{\geq 0}$ -Lipschitz doctrine $\langle \text{Lip}, \mathbf{m}, \mathbf{d} \rangle$ of Example 3.5 is an $\mathbb{R}_{\geq 0}$ -quantitative hyperdoctrine because it is the Lipschitz completion of $\langle \mathcal{P}_{[0, \infty]}, \mathbf{m} \rangle$ (see Example 5.7(1)), which is first order linear as $[0, \infty]$ is a quantale, and $\mathbb{R}_{\geq 0}$ has finite suprema.

Finally, note that any R -quantitative hyperdoctrine provides a domain for interpreting the ILL_R calculus, it suffices to extend Lemma 3.14 to all the other connectives given the definition of gr at the beginning of this section.

7. RELATED AND FUTURE WORKS

A syntactic study of equality in non-modal Linear Logic can be found in [Doz96]. Affineness and replicability of equality are derived from the (non-quantitative) substitution rule. In [CM96], full Linear Logic is considered. Equality is required to be intuitionistic, in the sense that $x = y \vdash !(x = y)$ holds, implying again affineness and replicability. A solution to recover linearity is in [Doz96] and is an instance of our calculus, when the semiring is $\mathbb{N}^=$ and the grade of each symbol is 1.

The construction in Section 5.1, like that of [Pas15, EPR20], originates from a series of works [MR13b, MR15, MR13a] on the notion of quotient in an elementary doctrine. An interesting direction for future work is the development of a quantitative counterpart of quotients in Lipschitz doctrines.

Any QET induces a monad on the category of metric spaces and non-expansive maps. For simple QET, models are proved to be the algebras for such a monad [BMPP18, MPP16, MPP17, Adá22]. This is a strong result attesting that QETs axiomatise the algebras of the monad induced by themselves. Similarly, in our setting, we would like to show that, under suitable assumptions, an equational theory in $LPLL_R$ induces a monad on categories of the form $\mathcal{L}_{\langle P, b \rangle}$, and the category of models in $D_{\langle P, b \rangle}$ for such a theory is (equivalent to) the category of algebras for such a monad. That is, equational theories in $LPLL_R$ axiomatise quantitative algebras where operations are Lipschitz maps.

In [DN23b, DN23a] the authors propose quantitative equational theories for linear λ -calculi also with graded modal types, proving these provide internal languages for autonomous categories enriched over certain metric spaces. We would like to see whether we can describe these theories in our calculus as well, possibly starting from the example of graded combinators in Section 4.

Another possible application of Lipschitz doctrines is the denotational semantics of various forms of “*graded*” λ -calculi (see, e.g., [POM14, AB20]). In these calculi variables in typing contexts are annotated by grades from a semiring, describing how much the term uses each variable. Consequently, arrow types are labelled by a grade describing how much the function uses its input. These data can be interpreted in (the base of) a Lipschitz doctrine: types are objects, contexts are products and terms are arrows whose Lipschitz constants agree with the grades reported in the context. To interpret labelled arrow type, we can exploit the structure of a Lipschitz doctrine to define a *graded exponential*, that intuitively collects all Lipschitz arrows having a fixed Lipschitz constant.

We also plan to move the ideas discussed in this paper to a proof-relevant setting, studying connections with type theories. It is known that every type theory gives rise to a doctrine [MR13b, MR15]. Similarly, linear dependent type theories, where linear and intuitionistic types are kept separated in contexts [CP02], give rise to a primary linear doctrine. Studies of Id-Types in linear type theories are available in the literature, but, to our knowledge, they treat Id-Types as intuitionistic types (see e.g., [KPB15, Vák15]). There are also dependent type theories using grading [Atk18, MIO21], but they do not consider Id-Types. Therefore, an interesting direction will be to use the machinery of the present paper, possibly extended to fibrations, to study quantitative Id-Types in linear type theories.

Finally, in [PON21] it is shown that certain proof systems of intuitionistic Linear Logic with subexponentials can be used to model and reason about concurrent programming under the processes-as-formulas interpretation. It would be interesting to investigate to which extent this paradigm applies to our calculus for quantitative equality and whether R -Lipschitz doctrines can be used to build new models for the associated computational paradigm.

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