

A GAME OF PAWNS

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ABSTRACT. We introduce and study *pawn games*, a class of two-player zero-sum turn-based graph games. A turn-based graph game proceeds by placing a token on an initial vertex, and whoever *controls* the vertex on which the token is located, chooses its next location. This leads to a path in the graph, which determines the winner. Traditionally, the control of vertices is predetermined and fixed. The novelty of pawn games is that control of vertices changes dynamically throughout the game as follows. Each vertex of a pawn game is *owned* by a *pawn*. In each turn, the pawns are partitioned between the two players, and the player who *controls* the pawn that owns the vertex on which the token is located, chooses the next location of the token. Control of pawns changes dynamically throughout the game according to a fixed mechanism. Specifically, we define several *grabbing*-based mechanisms in which control of at most one pawn transfers at the end of each turn. We study the complexity of solving pawn games, where we focus on reachability objectives and parameterize the problem by the mechanism that is being used and by restrictions on pawn ownership of vertices. On the positive side, even though pawn games are exponentially-succinct turn-based games, we identify several natural classes that can be solved in PTIME. On the negative side, we identify several EXPTIME-complete classes, where our hardness proofs are based on a new class of games called Lock & Key games, which may be of independent interest.

1. INTRODUCTION

Two-player zero-sum *graph games* constitute a fundamental class of games [AG11] with applications, e.g., in *reactive synthesis* [PR89], multi-agent systems [AHK02], a deep connections to foundations of logic [Rab69], and more. A graph game is played on a directed graph $\langle V, E \rangle$, where $V = V_1 \cup V_2$ is a fixed partition of the vertices. The game proceeds as follows. A token is initially placed on some vertex. When the token is placed on $v \in V_i$, for $i \in \{1, 2\}$, Player i chooses u with $\langle v, u \rangle \in E$ to move the token to. The outcome of the

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game is an infinite path, called a *play*. We focus on *reachability* games: Player 1 wins a play iff it visits a set of target vertices $T \subseteq V$.

In this paper, we introduce *pawn games*, which are graph games in which the control of vertices changes dynamically throughout the game as follows. The arena consists of d *pawns*. For $1 \leq j \leq d$, Pawn j *owns* a set of vertices V_j . Throughout the game, the pawns are distributed between the two players, and in each turn, the control of pawns determines which player moves the token. Pawn control may be updated after moving the token by running a predetermined *mechanism*. Formally, a *configuration* of a pawn game is a pair $\langle v, P \rangle$, where v denotes the position of the token and P the set of pawns that Player 1 controls. The player who moves the token is determined according to P : if Player 1 controls a pawn that owns v , then Player 1 moves. Specifically, when each vertex is owned by a unique pawn, i.e., V_1, \dots, V_d partitions V , then Player 1 moves iff he controls the pawn that owns v . We consider the following mechanisms for exchanging control of pawns. For $i \in \{1, 2\}$, we denote by $-i = 3 - i$ the “other player”.

- **Optional grabbing.** For $i \in \{1, 2\}$, following a Player i move, Player $-i$ has the *option* to *grab* one of Player i 's pawns; namely, transfer one of the pawns that Player $-i$ to his control.
- **Always grabbing.** For $i \in \{1, 2\}$, following *every* Player i move, Player $-i$ grabs one of Player i 's pawns.
- **Always grabbing or giving.** Following a Player i move, Player $-i$ either grabs one of Player i 's pawns or gives her one of his pawns.
- **k -grabbing.** For $k \in \mathbb{N}$, Player 1 can grab at most k pawns from Player 2 throughout the game. In each round, after moving the token, Player 1 has the option of grabbing one of the pawns that is controlled by Player 2. A grabbed pawn stays in the control of Player 1 for the remainder of the game. Note the asymmetry: only Player 1 grabs pawns.

Note that players in pawn games have two types of actions: moving the token and transferring control of pawns. We illustrate the model and some interesting properties of it.

Example 1.1. Consider the game \mathcal{G}_1 in Fig. 1(left). We consider optional-grabbing and the same reasoning applies for always-grabbing. Each vertex is owned by a unique pawn, and Player 1's target is t . Note that Player 2 wins if the game reaches s . We claim that \mathcal{G}_1 is *non-monotonic*: increasing the set of pawns that Player 1 initially controls is “harmful” for him. Formally, Player 1 wins from configuration $\langle v_0, \emptyset \rangle$, i.e., when he initially does not control any pawns, but loses from $\langle v_0, \{v_0\} \rangle$, i.e., when controlling v_0 . Indeed, from $\langle v_0, \emptyset \rangle$, Player 2 initially moves the token from v_0 to v_1 , Player 1 then uses his option to grab v_1 , and wins by proceeding to t . Second, from $\langle v_0, \{v_0\} \rangle$, Player 1 makes the first move and thus cannot grab v_1 . Since Player 2 controls v_1 , she wins by proceeding to s . In Thm. 3.1 and 4.12, we generalize this observation and show, somewhat surprisingly, that if a player wins from the current vertex v , then he wins from v with fewer pawns as long as if he controlled v previously, then he maintains control of v .

Consider the game \mathcal{G}_2 in Fig. 1 (right). We consider optional-grabbing, each vertex is owned by a unique pawn, and Player 1's target is t . We claim that Player 1 wins from configuration $\langle v_0, \{v_0, v_2\} \rangle$ and Player 2 can force the game to visit v_1 twice. This differs from turn-based games in which if Player 1 wins, he can force winning while visiting each vertex at most once. To illustrate, consider the following outcome. Player 1 makes the first move, so he cannot grab v_1 . Player 2 avoids losing by moving to v_2 . Player 1 will not grab,



Figure 1: Left: The pawn game \mathcal{G}_1 ; a non-monotonic game under optional-grabbing. Right: The pawn game \mathcal{G}_2 in which Player 1 wins from $\langle v_0, \{v_0, v_1\} \rangle$, but must visit v_1 twice.

move to v_3 , Player 2 moves to v_1 , then Player 1 grabs v_1 and proceeds to t . We point out that no loop is closed in the explicit *configuration graph* that corresponds to \mathcal{G}_2 .

Applications. Pawn games model multi-agent settings in which the agent who acts in each turn is not predetermined. We argue that such settings arise naturally.

Quantitative shield synthesis. It is common practice to model an *environment* as a Kripke structure (e.g. [SST18]), which for sake of simplicity, we will think of as a graph in which vertices model environment states and edges model actions. A *policy* chooses an outgoing edge from each vertex. A popular technique to obtain policies is *reinforcement learning* (RL) [SB98] whose main drawback is lack of worst-case guarantees [BCG⁺21]. In order to regain safety at runtime, a *shield* [KAB⁺17, ABC⁺19, BCG⁺21] is placed as a proxy: in each point in time, it can alter the action of a policy. The goal in *shield synthesis* is to synthesize a shield *offline* that ensures safety at runtime while minimizing interventions. We suggest a procedure to synthesize shields based on k -grabbing pawn games. Player 2 models an unknown policy. We set his goal to reaching an unsafe state. Player 1 (the shield) ensures safety by grabbing at most k times. Grabbing is associated with a shield intervention. Note that once the shield intervenes in a vertex v , it will choose the action at v in subsequent turns. An optimal shield is obtained by finding the minimal k for which Player 1 has a winning strategy.

We describe other examples that can be captured by a k -grabbing pawn game in which, as in the shield application, Player 1 models an “authority” that has the “upper hand”, and aims to maximize freedom of action for Player 2 while using grabs to ensure safety. Consider a concurrent system in which Player 2 models a scheduler and Player 1 can force synchronization, e.g., by means of “locks” or “fences” in order to maintain correctness (see [CCH⁺17]). Synchronization is minimized in order to maximize parallelism and speed. As another example, Player 1 might model an operating system that allows freedom to an application and blocks only unsafe actions. As a final example, in [AAHL20], synthesis for a safety specification was enriched with “advice” given by an external policy for optimizing a soft quantitative objective. Again, the challenge is how to maximize accepting advice while maintaining safety. **Modelling crashes.** A *sabotage game* [vB05] is a two-player game which is played on a graph. Player 1 (the Runner) moves a token throughout the graph with the goal of reaching a target set. In each round, Player 2 (the Saboteur) crashes an edge from the graph with the goal of preventing Player 1 from reaching his target. Crashes are a simple type of fault that restrict Player 1’s actions. A malicious fault (called *byzantine faults* [LSP82]) actively tries to harm the network, e.g., by moving away from the target. Pawn games can model sabotage games with byzantine faults: each vertex (router) is owned by a unique pawn, all pawns are initially owned by Player 1, and a Player 2 grab corresponds to a byzantine fault. Several grabbing mechanisms are appealing in this context: k -grabbing

restricts the number of faults and optional- and always-grabbing accommodate repairs of routers.

Our results. We distinguish between three types of ownership of vertices. Let $V = V_1 \cup \dots \cup V_d$ be a set of vertices, where for $j \in \{1, \dots, d\}$, Pawn j owns the vertices in V_j . In *one vertex per pawn* (OVPP) games, each pawn owns exactly one vertex, thus V_j is a singleton, for all $j \in \{1, \dots, d\}$. In *multiple vertices per pawn* (MVPP) games, V_1, \dots, V_d consists of a partition of V , where the sets might contain more than one vertex. In *overlapping multiple vertices per pawn* (OMVPP) games, the sets might overlap. For example, in the shield synthesis application above, the type of ownership translates to dependencies between interventions: OVPP models no dependencies, MVPP models cases in which interventions come in “batches”, e.g., grabbing control in all states labeled by some predicate, and OMVPP models the case when the batches overlap. We define that Player 1 moves the token from a vertex v iff he controls at least one of the pawns that owns v . Clearly, OMVPP generalizes MVPP, which in turn generalizes OVPP.

We consider the problem of deciding whether Player 1 wins a reachability pawn game from an initial configuration of the game. Our results are summarized below.

Mechanisms	OVPP	MVPP	OMVPP
k -grabbing	PTIME (Thm. 6.1)	NP-hard (Thm. 6.2)	PSPACE-C (Thm. 6.6)
Optional-grabbing	PTIME (Thm. 3.3)	EXPTIME-C (Thm. 3.10)	EXPTIME-C (Thm. 3.10)
Always	PTIME (grab or give; Thm. 5.4)	PTIME (grab or give; Thm. 5.4) EXPTIME-C (grab; Thm. 4.11)	EXPTIME-C (grab; Thm. 4.11)

Pawn games are succinctly-represented turn-based games. A naive algorithm to solve a pawn game constructs and solves an explicit turn-based game on its configuration graph leading to membership in EXPTIME. We thus find the positive results to be pleasantly surprising; we identify classes of succinctly-represented games that can be solved in PTIME. Each of these algorithms is obtained by a careful and tailored modification to the *attractor-computation* algorithm for turn-based reachability games. For OMVPP k -grabbing, the PSPACE upper bound is obtained by observing that grabs in a winning strategy must be spaced by at most $|V|$ turns, implying that a game ends within polynomial-many rounds (Lem. 6.4).

Our EXPTIME-hardness proofs are based on a new class of games called *Lock & Key* games and may be of independent interest. A Lock & Key game is a turn-based game that is enriched with a set of locks, where each lock is associated with a key. Each edge is labeled by a subset of locks and keys. A lock can either be *closed* or *open*. An edge that is labeled with a closed lock cannot be crossed. A lock changes state once an edge labeled by its key is traversed. We show two reductions. The first shows that deciding the winner in Lock & Key games is EXPTIME-hardness. Second, we reduce Lock & Key games to MVPP optional-grabbing pawn games. The core of the reduction consists of gadgets that simulate the operation of locks and keys using pawns. Then, we carefully analyze the pawn games that result from applying both reductions one after the other, and show that the guarantees are maintained when using always grabbing instead of optional grabbing. The main difficulty in constructing a winning Player i strategy under always-grabbing from a winning Player i strategy under optional-grabbing is to ensure that throughout the game, both players have *sufficient* and the *correct* pawns to grab (Lem. 4.9).

Related work. The semantics of pawn games is inspired by the seminal paper [AHK02]. There, the goal is, given a game, an objective O , and a set C of pawns (called “players” there), to decide whether Player 1 (called a “coalition” there) can ensure O when he controls the pawns in C . A key distinction from pawn games is that the set C that is controlled by Player 1 is fixed. The paper introduced a logic called *alternating time temporal logic*, which was later significantly extended and generalized to *strategy logic* [CHP10, MMPV14, MMV10]. Multi-player games with rational players have been widely studied; e.g., finding Nash equilibrium [Umm11] or subgame perfect equilibrium [BRvdB21], and rational synthesis [FKL10, KPV16, WGH⁺16, BCH⁺16]. A key distinction from pawn games is that, in pawn games, as the name suggests, the owners of the resources (pawns) have no individual goals and act as pawns in the control of the players. Changes to multi-player graph games in order to guarantee existence or improve the quality of an equilibrium have been studied [AAK15, Per19, BK20, KS23]. The key difference from our approach is that there, changes occur *offline*, before the game starts, whereas in pawn games, the transfer of vertex ownership occurs *online*. In *bidding games* [LLP⁺99, AHC19] (see in particular, *discrete-bidding games* [DP10, AAH21, AS25]) control of vertices changes online: players have budgets, and in each turn, a *bidding* determines which player moves the token. Bidding games are technically very different from pawn games. While pawn games allow varied and fine-grained mechanisms for transfer of control, bidding games only consider strict auction-based mechanisms, which lead to specialized proof techniques that cannot be applied to pawn games. For example, bidding games are monotonic – more budget cannot harm a player – whereas pawn games are not (see Ex. 1.1).

2. PRELIMINARIES

For $k \in \mathbb{N}$, we use $[k]$ to denote the set $\{1, \dots, k\}$. For $i \in \{1, 2\}$, we use $-i = 3 - i$ to refer to the “other player”.

Turn-based games. Throughout this paper we consider *reachability* objectives. For general graph games, see for example [AG11]. A *turn-based game* is $\mathcal{G} = \langle V, E, T \rangle$, where $V = V_1 \cup V_2$ is a set of vertices that is partitioned among the players, $E \subseteq V \times V$ is a set of directed edges, and $T \subseteq V$ is a set of target vertices for Player 1. Player 1’s goal is to reach T and Player 2’s goal is to avoid T . For $v \in V$, we denote the *neighbors* of v by $N(v) = \{u \in V : E(v, u)\}$. Intuitively, a *strategy* is a recipe for playing a game: in each vertex it prescribes a neighbor to move the token to. Formally, for $i \in \{1, 2\}$, a strategy for Player i is a function $f : V_i \rightarrow V$ such that for every $v \in V_i$, we have $f(v) \in N(v)$.¹ An initial vertex $v_0 \in V$ together with two strategies f_1 and f_2 for the players, give rise to a unique *play*, denoted $\pi(v_0, f_1, f_2)$, which is a finite or infinite path in \mathcal{G} and is defined inductively as follows. The first vertex is v_0 . For $j \geq 0$, assuming v_0, \dots, v_j has been defined, then $v_{j+1} = f_i(v_j)$, where $v_j \in V_i$, for $i \in \{1, 2\}$. A Player 1 strategy f_1 is *winning* from $v_0 \in V$ if for every Player 2 strategy f_2 , the play $\pi(v_0, f_1, f_2)$ ends in T . Dually, a Player 2 strategy f_2 is winning from $v_0 \in V$ if for every Player 1 strategy f_1 , the play $\pi(v_0, f_1, f_2)$ does not visit T .

Theorem 2.1 [GTW02]. *Turn based games are determined: from each vertex, one of the players has a (memoryless) winning strategy. Deciding the winner of a game is in PTIME.*

¹We restrict to *memoryless* strategies since these suffice for reachability objectives.

Proof sketch. For completeness, we briefly describe the classic *attractor-computation* algorithm. Consider a game $\langle V, E, T \rangle$. Let $W_0 = T$. For $i \geq 1$, let $W_i = W_{i-1} \cup \{v \in V_1 : N(v) \cap W_i \neq \emptyset\} \cup \{v \in V_2 : N(v) \subseteq W_i\}$. One can prove by induction that W_i consists of the vertices from which Player 1 can force reaching T within i turns. The sequence necessarily reaches a *fixed point* $W^1 = \bigcup_{i \geq 1} W_i$, which can be computed in linear time. Finally, one can show that Player 2 has a winning strategy from each $v \notin W^1$. \square

Pawn games. A *pawn game* with $d \in \mathbb{N}$ pawns is $\mathcal{P} = \langle V, E, T, \mathcal{M} \rangle$, where $V = V_1 \cup \dots \cup V_d$ and for $j \in [d]$, V_j denotes the vertices that Pawn j *owns*, E and T are as in turn-based games, and \mathcal{M} is a mechanism for exchanging pawns as we elaborate later. Player 1 wins a play if it reaches T . We stress that the set of pawns that he controls when reaching T is irrelevant. We omit \mathcal{M} when it is clear from the context. We distinguish between classes of pawn games based on the type of ownership of vertices:

- **One Vertex Per Pawn (OVPP).** There is a one-to-one correspondence between pawns and vertices; namely, $|V| = d$ and each V_j is singleton, for $j \in [d]$. For $j \in [d]$ and $\{v_j\} = V_j$, we sometimes abuse notation by referring to Pawn j as v_j .
- **Multiple Vertices Per Pawn (MVPP).** Each vertex is owned by a unique pawn but a pawn can own multiple vertices, thus V_1, \dots, V_d is a partition of V .
- **Overlapping Multiple Vertices Per Pawn (OMVPP).** Each pawn can own multiple vertices and a vertex can be owned by multiple pawns, i.e., we allow $V_i \cap V_j \neq \emptyset$, for $i \neq j$.

Clearly OMVPP generalizes MVPP, which generalizes OVPP. In MVPP too, we sometimes abuse notation and refer to a pawn by a vertex that it owns.

A *configuration* of a pawn game is $\langle v, P \rangle$, meaning that the token is placed on a vertex $v \in V$ and $P \subseteq [d]$ is the set of pawns that Player 1 *controls*. Implicitly, Player 2 controls the complement set $\bar{P} = [d] \setminus P$. Player 1 moves the token from $\langle v, P \rangle$ iff he controls at least one pawn that owns v . Note that in OVPP and MVPP, let $j \in [d]$ with $v \in V_j$, then Player 1 moves iff $i \in P$. Once the token moves, we update the control of the pawns by applying \mathcal{M} .

From pawn games to turn-based games. We describe the formal semantics of pawn games together with the pawn-exchanging mechanisms by describing the explicit turn-based game that corresponds to a pawn game. For a pawn game $\mathcal{G} = \langle V, E, T, \mathcal{M} \rangle$, we construct the turn-based game $\mathcal{G}' = \langle V', E', T' \rangle$. For $i \in \{1, 2\}$, denote by V'_i Player i 's vertices in \mathcal{G}' . The vertices of \mathcal{G}' consist of two types of vertices: configuration vertices $C = V \times 2^{[d]}$, and *intermediate* vertices $V \times C$. When \mathcal{M} is k -grabbing, configuration vertices include the remaining number of pawns that Player 1 can grab, as we elaborate below. The target vertices are $T' = \{\langle v, P \rangle : v \in T\}$. We describe E' next. For a configuration vertex $c = \langle v, P \rangle$, we define $c \in V'_1$ iff there exists $j \in P$ such that $v \in V_j$. That is, Player 1 moves from c in \mathcal{G}' iff he moves from c in \mathcal{G} . We define the neighbors of c to be the intermediate vertices $\{\langle v', c \rangle : v' \in N(v)\}$. That is, moving the token in \mathcal{G}' from c to $\langle v', c \rangle$ corresponds to moving the token from v to v' in \mathcal{G} . Moves from intermediate vertices represent an application of \mathcal{M} . We consider the following mechanisms.

Optional grabbing. For $i \in \{1, 2\}$, following a Player i move, Player $-i$ has the option to grab one of Player i 's pawns. Formally, for a configuration vertex $c = \langle v, P \rangle \in V'_1$, we have $N(c) \subseteq V'_2$. From $\langle v', c \rangle \in N(c)$, Player 2 has two options: (1) do not grab and proceed to $\langle v', P \rangle$, or (2) grab $j \in P$, and proceed to $\langle v', P \setminus \{j\} \rangle$. The definition for Player 2 is dual.

Always grabbing. For $i \in \{1, 2\}$, following a Player i move, Player $-i$ always has to grab one of Player i 's pawns. The formal definition is similar to optional grabbing with the difference that option (1) of not grabbing is not available to the players. We point out that Player $-i$ grabs only after Player i has moved, which in particular implies that Player i controls at least one pawn that Player $-i$ can grab.

Always grabbing or giving. Following a Player i move, Player $-i$ must either grab one of Player i 's pawns or *give* her a pawn. The formal definition is similar to always grabbing with the difference that, for an intermediate vertex $\langle v', \langle v, P \rangle \rangle$, there are both neighbors of the form $\langle v', P \setminus \{j\} \rangle$, for $j \in P$, and neighbors of the form $\langle v', P \cup \{j\} \rangle$, for $j \notin P$.

k -grabbing. After each round, Player 1 has the option of grabbing a pawn from Player 2, and at most k grabs are allowed in a play. A configuration vertex in k -grabbing is $c = \langle v, P, r \rangle$, where $r \in [k] \cup \{0\}$ denotes the number of grabs remaining. Intermediate vertices are Player 1 vertices. Let $\langle v', c \rangle \in V_1'$. Player 1 has two options: (1) do not grab and proceed to the configuration vertex $\langle v', P, r \rangle$, or (2) grab $j \notin P$, and proceed to $\langle v', P \cup \{j\}, r - 1 \rangle$ when $r > 0$. Note that grabs are not allowed when $r = 0$ and that Pawn j stays at the control of Player 1 for the remainder of the game.

Since pawn games are succinctly-represented turn-based games, Thm. 2.1 implies determinacy; namely, one of the players wins from each initial configuration. We study the problem of determining the winner of a pawn game, formally defined as follows.

Definition 2.2. Let $\alpha \in \{\text{OVPP}, \text{MVPP}, \text{OMVPP}\}$ and β be a pawn-grabbing mechanism. The problem $\alpha \beta$ PAWN-GAMES takes as input an $\alpha \beta$ pawn game \mathcal{G} and an initial configuration c , and the goal is to decide whether Player 1 wins from c in \mathcal{G} .

A naive algorithm to solve a pawn game applies attractor computation on the explicit turn-based game, which implies the following theorem.

Theorem 2.3. $\alpha \beta$ PAWN-GAMES is in EXPTIME, for all values of α and β .

3. OPTIONAL-GRABBING PAWN GAMES

Before describing our complexity results, we identify a somewhat unexpected property of MVPP optional-grabbing games. Consider a vertex v and two sets of pawns P and P' having $P' \subseteq P$. Intuitively, it is tempting to believe that Player 1 “prefers” configuration $c = \langle v, P \rangle$ over $c' = \langle v, P' \rangle$ since he controls more pawns in c . Somewhat surprisingly, the following theorem shows that the reverse holds (see also Ex. 1.1). More formally, the theorem states that if Player 1 wins from c , then he also wins from c' , under the restriction that if he makes the first move at c (i.e., he controls v in c), then he also makes the first move in c' (i.e., he controls v in c').

Theorem 3.1. Consider a configuration $\langle v, P \rangle$ of an MVPP optional-grabbing pawn game \mathcal{G} . Let $j \in [d]$ such that $v \in V_j$ and $P' \subseteq P$. Assuming that $j \in P$ implies $j \in P'$, if Player 1 wins from $\langle v, P \rangle$, he wins from $\langle v, P' \rangle$. Assuming that $j \in \overline{P'}$ implies $j \in \overline{P}$, if Player 2 wins from $\langle v, P' \rangle$, she wins from $\langle v, P \rangle$.

Proof. We prove for Player 1 and the proof for Player 2 follows from determinacy. Let \mathcal{G} , P , P' , $c = \langle v, P \rangle$, and $c' = \langle v, P' \rangle$ be as in the theorem statement. Let \mathcal{G}' be the turn-based game corresponding to \mathcal{G} . For $i \geq 0$, let W_i be the set of vertices in \mathcal{G}' from which Player 1

can win in at most i rounds (see Thm. 2.1). The following claim clearly implies the theorem. Its proof proceeds by a careful induction.

Claim: Configuration vertices: for $i \geq 0$, if $\langle v, P \rangle \in W_i$, then $\langle v, P' \rangle \in W_i$. Intermediate vertices: for $i \geq 1$ and every vertex $u \in N(v)$, if $\langle u, c \rangle \in W_{i-1}$, then $\langle u, c' \rangle \in W_{i-1}$.

For the base case, we prove for both $i = 0$ and $i = 1$. Recall that the target set of \mathcal{G}' , which coincides with W_0 , consists of configuration vertices whose V component is in T . Thus, for the first part of the claim, since $c \in W_0$, then $v \in T$, and thus $\langle v, P' \rangle \in W_0$. Recall that an intermediate vertex is of the form $\langle u, b \rangle$, where u denotes the “next” location of the token and b denotes the “previous” configuration. In the case of W_1 , since Player 1 wins in one turn, each vertex in W_1 is of the form $\langle u, b \rangle$, where $u \in T$. Thus, for $\langle u, c \rangle \in W_1$, we clearly have $\langle u, c' \rangle \in W_1$.

For the induction hypothesis, assume that the claim holds for $i = n$, and we prove for $i = n + 1$. We start with configuration vertices. Assume $c = \langle v, P \rangle \in W_{n+1}$. We will show that $c' = \langle v, P' \rangle \in W_{n+1}$. Recall that $N(c)$ consist of intermediate vertices of the form $\langle u, c \rangle$, where $u \in N(v)$, thus $\langle u, c \rangle \in N(c)$ iff $\langle u, c' \rangle \in N(c')$. Note that if $\langle u, c \rangle \in N(c)$ is winning for Player 1, i.e., $\langle u, c \rangle \in W_n$, then by the induction hypothesis, $\langle u, c' \rangle \in W_n$. We distinguish between two cases. In the first case, Player 1 controls c , i.e., $j \in P$ and c is a Player 1 vertex. In this case, our assumption is that c' is also a Player 1 vertex. Since $c \in W_{n+1}$, there must exist a neighbor $\langle u, c \rangle$ of c having $\langle u, c \rangle \in W_n$. By the above, $\langle u, c' \rangle \in (N(c') \cap W_n)$, thus $c' \in W_{n+1}$. In the second case, Player 2 controls c , thus $j \notin P$. Recall that $P' \subseteq P$, thus Player 2 controls c' as well. Note that $c \in W_{n+1}$ implies that Player 1 wins from all of its neighbors of the form $\langle u, c \rangle$. Now consider the contrapositive of the assumption that states that $j \notin P'$ implies that $j \notin P$ which holds here and since $\langle u, c \rangle \in W_n$, by induction hypothesis, it follows that $\langle u, c' \rangle \in W_n$. Thus, Player 1 wins from all the neighbors of c' , thus $c' \in W_{n+1}$.

We turn to the second part of the claim that addresses intermediate vertices. We again distinguish between two cases: $j \in P$ and $j \notin P$. Consider an intermediate vertex $\langle u, c \rangle \in W_{n+1}$. We will show that $\langle u, c' \rangle \in W_{n+1}$. We denote by ℓ , the pawn that owns u .

First consider the case $j \in P$. Recall that in optional grabbing, Player 2 has the option of grabbing after Player 1 moves, thus $\langle u, c \rangle$ is a Player 2 vertex. Since the claim requires that if $j \in P$, then $j \in P'$, we conclude that $\langle u, c' \rangle$ is also a Player 2 vertex. Consider a neighbor of $\langle u, c' \rangle$, a configuration vertex $\langle u, Q' \rangle$. Note that either Player 2 does not use the option to grab, then $Q' = P'$, or she grabs a pawn r from Player 1, then $Q' = P' \setminus \{r\}$. Note that in order to apply the induction hypothesis on $\langle u, Q' \rangle$, we need to find a neighbor $\langle u, Q \rangle$ of $\langle u, c \rangle$ such that if $\ell \notin Q'$, then $\ell \notin Q$. Note that at $\langle u, c \rangle$, Player 2 has the option of not grabbing as well as of grabbing ℓ , thus both $\langle u, P \rangle$ and $\langle u, P \setminus \{\ell\} \rangle$ are neighbors of $\langle u, c \rangle$. Note that $P = P \setminus \{\ell\}$ when $\ell \notin P$. That is, Player 2 cannot grab ℓ when she already owns it. Since $\langle u, c \rangle \in W_{n+1}$ and it is a Player 2 vertex, both $\langle u, P \rangle \in W_n$ and $\langle u, P \setminus \{\ell\} \rangle \in W_n$. Finally, if $\ell \in Q'$, define $Q := P$, and if $\ell \notin Q'$, define $Q := P \setminus \{\ell\}$. In both cases, since $Q' \subseteq P'$ and $P' \subseteq P$, we have $Q' \subseteq Q$ and meets the requirement in the claim on ℓ . Thus, since $\langle u, Q \rangle \in W_n$, by the induction hypothesis, $\langle u, Q' \rangle \in W_n$. Since $\langle u, Q' \rangle$ is any arbitrary neighbour of the Player 2 vertex $\langle u, c' \rangle$, we have that $\langle u, c' \rangle \in W_{n+1}$.

In the second case $j \notin P$. That is, Player 2 makes the move in $\langle v, P \rangle$, and thus $\langle u, c \rangle$ is a Player 1 vertex. Since $P' \subseteq P$, we have $j \notin P'$, thus $\langle u, c' \rangle$ is also a Player 1 vertex. Since $\langle u, c \rangle \in W_{n+1}$, it has a neighbor $\langle u, Q \rangle \in W_n$. Note that since Player 1 either grabs or does not grab in $\langle u, c \rangle$, we have $P \subseteq Q$. We find a neighbor $\langle u, Q' \rangle$ of $\langle u, c' \rangle$ to apply the induction hypothesis on. Note that Player 1 has the option to grab at $\langle u, c' \rangle$, and, among

his choices, he can choose not to grab or to grab ℓ . If $\ell \in Q$, then we choose $Q' := P' \cup \{\ell\}$, and if $\ell \notin Q$, we choose $Q' := P'$. In both cases, $Q' \subseteq Q$ and meets the requirement in the claim. Thus, since $\langle u, Q \rangle \in W_n$, by the induction hypothesis, we have $\langle u, Q' \rangle \in W_n$, and hence $\langle u, c' \rangle \in W_{n+1}$. \square

The following corollary of Thm. 3.1 shows that we can restrict attention to “locally-grabbing” strategies that only grab the pawn that owns the vertex on which the token is placed. In other words, a locally-grabbing strategy will not grab a pawn if the token is not placed on the vertex that it owns.

Corollary 3.2. *Consider an MVPP optional-grabbing game. Suppose that Player 1 controls $P \subseteq [d]$, and that Player 2 moves the token to a vertex v owned by Pawn j , i.e., $v \in V_j$. Player 1 has the option to grab. If Player 1 can win by grabbing a pawn $j' \neq j$, i.e., a pawn that does not own the next vertex, he can win by not grabbing at all. Formally, if Player 1 wins from $\langle v, P \cup \{j'\} \rangle$, he also wins from $\langle v, P \rangle$. And dually for Player 2.*

We point out that Thm. 3.1 and Cor. 3.2 do not hold for OMVPP optional-grabbing games.

3.1. OVPP: A PTIME algorithm. We turn to study complexity results, and start with the following positive result.

Theorem 3.3. *OVPP optional-grabbing PAWN-GAMES is in PTIME.*

Proof. We describe the intuition of the algorithm, the pseudo-code can be found in Alg. 1. Recall that in turn-based games (see Thm. 2.1), the attractor computation iteratively “grows” the set of states from which Player 1 wins: initially $W_0 = T$, and in each iteration, a vertex u is added to W_i if (1) u belongs to Player 2 and all its neighbors belong to W_i or (2) u belongs to Player 1 and it has a neighbor in W_i . In optional-grabbing games, applying attractor computation is intricate since vertex ownership is dynamic. Note that the reasoning behind (1) above holds; namely, if $N(u) \subseteq W_i$, no matter who controls u , necessarily W_i is reached in the next turn. However, the reasoning behind (2) fails. Consider a Player 1 vertex u that has two neighbors $v_1 \in W_i$ and $v_2 \notin W_i$. While u would be in W_{i+1} according to (2), under optional-grabbing, when Player 1 makes the move into u , Player 2 can avoid W_i by grabbing u and proceeding to v_2 .

In order to overcome this, our algorithm operates as follows. Vertices that satisfy (1) are added independent of their owner (Line 4). The counterpart of (2) can be seen as two careful steps of attractor computation. First, let B denote the *border* of W_i , namely the vertices who have a neighbor in W_i (Line 6). Second, a vertex u is in W_{i+1} in one of two cases. (i) $u \in B$ and all of its neighbors are in $B \cup W_i$ (Line 10). Indeed, if Player 1 controls u he wins by proceeding to W_i and if Player 2 owns u , she can avoid W_i by moving to B , then Player 1 grabs and proceeds to W_i . (ii) Player 2 controls u in the initial configuration and all of its neighbors are in B (Line 12). Indeed, Player 2 cannot avoid proceeding into B , and following Player 2’s move, Player 1 grabs and proceeds to W_i .

More formally, consider an input OVPP optional-grabbing game $\mathcal{G} = \langle V, E, T \rangle$ and an initial configuration $\langle v_0, P_0 \rangle$. Correctness of the algorithm follows from the two following claims. First, we show soundness; namely, whenever the algorithm returns that Player 1 wins, he does indeed have a winning strategy from $\langle v_0, P_0 \rangle$.

Claim 3.4. For $i \geq 0$, Player 1 wins from every vertex $v \in W_i$ no matter who makes the last step into v .

Proof. The proof is by induction on i . The base case is trivial since $W_0 = T$. For the inductive step, suppose that the claim holds for W_i and we show that it holds in all ways that it can be extended.

Case 1 – Line 4: for a vertex u having $N(u) \subseteq W_i$, Player 1 wins no matter who controls u since in the next turn, W_i is necessarily reached.

Case 2 – Line 10: We claim that Player 1 wins from $u \in B'$ no matter who controls u . Indeed, if Player 1 controls u , he wins by proceeding to W_i . If Player 2 controls u and avoids W_i , then the next vertex is necessarily in B . Player 1 grabs and proceeds to W_i .

Case 3 – Line 12: Let $v \in R \setminus P_0$. Player 1 can always ensure that Player 2 controls v by never grabbing v . Also in OVPP optional-grabbing, Player 1 does not need to grab v in order to reach v . Thus if Player 1 has a strategy such that the token reaches v , then he can ensure that the token reaches v and is controlled by Player 2. Hence, once v is reached, since it is controlled by Player 2, she is forced to move from v to a vertex v' in B . Player 1 then grabs v' unless he already controls it and moves the token to W_i . \square

We turn to prove completeness.

Claim 3.5. Suppose that the algorithm terminates at iteration i in Lines 7 or 13. Then, Player 2 has a winning strategy from $\langle v_0, P_0 \rangle$.

Proof. We first consider the case of terminating in Line 7. Thus, every vertex not in W_i , including v_0 , does not have a path to W_i . Clearly Player 2 wins.

Second, suppose that the algorithm terminates in Line 13. We describe a winning Player 2 strategy from $\langle v_0, P_0 \rangle$ that avoids W_i . Whenever Player 2 moves the token, she moves it to an arbitrary vertex not in $W_i \cup B$. When the token reaches $u \in B$, it is following a Player 1 move, and Player 2 grabs u , and moves to $W_i \cup B$.

We claim that Player 2 wins when the game starts from $\langle v_0, P_0 \rangle$. Assume towards contradiction that W_i is reached. Since $v_0 \notin W_i$, a visit to W_i must first visit a vertex $u \in B$. We claim that (i) u is in the control of Player 2 and that (ii) she can move away from W_i . Combining (i) and (ii) we reach a contradicting to the assumption that W_i is reached. We conclude the proof by showing that (i) and (ii) hold. For (ii), note that u has a neighbor not in $W_i \cup B$ otherwise u would have been added to W_{i+1} in Line 4 or Line 12. We turn to prove (i). Suppose first that $u = v_0$. If Player 1 controls v_0 , then the algorithm terminates in Line 8. Since it does not terminate there, Player 2 controls u . Next we consider the case that $u \neq v_0$, thus u has a predecessor u' in the play. We distinguish between two cases. If $u' \in R$, then $u' \in P_0$ otherwise it would have been added to W_{i+1} in Line 12. Thus, Player 1 makes the move from u' to u , and Player 2 grabs u following Player 1's move (if she does not control it already). The second case is when $u' \notin R$. Recall that a vertex is in R if all of its neighbors lead to B , thus $u' \notin R$ implies that it has at least one neighbor not in B . Note that if u' was in the control of Player 2, her strategy would have dictated to proceed to a vertex not in B . Thus, if Player 1 makes the move from u' to u , and Player 2 grabs u following Player 1's move.

Finally, note that the algorithm terminates once a fixed point is reached, thus it runs for at most $|V|$ iterations. \square

This concludes the proof of the theorem. \square

Algorithm 1 Given an OVPP optional-grabbing pawn game $\mathcal{G} = \langle V, E, T \rangle$ and an initial configuration $c = \langle v, P_0 \rangle$, determines which player wins \mathcal{G} from c .

```

1:  $W_0 = T, i = 0$ 
2: while True do
3:   if  $v_0 \in W_i$  then return Player 1
4:    $W_{i+1} = W_i \cup \{u : N(u) \subseteq W_i\}$ 
5:   if  $W_i \neq W_{i+1}$  then  $i := i + 1$ ; Continue
6:    $B := \{u : N(u) \cap W_i \neq \emptyset\}$ 
7:   if  $B = \emptyset$  then return Player 2
8:   if  $v_0 \in B$  and  $v_0 \in P_0$  then return Player 1
9:    $B' := \{u \in B : N(u) \subseteq B \cup W_i\}$ 
10:  if  $B' \neq \emptyset$  then  $W_{i+1} := W_i \cup B'$ ;  $i := i + 1$ ; Continue
11:   $R = \{u : N(u) \subseteq B\}$ 
12:  if  $R \setminus P_0 \neq \emptyset$  then  $W_{i+1} = W_i \cup (R \setminus P_0)$ ;  $i := i + 1$ 
13: else return Player 2

```

3.2. MVPP: EXPTIME-hardness via Lock & Key games. We prove hardness of MVPP optional-grabbing pawn games by reduction through a class of games that we introduce and call *Lock & Key* games, and may be of independent interest. A Lock & Key game is $\mathcal{G} = \langle V, E, T, L, K, \lambda, \kappa \rangle$, where $\langle V, E, T \rangle$ is a turn-based game, $L = \{\ell_1, \dots, \ell_n\}$ is a set of locks $K = \{k_1, \dots, k_n\}$ is a set of keys, each ℓ_j is associated to key $k_j \in K$ for $j \in [n]$, and each edge is labeled by a set of locks and keys respectively given by $\lambda : E \rightarrow 2^L$ and $\kappa : E \rightarrow 2^K$. Note that a lock and a key can appear on multiple edges.

Intuitively, a Lock & Key game is a turn-based game, only that the locks impose restrictions on the edges that a player is allowed to cross. Formally, a configuration of a Lock & Key game is $c = \langle v, A \rangle \in V \times 2^L$, meaning that the token is placed on v and each lock in A is *closed* (all other locks are *open*). When $v \in V_i$, for $i \in \{1, 2\}$, then Player i moves the token as in turn-based games with the restriction that he cannot choose an edge that is labeled by a closed lock, thus $e = \langle v, u \rangle \in E$ is a legal move at c when $\lambda(e) \subseteq (L \setminus A)$. Crossing e updates the configuration of the locks by “turning” all keys that e is labeled with. Formally, let $\langle u, A' \rangle$ be the configuration after crossing e . For $k_j \in \kappa(e)$ (“key k_j is turned”), we have $\ell_j \in A$ iff $\ell_j \notin A'$. For $k_j \notin \kappa(e)$ (“key k_j is unchanged”), we have $\ell_j \in A$ iff $\ell_j \in A'$.

Note that, similar to pawn games, each Lock & Key game corresponds to an exponentially sized two-player turn-based game. Thus, membership in EXPTIME is immediate. For the lower bound, we show a reduction for the problem of deciding whether an *alternating polynomial-space Turing machine* (ATM) accepts a given word.

Theorem 3.6. *Given a Lock & Key game \mathcal{G} and an initial configuration c , deciding whether Player 1 wins from c in \mathcal{G} is EXPTIME-complete.*

Proof. We briefly describe the syntax and semantics of ATMs, see for example [Sip13], for more details. An ATM is $\mathcal{A} = \langle Q, \Gamma, \delta, q_0, q_{acc}, q_{rej} \rangle$, where Q is a collection of states that is partitioned into $Q = Q_1 \cup Q_2$ owned by Player 1 and Player 2 respectively, Γ is a tape alphabet, $\delta : Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}$ is a transition function, $q_0, q_{acc}, q_{rej} \in Q$ are respectively an initial, accepting, and rejecting states. A configuration of \mathcal{A} is $c = \langle q, i, \langle \gamma_1, \dots, \gamma_m \rangle \rangle$, meaning that the control state is q , the head position is i , and $\langle \gamma_1, \dots, \gamma_m \rangle$ is the tape content, where m is polynomial in the length of the input word w . In order to determine

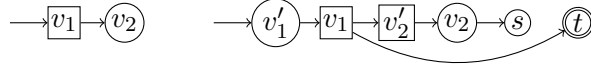


Figure 2: From turn-based to optional-grabbing games.

whether \mathcal{A} accepts w we construct a (succinctly-represented) turn-based game over the possible configurations of \mathcal{A} , the neighbors of a configuration are determined according to δ , and, for $i \in \{1, 2\}$, Player i moves from states in Q_i . We say that \mathcal{A} accepts w iff Player 1 has a winning strategy from the initial configuration for the target state q_{acc} .

Given \mathcal{A} and w , we construct a Lock & Key game $\mathcal{G} = \langle V, E, T, L, K, \lambda, \kappa \rangle$ and an initial configuration $\langle v_0, A \rangle$ such that Player 1 wins from $\langle v, A \rangle$ in \mathcal{G} iff w is accepted by \mathcal{A} . The vertices of \mathcal{G} consist of *main* and *intermediate* vertices. Consider a configuration $c = \langle q, i, \langle \gamma_1, \dots, \gamma_m \rangle \rangle$ of \mathcal{A} . We simulate c in \mathcal{G} using $c' = \langle v, A \rangle$ as follows. First, the main vertices are $Q \times \{1, \dots, m\} \times \Gamma$ and keep track of the control state and position on the tape. The main vertex that simulates $c = \langle q, i, \langle \gamma_1, \dots, \gamma_m \rangle \rangle$ is $v = \langle q, i, \gamma_i \rangle$. We define $v \in V_i$ iff $q \in Q_i$. Second, we use locks to keep track of the tape contents. For each $1 \leq i \leq m$ and $\gamma \in \Gamma$, we introduce a lock $\ell_{i, \gamma}$. Then, in the configuration $c' = \langle v, A \rangle$ that simulates c , the only locks that are open are ℓ_{i, γ_i} , for $i \in \{1, \dots, m\}$. Next, we describe the transitions, where intermediate vertices are used for book-keeping. The neighbors of a main vertex v are the intermediate vertices $\{\langle v, t \rangle : t \in \delta(q, \gamma)\}$, where a transition of \mathcal{A} is $t = \langle q', \gamma', B \rangle$, meaning that the next control state is q' , the tape head moves to $i + 1$ if $B = R$ and to $i - 1$ if $B = L$, and the i -th tape content changes from γ to γ' . We update the state of the locks so that they reflect the tape contents: for the edge $\langle v, \langle v, t \rangle \rangle$, we have $\kappa(\langle v, \langle v, t \rangle \rangle) = \{k_{i, \gamma}, k_{i, \gamma'}\}$. That is, traversing the edge turn the keys to close $\ell_{i, \gamma}$ and open $\ell_{i, \gamma'}$. The neighbors of $\langle v, t \rangle$ are main vertices having control state q' and head position i' . Recall that the third component of a main vertex is the tape content at the current position. We use the locks' state to prevent moving to main vertices with incorrect tape content: outgoing edges from $\langle v, t \rangle$ are of the form $\langle \langle v, t \rangle, \langle q', i', \gamma'' \rangle \rangle$ and is labeled by the lock $\ell_{i', \gamma''}$. That is, the edge can only be traversed when the i' -th tape position is γ'' . It is not hard to verify that there is a one-to-one correspondence between runs of \mathcal{A} and plays of \mathcal{G} . Thus, Player 1 forces \mathcal{A} to reach a configuration with control state q_{acc} iff Player 1 forces to reach a main vertex with control state q_{acc} . Note that the construction is clearly polynomial since \mathcal{G} has $|Q| \cdot m \cdot |\Gamma|$ main vertices. \square

3.2.1. *From Lock & Key games to optional-grabbing pawn games.* Throughout this section, fix a Lock & Key game \mathcal{G} and an initial configuration c . We construct an optional-grabbing pawn game \mathcal{G}' over a set of pawns $[d]$, and identify an initial configuration c' such that Player 1 wins in \mathcal{G} from c iff Player 1 wins from c' in \mathcal{G}' .

From turn-based games to optional-grabbing games. In this section, we consider the case in which \mathcal{G} has no keys or locks, thus \mathcal{G} is a turn-based game. The reduction is depicted in Fig. 2. Denote the turn-based game $\mathcal{G} = \langle V, E, T \rangle$ with $V = V_1 \cup V_2$ and initial vertex v_0 . We construct an OVPP optional-grabbing $\mathcal{G}' = \langle V', E', T' \rangle$, where $V' = V \cup \{v' : v \in V\} \cup \{s, t\}$. We add edges to ensure that the player who owns a vertex $v \in V$ is the player who moves from v in \mathcal{G}' : we have $\langle v', v \rangle \in E'$, and if $v \in V_1$, then $\langle v, s \rangle \in E'$, and if $v \in V_2$, then $\langle v, t \rangle \in E'$. We redirect each edge $\langle u, v \rangle$ in \mathcal{G} to $\langle u, v' \rangle$ in \mathcal{G}' . Intuitively, for $v \in V_1$, a

Player 1 winning strategy will guarantee that v' is always in the control of Player 2, and following her move at v' , Player 1 must grab v otherwise Player 2 wins and choose the next location. And dually for $v \in V_2$. Let $V'_1 = V_1 \cup \{v' : v \in V_2\}$, the initial configuration of \mathcal{G}' is $\langle v_0, V'_1 \rangle$, that is Player 2 controls $V_2 \cup \{v' : v \in V_1\}$. We thus have the following:

Lemma 3.7. *For a turn-based game \mathcal{G} , Player 1 wins \mathcal{G} from a vertex $v_0 \in V$ iff Player 1 wins the optional-grabbing game \mathcal{G}' from configuration $\langle v_0, V'_1 \rangle$.*

Proof. Let f be a Player 1 winning strategy in \mathcal{G} . We describe a Player 1 winning strategy f' in \mathcal{G}' . The proof is dual when Player 2 wins in \mathcal{G} . Player 1's strategy f' simulates the execution of f on \mathcal{G} so that when the token is placed on $v \in V$ in \mathcal{G}' it is also placed on v in \mathcal{G} . Moreover, if $v \in V_1$ in \mathcal{G} , then Player 1 controls v in \mathcal{G}' .

Initially, the invariant clearly holds. Suppose that Player 2 is playing \mathcal{G}' according to some strategy g' . We show that either f' wins by reaching t , or that the invariant is maintained. Suppose that the token is placed on $v \in V$. We distinguish between three cases. (1) If $v \in V_2$ and Player 1 owns v in \mathcal{G}' , then he moves to t to win the game. (2) Suppose that $v \in V_2$, that Player 2 owns v , and that she moves to u' . The simulation in \mathcal{G} proceeds to vertex u . In order to maintain the second part of the invariant, Player 1 does not grab u' but grabs u when $u \in V_1$, which is possible because in such a case we define $u' \in V'_2$. (3) When $v \in V_1$, the invariant implies that Player 1 controls v in \mathcal{G}' . Player 1 then copies the move of f in \mathcal{G} ; namely, let $u = f(v)$, then $u' = f'(v)$. The move from u' is as in case (2).

Let v_0, v_1, \dots be the play in \mathcal{G} . Clearly, if case (1) occurs at some point, Player 1 wins \mathcal{G}' . Assume that it does not occur. Then, the invariant implies that the play in \mathcal{G}' is $v_0, v'_1, v_1, v'_2, v_2, \dots$. Since f is winning in \mathcal{G} , there is an $i \geq 0$ such that $v_i = t$, thus the play is winning in \mathcal{G}' as well. \square

Gadgets for simulating locks and keys. The core of the reduction is to construct gadgets that simulate locks and keys. Let \mathcal{G}' denote the optional-grabbing pawn game that we construct, and let $[d]$ denote its set of pawns. For each lock $\ell \in L$ and its corresponding key $k \in K$, we construct gadgets \mathcal{G}_ℓ and \mathcal{G}_k that simulate the operations of ℓ and k in \mathcal{G}' . The gadgets in two *states* are depicted in Fig. 3. We highlight three pawns colored blue, green, and red, respectively owning, $\{v_1^\ell, v_1^k\}$, $\{v_2^\ell, v_2^k, v_7^k, v_8^k\}$, and $\{v_{\text{in}}^k, v_4^k, v_5^k, v_6^k\}$. Each of the other vertices (colored white) is owned by a fresh pawn. Intuitively, for each lock ℓ , we identify two sets $\mathcal{P}_O^\ell, \mathcal{P}_C^\ell \subseteq 2^{[d]}$, respectively representing an open and closed state of ℓ . We will ensure that when entering and exiting a gadget, the configuration is in one of these sets. When the set of pawns that Player 1 controls is in \mathcal{P}_O^ℓ and \mathcal{P}_C^ℓ , we respectively say that \mathcal{G}_ℓ is in open and closed state, and similarly for \mathcal{G}_k as stated below. We define $\mathcal{P}_O^\ell = \{P \in 2^{[d]} : v_1^\ell \notin P \wedge v_2^\ell \in P\}$ and $\mathcal{P}_C^\ell = \{P \in 2^{[d]} : v_1^\ell \in P \wedge v_2^\ell \notin P\}$. Formally, we define $\mathcal{P}_O^\ell = \{P \in 2^{[d]} : v_1^\ell \notin P \wedge v_2^\ell \in P\}$ and $\mathcal{P}_C^\ell = \{P \in 2^{[d]} : v_1^\ell \in P \wedge v_2^\ell \notin P\}$.

Lemma 3.8. *Let $i \in \{1, 2\}$. An open lock stays open: If Player i enters \mathcal{G}_ℓ in \mathcal{P}_O^ℓ , then he has a strategy that guarantees that either he wins \mathcal{G}' or \mathcal{G}_ℓ is exited in \mathcal{P}_O^ℓ . A closed lock cannot be crossed: If Player i enters \mathcal{G}_ℓ in \mathcal{P}_C^ℓ , then Player $-i$ has a strategy that guarantees that Player i loses \mathcal{G}' .*

Proof. We prove for Player 1 and the proof is dual for Player 2. First, suppose Player 1 enters \mathcal{G}_ℓ in \mathcal{P}_O^ℓ . Player 2 may or may not grab v_{in}^ℓ , and the game can proceed to either v_1^ℓ or v_2^ℓ . We argue that if the game proceeds to v_1^ℓ , then Player 1 will not grab v_1^ℓ . We can also similarly show that if the game proceeds to v_2^ℓ , then Player 2 will not grab v_2^ℓ . Player 2

controls v_1^ℓ . We claim that if Player 1 grabs v_1^ℓ , he will lose the game. Indeed, following Player 1's move in v_1^ℓ , Player 2 will grab v_3^ℓ and move the token to the sink vertex s to win the game. Thus, Player 1 does not grab v_1^ℓ and keeps it in the control of Player 2. Following Player 2's move in v_3^ℓ , Player 1 grabs v_2^ℓ and proceeds to exit \mathcal{G}_ℓ . Note that when \mathcal{G}_ℓ is exited, Player 1 maintains control of v_2^ℓ and Player 2 maintains control of v_1^ℓ , thus the configuration is in \mathcal{P}_O^ℓ . Second, suppose that Player 1 enters \mathcal{G}_ℓ in \mathcal{P}_C^ℓ . Then, Player 2 grabs v_{in}^ℓ and moves the token to v_1^ℓ . Since Player 1 controls v_1^ℓ he must make the next move. Player 2 then grabs v_3^ℓ and moves the token to s to win the game. \square

Next, we present the gadget \mathcal{G}_k for simulating the operation of a key k (see Fig. 3). Intuitively, we maintain that \mathcal{G}_k is in open state iff \mathcal{G}_ℓ is in open state, and traversing \mathcal{G}_k swaps the state of both. We define sets of configurations $\mathcal{P}_O^k = \{P \in 2^{[d]} : \{v_{in}^k, v_1^k, v_4^k, v_5^k, v_6^k\} \cap P = \emptyset \wedge \{v_2^k, v_7^k, v_8^k\} \subseteq P\}$ and $\mathcal{P}_C^k = \{P \in 2^{[d]} : \{v_{in}^k, v_1^k, v_4^k, v_5^k, v_6^k\} \subseteq P \wedge \{v_2^k, v_7^k, v_8^k\} \cap P = \emptyset\}$ (see Fig. 3). Note that $\mathcal{P}_O^k \subseteq \mathcal{P}_O^\ell$ and $\mathcal{P}_C^k \subseteq \mathcal{P}_C^\ell$ since v_i^k and v_i^ℓ are owned by the same pawn for $i \in [2]$.

Lemma 3.9. *Turning k closes an open ℓ : Let $i \in \{1, 2\}$. If Player i enters \mathcal{G}_k in \mathcal{P}_O^k , then he has a strategy that ensures that either Player i wins \mathcal{G}' or \mathcal{G}_k is exited in \mathcal{P}_C^k . Turning k opens a closed ℓ : when Player i enters \mathcal{G}_k in \mathcal{P}_C^k , Player i ensures that either he wins \mathcal{G}' or \mathcal{G}_k is exited in \mathcal{P}_O^k .*

Proof. We depict \mathcal{G}_k in two configurations in Fig. 3. The vertices v_3^k and v_{out}^k are controlled by one-vertex pawns. The rest of the vertices in \mathcal{G}_k , not including targets, are controlled by three pawns and are depicted using colors. Vertex v_1^k is controlled by a “blue” pawn, who also owns v_1^ℓ . Vertices v_2^k, v_7^k, v_8^k are all owned by a “green” pawn who also controls v_2^ℓ . The other vertices are owned by a “red” pawn.

We simulate the properties of a key using the configurations, where we prove the claim for an entry configuration in \mathcal{P}_C^k and the proof for \mathcal{P}_O^k is dual. We claim that for $i \in \{1, 2\}$, if the token lands on v_2^k when Player i controls that vertex, then Player i loses. Indeed, following Player i 's move at v_2^k , Player $-i$ grabs v_3^k and directs the token to the winning vertex. It follows that Player 1 does not grab v_{in}^k upon entry to \mathcal{G}_k and allows Player 2 to move. Following Player 2's move at v_{in}^k , Player 1 grabs v_1^k , and proceeds to v_4^k . Since Player 1 moves and v_4^k is in the control of Player 2, no change of pawns occurs. We claim that Player 2 now proceeds to v_5^k . Indeed, if she proceeds to v_6^k , Player 1 will grab v_6^k and win by moving to t . Observe that Player 1 grabs the red pawn since it controls v_5^k following Player 2's move from v_4^k to v_5^k . Indeed, otherwise Player 2 proceeds to s to win the game. Finally, following Player 1's move from v_5^k , Player 2 must grab the green pawn since it grabs v_7^k , otherwise Player 1 moves from v_7^k to t to win the game. To conclude, if the token exits \mathcal{G}_k , Player 1 has grabbed the blue and red pawns and Player 2 has grabbed the green pawn. The configuration upon exiting \mathcal{G}_k is thus in \mathcal{P}_O^k , and we are done. \square

Putting it all together. We describe the construction of a pawn game \mathcal{G}' from a Lock & Key game \mathcal{G} . We assume w.l.o.g. that each edge $\langle u, v \rangle$ in \mathcal{G} is labeled by at most one lock or key since an edge that is labeled by multiple locks or keys can be split into a chain of edges, each labeled by a single lock or a key. We describe the construction of \mathcal{G}' . We first apply the construction for turn-based games on \mathcal{G} while “ignoring” the locks and keys. Recall that the construction introduces fresh vertices so that an edge $e = \langle u, v \rangle$ in \mathcal{G} is mapped to an edge

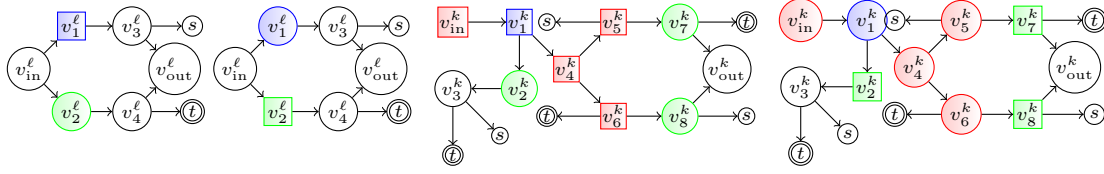


Figure 3: From left to right: \mathcal{G}_ℓ in open and closed state and \mathcal{G}_k in open and closed state.

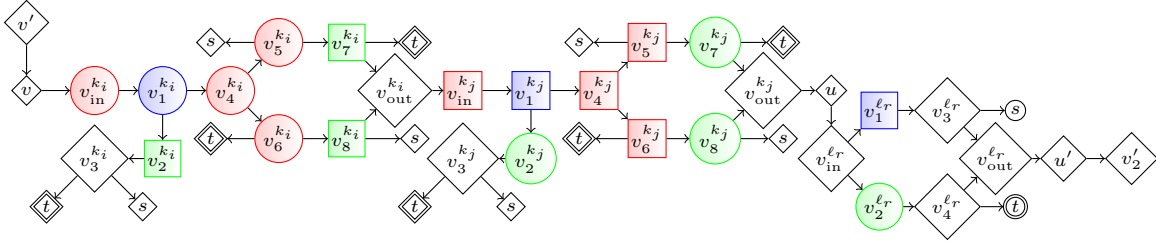


Figure 4: A δ -path is a path between two primed main vertices in an optional- or always-grabbing game, and it crosses two key gadgets and one lock gadget.

$e' = \langle u', v \rangle$ in \mathcal{G}' . We re-introduce the locks and keys so that the labeling of e' coincides with the labeling of e . Next, we replace an edge e' that is labeled by a lock ℓ , by a copy of \mathcal{G}_ℓ , and if e is labeled by a key k , we replace e' by a copy of \mathcal{G}_k . Note that multiple edges could be labeled by the same lock ℓ . In such a case we use fresh vertices in each copy of \mathcal{G}_ℓ , but crucially, all gadgets share the same pawns so that they share the same state. And similarly for keys. For an illustration of this construction, see Fig. 4, which applies the construction on a Lock & Key game that is output from the reduction in Thm. 3.6.

Finally, given an initial configuration $c = \langle v, A \rangle$ of \mathcal{G} we define an initial configuration $c' = \langle v, P \rangle$ of \mathcal{G}' . Note that the initial vertex is the entry point of the gadget that simulates v in \mathcal{G}' . For each lock ℓ and corresponding key k , if ℓ is open according to A , then $P \in \mathcal{P}_O^\ell$, i.e., both \mathcal{G}_ℓ and \mathcal{G}_k are initially in open state. And similarly when ℓ is closed according to A . Combining the properties in Lemmas 3.7, 3.8, and 3.9 implies that Player 1 wins \mathcal{G} from c iff Player 1 wins \mathcal{G}' from c' . Thus, by Thm. 3.6, we have the following.

Theorem 3.10. *MVPP optional-grabbing PAWN-GAMES is EXPTIME-complete.*

4. ALWAYS-GRABBING PAWN GAMES

In this section, we study always-grabbing pawn games and show that MVPP always-grabbing pawn-games are EXPTIME-complete. The main challenge is proving the lower bound. We proceed as follows. Let \mathcal{M} be an ATM. Apply the reduction in Thm 3.6 and the one in Thm. 3.10 to obtain pairs $\langle \mathcal{G}, c \rangle$ and $\langle \mathcal{G}', c' \rangle$, where \mathcal{G} and \mathcal{G}' are respectively Lock & Key and optional-grabbing games with initial configurations c and c' respectively. We devise a construction that takes $\langle \mathcal{G}', c' \rangle$ and produces an always-grabbing game \mathcal{G}'' and a configuration c'' such that Player 1 wins from c'' in \mathcal{G}'' iff he wins from c in \mathcal{G} .

Our analysis heavily depends on the special structure of \mathcal{G}' . The construction in Thm. 3.6 outputs a game \mathcal{G} with main vertices of the form $\langle q, i, \gamma \rangle$ (q is a state, i is a tape position, and γ is a letter in the tape alphabet). A play of \mathcal{G} can be partitioned into paths between main

vertices. Each such path corresponds to one transition of the Turing machine and traverses two keys and a lock before again reaching a main vertex. Recall that when constructing \mathcal{G}' from \mathcal{G} , we replace locks and keys with their respective gadgets, and for every vertex v that belongs to \mathcal{G} , we add a new primed vertex v' such that if v is controlled by Player i then v' is controlled by Player $-i$. We call a path in \mathcal{G}' that corresponds to a path in \mathcal{G} between two successive main vertices, say v and v' , a δ -path. Fig. 4 depicts a δ -path. The δ -path shown here is along a closed key k_i , an open key k_j and an open lock ℓ_r such that $r \neq i$. An open key represents that the corresponding lock is currently open, and going through this key closes the lock and also the state of the key changes from open to closed. Similarly, a closed key represents that the corresponding lock is currently closed, and going through this key opens the lock and also the state of the key changes from closed to open. Recall from the proof of Thm 3.6, that we go through the open key for lock $\ell_{i,\gamma'}$ and the closed key for lock $\ell_{i,\gamma}$ and finally, the open lock $\ell_{i',\gamma''}$. For simplicity of notation, here, we refer to the keys corresponding to the locks $\ell_{i,\gamma'}$ and $\ell_{i,\gamma}$ as k_i and k_j respectively, while we denote by ℓ_r the lock $\ell_{i',\gamma''}$. Recall from Section 3.2 that the gadget for Key k_m and Lock ℓ_m mainly have vertices owned by three pawns $p_{Red_m}, p_{Blue_m}, p_{Green_m}$. p_{Red_m} owns vertices $\{v_{in}^{k_m}, v_4^{k_m}, v_5^{k_m}, v_6^{k_m}\}$, pawn p_{Blue_m} owns $\{v_1^{k_m}, v_1^{l_m}\}$, and pawn p_{Green_m} owns $\{v_2^{k_m}, v_7^{k_m}, v_8^{k_m}, v_2^{l_m}\}$. In this δ -path we have pawns $p_{Red_m}, p_{Blue_m}, p_{Green_m}$ for $m = i, j, r$ (The pawns $p_{Blue_i}, p_{Blue_j}, p_{Blue_r}$ are different pawns even though the vertices owned by them have the same colour in a δ -path and the same holds for vertices belonging to pawns p_{Red} and p_{Green}). The ownership of the diamond vertices is not fixed; they can be either controlled by Player 1 or Player 2.

An important property of the specific optional-grabbing game \mathcal{G}' that is constructed in Thm 3.6 from an ATM is that every play of \mathcal{G}' consists of a sequence of δ -paths. The following observation can easily be verified:

Observation 4.1. A δ -path from v' to v'_2 consists of 20 turns.

The following lemma is crucial for the construction of \mathcal{G}'' . The proof of this lemma is obtained by proving several claims made below.

Lemma 4.2. *For $i \in \{1, 2\}$, if Player i has a strategy in the optional-grabbing game \mathcal{G}' to cross a δ -path from v' to v'_2 , then Player i has a strategy that moves the token in at least 10 rounds and Player $-i$ moves the token in at most 10 rounds in the δ -path.*

Proof. We will prove this lemma for $i = 1$ that is for Player 1. The proof for Player 2 is similar. This follows from the fact that in Figure 4, the vertices controlled initially by Player 1 in Key k_i have their corresponding vertices in Key k_j that are initially controlled by Player 2. Further, the gadget for the Lock ℓ_r is symmetric for both players.

In order to prove the lemma we first prove some claims. Note that in all these claims, we talk about a particular player controlling a vertex, for example, in Claim 4.3, we say that Player 2 controls $v_1^{l_r}$, because Player 1 can ensure that the game reaches such configuration from vertex v' . We can indeed show that Player 1 has a strategy such that in the δ -path, Player 2 controls vertex $v_1^{l_r}$ when the token reaches this vertex. Refer to Figure 4 in the following claims.

Claim 4.3. If Player 2 controls vertex $v_1^{l_r}$ then Player 1 has a strategy from $v_1^{l_r}$ to reach v'_2 which ensures that out of the 4 rounds that moves the token to v'_2 , he moves the token in at least 2 rounds.

Proof. Suppose the token is at vertex v_1^{lr} and Player 2 controls v_1^{lr} . Then Player 2 moves the token from v_1^{lr} to v_3^{lr} . Player 1 can now grab a pawn. If Player 2 controls vertex v_3^{lr} , Player 1 grabs the pawn owning this vertex else he chooses to not grab a pawn. Note that under both the cases by the end of this round Player 1 controls vertex v_3^{lr} . He now moves the token to v_{out}^{lr} . Note that till now Player 1 has moved the token in one round. Now by the end of this round there are two possible cases:

- (1) Player 1 controls v_{out}^{lr} . In this case Player 1 moves the token to u' , and thus it Player 1 has moved the token in two rounds, and hence regardless of what happens in next round the claim holds.
- (2) Player 2 controls v_{out}^{lr} . In this case Player 2 moves the token to u' and now Player 1 can control u' and move the token to v_2' , which will be his second instance of moving the token and thus the claim holds.

Hence showed. □

Claim 4.4. If Player 1 controls vertex v_2^{lr} then he has a strategy from v_2^{lr} to reach v_2' which ensures that, out of the 4 rounds that moves the token from v_2^{lr} to v_2' , he moves the token in at least 2 rounds.

Proof. Suppose the token is at vertex v_2^{lr} and Player 1 controls v_2^{lr} . Then Player 1 moves the token from v_2^{lr} to v_4^{lr} . Now if Player 1 has the pawn owning vertex v_4^{lr} , then Player 2 is forced to grab this pawn in this round. Thus by the end of this round Player 2 controls v_4^{lr} . Now Player 2 moves the token from v_4^{lr} to v_{out}^{lr} . Player 1 now can grab a pawn. If Player 2 controls vertex v_{out}^{lr} , Player 1 grabs the pawn owning this vertex else he chooses to not grab a pawn. Note under both the cases by the end of this round Player 1 controls vertex v_{out}^{lr} . He now moves the token to u' , which will be his second instance of moving the token and thus the claim holds. □

Claim 4.5. If Player 1 controls vertex v_{out}^{kj} then he has a strategy from v_{out}^{kj} to reach v_2' which ensures that, out of the 7 rounds that moves the token from v_{out}^{kj} to v_2' he moves the token in at least 4 rounds.

Proof. Suppose the token is at vertex v_{out}^{kj} and Player 1 controls v_{out}^{kj} . Then Player 1 moves the token from v_{out}^{kj} to u . Now by the end of this round there are two possible cases:

- (1) Player 1 controls u . In this case Player 1 moves the token to v_{in}^{lr} . Note that till now Player 1 has moved the token in two rounds. Now again by the end of this round there are two possible cases:
 - (a) Player 1 controls v_{in}^{lr} . In this case Player 1 moves the token to v_1^{lr} . Now note that the token is at vertex v_1^{lr} and Player 2 controls v_1^{lr} . Thus by Claim 4.3 we know that from here Player 1 moves the token in at least 2 rounds before reaching v_2' . As till now when the token is at v_1^{lr} Player 1 has moved the token in three rounds, thus overall under this case Player 1 moves the token in at least 5 rounds.
 - (b) Player 2 controls v_{in}^{lr} . In this case Player 2 either moves the token to the vertex v_1^{lr} in which case Player 1 chooses to not grab. Now note that the token is at vertex v_1^{lr} and Player 2 controls v_1^{lr} . Thus by Claim 4.3 we know that from here Player 1 moves the token in at least 2 rounds before reaching v_2' . Otherwise suppose from v_{in}^{lr} Player 2 moves the token to vertex v_2^{lr} , in which case Player 1 chooses to not grab. Now note that here the token is at vertex v_2^{lr} and Player 1 controls v_2^{lr} . Thus

by Claim 4.4 we know that from here Player 1 moves the token in at least 2 rounds before reaching v'_2 . Thus in this case from v_{in}^{lr} Player 1 moves the token in at least two rounds. As till reaching v_{in}^{lr} Player 1 has moved the token in two rounds, thus overall under this case Player 1 moves the token in at least 4 rounds as the token reaches v'_2 .

- (2) Player 2 controls u . In this case Player 2 moves the token to v_{in}^{lr} . Player 1 now can grab a pawn. If Player 2 controls the vertex v_{in}^{lr} , Player 1 grabs the pawn owning this vertex else he choose to not grab a pawn. Note under both the cases by the end of this round Player 1 controls vertex v_{in}^{lr} . He now moves the token to v_1^{lr} . Now note that the token is at vertex v_1^{lr} and Player 2 controls v_1^{lr} thus by Claim 4.3 we know that from here Player 1 moves the token in at least two rounds before reaching v'_2 . As till reaching v_1^{lr} Player 1 has moved the token in two rounds, thus overall under this case Player 1 moves the token in at least 4 rounds.

Hence showed. □

Claim 4.6. If Player 2 controls vertex v_{in}^{kj} , then Player 1 has a strategy from v_{in}^{kj} to reach v'_2 which ensures that, out of the 12 rounds that moves the token from v_{in}^{kj} to v'_2 he moves the token in at least 6 rounds.

Proof. Suppose the token is at vertex v_{in}^{kj} and Player 2 controls v_{in}^{kj} . Then Player 2 moves the token from v_{in}^{kj} to v_1^{kj} . In this round Player 1 grabs pawn p_{Blue_j} in order to control v_1^{kj} . Now Player 1 controls v_1^{kj} , he moves the token to vertex v_4^{kj} . In the next round Player 2 moves the token from v_4^{kj} to v_5^{kj} . Player 1 in this round grabs p_{Red_j} in order to control v_5^{kj} . Now Player 1 moves the token from v_5^{kj} to v_7^{kj} . Here Player 2 grabs p_{Green_j} in order to control v_7^{kj} . Now Player 2 moves the token from v_7^{kj} to v_{out}^{kj} . Player 1 now can grab a pawn. If Player 2 controls vertex v_{out}^{kj} , Player 1 grabs the pawn owning this vertex else he choose to not grab a pawn. Note under both the cases by the end of this round Player 1 controls vertex v_{out}^{kj} . Now note that the token is at vertex v_{out}^{kj} and Player 1 controls v_{out}^{kj} . Thus by Claim 4.5 we know that from here Player 1 moves the token in at least 4 rounds before reaching v'_2 . As till reaching v_{out}^{kj} Player 1 has moved the token in two rounds, thus overall under this case Player 1 moves the token in at least 6 rounds. □

Claim 4.7. If Player 1 controls vertex v_{in}^{ki} , then he has a strategy from v_{in}^{ki} to reach v'_2 which ensures that, out of the 18 rounds that moves the token to v'_2 he moves the token in at least 9 rounds.

Proof. Player 1 controls vertex v_{in}^{ki} , he moves the token to v_1^{ki} , now Player 2 grabs p_{Blue_i} in order to control v_1^{ki} . In the next round Player 2 moves the token to v_4^{ki} . In this round Player 1 does not grab a pawn. Now since p_{Red_i} belongs to Player 1 and p_{Red_i} owns v_4^{ki} , Player 1 moves the token to v_6^{ki} . In this round Player 2 is forced to grab p_{Red_i} in order to control v_6^{ki} . Now since here Player 2 controls v_6^{ki} , she moves the token to v_8^{ki} . Now Player 1 grabs p_{Green_i} and controls v_8^{ki} . In the next round Player 1 moves the token to v_{out}^{ki} . Now observe that till now Player 1 has moved the token in three rounds. Now by the end of this round there two possible cases:

- (1) Player 2 controls $v_{out}^{k_i}$. In this case Player 2 moves the token to $v_{in}^{k_j}$, Player 1 here chooses to not grab a pawn. Now observe that the token is at vertex $v_{in}^{k_j}$ and Player 2 controls vertex $v_{in}^{k_j}$ thus by claim 4.6 we know that from here Player 1 moves the token in at least 6 rounds before reaching v'_2 . As until reaching $v_{in}^{k_j}$ Player 1 has moved the token in three rounds, thus overall under this case Player 1 moves the token in at least 9 rounds.
- (2) Player 1 controls $v_{out}^{k_i}$. In this case Player 1 moves the token to $v_{in}^{k_j}$. Now note that Player 2 controls $v_{in}^{k_j}$. Thus the token is at the vertex $v_{in}^{k_j}$ and Player 2 controls vertex $v_{in}^{k_j}$. Now by Claim 4.6, we know that from here Player 1 moves the token in at least 6 rounds before reaching v'_2 . As until reaching $v_{in}^{k_j}$ Player 1 has moved the token in four rounds, thus overall under this case Player 1 moves the token in at least 10 rounds.

Hence showed. \square

Now let us prove the lemma. So the token is at vertex v' . There are two possible cases:

- (1) Player 1 controls v' . In this case Player 1 moves the token to v . Now regardless of who controls, v the token reaches $v_{in}^{k_i}$ with Player 1 controlling vertex $v_{in}^{k_i}$. Note that Player 2 will not control $v_{in}^{k_i}$ as he would like to control $v_1^{k_i}$. As till reaching $v_{in}^{k_i}$ Player 1 moved the token in atleast one round and by Claim 4.7 we know that from here Player 1 moves the token in at least 9 rounds before reaching v'_2 , thus overall Player 1 moves in at least 10 rounds before reaching v'_2 .
- (2) Player 2 control v' . In this case Player 2 moves the token to v . Player 1 now can grab a pawn. If Player 2 controls vertex v , Player 1 grabs the pawn owning vertex v else he chooses to not grab a pawn. Note under both the cases by the end of this round Player 1 controls the vertex v' . He now moves the token to $v_{in}^{k_i}$. Again note that Player 2 will not control $v_{in}^{k_i}$ as he would like to control $v_1^{k_i}$. As until reaching $v_{in}^{k_i}$ Player 1 moved the token in one round and by Claim 4.7 we know that from here Player 1 moves the token in at least 9 rounds before reaching v'_2 . Thus overall Player 1 moves in at least 10 rounds before reaching v'_2 .

This concludes the proof of the lemma. \square

Let $\mathcal{G}' = \langle V', E', T' \rangle$ with d pawns. The game \mathcal{G}'' is constructed from \mathcal{G}' by adding $2(d + 10)$ fresh isolated vertices each owned by a fresh unique pawn. Formally, $\mathcal{G}'' = \langle V'', E', T' \rangle$, where $V'' = V' \cup \{v_1, v_2, \dots, v_{2(d+10)}\}$ such that $v_j \notin V'$, for $j \in [2(d + 10)]$. Consider a configuration $c' = \langle v, P \rangle$ in \mathcal{G}' . Let $c'' = \langle v, P \cup \{1, 2, \dots, d + 10\} \rangle$ be a configuration in \mathcal{G}'' . Note that Lemma 4.2 also applies to the always-grabbing game \mathcal{G}'' , and we get the following.

Corollary 4.8. *For $i \in \{1, 2\}$, if Player i has a strategy in the always-grabbing game \mathcal{G}'' to cross a δ -path from v' to v'_2 , then Player i has a strategy such that out of the 20 rounds in the δ -path, the following hold.*

- (1) Player $-i$ grabs a pawn in at least 10 rounds, and
- (2) Player i grabs a pawn in at most 10 rounds.

Corollary 4.8 follows directly from Lemma 4.2 since in an always-grabbing game, the number of times Player $-i$ grabs equals the number of times Player i moves. In the remaining part of this section, we show that Player 1 wins \mathcal{G}' from c' iff Player 1 wins \mathcal{G}'' from the configuration c'' described above.

Lemma 4.9. *For $i \in \{1, 2\}$, Player i wins from c' in the optional-grabbing game \mathcal{G}' iff he wins from c'' in the always-grabbing game \mathcal{G}'' .*

Proof. We first give an outline of the proof before proceeding to proving the lemma formally. We prove that if Player 1 has a winning strategy f' in \mathcal{G}' from c' , then he has a winning strategy f'' from c'' in \mathcal{G}'' . The case for Player 2 is analogous and the other direction follows from determinacy (Thm. 2.1). We construct f'' to mimic f' with the following difference. Whenever f' chooses not to grab, in order to follow the rules of the always-grabbing mechanism, f'' grabs a pawn owning an isolated vertex. This is possible since we show that we maintain the invariant that along a play in \mathcal{G}'' that consists of sequences of δ -paths, at the beginning of each δ -path, Player 2 has at least 10 isolated pawns. Note that the invariant holds initially due to the definition of c'' . We show that it is maintained. Recall from the proof of Theorem 3.6 that crossing a δ -path simulates a transition in the Turing machine. Since Player 1 has a winning strategy in \mathcal{G}' , in a winning play, the strategy enables her to cross the δ -path. Thus, by Lem. 4.2, Player 1 moves in at least 10 rounds. Thus, Player 2 moves in at most 10 rounds, and during each such round, Player 1 grabs a pawn. Hence, Player 1 grabs at most 10 times which thus maintains the invariant. We show that f'' is a winning Player 1 strategy.

We now formally prove each of the claims stated above. Now since for Player 1, grabbing an isolated pawn serves no extra purpose than grabbing nothing, the action of not grabbing in the optional-grabbing game can be replaced with grabbing a pawn owning an isolated vertex in the always-grabbing game \mathcal{G}'' . Now assuming that Player 1 has a winning strategy f' in the optional-grabbing game \mathcal{G}' , consider a winning play π' for Player 1 in \mathcal{G}' . Now assume that in every round in which Player 1 does not grab a pawn in \mathcal{G}' can be replaced with Player 1 grabbing a pawn owning an isolated vertex in the always-grabbing game \mathcal{G}'' . Thus consider a strategy f'' of Player 1 and a strategy of Player 2 such that in the resulting play Player 1 grabs a pawn owning an isolated vertex from Player 2 if in the corresponding round in the optional-grabbing game he does not grab anything, otherwise, f'' follows the moves of f' . Now consider that Player 1 chooses strategy f'' . Recall that f' is winning for Player 1 in \mathcal{G}' . We argue that Player 2 cannot win in \mathcal{G}'' against the strategy f'' of Player 1. Suppose Player 2 uses a strategy and grabs some pawns owning non-isolated vertices while playing against strategy f'' of Player 1 in rounds other than those rounds in \mathcal{G}' in which he grabs the pawns owning non-isolated vertices while playing against strategy f' of Player 1. Then the resulting play in the always-grabbing game will still be losing for Player 2, since otherwise, Player 2 could have followed the same order of grabbing the non-isolated pawns in the optional-grabbing game \mathcal{G}' and would have won the optional-grabbing game. This contradicts that f' is winning for Player 1 in the optional-grabbing game. This implies that if Player 1 has a winning strategy in the optional-grabbing game \mathcal{G}' from the configuration $\langle v, P \rangle$, then he has a winning strategy in the always-grabbing game \mathcal{G}'' from the configuration $\langle v, P \cup \{1, \dots, d + 10\} \rangle$.

We now only need to show that in every round in \mathcal{G}'' where Player 2 moves the token, Player 2 has an isolated pawn for Player 1 to grab. We consider the following claim.

Claim 4.10. In the always-grabbing game \mathcal{G}'' , for strategy f'' of Player 1 and some strategy of Player 2, every time the token is at a primed main vertex, that is, at the beginning of a δ -path, Player 2 owns at least 10 pawns owning the isolated vertices.

Proof. Consider the case in which at a primed main vertex v , Player 2 has r pawns and the token is at v . Now suppose that the token is moved along a δ -path to the next primed

main vertex v' . Note that at this next primed vertex v' , Player 2 has at least r pawns. This clearly holds since by Corollary 4.8, along the δ -path, *Player 1* grabs at most 10 pawns and *Player 2* grabs at least 10 pawns. Thus, after traversing a δ -path, the number of pawns that *Player 2* controls does not decrease. Now, in a play, every time the token is at a primed main vertex in \mathcal{G}'' , it goes along a δ -path to reach the next primed main vertex again. Now the initial vertex is a primed main vertex and *Player 2* has $(d - |P|) + (d + 10)$ pawns at the beginning. Here P is the initial set of pawns owned by *Player 1* in optional-grabbing game \mathcal{G}' . Thus, every time the token reaches a primed main vertex afterwards, *Player 2* has at least $(d - |P|) + (d + 10)$ pawns. Now since there are exactly d non-isolated pawns, and $(d - |P|) + (d + 10) \geq (d + 10)$, we have that out of the pawns that *Player 2* has at a primed main vertex, at least 10 are isolated pawns. Hence proved. \square

Now to prove Lemma 4.9, since by Corollary 4.8, we know that *Player 1* has a strategy in the always-grabbing game such that from a primed main vertex to reach the next primed main vertex, he needs to grab in at most 10 rounds, and since by Claim 4.10, *Player 2* has at least 10 pawns owning isolated vertices, whenever the token is at primed main vertex, in every round between primed main vertices where *Player 1* needs to grab, *Player 2* has an isolated pawn that *Player 1* can grab. Hence by the argument above, if *Player 1* has a winning strategy in the optional-grabbing game \mathcal{G}' , we have that *Player 1* also has a winning strategy in the always-grabbing game \mathcal{G}'' .

Note that the proof of this direction is also analogous for *Player 2*. We show that if *Player 2* has a winning strategy in \mathcal{G}' , then she also has a winning strategy in \mathcal{G}'' . In particular, from Corollary 4.8, along a δ -path, *Player 1* grabs in at least 10 rounds and *Player 2* grabs in at most 10 rounds. Also, similar to Claim 4.10, we can show that every time the token is at a primed main vertex, *Player 1* owns at least 10 pawns owning isolated vertices. This is because initially, *Player 1* controls $|P| + d + 10$ vertices and this invariant is maintained throughout the play whenever the token reaches a primed main vertex. Now $|P| + d + 10 \geq d + 10$, and hence, at the beginning of a δ -path, *Player 1* controls at least 10 pawns owning isolated vertices.

For the converse direction in Lemma 4.9, we use the determinacy argument as follows. Again, we show the proof for *Player 1* that if *Player 1* wins in the always-grabbing game \mathcal{G}'' , then he also wins the optional-grabbing game \mathcal{G}' . The proof for *Player 2* is analogous.

Let *Player 1* has a winning strategy in \mathcal{G}'' . Then, by determinacy, *Player 2* loses in \mathcal{G}'' . Now from Lemma 4.9, by taking the contrapositive for *Player 2*, we have that *Player 2* also loses \mathcal{G}' . Hence, again by determinacy of pawn games, we have that *Player 1* wins \mathcal{G}' . \square

We now state the following theorem. While the lower bound follows from Thm. 3.10 and Lem. 4.9, the upper bound follows from Thm. 2.3.

Theorem 4.11. *MVPP always-grabbing PAWN-GAMES is EXPTIME-complete.*

We conclude this section by adapting Thm. 3.1 to always-grabbing. Namely, we show that adding pawns to a player is never beneficial in MVPP always-grabbing games (with the exception of the pawn that owns the current vertex).

Theorem 4.12. *Consider a configuration $\langle v, P \rangle$ of an MVPP always-grabbing pawn game \mathcal{G} . Let $j \in [d]$ such that $v \in V_j$ and $P' \subseteq P$. Assuming that $j \in P$ implies $j \in P'$, if *Player 1* wins from $\langle v, P \rangle$, he wins from $\langle v, P' \rangle$. Assuming that $j \in \overline{P'}$ implies $j \in \overline{P}$, if *Player 2* wins from $\langle v, P' \rangle$, she wins from $\langle v, P \rangle$.*

Proof. We show the case when Player 1 has a winning strategy. The case for Player 2 having a winning strategy is analogous. Recall from the proof of Thm. 3.1 that the cases were argued for both configuration vertices and intermediate vertices. For configuration vertices, The proof of this theorem is exactly the same as in the proof of Thm. 3.1. We detail below the case for intermediate vertices in the case of always-grabbing games. We use the same notations as in the proof of Thm. 3.1.

As in the proof of Thm. 3.1, we again distinguish between two cases: $j \in P$ and $j \notin P$. Consider an intermediate vertex $\langle u, c \rangle \in W_{n+1}$. We will show that $\langle u, c' \rangle \in W_{n+1}$. We denote by ℓ , the pawn that owns u .

First consider $j \in P$, thus Player 1 controls c . We have as in the proof of Thm. 3.1 that both $\langle u, c \rangle$ and $\langle u, c' \rangle$ are Player 2 vertices. Consider a neighbor of $\langle u, c' \rangle$, a configuration vertex $\langle u, Q' \rangle$. Note that in always-grabbing Player 2 has to grab a pawn r from Player 1, thus $Q' = P' \setminus \{r\}$. There are two cases, either $\ell \in Q'$ or $\ell \notin Q'$. Recall that in order to apply the induction hypothesis on $\langle u, Q' \rangle$, we need to find a neighbor $\langle u, Q \rangle$ of $\langle u, c \rangle$ such that if $\ell \notin Q'$, then $\ell \notin Q$. If $\ell \notin Q'$, then we set $Q = P \setminus \{\ell\}$ if $\ell \in P$ and we set $Q = P \setminus \{r\}$ when $\ell \notin P$. If $\ell \in Q'$, we set $Q = P \setminus \{r\}$. The rest of the argument is as in the proof of Theorem 3.1.

Next consider the case when $j \notin P$. In this case, both $\langle u, c \rangle$ and $\langle u, c' \rangle$ are Player 1 vertices. In always-grabbing, Player 1 grabs a pawn r in $\langle u, c \rangle$, thus we have $Q = P \cup \{r\}$. We find a neighbor $\langle u, Q' \rangle$ of $\langle u, c' \rangle$ to apply the induction hypothesis on. If $\ell \in P$ and $\ell \in P'$, then we set $Q' = P' \cup \{r\}$. If $\ell \in P$ and $\ell \notin P'$, then we set $Q' = P' \cup \{\ell\}$. If $\ell \notin P$ and $\ell \notin P'$, then we set $Q' = P' \cup \{r\}$. Again, the remaining argument is as in the proof of Theorem 3.1 and we are done. \square

5. ALWAYS GRABBING-OR-GIVING PAWN GAMES

In this section, we show that MVPP always grabbing or giving games are in PTIME. We find it intriguing that a seemingly small change in the mechanism – allowing a choice between grabbing and giving instead of only grabbing – reduces the complexity to PTIME from EXPTIME-complete. We make the following simple observation.

Observation 5.1. In an always grabbing or giving game, every time Player i makes a move from a vertex v to a vertex u , Player $-i$ can decide which player controls u .

If Player $-i$ does not control p_u that owns u and he wants to control u , he can grab p_u from Player i . If he does not want to control u and if he has p_u , he can give it to Player i .

Consider an always-grabbing-or-giving game $\mathcal{G} = \langle V, E, T \rangle$ and an initial configuration c . We construct a turn-based game \mathcal{G}' and an initial vertex v_0 so that Player 1 wins in \mathcal{G} from c iff he wins in \mathcal{G}' from v_0 . Let $\mathcal{G}' = \langle V', E', T' \rangle$, where $V' = \{\langle v, i \rangle, \langle \hat{v}, i \rangle \mid v \in V, i \in \{1, 2\}\}$ with $V'_1 = \{\langle v, 1 \rangle, \langle \hat{v}, 1 \rangle \mid v \in V\}$ and $V'_2 = \{\langle v, 2 \rangle, \langle \hat{v}, 2 \rangle \mid v \in V\}$, $T' = T \times \{1, 2\}$, and $E' = \{(\langle v, i \rangle, \langle \hat{u}, 3 - i \rangle), (\langle \hat{u}, 3 - i \rangle, \langle u, i \rangle), (\langle \hat{u}, 3 - i \rangle, \langle u, 3 - i \rangle) \mid (v, u) \in E, i \in \{1, 2\}\}$. We call each vertex $\langle v, i \rangle$ a *main* vertex and each $\langle \hat{v}, i \rangle$ an *intermediate* vertex. Suppose that Player i moves the token from v to u in \mathcal{G} . If Player $-i$ decides to control u , then in \mathcal{G}' , the token moves from the main vertex $\langle v, i \rangle$ to the main vertex $\langle u, 3 - i \rangle$, else from $\langle v, i \rangle$ to the main vertex $\langle u, i \rangle$, and in each case, through the intermediate vertex $\langle \hat{u}, 3 - i \rangle$ that models the decision of Player $-i$ on the control of u . The target vertices T' are main vertices.

We can prove the following lemma.

Lemma 5.2. *Suppose Player 1 wins from configuration $\langle v, P \rangle$ in \mathcal{G} . If he controls v , he wins from $\langle v, 1 \rangle$ in \mathcal{G}' , and if Player 2 controls v , Player 1 wins from $\langle v, 2 \rangle$ in \mathcal{G}' . Dually, suppose that Player 2 wins from $\langle v, P \rangle$ in \mathcal{G} . If she controls v , then she wins from $\langle v, 2 \rangle$ in \mathcal{G}' , and if Player 1 controls v , Player 2 wins from $\langle v, 1 \rangle$ in \mathcal{G}' .*

Proof. Suppose Player i has a winning strategy in the game \mathcal{G} from configuration $\langle v, P \rangle$. We show that Player i has a winning strategy in the game \mathcal{G}' from vertex $\langle v, i \rangle$. The other direction of the lemma follows from determinacy of two-player reachability games. We prove the above for Player 1. The proof for Player 2 is analogous.

Let W_j be the set of configurations in \mathcal{G} such that Player 1 has a strategy to reach T in at most j rounds. Let A_j be the set of vertices in \mathcal{G}' such that Player 1 has a strategy to reach T' in at most j rounds. We prove the following claim.

Claim 5.3. If $\langle v, P \rangle \in W_j$, with Player 1 (Player 2) controlling v then $\langle v, 1 \rangle$ ($\langle v, 2 \rangle$) belongs to A_{2j} .

Proof. We prove this by induction on j . For the base case with $j = 0$, this clearly holds. Suppose the claim holds for $j = h$, and we now show that the claim holds for $j = h + 1$.

Consider a configuration $\langle v, P \rangle \in W_{h+1}$. We first look at the case when Player 1 controls v . We show that $\langle v, 1 \rangle \in A_{2(h+1)}$. Note that, by definition of \mathcal{G} and \mathcal{G}' , both $\langle v, P \rangle$ and $\langle v, 1 \rangle$ are controlled by Player 1 in \mathcal{G} and \mathcal{G}' respectively.

Since $\langle v, P \rangle \in W_{h+1}$ and Player 1 controls vertex v , there is a strategy of Player 1 that takes the token starting from v with pawns P to a target vertex in T in at most $h + 1$ steps. Let under this strategy, Player 1 moves the token from v to u . Note that $\langle u, P' \rangle \in W_h$ for all configurations $\langle u, P' \rangle$ that Player 2 can reach after grabbing from or giving Player 1 a pawn after Player 1 moves the token from v to u . Let P_u be the pawn owning vertex u . Now from Observation 5.1, we know that once Player 1 moves the token from vertex v to a vertex u , then Player 2 can reach a configuration $\langle u, P' \rangle$ with $p_u \in P'$ as well $\langle u, P'' \rangle$ with $p_u \notin P''$. Thus there exist configurations $\langle u, P' \rangle, \langle u, P'' \rangle$ in W_h such that $p_u \in P'$ and $p_u \notin P''$. Now suppose the token is at vertex $\langle v, 1 \rangle$ in \mathcal{G}' . Since u is a neighbour of v , by the definition of \mathcal{G}' , we have that vertex $\langle \hat{u}, 2 \rangle$ is a neighbour of $\langle u, 1 \rangle$ in \mathcal{G}' . We show that $\langle \hat{u}, 2 \rangle \in A_{2h+1}$. Recall that vertex $\langle \hat{u}, 2 \rangle$ is controlled by Player 2 and the only neighbours of this vertex are $\langle u, 1 \rangle$ and $\langle u, 2 \rangle$. Now if we show that both $\langle u, 1 \rangle$ and $\langle u, 2 \rangle$ are in A_{2h} , then we are done. Since we know that there exists $\langle u, P' \rangle$ in W_h such that $p_u \in P'$, by the induction hypothesis, we have that $\langle u, 1 \rangle \in A_{2h}$, and similarly since there exists $\langle u, P'' \rangle$ in W_h such that $p_u \notin P''$, we have that $\langle u, 2 \rangle \in A_{2h}$.

The case for which Player 1 does not control v can also be proved similarly. \square

We show that Player i wins in \mathcal{G} from $\langle v, P \rangle$ when Player i (Player $-i$) controls v implies that Player i wins from $\langle v, i \rangle$ ($\langle v, 3 - i \rangle$) in \mathcal{G}' . By taking contrapositive, we have that Player i loses from $\langle v, i \rangle$ ($\langle v, 3 - i \rangle$) in \mathcal{G}' implies that Player i loses in \mathcal{G} from $\langle v, P \rangle$ when Player i (Player $-i$) controls v . By determinacy, we have that Player $-i$ wins from $\langle v, i \rangle$ ($\langle v, 3 - i \rangle$) in \mathcal{G}' implies that Player $-i$ wins in \mathcal{G} from $\langle v, P \rangle$ when Player i (Player $-i$) controls v .

Hence showed. \square

Since the size of \mathcal{G}' is polynomial in the size of \mathcal{G} , Thm. 2.1 implies the following.

Theorem 5.4. *MVPP always-grab-or-give PAWN-GAMES is in PTIME.*

Algorithm 2 Given an OVPP pawn game $\mathcal{G} = \langle V, E, T \rangle$ and a set of pawns $P_0 \subseteq V$ that Player 1 controls, the algorithm returns a minimum-grabbing function $\eta : V \rightarrow \mathbb{N}$.

```

1:  $\ell = 0$ 
2: while  $\exists u \in V$  that is not labeled by  $\eta$  do
3:    $W_\ell^1 = \text{SOLVE-TURN-BASED-GAME}(V_1 = P_0, V_2 = V \setminus P_0, E, T)$ 
4:   Define  $\eta(u) = \ell$ , for all  $u \in W_\ell^1$  that is not yet labeled.
5:    $B_\ell = \{u \in V \setminus W_\ell^1 : N(u) \cap W_\ell^1 \neq \emptyset\}$ 
6:    $T = B_\ell$  and  $\ell = \ell + 1$ .
```

6. K-GRABBING PAWN GAMES

In this section, we consider pawn games under k -grabbing in increasing level of generality of the mechanisms. We start with positive news.

Theorem 6.1. *OVPP k -grabbing PAWN-GAMES is in PTIME.*

Proof. Let $k \in \mathbb{N}$, an OVPP k -grabbing game $\mathcal{G} = \langle V, E, T \rangle$, and an initial configuration $c = \langle v_0, P_0 \rangle$, where we refer to P_0 as a set of vertices rather than pawns. Our algorithm solves a harder problem. It computes a *minimum-grabbing* function $\eta : V \rightarrow \mathbb{N}$ that labels each vertex $u \in V$ with the necessary and sufficient number grabs needed from u to win. Formally, $\eta(u)$ is such that Player 1 wins an $\eta(u)$ -grabbing game played on \mathcal{G} from configuration $\langle u, P_0 \rangle$ but loses an $(\eta(u) - 1)$ -grabbing game from configuration $\langle u, P_0 \rangle$. The algorithm is depicted in Alg. 2. It calls the `SOLVE-TURN-BASED-GAME()`, which returns the set of vertices that are winning for Player 1 in a turn-based reachability game, e.g., using attractor-based computation as in Thm. 2.1.

For the base case, consider the turn-based game $\mathcal{G}_0 = \langle V, E, T \rangle$ with $V_1 = P_0$. Let $W_0^1 \subseteq V$ denote Player 1's winning region in \mathcal{G}_0 . Clearly, for every $u \in W_0^1$, we have $\eta(v_0) = 0$, and for every $u \notin W_0^1$, we have $\eta(v_0) \geq 1$. For the inductive step, suppose that for $\ell \geq 0$, the set $W_\ell^1 = \{u \in V : \eta(u) \leq \ell\}$ has been found. That is, for every $u \notin W_\ell^1$, Player 2 has a strategy that wins the ℓ -grabbing pawn game \mathcal{G} from configuration $\langle u, P_0 \rangle$. We show how to find $W_{\ell+1}^1$ in linear time. Let the *border* of W_ℓ^1 , denoted B_ℓ , be the set of vertices from which W_ℓ^1 can be reached in one step, thus $B_\ell = \{v \in V : v \notin W_\ell^1 : N(v) \cap W_\ell^1 \neq \emptyset\}$. Note that the vertices in B_ℓ are all controlled by Player 2 since otherwise, such vertices will be in the set W_ℓ^1 . Now we show that a vertex $u \notin W_\ell^1$ has $\eta(u) = \ell + 1$ iff Player 1 can force the game from configuration $\langle u, P_0 \rangle$ to a vertex in B_ℓ in one or more rounds without making any grab. Player 1 wins from such a vertex u by forcing the game into B_ℓ , grabbing the pawn in B_ℓ , and proceeding to W_ℓ , where by the induction hypothesis, he wins with the remaining grabs. Computing $W_{\ell+1}^1$ roughly entails a solution to a turn-based game with target set $B_\ell \cup W_\ell^1$.

Intuitively, we show that a vertex $u \notin W_\ell^1$ has $\eta(u) = \ell + 1$ iff Player 1 can force the game from configuration $\langle u, P_0 \rangle$ to a vertex in B_ℓ without making any grabs and in a *non-trivial* manner. We compute $W_{\ell+1}^1$ as follows. Consider the turn-based game $\mathcal{G}_\ell = \langle V, E, B_\ell \cup W_\ell^1 \rangle$ with $V_1 = P_0$. We say that Player 1 wins non-trivially from $u \in V$ if he has a strategy f that guarantees the following against any Player 2 strategy: the resulting play is of the form $u = v_0, v_1, v_2, \dots, v_m$ with $v_m \in (B_\ell \cup W_\ell^1)$ and it is non-trivial, meaning that $m \geq 1$. Note the difference from standard turn-based games: a target vertex $u \in B_\ell \cup W_\ell^1$ might be losing for Player 1 when considering winning non trivially, e.g., when u has an outgoing edge to a sink, no non-trivial play that starts in u ends in a target vertex.

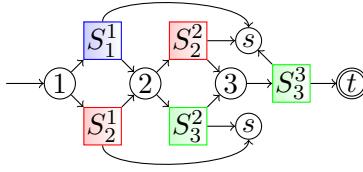


Figure 5: Consider the input to SET-COVER $U = [3]$ and $\mathcal{S} = \{\{1\}, \{1, 2\}, \{2, 3\}\}$. The figure depicts the output on this input of the reduction in Thm. 6.2

Let U_ℓ^1 denote the set of vertices from which Player 1 wins in \mathcal{G}_ℓ non-trivially. We describe an algorithm for finding U_ℓ^1 . We construct a game \mathcal{G}'_ℓ in which we make a copy u' of each vertex $u \in B_\ell$, and set the neighbors of u' to be $\{v \in N(u) : v \notin W_\ell^1\}$. Intuitively, if Player 1 wins from u' in \mathcal{G}'_ℓ , he can win in \mathcal{G}_ℓ from u in a non-trivial manner: since u' is not winning in \mathcal{G}'_ℓ , Player 1 must make at least one step before entering B_ℓ . Let U' be the winning set for Player 1 in \mathcal{G}'_ℓ . It is not hard to see that $U_\ell^1 = \{u \notin B_\ell : u \in U'\} \cup \{u \in B_\ell : u' \in U'\}$.

Let $u \in U_\ell^1$. We claim that if $u \notin W_\ell^1$ then $\eta(u) = \ell + 1$. First, since $u \notin W_\ell^1$, by the induction hypothesis, $\eta(u) \geq \ell + 1$. Next, we show that $\eta(u) = \ell + 1$ by showing that Player 1 wins the $(\ell + 1)$ -grabbing pawn game \mathcal{G} from configuration $\langle u, P_0 \rangle$. Player 1 initially plays according to a strategy that forces the game to B_ℓ non-trivially without grabbing. Suppose that Player 2 is playing according to some strategy and the game visits u' before visiting $u'' \in B_\ell$. Then, Player 1 grabs u'' , proceeds to W_ℓ^1 , and uses his winning ℓ -grabbing strategy from there.

Let $u \notin U_\ell^1 \cup W_\ell^1$. We claim that $\eta(u) > \ell + 1$. Note that in order to reach W_ℓ^1 and in particular reach T , the game must first visit B_ℓ . Suppose that Player 2 is following a winning strategy in \mathcal{G}_ℓ . Thus, in order to reach B_ℓ , Player 1 must grab $u' \notin B_\ell \cup W_\ell^1$. Suppose that the game reaches a configuration $\langle u'', P_0 \cup \{u'\} \rangle$, where $u'' \in B_\ell$. By the above, $\eta(u'') \geq \ell + 1$. That is, any Player 1 strategy that wins from u'' must make at least $\ell + 1$ grabs. Hence, in order to win from u , Player 1 makes at least $\ell + 2$ grabs.

Let $n = |V|$. Note that $W_n^1 = V$. Computing $W_{\ell+1}^1$ from W_ℓ^1 requires a call to an algorithm that solves a reachability game, which can be done in linear time. Hence, the total running time of the algorithm is polynomial in the size of \mathcal{G} . \square

The proof of the following theorem, which is obtained by a reduction from SET-COVER.

Theorem 6.2. *MVPP k -grabbing game PAWN-GAMES is NP-hard.*

Proof. Given an input $\langle U, \mathcal{S}, k \rangle$ to SET-COVER, we construct an MVPP k -grabbing pawn game $\mathcal{G} = \langle V, E, T \rangle$ in which Player 1 grabs (see Fig. 5). Intuitively, \mathcal{G} has a “chain-like” structure, and in order to win, Player 1 must cross the chain. Certain positions in \mathcal{G} corresponds to $i \in U$. Each neighbor of i correspond to a set $S \in \mathcal{S}$ with $i \in S$. Thus, a choice of Player 1 at i can be thought of as assigning a set in \mathcal{S} that covers i . We construct \mathcal{G} so that before moving to a neighbor S of i , Player 1 must grab the pawn that owns S . All vertices that correspond to S are controlled by the same pawn, thus if Player 1 moves from $i' > i$ to a neighbor that corresponds to S , there is no need to grab again. Since Player 1 is allowed only k grabs, there is a one-to-one correspondence between winning Player 1 strategies and set covers of size k .

We describe the construction of \mathcal{G} formally. Let $V = U \cup (\mathcal{S} \times U) \cup \{s, t\}$. Player 1's target is t and s is a vertex with no path to t , thus it can be thought of as a target for Player 2. There are $m + 1$ pawns. For $j \in [m]$, Pawn j owns all vertices in $\{\langle S_j, i \rangle : i \in U\}$. Pawn 0 owns the vertices in U . We describe the edges. For $i \in U$, we define $N(i) = \{\langle S_j, i \rangle : i \in S_j\}$. For $1 \leq i \leq n - 1$ and $j \in [m]$, we define $N(\langle S_j, i \rangle) = \{i + 1, s\}$ and $N(\langle S_j, n \rangle) = \{t, s\}$. That is, if Player 1 moves from i to its neighbor S_j without controlling Pawn j , then Player 2 will proceed to s and win the game.

We claim that a set cover \mathcal{S}' of size k gives rise to a winning k -grabbing Player 1 strategy. Indeed, by choosing, for each $i \in U$, to move to $S \in \mathcal{S}'$ and grabbing it if it has not been grabbed previously, Player 1 guarantees that t is reached using at most k grabs. On the other hand, observe a play of a winning k -grabbing Player 1 strategy against a *reasonable* Player 2 strategy; namely, a strategy that always moves to s from a neighboring vertex u when Player 2 controls u . Suppose that the set of pawns \mathcal{S}' is grabbed. It is not hard to see that \mathcal{S}' is a set cover of size at most k , and we are done. \square

We conclude this section by studying OMVPP games.

Lemma 6.3. *OMVPP k -grabbing PAWN-GAMES is PSPACE-hard.*

Proof. Consider an input $\phi = Q_1x_1 \dots Q_nx_n C_1 \wedge \dots \wedge C_m$ to TQBF, where $Q_i \in \{\exists, \forall\}$, for $1 \leq i \leq n$, each C_j , for $1 \leq j \leq m$, is a clause over the variables x_1, \dots, x_n . We construct an OMVPP n -grabbing pawn game $\mathcal{G} = \langle V, E, T \rangle$ such that Player 1 wins iff ϕ is true.

The intuition is similar to Thm. 6.2. We construct \mathcal{G} to have a chain-like structure that Player 1 must cross in order to win. The chain consists of two parts. In the first part, each position corresponds to a variable x_i , for $i \in [n]$. We construct \mathcal{G} so that if x_i is existentially quantified, then Player 1 controls x_i , and if x_i is universally quantified, then Player 2 controls x_i and Player 1 cannot win by grabbing it. We associate a move in x_i with an assignment to x_i . Technically, x_i has two neighbors v_i and $\neg v_i$, initially at the control of Player 2. If the token reaches a neighbor of x_i without Player 1 grabbing it, then Player 1 loses the game. Player 1 is allowed n grabs, thus in order to win, it is necessary for him to grab, for each i , one of the neighbors of x_i . It follows that a play that crosses the first part of the chain gives rise to an assignment f to x_1, \dots, x_n .

In the second part of \mathcal{G} , we verify that f is valid. Each position of \mathcal{G} corresponds to a clause C_j , for $j \in [m]$, all of which are at the control of Player 2 in the initial configuration of \mathcal{G} . Note that once the first part of \mathcal{G} is crossed, Player 1 is not allowed any more grabs. The idea is that when arriving at C_j , for $j \in [m]$, Player 1 must control C_j , otherwise he loses. Recall that in OMVPP, it suffices to control one of the pawns that owns C_j in order to control it. We define the pawns that own C_j as follows. For $i \in [n]$, call p_i the pawn that owns v_i and $\neg p_i$ the pawn that owns $\neg v_i$. Thus, grabbing p_i and $\neg p_i$ respectively corresponds to an assignment that sets x_i to true and false. If x_i appears in C_j , then p_i is an owner of C_j , and if $\neg x_i$ appears in C_j , then $\neg p_i$ is an owner of C_j . Thus, Player 1 controls C_j iff $f(C_j) = \text{true}$. It follows that Player 1 wins iff f is valid.

Formally, the vertices of \mathcal{G} are $V = \{s, t\} \cup \{x_i, v_i, \neg v_i : 1 \leq i \leq n\} \cup \{C_j : 1 \leq j \leq m\}$. The target vertex is t . The vertex s is a sink vertex that is winning for Player 2, i.e., there is no path from s to t . The game consists of $2n + 2$ pawns. For each $1 \leq i \leq n$, we associate with variable x_i two pawns, which we denote by p_i and $\neg p_i$. Intuitively, we will construct the game such that in order to win, Player 1 must grab either p_i or $\neg p_i$, and we associate grabbing p_i and $\neg p_i$ with an assignment that sets x_i to true and false, respectively. Formally, if x_i appears in C_j , then p_i is an owner of C_j , and if $\neg x_i$ appears in C_j , then $\neg p_i$ is an owner

of C_j . We use 1 and 2 to refer to the final two pawns. Pawn 1 owns every vertex x_i such that variable x_i is existentially quantified in ϕ and Pawn 2 owns every vertex x_i such that x_i is universally quantified in ϕ . In the initial configuration, Player 1 controls only Pawn 1.

We describe the edges in the game, which has a chain-like structure. The last vertex on the chain is the target t , thus Player 1 must cross the chain in order to win. We assume that Player 2 follows a *reasonable* strategy; namely, a strategy that proceeds to win at s given the option, i.e., when the game reaches a vertex u at her control and that neighbors s . The token is initially placed on x_0 . For $1 \leq i \leq n$, the neighbors of x_i are v_i and $\neg v_i$. We associate with the choice at x_i , an assignment to x_i in the expected manner. Suppose that the token is placed on $u \in \{v_i, \neg v_i\}$. We require Player 1 to grab the pawn that owns u by adding an edge from u to s . That is, since initially Player 2 controls the pawn that owns u , if Player 1 does not grab it, Player 2 will win from u . For $1 \leq i < n$, the neighbor of v_i and $\neg v_i$ is v_{i+1} . The neighbor of v_n and $\neg v_n$ is C_1 .

Suppose that Player 1 follows a strategy that leads the token to C_1 . We observe that since Pawn 1 is controlled by Player 1, Player 1 chooses the assignment of the existentially-quantified variables. We claim that Player 1's strategy does not grab Pawn 2, thus Player 2 chooses the assignment of the universally-quantified variables. Indeed, since Player 1 is restricted to grab at most n pawns and must grab either p_i or $\neg p_i$, for each $1 \leq i \leq n$. If he grabs Pawn 2, then there exists $1 \leq i \leq n$ such that he cannot grab any of p_i or $\neg p_i$, and the game will necessarily reach either v_i or $\neg v_i$ at the control of Player 2, from which she wins.

We describe the outgoing edges from clause vertices. For $1 \leq j < m$, the vertex C_j has two neighbors, s and C_{j+1} , and the neighbors of C_m are s and t . That is, in order to draw the game to t , Player 1 must cross all clause vertices. Recall that the definition of OMVPP dictates that Player 1 chooses how to move the token at a vertex u if he controls at least one of the pawns that owns u . Thus, a winning Player 1 strategy must grab, for each $1 \leq j \leq m$, at least one pawn that owns C_j . In turn, grabbing a pawn that owns C_j means that the assignment to x_1, \dots, x_n that arises from the players' strategies satisfies C_j . It follows that Player 1 wins the game iff ϕ is true. \square

We turn to study the upper bound. The following lemma bounds the provides a polynomial bound on the length of a winning play for Player 1. The core of the proof intuitively shows that we can restrict attention to Player 1 strategies that grab at least once in a sequence of $|V|$ rounds. Otherwise, the game enters a cycle that is winning for Player 2. We thus obtain a polynomial bound on the length of a winning play for Player 1.

Lemma 6.4. *Consider an OMVPP k -grabbing PAWN-GAME $\mathcal{G} = \langle V, E, T \rangle$, and an initial configuration c that is winning for Player 1. Then, Player 1 has a strategy such that, for every Player 2 strategy, a target in T is reached within $|V| \cdot (k + 1)$ rounds.*

Proof. Consider the turn-based game \mathcal{G}' that corresponds to \mathcal{G} . Recall that c is a vertex in \mathcal{G}' . Fix a Player 1 memoryless winning Player 1 strategy in \mathcal{G}' , which exists since \mathcal{G}' is a turn-based reachability game. Consider some Player 2 strategy and let π be the resulting play. We make three observations. (1) Each vertex in \mathcal{G}' is visited at most once by π . Otherwise, π enters a loop, which is losing for Player 1. (2) The configurations can be partially ordered: a configuration (v, P) can only be visited before a configuration (u, P') , for $P \subseteq P'$. Indeed, the only manner in which control of pawns changes is by Player 1 grabbing, which only adds pawns to the set that he already controls. (3) If the game starts from (v, P) , then it ends in a configuration (u, P') with $|P'| \leq |P| + k$. Indeed, Player 1 can only grab k times. Combining (1)-(3) implies that π visits T within $|V| \cdot (k + 1)$ rounds. \square

For the upper bound, we describe an algorithm performing a depth-first traversal of the configuration graph of a game while storing, at a time, only a branch in PSPACE.

Lemma 6.5. *OMVPP k -grabbing PAWN-GAMES is in PSPACE.*

Proof. We describe a PSPACE algorithm for OMVPP k -grabbing PAWN-GAMES as follows. The algorithm explores an unwinding of the configuration graph of the k -grabbing pawn game in a depth-first manner. We call this unwinding of the configuration graph a *game tree*. Recall that after each move by a player in the game, Player 1 chooses to grab a pawn and if Player 1 is controlling the current vertex, that is, where the token lies, then Player 1 chooses a successor, otherwise Player 2 chooses a successor. A vertex v that is controlled by Player 1 is an OR-vertex in the game tree in the sense that there should exist a successor of v which accepts. On the other hand, a vertex v that is controlled by Player 2 is an AND-vertex in the game tree in the sense that all the successors of v should accept.

Since by Lemma 6.4, if Player 1 has a winning strategy then he has one such that for all strategies of Player 2, he wins in $n \cdot (k + 1)$ steps, it is sufficient to unwind the configuration graph such that the length of a path in the game tree starting from the initial vertex does not exceed $n \cdot (k + 1)$. At any time during exploring the game tree in a depth-first manner, the algorithm stores the path that is being currently traversed from the initial vertex, and the length of the path is thus at most $n \cdot (k + 1)$. In a depth-first traversal, from each vertex, its successors are visited in a particular order. At each level of the path that has been currently traversed, the algorithm also keeps count of how many successors of the vertex at that level have been visited, and also the depth of the level from the root of the tree. The latter ensures that the algorithm does not unwind the configuration graph to an extent so that the length of a path in the game tree exceeds $n \cdot (k + 1)$. Since in the configuration graph, there are at most exponentially many vertices of the form $\langle v, P \rangle$ where v is a vertex of the k -grabbing pawn game and P is a set of pawns, at each level, the count of number of successors that have been visited so far can be stored in PSPACE. Since the number of levels of the current path is bounded by $n \cdot (k + 1)$ which is polynomial and each level uses polynomial space, the algorithm is in PSPACE. \square

By Lem. 6.4, each branch of such a traversal has polynomial length, leading to the PSPACE upper bound. We thus have the following.

Theorem 6.6. *OMVPP k -grabbing PAWN-GAMES is PSPACE-complete.*

Remark 6.7. We also note that in the case of k -grabbing mechanism, unlike optional-grabbing or always-grabbing mechanisms, it is always the case that for Player 1, at every vertex, controlling more pawns is at least as good as controlling fewer pawns. This is because, if Player 1 needs to control a particular vertex v in any round of the game in order to win, he can grab the pawn controlling the vertex v if the pawn does not belong to him regardless of which player moves the token provided he has not already grabbed k pawns.

7. DISCUSSION

We introduce pawn games, a class of two-player turn-based games in which control of vertices changes dynamically throughout the game. Pawn games constitute a class of succinctly-represented turn-based games. We identify natural classes that are in PTIME. Our EXPTIME-hardness results are based on Lock & Key games, which we hope will serve

as a framework for proving lower bounds. We mention directions for future research. First, we leave several open problems; e.g., for MVPP k -grabbing pawn games, we only show NP-hardness and membership in PSPACE. Second, we focused on reachability games. It is interesting to study pawn games with richer objectives such as parity or quantitative objectives. Quantitative objectives are especially appealing since one can quantify the benefit of controlling a pawn. Third, it is interesting to consider other pawn-transferring mechanisms and to identify properties of mechanisms that admit low-complexity results. Finally, grabbing pawns is a general concept and can be applied to more involved games like stochastic or concurrent games.

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