

LEFT-LINEAR COMPLETION WITH AC AXIOMS

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ABSTRACT. We revisit completion modulo equational theories for left-linear term rewrite systems where unification modulo the theory is avoided and the normal rewrite relation can be used in order to decide validity questions. To that end, we give a new correctness proof for finite runs and establish a simulation result between the two inference systems known from the literature. Given a concrete reduction order, novel canonicity results show that the resulting complete systems are unique up to the representation of their rules' right-hand sides. Furthermore, we show how left-linear AC completion can be simulated by general AC completion. In particular, this result allows us to switch from the former to the latter at any point during a completion process.

1. INTRODUCTION

Completion has been extensively studied since its introduction in the seminal paper by Knuth and Bendix [KB70]. One of the main limitations of the original formulation is its inability to deal with equations which cannot be oriented into a terminating rule such as the commutativity axiom. This shortcoming can be resolved by completion modulo an equational theory \mathcal{E} . In the literature, there are two different approaches of achieving this. The general approach [JK86, Bac91] requires \mathcal{E} -unification and allows us to decide validity problems using the rewrite relation $\rightarrow_{\mathcal{R}/\mathcal{E}}$ which is defined as $\leftrightarrow_{\mathcal{E}}^* \cdot \rightarrow_{\mathcal{R}} \cdot \leftrightarrow_{\mathcal{E}}^*$. For left-linear term rewrite systems, however, there is Huet's approach [Hue80] which avoids \mathcal{E} -unification. In particular, Huet's approach does not consider local peaks modulo \mathcal{E} ($\mathcal{R}\leftarrow \cdot \sim_{\mathcal{E}} \cdot \rightarrow_{\mathcal{R}}$) but works with ordinary local peaks ($\mathcal{R}\leftarrow \cdot \rightarrow_{\mathcal{R}}$) as well as local cliffs of the form $\leftrightarrow_{\mathcal{E}} \cdot \rightarrow_{\mathcal{R}}$. Hence, instead of \mathcal{E} -critical pairs we can use normal critical pairs when we also take overlaps between \mathcal{R} and \mathcal{E} into account. We call this approach *left-linear completion modulo an equational theory*. If we have a complete TRS \mathcal{R} in the general sense, we can decide validity problems $s \approx t$ by rewriting both terms to normal form with $\rightarrow_{\mathcal{R}/\mathcal{E}}$ and then checking the result for \mathcal{E} -equivalence. For complete systems stemming from left-linear \mathcal{E} -completion,

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however, it suffices to rewrite both s and t to normal form using the normal rewrite relation $\rightarrow_{\mathcal{R}}$ and then perform just one \mathcal{E} -equivalence check on these normal forms.

In their respective books, Avenhaus [Ave95] and Bachmair [Bac91] present inference systems for left-linear completion modulo an equational theory. This article gives a detailed account of the nature of and relation between these two systems for finite runs. For the concrete case of AC (associative and commutative function symbols), we compare left-linear completion modulo equational theories with the general approach by presenting an implementation of left-linear AC completion and comparing it with the state of the art concerning general AC completion. After setting the stage in Section 2, we present a new criterion for the Church–Rosser modulo property of left-linear TRSs based on prime critical pairs in Section 3. Slightly modified versions (A and B) of the inference systems due to Avenhaus and Bachmair are discussed in Sections 4 and 5, respectively. Both sections include a new correctness proof of the given inference system for finite runs. For A, this is done from scratch by using the criterion from Section 3 in the spirit of [HMSW19]. Correctness of B is then reduced to the correctness of A by establishing a simulation result between finite runs in these systems. Furthermore, Section 6 reports on novel results on canonicity for this setting. For the concrete equational theory of associative and commutative (AC) function symbols, we also show the connection between the inference system A and general AC completion by means of another simulation result (Section 7). Finally, we describe our implementation of A in the tool `accompII` and present experimental results which show that the avoidance of AC unification and AC matching can result in significant performance improvements over general AC completion (Sections 8 and 9).

This article extends our previous papers [NHM23b] and [NHM23a] by including full proof details, more examples and experimental data as well as novel results on canonicity.

2. PRELIMINARIES

We assume familiarity with term rewriting and completion as described e.g. in [BN98, Ter03] but recall some central notions in Section 2.1. The concept of (prime) critical peaks which leads to the definition of (prime) critical pairs is introduced in Section 2.2. Finally, Section 2.3 provides the necessary background for rewriting modulo equational theories.

2.1. Rewrite Systems. An *abstract rewrite system* or *abstract reduction system* (ARS) $\mathcal{A} = \langle A, \rightarrow \rangle$ is a set A together with a binary relation \rightarrow on A . If $a \rightarrow b$ for no b , then a is called a *normal form* of \mathcal{A} . Otherwise, we say that a is *reducible*. The set of normal forms is denoted by $\text{NF}(\mathcal{A})$. Given an arbitrary binary relation \rightarrow , we write \leftarrow , \leftrightarrow , $\rightarrow^=$, \rightarrow^+ and \rightarrow^* to denote its *inverse*, its *symmetric closure*, its *reflexive closure*, its *transitive closure* and its *symmetric and transitive closure*, respectively. Hence, the relation \leftrightarrow^* denotes the symmetric, reflexive and transitive closure of \rightarrow and is called *conversion*. The relation $a \rightarrow^! b$ is defined as $a \rightarrow^* b$ and $b \in \text{NF}(\mathcal{A})$ and is used to denote rewriting to a normal form. Finally, \downarrow abbreviates the *joinability relation* $\rightarrow^* \cdot \ast \leftarrow$ where \cdot denotes the composition of binary relations which is defined as follows: $R_1 \cdot R_2 = \{(a, c) \mid (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$.

Given a signature \mathcal{F} and a set of variables \mathcal{V} , we consider the set of terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$ which is defined as usual. The set of variables in a term t is written as $\text{Var}(t)$. Terms which do not contain the same variable more than once are referred to as *linear* terms. Each subterm of a term t has a unique *position* which is a finite sequence of positive integers where the empty sequence representing the root position is written as ϵ . The set of positions in a term t is

denoted by $\mathcal{Pos}(t)$ and further divided into the subset $\mathcal{Pos}_{\mathcal{F}}(t)$ of positions which address function symbols and the subset $\mathcal{Pos}_{\mathcal{V}}(t) = \mathcal{Pos}(t) \setminus \mathcal{Pos}_{\mathcal{F}}(t)$ of variable positions. If a position p is a prefix of the position q we write $p \leq q$. Positions p and q are parallel, denoted by $p \parallel q$, if neither $p \leq q$ nor $q \leq p$. If $p \leq q$ then $q \setminus p$ denotes the unique position r such that $pr = q$. By $s|_p$ we denote the subterm of s at position p . Mappings σ from variables to terms with a finite domain ($\{x \in \mathcal{V} \mid x \neq \sigma(x)\}$) are called *substitutions*. The application of a substitution σ to a term t is denoted by $t\sigma$. A *renaming* is a bijective substitution from \mathcal{V} to \mathcal{V} . A term s is a *variant* of a term t ($s \doteq t$) if there exists a renaming σ such that $s = t\sigma$. A context C is a term with exactly one occurrence of the special symbol $\square \notin \mathcal{F} \cup \mathcal{V}$ called *hole* which acts as a placeholder for concrete terms. Replacing the hole with a term t results in a term which we denote by $C[t]$. A term s *encompasses* a term t ($s \triangleright t$) if $s = C[t\sigma]$ for some context C and substitution σ . It is known that \triangleright is a quasi-order on terms and its strict part \triangleright is a well-founded order with $\triangleright = \triangleright \setminus \doteq$. We write $s[t]_p$ for the term which is created from s by replacing its subterm at position p by t .

A pair of terms (s, t) can be viewed as an *equation* ($s \approx t$) or a *rule* ($s \rightarrow t$). In the latter case we assume that s is not a variable and $\mathcal{Var}(s) \supseteq \mathcal{Var}(t)$. *Equational systems* (ESs) are sets of equations while *term rewrite systems* (TRSs) are sets of rules. Given a set of pairs of terms \mathcal{E} , we define a *rewrite relation* $\rightarrow_{\mathcal{E}}$ as the closure of its pairs under substitutions and contexts. More formally, $s \rightarrow_{\mathcal{E}} t$ if there is a pair $(\ell, r) \in \mathcal{E}$, a position p and a substitution σ such that $s|_p = \ell\sigma$ and $t = s[r\sigma]_p$. We sometimes make the position p explicit by writing $s \rightarrow_{\mathcal{E}}^p t$ and define $\leftrightarrow_{\mathcal{E}}^p$ as $\overset{p}{\leftarrow}_{\mathcal{E}} \cup \overset{p}{\rightarrow}_{\mathcal{E}}$. The *equational theory* induced by \mathcal{E} consists of all pairs of terms (s, t) such that $s \leftrightarrow_{\mathcal{E}}^* t$. A TRS \mathcal{R} *represents* an ES \mathcal{E} if $\leftrightarrow_{\mathcal{E}}^* = \leftrightarrow_{\mathcal{R}}^*$. Two rules $\ell \rightarrow r$ and $\ell' \rightarrow r'$ are *variants* if there exists a renaming σ such that $\ell\sigma = \ell'$ and $r\sigma = r'$. Two TRSs \mathcal{R}_1 and \mathcal{R}_2 over the same signature \mathcal{F} are *literally similar* ($\mathcal{R}_1 \doteq \mathcal{R}_2$) if every rule in \mathcal{R}_1 has a variant in \mathcal{R}_2 and vice versa. A TRS is *left-linear* if ℓ is a linear term for every rule $\ell \rightarrow r \in \mathcal{R}$. Given an ES \mathcal{E} , \mathcal{E}^{\pm} denotes $\mathcal{E} \cup \{t \approx s \mid s \approx t \in \mathcal{E}\}$.

A TRS \mathcal{R} is *terminating* if there is no infinite rewrite sequence $s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} \dots$. If $\overset{*}{\leftarrow}_{\mathcal{R}} \cdot \overset{*}{\rightarrow}_{\mathcal{R}} \subseteq \overset{*}{\rightarrow}_{\mathcal{R}} \cdot \overset{*}{\leftarrow}_{\mathcal{R}}$ then \mathcal{R} is *confluent*. A TRS is *complete* if it is terminating and confluent. For standard rewriting, the *Church-Rosser property* ($\leftrightarrow_{\mathcal{R}}^* \subseteq \downarrow_{\mathcal{R}}$) coincides with confluence. Hence, complete presentations \mathcal{R} of an ES \mathcal{E} can be used to decide the validity problem for \mathcal{E} : $s \leftrightarrow_{\mathcal{E}}^* t$ if and only if $s \rightarrow_{\mathcal{R}}^! \cdot \overset{!}{\leftarrow}_{\mathcal{R}} t$.

2.2. Critical Peaks. Confluence of terminating TRSs is characterized by joinability of critical pairs. Critical pairs are computed from overlaps which are the potentially dangerous cases of nondeterminism in rewriting as long as termination holds.

Definition 2.1. Let \mathcal{R} be a TRS. An *overlap* is a triple $\langle \ell_1 \rightarrow r_1, p, \ell_2 \rightarrow r_2 \rangle$ satisfying the following properties:

- ▷ $\ell_1 \rightarrow r_1$ and $\ell_2 \rightarrow r_2$ are variants of rewrite rules of \mathcal{R} without common variables,
- ▷ $p \in \mathcal{Pos}_{\mathcal{F}}(\ell_2)$,
- ▷ ℓ_1 and $\ell_2|_p$ are unifiable and
- ▷ if $p = \epsilon$ then $\ell_1 \rightarrow r_1$ and $\ell_2 \rightarrow r_2$ are not variants.

Overlaps give rise to *critical peaks* from which the critical pairs can then be extracted.

Definition 2.2. Let $\langle \ell_1 \rightarrow r_1, p, \ell_2 \rightarrow r_2 \rangle$ be an overlap of a TRS \mathcal{R} and σ a most general unifier of ℓ_1 and $\ell_2|_p$. The term $\ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma$ can be rewritten in two different ways which gives rise to a critical peak, here illustrated graphically:

$$\begin{array}{ccc}
 & \ell_2\sigma[\ell_1\sigma]_p = \ell_2\sigma & \\
 \swarrow \scriptstyle p & & \searrow \scriptstyle \epsilon \\
 \ell_2\sigma[r_1\sigma]_p & & r_2\sigma
 \end{array}
 \begin{array}{l}
 \leftarrow \scriptstyle r_1 \leftarrow \ell_1 \\
 \leftarrow \scriptstyle \ell_2 \rightarrow r_2
 \end{array}$$

More formally, a critical peak is a quadruple $\langle \ell_2\sigma[r_1\sigma]_p, p, \ell_2\sigma, r_2\sigma \rangle$ and the equation $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ is a critical pair of \mathcal{R} obtained from the original overlap.

Usually, we denote a critical peak $\langle t, p, s, u \rangle$ more illustratively by $t \xleftarrow{p} s \xrightarrow{\epsilon} u$. The set of all critical pairs of a TRS \mathcal{R} (obtained from all possible overlaps) is denoted by $\text{CP}(\mathcal{R})$. Knuth and Bendix' criterion [KB70] states that a terminating TRS \mathcal{R} is confluent if and only if $\text{CP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$. Kapur et al. [KMN88] showed that joinability of prime critical pairs still guarantees confluence for terminating TRSs.

Definition 2.3. A critical peak $t \xleftarrow{p} s \xrightarrow{\epsilon} u$ is *prime* if all proper subterms of $s|_p$ are in normal form. Critical pairs derived from prime critical peaks are called prime. The set of all prime critical pairs of a TRS \mathcal{R} is denoted by $\text{PCP}(\mathcal{R})$.

Theorem 2.4 [KMN88]. *Let \mathcal{R} be a terminating TRS. The TRS \mathcal{R} is confluent if and only if $\text{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$. \square*

Example 2.5. Using Theorem 2.4, we show confluence of the following TRS \mathcal{R} :

$$f(a+x) \rightarrow x \quad f(x+a) \rightarrow x \quad f(b+x) \rightarrow x \quad f(x+b) \rightarrow x \quad a \rightarrow b$$

Since every rewrite step reduces the total number of f and a , termination of \mathcal{R} follows. Observe that \mathcal{R} admits six (modulo symmetry) critical peaks of the form $t \xleftarrow{p} s \xrightarrow{\epsilon} u$:

$$\begin{array}{cccccc}
 \underline{f(a+a)} & \underline{f(a+b)} & \underline{f(b+a)} & \underline{f(b+b)} & f(\underline{a}+x) & f(x+\underline{a}) \\
 \swarrow \searrow & \swarrow \searrow & \swarrow \searrow & \swarrow \searrow & \swarrow \searrow & \swarrow \searrow \\
 a \quad a & a \quad b & a \quad b & b \quad b & f(b+x) \quad x & f(x+b) \quad x
 \end{array}$$

Here the positions p in s are indicated by underlining. The first three critical peaks are not prime due to the reducible proper subterm a in $s|_p$, while the others are prime. Therefore, $\text{PCP}(\mathcal{R}) = \{b \approx a, f(b+x) \approx x, f(x+b) \approx x\}$. Since $\text{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}$ holds, \mathcal{R} is confluent.

2.3. Rewriting Modulo. We now turn our attention to rewriting modulo an equational theory. To that end, we start by giving general definitions for abstract rewrite systems (ARSs). Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS and \sim an equivalence relation on A . We write \Leftrightarrow for $\leftarrow \cup \rightarrow \cup \sim$ (conversion modulo \sim), \rightarrow/\sim for $\sim \cdot \rightarrow \cdot \sim$ (rewriting modulo \sim) and \downarrow^{\sim} for $\rightarrow^* \cdot \sim \cdot \leftarrow^*$ (valley modulo \sim). In order to simplify our terminology, we sometimes refer to conversions modulo \sim as conversions. Hence, whether conversions include \sim -steps or not depends on the type of rewriting we consider. Given \mathcal{A} , we denote $\langle A, \rightarrow/\sim \rangle$ by \mathcal{A}/\sim . The ARS \mathcal{A} is *terminating modulo \sim* if there are no infinite rewrite sequences with \rightarrow/\sim and *Church–Rosser modulo \sim* if $\Leftrightarrow^* \subseteq \downarrow^{\sim}$.

Confluence and the Church–Rosser property do not coincide for rewriting modulo equational theories [Ohl98]. Therefore, a notion of completeness for this setting has to depend on the Church–Rosser property instead of confluence in order to facilitate a decision procedure for validity problems of equational theories with complete presentations. Hence, we define an ARS \mathcal{A} to be *complete modulo \sim* if it is terminating modulo \sim and Church–Rosser

modulo \sim . While there is no distinction for termination modulo \sim between \mathcal{A} and \mathcal{A}/\sim ($\sim \cdot \sim = \sim$ by transitivity), it makes a considerable difference whether we talk about the Church–Rosser modulo \sim property and therefore completeness modulo \sim of \mathcal{A} or \mathcal{A}/\sim .

Example 2.6. Consider the ARS \mathcal{A} together with the equivalence relation \sim defined as follows:

$$a \sim b \rightarrow c$$

Since $a \Leftrightarrow^* c$ and $a \rightarrow/\sim c$, the ARS \mathcal{A}/\sim is Church–Rosser modulo \sim . However, \mathcal{A} is not Church–Rosser modulo \sim as a and c are different normal forms of \rightarrow which are not equivalent in \sim .

The following lemma is taken from [Ave95, Lemma 4.1.12]. It establishes an important connection between the Church–Rosser modulo \sim property of an ARS \mathcal{A} and \mathcal{A}/\sim .

Lemma 2.7 [Ave95]. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ and $\mathcal{A}' = \langle A, \dashv \rangle$ be ARSs and \sim an equivalence relation on A such that $\rightarrow \subseteq \dashv \subseteq \rightarrow/\sim$. If \mathcal{A}' is Church–Rosser modulo \sim then \mathcal{A}/\sim is Church–Rosser modulo \sim . \square*

Note that the implication cannot be strengthened to an equivalence due to Example 2.6. The following example illustrates a successful use of Lemma 2.7.

Example 2.8. Consider the ARS $\mathcal{A} = \langle A, \rightarrow \rangle$ from Example 2.6. For $\mathcal{A}' = \langle A, \dashv \rangle$ with $\dashv = \sim \cdot \rightarrow$ we can easily establish the Church–Rosser modulo \sim property. Hence, by Lemma 2.7 also \mathcal{A}/\sim is Church–Rosser modulo \sim .

The definitions and results for ARSs carry over to TRSs by replacing the equivalence relation \sim by the equational theory $\leftrightarrow_{\mathcal{B}}^*$ of an ES \mathcal{B} . Most theoretical results of this article are not specific to AC but hold for an arbitrary base theory \mathcal{B} of which we only demand that $\text{Var}(\ell) = \text{Var}(r)$ for all $\ell \approx r \in \mathcal{B}$. We abbreviate $\leftrightarrow_{\mathcal{B}}^*$ by $\sim_{\mathcal{B}}$ and the rewrite relation $\rightarrow_{\mathcal{R}/\mathcal{B}}$ is defined as $\sim_{\mathcal{B}} \cdot \rightarrow_{\mathcal{R}} \cdot \sim_{\mathcal{B}}$. Furthermore, we write $\Downarrow_{\mathcal{R}}^{\sim}$ for the relation $\rightarrow_{\mathcal{R}}^* \cdot \sim_{\mathcal{B}} \cdot \mathcal{R}^* \leftarrow$. Note that the omission of \mathcal{B} in the notation $\Downarrow_{\mathcal{R}}^{\sim}$ poses no problem as \mathcal{B} is usually fixed and can be inferred from the context in all other cases. Termination modulo \mathcal{B} is shown by \mathcal{B} -compatible reduction orders $>$, i.e., $>$ is well-founded, closed under contexts and substitutions and $\sim_{\mathcal{B}} \cdot > \cdot \sim_{\mathcal{B}} \subseteq >$. Note that $\rightarrow_{\mathcal{R}/\mathcal{B}}$ is not a very practical rewrite relation: It is undecidable in general, and even if the equational theory of \mathcal{B} is decidable, rewriting a term t requires to check every member of its \mathcal{B} -equivalence class. A more practical alternative due to Peterson and Stickel [PS81] is the relation $\rightarrow_{\mathcal{R},\mathcal{B}}$ defined as follows: $s \rightarrow_{\mathcal{R},\mathcal{B}} t$ if there exist a rule $\ell \rightarrow r \in \mathcal{R}$, a substitution σ and a position p such that $s|_p \sim_{\mathcal{B}} \ell\sigma$ and $t = s[r\sigma]_p$. This definition is very similar to the definition of standard rewriting but with \mathcal{B} -matching instead of normal matching. It is immediate from the respective definitions that the inclusions $\rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R},\mathcal{B}} \subseteq \rightarrow_{\mathcal{R}/\mathcal{B}}$ hold.

Example 2.9. Consider the TRS \mathcal{R} consisting of the rules

$$0 + y \rightarrow y \qquad \mathfrak{s}(x) + y \rightarrow \mathfrak{s}(x + y)$$

where $+$ is an AC symbol. We have $y + \mathfrak{s}(x) \rightarrow_{\mathcal{R},\text{AC}} \mathfrak{s}(x + y)$ as $y + \mathfrak{s}(x) \sim_{\text{AC}} \mathfrak{s}(x) + y$ but $y + \mathfrak{s}(x)$ is a normal form with respect to $\rightarrow_{\mathcal{R}}$. Furthermore, $(y + z) + \mathfrak{s}(x) \rightarrow_{\mathcal{R}/\text{AC}} \mathfrak{s}(x + y) + z$ since $(y + z) + \mathfrak{s}(x) \sim_{\text{AC}} (\mathfrak{s}(x) + y) + z$ but this step is not possible with $\rightarrow_{\mathcal{R},\text{AC}}$ as the rewrite step takes place at position 1 whereas we need to search for AC equivalent terms at the root position.

3. CHURCH–ROSSER CRITERION

In this section we present a new characterization of the Church–Rosser property modulo an equational theory \mathcal{B} for left-linear TRSs which are terminating modulo \mathcal{B} . The original left-linear completion procedure [Ave95, Bac91] relies on the following theorem. We use critical pairs between two different TRSs \mathcal{R}_1 and \mathcal{R}_2 with the same signature. We write $\text{CP}(\mathcal{R}_1, \mathcal{R}_2)$ for the set of all critical pairs originating from overlaps $\langle \rho_1, p, \rho_2 \rangle$ where $\rho_1 \in \mathcal{R}_1$ and $\rho_2 \in \mathcal{R}_2$. The union of $\text{CP}(\mathcal{R}_1, \mathcal{R}_2)$ and $\text{CP}(\mathcal{R}_2, \mathcal{R}_1)$ is denoted by $\text{CP}^\pm(\mathcal{R}_1, \mathcal{R}_2)$.

Theorem 3.1 [Hue80]. *A left-linear TRS \mathcal{R} which is terminating modulo \mathcal{B} is Church–Rosser modulo \mathcal{B} if and only if $\text{CP}(\mathcal{R}) \cup \text{CP}^\pm(\mathcal{R}, \mathcal{B}^\pm) \subseteq \downarrow_{\mathcal{R}}^\sim$. \square*

Example 3.2 (continued from Example 2.5). Let $\mathcal{B} = \{x + y \approx y + x\}$. We show that the left-linear TRS \mathcal{R} of Example 2.5 is Church–Rosser modulo \mathcal{B} . As before, the termination of \mathcal{R}/\mathcal{B} follows from the fact that every rewrite step reduces the total number of f and a . The six critical peaks result in $\text{CP}(\mathcal{R}) = \{a \approx a, a \approx b, b \approx a, b \approx b, f(b + x) \approx x, f(x + b) \approx x\}$. Observe that \mathcal{R} and \mathcal{B} admit four critical peaks of the forms $t \xrightarrow{\mathcal{R}}^p s \leftrightarrow_{\mathcal{B}}^\epsilon u$ or $t \leftrightarrow_{\mathcal{B}}^p s \xrightarrow{\mathcal{R}}^\epsilon u$:

$$\begin{array}{cccc}
 \begin{array}{c} f(a + x) \\ \swarrow \searrow \\ f(x + a) \quad x \end{array} &
 \begin{array}{c} f(x + a) \\ \swarrow \searrow \\ f(a + x) \quad x \end{array} &
 \begin{array}{c} f(b + x) \\ \swarrow \searrow \\ f(x + b) \quad x \end{array} &
 \begin{array}{c} f(x + b) \\ \swarrow \searrow \\ f(b + x) \quad x \end{array}
 \end{array}$$

Thus, $\text{CP}^\pm(\mathcal{R}, \mathcal{B}^\pm) = \{f(x + a) \approx x, f(a + x) \approx x, f(x + b) \approx x, f(b + x) \approx x\}$. It is easy to see that $t \downarrow_{\mathcal{R}}^\sim u$ for all critical pairs $t \approx u$ in $\text{CP}(\mathcal{R}) \cup \text{CP}^\pm(\mathcal{R}, \mathcal{B}^\pm)$. Hence, by Theorem 3.1, the TRS \mathcal{R} is Church–Rosser modulo \mathcal{B} .

Theorem 3.1 allows us to use ordinary critical pairs instead of \mathcal{B} -critical pairs, i.e., critical pairs stemming from local peaks modulo \mathcal{B} ($\mathcal{R} \leftarrow \cdot \sim_{\mathcal{B}} \cdot \rightarrow_{\mathcal{R}}$) [JK86]. In particular, equational unification modulo \mathcal{B} can be replaced by syntactic unification which improves efficiency. Furthermore, the form of the joining sequence ($\downarrow_{\mathcal{R}}^\sim$) is advantageous as it uses the normal rewrite relation and just one \mathcal{B} -equality check in the end as opposed to rewrite steps modulo the theory ($\sim_{\mathcal{B}} \cdot \rightarrow_{\mathcal{R}} \cdot \sim_{\mathcal{B}}$). However, left-linearity is necessary in Theorem 3.1 as the following example illustrates.

Example 3.3. Consider the AC-terminating TRS \mathcal{R} consisting of the single rule $f(x, y) \rightarrow x + y$ with $+$ as an additional AC function symbol as well as the following conversion:

$$x + y \xrightarrow{\mathcal{R}} f(x + y, x + y) \sim_{\text{AC}} f(x + y, y + x)$$

There are no critical pairs in \mathcal{R} and between \mathcal{R} and AC^\pm , so $\text{CP}(\mathcal{R}) = \text{CP}^\pm(\mathcal{R}, \text{AC}^\pm) = \emptyset$. However, $x + y \not\downarrow_{\mathcal{R}}^\sim f(x + y, y + x)$ does not hold because $x + y$ and $f(x + y, y + x)$ are \mathcal{R} -normal forms which are not AC equivalent. Thus, \mathcal{R} is not Church–Rosser modulo AC.

Even though most results in this article only apply to left-linear TRSs, we do not demand that \mathcal{B} is linear (i.e., having equations only consisting of linear terms). The reason for that can be found in the proof of Theorem 3.1 where left-linearity is crucial but only needed for rewrite rules: If \mathcal{R} is not left-linear, the proof reveals that we also have to consider variable overlaps (i.e., overlaps at variable positions) between \mathcal{R} and \mathcal{B} . This would make a confluence analysis solely based on critical pairs impossible. In the remainder of this section we show that joinability of prime critical pairs suffices for the characterization of Theorem 3.1.

3.1. Peak-and-Cliff Decreasingness. We present a new Church–Rosser modulo criterion, dubbed *peak-and-cliff decreasingness*. This is an extension of peak decreasingness [HMSW19] which is a simple confluence criterion for ARSs designed to replace complicated proof orderings in the correctness proofs of completion procedures. As such, peak-and-cliff decreasingness will be a crucial ingredient in the correctness proof we give for Avenhaus’ inference system in Section 4.

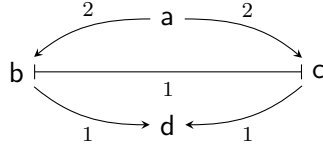
In the following, we assume that equivalence relations \sim are defined as the reflexive and transitive closure of a symmetric relation \vdash , so $\sim = \vdash^*$. We refer to conversions of the form $\leftarrow \cdot \vdash$ or $\vdash \cdot \rightarrow$ as *local cliffs* and conversions of the form $\leftarrow \cdot \rightarrow$ as *local peaks*. Furthermore, we assume that rewrite and equality steps are labeled with labels from the same set I , so let $\mathcal{A} = \langle A, \{\rightarrow_\alpha \mid \alpha \in I\} \rangle$ be an ARS and $\sim = (\bigcup \{\vdash_\alpha \mid \alpha \in I\})^*$ an equivalence relation on A . Note that several different steps can have the same label. Furthermore, for the sake of better readability, we allow ourselves some freedom of where to annotate our arrow relations with labels, closure operators (reflexive, transitive, ...) and the like. This will cause no confusion.

Definition 3.4. An ARS \mathcal{A} is *peak-and-cliff decreasing* if there is a well-founded order $>$ on I such that for all $\alpha, \beta \in I$ the inclusions

$$\alpha \leftarrow \cdot \rightarrow \beta \subseteq \overset{*}{\longleftarrow}_{\forall \alpha \beta} \qquad \alpha \leftarrow \cdot \vdash \beta \subseteq \overset{*}{\longleftarrow}_{\forall \alpha} \cdot \overset{=}{\longleftarrow}_{\beta}$$

hold. Here $\forall \alpha \beta$ denotes the set $\{\gamma \in I \mid \alpha > \gamma \text{ or } \beta > \gamma\}$ and if $J \subseteq I$ then \rightarrow_J denotes $\bigcup \{\rightarrow_\gamma \mid \gamma \in J\}$. We abbreviate $\forall \alpha$ to $\forall \alpha$.

Example 3.5. Let $I = \{1, 2\}$ and consider the following I -labeled ARS $\mathcal{A} = \langle A, \{\rightarrow_1, \rightarrow_2\} \rangle$ equipped with the equivalence relation $\sim = (\vdash_1 \cup \vdash_2)^*$:



Using the well-founded order $2 > 1$ it is easily established that \mathcal{A} is peak-and-cliff decreasing: For the only local peak $b \xrightarrow{2} a \xrightarrow{2} c$ we have $b \vdash_1 c$. The two local cliffs $d \xrightarrow{1} b \vdash_1 c$ and $d \xrightarrow{1} c \vdash_1 b$ are handled by $d \xrightarrow{1} c$ and $d \xrightarrow{1} b$, respectively.

We show that peak-and-cliff decreasingness is a sufficient condition for the Church–Rosser modulo property.

Lemma 3.6. *Every conversion modulo \sim is a valley modulo \sim or contains a local peak or cliff:*

$$\Leftrightarrow^* \subseteq \Downarrow \sim \cup \Leftrightarrow^* \cdot \leftarrow \cdot \rightarrow \cdot \Leftrightarrow^* \cup \Leftrightarrow^* \cdot \vdash \cdot \rightarrow \cdot \Leftrightarrow^* \cup \Leftrightarrow^* \cdot \leftarrow \cdot \vdash \cdot \Leftrightarrow^*$$

Proof. We abbreviate $\Leftrightarrow^* \cdot \leftarrow \cdot \rightarrow \cdot \Leftrightarrow^* \cup \Leftrightarrow^* \cdot \vdash \cdot \rightarrow \cdot \Leftrightarrow^* \cup \Leftrightarrow^* \cdot \leftarrow \cdot \vdash \cdot \Leftrightarrow^*$ to \Leftarrow . Suppose $a \Leftrightarrow^n b$. We show $a \Downarrow \sim b$ or $a \Leftarrow b$ by induction on n . If $n = 0$ then $a = b$ and therefore also $a \Downarrow \sim b$. If $n > 0$ then $a \Leftrightarrow c \Leftrightarrow^{n-1} b$ for some c . The induction hypothesis yields $c \Downarrow \sim b$ or $c \Leftarrow b$. In the latter case we are already done because $\Leftrightarrow \cdot \Leftarrow \subseteq \Leftarrow$. In the former case, note that there exists a k such that $c \rightarrow^k \cdot \sim \cdot \xrightarrow{*} b$. We continue with a case analysis on k :

$\triangleright k = 0$: From $a \Leftrightarrow c$ we obtain $a \rightarrow c$, $a \leftarrow c$ or $a \sim c$. If $a \rightarrow c$ we immediately obtain $a \Downarrow \sim b$. If $a \leftarrow c$ we have $a \leftarrow c \sim c' \xrightarrow{*} b$ for some c' . Now $c = c'$ and hence $a \Downarrow \sim b$ or $c \vdash \cdot \sim c'$ and therefore $a \Leftarrow b$. If $a \sim c$ we have $a \Downarrow \sim b$ because \sim is transitive.

▷ $k > 0$: From $a \Leftrightarrow c$ we obtain $a \rightarrow c$, $a \leftarrow c$ or $a \sim c$. If $a \rightarrow c$ we immediately obtain $a \downarrow \sim b$. If $a \leftarrow c$ then there exists a c' such that $a \leftarrow c \rightarrow c' \Leftrightarrow^* b$ and therefore $a \Leftarrow b$. Finally, if $a \sim c$ then $a \sim c \rightarrow c' \Leftrightarrow^* b$ for some c' . If $a = c$ then we obtain $a \downarrow \sim b$ from the induction hypothesis as there is a conversion between a and b of length $n - 1$. Otherwise, $a \sim \cdot \vdash c$ and therefore $a \Leftarrow b$. \square

The proof of the following theorem relies on the fact that the well-founded order on an index set obtained from peak-and-cliff decreasingness can be extended to a well-founded order on multisets of labels. Here, the multiset extension of an order $>$ is defined as follows: $M_1 >_{\text{mul}} M_2$ if $M_2 = (M_1 \setminus X) \uplus Y$ where $\emptyset \neq X \subseteq M_1$ and for all $y \in Y$ there exists an $x \in X$ such that $x > y$. It is well-known that the multiset extension of a well-founded order is also well-founded [DM79].

Theorem 3.7. *If \mathcal{A} is a peak-and-cliff decreasing ARS then \mathcal{A} is Church–Rosser modulo \sim .*

Proof. With every conversion C we associate a multiset M_C consisting of labels of its rewrite and equivalence relation steps. Since \mathcal{A} is peak-and-cliff decreasing, there is a well-founded order $>$ on I which allows us to replace conversions C of the forms $\alpha \leftarrow \cdot \rightarrow \beta$, $\alpha \leftarrow \cdot \vdash \beta$ and $\vdash \beta \cdot \rightarrow \alpha$ by conversions C' where $M_C >_{\text{mul}} M_{C'}$. Hence, we prove that \mathcal{A} is Church–Rosser modulo \sim , i.e., $\Leftrightarrow^* \subseteq \downarrow \sim$, by well-founded induction on $>_{\text{mul}}$. Consider a conversion $a \Leftrightarrow^* b$ which we call C . By Lemma 3.6 we have $a \downarrow \sim b$ (which includes the case that C is empty) or one of the following cases holds:

$$a \Leftrightarrow^* \cdot \leftarrow \cdot \rightarrow \cdot \Leftrightarrow^* b \quad a \Leftrightarrow^* \cdot \leftarrow \cdot \vdash \cdot \Leftrightarrow^* b \quad a \Leftrightarrow^* \cdot \vdash \cdot \rightarrow \cdot \Leftrightarrow^* b$$

If $a \downarrow \sim b$ we are immediately done. In the remaining cases, we have a local peak or cliff with concrete labels α and β , so $M_C = \Gamma_1 \uplus \{\alpha, \beta\} \uplus \Gamma_2$. Since \mathcal{A} is peak-and-cliff decreasing, there is a conversion C' with $M_{C'} = \Gamma_1 \uplus \Gamma \uplus \Gamma_3$ where $\{\alpha, \beta\} >_{\text{mul}} \Gamma$. Hence, $M_C >_{\text{mul}} M_{C'}$ and we finish the proof by applying the induction hypothesis. \square

The above theorem can also be shown by verifying that it is a special case of the Church–Rosser modulo criterion known as decreasing diagrams [FvO13, Theorem 31]. Note, however, that it is not as obvious as the fact that peak decreasingness [HMSW19] is an instance of decreasing diagrams for confluence [vO94].¹

For the main result of this section, a simpler version of peak-and-cliff decreasingness suffices. The full power of peak-and-cliff decreasingness will be needed in the correctness proof of Avenhaus' inference system in Section 4.

Definition 3.8. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS equipped with a \sim -compatible well-founded order $>$ on A and $\sim = \vdash^*$ an equivalence relation on A . We write $b \xrightarrow{a} c$ ($b \vdash^a c$) if $b \rightarrow c$ ($b \vdash c$) and $b \sim a$, i.e., steps are labeled with elements of A as indices. We say that \mathcal{A} is *source decreasing modulo \sim* if the inclusions

$$\leftarrow a \rightarrow \subseteq \xleftrightarrow[\vee a]{*} \quad \leftarrow a \vdash \subseteq \xleftrightarrow[\vee a]{*} \cdot \xleftarrow{a}$$

hold for all $a \in A$. Here $\leftarrow a \rightarrow$ ($\leftarrow a \vdash$) denotes the binary relation consisting of all pairs (b, c) such that $a \rightarrow b$ and $a \rightarrow c$ ($a \vdash c$). Furthermore, $\xleftrightarrow[\vee a]{*}$ denotes the binary relation consisting of all pairs of elements which are connected by a conversion where each step is labeled by an element smaller than a .

Corollary 3.9. *Every ARS that is source decreasing modulo \sim is Church–Rosser modulo \sim .*

¹Pointed out by Vincent van Oostrom (personal communication).

Proof. In the definition of peak-and-cliff decreasingness we set $I = A$. Note that this implies $\alpha = \beta$ for all local peaks and cliffs. Hence, the ARS is peak-and-cliff decreasing and we can conclude by Theorem 3.7. \square

3.2. Prime Critical Pairs. We show that joinability of prime critical pairs is enough for characterizing the Church–Rosser modulo property. In the following, $\text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm)$ denotes the restriction of $\text{CP}^\pm(\mathcal{R}, \mathcal{B}^\pm)$ to prime critical pairs but where irreducibility is always checked with respect to \mathcal{R} , i.e., the critical peaks $t \xrightarrow{\mathcal{R}}^p s \xleftrightarrow{\mathcal{B}}^\epsilon u$ and $t' \xleftrightarrow{\mathcal{B}}^p s \xrightarrow{\mathcal{R}}^\epsilon u'$ are both prime if proper subterms of $s|_p$ are irreducible with respect to \mathcal{R} .

Example 3.10 (continued from Example 3.2). Recall the four critical peaks between \mathcal{R} and \mathcal{B} . The first two peaks $f(x + a) \xrightarrow{\mathcal{B}}^1 f(a + x) \rightarrow_{\mathcal{R}} x$ and $f(a + x) \xrightarrow{\mathcal{B}}^1 f(x + a) \rightarrow_{\mathcal{R}} x$ are not prime due to the reducible proper subterm a . The other two are prime. Hence, $\text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm) = \{f(b + x) \approx x, f(x + b) \approx x\}$.

Correctness of Theorem 3.1 can be shown by the combination of Corollary 3.9 with the following lemma.

Lemma 3.11 [Hue80]. *For left-linear TRSs \mathcal{R} , the following inclusion holds:*

$$\mathcal{R} \leftarrow \cdot \leftrightarrow_{\mathcal{B}} \subseteq \downarrow_{\mathcal{R}}^{\sim} \cup \leftrightarrow_{\text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm)} \quad \square$$

In order to integrate the refinement by prime critical pairs some more observations are required. Note that for the original refinement by Kapur et al. [KMN88], correctness is shown in the context of general AC rewriting by flattening terms with AC symbols. We employ our novel notion of peak-and-cliff decreasingness instead. Our proof can be seen as an extension of the corresponding proof given for the ordinary Church–Rosser property in [HMSW19].

Definition 3.12. Given a TRS \mathcal{R} and terms s, t and u , we write $t \nabla_s u$ if $s \rightarrow_{\mathcal{R}}^+ t$, $s \rightarrow_{\mathcal{R}}^+ u$ and $t \downarrow_{\mathcal{R}} u$ or $t \leftrightarrow_{\text{PCP}(\mathcal{R})} u$. We write $t \nabla_s^{\sim} u$ if $s \rightarrow_{\mathcal{R}}^+ t$, $s \sim u$ and $t \downarrow_{\mathcal{R}}^{\sim} u$ or $t \leftrightarrow_{\text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm)} u$. Furthermore, $\sim_s \nabla = \{(u, t) \mid t \nabla_s^{\sim} u\}$.

Note that the joinability of ordinary critical peaks is not affected by incorporating \mathcal{B} into conversions. Hence, the following result is taken from [HMSW19, Lemma 2.15] and therefore stated without proof. Here, $t \nabla_s^2 u$ means that there is a term v with $t \nabla_s v$ and $v \nabla_s u$.

Lemma 3.13 [HMSW19]. *If $t \xrightarrow{\mathcal{R}}^p s \xrightarrow{\mathcal{R}}^\epsilon u$ is a critical peak of a TRS \mathcal{R} then $t \nabla_s^2 u$. \square*

Lemma 3.14. *Let \mathcal{R} be a left-linear TRS.*

- (1) *If $t \xrightarrow{\mathcal{R}}^p s \xrightarrow{\mathcal{B}}^\epsilon u$ is a critical peak then $t \nabla_s \cdot \nabla_s^{\sim} u$.*
- (2) *If $t \xrightarrow{\mathcal{B}}^p s \xrightarrow{\mathcal{R}}^\epsilon u$ is a critical peak then $t \sim_s \nabla \cdot \nabla_s u$.*

Proof. We only prove (1) as the other statement is symmetrical. If all proper subterms of $s|_p$ are in normal form with respect to $\rightarrow_{\mathcal{R}}$, $t \approx u \in \text{PCP}(\mathcal{R}, \mathcal{B}^\pm)$ which establishes $t \nabla_s^{\sim} u$. Since also $t \nabla_s t$, we obtain the desired result. Otherwise, there are a position $q > p$ and a term v such that $s \xrightarrow{\mathcal{R}}^q v$ and all proper subterms of $s|_q$ are in normal form with respect to $\rightarrow_{\mathcal{R}}$. Together with Lemma 3.11 we obtain $v \downarrow_{\mathcal{R}}^{\sim} u$ or $v \leftrightarrow_{\text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm)} u$. In both cases

$v \nabla_s^{\sim} u$ holds. A similar case analysis applies to the local peak $t \xleftarrow{\frac{p}{\mathcal{R}}} s \xrightarrow{\frac{q}{\mathcal{R}}} v$: By the Critical Pair Lemma, either $t \downarrow_{\mathcal{R}} v$ or $t \leftrightarrow_{\text{CP}(\mathcal{R})} v$. In the latter case

$$v|_p \xleftarrow{\frac{q \wedge p}{\mathcal{R}}} s|_p \xrightarrow{\frac{\epsilon}{\mathcal{R}}} t|_p$$

is an instance of a prime critical peak as $q > p$ and all proper subterms of $s|_q$ are in normal form with respect to $\rightarrow_{\mathcal{R}}$. Closure of rewriting under contexts and substitutions yields $t \leftrightarrow_{\text{PCP}(\mathcal{R})} v$. Therefore, we have $t \nabla_s v$ in both cases, concluding the proof. \square

The following lemma generalizes the previous results of this section to arbitrary local peaks and cliffs.

Lemma 3.15. *Let \mathcal{R} be a left-linear TRS.*

- (1) *If $t \xrightarrow{\mathcal{R}} s \rightarrow_{\mathcal{R}} u$ then $t \nabla_s^2 u$.*
- (2) *If $t \xrightarrow{\mathcal{R}} s \leftrightarrow_{\mathcal{B}} u$ then $t \nabla_s \cdot \nabla_s^{\sim} u$.*

Proof. We only prove (2) as the proof of (1) (which depends on the Critical Pair Lemma) can be found in [HMSW19, Lemma 2.16]. Let $t \xrightarrow{\mathcal{R}} s \leftrightarrow_{\mathcal{B}} u$. From Lemma 3.11 we obtain $t \downarrow_{\mathcal{R}}^{\sim} u$ or $t \leftrightarrow_{\text{CP}^{\pm}(\mathcal{R}, \mathcal{B}^{\pm})} u$. In the former case we are done as $t \nabla_s u \nabla_s u$. For the latter case we further distinguish between the two subcases $t \rightarrow_{\text{CP}(\mathcal{R}, \mathcal{B}^{\pm})} u$ and $u \rightarrow_{\text{CP}(\mathcal{B}^{\pm}, \mathcal{R})} t$. Note that this list of subcases is exhaustive due to the direction of the local cliff. If $t \rightarrow_{\text{CP}(\mathcal{R}, \mathcal{B}^{\pm})} u$, $t \nabla_s \cdot \nabla_s^{\sim} u$ follows from Lemma 3.14(1) and closure of rewriting under contexts and substitutions. If $u \rightarrow_{\text{CP}(\mathcal{B}^{\pm}, \mathcal{R})} t$, $u \xrightarrow{\sim} \nabla_s \cdot \nabla_s t$ and therefore $t \nabla_s \cdot \nabla_s^{\sim} u$ follows from Lemma 3.14(2) as well as closure of rewriting under contexts and substitutions. \square

Now, we are able to prove the main result of this section, a novel necessary and sufficient condition for the Church–Rosser property modulo an equational theory \mathcal{B} which strengthens the original result from [Hue80] to prime critical pairs.

Theorem 3.16. *A left-linear TRS \mathcal{R} which is terminating modulo \mathcal{B} is Church–Rosser modulo \mathcal{B} if and only if $\text{PCP}(\mathcal{R}) \cup \text{PCP}^{\pm}(\mathcal{R}, \mathcal{B}^{\pm}) \subseteq \downarrow_{\mathcal{R}}^{\sim}$.*

Proof. The only-if direction is trivial. For a proof of the if direction, we show that \mathcal{R} is source decreasing modulo \mathcal{B} ; the Church–Rosser property modulo \mathcal{B} is then an immediate consequence of Corollary 3.9. From the termination of \mathcal{R} modulo \mathcal{B} we obtain the well-founded order $> = \rightarrow_{\mathcal{R}/\mathcal{B}}^{+}$.

Consider an arbitrary local peak $t \xrightarrow{\mathcal{R}} s \rightarrow_{\mathcal{R}} u$. Lemma 3.15(1) yields a term v such that $t \nabla_s v \nabla_s u$. Together with $\text{PCP}(\mathcal{R}) \subseteq \downarrow_{\mathcal{R}}^{\sim}$ we obtain $t \downarrow_{\mathcal{R}}^{\sim} v \downarrow_{\mathcal{R}}^{\sim} u$. By definition, $s > t, v, u$ so the corresponding condition required by source decreasingness modulo \mathcal{B} is fulfilled.

Now consider an arbitrary local cliff $t \xrightarrow{\mathcal{R}} s \leftrightarrow_{\mathcal{B}} u$. Lemma 3.15(2) yields a term v such that $t \nabla_s v \nabla_s^{\sim} u$. Together with $\text{PCP}(\mathcal{R}) \cup \text{PCP}^{\pm}(\mathcal{R}, \mathcal{B}^{\pm}) \subseteq \downarrow_{\mathcal{R}}^{\sim}$ we obtain $t \downarrow_{\mathcal{R}}^{\sim} v \downarrow_{\mathcal{R}}^{\sim} u$. By definition, $s > t, v$ and $s \sim u$. The conversion between v and u is of the form $v \rightarrow_{\mathcal{R}}^* \cdot \sim \cdot \frac{k}{\mathcal{R}} \leftarrow u$ for some k . If $k = 0$ then all steps between v and u can be labeled with terms which are smaller than s . If $k > 0$ then there exists a $w < s$ such that $v \rightarrow_{\mathcal{R}}^* \cdot \sim \cdot \frac{k-1}{\mathcal{R}} \leftarrow w \xrightarrow{\mathcal{R}} u$. In this case all steps of the conversion are labeled with terms which are smaller than s except for the rightmost step which we may label with s . Hence, the corresponding condition required by source decreasingness modulo \mathcal{B} is fulfilled in all cases. \square

Example 3.17 (continued from Example 3.10). One can verify the termination of \mathcal{R}/\mathcal{B} and the inclusion $\text{PCP}(\mathcal{R}) \cup \text{PCP}^\pm(\mathcal{R}, \mathcal{B}^\pm) \subseteq \downarrow_{\mathcal{R}}^\sim$. By Theorem 3.16 the Church–Rosser modulo property holds.

Finally, we show that the previous result does not hold if we just demand termination of \mathcal{R} . The counterexample shows this for the concrete case of AC and is based on Example 4.1.8 from [Ave95] which uses an ARS. Note that the usage of prime critical pairs instead of critical pairs has no effect.

Example 3.18. Consider the TRS \mathcal{R} consisting of the rules

$$\begin{aligned} (b+a)+a &\rightarrow a+(a+b) & (a+b)+a &\rightarrow a+(a+b) & (a+a)+b &\rightarrow a+(a+b) \\ a+(a+b) &\rightarrow b+(a+a) & b+(a+a) &\rightarrow c \\ a+(a+b) &\rightarrow a+(b+a) & a+(b+a) &\rightarrow d \end{aligned}$$

where $+$ is an AC function symbol. Clearly, the (prime) critical pairs of \mathcal{R} are joinable modulo AC because $b+(a+a) \sim_{\text{AC}} a+(b+a)$. For $\text{PCP}^\pm(\mathcal{R}, \text{AC}^\pm)$ we only have to consider the rules which rewrite to c and d respectively since all other rules only involve AC equivalent terms. Modulo symmetry, these (prime) critical pairs are:

$$\begin{aligned} c &\approx b+(a+a) & c &\approx (a+a)+b & c &\approx (b+a)+a \\ c+x &\approx b+((a+a)+x) & x+c &\approx (x+b)+(a+a) \\ d &\approx a+(a+b) & d &\approx (b+a)+a & d &\approx (a+b)+a \\ d+x &\approx a+((b+a)+x) & x+d &\approx (x+a)+(b+a) \end{aligned}$$

Removing the joinable (prime) critical pairs leaves us with

$$\begin{aligned} c+x &\approx b+((a+a)+x) & x+c &\approx (x+b)+(a+a) \\ d+x &\approx a+((b+a)+x) & x+d &\approx (x+a)+(b+a) \end{aligned}$$

which are not joinable at the moment. However, we can extend \mathcal{R} by the rewrite rules

$$\begin{aligned} b+((a+a)+x) &\rightarrow (b+(a+a))+x & (x+b)+(a+a) &\rightarrow x+(b+(a+a)) \\ a+((b+a)+x) &\rightarrow (a+(b+a))+x & (x+a)+(b+a) &\rightarrow x+(a+(b+a)) \end{aligned}$$

in order to make them joinable. Note that the additional (prime) critical pairs in $\text{PCP}(\mathcal{R}) \cup \text{PCP}^\pm(\mathcal{R}, \text{AC}^\pm)$ caused by adding the new rules are trivially joinable modulo AC as all of these new critical pairs are AC equivalent. To sum up, $\text{PCP}(\mathcal{R}) \cup \text{PCP}^\pm(\mathcal{R}, \text{AC}^\pm) \subseteq \downarrow_{\mathcal{R}}^\sim$. Termination of \mathcal{R} can be checked by e.g. the termination tool $\text{T}\overline{\text{T}}\text{T}_2$ [KSZM09], but the loop

$$a+(a+b) \rightarrow_{\mathcal{R}} a+(b+a) \sim_{\text{AC}} a+(a+b)$$

shows that \mathcal{R} is not AC terminating. We have $c \Leftrightarrow^* d$ but not $c \downarrow_{\mathcal{R}}^\sim d$ as the terms are normal forms and not AC equivalent. Hence, \mathcal{R} is not Church–Rosser modulo AC.

4. AVENHAUS' INFERENCE SYSTEM

The idea of completion modulo an equational theory \mathcal{B} for left-linear systems where the normal rewrite relation can be used to decide validity problems has been put forward by Huet [Hue80]. To the best of our knowledge, inference systems for this approach are only presented in the books by Avenhaus [Ave95] and Bachmair [Bac91]. This section presents a new correctness proof of a version of Avenhaus' inference system for finite runs in the spirit

of [HMSW19] which does not rely on proof orderings. Correctness of Bachmair's system is established by a simulation result in Section 5.

4.1. Inference System.

Definition 4.1. The inference system \mathbf{A} is parameterized by a fixed \mathcal{B} -compatible reduction order $>$. It transforms pairs consisting of an ES \mathcal{E} and a TRS \mathcal{R} over the common signature \mathcal{F} according to the following inference rules where $s \approx^\pm t$ denotes either $s \approx t$ or $t \approx s$:

$$\begin{array}{ll}
\text{deduce} & \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \mathcal{R} \leftarrow \cdot \rightarrow_{\mathcal{R}} t & \text{orient} & \frac{\mathcal{E} \uplus \{s \approx^\pm t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}} \quad \text{if } s > t \\
\text{deduce} & \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \rightarrow s\}} \quad \text{if } s \mathcal{R} \leftarrow \cdot \leftrightarrow_{\mathcal{B}} t & \text{delete} & \frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}} \quad \text{if } s \sim_{\mathcal{B}} t \\
\text{simplify} & \frac{\mathcal{E} \uplus \{s \approx^\pm t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \rightarrow_{\mathcal{R}/\mathcal{B}} u & \text{collapse} & \frac{\mathcal{E}, \mathcal{R} \uplus \{s \rightarrow t\}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \rightarrow_{\mathcal{R}} u \\
& & \text{compose} & \frac{\mathcal{E}, \mathcal{R} \uplus \{s \rightarrow t\}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow u\}} \quad \text{if } t \rightarrow_{\mathcal{R}/\mathcal{B}} u
\end{array}$$

A step in an inference system \mathbf{I} from an ES \mathcal{E} and a TRS \mathcal{R} to an ES \mathcal{E}' and a TRS \mathcal{R}' is denoted by $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{I}} (\mathcal{E}', \mathcal{R}')$. The parentheses of the pairs are only used when the expression is surrounded by text in order to increase readability.

Definition 4.2. Let \mathcal{E} be an ES. A finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathbf{A}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathbf{A}} \cdots \vdash_{\mathbf{A}} \mathcal{E}_n, \mathcal{R}_n$$

with $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \emptyset$ is a *run* for \mathcal{E} . If $\mathcal{E}_n \neq \emptyset$, the run *fails*. The run is *fair* if \mathcal{R}_n is left-linear and the following inclusions hold:

$$\text{PCP}(\mathcal{R}_n) \subseteq \downarrow_{\tilde{\mathcal{R}}_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{E}_i \cup \mathcal{R}_i} \quad \text{PCP}^\pm(\mathcal{R}_n, \mathcal{B}^\pm) \subseteq \downarrow_{\tilde{\mathcal{R}}_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{R}_i}$$

Intuitively, fair and non-failing runs yield a \mathcal{B} -complete presentation \mathcal{R}_n of the initial set of equations \mathcal{E} , i.e., $\leftrightarrow_{\mathcal{E} \cup \mathcal{B}}^* = \leftrightarrow_{\mathcal{R}_n \cup \mathcal{B}}^* \subseteq \downarrow_{\tilde{\mathcal{R}}_n}$. In particular, the inference rules are designed to preserve the equational theory augmented by \mathcal{B} .

Example 4.3. In this example we illustrate a successful run for the ES \mathcal{E} consisting of the equations

$$f(x+y) \stackrel{1}{\approx} f(x) + f(y) \quad f(0) \stackrel{2}{\approx} 0 \quad x+0 \stackrel{3}{\approx} x$$

where $+$ is an AC function symbol. This example is taken from [Ave95, Example 4.2.15(b)]. As suggested by Definition 4.2 we only consider prime critical pairs. As AC-compatible reduction order we use the polynomial interpretation [BL87]

$$+_{\mathbb{N}}(x, y) = x + y + 1 \quad f_{\mathbb{N}}(x) = x^2 + x \quad 0_{\mathbb{N}} = 1$$

and start by orienting equations 2 and 3 into the rules

$$f(0) \xrightarrow{2'} 0 \quad x+0 \xrightarrow{3'} x$$

which only leads to prime critical pairs between rule 3' and AC^\pm . We add these prime critical pairs as rules by applying **deduce**:

$$\begin{array}{ccc} 0 + x \xrightarrow{4} x & & x + (y + 0) \xrightarrow{6} x + y \\ x + (0 + y) \xrightarrow{5} x + y & & (x + y) + 0 \xrightarrow{7} x + y \end{array}$$

Keeping rule 4 enables us to collapse the remaining three rules to the equations

$$x + y \overset{5'}{\approx} x + y \qquad x + y \overset{6'}{\approx} x + y \qquad x + y \overset{7'}{\approx} x + y$$

which can all be deleted. Next, we deduce the prime critical pair stemming from rules 3' and 4 which is just the trivial equation $0 \approx 0$ and therefore can be deleted. We continue by deducing the prime critical pairs between rule 4 and AC^\pm which adds the new rules

$$0 + (x + y) \xrightarrow{8} x + y \qquad (0 + x) + y \xrightarrow{9} x + y \qquad (x + 0) + y \xrightarrow{10} x + y$$

which can all be collapsed to trivial equations

$$x + y \overset{8'}{\approx} x + y \qquad x + y \overset{9'}{\approx} x + y \qquad x + y \overset{10'}{\approx} x + y$$

and therefore deleted. Now we orient the only remaining original equation 1 to

$$f(x + y) \xrightarrow{1'} f(x) + f(y)$$

which gives rise to two prime critical pairs between rules 1' and 3' as well as rule 4

$$f(x) + f(0) \overset{11}{\approx} f(x) \qquad f(0) + f(x) \overset{12}{\approx} f(x)$$

which can be simplified to the trivial equations

$$f(x) \overset{11'}{\approx} f(x) \qquad f(x) \overset{12'}{\approx} f(x)$$

and therefore deleted. Finally, we deduce rules corresponding to the prime critical pairs between rule 1' and AC^\pm :

$$\begin{array}{ccc} f(y + x) \xrightarrow{13} f(x) + f(y) & & f(x + (y + z)) \xrightarrow{14} f(x + y) + f(z) \\ & & f((x + y) + z) \xrightarrow{15} f(x) + f(y + z) \end{array}$$

Applications of **collapse** and **simplify** transform these rules to AC equivalent equations

$$\begin{array}{ccc} f(y) + f(x) \overset{13'}{\approx} f(x) + f(y) & & f(x) + (f(y) + f(z)) \overset{14'}{\approx} (f(x) + f(y)) + f(z) \\ & & (f(x) + f(y)) + f(z) \overset{15'}{\approx} f(x) + (f(y) + f(z)) \end{array}$$

which can be deleted. Thus, the TRS consisting of the rules 1', 2', 3' and 4 is the result of a fair and non-failing run which is an AC complete presentation of the original equations as we will show in the correctness proof.

The next example shows that deducing local cliffs as rules instead of equations as well as the restriction to $\rightarrow_{\mathcal{R}}$ in the **collapse** rule are crucial properties of the inference system.

Example 4.4. Consider the ES \mathcal{E} consisting of the single equation

$$x + 0 \approx x$$

where $+$ is an AC function symbol. We clearly have $0 + x \leftrightarrow_{\mathcal{E} \cup \text{AC}}^* x$, so an AC complete system \mathcal{C} representing \mathcal{E} has to satisfy $0 + x \downarrow_{\mathcal{C}}^{\sim} x$. There is just one way to orient the only

equation in \mathcal{E} which results in the rule $x + 0 \rightarrow x$. Since we want our run to be fair, we add the rules stemming from the prime critical pairs between $x + 0 \rightarrow x$ and AC^\pm :

$$0 + x \rightarrow x \quad x + (0 + y) \rightarrow x + y \quad x + (y + 0) \rightarrow x + y \quad (x + y) + 0 \rightarrow x + y$$

If collapsing with $\rightarrow_{\mathcal{R}/\text{AC}}$ is allowed, all these rules become trivial equations and can therefore be deleted. Thus, the modified inference system allows for a fair run which is not complete as $0 + x \not\downarrow_{\mathcal{R}} x$ does not hold for $\mathcal{R} = \{x + 0 \rightarrow x\}$. Furthermore, if we add pairs of terms stemming from local cliffs as equations, we get the same result by applications of `simplify`.

The inference system presented in Definition 4.1 is almost the same as the one presented by Avenhaus in [Ave95]. However, since we only consider finite runs, the encompassment condition for the `collapse` rule has been removed in the spirit of [ST13]. (The original side condition is $s \rightarrow_{\mathcal{R}} u$ with $\ell \rightarrow r \in \mathcal{R}$ and $s \triangleright l$.) The following example shows that this can lead to smaller \mathcal{B} -complete systems.

Example 4.5. Consider the ES \mathcal{E} consisting of the single equation

$$f(x + y) \approx f(x) + f(y)$$

where $+$ is an AC function symbol. The inference system presented in [Ave95] produces the AC complete system

$$f(x + y) \rightarrow f(x) + f(y) \qquad f(y + x) \rightarrow f(x) + f(y)$$

in which either of the rules could be collapsed if it was allowed to collapse with the other rule. In [Ave95] this is prevented by an encompassment condition which essentially forbids to collapse at the root position with a rewrite rule whose left-hand side is a variant of the left-hand side of the rule which should be collapsed. However, this is possible with the system presented in this article, so for an AC complete representation just one of the two rules suffices.

4.2. Correctness Proof. We show that every fair and non-failing finite run results in a \mathcal{B} -complete presentation. To this end, we first verify that inference steps in A preserve convertibility. We abbreviate $\mathcal{E} \cup \mathcal{R} \cup \mathcal{B}$ to \mathcal{ERB} and $\mathcal{E}' \cup \mathcal{R}' \cup \mathcal{B}$ to \mathcal{ERB}' .

Lemma 4.6. *If $(\mathcal{E}, \mathcal{R}) \vdash_A (\mathcal{E}', \mathcal{R}')$ then the following inclusions hold:*

- (1) $\xrightarrow{\mathcal{ERB}} \subseteq \xrightarrow{\mathcal{R}'/\mathcal{B}} \cdot (\xrightarrow{\mathcal{E}\mathcal{R}'} \cup \xrightarrow{\mathcal{B}'}^*) \cdot \xrightarrow{\mathcal{R}'/\mathcal{B}}$
- (2) $\xrightarrow{\mathcal{ERB}'} \subseteq \xrightarrow{\mathcal{ERB}'}^*$

Proof. By inspecting the inference rules of A we obtain the following inclusions:

deduce

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xrightarrow{\mathcal{R}} \cdot \xrightarrow{\mathcal{R}} \cup \xrightarrow{\mathcal{B}} \cdot \xrightarrow{\mathcal{R}} \cup \xrightarrow{\mathcal{R}} \cdot \xrightarrow{\mathcal{B}}$$

orient

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup (\mathcal{R}')^{-1} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup (\mathcal{E})^{-1} \cup \mathcal{R}$$

delete

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \sim_{\mathcal{B}} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R}$$

compose

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow{\mathcal{R}'} \cdot \overleftarrow{\mathcal{R}'/\mathcal{B}} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \xrightarrow{\mathcal{R}} \cdot \overleftarrow{\mathcal{R}/\mathcal{B}}$$

collapse

$$\mathcal{E} \cup \mathcal{R} \subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow{\mathcal{R}'} \cdot \xrightarrow{\mathcal{E}'} \qquad \mathcal{E}' \cup \mathcal{R}' \subseteq \mathcal{E} \cup \mathcal{R} \cup \overleftarrow{\mathcal{R}} \cdot \xrightarrow{\mathcal{R}}$$

simplify

$$\begin{aligned} \mathcal{E} \cup \mathcal{R} &\subseteq \mathcal{E}' \cup \mathcal{R}' \cup \xrightarrow{\mathcal{R}'/\mathcal{B}} \cdot \xrightarrow{\mathcal{E}'} \cup \xrightarrow{\mathcal{E}'} \cdot \overleftarrow{\mathcal{R}'/\mathcal{B}} \\ \mathcal{E}' \cup \mathcal{R}' &\subseteq \mathcal{E} \cup \mathcal{R} \cup \overleftarrow{\mathcal{R}/\mathcal{B}} \cdot \xrightarrow{\mathcal{E}} \cup \xrightarrow{\mathcal{E}} \cdot \overleftarrow{\mathcal{R}/\mathcal{B}} \end{aligned}$$

Then, inclusion (2) follows directly from the closure of rewrite relations under contexts and substitutions. Statement (1) holds since it is a generalization that all cases have in common. \square

Corollary 4.7. *If $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{A}}^* (\mathcal{E}', \mathcal{R}')$ then $\overleftarrow{\mathcal{E}\mathcal{R}\mathcal{B}}^* = \overleftarrow{\mathcal{E}\mathcal{R}\mathcal{B}'}^*$.*

Lemma 4.8. *If $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{A}}^* (\mathcal{E}', \mathcal{R}')$ and $\mathcal{R} \subseteq >$ then $\mathcal{R}' \subseteq >$.*

Proof. According to the assumption we have $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{A}}^n (\mathcal{E}', \mathcal{R}')$ for some natural number n . We proceed by induction on n . If $n = 0$, the statement is trivial since $\mathcal{R} = \mathcal{R}'$. Let $n > 0$ and consider $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{A}}^{n-1} (\mathcal{E}'', \mathcal{R}'') \vdash_{\mathbf{A}} (\mathcal{E}', \mathcal{R}')$. The induction hypothesis yields $\mathcal{R}'' \subseteq >$. We continue with a case analysis on the rule applied in the step $(\mathcal{E}'', \mathcal{R}'') \vdash_{\mathbf{A}} (\mathcal{E}', \mathcal{R}')$. For the rules *delete* and *simplify* there is nothing to show as the set of rewrite rules is not changed. If *deduce* is applied to a local peak there is nothing to show. Otherwise, we have $\mathcal{R}' = \mathcal{R}'' \cup \{s \rightarrow t\}$ where $s \leftrightarrow_{\mathcal{B}} \cdot \rightarrow_{\mathcal{R}} t$. From the fact that $>$ is \mathcal{B} -compatible we obtain $s > t$ and therefore $\mathcal{R}' \subseteq >$. For *orient* we have $\mathcal{R}' = \mathcal{R}'' \cup \{s \rightarrow t\}$ with $s > t$, so $\mathcal{R}' \subseteq >$. In the case of *compose* we have $\mathcal{R}' = (\mathcal{R}'' \setminus \{s \rightarrow t\}) \cup \{s \rightarrow u\}$ with $t \rightarrow_{\mathcal{R}/\mathcal{B}} u$. Since $>$ is a \mathcal{B} -compatible reduction order, $t \rightarrow_{\mathcal{R}/\mathcal{B}} u$ implies $t > u$. From the induction hypothesis we obtain $s > t$. Now $\mathcal{R}' \subseteq >$ follows by the transitivity of $>$. Finally, for *collapse* we have $\mathcal{R}' \subsetneq \mathcal{R}'' \subseteq >$ which establishes $\mathcal{R}' \subseteq >$. \square

In order to use peak-and-cliff decreasingness in the correctness proof, we have to define an appropriate notion of labeled rewriting. Intuitively, we want to annotate a step $s \rightarrow_{\mathcal{R}} t$ or $s \sim_{\mathcal{B}} t$ with a collection of terms in such a way that the collection contains terms which are \mathcal{B} -equivalent or rewrite in a positive number of steps using $\rightarrow_{\mathcal{R}/\mathcal{B}}$ to s and t , respectively.

Definition 4.9. Let \leftrightarrow be a rewrite relation or equivalence relation, M a finite multiset of terms and $>$ a \mathcal{B} -compatible reduction order. We write $s \xleftrightarrow{M} t$ if $s \leftrightarrow t$ and there exist terms $s', t' \in M$ such that $s' \gtrsim s$ and $t' \gtrsim t$ for $\gtrsim = > \cup \sim_{\mathcal{B}}$.

We follow the convention that if a conversion is labeled with M , all single steps can be labeled with M .

Example 4.10. Consider the TRS \mathcal{R} consisting of the rules

$$0 + y \rightarrow y \qquad \mathbf{s}(x) + y \rightarrow \mathbf{s}(x + y)$$

and the equational theory $\mathcal{B} = \{x + y \approx y + x\}$. Let $> = \rightarrow_{\mathcal{R}/\mathcal{B}}$ which is a \mathcal{B} -compatible reduction order by definition as \mathcal{R} is \mathcal{B} -terminating. We have

$$\mathbf{s}(0) + x \xrightarrow{\{x+\mathbf{s}(0)\}}_{\mathcal{R}} \mathbf{s}(0 + x) \quad \text{and} \quad \mathbf{s}(0 + x) \xrightarrow{\{\mathbf{s}(0+x)\}}_{\mathcal{R}} \mathbf{s}(x)$$

as well as

$$s(0) + x \xrightarrow[\mathcal{R}]{\{x+s(0)\}}^* s(x) \quad \text{but not} \quad s(0) + x \xrightarrow[\mathcal{R}]{\{s(0+x)\}}^* s(x).$$

Lemma 4.11. *Let $(\mathcal{E}, \mathcal{R}) \vdash_{\mathbf{A}} (\mathcal{E}', \mathcal{R}')$ and $\mathcal{R}' \subseteq >$.*

- (1) *For any finite multiset M we have $\xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}]{M}^* \subseteq \xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}']{M}^*$.*
- (2) *If $s \xrightarrow[\mathcal{R}]{M} t$ then $s \xrightarrow[\mathcal{R}']{M} \cdot \xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}']{N}^* t$ with $\{s\} >_{\text{mul}} N$.*

Proof. For (1) it suffices to show that $\xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}]{M} \subseteq \xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}']{M}^*$. Let $s \xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}]{M} t$. By definition, there exist terms $s', t' \in M$ with $s' \gtrsim s$ and $t' \gtrsim t$. According to Lemma 4.6 there exist terms u and v such that

$$s \xrightarrow[\mathcal{R}'/\mathcal{B}]{=} u \left(\xrightarrow[\mathcal{E}' \cup \mathcal{R}']{=} \cup \xrightarrow[\mathcal{B}]{*} \right) v \xrightarrow[\mathcal{R}'/\mathcal{B}]{=} t$$

Since $\mathcal{R}' \subseteq >$ we have $s \gtrsim u$ and $t \gtrsim v$ and therefore $s' \gtrsim u$ and $t' \gtrsim v$. Hence, every non-empty step can be labeled with M and we obtain $s \xrightarrow[\mathcal{E}\mathcal{R}\mathcal{B}']{M}^* t$ as desired.

For a proof of (2), let $s \xrightarrow[\mathcal{R}]{M} t$. By definition, there exists an $s' \in M$ such that $s' \gtrsim s$. We proceed by case analysis on the rule applied in the inference step. For **deduce**, **orient**, **delete** and **simplify** there is nothing to show since $\mathcal{R} \subseteq \mathcal{R}'$.

Suppose the step is an application of **compose**. If the rule used in the step $s \xrightarrow[\mathcal{R}]{M} t$ is not altered, we are done. Otherwise, the step was performed with a rule $\ell \rightarrow r \in \mathcal{R}$ which is changed to $\ell \rightarrow r' \in \mathcal{R}'$ with $r \rightarrow_{\mathcal{R}'/\mathcal{B}} r'$. There exist a substitution σ and a position p such that $s = s[\ell\sigma]_p$ and $t = s[r\sigma]_p$. The new step is $s \xrightarrow[\mathcal{R}']{M} t'$ where $t' = s[r'\sigma]_p$. Since $>$ is a \mathcal{B} -compatible reduction order, we have $s' > t'$ and therefore the label M is still valid. From t' we can reach t with

$$t' \xrightarrow[\mathcal{B}]{\{t\}}^* \cdot \xrightarrow[\mathcal{R}']{\{t\}} \cdot \xrightarrow[\mathcal{B}]{\{t\}}^* t$$

From $s > t$ we obtain $\{s\} >_{\text{mul}} \{t\}$ which means that the new conversion between s and t is of the desired form.

Finally, suppose the step is an application of **collapse**. If the rule used in the step $s \xrightarrow[\mathcal{R}]{M} t$ is not altered, we are done immediately. Otherwise, the step was performed with a rule $\ell \rightarrow r \in \mathcal{R}$ which is changed to an equation $\ell' \approx r \in \mathcal{E}'$ with $\ell \rightarrow_{\mathcal{R}'} \ell'$. There exist a substitution σ and a position p such that $s = s[\ell\sigma]_p$ and $t = s[r\sigma]_p$. The new step is $s \xrightarrow[\mathcal{R}']{M} t'$ where $t' = s[\ell'\sigma]_p$. Since $>$ is a \mathcal{B} -compatible reduction order we have $s' > t'$ and therefore the label M is still valid. From t' we can reach t with $t' \xrightarrow[\mathcal{E}']{N} t$ where $N = \{t', t\}$. From $s > t'$ and $s > t$ we obtain $\{s\} >_{\text{mul}} N$ which means that the new conversion between s and t is of the desired form. \square

Finally, we are able to prove the correctness result for \mathbf{A} , i.e., all finite fair and non-failing runs produce a \mathcal{B} -complete TRS which represents the original set of equations. In contrast to [Ave95] and [Bac91], the proof shows that it suffices to consider prime critical pairs. This is achieved by showing peak-and-cliff decreasingness and using Theorem 3.7 instead of directly using the main theorem of Section 3 (Theorem 3.16). The usage of peak-and-cliff decreasingness makes the proof more modular and easier to formalize than the proof in [Ave95] because it is split up into the preceding auxiliary lemmata of this section which are mostly independent from each other and can use different suitable proof methods. This is very different from the approach in [Ave95] where all the necessary information and induction hypotheses are incorporated into one large proof ordering. The results presented in

[HMSW19] for standard rewriting suggest that our approach still has merits when compared to the original proofs in [Ave95] and [Bac91] if infinite runs are considered.

Theorem 4.12. *Let \mathcal{E} be an ES. For every fair and non-failing run*

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathbf{A}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathbf{A}} \cdots \vdash_{\mathbf{A}} \mathcal{E}_n, \mathcal{R}_n$$

for \mathcal{E} , the TRS \mathcal{R}_n is a \mathcal{B} -complete representation of \mathcal{E} .

Proof. Let $>$ be the \mathcal{B} -compatible reduction order used in the run. From fairness we obtain $\mathcal{E}_n = \emptyset$ as well as the fact that \mathcal{R}_n is left-linear. Corollary 4.7 establishes $\leftrightarrow_{\mathcal{E} \cup \mathcal{B}}^* = \leftrightarrow_{\mathcal{R}_n \cup \mathcal{B}}^*$ and termination modulo \mathcal{B} of \mathcal{R}_n follows from Lemma 4.8. It remains to prove that \mathcal{R}_n is Church–Rosser modulo \mathcal{B} which we do by showing peak-and-cliff decreasingness. So consider a labeled local peak $t \xrightarrow{\mathcal{R}_n}^{M_1} s \xrightarrow{\mathcal{R}_n}^{M_2} u$. Lemma 3.15(1) yields $t \nabla_s^2 u$. Let $v \nabla_s w$ appear in this sequence (so $v = t$ or $w = u$). By definition, $v \downarrow_{\mathcal{R}_n} w$ or $v \leftrightarrow_{\text{PCP}(\mathcal{R}_n)} w$. Together with fairness, the fact that $\sim_{\mathcal{B}}$ is reflexive as well as closure of rewriting under contexts and substitutions we obtain $v \downarrow_{\mathcal{R}_n}^{\sim} w$ or $(v, w) \in \bigcup_{i=0}^n \leftrightarrow_{\mathcal{E}_i \cup \mathcal{R}_i}$. In both cases, it is possible to label all steps between v and w with $\{v, w\}$. Since $s > v$ and $s > w$ we have $M_1 >_{\text{mul}} \{v, w\}$ and $M_2 >_{\text{mul}} \{v, w\}$. Repeated applications of Lemma 4.11(1) therefore yield a conversion in $\mathcal{R}_n \cup \mathcal{B}$ between v and w where every step is labeled with a multiset that is smaller than both M_1 and M_2 . Hence, the corresponding condition required by peak-and-cliff decreasingness is fulfilled.

Next consider a labeled local cliff $t \xrightarrow{\mathcal{R}_n}^{M_1} s \leftrightarrow_{\mathcal{B}}^{M_2} u$. From Lemma 3.15(2) we obtain a term v such that $t \nabla_s v \nabla_s^{\sim} u$. As in the case for local peaks we obtain a conversion between t and v where each step can be labeled with $\{t, v\} <_{\text{mul}} M_1$. Together with fairness, $v \nabla_s^{\sim} u$ yields $v \downarrow_{\mathcal{R}_n}^{\sim} u$ or $(v, u) \in \bigcup_{i=0}^n \leftrightarrow_{\mathcal{R}_i}$. In the former case there exists a k such that $v \xrightarrow{\mathcal{R}_n}^* \cdot \sim_{\mathcal{B}} \cdot \mathcal{R}_n^k \leftarrow u$. If $k = 0$ we can label all steps with $\{v\}$. If $k > 0$ the conversion is of the form $v \xrightarrow{\mathcal{R}_n}^* \cdot \sim_{\mathcal{B}} \cdot \mathcal{R}_n^{k-1} \leftarrow w \mathcal{R}_n \leftarrow u$. We can label the rightmost step with M_2 and the remaining steps with $\{v, w\}$. Note that $s > v$. Since $>$ is a \mathcal{B} -compatible reduction order we also have $s > w$. Thus, $M_1 >_{\text{mul}} \{v, w\}$ which establishes the corresponding condition required by peak-and-cliff decreasingness for all k . In the remaining case we have $(v, u) \in \bigcup_{i=0}^n \leftrightarrow_{\mathcal{R}_i}$, so there is some $i \leq n$ such that $v \leftrightarrow_{\mathcal{R}_i} u$. Actually, we know that $u \xrightarrow{\mathcal{R}_i}^{M_2} v$ since otherwise we would have both $s > v$ and $v > s$ by the \mathcal{B} -compatibility of $>$. Repeated applications of Lemma 4.11(1,2) therefore yield a conversion between u and v of the form

$$u \xrightarrow{\mathcal{R}_n}^{M_2} \cdot \xleftarrow{\mathcal{R}_n \cup \mathcal{B}}^* N v$$

where $\{u\} >_{\text{mul}} N$. By definition, $s' \succsim u$ for some $s' \in M_1$ and therefore $M_1 >_{\text{mul}} N$, which means that the corresponding condition required by peak-and-cliff decreasingness is fulfilled. Overall, it follows that \mathcal{R}_n is peak-and-cliff decreasing and therefore Church–Rosser modulo \mathcal{B} . \square

Note that the proofs of the previous theorem and Theorem 3.7 do not require multiset orders induced by quasi-orders but use multiset extensions of proper \mathcal{B} -compatible reduction orders which are easier to work with. This could be achieved by defining peak-and-cliff decreasingness in such a way that well-founded orders suffice for the abstract setting. However, the usage of multiset orders based on \mathcal{B} -compatible reduction orders as well as a notion of labeled rewriting which allows us to label steps with \mathcal{B} -equivalent terms are crucial in order to establish peak-and-cliff decreasingness for TRSs.

As we have established correctness of \mathbf{A} , it is natural to ask the question whether \mathbf{A} is also *complete*, i.e., can \mathbf{A} generate a complete presentation whenever there exists one. Contrary to ordered completion where there are known completeness results (see e.g. [BDP89, Dev91]), this is not possible in our setting. Consider the ES \mathcal{E} [BD94] consisting of the three equations

$$1 \cdot (-x + x) \approx 0 \qquad 1 \cdot (x + -x) \approx x + -x \qquad -x + x \approx y + -y$$

which admits the following complete presentation \mathcal{R} :

$$1 \cdot 0 \rightarrow 0 \qquad x + -x \rightarrow 0 \qquad -x + x \rightarrow 0$$

In standard completion, only the first two equations can be oriented from left to right but no further step is possible. In \mathbf{A} , we have the same situation if $\mathcal{B} = \emptyset$, but we can choose \mathcal{B} as some nonempty subset of \mathcal{E} . Note that the first and third equation are not eligible since they violate the assumption $\text{Var}(\ell) = \text{Var}(r)$. Hence, $\mathcal{B} = \{1 \cdot (x + -x) \approx x + -x\}$ is the only other option. After orienting the first equation, again no further step is possible since there is also no overlap between $1 \cdot (-x + x)$ and $x + -x$.

5. BACHMAIR'S INFERENCE SYSTEM

As already mentioned, the inference system proposed by Avenhaus [Ave95] is essentially the same as \mathbf{A} . The only other inference system for \mathcal{B} -completion for left-linear TRSs is due to Bachmair [Bac91]. We investigate a slightly modified version of this inference system where arbitrary local peaks are deducible and the encompassment condition from the `collapse` rule is removed as we only consider finite runs. The resulting system will be called \mathbf{B} . Note that the purpose of the change in the `deduce` rule of Bachmair's system is to eliminate this unimportant difference to \mathbf{A} . The following results would still hold if we limited `deduce` in \mathbf{B} to (prime) critical pairs but aligning \mathbf{A} and \mathbf{B} would be unnecessarily complicated. Furthermore, deducing arbitrary local peaks offers a simpler definition of `deduce` which gives implementations more freedom. However, we are not aware of any work that has investigated whether deducing non-critical peaks can be beneficial in completion.

The main difference between \mathbf{A} and \mathbf{B} is that in \mathbf{B} one may only use the standard rewrite relation $\rightarrow_{\mathcal{R}}$ for simplifying equations and composing rules. This allows us to deduce local cliffs as equations. The goal of this section is to establish correctness of \mathbf{B} via a simulation by \mathbf{A} .

Definition 5.1. The inference system \mathbf{B} is the same as \mathbf{A} but with rewriting in `compose` and `simplify` restricted to $\rightarrow_{\mathcal{R}}$ and the following rule which replaces the two deduction rules of \mathbf{A} :

$$\text{deduce} \quad \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \mathcal{R} \leftarrow \cdot \rightarrow_{\mathcal{R} \cup \mathcal{B}^\pm} t$$

Example 5.2. Recall the ES $\mathcal{E} = \{x + 0 \approx x\}$ of Example 4.4, where $+$ is an AC function symbol. AC completion based on \mathbf{B} proceeds as follows:

$$(\mathcal{E}, \emptyset) \vdash_{\mathbf{B}} (\emptyset, \{x + 0 \rightarrow x\}) \vdash_{\mathbf{B}} (\{0 + x \approx x\}, \{x + 0 \rightarrow x\}) \vdash_{\mathbf{B}} \dots$$

The second step obtained by `deduce` reveals a main difference between \mathbf{A} and \mathbf{B} . If `deduce` in \mathbf{A} is employed, $(\emptyset, \{x + 0 \rightarrow x\}) \vdash_{\mathbf{A}} (\emptyset, \{0 + x \rightarrow x, 0 + x \rightarrow x\})$ is obtained.

Definition 5.3. Let \mathcal{E} be an ES. A finite sequence

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_{\mathbf{B}} \mathcal{E}_1, \mathcal{R}_1 \vdash_{\mathbf{B}} \dots \vdash_{\mathbf{B}} \mathcal{E}_n, \mathcal{R}_n$$

with $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{R}_0 = \emptyset$ is a *run* for \mathcal{E} . If $\mathcal{E}_n \neq \emptyset$, the run *fails*. The run is *fair* if \mathcal{R}_n is left-linear and the following inclusion holds:

$$\text{PCP}(\mathcal{R}_n) \cup \text{PCP}^\pm(\mathcal{R}_n, \mathcal{B}^\pm) \subseteq \downarrow_{\tilde{\mathcal{R}}_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{E}_i}$$

In contrast to Definition 4.2, the fairness condition is the same for all prime critical pairs since the inference rule **deduce** of **B** never produces rewrite rules. In that sense, **B** is closer to known completion procedures, but as shown before, this comes at the expense of not being allowed to apply **simplify**, **collapse** and **compose** with $\rightarrow_{\mathcal{R}/\mathcal{B}}$. If we want to allow this more general kind of simplification as it is the case for **A**, local cliffs have to be deduced as rules. Note that this possibly leads to an increase in critical pairs which one has to consider in practice, but it can also reduce the number of **orient** steps one has to perform. Since **compose** and **collapse** (without the encompassment condition) can emulate the behavior of **simplify**, it is possible to get the best of both worlds by using **A** and deferring the computation of critical pairs with rules which stem from local cliffs until it is needed to proceed in the completion process. We will now show that for finite runs, **A** is at least as powerful as **B** which is the main motivation for the focus on **A** instead of **B** in this article. Moreover, the simulation result actually allows us to reduce correctness of **B** to correctness of **A**, so we get this property without any additional effort.

In order to prove that fair and non-failing runs in **B** can be simulated in **A**, we start with the following technical lemma which intuitively states that a step in **B** can be simulated by at most one step in **A** in such a way that the results only differ in the number of oriented equations. We denote an application of the rule **orient** in an inference system **I** by $\overset{\circ}{\vdash}_I$.

Lemma 5.4. *If $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}} (\mathcal{E}_2, \mathcal{R}_2)$ and $(\mathcal{E}_1, \mathcal{R}_1) \overset{\circ}{\vdash}_{\mathbf{B}}^* (\mathcal{E}'_1, \mathcal{R}'_1)$ then there exists a pair $(\mathcal{E}'_2, \mathcal{R}'_2)$ such that $(\mathcal{E}'_1, \mathcal{R}'_1) \vdash_{\mathbf{A}} (\mathcal{E}'_2, \mathcal{R}'_2)$ and $(\mathcal{E}_2, \mathcal{R}_2) \overset{\circ}{\vdash}_{\mathbf{B}}^* (\mathcal{E}'_2, \mathcal{R}'_2)$. In a picture:*

$$\begin{array}{ccc} \mathcal{E}_1, \mathcal{R}_1 & \vdash_{\mathbf{B}} & \mathcal{E}_2, \mathcal{R}_2 \\ \Downarrow_{\varpi^*}^{\circ} & & \Downarrow_{\varpi^*}^{\circ} \\ \mathcal{E}'_1, \mathcal{R}'_1 & \vdash_{\mathbf{A}} & \mathcal{E}'_2, \mathcal{R}'_2 \end{array}$$

Proof. Let $>$ be a fixed **B**-compatible reduction order which is used in both **A** and **B**. From $(\mathcal{E}_1, \mathcal{R}_1) \overset{\circ}{\vdash}_{\mathbf{B}}^* (\mathcal{E}'_1, \mathcal{R}'_1)$ we obtain $\mathcal{E}'_1 \subseteq \mathcal{E}_1$, $\mathcal{R}_1 \subseteq \mathcal{R}'_1$ and $\mathcal{E}_1 \setminus \mathcal{E}'_1 \subseteq \mathcal{R}'_1 \cup \mathcal{R}_1^{-1}$. In order to simplify the formulation, we will refer to the sequence of **orient** steps between a pair and its primed variant as the invariant. We proceed by a case analysis on the rule applied in the inference step $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}} (\mathcal{E}_2, \mathcal{R}_2)$.

- ▷ In the case of **deduce**, we apply the same rule in **A**. For local peaks $s \mathcal{R}_1 \leftarrow \cdot \rightarrow_{\mathcal{R}_1} t$ we have $\mathcal{E}_2 = \mathcal{E}_1 \cup \{s \approx t\}$, $\mathcal{R}_2 = \mathcal{R}_1$, $\mathcal{E}'_2 = \mathcal{E}'_1 \cup \{s \approx t\}$ and $\mathcal{R}'_2 = \mathcal{R}'_1$. For local cliffs $s \mathcal{R}_1 \leftarrow \cdot \leftrightarrow_{\mathcal{B}} t$ we have $\mathcal{E}_2 = \mathcal{E}_1 \cup \{s \approx t\}$, $\mathcal{R}_2 = \mathcal{R}_1$ as well as $\mathcal{R}'_2 = \mathcal{R}'_1 \cup \{t \rightarrow s\}$, $\mathcal{E}'_2 = \mathcal{E}'_1$. In both cases, the invariant is preserved.
- ▷ Suppose the inference step in **B** orients an equation $s \approx t$. If $s \approx t \in \mathcal{E}'_1$ we perform the same step in **A** which preserves the invariant. Otherwise, $s \approx t \in \mathcal{E}_1 \setminus \mathcal{E}'_1$ and hence $v \rightarrow w \in \mathcal{R}'_1$ where $\{v, w\} = \{s, t\}$. In this case, an empty step in **A** preserves the invariant.

- ▷ If the inference step in \mathbf{B} deletes an equation $s \approx t$, it has to be in \mathcal{E}'_1 which enables us to perform the same step in \mathbf{A} while preserving the invariant: Suppose that $s \approx t \in \mathcal{E}_1 \setminus \mathcal{E}'_1$. Since the equation is deleted, we have $s \sim_{\mathcal{B}} t$. Neither $s > t$ nor $t > s$ can hold as $>$ is \mathcal{B} -compatible and irreflexive. Hence, the equation cannot be oriented which contradicts the assumption $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}}^{\circ*} (\mathcal{E}'_1, \mathcal{R}'_1)$.
- ▷ If the inference step in \mathbf{B} is **compose** or **collapse**, the same step can be performed in \mathbf{A} while preserving the invariant as $\mathcal{R}_1 \subseteq \mathcal{R}'_1$.
- ▷ Finally, suppose that the inference step in \mathbf{B} simplifies an equation $s \approx t$. Since the orientation of equations does not matter in completion, we may assume that the simplification transforms the equation to $s' \approx t$ without loss of generality. If $s \approx t \in \mathcal{E}'_1$, we perform the same step in \mathbf{A} which preserves the invariant. Otherwise, there is a rule $v \rightarrow w \in \mathcal{R}'_1$ such that $\{v, w\} = \{s, t\}$. If $v = s$ and $w = t$, we use **collapse** for the inference step in \mathbf{A} which produces the same equation $s' \approx t \in \mathcal{E}'_2$ which means that the invariant is preserved. If $v = t$ and $w = s$, we use **compose** for the inference step in \mathbf{A} in order to obtain the rule $t \rightarrow s'$. From $t > s$ and $s > s'$ we obtain $t > s'$ and therefore the equation $s' \approx t$ can be oriented into the rule $t \rightarrow s'$. Thus, the invariant is preserved. \square

For the proof of the simulation result, we need a slightly different form of the previous lemma. Analogous to the notation for rewrite relations, the relation $\vdash_{\mathbf{B}}^{\circ!}$ denotes the exhaustive application of the inference rule **orient**.

Corollary 5.5. *If $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}} (\mathcal{E}_2, \mathcal{R}_2)$ and $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}}^{\circ!} (\mathcal{E}'_1, \mathcal{R}'_1)$ then $(\mathcal{E}'_1, \mathcal{R}'_1) \vdash_{\mathbf{A}}^* (\mathcal{E}'_2, \mathcal{R}'_2)$ where $(\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathbf{B}}^{\circ!} (\mathcal{E}'_2, \mathcal{R}'_2)$.*

Proof. Let $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}} (\mathcal{E}_2, \mathcal{R}_2)$ and $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{B}}^{\circ!} (\mathcal{E}'_1, \mathcal{R}'_1)$. Lemma 5.4 yields $(\mathcal{E}'_1, \mathcal{R}'_1) \vdash_{\mathbf{A}}^{\bar{=}} (\mathcal{E}'_2, \mathcal{R}'_2)$ and $(\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathbf{B}}^{\circ*} (\mathcal{E}'_2, \mathcal{R}'_2)$. It follows that the orientable equations in \mathcal{E}'_2 are also included in \mathcal{E}_2 . Hence, we can compute $(\mathcal{E}'_2, \mathcal{R}'_2) \vdash_{\mathbf{A}}^{\circ!} (\mathcal{E}'_3, \mathcal{R}'_3)$, satisfying $(\mathcal{E}'_1, \mathcal{R}'_1) \vdash_{\mathbf{A}}^* (\mathcal{E}'_3, \mathcal{R}'_3)$ and $(\mathcal{E}_2, \mathcal{R}_2) \vdash_{\mathbf{B}}^{\circ!} (\mathcal{E}'_3, \mathcal{R}'_3)$ as desired. \square

Theorem 5.6. *For every fair run $(\mathcal{E}, \emptyset) \vdash_{\mathbf{B}}^* (\emptyset, \mathcal{R})$ there exists a fair run $(\mathcal{E}, \emptyset) \vdash_{\mathbf{A}}^* (\emptyset, \mathcal{R})$.*

Proof. Assume $(\mathcal{E}_0, \mathcal{R}_0) \vdash_{\mathbf{B}}^n (\mathcal{E}_n, \mathcal{R}_n)$ where $\mathcal{R}_0 = \mathcal{E}_n = \emptyset$. By n applications of Corollary 5.5 we arrive at the following situation:

$$\begin{array}{ccccccc}
 \mathcal{E}_0, \mathcal{R}_0 & \vdash_{\mathbf{B}} & \mathcal{E}_1, \mathcal{R}_1 & \vdash_{\mathbf{B}} & \cdots & \vdash_{\mathbf{B}} & \mathcal{E}_n, \mathcal{R}_n \\
 \Downarrow_{\circ} & & \Downarrow_{\circ} & & & & \Downarrow_{\circ} \\
 \mathcal{E}_0, \mathcal{R}_0 & \vdash_{\mathbf{A}}^{\circ!} & \mathcal{E}'_0, \mathcal{R}'_0 & \vdash_{\mathbf{A}}^* & \mathcal{E}'_1, \mathcal{R}'_1 & \vdash_{\mathbf{A}}^* & \cdots & \vdash_{\mathbf{A}}^* & \mathcal{E}'_n, \mathcal{R}'_n
 \end{array}$$

The following two statements hold:

- (1) For $0 \leq i \leq n$, all orientable equations in \mathcal{E}_i are in \mathcal{R}'_i (possibly reversed) and the other equations are in \mathcal{E}'_i .
- (2) $\text{PCP}^{\pm}(\mathcal{R}'_n, \mathcal{B}^{\pm})$ is a set of orientable equations.

Statement (1) is immediate from the simulation relation $\vdash_{\mathbf{B}}^{\circ!}$ and statement (2) follows from \mathcal{B} -compatibility of the used reduction order together with the fact that every (prime) critical pair is connected by one \mathcal{R}_n -step and one \mathcal{B} -step. Furthermore, $\mathcal{E}_n = \emptyset$ implies $\mathcal{E}'_n = \emptyset$

as well as $\mathcal{R}_n = \mathcal{R}'_n$. Hence, we obtain fairness of the run in **A** by showing the following inclusions:

$$\text{PCP}(\mathcal{R}'_n) \subseteq \downarrow_{\tilde{\mathcal{R}}'_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{E}'_i \cup \mathcal{R}'_i} \quad \text{PCP}^\pm(\mathcal{R}'_n, \mathcal{B}^\pm) \subseteq \downarrow_{\tilde{\mathcal{R}}'_n} \cup \bigcup_{i=0}^n \leftrightarrow_{\mathcal{R}'_i}$$

Let $s \approx t \in \text{PCP}(\mathcal{R}'_n)$. By fairness of the run in **B** we obtain $s \downarrow_{\tilde{\mathcal{R}}'_n} t$ or $s \leftrightarrow_{\mathcal{E}_k} t$ for some $k \leq n$. In the former case, we are immediately done. In the latter case we obtain $s \leftrightarrow_{\mathcal{E}'_k \cup \mathcal{R}'_k} t$ from (1) as desired. Now, let $s \approx t \in \text{PCP}^\pm(\mathcal{R}'_n, \mathcal{B}^\pm)$. By fairness of the run in **B** we obtain $s \downarrow_{\tilde{\mathcal{R}}'_n} t$ or $s \leftrightarrow_{\mathcal{E}_k} t$ for some $k \leq n$. Again, we are immediately done in the former case. In the latter case we have $s \leftrightarrow_{\mathcal{R}'_k} t$ because of (1) and (2). Therefore, the run in **A** is fair. \square

The previous theorem is an important simulation result which justifies the emphasis on **A** in this article. Moreover, together with Theorem 4.12 the correctness of the inference system **B** is an easy consequence.

Corollary 5.7. *Every fair and non-failing run for \mathcal{E} in **B** produces a \mathcal{B} -complete presentation of \mathcal{E} .*

6. CANONICITY

Complete representations resulting from completion may have redundant rules which do not contribute to the computation of normal forms. The notion of *canonicity* addresses this issue by defining a minimal and unique representation of a complete TRS for a given reduction order. In this section, canonicity results for \mathcal{B} -complete TRSs are presented. After establishing results for abstract rewriting, means to compute \mathcal{B} -canonical TRSs and the uniqueness of \mathcal{B} -canonical TRSs are discussed. The results and proofs in this section closely follow the presentation of canonicity results for standard rewriting in [HMSW19] by carefully lifting definitions and results to rewriting modulo \mathcal{B} . To the best of our knowledge, this section presents the first account of canonicity results for \mathcal{B} -complete TRSs.

6.1. Results for Abstract Rewriting Modulo. In the following we assume that $\mathcal{A}_1 = \langle A, \rightarrow_{\mathcal{A}_1} \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_{\mathcal{A}_2} \rangle$ are ARSs with the same underlying set A and \sim is an equivalence relation on A . The upcoming definition introduces two different notions of equivalence between ARSs. Both are needed in order to relate different representations of the same equational theory, in particular the relation between complete presentations and their canonical forms.

Definition 6.1. The ARSs \mathcal{A}_1 and \mathcal{A}_2 are (*conversion*) *equivalent modulo \sim* if $\Leftrightarrow_{\mathcal{A}_1}^* = \Leftrightarrow_{\mathcal{A}_2}^*$ and (*normalization*) *equivalent modulo \sim* if $\rightarrow_{\mathcal{A}_1}^! \cdot \sim = \rightarrow_{\mathcal{A}_2}^! \cdot \sim$.

Example 6.2. Let $A = \{a, b, c\}$, $a \sim b$, $\rightarrow_{\mathcal{A}_1} = \{(a, c)\}$ and $\rightarrow_{\mathcal{A}_2} = \{(b, c)\}$. Then, \mathcal{A}_1 and \mathcal{A}_2 are conversion equivalent modulo \sim but not normalization equivalent modulo \sim : We have $a \rightarrow_{\mathcal{A}_1}^! c$ but $a \in \text{NF}(\mathcal{A}_2)$ and $a \not\sim c$.

Next we show two fundamental properties of normalization equivalence modulo \sim which are needed in Theorem 6.12.

Lemma 6.3. *If \mathcal{A}_1 and \mathcal{A}_2 are normalization equivalent modulo \sim and terminating, they are equivalent modulo \sim .*

Proof. We prove $\Leftrightarrow_{\mathcal{A}_1}^* \subseteq \Leftrightarrow_{\mathcal{A}_2}^*$ by induction on the length of conversions in \mathcal{A}_1 . The claim follows by symmetry. If $a \Leftrightarrow_{\mathcal{A}_1}^0 b$ then $a = b$ and therefore also $a \Leftrightarrow_{\mathcal{A}_2}^0 b$. For $a \Leftrightarrow_{\mathcal{A}_1}^n a' \Leftrightarrow_{\mathcal{A}_1} b$ the induction hypothesis yields $a \Leftrightarrow_{\mathcal{A}_2}^* a'$. If $a' \sim b$, the result follows immediately. Otherwise, either $a' \rightarrow_{\mathcal{A}_1} b$ or $b \rightarrow_{\mathcal{A}_1} a'$. We only consider the first case by again exploiting symmetry. Since \mathcal{A}_1 and \mathcal{A}_2 are normalization equivalent and terminating, there is an element c such that $b \rightarrow_{\mathcal{A}_1}^! \cdot \sim c$ and $b \rightarrow_{\mathcal{A}_2}^! \cdot \sim c$ and therefore also $a' \rightarrow_{\mathcal{A}_1}^! \cdot \sim c$ and $a' \rightarrow_{\mathcal{A}_2}^! \cdot \sim c$. Hence, we can connect a' and b by a conversion in \mathcal{A}_2 as desired. \square

Lemma 6.4. *Let $\text{NF}(\mathcal{A}_2) \subseteq \text{NF}(\mathcal{A}_1)$ and $\rightarrow_{\mathcal{A}_2} \subseteq (\rightarrow_{\mathcal{A}_1}/\sim)^+$. If \mathcal{A}_1 is complete modulo \sim then \mathcal{A}_2 is complete modulo \sim and normalization equivalent modulo \sim to \mathcal{A}_1 .*

Proof. From the inclusion $\rightarrow_{\mathcal{A}_2} \subseteq (\rightarrow_{\mathcal{A}_1}/\sim)^+$ as well as the termination of \mathcal{A}_1 modulo \sim we obtain that \mathcal{A}_2 is terminating modulo \sim . Next, we show $\rightarrow_{\mathcal{A}_2}^! \cdot \sim \subseteq \rightarrow_{\mathcal{A}_1}^! \cdot \sim$. The inclusion $\rightarrow_{\mathcal{A}_2} \subseteq (\rightarrow_{\mathcal{A}_1}/\sim)^+$ gives rise to $\rightarrow_{\mathcal{A}_2}^! \subseteq \Leftrightarrow_{\mathcal{A}_1}^*$. Since \mathcal{A}_1 is Church–Rosser modulo \sim and $\text{NF}(\mathcal{A}_2) \subseteq \text{NF}(\mathcal{A}_1)$ we have $\rightarrow_{\mathcal{A}_2}^! \subseteq \rightarrow_{\mathcal{A}_1}^! \cdot \sim$ and therefore $\rightarrow_{\mathcal{A}_2}^! \cdot \sim \subseteq \rightarrow_{\mathcal{A}_1}^! \cdot \sim$ as desired. For the reverse inclusion consider $a \rightarrow_{\mathcal{A}_1}^! b' \sim b$. The termination of \mathcal{A}_2 modulo \sim implies the termination of \mathcal{A}_2 , so there is some c such that $a \rightarrow_{\mathcal{A}_2}^! \cdot \sim c$ and thus $a \rightarrow_{\mathcal{A}_1}^! c' \sim c$. Now, the Church–Rosser modulo \sim property of \mathcal{A}_1 yields $b' \sim c'$. We obtain $b \sim c$ and therefore $a \rightarrow_{\mathcal{A}_2}^! \cdot \sim b$. Hence, \mathcal{A}_1 and \mathcal{A}_2 are normalization equivalent modulo \sim . Finally, the Church–Rosser modulo \sim property of \mathcal{A}_2 follows from the sequence of inclusions

$$\Leftrightarrow_{\mathcal{A}_2}^* \subseteq \Leftrightarrow_{\mathcal{A}_1}^* \subseteq \rightarrow_{\mathcal{A}_1}^! \cdot \sim \cdot \mathcal{A}_1^! \leftarrow \subseteq \rightarrow_{\mathcal{A}_2}^! \cdot \sim \cdot \mathcal{A}_2^! \leftarrow$$

which are justified by $\rightarrow_{\mathcal{A}_2} \subseteq (\rightarrow_{\mathcal{A}_1}/\sim)^+$, the fact that \mathcal{A}_1 is complete modulo \sim and normalization equivalence modulo \sim of \mathcal{A}_1 and \mathcal{A}_2 , respectively. \square

6.2. Computing \mathcal{B} -Canonical Term Rewrite Systems. Intuitively, \mathcal{B} -complete TRSs need to have a variety of left-hand sides of rules in order to match every possible \mathcal{B} -equivalent term of a reducible term. However, for the right-hand sides of rules only one representative of the \mathcal{B} -equivalence class suffices. This rationale is reflected in the following series of definitions.

Definition 6.5. Two rules $\ell \rightarrow r$ and $\ell' \rightarrow r'$ are *right- \mathcal{B} -equivalent variants* if there exists a renaming σ such that $\ell\sigma = \ell'$ and $r\sigma \sim_{\mathcal{B}} r'$. We write $\mathcal{R}_1 \doteq_{\sim} \mathcal{R}_2$ if every rule of \mathcal{R}_1 has a right- \mathcal{B} -equivalent variant in \mathcal{R}_2 and vice versa. For any TRS \mathcal{R} , the TRS $\mathcal{R}_{\doteq_{\sim}}$ denotes a right- \mathcal{B} -equivalent variant-free version, i.e., only one representative of every equivalence class of right- \mathcal{B} -equivalent variants is present.

Example 6.6. From the TRS \mathcal{R} consisting of the rules

$$s(x) \times y \rightarrow (x \times y) + y \quad y \times s(x) \rightarrow (x \times y) + y \quad s(x) \times y \rightarrow y + (x \times y)$$

where $+$ and \times are AC function symbols we obtain $\mathcal{R}_{\doteq_{\sim}}$ by removing the third rule:

$$s(x) \times y \rightarrow (x \times y) + y \quad y \times s(x) \rightarrow (x \times y) + y$$

The relation \doteq_{\sim} weakens the notion of literal similarity of TRSs (\doteq) to the setting of the inference system \mathbf{A} : While composing with $\rightarrow_{\mathcal{R}/\mathcal{B}}$ is allowed, left-hand sides may only be rewritten with $\rightarrow_{\mathcal{R}}$.

Definition 6.7. A TRS \mathcal{R} is *left-reduced* if for every rewrite rule $\ell \rightarrow r \in \mathcal{R}$, ℓ is a normal form of $\mathcal{R} \setminus \{\ell \rightarrow r\}$ and *right- \mathcal{B} -reduced* if for every rewrite rule $\ell \rightarrow r \in \mathcal{R}$, r is a normal form with respect to $\rightarrow_{\mathcal{R}/\mathcal{B}}$. We say that \mathcal{R} is *canonical modulo \mathcal{B}* if it is complete modulo \mathcal{B} , left-reduced and right- \mathcal{B} -reduced.

For standard rewriting, canonical systems are defined in [Mét83] as TRSs which are complete, left-reduced and *right-reduced*. Here, right-reduced is defined just as right- \mathcal{B} -reduced but with $\rightarrow_{\mathcal{R}}$ instead of $\rightarrow_{\mathcal{R}/\mathcal{B}}$. Intuitively, canonical systems are minimal complete systems because the rules cannot be used to simplify (other) rules anymore. In our setting, it is important to note that while right-hand sides can be simplified with $\rightarrow_{\mathcal{R}/\mathcal{B}}$, for left-hand sides only simplification with $\rightarrow_{\mathcal{R}}$ is considered. This ensures that all terms which are reducible with respect to $\rightarrow_{\mathcal{R}/\mathcal{B}}$ can also be rewritten with $\rightarrow_{\mathcal{R}}$. Note that this also reflects the definition of *collapse* and *compose* in A.

The next definition is an extension of Métivier's proposed procedure [Mét83, Theorem 7] to transform a complete system into a canonical system. As in the original definition, a \mathcal{B} -complete TRS is transformed into a \mathcal{B} -canonical TRSs in two stages: First, a right- \mathcal{B} -reduced version without right- \mathcal{B} -equivalent variants is constructed. After that, superfluous rules are removed in order to obtain a left-reduced system. Hence, the overall result is canonical modulo \mathcal{B} . Needless to say, this is only computable if \mathcal{B} has a decidable equational theory. A presentation of Métivier's concept for standard rewriting in the style of our upcoming definition can be found in [HMSW19].

Definition 6.8. Given a \mathcal{B} -terminating TRS \mathcal{R} , the TRSs $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are defined as follows:

$$\begin{aligned}\dot{\mathcal{R}} &= \{\ell \rightarrow r \downarrow_{\mathcal{R}/\mathcal{B}} \mid \ell \rightarrow r \in \mathcal{R}\}_{\doteq\sim} \\ \ddot{\mathcal{R}} &= \{\ell \rightarrow r \in \dot{\mathcal{R}} \mid \ell \in \text{NF}(\dot{\mathcal{R}} \setminus \{\ell \rightarrow r\})\}\end{aligned}$$

Here, $r \downarrow_{\mathcal{R}/\mathcal{B}}$ denotes an arbitrary normal form of r with respect to $\rightarrow_{\mathcal{R}/\mathcal{B}}$.

Example 6.9. Consider again the ES \mathcal{E} from Example 4.3. If neither *collapse* nor *compose* is used, AC completion results in the complete presentation $\mathcal{R} = \{1', 2', 3', 4-10, 13, 14, 15\}$. Because the right-hand sides of 14 and 15 are reducible by \mathcal{R}/\mathcal{B} and also because 13 is a right- \mathcal{B} -equivalent variant of $1'$, the TRS $\dot{\mathcal{R}}$ consists of $1', 2', 3', 4-10$ and the modified rules:

$$f(x + (y + z)) \xrightarrow{14''} (f(x) + f(y)) + f(z) \quad f((x + y) + z) \xrightarrow{15''} f(x) + (f(y) + f(z))$$

Since the left-hand sides of 5-10, $14''$, and $15''$ are reducible by $1'$, $3'$ or 4 , they are excluded from $\ddot{\mathcal{R}}$. Thus, $\ddot{\mathcal{R}} = \{1', 2', 3', 4\}$ is obtained. As expected, it is the result of performing completion with *collapse* and *compose*.

The following example shows that for preserving the equational theory, it does not suffice to make $\dot{\mathcal{R}}$ variant-free in the sense of \doteq .

Example 6.10. Suppose we change the definition to

$$\dot{\mathcal{R}} = \{\ell \rightarrow r \downarrow_{\mathcal{R}/\mathcal{B}} \mid \ell \rightarrow r \in \mathcal{R}\}$$

where variant-freeness is implicit. Consider the TRS \mathcal{R} consisting of the rules

$$f(x + y) \rightarrow f(x) + f(y) \quad f(x + y) \rightarrow f(y) + f(x)$$

where $+$ is an AC function symbol. We have $\dot{\mathcal{R}} = \mathcal{R}$ since the two rules are not variants and therefore $\ddot{\mathcal{R}} = \emptyset$ which obviously induces a different equational theory than \mathcal{R} . However, by using Definition 6.8 we obtain e.g. the following \mathcal{B} -canonical TRS:

$$\ddot{\mathcal{R}} = \dot{\mathcal{R}} = \{f(x + y) \rightarrow f(x) + f(y)\}$$

Before we can prove the main result of this section, we need the following technical lemma.

Lemma 6.11. *If \mathcal{R} is a \mathcal{B} -complete TRS and $\dot{\mathcal{R}}$ is Church–Rosser modulo \mathcal{B} then $\text{NF}(\ddot{\mathcal{R}}) \subseteq \text{NF}(\dot{\mathcal{R}})$.*

Proof. Suppose that \mathcal{R} is \mathcal{B} -complete and $\dot{\mathcal{R}}$ is Church–Rosser modulo \mathcal{B} . We prove the contrapositive $s \notin \text{NF}(\dot{\mathcal{R}}) \implies s \notin \text{NF}(\ddot{\mathcal{R}})$ for which it suffices to show that $l \notin \text{NF}(\ddot{\mathcal{R}})$ whenever $l \rightarrow r \in \dot{\mathcal{R}}$ due to closure of rewriting under contexts and substitutions. We prove this by induction on l with respect to the well-founded order \triangleright . If $l \rightarrow r \in \ddot{\mathcal{R}}$, the claim follows immediately. Otherwise, $l \notin \text{NF}(\dot{\mathcal{R}} \setminus \{l \rightarrow r\})$, i.e., there is a rule $l' \rightarrow r' \in \dot{\mathcal{R}}$ with $l \triangleright l'$. Furthermore, the rules $l \rightarrow r$ and $l' \rightarrow r'$ are not right- \mathcal{B} -equivalent variants by the definition of $\dot{\mathcal{R}}$. By definition, $\triangleright = \triangleright \cup \dot{=}$, so we continue by a case analysis on $l \triangleright l'$.

- ▷ If $l \triangleright l'$, the induction hypothesis yields $l' \notin \text{NF}(\ddot{\mathcal{R}})$ and therefore $l \notin \text{NF}(\ddot{\mathcal{R}})$ as desired.
- ▷ If $l \dot{=} l'$ then by definition there exists a renaming σ such that $l = l'\sigma$. From the right- \mathcal{B} -reducedness of $\dot{\mathcal{R}}$ we conclude that r and r' are normal forms with respect to $\rightarrow_{\dot{\mathcal{R}}/\mathcal{B}}$. Since normal forms are closed under renamings, this also holds for $r'\sigma$. Together with $r \xrightarrow{\dot{\mathcal{R}}} l = l'\sigma \xrightarrow{\dot{\mathcal{R}}} r'\sigma$ and the Church–Rosser modulo \mathcal{B} property of $\dot{\mathcal{R}}$ we obtain $r \sim_{\mathcal{B}} r'\sigma$. Thus, $l \rightarrow r$ and $l' \rightarrow r'$ are right- \mathcal{B} -equivalent variants. This contradicts the definition of $\dot{\mathcal{R}}$. Therefore, this case cannot happen. \square

Theorem 6.12. *If \mathcal{R} is a \mathcal{B} -complete TRS then $\ddot{\mathcal{R}}$ is a TRS which is normalization equivalent modulo \mathcal{B} to \mathcal{R} , conversion equivalent modulo \mathcal{B} to \mathcal{R} and canonical modulo \mathcal{B} .*

Proof. Let \mathcal{R} be a \mathcal{B} -complete TRS. By definition, $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}} \subseteq \rightarrow_{\dot{\mathcal{R}}/\mathcal{B}}^+$. Since \mathcal{R} and $\dot{\mathcal{R}}$ have the same left-hand sides, their normal forms coincide. An application of Lemma 6.4 yields that $\dot{\mathcal{R}}$ is \mathcal{B} -complete and normalization equivalent modulo \mathcal{B} to \mathcal{R} . Furthermore, $\text{NF}(\dot{\mathcal{R}}) = \text{NF}(\mathcal{R})$: The inclusion $\text{NF}(\ddot{\mathcal{R}}) \subseteq \text{NF}(\dot{\mathcal{R}})$ follows from Lemma 6.11 and the inclusion $\text{NF}(\dot{\mathcal{R}}) \subseteq \text{NF}(\ddot{\mathcal{R}})$ is a consequence of $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}}$. Now, another application of Lemma 6.4 yields that $\ddot{\mathcal{R}}$ is also \mathcal{B} -complete and normalization equivalent modulo \mathcal{B} to \mathcal{R} . The TRS $\dot{\mathcal{R}}$ is right- \mathcal{B} -reduced by definition. Since $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}}$, $\ddot{\mathcal{R}}$ is also right- \mathcal{B} -reduced. Together with the already established fact that $\text{NF}(\dot{\mathcal{R}}) = \text{NF}(\ddot{\mathcal{R}})$, left-reducedness of $\ddot{\mathcal{R}}$ follows from its definition. Hence, $\ddot{\mathcal{R}}$ is canonical modulo \mathcal{B} . Finally, Lemma 6.3 establishes that $\ddot{\mathcal{R}}$ is not only normalization equivalent modulo \mathcal{B} but also (conversion) equivalent modulo \mathcal{B} to \mathcal{R} . \square

Due to the constructive nature of Definition 6.8, the previous theorem states that given a \mathcal{B} -complete TRS \mathcal{R} it is possible to compute (if the equational theory of \mathcal{B} is decidable) a \mathcal{B} -canonical TRS which is equivalent modulo \mathcal{B} to \mathcal{R} . An inspection of Definition 6.7 further reveals that the inference system \mathbf{A} can produce \mathcal{B} -canonical TRSs: If $(\mathcal{E}, \emptyset) \vdash_{\mathbf{A}}^* (\emptyset, \mathcal{R})$ is a fair run and neither **compose** nor **collapse** are applicable, then \mathcal{R} is not only complete but also canonical modulo \mathcal{B} . The following lemma shows that this also holds for the inference system \mathbf{B} .

Lemma 6.13. *If $(\mathcal{E}, \emptyset) \vdash_{\mathbf{B}}^* (\emptyset, \mathcal{R})$ is a fair run and neither **compose** nor **collapse** are applicable, then \mathcal{R} is \mathcal{B} -canonical.*

Proof. According to Corollary 5.7, the TRS \mathcal{R} is a \mathcal{B} -complete presentation of \mathcal{E} . Furthermore, \mathcal{R} is left-reduced by definition as **collapse** is not applicable. For a proof by contradiction, suppose that \mathcal{R} is not right- \mathcal{B} -reduced. Hence, there exists a rule $\ell \rightarrow r$ in \mathcal{R} which can be modified to $\ell \rightarrow r'$ where $r \rightarrow_{\mathcal{R}/\mathcal{B}}^! r'$. Note that r and r' are not equivalent modulo \mathcal{B} as \mathcal{R} is terminating modulo \mathcal{B} . From the \mathcal{B} -completeness of \mathcal{R} we obtain $r \rightarrow_{\mathcal{R}}^! \cdot \sim_{\mathcal{B}} \cdot \mathcal{R}^! \leftarrow r'$. Since r' is a normal form with respect to $\rightarrow_{\mathcal{R}/\mathcal{B}}$, we have $r \rightarrow_{\mathcal{R}}^+ \cdot \sim_{\mathcal{B}} r'$ which contradicts our assumption that **compose** is not applicable. \square

Therefore, both A and B can produce canonical systems due to the availability of **compose** and **collapse** which is also referred to as *inter-reduction*. If a given completion procedure lacks inter-reduction, it is an instance of *elementary completion*. Note that the original completion procedure by Knuth and Bendix [KB70] performs elementary completion. The next example shows that there is a big difference between A and B with respect to elementary completion. In particular, A cannot simulate B if both are restricted to elementary completion: In order to simulate B's **simplify** in A, **compose** and **collapse** are needed as can be seen in the proof of Lemma 5.4.

Example 6.14. Consider the ESs $\mathcal{E} = \{x \cdot 1 \approx x\}$ and $\mathcal{B} = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z)\}$ as well as a corresponding run in A. There is only one option to orient the only equation into a terminating rewrite rule, namely $x \cdot 1 \rightarrow x$. From the critical peak $x \cdot y \xleftarrow{\epsilon} (x \cdot y) \cdot 1 \xrightarrow{\epsilon} x \cdot (y \cdot 1)$ we can deduce the rule $x \cdot (y \cdot 1) \rightarrow x \cdot y$. In B restricted to elementary completion, the result of deduce would be the corresponding equation which can be simplified to a trivial equation and therefore deleted. In A restricted to elementary completion, however, there are no means of removing or even altering the deduced rule. Hence, the rule will be used to deduce more critical pairs with the associativity axiom which are again kept as rules. Clearly, this process cannot terminate as overlaps with the associativity axioms can increase the size of the left-hand side of the rules without bound.

The previous example shows that for many ESs, there will not be a finite run in A restricted to elementary completion. Hence, the usage of inter-reduction and the resulting definition of canonicity is crucial for the success of A in generating finite solutions to validity problems. We close the section by stating the fact that even for reduced TRSs which are not complete, there may be fewer prime critical pairs than ordinary critical pairs. In Examples 3.2 and 3.10 we already observed that for a complete TRS.

Example 6.15. Consider the left-linear reduced TRS \mathcal{R} :

$$f(g(x, a)) \rightarrow x \qquad f(g(a, x)) \rightarrow f(f(a)) \qquad g(a, a) \rightarrow a$$

The TRS admits the three critical pairs stemming from the following critical peaks:

$$\begin{array}{cccc} \frac{f(g(a, a))}{\swarrow \searrow} & \frac{f(g(a, a))}{\swarrow \searrow} & \frac{f(g(a, a))}{\swarrow \searrow} & \frac{f(g(a, a))}{\swarrow \searrow} \\ f(f(a)) \quad a & a \quad f(f(a)) & f(a) \quad a & f(a) \quad f(f(a)) \end{array}$$

As the first two are not prime, $\text{PCP}(\mathcal{R})$ only consists of the two equations $f(a) \approx a$ and $f(f(a)) \approx a$. Note that the TRS is completed by adding the rule $f(a) \rightarrow a$ corresponding to the former equation.

6.3. Uniqueness of \mathcal{B} -Canonical Term Rewrite Systems. The main result of this section states that \mathcal{B} -canonical TRSs which are compatible with the same \mathcal{B} -compatible reduction order are unique up to right- \mathcal{B} -equivalent variants. This property justifies the usage of the term *canonical* and motivates the computation of TRSs which are canonical modulo \mathcal{B} as discussed in the last section. In order to prove this result, we start with an auxiliary lemma.

Lemma 6.16. *Let \mathcal{R} be a TRS which is right- \mathcal{B} -reduced and let s be a reducible term which is minimal with respect to \triangleright . If $s \rightarrow_{\mathcal{R}}^+ \cdot \sim_{\mathcal{B}} t$ then $s \rightarrow t$ is a right- \mathcal{B} -equivalent variant of a rule in \mathcal{R} .*

Proof. Let $\ell \rightarrow r$ be the first rule which is applied in the sequence $s \rightarrow_{\mathcal{R}}^+ \cdot \sim_{\mathcal{B}} t$, so $s \triangleright \ell$. Since s is a minimal reducible term with respect to \triangleright , we have $s \doteq \ell$. By definition, there is a renaming σ such that $s = \ell\sigma$. Since \mathcal{R} is right- \mathcal{B} -reduced, r is a normal form with respect to $\rightarrow_{\mathcal{R}/\mathcal{B}}$ and therefore also $\rightarrow_{\mathcal{R}}$. Closure of normal forms under renamings yields $r\sigma \in \text{NF}(\mathcal{R})$. Hence, $t \sim_{\mathcal{B}} r\sigma$ and we conclude that $s \rightarrow t$ and $\ell \rightarrow r$ are right- \mathcal{B} -equivalent variants. \square

In contrast to its counterpart for standard rewriting in [HMSW19, Theorem 4.9], the following lemma needs termination modulo \mathcal{B} as an additional precondition. Note that it still treats a more general case than the main result of this subsection (Theorem 6.18) as the Church–Rosser modulo \mathcal{B} property is not required.

Lemma 6.17. *TRSs which are normalization equivalent modulo \mathcal{B} , terminating modulo \mathcal{B} , left-reduced and right- \mathcal{B} -reduced are unique up to right- \mathcal{B} -equivalent variants.*

Proof. Let \mathcal{R} and \mathcal{S} be TRSs which are normalization equivalent modulo \mathcal{B} , terminating modulo \mathcal{B} , left-reduced and right- \mathcal{B} -reduced. As the argument is symmetric, we only show that every rule of \mathcal{R} has a right- \mathcal{B} -equivalent variant in \mathcal{S} . Consider $\ell \rightarrow r \in \mathcal{R}$. Note that ℓ and r are not \mathcal{B} -equivalent since otherwise the cycle $r \sim_{\mathcal{B}} \ell \rightarrow_{\mathcal{R}} r$ contradicts the fact that \mathcal{R} is terminating modulo \mathcal{B} . Furthermore, $r \in \text{NF}(\mathcal{R})$ as \mathcal{R} is right- \mathcal{B} -reduced. Hence, normalization equivalence modulo \mathcal{B} of \mathcal{R} and \mathcal{S} yields $\ell \rightarrow_{\mathcal{S}}^+ \cdot \sim_{\mathcal{B}} r$. Moreover, the left-reducedness of \mathcal{R} yields that ℓ is a minimal \mathcal{R} -reducible term with respect to \triangleright . Now suppose there exists a rule $\ell' \rightarrow r' \in \mathcal{S}$ such that $\ell \triangleright \ell'$. Right- \mathcal{B} -reducedness of \mathcal{S} yields $r' \in \text{NF}(\mathcal{S})$ and termination modulo \mathcal{B} of \mathcal{S} implies that ℓ' and r' are not \mathcal{B} -equivalent. Together with normalization equivalence modulo \mathcal{B} of \mathcal{R} and \mathcal{S} we obtain $\ell' \rightarrow_{\mathcal{R}}^+ \cdot \sim_{\mathcal{B}} r'$ which contradicts the fact that ℓ is a minimal \mathcal{R} -reducible term with respect to \triangleright . Therefore, ℓ is a minimal \mathcal{S} -reducible term with respect to \triangleright and from Lemma 6.16 we obtain that $\ell \rightarrow r$ is a right- \mathcal{B} -equivalent variant of a rule in \mathcal{S} which concludes the proof. \square

In the proof of the following theorem we now just need to establish the preconditions of the previous lemma.

Theorem 6.18. *Let \mathcal{R} and \mathcal{S} be TRSs which are equivalent modulo \mathcal{B} and canonical modulo \mathcal{B} . If \mathcal{R} and \mathcal{S} are compatible with the same \mathcal{B} -compatible reduction order then $\mathcal{R} \doteq_{\sim} \mathcal{S}$.*

Proof. Let \mathcal{R} and \mathcal{S} be compatible with the \mathcal{B} -compatible reduction order $>$. We show that \mathcal{R} and \mathcal{S} are normalization equivalent modulo \mathcal{B} which allows us to conclude the proof by Lemma 6.17. As the argument is symmetric, we only show $\rightarrow_{\mathcal{R}}^! \cdot \sim_{\mathcal{B}} \subseteq \rightarrow_{\mathcal{S}}^! \cdot \sim_{\mathcal{B}}$. Consider $s \rightarrow_{\mathcal{R}}^! t' \sim_{\mathcal{B}} t$. Since \mathcal{S} is terminating, there exists a term u such that $t' \rightarrow_{\mathcal{S}}^! u$. From the equivalence modulo \mathcal{B} of \mathcal{R} and \mathcal{S} we obtain $t' \xrightarrow[\mathcal{R} \cup \mathcal{B}]{}^* u$. As \mathcal{R} is Church–Rosser modulo \mathcal{B}

and $t' \in \text{NF}(\mathcal{R})$ we have $u \rightarrow_{\mathcal{R}}^! \cdot \sim_{\mathcal{B}} t'$. If t' and u are not \mathcal{B} -equivalent then both $u > t'$ (as $u \rightarrow_{\mathcal{R}}^! \cdot \sim_{\mathcal{B}} t'$) and $t' > u$ (as $t' \rightarrow_{\mathcal{S}}^! u$), which contradicts the irreflexivity of $>$. If $t' \neq u$, we also obtain a contradiction to the irreflexivity of $>$ from $u \sim_{\mathcal{B}} t' \rightarrow_{\mathcal{S}}^! u$ together with termination modulo \mathcal{B} of \mathcal{S} . Hence, $t' = u$ which means that $t' \in \text{NF}(\mathcal{S})$. Equivalence of \mathcal{R} and \mathcal{S} modulo \mathcal{B} yields $s \xrightarrow[\mathcal{S} \cup \mathcal{B}]{*} t'$ from which we obtain $s \rightarrow_{\mathcal{S}}^! \cdot \sim_{\mathcal{B}} t'$ by \mathcal{B} -completeness of \mathcal{S} and $t' \in \text{NF}(\mathcal{S})$. Finally, $t \sim_{\mathcal{B}} t'$ establishes $s \rightarrow_{\mathcal{S}}^! \cdot \sim_{\mathcal{B}} t$ as desired. \square

The previous result cannot be strengthened to literal similarity as the following counterexample shows.

Example 6.19. Consider the ES \mathcal{E} consisting of the single equation

$$f(x + y) \approx f(x) + f(y)$$

where $+$ is an AC function symbol. There are two \mathcal{B} -complete presentations of \mathcal{E} consisting of one rule each:

$$\mathcal{R} = \{f(x + y) \rightarrow f(x) + f(y)\} \quad \mathcal{R}' = \{f(x + y) \rightarrow f(y) + f(x)\}$$

While the rules of the two TRSs are right- \mathcal{B} -equivalent variants, they are not variants. Note that both systems are compatible with the same \mathcal{B} -compatible reduction order.

Note that \mathcal{R} and \mathcal{R}' in the previous example can also be obtained with the inference system \mathcal{B} . This shows that while the relation $\dot{\sim}$ is motivated by the definition of the inference system \mathcal{A} , the notion of right- \mathcal{B} -equivalent variants naturally arises in completion modulo \mathcal{B} for left-linear TRSs.

7. AC COMPLETION

So far, the theoretical results have been generalized by using an arbitrary equational theory \mathcal{B} . In practice, however, this article is concerned with the particular theory AC. The results of this section allow us to assess the effectiveness of the inference system \mathcal{A} in the setting of AC completion.

7.1. Limitations of Left-Linear AC Completion. In addition to the restriction to left-linear rewrite rules, the following example demonstrates another severe limitation of the inference system \mathcal{A} previously unmentioned in the literature.

Example 7.1. Consider the ES \mathcal{E} consisting of the equations

$$\text{and}(0, 0) \approx 0 \quad \text{and}(1, 1) \approx 1 \quad \text{and}(0, 1) \approx 0$$

where and is an AC function symbol. There is only one way to orient each equation. Furthermore, there are no critical pairs between the resulting rewrite rules. Hence, using the inference system \mathcal{A} we arrive at the intermediate TRS

$$\text{and}(0, 0) \rightarrow 0 \quad \text{and}(1, 1) \rightarrow 1 \quad \text{and}(0, 1) \rightarrow 0$$

where the only possible next step is to deduce local cliffs. We will now show that this has to be done infinitely many times. Note that an AC-complete presentation \mathcal{R} of \mathcal{E} has to be able to rewrite any term that is AC-equivalent to a reducible term. Consider the infinite family of terms

$$s_0 = \text{and}(0, 1) \quad s_1 = \text{and}(\text{and}(0, x_1), 1) \quad s_2 = \text{and}(\text{and}(\text{and}(0, x_1), x_2), 1) \quad \dots$$

as well as

$$t_0 = 0 \quad t_1 = \text{and}(0, x_1) \quad t_2 = \text{and}(\text{and}(0, x_1), x_2) \quad \dots$$

Clearly, $s_n \leftrightarrow_{\mathcal{E} \cup \text{AC}}^* t_n$ for all $n \in \mathbb{N}$ and therefore also $s_n \downarrow_{\mathcal{R}}^{\sim} t_n$ for all $n \in \mathbb{N}$, but this demands infinitely many rules in \mathcal{R} : For each s_n there is an AC-equivalent term such that the constants 0 and 1 are next to each other which allows us to rewrite it using the rule $\text{and}(0, 1) \rightarrow 0$. However, with n also the number of variables between these constants increases which requires \mathcal{R} to have infinitely many rules since rewrite rules can only be applied before the representation modulo AC is changed.

Note that there is nothing special about this example except the fact that it contains at least one equation which can only be oriented such that the left-hand side contains an AC function symbol where both arguments have “structure”, i.e., both arguments contain a function symbol which is different from the original AC function symbol. As a consequence, the necessity of infinitely many rules applies to all equational systems which have this property. Needless to say, this means that for a large class of equational systems the corresponding AC-canonical presentation (in the left-linear sense) is infinite if it exists. This observation is in stark contrast to the properties of general AC completion as presented in the next section which can complete the ES \mathcal{E} from Example 7.1 into a finite AC-canonical TRS by simply orienting all rules from left to right. The following example shows that given a nonempty context, the same effect can also be seen even when only one argument contains a different function symbol.

Example 7.2. Consider the ES \mathcal{E} consisting of the equation $s(\text{p}(x) + y) = x + y$ where $+$ is an AC function symbol. Note that orienting the equation from left to right is the only way to convert \mathcal{E} into a terminating TRS \mathcal{R} . The rule has no critical pairs with itself, but as in the previous example, adding critical pairs with AC does not terminate. This can be seen by considering the infinite family of terms

$$s_0 = s(\text{p}(x) + y_1) \quad s_1 = s(\text{p}(x) + y_1) + y_2 \quad s_2 = s(((\text{p}(x) + y_1) + y_2) + y_3) \quad \dots$$

as well as

$$t_0 = x + y_1 \quad t_1 = (x + y_1) + y_2 \quad t_2 = ((x + y_1) + y_2) + y_3 \quad \dots$$

Again, $s_n \leftrightarrow_{\mathcal{E} \cup \text{AC}}^* t_n$ for all $n \in \mathbb{N}$. For a complete presentation \mathcal{R} of $\mathcal{E} \cup \text{AC}$ we have $s_n \downarrow_{\mathcal{R}}^{\sim} t_n$ for all $n \in \mathbb{N}$, but this demands infinitely many rules in \mathcal{R} as before.

7.2. General AC Completion. Inference systems for completion modulo an equational theory which are not restricted to the left-linear case usually need more inference rules than the ones already covered in this article. For general AC completion, however, there exists a particularly simple inference system which constitutes a special case of normalized completion [Mar96] and can be found in Sarah Winkler’s PhD thesis [Win13, p. 109].

Definition 7.3. The inference system KB_{AC} is the same as A for the fixed theory AC but with a modified collapse rule which allows us to rewrite with $\rightarrow_{\mathcal{R}/\text{AC}}$ and the following rule which replaces the two deduction rules of A :

$$\text{deduce} \quad \frac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \mathcal{R} \leftarrow \cdot \sim_{\text{AC}} \cdot \rightarrow_{\mathcal{R}} t$$

The purpose of this section is to show how \mathbf{A} can be simulated by $\mathbf{KB}_{\mathbf{AC}}$ in the case of $\mathcal{B} = \mathbf{AC}$. In addition to its theoretical significance, this simulation result is also of practical importance as it facilitates switching from \mathbf{A} to $\mathbf{KB}_{\mathbf{AC}}$ in the middle of a completion process instead of starting from scratch. We have more to say about this at the end of this section.

Since local cliffs cannot be deduced in $\mathbf{KB}_{\mathbf{AC}}$, the simulation has to work with a potentially smaller set of rewrite rules. Furthermore, during a run, the variants of rules stemming from local cliffs may be in different states with respect to inter-reduction (collapse and compose). Given an intermediate TRS \mathcal{R} of a run in \mathbf{A} as well as an intermediate TRS \mathcal{R}' of a run in $\mathbf{KB}_{\mathbf{AC}}$, the invariant $\mathcal{R} \subseteq \rightarrow_{\mathcal{R}'/\mathbf{AC}}^+$ resolves both of the aforementioned problems. The main motivation behind this invariant is the avoidance of *compose* and *collapse* in the $\mathbf{KB}_{\mathbf{AC}}$ run.

Lemma 7.4. *If $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{A}} (\mathcal{E}_2, \mathcal{R}_2)$ and $\mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ then there exists a TRS \mathcal{R}'_2 such that $(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^* (\mathcal{E}_2, \mathcal{R}'_2)$ and $\mathcal{R}_2 \subseteq \rightarrow_{\mathcal{R}'_2/\mathbf{AC}}^+$.*

Proof. Let $>$ be a fixed AC-compatible reduction order which is used in both \mathbf{A} and $\mathbf{KB}_{\mathbf{AC}}$. Suppose $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{A}} (\mathcal{E}_2, \mathcal{R}_2)$ and $\mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$. We perform a case analysis on the inference rule applied in $(\mathcal{E}_1, \mathcal{R}_1) \vdash_{\mathbf{A}} (\mathcal{E}_2, \mathcal{R}_2)$. The only interesting cases are when *deduce*, *simplify*, *compose* or *collapse* are applied.

- ▷ If *deduce* is applied, we further distinguish whether it was applied to a local peak or cliff. In the case of a local cliff, we have $\mathcal{E}_1 = \mathcal{E}_2$ and $\mathcal{R}_2 = \mathcal{R}_1 \cup \{\ell \rightarrow r\}$ with $\ell \rightarrow_{\mathcal{R}_1/\mathbf{AC}} r$. From $\ell \rightarrow_{\mathcal{R}_1/\mathbf{AC}} r$ and $\mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ we obtain $\ell \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+ r$. Thus, $\mathcal{R}_2 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ holds. As $(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^0 (\mathcal{E}_2, \mathcal{R}'_1)$ is trivial, the claim follows. In the case of a local peak, we have $\mathcal{R}_1 = \mathcal{R}_2$ and $\mathcal{E}_2 = \mathcal{E}_1 \cup \{t \approx u\}$ with $t \mathcal{R}_1 \leftarrow s \rightarrow_{\mathcal{R}_1} u$. Since $\mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ holds, we have

$$t \mathcal{R}'_1/\mathbf{AC} \leftarrow^* v \mathcal{R}'_1 \leftarrow \cdot \sim_{\mathbf{AC}} s \sim_{\mathbf{AC}} \cdot \rightarrow_{\mathcal{R}'_1} w \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^* u$$

for some v and w . By performing *deduce* and *simplify* steps

$$(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}} (\mathcal{E}_1 \cup \{v \approx w\}, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^* (\mathcal{E}_1 \cup \{t \approx u\}, \mathcal{R}'_1) = (\mathcal{E}_2, \mathcal{R}'_1)$$

is obtained. As $\mathcal{R}_1 = \mathcal{R}_2$, the inclusion $\mathcal{R}_2 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ is trivial. Hence, the claim holds.

- ▷ If *simplify* is applied, we have $\mathcal{R}_1 = \mathcal{R}_2$, $\mathcal{E}_1 = \mathcal{E}_0 \cup \{s \approx t\}$ and $\mathcal{E}_2 = \mathcal{E}_0 \cup \{s' \approx t'\}$ with $s \rightarrow_{\mathcal{R}_1} s'$ and $t \rightarrow_{\mathcal{R}_1} t'$. By $\mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ we have $s \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^* s'$ and $t \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^* t'$. Therefore, performing *simplify*, we obtain $(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^* (\mathcal{E}_2, \mathcal{R}'_1)$. As $\mathcal{R}_1 = \mathcal{R}_2$, the inclusion $\mathcal{R}_2 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ is trivial.
- ▷ If *compose* is applied, we have $\mathcal{E}_1 = \mathcal{E}_2$, $\mathcal{R}_1 = \mathcal{R}_0 \cup \{\ell \rightarrow r\}$ and $\mathcal{R}_2 = \mathcal{R}_0 \cup \{\ell \rightarrow r'\}$ with $r \rightarrow_{\mathcal{R}_0/\mathbf{AC}} r'$. We have $(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^0 (\mathcal{E}_2, \mathcal{R}'_1)$. Since the inclusions $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ yield $\ell \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+ r \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+ r'$, we obtain $\mathcal{R}_2 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$.
- ▷ If *collapse* is applied, we have $\mathcal{E}_2 = \mathcal{E}_1 \cup \{\ell' \approx r\}$ and $\mathcal{R}_1 = \mathcal{R}_2 \uplus \{\ell \rightarrow r\}$ with $\ell \rightarrow_{\mathcal{R}_2} \ell'$. By $\mathcal{R}_2 \subseteq \mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ we have

$$\ell' \mathcal{R}'_1/\mathbf{AC} \leftarrow^* t \mathcal{R}'_1 \leftarrow \cdot \sim_{\mathbf{AC}} \ell \sim_{\mathbf{AC}} \cdot \rightarrow_{\mathcal{R}'_1} u \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^* r$$

for some t and u . Performing *deduce* and *simplify*, we obtain:

$$(\mathcal{E}_1, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}} (\mathcal{E}_1 \cup \{t \approx u\}, \mathcal{R}'_1) \vdash_{\mathbf{KB}_{\mathbf{AC}}}^* (\mathcal{E}_1 \cup \{\ell' \approx r\}, \mathcal{R}'_1) = (\mathcal{E}_2, \mathcal{R}'_1)$$

By $\mathcal{R}_2 \subseteq \mathcal{R}_1 \subseteq \rightarrow_{\mathcal{R}'_1/\mathbf{AC}}^+$ the claim is concluded. \square

Theorem 7.5. *For every fair run $(\mathcal{E}, \emptyset) \vdash_{\mathbf{A}}^* (\emptyset, \mathcal{R})$ there exists a run $(\mathcal{E}, \emptyset) \vdash_{\text{KB}_{\text{AC}}}^* (\emptyset, \mathcal{R}')$ such that \mathcal{R}'/AC is an AC-complete presentation of \mathcal{E} .*

Proof. With a straightforward induction argument, we obtain the run $(\mathcal{E}, \emptyset) \vdash_{\text{KB}_{\text{AC}}}^* (\emptyset, \mathcal{R}')$ as well as $\mathcal{R} \subseteq \rightarrow_{\mathcal{R}'/\text{AC}}^+$ (*) from Lemma 7.4. Furthermore, AC termination of \mathcal{R}' and $\leftrightarrow_{\mathcal{E} \cup \text{AC}}^* = \leftrightarrow_{\mathcal{R}' \cup \text{AC}}^*$ (**) are easy consequences of the definition of KB_{AC} . AC-completeness of \mathcal{R} follows from fairness of the run in \mathbf{A} and Theorem 4.12. For the Church–Rosser modulo AC property of \mathcal{R}'/AC , consider a conversion $s \leftrightarrow_{\mathcal{R}' \cup \text{AC}}^* t$. From (**) we obtain $s \leftrightarrow_{\mathcal{E} \cup \text{AC}}^*$ and therefore $s \rightarrow_{\mathcal{R}}^* \cdot \sim_{\text{AC}} \cdot \mathcal{R} \leftarrow t$ by the fact that \mathcal{R} is an AC-complete presentation of \mathcal{E} . Finally, (*) yields $s \rightarrow_{\mathcal{R}'/\text{AC}}^* \cdot \sim_{\text{AC}} \cdot \mathcal{R}'/\text{AC} \leftarrow t$ as desired. Thus, \mathcal{R}'/AC is an AC-complete presentation of \mathcal{E} . \square

In addition to the result of the previous theorem, the proof of Lemma 7.4 provides a procedure to construct a KB_{AC} run which “corresponds” to a given \mathbf{A} run. In particular, this means that it is possible to switch from \mathbf{A} to KB_{AC} at any point while performing AC completion. This is of practical relevance: Assume that AC completion is started with \mathbf{A} in order to avoid AC unification. If \mathbf{A} gets stuck due to simplified equations which are not orientable into a left-linear rule or it seems to be the case that the procedure diverges due to the problem described in Example 7.1, starting from scratch with KB_{AC} is not necessary. We conclude the section by illustrating the practical relevance of the simulation result with an example.

Example 7.6. Consider the ES \mathcal{E} for abelian groups consisting of the equations

$$e \cdot x \approx x \qquad x^- \cdot x \approx e$$

where \cdot is an AC symbol. Note that the well-known completion run for non-abelian group theory is also a run in \mathbf{A} : Critical pairs with respect to the associativity axiom are deducible via local cliffs, non-left-linear intermediate rules are allowed and all (intermediate) rules are orientable with e.g. AC-KBO. Hence, we obtain the TRS \mathcal{R}' consisting of the rules

$$\begin{array}{lll} e \cdot x \xrightarrow{1} x & x^- \cdot (x \cdot y) \xrightarrow{4} y & x \cdot x^- \xrightarrow{7} e \\ x^- \cdot x \xrightarrow{2} e & (x \cdot y)^- \xrightarrow{5} y^- \cdot x^- & e^- \xrightarrow{8} e \\ x^- \xrightarrow{3} x & x \cdot e \xrightarrow{6} x & x \cdot (x^- \cdot y) \xrightarrow{9} y \end{array}$$

and switch to KB_{AC} where we can collapse the redundant rules 4, 6, 7 and 9. A final joinability check of all AC critical pairs reveals that the resulting TRS \mathcal{R} is an AC-complete presentation of abelian groups. Hence, the simulation result allows us to make progress with \mathbf{A} even when it is doomed to fail. In particular, critical pairs between rules whose left-hand sides do not contain AC symbols do not need to be recomputed.

8. IMPLEMENTATION

The command-line tool `accomp11` implements AC completion for left-linear TRSs based on the inference system \mathbf{A} (Definition 4.1). It uses external termination tools instead of a fixed AC-compatible reduction order and is written in the programming language Haskell. The source code of the tool `accomp11` is available on GitHub². As input, the tool expects a file in the WST³ format describing the equational theory on which left-linear AC completion should

²<https://github.com/niedjoh/accomp11>

³<https://www.lri.fr/~marche/tpdb/format.html>

be performed. The user can choose whether $\rightarrow_{\mathcal{R}}$, $\rightarrow_{\mathcal{R},\text{AC}}$ or $\rightarrow_{\mathcal{R}/\text{AC}}$ is used for rewriting in the inference rules `simplify` and `compose`. Furthermore, the generation of critical pairs can be restricted to the primality criterion. Note that most of these options are facilitated by the theoretical results in the previous sections. A discussion of $\rightarrow_{\mathcal{R},\text{AC}}$ will follow in Section 8.3.

Another feature is the validity problem solving mode which solves a given instance of the validity problem for an equational theory \mathcal{E} upon successful completion of \mathcal{E} . This mode can be triggered by supplying a concrete equation $s \approx t$ as a command line argument in addition to the file describing \mathcal{E} .

In the tool `accompII`, external termination tools do much of the heavy lifting. In particular, the user can supply the executable of an arbitrary termination tool as long as the output starts with `YES`, `MAYBE`, `NO` or `TIMEOUT` (all other cases are treated as an error). The input format for the termination tool can be set by a command line argument. The available options are the WST format as well as the XML format of the Nagoya Termination Tool [YKS14].⁴

Since starting a new process for every call of the termination tool causes a lot of operating system overhead, the tool supports an interactive mode which allows it to communicate with a single process of the termination tool in a dialogue style. Here, the only constraint for the termination tool is that it accepts a sequence of termination problems separated by the keyword `(RUN)`. This is currently only implemented in an experimental version of Tyrolean Termination Tool 2 ($\top\top\top_2$) [KSZM09], but we hope that more termination tools will follow as this approach has a positive effect on the runtime of completion with termination tools while demanding comparatively little implementation effort. The remainder of this section discusses important aspects and properties of the implementation.

8.1. Termination Tools. The reduction order is a critical input parameter of a completion procedure. Finding an appropriate order can be very challenging as the equations which are generated during a completion run are usually not known in advance. Hence, the nature of this problem is different from the standard termination problem where the input is one specific TRS.

While it is well-known that termination is an undecidable property of TRSs, a number of termination tools have been developed which can solve the termination problem automatically in many practical cases. The usage of termination tools in completion has been pioneered by Wehrman et al. [WSW06]. Intuitively, the difficulty of finding an appropriate reduction order is replaced by a sequence of calls to a termination tool. This allows us to define a completion procedure which does not depend on a reduction order as input. An important ingredient for completion with termination tools is a *constraint system* which keeps track of all rules which have been produced during the run. Checking termination for the constraint system as opposed to the current TRS guarantees that there exists a single reduction order which can be used for the whole run. This is important as correctness is lost when the reduction order is changed during a completion run [SK94]. The next definition extends the inference system \mathbf{A} by a constraint system.

Definition 8.1. The inference system $\mathbf{A}_{\top\top}$ transforms triples consisting of an ES \mathcal{E} and TRSs \mathcal{R} , \mathcal{C} over the common signature \mathcal{F} . Except for `orient`, the inference rules are trivial

⁴<https://www.trs.cm.is.nagoya-u.ac.jp/NaTT/natt-xml.html>

extensions of the rules in $A_{\top\top}$ where the constraint system \mathcal{C} is not changed.

$$\text{orient} \quad \frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{s \rightarrow t\}, \mathcal{C} \cup \{s \rightarrow t\}} \quad \text{if } \mathcal{C} \cup \{s \rightarrow t\} \text{ is } \mathcal{B}\text{-terminating}$$

$$\frac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}, \mathcal{C}}{\mathcal{E}, \mathcal{R} \cup \{t \rightarrow s\}, \mathcal{C} \cup \{t \rightarrow s\}} \quad \text{if } \mathcal{C} \cup \{t \rightarrow s\} \text{ is } \mathcal{B}\text{-terminating}$$

Note that \mathcal{R} is subject to insertion as well as removal of rules while the constraint system \mathcal{C} just acts as an accumulator of all orientations for the termination check. As usual, $(\mathcal{E}, \mathcal{R}, \mathcal{C}) \vdash_{A_{\top\top}} (\mathcal{E}', \mathcal{R}', \mathcal{C}')$ denotes a step in the inference system $A_{\top\top}$ and a sequence of steps starting with $(\mathcal{E}, \emptyset, \emptyset)$ constitutes a run for \mathcal{E} . We will now prove that $A_{\top\top}$ is sound and complete with respect to A .

Lemma 8.2. *For every run*

$$\mathcal{E}_0, \mathcal{R}_0, \mathcal{C}_0 \vdash_{A_{\top\top}} \mathcal{E}_1, \mathcal{R}_1, \mathcal{C}_1 \vdash_{A_{\top\top}} \cdots \vdash_{A_{\top\top}} \mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n$$

there exists an equivalent run

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_A \mathcal{E}_1, \mathcal{R}_1 \vdash_A \cdots \vdash_A \mathcal{E}_n, \mathcal{R}_n$$

which uses the reduction order $\rightarrow_{\mathcal{C}_n/\mathcal{B}}^+$.

Proof. First of all, note that $\rightarrow_{\mathcal{C}_n/\mathcal{B}}^+$ is a \mathcal{B} -compatible reduction order as it is transitive, closed under contexts and substitutions and the \mathcal{B} -termination of \mathcal{C}_n is established by induction on the length of the run in $A_{\top\top}$. Since the constraint system is only altered and used in the orient rule, all other steps are valid steps in A by just removing the constraint system. Since \mathcal{C}_n includes all rules which are oriented during the run, the corresponding orient steps in A also succeed. \square

Lemma 8.3. *For every run*

$$\mathcal{E}_0, \mathcal{R}_0 \vdash_A \mathcal{E}_1, \mathcal{R}_1 \vdash_A \cdots \vdash_A \mathcal{E}_n, \mathcal{R}_n$$

using the \mathcal{B} -compatible reduction order $>$ there exists an equivalent run

$$\mathcal{E}_0, \mathcal{R}_0, \mathcal{C}_0 \vdash_{A_{\top\top}} \mathcal{E}_1, \mathcal{R}_1, \mathcal{C}_1 \vdash_{A_{\top\top}} \cdots \vdash_{A_{\top\top}} \mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n$$

where $\rightarrow_{\mathcal{C}_n/\mathcal{B}}^+ \subseteq >$.

Proof. We perform induction on n . For $n = 0$ we just translate $(\mathcal{E}_0, \emptyset)$ to $(\mathcal{E}_0, \emptyset, \emptyset)$. For $n > 0$ we obtain $(\mathcal{E}_0, \mathcal{R}_0, \mathcal{C}_0) \vdash_{A_{\top\top}}^* (\mathcal{E}_{n-1}, \mathcal{R}_{n-1}, \mathcal{C}_{n-1})$ and $\rightarrow_{\mathcal{C}_{n-1}/\mathcal{B}}^+ \subseteq >$ from the induction hypothesis. If the step $(\mathcal{E}_{n-1}, \mathcal{R}_{n-1}) \vdash_A (\mathcal{E}_n, \mathcal{R}_n)$ is not an application of orient we have $(\mathcal{E}_{n-1}, \mathcal{R}_{n-1}, \mathcal{C}_{n-1}) \vdash_{A_{\top\top}} (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ where $\mathcal{C}_{n-1} = \mathcal{C}_n$ and therefore $\rightarrow_{\mathcal{C}_n/\mathcal{B}}^+ \subseteq >$ as the constraint system is just carried along without any further restrictions in all rules except orient. Now assume that the step is an application of orient on the equation $s \approx t$. We have $\mathcal{E}_n = \mathcal{E}_{n-1} \setminus \{s \approx t\}$ and $\mathcal{R}_n = \mathcal{R}_{n-1} \cup \{v \rightarrow w\}$ where $v > w$ and $\{v, w\} = \{s, t\}$. Let $\mathcal{C}_n = \mathcal{C}_{n-1} \cup \{v \rightarrow w\}$. Thus, $(\mathcal{E}_{n-1}, \mathcal{R}_{n-1}, \mathcal{C}_{n-1}) \vdash_{A_{\top\top}} (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ by an application of orient and we obtain $(\mathcal{E}_0, \mathcal{R}_0, \mathcal{C}_0) \vdash_{A_{\top\top}}^* (\mathcal{E}_n, \mathcal{R}_n, \mathcal{C}_n)$ as well as $\rightarrow_{\mathcal{C}_n/\mathcal{B}}^+ \subseteq >$ from the induction hypothesis together with $v > w$ and the definition of \mathcal{C}_n . \square

Note that the presented translation between runs of A and A_{TT} allows us to speak about the fairness of runs in A_{TT} since this simply means that the corresponding run in A is fair.

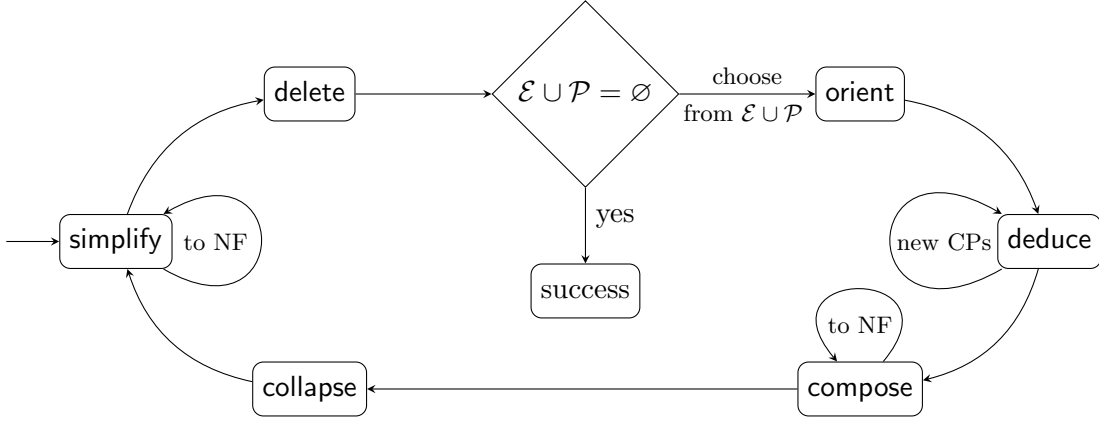
Lemma 8.2 shows that the usage of termination tools is a sound extension which allows us to perform completion without constructing an appropriate reduction order beforehand. Furthermore, the usage of termination tools does not affect the applicability of our completion procedure due to the previous completeness result (Lemma 8.3). However, implementing completion with termination tools such that it does not affect the applicability in practice is a highly nontrivial task. The reason for this is that without a concrete reduction order, both versions of the `orient` rule may be applicable. This yields a potentially huge search space as orienting a rule in the “wrong” direction may cause completion to fail or diverge. In [WSW06] this problem is solved by traversing the search space which is a binary tree with some best-first strategy based on a cost function which takes the number of equations, rules and critical pairs into account.

The state of the art for solving this problem efficiently is *multi-completion with termination tools* due to Winkler et al. [WSMK13]. This method traverses the whole search space while being as efficient as possible by sharing computations between different nodes in the binary tree which represents the search space. The method is implemented in the tool `mkbTT`. As the implementation of this approach is a major effort, we adopt the strategy used by the automatic mode of the Knuth–Bendix Completion Visualizer (KBCV) [SZ12]. Instead of traversing the whole search space, KBCV runs two threads in parallel where one thread prefers to orient equations from left to right while the other one prefers to orient from right to left. If one of the threads finishes successfully, the corresponding result is reported. Completion fails if both threads fail. Needless to say, this compromises completeness, but it is a trade-off which works well in many practical cases. In particular, KBCV can also complete systems which `mkbTT` cannot within a given time constraint [SZ12].

8.2. Strategy. The presentation of A_{TT} as an inference system allows implementations to use the given rules in any order as long as the produced run is fair. This section is about the strategy for applying rules of A_{TT} which is employed by the tool `accompII`. From now on, we specialize to the equational theory AC as `accompII` was developed for this specific case.

The used strategy is based on Huet’s completion procedure [BN98, Section 7.4]. An important property of this procedure is that the rules `simplify`, `delete`, `compose` and `collapse` are applied eagerly in order to keep the intermediate ESs and TRSs as small as possible. Usually, this has a positive effect on the runtime of the completion process. Unlike Huet’s completion procedure, in the employed strategy the orientation of a rule is always directly followed by the computation of all possible critical pairs which involve this rule. Furthermore, `orient` is only applied once per iteration to the smallest equation with respect to the term size. This modified version of Huet’s completion procedure is implemented e.g. in KBCV and the flow chart depicted in Figure 1 is based on [SZ12]. However, in contrast to standard completion, we have to add some of the critical pairs as rules in A_{TT} . Hence, we apply `deduce` on the selected equation before `compose` and `collapse` are applied exhaustively. Moreover, we need to keep track of a separate set of pending rules (\mathcal{P}) as the eager recursive computation of critical pairs between rules and AC axioms might unnecessarily lead to non-termination of the completion procedure.

We will now describe the flow chart depicted in Figure 1 in detail. As already mentioned, `accompII` also keeps track of a list of pending rules \mathcal{P} . Intuitively, a pending rule is like a normal rule with the exception that critical pairs involving it have not yet been computed

Figure 1: Flow chart for `accompl`'s completion procedure.

and it is not used to collapse other rules. Hence, `accompl` works on a quadruple of an ES and three TRSs $(\mathcal{E}, \mathcal{P}, \mathcal{R}, \mathcal{C})$ which are processed by one main loop. First of all, `simplify` is applied exhaustively such that both sides of every equation are normal forms with respect to \mathcal{E} and \mathcal{P} . This preliminary step may be already enough to join some equations, so after that every AC equivalent equation is removed with `delete`. If no equations or pending rules are left, we are done. Otherwise, an equation $\ell \approx r$ or pending rule $\ell \rightarrow r$ which is minimal with respect to $|\ell| + |r|$ where $|\cdot|$ denotes the size of a term is selected. If it is a rule, `orient` is just the identity function. Otherwise, `orient` is applied with the orientation preference of the given thread, i.e., it first orients the equation in the preferred direction and only tries the other option in case of failure. An important feature of the implementation of `orient` is that it can be postponed, i.e., if one equation is not orientable in either direction, the next one is tried. If `orient` never succeeds, the thread terminates in a failure state.

In any case, a successful application of `orient` yields some rule ρ . Next, `deduce` is used to produce the sets $\text{CP}(\{\rho\})$ and $\text{CP}^\pm(\mathcal{R}, \{\rho\})$ which are added to \mathcal{E} as well as $\text{CP}^\pm(\text{AC}^\pm, \{\rho\})$ which is added to \mathcal{P} . The computation of critical pairs is of course restricted to the primality criterion if the corresponding option is set.

After that, all rules in $\mathcal{P} \cup \mathcal{R}$ are exhaustively composed such that their right-hand sides are normal forms with respect to $\mathcal{P} \cup \mathcal{R}$. Finally, `collapse` is applied exhaustively in \mathcal{R} with respect to ρ and in \mathcal{P} with respect to $\{\rho\} \cup \mathcal{R}$. Here, we distinguish between \mathcal{R} and \mathcal{P} because at this point, the left-hand sides of \mathcal{R} are known to be normal forms of the remaining rules of \mathcal{R} while this can not be assured for the deduced rules \mathcal{P} . Note that we do not collapse \mathcal{P} with respect to \mathcal{P} as it would require checking rules for equality.

In order to produce a left-linear system, `orient` is only applied if the left-hand side of the resulting rule is linear which makes all intermediate TRSs left-linear. Fairness only demands left-linearity for the resulting TRS, but improving this aspect was not considered as it is unclear based on which criteria non-left-linear rules should be turned back into equations again. Moreover, keeping each intermediate TRS left-linear makes intermediate rules stemming from critical pairs left-linear by definition: Since the AC axioms are linear, overlaps between the AC axioms and a left-linear TRS always produce linear terms. Thus, the left-linearity of pending rules does not have to be checked as rewriting with linear rules preserves linearity. We are now ready to prove the main result of this section.

Theorem 8.4. *The TRSs produced by the tool `accompII` are AC-canonical presentations of the ES provided as input.*

Proof. Suppose that given an ES \mathcal{E} , `accompII` outputs a TRS \mathcal{R}_n and consider the thread which produced this result in n iterations of the main loop as depicted in the flow chart (Figure 1). In particular, let $\mathcal{E}_i, \mathcal{P}_i, \mathcal{R}_i$ and \mathcal{C}_i denote the respective values of $\mathcal{E}, \mathcal{P}, \mathcal{R}$ and \mathcal{C} at the decision node in the flow chart in the i -th iteration. Note that $\mathcal{E}_0 = \mathcal{E}, \mathcal{P}_0 = \mathcal{R}_0 = \mathcal{C}_0 = \emptyset$ and $\mathcal{E}_n = \mathcal{P}_n = \emptyset$. It is immediate from the flow chart and its textual description that

$$(\mathcal{E}_0, \mathcal{R}_0 \cup \mathcal{P}_0, \mathcal{C}_0) \vdash_{\text{ATT}}^* (\mathcal{E}_1, \mathcal{R}_1 \cup \mathcal{P}_1, \mathcal{C}_1) \vdash_{\text{ATT}}^* \cdots \vdash_{\text{ATT}}^* (\mathcal{E}_n, \mathcal{R}_n \cup \mathcal{P}_n, \mathcal{C}_n)$$

is a run for \mathcal{E} . Furthermore, all prime critical pairs of \mathcal{R}_n have been considered as an intermediate equation and all prime critical pairs between \mathcal{R}_n and AC^\pm have been considered as an intermediate rule. The fact that \mathcal{R}_n is an AC-complete presentation of \mathcal{E} now follows from fairness (Definition 4.2) together with Lemma 8.2 and Theorem 4.12. Finally, a straightforward induction argument shows that the statements

- (1) $\ell \in \text{NF}(\mathcal{R}_i)$ for every $\ell \rightarrow r \in \mathcal{R}_i \cup \mathcal{P}_i$ and
- (2) $r \in \text{NF}(\mathcal{R}_i \cup \mathcal{P}_i)$ for every $\ell \rightarrow r \in \mathcal{R}_i \cup \mathcal{P}_i$

hold for $0 \leq i \leq n$. Together, (1) and (2) show that \mathcal{R}_n is also AC-canonical: Just like in the proof of Lemma 6.13 we conclude that if each rule of an AC-complete system is right-reduced, it is also right-AC-reduced. \square

8.3. Implementation Details. The implementation of the tool `accompII` is based on the Haskell `term-rewriting` library [FAS13] which takes care of most of our needs regarding term rewriting except rewriting modulo AC, prime critical pairs and the computation of normal forms. The following paragraphs describe selected implementation details.

Rewriting to normal forms. A naive implementation of the computation of normal forms is highly inefficient as in general, the whole term has to be traversed for every rewrite step. As a trade-off between the time-consuming implementation of sophisticated term indexing techniques and the naive implementation we employ a bottom-up construction of normal forms by innermost rewriting of marked terms where normal form positions are remembered.

AC equivalence of terms. Checking whether two terms are equivalent modulo AC is needed for the implementation of the inference rule `deduce`. To this end, an equation $s \approx t$ is transformed into a canonical form $s' \approx t'$ where nested applications of AC symbols are flattened out to just one n -ary application for arbitrary n and their arguments are ordered with respect to some total order on terms. Then, $s \sim_{\text{AC}} t$ if and only if $s' = t'$. As an example, the terms $f(x, f(y, g(f(z, a))))$ and $f(f(y, x), g(f(a, z)))$ with $f \in \text{AC}$ are AC equivalent because they have the same canonical form $f(g(f(a, z)), x, y)$ with respect to the lexicographic order on the representation of terms as strings of ASCII symbols.

Normal forms in rewriting modulo AC. As already mentioned in Section 2, a direct implementation of $\rightarrow_{\mathcal{R}/\text{AC}}$ cannot be efficient. In the following, we will prove that it suffices to implement the relation $\rightarrow_{\mathcal{R},\text{AC}}$ from Peterson and Stickel [PS81] which is easier to compute but relies on AC matching. Although there are more efficient implementations, we used the AC matching algorithm due to Contejean [Con04] as it has been certified. The result we are about to prove (Corollary 8.8) is a special case of more general results for conditional rewriting modulo an equational theory by Meseguer [Mes17, Corollary 3]. However, to the best of our knowledge, the literature does not contain a direct proof of the required result despite its importance for effective implementations also in the unconditional case. Therefore, in the following, we give a detailed and direct proof of the result.

We start with recalling the definition of the concept of extended rules due to [PS81]. Let \mathcal{R} be a TRS and let $f(u, v) \rightarrow r$ be a rule in \mathcal{R} where f is an AC function symbol. The rule $f(f(u, v), x) \rightarrow f(r, x)$ with $x \in \mathcal{V} \setminus \mathcal{V}\text{ar}(f(u, v))$ is an *extension* of $f(u, v) \rightarrow r$. The TRS consisting of \mathcal{R} together with all extensions of rules in \mathcal{R} is denoted by \mathcal{R}^e . We will now prove that the relations $\rightarrow_{\mathcal{R}/\text{AC}}^* \cdot \sim_{\text{AC}}$ and $\rightarrow_{\mathcal{R}^e, \text{AC}}^* \cdot \sim_{\text{AC}}$ coincide. To this end, we show $\sim_{\text{AC}} \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}^e, \text{AC}} \cdot \sim_{\text{AC}}$, which is called *strict coherence* in [Mes17].

Lemma 8.5. $\text{AC}^{\leftarrow} \cdot \rightarrow_{\mathcal{R}^e, \text{AC}} \subseteq \rightarrow_{\mathcal{R}^e, \text{AC}} \cdot \overline{\text{AC}}^{\leftarrow}$

Proof. Suppose $t \text{AC}^{\leftarrow} s \rightarrow_{\mathcal{R}^e, \text{AC}}^q u$. We show $t \rightarrow_{\mathcal{R}^e, \text{AC}} \cdot \overline{\text{AC}}^{\leftarrow} u$ by induction on s . Let $\ell_1 \rightarrow r_1$ and $\ell_2 \rightarrow r_2$ be the rules employed in the left and right steps. We distinguish four cases. We use the facts that $\sim_{\text{AC}} \cdot \rightarrow_{\mathcal{R}, \text{AC}}^\epsilon$ and $\rightarrow_{\mathcal{R}, \text{AC}}^\epsilon$ coincide and AC^{\leftarrow} is contained in \sim_{AC} .

- (1) If $q = \epsilon$ then $t \rightarrow_{\mathcal{R}^e, \text{AC}} u$. Thus, the claim holds.
- (2) If $p = \epsilon$ and $q \notin \text{Pos}_{\mathcal{F}}(\ell_1)$ then $s = \ell_1\sigma$ and $t = r_1\sigma$ for some substitution σ and there exists a variable position p' in ℓ_1 with $p' \leq q$. We have $\ell_1\sigma|_{p'} \rightarrow_{\mathcal{R}^e, \text{AC}} u|_{p'}$. Define the substitution τ as follows:

$$\tau(x) = \begin{cases} u|_{p'} & \text{if } x = \ell_1|_{p'} \\ \sigma(x) & \text{otherwise} \end{cases}$$

As r_1 is linear and $\mathcal{V}\text{ar}(\ell_1) = \mathcal{V}\text{ar}(r_1)$, the variable $\ell_1|_{p'}$ has one occurrence in r_1 . Hence, we obtain $t = r_1\sigma \rightarrow_{\mathcal{R}^e, \text{AC}} r_1\tau \text{AC}^{\leftarrow} \ell_1\tau = u$.

- (3) If $p = ip'$ and $q = jq'$ with $i, j \in \mathbb{N}$ then we further distinguish two subcases. If $i = j$ then $t|_i \text{AC}^{\leftarrow} s|_i \rightarrow_{\mathcal{R}^e, \text{AC}} u|_i$ and we obtain $t \rightarrow_{\mathcal{R}^e, \text{AC}} \cdot \overline{\text{AC}}^{\leftarrow} u$ with help of the induction hypothesis. If $i \neq j$ then we clearly have $t \rightarrow_{\mathcal{R}^e, \text{AC}}^q \cdot \text{AC}^{\leftarrow} u$.
- (4) In the final case, $p = \epsilon$ and $q \in \text{Pos}_{\mathcal{F}}(\ell_1)$ with $q \neq \epsilon$. Since $\text{Pos}_{\mathcal{F}}(\ell_1) \subseteq \{\epsilon, 1\}$, the rule $\ell_1 \rightarrow r_1$ is an associativity rule $f(f(x, y), z) \rightarrow f(x, f(y, z))$ and $q = 1$. So $s|_1 \sim_{\text{AC}} \ell_2\sigma$ and $u = f(r_2\sigma, s|_2)$. If $\ell_2 \rightarrow r_2 \in \mathcal{R}$ then the corresponding extended rule results in $f(\ell_2\sigma, s|_2) \rightarrow_{\mathcal{R}^e}^\epsilon f(r_2\sigma, s|_2)$ and thus $t \sim_{\text{AC}} s \sim_{\text{AC}} f(\ell_2\sigma, s|_2) \rightarrow_{\mathcal{R}^e}^\epsilon u$, which entails $t \rightarrow_{\mathcal{R}^e, \text{AC}} u$. Otherwise, $\ell_2 \rightarrow r_2$ is the extension $f(\ell, x') \rightarrow f(r, x')$ of some rule $\ell \rightarrow r \in \mathcal{R}$ with $x' \notin \mathcal{V}\text{ar}(\ell)$. Define the substitution τ as follows:

$$\tau(x) = \begin{cases} f(x'\sigma, s|_2) & \text{if } x = x' \\ \sigma(x) & \text{otherwise} \end{cases}$$

Since $t \sim_{\text{AC}} s \sim_{\text{AC}} f(\ell_2\sigma, s|_2) \sim_{\text{AC}} f(\ell\sigma, f(x'\sigma, s|_2)) = \ell_2\tau$, we obtain $t \rightarrow_{\mathcal{R}^e, \text{AC}}^\epsilon r_2\tau$. Moreover, $r_2\tau = f(r\sigma, f(x'\sigma, s|_2)) \text{AC}^{\leftarrow} f(f(r\sigma, x'\sigma), s|_2) = u$. \square

Lemma 8.6. $\sim_{AC} \cdot \rightarrow_{\mathcal{R}^e, AC} \subseteq \rightarrow_{\mathcal{R}^e, AC} \cdot \sim_{AC}$

Proof. Lemma 8.5 generalizes to $AC^{*\leftarrow} \cdot \rightarrow_{\mathcal{R}^e, AC} \subseteq \rightarrow_{\mathcal{R}^e, AC} \cdot AC^{*\leftarrow}$ by a straightforward induction argument. From $AC \subseteq AC^{*\leftarrow}$ we obtain $\sim_{AC} \subseteq AC^{*\leftarrow}$ and hence the claim follows. \square

Theorem 8.7. *The relations $\rightarrow_{\mathcal{R}/AC}$ and $\rightarrow_{\mathcal{R}^e, AC} \cdot \sim_{AC}$ coincide.*

Proof. Using Lemma 8.6 we obtain

$$\rightarrow_{\mathcal{R}/AC} = \sim_{AC} \cdot \rightarrow_{\mathcal{R}} \cdot \sim_{AC} \subseteq \rightarrow_{\mathcal{R}^e, AC} \cdot \sim_{AC} \cdot \sim_{AC} = \rightarrow_{\mathcal{R}^e, AC} \cdot \sim_{AC}$$

The other direction follows from the definition of $\rightarrow_{\mathcal{R}, AC}$ as well as the fact that $\rightarrow_{\mathcal{R}} = \rightarrow_{\mathcal{R}^e}$:

$$\rightarrow_{\mathcal{R}^e, AC} \subseteq \sim_{AC} \cdot \rightarrow_{\mathcal{R}^e} = \sim_{AC} \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}/AC} \quad \square$$

Corollary 8.8. *The relations $\rightarrow_{\mathcal{R}/AC}^* \cdot \sim_{AC}$ and $\rightarrow_{\mathcal{R}^e, AC}^* \cdot \sim_{AC}$ coincide.*

Proof. It is sufficient to show $\rightarrow_{\mathcal{R}/AC}^n \cdot \sim_{AC} = \rightarrow_{\mathcal{R}^e, AC}^n \cdot \sim_{AC}$ for all $n \geq 0$. We use induction on n . If $n = 0$ then the claim is trivial. If $n > 0$ then the claim is verified as follows:

$$\begin{aligned} \rightarrow_{\mathcal{R}/AC}^n \cdot \sim_{AC} &= \rightarrow_{\mathcal{R}/AC}^{n-1} \cdot \sim_{AC} \cdot \rightarrow_{\mathcal{R}/AC} && (\rightarrow_{\mathcal{R}/AC} \cdot \sim_{AC} = \sim_{AC} \cdot \rightarrow_{\mathcal{R}/AC}) \\ &= \rightarrow_{\mathcal{R}^e, AC}^{n-1} \cdot \sim_{AC} \cdot \rightarrow_{\mathcal{R}/AC} && (\text{induction hypothesis}) \\ &= \rightarrow_{\mathcal{R}^e, AC}^{n-1} \cdot \rightarrow_{\mathcal{R}/AC} && (\sim_{AC} \cdot \rightarrow_{\mathcal{R}/AC} = \rightarrow_{\mathcal{R}/AC}) \\ &= \rightarrow_{\mathcal{R}^e, AC}^{n-1} \cdot \rightarrow_{\mathcal{R}^e, AC} \cdot \sim_{AC} && (\text{Theorem 8.7}) \\ &= \rightarrow_{\mathcal{R}^e, AC}^n \cdot \sim_{AC} && \square \end{aligned}$$

9. EXPERIMENTAL RESULTS

We evaluated the performance of our tool `accompII` on a problem set containing 52 ESs. The problem set is based on the one used in [WM11] and has been extended by further examples from the literature as well as handcrafted examples. From the 52 ESs, 10 contain equations which cannot be oriented into a left-linear rule. Furthermore, 6 problems are ground which means that `accompII` cannot find a finite solution by Example 7.1. This leaves us with 36 problems for which `accompII` may find an AC complete presentation. However, for some of these ESs, simplified equations where both terms are non-linear may be deduced which causes `accompII` to get stuck. Furthermore, many interesting problems exhibit the properties described in Examples 7.1 and 7.2 and are therefore out of reach for our method. Nevertheless, there are 7 examples in the problem set where left-linear completion with AC axioms is preferable to general AC completion due to significantly better performance.

The experiments were performed on an Intel Core i7-7500U running at a clock rate of 2.7 GHz with 15.5 GiB of main memory. Our tool `accompII` was used with the termination tool `T1T2` as well as an experimental version (denoted by `T1T2e`) which allows our tool to communicate a sequence of termination problems without having to start a new process all the time, as described in the preceding section.

Table 1 compares the two configurations of `accompII` with the normalized completion [Mar96] mode of `mkbTT` [WM13] and the AC completion mode of `MædMax` [Win19] on selected examples. The tool `mkbTT` is the original implementation of multi-completion with termination tools [WSMK13]. `MædMax`, on the other hand, implements *maximal completion* [KH11] which makes use of MaxSAT/MaxSMT solvers instead of termination tools in order

Table 1: Experimental results on 52 problems (excerpt)

	accompII ($\top\top\top_2$)		accompII ($\top\top\top_2e$)		MædMax		mkbTT	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
Example 4.3	0.54	4	0.31	4	0.02	3	0.23	2
Example 7.1	∞		∞		0.01	3	0.16	3
Example 7.6	\perp		\perp		0.16	5	0.16	0
Example 9.1	1.60	4	0.48	4	∞		13.66	3
Example 9.2	2.62	14	0.54	14	\perp		60.03	11
Example 9.3	1.99	15	0.48	15	∞		9.14	10
problems solved		17		17		23		38

to avoid using concrete reduction orders as input. To the best of our knowledge, there is no other comparable completion tool which supports AC axioms. Since normalized completion subsumes general AC completion, a comparison with the aforementioned modes of both systems allows us to assess the effectiveness of `accompII` with respect to the state of the art in AC completion. Note that both normalized completion and general AC completion use AC unification.

In Table 1, columns (1) show the execution time in seconds where ∞ denotes that the timeout of 60 seconds has been reached and \perp denotes failure of completion. Columns (2) state the number of rules of the completed TRS. In Example 4.3, the equations just have to be oriented and one additional rule has to be added in the case of left-linear AC completion. Hence, all three systems can handle this problem easily. Examples 7.1 and 7.6 show the two main limitations of left-linear AC completion: it diverges on problems which contain an AC symbol where both arguments have “structure” and non-left-linear presentations are out of reach. For general AC completion and normalized completion, respectively, these examples pose no problem. The remaining examples show that the absence of AC unification can make left-linear completion more practical than general AC completion.

Example 9.1. The Eckmann–Hilton argument [EH62] considers the following equational theory \mathcal{E}

$$\begin{array}{lll} 0 \oplus a \approx a & 1 \otimes a \approx a & (a \oplus b) \otimes (c \oplus d) \approx (a \otimes c) \oplus (b \otimes d) \\ a \oplus 0 \approx a & a \otimes 1 \approx a & \end{array}$$

and proves $0 \approx 1$.⁵ Moreover, it establishes that \oplus and \otimes coincide and are associative as well as commutative. If we assume that \oplus and \otimes are AC symbols, `accompII` produces the following complete presentation \mathcal{R} of \mathcal{E} in less than a second:

$$\begin{array}{ll} b \oplus a \rightarrow a \otimes b & 1 \otimes a \rightarrow a \\ 0 \rightarrow 1 & a \otimes 1 \rightarrow a \end{array}$$

In Table 1, we can see that `mkbTT` needs considerably more time to complete \mathcal{E} and `MædMax` even times out after 60 seconds. By inspecting \mathcal{R} , it is easily seen that the operations as well as the unit elements are equivalent. Furthermore, the concise presentation of \mathcal{E} as a complete

⁵This example was brought to our attention by Vincent van Oostrom.

TRS is facilitated by our canonicity results as well as the implementation of inter-reduction (collapse and compose). Using the inference system **B** without inter-reduction and with the same AC compatible reduction order, we obtain a much larger complete but non-canonical presentation of \mathcal{E} which extends \mathcal{R} :

$$\begin{array}{lll}
b \oplus a \rightarrow a \otimes b & 0 \oplus a \rightarrow a & (a \oplus b) \otimes (c \oplus d) \rightarrow (a \otimes c) \oplus (b \otimes d) \\
0 \rightarrow 1 & a \oplus 0 \rightarrow a & a \otimes (b \oplus c) \rightarrow (0 \otimes b) \oplus (a \otimes c) \\
1 \otimes a \rightarrow a & 0 \otimes a \rightarrow a & (0 \otimes a) \oplus b \rightarrow a \oplus b \\
a \otimes 1 \rightarrow a & &
\end{array}$$

Example 9.2. Consider the ES \mathcal{E}

$$\begin{array}{ll}
x + 0 \approx x & \text{append}(\text{nil}, l) \approx l \\
x + \text{s}(y) \approx \text{s}(x + y) & \text{append}(c(x, l_1), l_2) \approx c(x, \text{append}(l_1, l_2)) \\
\text{add}(\text{nil}, \text{nil}) \approx 0 & \text{rev}(\text{nil}) \approx \text{nil} \\
\text{add}(c(x, l), \text{nil}) \approx x + \text{add}(l, \text{nil}) & \text{rev}(c(x, l)) \approx \text{append}(\text{rev}(l), c(x, \text{nil})) \\
\text{add}(\text{nil}, c(x, l)) \approx x + \text{add}(\text{nil}, l) & \text{rev}(\text{rev}(l)) \approx l \\
\text{add}(c(x, l_1), c(y, l_2)) \approx (x + y) + \text{add}(l_1, l_2) &
\end{array}$$

together with the AC axioms for $+$. This is an extension of [Ave95, Exercise 4.2.4(b)] (the addition operation on lists) with the standard append and reverse functions on lists. We added the involution axiom for list reversal in order to generate critical pairs. Our tool `accompII` produces a complete presentation in less than a second by orienting all equations in \mathcal{E} from left to right and adding the following rules:

$$\begin{array}{ll}
0 + x \rightarrow x & \text{rev}(\text{append}(l, c(x, \text{nil}))) \rightarrow c(x, \text{rev}(l)) \\
\text{s}(x) + y \rightarrow \text{s}(y + x) &
\end{array}$$

It takes `mkbTT` more than 60 seconds to solve this problem and `MædMax` terminates with an error message.

Example 9.3. Consider the ES \mathcal{E}

$$\begin{array}{ll}
0 + x \approx x & 0 \times x \approx 0 \\
\text{s}(x) + y \approx \text{s}(x + y) & \text{s}(x) \times y \approx (x \times y) + y \\
0 - x \approx 0 & (x + y) \times z \approx (x \times z) + (y \times z) \\
x - 0 \approx x & \text{div}(0, \text{s}(y)) \approx 0 \\
\text{s}(x) - \text{s}(y) \approx x - y & \text{div}(\text{s}(x), \text{s}(y)) \approx \text{s}(\text{div}(x - y, \text{s}(y)))
\end{array}$$

together with the AC axioms for $+$ and \times , defining addition, multiplication, cutoff subtraction and round-up division on the natural numbers. Our tool `accompII` produced a complete presentation in less than a second by orienting all equations in \mathcal{E} from left to right and adding the following rules:

$$\begin{array}{ll}
x + 0 \rightarrow x & x \times 0 \rightarrow 0 \\
x + \text{s}(y) \rightarrow \text{s}(y + x) & x \times \text{s}(y) \rightarrow (y \times x) + x
\end{array}$$

Since round-up division cannot be handled by simplification orders [Der79], this example also shows the merits of using termination tools in completion. Note that it takes `mkbTT`

much longer to complete \mathcal{E} (more than 9 seconds) and **MædMax** times out on this problem after 60 seconds.

Despite the successes in solving the previous three examples quickly, the severity of the limitations of left-linear AC completion is reflected in the total number of solved problems as shown in Table 1. In particular, the problem set does not contain an ES which is only completed by **accomp11**. However, given Theorem 7.5, this is not unexpected. Another noteworthy but unsurprising fact is that complete systems produced by **accomp11** tend to have more rules since every rule needs different versions of left-hand sides to facilitate rewriting without AC matching. The complete results are available on the tool’s website.⁶ In addition to full details for the experiments with $\mathsf{T}\mathsf{T}\mathsf{T}_2$ as shown in Table 1, the website also contains additional experiments with the termination tool **MU-TERM** [GL20]. For **accomp11**, using **MU-TERM** instead of $\mathsf{T}\mathsf{T}\mathsf{T}_2$ does not change the set of solved problems but it usually takes more time to complete ESs. The situation is different for **mkbTT**: Using **MU-TERM** instead of $\mathsf{T}\mathsf{T}\mathsf{T}_2$ internally, it can complete fewer systems overall (26 instead of 38 out of 52). However, five of these systems are not completed by **mkbTT** with $\mathsf{T}\mathsf{T}\mathsf{T}_2$ and three of them are not completed by any other tool configuration. We conclude with some additional comments on the results.

- ▷ The results are not cluttered with detailed results for the available options regarding prime critical pairs and the concrete rewrite relation used for **simplify** and **compose** since they did not lead to significant runtime differences. Instead, the default options (no prime critical pairs and the rewrite relation $\rightarrow_{\mathcal{R}}$) were used for the experiments.
- ▷ The restriction to prime critical pairs did not pay off in the experimental results. However, it may reduce the number of critical pairs which have to be considered even if inter-reduction is applied eagerly (see Example 6.15). Therefore, apart from the theoretical relevance of its feasibility, we consider it an important improvement over the status quo which could be beneficial for users of **accomp11** in the future.
- ▷ The possibility to choose the concrete rewrite relation used for **simplify** and **compose** also did not lead to significant improvements in our experiments. However, simplifying with $\rightarrow_{\mathcal{R}/\text{AC}}$ can boost performance since the problem can be simplified even before a rule with a fitting left-hand side is derived.
- ▷ Due to the incompleteness of the used approach for completion with termination tools, some equations in the problems **A95_ex4_2_4a.trs** as well as **sp.trs** had to be reversed in order to get appropriate results. Note that this does not distort the experimental results for left-linear AC completion in general as the problem lies in the particular implementation of completion with termination tools.

10. CONCLUSION

This article consolidates and extends existing work on left-linear \mathcal{B} -completion [Hue80, Bac91, Ave95] by using and adapting elegant proof techniques which have been put forward for standard completion in [HMSW19]. This approach allowed for improvements of existing results: Huet’s result could be strengthened to prime critical pairs in Theorem 3.16 which in turn improved the definition of fair runs in the inference system **A** (Definition 4.2). Furthermore, the usage of peak-and-cliff decreasingness instead of proof orderings simplified

⁶<http://cl-informatik.uibk.ac.at/software/accomp11/>

the proof of the correctness result for \mathbf{A} (Theorem 4.12). The limitation to finite runs also facilitated the removal of the encompassment condition which allows the inference system to produce smaller \mathcal{B} -complete systems as we showed in Example 4.5.

In addition, the relationship between the two existing inference systems from the literature has been investigated thoroughly, its core part being a simulation result (Theorem 5.6) which states that any fair run in \mathbf{B} can be simulated by a fair run in \mathbf{A} . With the novel simulation result, completion with \mathbf{B} can be reduced to completion with \mathbf{A} in the case of finite runs. Furthermore, the presentation of novel canonicity results facilitates a concrete definition of minimal complete systems in our setting. Since the inference system \mathbf{A} adds critical pairs stemming from overlaps between (intermediate) rules and equations in \mathcal{B} , its termination relies heavily on inter-reduction and therefore canonicity (Example 6.14). The final theoretical contribution of this article is a formal presentation of the correspondence between left-linear AC completion and general AC completion through another simulation result (Theorem 7.5).

The tool `accomp11` is the first implementation of left-linear AC completion. Our novel results on canonicity define in which way the systems produced by `accomp11` are minimal and unique. Unfortunately, the experimental results show that despite the practical advantages of avoiding AC unification and deciding validity problems with the normal rewrite relation, left-linear AC completion often needs infinitely many rules and therefore diverges. This problem does not seem to be mentioned in the literature. However, the possible speedup due to the avoidance of AC unification reported in the experimental results as well as the already mentioned simulation result for general AC completion show that left-linear AC completion also has merits in practice.

We conclude by giving some pointers for future work. First of all, the merits of our novel simulation result for general AC completion could be evaluated experimentally by providing an implementation. While `accomp11` adopts the two-thread version of multi-completion [SZ12], we anticipate that left-linear AC completion can also be effectively implemented by a variant of maximal completion that aims to find a canonical system [SW15, Hir21]. Another interesting research direction is normalized completion for the left-linear case. If successful, this would facilitate the treatment of important cases such as abelian groups despite the restriction to left-linear TRSs. Furthermore, a formalization of the established theoretical results is desirable. To that end, the existing Isabelle/HOL formalization from [HMSW19] is a perfect starting point as some results of this article are extensions of the results for standard rewriting presented there.

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REFERENCES

- [Ave95] Jürgen Avenhaus. *Reduktionssysteme*. Springer Berlin Heidelberg, 1995. In German. doi:10.1007/978-3-642-79351-6.
- [Bac91] Leo Bachmair. *Canonical Equational Proofs*. Progress in Theoretical Computer Science. Birkhäuser Boston, 1991. doi:10.1007/978-1-4684-7118-2.
- [BD94] Leo Bachmair and Nachum Dershowitz. Equational inference, canonical proofs, and proof orderings. *Journal of the ACM*, 41(2):236–276, 1994. doi:10.1145/174652.174655.

- [BDP89] Leo Bachmair, Nachum Dershowitz, and David A. Plaisted. Completion without failure. In Hassan Aït-Kaci and Maurice Nivat, editors, *Resolution of Equations in Algebraic Structures*, volume 2: Rewriting Techniques, pages 1–30. Academic Press, 1989. doi:10.1016/B978-0-12-046371-8.50007-9.
- [BL87] Ahlem Ben Cherifa and Pierre Lescanne. Termination of rewriting systems by polynomial interpretations and its implementation. *Science of Computer Programming*, 9(2):137–159, 1987. doi:10.1016/0167-6423(87)90030-X.
- [BN98] Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998. doi:10.1017/CB09781139172752.
- [Con04] Evelyne Contejean. A certified AC matching algorithm. In Vincent van Oostrom, editor, *Proc. 15th International Conference on Rewriting Techniques and Applications*, volume 3091 of *Lecture Notes in Computer Science*, pages 70–84, 2004. doi:10.1007/978-3-540-25979-4_5.
- [Der79] Nachum Dershowitz. A note on simplification orderings. *Information Processing Letters*, 9(5):212–215, 1979. doi:10.1016/0020-0190(79)90071-1.
- [Dev91] Hervé Devie. Linear completion. In Stéphane Kaplan and Mitsuhiro Okada, editors, *Proc. 2nd Workshop on Conditional and Typed Rewriting Systems*, volume 516 of *Lecture Notes in Computer Science*, pages 233–245, 1991. doi:10.1007/3-540-54317-1_94.
- [DM79] Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476, 1979. doi:10.1145/359138.359142.
- [EH62] Beno Eckmann and Peter John Hilton. Group-like structures in general categories I multiplications and comultiplications. *Mathematische Annalen*, 145(3):227–255, 1962. doi:10.1007/BF01451367.
- [FAS13] Bertram Felgenhauer, Martin Avanzini, and Christian Sternagel. A Haskell library for term rewriting. In *Proc. 1st International Workshop on Haskell and Rewriting Techniques*, 2013. doi:10.48550/ARXIV.1307.2328.
- [FvO13] Bertram Felgenhauer and Vincent van Oostrom. Proof orders for decreasing diagrams. In Femke van Raamsdonk, editor, *Proc. 24th International Conference on Rewriting Techniques and Applications*, volume 21 of *Leibniz International Proceedings in Informatics*, pages 174–189, 2013. doi:10.4230/LIPIcs.RTA.2013.174.
- [GL20] Raúl Gutiérrez and Salvador Lucas. MU-TERM: Verify termination properties automatically (system description). In Nicolas Peltier and Viorica Sofronie-Stokkermans, editors, *Proc. 10th International Joint Conference on Automated Reasoning*, volume 12167 of *Lecture Notes in Artificial Intelligence*, pages 436–447, 2020. doi:10.1007/978-3-030-51054-1_28.
- [Hir21] Nao Hirokawa. Completion and reduction orders. In Naoki Kobayashi, editor, *Proc. 6th International Conference on Formal Structures for Computation and Deduction*, volume 195 of *Leibniz International Proceedings in Informatics*, pages 2:1–2:9, 2021. doi:10.4230/LIPIcs.FSCD.2021.2.
- [HMSW19] Nao Hirokawa, Aart Middeldorp, Christian Sternagel, and Sarah Winkler. Abstract completion, formalized. *Logical Methods in Computer Science*, 15(3):19:1–19:42, 2019. doi:10.23638/LMCS-15(3:19)2019.
- [Hue80] Gérard Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *Journal of the ACM*, 27(4):797–821, 1980. doi:10.1145/322217.322230.
- [JK86] Jean-Pierre Jouannaud and Hélène Kirchner. Completion of a set of rules modulo a set of equations. *SIAM Journal on Computing*, 15(4):1155–1194, 1986. doi:10.1137/0215084.
- [KB70] Donald E. Knuth and Peter B. Bendix. Simple word problems in universal algebras. In John Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, 1970. doi:10.1016/B978-0-08-012975-4.50028-X.
- [KH11] Dominik Klein and Nao Hirokawa. Maximal completion. In Manfred Schmidt-Schauß, editor, *Proc. 22nd International Conference on Rewriting Techniques and Applications*, volume 10 of *Leibniz International Proceedings in Informatics*, pages 71–80, 2011. doi:10.4230/LIPIcs.RTA.2011.71.
- [KMN88] Deepak Kapur, David R. Musser, and Paliath Narendran. Only prime superpositions need be considered in the Knuth–Bendix completion procedure. *Journal of Symbolic Computation*, 6(1):19–36, 1988. doi:10.1016/S0747-7171(88)80019-1.
- [KSZM09] Martin Korp, Christian Sternagel, Harald Zankl, and Aart Middeldorp. Tyrolean Termination Tool 2. In Ralf Treinen, editor, *Proc. 20th International Conference on Rewriting Techniques and Applications*, volume 5595 of *Lecture Notes in Computer Science*, pages 295–304, 2009. doi:10.1007/978-3-642-02348-4_21.

- [Mar96] Claude Marché. Normalized rewriting: An alternative to rewriting modulo a set of equations. *Journal of Symbolic Computation*, 21(3):253–288, 1996. doi:10.1006/j.sco.1996.0011.
- [Mes17] José Meseguer. Strict coherence of conditional rewriting modulo axioms. *Theoretical Computer Science*, 672:1–35, 2017. doi:10.1016/j.tcs.2016.12.026.
- [Mét83] Yves Métivier. About the rewriting systems produced by the Knuth–Bendix completion algorithm. *Information Processing Letters*, 16(1):31–34, 1983. doi:10.1016/0020-0190(83)90009-1.
- [NHM23a] Johannes Niederhauser, Nao Hirokawa, and Aart Middeldorp. Church–Rosser modulo for left-linear TRSs revisited. In Cyrille Chenavier and Sarah Winkler, editors, *Proc. 12th International Workshop on Confluence*, pages 14–19, 2023.
- [NHM23b] Johannes Niederhauser, Nao Hirokawa, and Aart Middeldorp. Left-linear completion with AC axioms. In Brigitte Pientka and Cesare Tinelli, editors, *Proc. 29th International Conference on Automated Deduction*, volume 14132 of *Lecture Notes in Artificial Intelligence*, pages 401–418, 2023. doi:10.1007/978-3-031-38499-8_23.
- [Ohl98] Enno Ohlebusch. Church–Rosser theorems for abstract reduction modulo an equivalence relation. In Tobias Nipkow, editor, *Proc. 9th International Conference on Rewriting Techniques and Applications*, volume 1379 of *Lecture Notes in Computer Science*, pages 17–31, 1998. doi:10.1007/BFb0052358.
- [PS81] Gerald E. Peterson and Mark E. Stickel. Complete sets of reductions for some equational theories. *Journal of the ACM*, 28(2):233–264, 1981. doi:10.1145/322248.322251.
- [SK94] Andrea Sattler-Klein. About changing the ordering during Knuth–Bendix completion. In Patrice Enjalbert, Ernst W. Mayr, and Klaus W. Wagner, editors, *Proc. 11th Annual Symposium on Theoretical Aspects of Computer Science*, volume 775 of *Lecture Notes in Computer Science*, pages 175–186, 1994. doi:10.1007/3-540-57785-8_140.
- [ST13] Christian Sternagel and René Thiemann. Formalizing Knuth–Bendix orders and Knuth–Bendix completion. In Femke van Raamsdonk, editor, *Proc. 24th International Conference on Rewriting Techniques and Applications*, volume 21 of *Leibniz International Proceedings in Informatics*, pages 286–301, 2013. doi:10.4230/LIPIcs.RTA.2013.287.
- [SW15] Haruhiko Sato and Sarah Winkler. Encoding dependency pair techniques and control strategies for maximal completion. In Amy P. Felty and Aart Middeldorp, editors, *Proc. 25th International Conference on Automated Deduction*, volume 9195 of *Lecture Notes in Computer Science*, pages 152–162, 2015. doi:10.1007/978-3-319-21401-6_10.
- [SZ12] Thomas Sternagel and Harald Zankl. KBCV – Knuth–Bendix completion visualizer. In Bernhard Gramlich, Dale Miller, and Uli Sattler, editors, *Proc. 6th International Joint Conference on Automated Reasoning*, volume 7364 of *Lecture Notes in Artificial Intelligence*, pages 530–536, 2012. doi:10.1007/978-3-642-31365-3_41.
- [Ter03] Terese, editor. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.
- [vO94] Vincent van Oostrom. Confluence by decreasing diagrams. *Theoretical Computer Science*, 126(2):259–280, 1994. doi:10.1016/0304-3975(92)00023-K.
- [Win13] Sarah Winkler. *Termination Tools in Automated Reasoning*. PhD thesis, University of Innsbruck, 2013.
- [Win19] Sarah Winkler. Extending maximal completion. In Herman Geuvers, editor, *Proc. 4th International Conference on Formal Structures for Computation and Deduction*, volume 131 of *Leibniz International Proceedings in Informatics*, pages 3:1–3:15, 2019. doi:10.4230/LIPIcs.FSCD.2019.3.
- [WM11] Sarah Winkler and Aart Middeldorp. AC completion with termination tools. In Nikolaj Bjørner and Viorica Sofronie-Stokkermans, editors, *Proc. 23rd International Conference on Automated Deduction*, volume 6803 of *Lecture Notes in Artificial Intelligence*, pages 492–498, 2011. doi:10.1007/978-3-642-22438-6_37.
- [WM13] Sarah Winkler and Aart Middeldorp. Normalized completion revisited. In Femke van Raamsdonk, editor, *Proc. 24th International Conference on Rewriting Techniques and Applications*, volume 21 of *Leibniz International Proceedings in Informatics*, pages 318–333, 2013. doi:10.4230/LIPIcs.RTA.2013.319.
- [WSMK13] Sarah Winkler, Haruhiko Sato, Aart Middeldorp, and Masahito Kurihara. Multi-completion with termination tools. *Journal of Automated Reasoning*, 50(3):317–354, 2013. doi:10.1007/s10817-012-9249-2.

- [WSW06] Ian Wehrman, Aaron Stump, and Edwin M. Westbrook. Slothrop: Knuth–Bendix completion with a modern termination checker. In Frank Pfenning, editor, *Proc. 17th International Conference on Rewriting Techniques and Applications*, volume 4098 of *Lecture Notes in Computer Science*, pages 287–296, 2006. doi:10.1007/11805618_22.
- [YKS14] Akihisa Yamada, Keiichirou Kusakari, and Toshiki Sakabe. Nagoya Termination Tool. In Gilles Dowek, editor, *Proc. 25th International Conference on Rewriting Techniques and Applications and 12th International Conference on Typed Lambda Calculi and Applications*, volume 8560 of *Lecture Notes in Computer Science*, pages 466–475, 2014. doi:10.1007/978-3-319-08918-8_32.