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FROM KLEISLI CATEGORIES TO COMMUTATIVE C^*-ALGEBRAS: PROBABILISTIC GELFAND DUALITY

ROBERT FURBER AND BART JACOBS

Institute for Computing and Information Sciences (iCIS), Radboud University Nijmegen, The Netherlands.
e-mail address: {r.furber,bart}@cs.ru.nl

Abstract. C^*-algebras form rather general and rich mathematical structures that can be studied with different morphisms (preserving multiplication, or not), and with different properties (commutative, or not). These various options can be used to incorporate various styles of computation (set-theoretic, probabilistic, quantum) inside categories of C^*-algebras. At first, this paper concentrates on the commutative case and shows that there are functors from several Kleisli categories, of monads that are relevant to model probabilistic computations, to categories of C^*-algebras. This yields a new probabilistic version of Gelfand duality, involving the "Radon" monad on the category of compact Hausdorff spaces. We then show that the state space functor from C^*-algebras to Eilenberg-Moore algebras of the Radon monad is full and faithful. This allows us to obtain an appropriately commuting state-and-effect triangle for C^*-algebras.

1. Introduction

There are several notions of computation. We have the classical notion of computation, probabilistic computation, where a computer may make random choices, and quantum computation, which uses quantum mechanical interference and measurement. Normally we would consider classical computation to be done on sets, probabilistic computation on spaces with a measure, and quantum computation on Hilbert spaces. We can instead use categories with C^*-algebras as objects and a choice of either *-homomorphisms (called MIU-map below) or positive unital maps as the morphisms. We note at this point that positive unital maps coincide with *completely* positive unital maps if either the domain or codomain of a map is a commutative C^*-algebra, but not in general. The general outline is represented in this table.

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	set-theoretic	**probabilistic**	**quantum**
C^*-algebras	commutative	commutative	non-commutative
maps preserve	multiplication involution unit	positivity unit	positivity unit
maps abbreviation	MIU	PU	PU

While the quantum case is an important source of motivation, we will deal more with the classical and probabilistic cases in this article. In particular, we will relate the alternative method of representing probabilistic computation, using monads, to the C^*-algebraic approach.

In recent years the methods and tools of category theory have been applied to Hilbert spaces — see *e.g.* [1] and the references there — and also to C^*-algebras, see for instance [32, 29]. In this paper we show that clearly distinguishing different types of homomorphisms of C^*-algebras already brings quite some clarity. Moreover, we demonstrate the relevance of monads (and their Kleisli and Eilenberg-Moore categories) in this field. The aforementioned paper [32] concerns itself with only the *-homomorphisms (*i.e.* with the MIU-maps in our terminology).

The main results of the paper can be summarised as follows. The well-known finite ('baby') version of Gelfand duality involves an equivalence between on the one hand the category of finite sets (and all functions between them), and on the other hand the opposite of the category of finite-dimensional commutative C^*-algebras with MIU-maps (*-homomorphisms) between them. Diagrammatically:

$$\mathbf{FinSets} \xrightarrow{\simeq} \big(\mathbf{FdCCstar}_{\mathrm{MIU}}\big)^{\mathrm{op}}$$

Our first observation is that if we generalise from MIU to PU (positive unital) maps we get an equivalence:

$$\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \xrightarrow{\simeq} \big(\mathbf{FdCCstar}_{\mathrm{PU}}\big)^{\mathrm{op}}$$

where $\mathcal{D}$ is the distribution monad on **Sets**, and $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$ is the Kleisli category of this monad, but with objects restricted to natural numbers. This shows that the category $\mathbf{FdCCstar}_{\mathrm{PU}}$ is the Lawvere theory of the distribution monad. Details are in Section 4.

The main contribution of the paper lies in a generalisation of the latter equivalence beyond the finite case, which can be summarised in a diagram:

$$\begin{array}{ccc}
\overset{\mathcal{R}}{\circlearrowleft}\;\mathbf{CH} & \xrightarrow[\text{Gelfand}]{\simeq} & \big(\mathbf{CCstar}_{\mathrm{MIU}}\big)^{\mathrm{op}} \\
\big\downarrow \dashv \big\uparrow & & \big\downarrow \\
\mathcal{K}\ell(\mathcal{R}) & \xrightarrow[\text{new}]{\simeq} & \big(\mathbf{CCstar}_{\mathrm{PU}}\big)^{\mathrm{op}}
\end{array} \tag{1.1}$$

At the top of this diagram we have the classical Gelfand duality between the category **CH** of compact Hausdorff spaces and the (opposite of the) category of commutative C^*-algebras with MIU-maps. Again, the generalisation to the computationally more interesting PU-maps involves a duality with a Kleisli category, namely the Kleisli category $\mathcal{K}\ell(\mathcal{R})$ of what we call the Radon monad $\mathcal{R}$ on compact Hausdorff spaces. Elements of $\mathcal{R}(X)$ can be described as so-called Radon probability measures, also known as inner regular probability measures (see [33]).

In the end, in Diagram (6.1) we show how the Kleisli category of the Radon monad gives rise to a 'state-and-effect' triangle that combines Kleisli computations for the Radon monad and their associated predicate transformers and state transformers. These predicate and state transformers correspond to the Heisenberg and Schrödinger picture, respectively.

Incidentally, the adjunction on the left in Diagram (1.1) can be transferred to the right, and then yields a right adjoint to the inclusion $\mathbf{CCstar}_{\mathrm{MIU}} \hookrightarrow \mathbf{CCstar}_{\mathrm{PU}}$. In [40] it is shown that such a right adjoint also exists in the general non-commutative case.

Giry [14, I.4] described how we can consider a stochastic process as being a diagram in the Kleisli category of the Giry monad on measure spaces. By using the Radon monad $\mathcal{R}$ on compact spaces instead, we can get a different category of stochastic processes on compact spaces as diagrams in the (opposite of the) category of *commutative* C^*-algebras with PU-maps. This allows the quantum generalization to taking diagrams in the category of *non-commutative* C^*-algebras, or by considering diagrams in the category $\mathcal{EM}(\mathcal{R})$ of Eilenberg-Moore algebras of the Radon monad $\mathcal{R}$, in which the category of C^*-algebras faithfully embeds. The relationship to quantum computation is that $B(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, is a C^*-algebra, and for every C^*-algebra A, there is a Hilbert space $\mathcal{H}$ such that A is isomorphic to a norm-closed *-subalgebra of $B(\mathcal{H})$. Unitary maps $U\colon \mathcal{H} \to \mathcal{H}$ define MIU maps $a \mapsto U^*aU\colon B(\mathcal{H}) \to B(\mathcal{H})$. The category of C^*-algebras allows us to represent measurement with maps from a commutative C^*-algebra to $B(\mathcal{H})$. We can also represent composite systems that are partly quantum and partly classical. Girard also used certain special C^*-algebras, von Neumann algebras, for his Geometry of Interaction [13].

2. Preliminaries on C^*-algebras

We write $\mathbf{Vect} = \mathbf{Vect}_{\mathbb{C}}$ for the category of vector spaces over the complex numbers $\mathbb{C}$. This category has direct product $V \oplus W$, forming a biproduct (both a product and a coproduct) and tensors $V \otimes W$, which distribute over $\oplus$. The tensor unit is the space $\mathbb{C}$ of complex numbers. The unit for $\oplus$ is the singleton (null) space 0. We write $\overline{V}$ for the vector space with the same vectors/elements as V, but with conjugate scalar product: $z \bullet_{\overline{V}} v = \overline{z} \bullet_V v$. This makes **Vect** an involutive category, see [19].

A **-algebra* is an involutive monoid A in the category **Vect**. Thus, A is itself a vector space, carries a multiplication $\cdot\colon A \otimes A \to A$, linear in each argument, and has a unit $1 \in A$. Moreover, there is an involution map $(-)^*\colon \overline{A} \to A$, preserving 0 and + and satisfying:

$$1^* = 1 \qquad (x \cdot y)^* = y^* \cdot x^* \qquad x^{**} = x \qquad (z \bullet x)^* = \overline{z} \bullet x^*.$$

Here we have written a fat dot $\bullet$ for scalar multiplication, to distinguish it from the algebra's multiplication $\cdot$. For $z = a + bi \in \mathbb{C}$ we have the conjugate $\overline{z} = a - bi$. Often we omit the multiplication dot $\cdot$ and simply write xy for $x \cdot y$. Similarly, the scalar multiplication $\bullet$ is often omitted. We then rely on the context to distinguish the two multiplications.

A *C^*-algebra* is a *-algebra A with a norm $\|-\|\colon A \to \mathbb{R}_{\geq 0}$ in which it is complete, satisfying the conditions $\|x\| = 0$ iff $x = 0$ and:

$$\begin{aligned} \|x + y\| &\leq \|x\| + \|y\| & \qquad \|z \bullet x\| &= |z| \cdot \|x\| \\ \|x \cdot y\| &\leq \|x\| \cdot \|y\| & \qquad \|x^* \cdot x\| &= \|x\|^2. \end{aligned}$$

The last equation $\|x^* \cdot x\| = \|x\|^2$, is the *$C^*$-identity* and distinguishes C^*-algebras from Banach *-algebras. We remark at this point that a Banach *-algebra admits at most one norm satisfying the C^*-identity. The reason for this is that the spectral radius $r(x)$ is definable in terms of the ring structure of the algebra, and for self-adjoint elements $r(x) = \|x\|$ [24, Proposition 4.1.1 (a)]. If x is an arbitrary element, $x^* \cdot x$ is self-adjoint, so $r(x^* \cdot x) = \|x^* \cdot x\| = \|x\|^2$. In the current setting, each C^*-algebra is unital, *i.e.* has a (multiplicative) unit 1. A consequence of the axioms above is that $\|1\| = 1$ unless the C^*-algebra is the unique one in which $0 = 1$. A C^*-algebra is called *commutative* if its multiplication is commutative, and *finite-dimensional* is it has finite dimension when considered as a vector space.

An element x in a C^*-algebra A is called *positive* if it can be written in the form $x = y^* \cdot y$. We write $A^+ \subseteq A$ for the subset of positive elements in A. This subset is a cone, which is to say it is closed under addition and scalar multiplication with positive real numbers. The multiplication $x \cdot y$ of two positive elements need not be positive in general (think of matrices). The square $x^2 = x \cdot x$ of a self-adjoint element $x = x^*$, however, is obviously positive. In a *commutative* C^*-algebra the positive elements are closed under multiplication. A cone A^+ in a vector space defines a partial order as follows.

$$x \leq y \iff y - x \in A^+. \tag{2.1}$$

This is defines an order on every C^*-algebra.

There are mainly two options when it comes to maps between C^*-algebras. The difference between them plays an important role in this paper.

Definition 2.1. We define two categories $\mathbf{Cstar}_{\mathrm{MIU}}$ and $\mathbf{Cstar}_{\mathrm{PU}}$ with C^*-algebras as objects, but with different morphisms.

(1) A morphism $f \colon A \to B$ in $\mathbf{Cstar}_{\mathrm{MIU}}$ is a linear map preserving multiplication (M), involution (I), and unit (U). Explicitly, this means for all $x, y \in A$,

$$f(x \cdot y) = f(x) \cdot f(y) \qquad f(x^*) = f(x)^* \qquad f(1) = 1.$$

Often such "MIU" maps are called *-homomorphisms.

(2) A morphism $f \colon A \to B$ in $\mathbf{Cstar}_{\mathrm{PU}}$ is a linear map that preserves positive elements and the unit. This means that f restricts to a function $A^+ \to B^+$. Alternatively, for each $x \in A$ there is an $y \in B$ with $f(x^*x) = y^*y$. □

For both $X = \mathrm{MIU}$ and $X = \mathrm{PU}$ there are obvious full subcategories of commutative and/or finite-dimensional C^*-algebras, as described in:

$$\begin{array}{ccccc} & & \mathbf{CCstar}_X & & \\ & \nearrow & & \searrow & \\ \mathbf{FdCCstar}_X & & & & \mathbf{Cstar}_X \\ & \searrow & & \nearrow & \\ & & \mathbf{FdCstar}_X & & \end{array}$$

Clearly, each "MIU" map is also a "PU" map, so that we have inclusions $\mathbf{Cstar}_{\mathrm{MIU}} \hookrightarrow \mathbf{Cstar}_{\mathrm{PU}}$, also for the various subcategories. A map that preserves positive elements is called positive itself; and a unit preserving map is called unital. Positive unital maps are the natural notion of morphism between order unit spaces and Riesz spaces.

For a category $\mathbf{B}$ one often writes $\mathbf{B}(X, Y)$ or $\mathrm{Hom}(X, Y)$ for the "homset" of morphisms $X \to Y$ in $\mathbf{B}$. For C^*-algebras A, B we write $\mathrm{Hom}_{\mathrm{MIU}}(A, B) = \mathbf{Cstar}_{\mathrm{MIU}}(A, B)$ and

$\mathrm{Hom}_{\mathrm{PU}}(A,B) = \mathbf{Cstar}_{\mathrm{PU}}(A,B)$ for the homsets of MIU- and PU-maps. For the special case where B is the algebra $\mathbb{C}$ of complex numbers we define sets of "states" and of "multiplicative states" as:

$$\mathrm{Stat}(A) = \mathrm{Hom}_{\mathrm{PU}}(A,\mathbb{C}) \qquad \text{and} \qquad \mathrm{MStat}(A) = \mathrm{Hom}_{\mathrm{MIU}}(A,\mathbb{C}).$$

There is also the commonly used notion of completely positive maps, which is a stronger condition than positivity but weaker than being MIU. These maps are important when defining the tensor of C^*-algebras as a functor, as the tensor of positive maps need not be positive. They are also widely considered to represent the physically realizable transformations. Positive, but non-completely positive maps of C^*-algebras also have their uses, as entanglement witnesses for example [17, theorem 2]. Since we mainly consider the commutative case, where positive and completely positive coincide, we do not consider the category of C^*-algebras with completely positive maps any further in this paper. However, since a completely positive unital map is what is known as a channel in quantum information, then theorem 5.1 shows that every channel in Mislove's sense [30] is a channel in this sense.

We collect some basic (standard) properties of PU-morphisms between C^*-algebras (see *e.g.* [35, 5]).

Lemma 2.2. *A PU-map, i.e. a morphism in the category $\mathbf{Cstar}_{PU}$, commutes with involution $(-)^*$, and preserves the partial order $\leq$ given by* (2.1).

Moreover, a PU-map f satisfies $\|f(x)\| \leq 4\|x\|$, so that $\|f(x) - f(y)\| \leq 4\|x - y\|$, making f continuous.

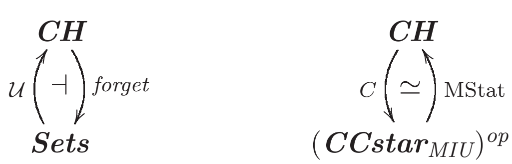
Proof. An element x is called self-adjoint if $x^* = x$. Each self-adjoint x can be written uniquely as a difference $x = x_p - x_n$ of positive elements x_p, x_n, with $x_p x_n = x_n x_p = 0$ and $\|x_p\|, \|x_n\| \leq \|x\|$, see [24, Proposition 4.2.3 (iii)]; as a result $f(x^*) = f(x) = f(x)^*$, for a PU-map f. Next, an arbitrary element y can be written uniquely as $y = y_r + iy_i$ for self-adjoint elements $y_r = \frac{1}{2}(y + y^*), y_i = \frac{1}{2i}(y - y^*)$, so that $\|y_r\|, \|y_i\| \leq \|y\|$. Then $f(y^*) = f(y)^*$. Preservation of the order is trivial.

For positive x we have $x \leq \|x\| \bullet 1$, and thus $f(x) \leq \|x\| \bullet 1$, which gives $\|f(x)\| \leq \|x\|$. An arbitrary element x can be written as linear combination of four positive elements x_i, as in $x = x_1 - x_2 + ix_3 - ix_4$, with $\|x_i\| \leq \|x\|$. Finally, $\|f(x)\| = \|f(x_1) - f(x_2) + if(x_3) - if(x_4)\| \leq \sum_i \|f(x_i)\| \leq \sum_i \|x_i\| \leq 4\|x\|$. □

In fact, it can be shown that $\|f(x)\| \leq \|x\|$ for all x, not just positive x, reducing the constant 4 in the inequality above to 1 (see [34, corollary 1]). But this sharpening is not needed here.

We next recall two famous adjunctions involving compact Hausdorff spaces. The first one is due to Manes [28] and describes compact Hausdorff spaces as monadic over **Sets**, via the ultrafilter monad. The second one is known as Gelfand duality, relating compact Hausdorff spaces and *commutative* C^*-algebras. Notice that this result involves the "MIU" maps.

Theorem 2.3. *Let $\mathbf{CH}$ be the category of compact Hausdorff spaces, with continuous maps between them. There are two fundamental adjunctions:*



On the left the functor $\mathcal{U}$ sends a set X to the ultrafilters on the powerset $\mathcal{P}(X)$. And on the right the equivalence of categories is given by sending a compact Hausdorff space X to the commutative C^*-algebra $C(X) = \mathrm{Cont}(X, \mathbb{C})$ of continuous functions $X \to \mathbb{C}$. The "weak- topology" on states will be discussed below.* □

The multiplicative states on a commutative C^*-algebra can equivalently be described as maximal ideals, or also as so-called pure states (see below).

Corollary 2.4. *For each finite-dimensional commutative C^*-algebra A there is an $n \in \mathbb{N}$ with $A \cong \mathbb{C}^n$ in $\mathbf{FdCCstar}_{MIU}$.*

Proof. By the previous theorem there is a compact Hausdorff space X such that A is MIU-isomorphic to the algebra of continuous maps $X \to \mathbb{C}$. This X must be finite, and since a finite Hausdorff space is discrete, all maps $X \to \mathbb{C}$ are continuous. Let $n \in \mathbb{N}$ be the number of elements in X; then we have an isomorphism $A \cong \mathbb{C}^n$. □

As we can already see in the above theorem, it is the *opposite* of a category of C^*-algebras that provides the most natural setting for computations. This is in line with what is often called the Heisenberg picture. In a logical setting it corresponds to computation of weakest preconditions, going backwards. The situation may be compared to the category of complete Heyting algebras, which is most usefully known in opposite form, as the category of locales, see [23].

The set of states $\mathrm{Stat}(A) = \mathrm{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ now can be equipped with the weak-* topology, defined as the coarsest (smallest) topology in which all evaluation maps $ev_x = \lambda s.\, s(x) \colon \mathrm{Hom}_{\mathrm{PU}}(A, \mathbb{C}) \to \mathbb{C}$, for $x \in A$, are continuous. We introduce the category **CCLcvx**, which first appeared in [39], in order to extend Stat to a functor.

The category **CCLcvx** has as its objects compact convex subsets of (Hausdorff) locally convex vector spaces. More accurately, the objects are pairs (V, X) where V is a (Hausdorff) locally convex space, and X is a compact convex subset of V. The maps $(V, X) \to (W, Y)$ are continuous, affine maps $X \to Y$. Note that if (V, X) and (W, Y) are isomorphic, while X is necessarily homeomorphic to Y, V need not bear any particular relation to W at all. We can see **CCLcvx** forms a category, as identity maps are affine and continuous and both of these attributes of a map are preserved under composition. We remark at this point that we have a forgetful functor $U \colon \mathbf{CCLcvx} \to \mathbf{CH}$, taking the underlying compact Hausdorff space of X.

Proposition 2.5. *For each C^*-algebra A, the set of states $\mathrm{Stat}(A) = \mathrm{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ is convex, and is a compact Hausdorff subspace of the dual space of A given the weak-* topology. Each PU-map $f \colon A \to B$ yields an affine continuous function $\mathrm{Stat}(f) = (-) \circ f \colon \mathrm{Stat}(B) \to \mathrm{Stat}(A)$. This defines a functor $\mathrm{Stat} \colon (\mathbf{Cstar}_{PU})^{op} \to \mathbf{CCLcvx}$.*

We recall that a function (between convex sets) is called *affine* if it preserves convex sums. We will see shortly that such affine maps are homomorphisms of Eilenberg-Moore algebras for the distribution monad $\mathcal{D}$.

Proof. For each finite collection $h_i \in \mathrm{Hom}_{\mathrm{PU}}(A, \mathbb{C})$ with $r_i \in [0, 1]$ satisfying $\sum_i r_i = 1$, the function $h = \sum_i r_i h_i$ is again a state. Moreover, such convex sums are preserved by precomposition, making the maps $(-) \circ f$ affine.

The fact that the dual space of A, given the weak-* topology, is a locally convex space is standard, and only uses that A is a Banach space [7, Example 1.8]. This implies that the space of states is Hausdorff. The space of states is closed since because the positive cone in a C^*-algebra is closed [24, Proposition 2.4.5 (i)][8, Proposition 1.6.1] and the set of linear functionals such that $\phi(1) = 1$ is weak-* closed, and the set of states is the intersection of the two. The space of states is also bounded as each state has norm 1. Therefore the state space is a closed and bounded and hence compact by the Banach-Alaoglu Theorem.

Precomposition $(-) \circ f$ is continuous, since for $x \in A$ and $U \subseteq \mathbb{C}$ open we get an open subset $\big((-) \circ f\big)^{-1}(ev_x^{-1}(U)) = \{h \mid ev_x(h \circ f) \in U\} = ev_{f(x)}^{-1}(U)$.

Precomposition with the identity map gives the same state again, so Stat preserves identity maps. Since composition of PU-maps is associative, Stat preserves composition, and hence is a functor. □

2.1. **Effect modules.** Effect algebras have been introduced in mathematical physics [10], in the investigation of quantum probability, see [9] for an overview. An *effect algebra* is a partial commutative monoid $(M, 0, \ovee)$ with an orthocomplement $(-)^\perp$. One writes $x \perp y$ if $x \ovee y$ is defined. The formulation of the commutativity and associativity requirements is a bit involved, but essentially straightforward. The orthocomplement satisfies $x^{\perp\perp} = x$ and $x \ovee x^\perp = 1$, where $1 = 0^\perp$. There is always a partial order, given by $x \leq y$ iff $x \ovee z = y$, for some z. The main example is the unit interval $[0, 1] \subseteq \mathbb{R}$, where addition $+$ is obviously partial, commutative, associative, and has 0 as unit; moreover, the orthocomplement is $r^\perp = 1 - r$. We write **EA** for the category of effect algebras, with morphism preserving $\ovee$ and 1 — and thus all other structure.

For each set X, the set $[0, 1]^X$ of fuzzy predicates on X is an effect algebra, via pointwise operations. Each Boolean algebra B is an effect algebra with $x \perp y$ iff $x \wedge y = \perp$; then $x \ovee y = x \vee y$. In a quantum setting, the main example is the set of effects $\mathcal{E}f(H) = \{E : H \to H \mid 0 \leq E \leq I\}$ on a Hilbert space H, see *e.g.* [9, 16].

An *effect module* is an "effect" version of a vector space. It involves an effect algebra M with a scalar multiplication $s \bullet x \in M$, where $s \in [0, 1]$ and $x \in M$. This scalar multiplication is required to be a suitable homomorphism in each variable separately. The algebras $[0, 1]^X$ and $\mathcal{E}f(H)$ are clearly such effect modules. Maps in **EMod** are **EA** maps that are additionally required to commute with scalar multiplication.

For a C^*-algebra A the subset $A^+ \hookrightarrow A$ of positive elements carries a partial order $\leq$ defined on self-adjoint elements in (2.1). We write $[0, 1]_A \subseteq A^+ \subseteq A$ for the subset of positive elements below the unit. The elements in $[0, 1]_A$ will be called effects (or sometimes also: predicates). For instance, for the C^*-algebra $B(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ the unit interval $[0, 1]_{B(\mathcal{H})} \subseteq B(H)$ contains the effects $\mathcal{E}f(\mathcal{H}) = \{A \in B(\mathcal{H}) \mid 0 \leq A \leq \mathrm{id}\}$ on $\mathcal{H}$.

We claim that $[0, 1]_A$ is an *effect algebra* and carries a $[0, 1] \subseteq \mathbb{R}$ scalar multiplication, thus making it an *effect module*.

- Since A with $0, +$ is a partially ordered Abelian group, $[0, 1]_A$ is a so-called interval effect algebra, with $x \perp y$ iff $x + y \leq 1$, and in that case $x \ovee y = x + y$. The orthocomplement $x^\perp$ is given by $1 - x$.
- For $r \in [0, 1]$ and $x \in [0, 1]_A$ the scalar multiplications rx and $(1 - r)x$ are positive, and their sum is $x \leq 1$. Hence $rx \leq 1$ and thus $rx \in [0, 1]_A$.

Each PU-map of C^*-algebras $f : A \to B$ preserves $\leq$ and thus restricts to $[0, 1]_A \to [0, 1]_B$. This restriction is a map of effect modules. Hence we get a "predicate" functor $\mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$.

Lemma 2.6. *The functor* $[0,1]_{(-)}\colon \boldsymbol{Cstar}_{PU} \to \boldsymbol{EMod}$ *is full and faithful.*

Proof. Any PU-map $f\colon A \to B$ is completely determined (and defined by) its action on $[0,1]_A$: for a non-zero positive element $x \in A$ we use $x \leq \|x\|\,1$ and thus $\frac{1}{\|x\|}\,x \in [0,1]_A$ to see that $f(x) = \|x\|\,f(\frac{1}{\|x\|}\,x)$. An arbitrary element $y \in A$ can be written uniquely as linear sum of four positive elements (see Lemma 2.2), determining $f(y)$. □

The (finite, discrete probability) distribution monad $\mathcal{D}\colon \mathbf{Sets} \to \mathbf{Sets}$ sends a set X to the set $\mathcal{D}(X) = \{\varphi\colon X \to [0,1] \mid \mathit{supp}(\varphi) \text{ is finite, and } \sum_x \varphi(x) = 1\}$, where $\mathit{supp}(\varphi) = \{x \mid \varphi(x) \neq 0\}$. Such an element $\varphi \in \mathcal{D}(X)$ may be identified with a finite, formal convex sum $\sum_i r_i x_i$ with $x_i \in X$ and $r_i \in [0,1]$ satisfying $\sum_i r_i = 1$. The unit $\eta\colon X \to \mathcal{D}(X)$ and multiplication $\mu\colon \mathcal{D}^2(X) \to \mathcal{D}(X)$ of this monad are given by singleton/Dirac convex sum and by matrix multiplication:

$$\eta(x) = 1x \qquad \mu(\Phi)(x) = \textstyle\sum_{\varphi} \Phi(\varphi)\cdot\varphi(x).$$

A *convex* set is an Eilenberg-Moore algebra of this monad: it consists of a carrier set X in which actual sums $\sum_i r_i x_i \in X$ exist for all convex combinations. We write $\mathbf{Conv} = \mathcal{EM}(\mathcal{D})$ for the category of convex sets, with "affine" functions preserving convex sums.

Effect modules and convex sets are related via a basic adjunction [22], obtained by "homming into $[0,1]$", as in:

$$\mathbf{EMod}^{\mathrm{op}} \underset{\mathbf{Conv}(-,[0,1])}{\overset{\mathbf{EMod}(-,[0,1])}{\underset{\top}{\rightleftarrows}}} \mathbf{Conv} \tag{2.2}$$

3. Set-theoretic computations in C^*-algebras

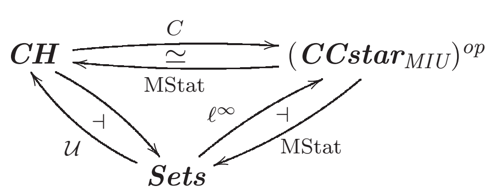
For a set X, a function $f\colon X \to \mathbb{C}$ is called *bounded* if $|f(x)| \leq s$, for some $s \in \mathbb{R}_{\geq 0}$. We write $\ell^\infty(X)$ for the set of such bounded functions. Notice that if X is finite, any function $X \to \mathbb{C}$ is bounded, so that $\ell^\infty(X) = \mathbb{C}^X$.

Each $\ell^\infty(X)$ is a commutative C^*-algebra, with pointwise addition, multiplication and involution, and with the uniform/supremum norm:

$$\|f\|_\infty = \inf\{s \in \mathbb{R}_{\geq 0} \mid \forall x.\, |f(x)| \leq s\}.$$

In fact it is a typical example of a commutative W^*-algebra, but we do not require this fact. This yields a functor $\ell^\infty\colon \mathbf{Sets} \to (\mathbf{CCstar}_{\mathrm{MIU}})^{\mathrm{op}}$, where for $h\colon X \to Y$ we have $\ell^\infty(h) = (-) \circ h\colon \ell^\infty(Y) \to \ell^\infty(X)$; it preserves the (pointwise) operations. We have the following result.

Proposition 3.1. *The functor* $\ell^\infty\colon \boldsymbol{Sets} \to (\boldsymbol{CCstar}_{MIU})^{op}$ *is left adjoint to the multiplicative states functor* $\mathrm{MStat}\colon (\boldsymbol{CCstar}_{MIU})^{op} \to \boldsymbol{Sets}$*. In combination with the adjunctions from Theorem 2.3 we get a situation:*



By composition and uniqueness of adjoints we get:

$$C \circ \mathcal{U} \cong \ell^\infty \qquad \textit{and also} \qquad \mathrm{MStat} \circ \ell^\infty \cong \mathcal{U}.$$

Proof. Note that MStat is used in two different senses in the above diagram, in one case with a compact Hausdorff topology, and in the other case simply as a set. The adjunction involving ℓ^∞ and MStat is for MStat as a set. We show this adjunction using the universal property of the unit of an adjunction. We define the unit $\eta_X \colon X \to \mathrm{MStat}(\ell^\infty(X))$, where $X \in \mathbf{Sets}$, as

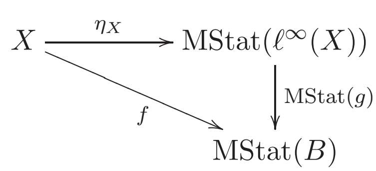
$$\eta_X(x)(a) = a(x),$$

where $a \in \ell^\infty(X)$. Then $\eta_X(x)$ is a multiplicative state on $\ell^\infty(X)$ because the vector space structure, multiplication and multiplicative unit are defined pointwise. To show the naturality square for η commutes, we must show that for all $f \colon X \to Y$ in **Sets**, $\mathrm{MStat}(\ell^\infty(f)) \circ \eta_X = \eta_Y \circ f$. If we take $x \in X$ and $b \in \ell^\infty(Y)$, we have:

$$\begin{aligned}
\big(\mathrm{MStat}(\ell^\infty(f)) \circ \eta_X\big)(x)(b) &= \mathrm{MStat}(\ell^\infty(f))(\eta_X(x))(b) \\
&= (\eta_X(x) \circ \ell^\infty(f))(b) \\
&= \eta_X(x)(\ell^\infty(f)(b)) \\
&= \eta_X(x)(b \circ f) \\
&= b(f(x)) \\
&= \eta_Y(f(x))(b) \\
&= (\eta_Y \circ f)(x)(b).
\end{aligned}$$

We now show this natural transformation satisfies the universal property making it the unit of the adjunction. Let $X \in \mathbf{Sets}$, $B \in \mathbf{CCstar}_{\mathrm{MIU}}$ and $f \colon X \to \mathrm{MStat}(B)$. Define $g : B \to \ell^\infty(X)$ as $g(b)(x) = f(x)(b)$. We must show that $g(b)$ is an element of $\ell^\infty(X)$, *i.e.* that it is bounded. For all $x \in X$, $f(x)$ is a multiplicative state, hence a state, so by [8, Proposition 2.1.4] we have $\|f(x)\| = 1$, and so $|g(b)(x) = |f(x)(b)| \leq \|f(x)\|\|b\| = \|b\|$. Therefore $\|b\|$ is a bound for $g(b)$, showing that it is a bounded function. The fact that g is an MIU map is easily deduced from the fact that $f(x)$ is a multiplicative state for all x (it would fail if $f(x)$ were only a state).

We must now show that



commutes. Taking $x \in X$ and $b \in B$, we see

$$\begin{aligned}
\mathrm{MStat}(g)(\eta_X(x))(b) &= (\eta_X(x) \circ g)(b) \\
&= \eta_X(x)(g(b)) \\
&= g(b)(x) \\
&= f(x)(b),
\end{aligned}$$

and hence the unit diagram commutes.

To show the uniqueness of g, suppose there were $h \colon B \to \ell^\infty(X)$ that also made the unit diagram commute. By evaluating $\mathrm{MStat}(h)(\eta_X(x))(b)$ we would obtain $g(b)(x) = h(b)(x)$.

Since $g(b)$ and $h(b)$ are elements of $\ell^\infty(X)$ and hence functions, this implies $g(b) = h(b)$ by extensionality, and we can then conclude that $g = h$, as required. We have now shown that ℓ^∞ is a left adjoint to MStat. The other two adjunctions are simply the Stone-Čech compactification of a set and Gelfand duality (which is even an equivalence).

Since the triangle consisting of MStat, in both forms, and the forgetful functor $\mathbf{CH} \to \mathbf{Sets}$ commutes, the triangle for ℓ^∞,$\mathcal{U}$ and C commutes up to isomorphism, *i.e.* $\ell^\infty \cong C \circ \mathcal{U}$ by uniqueness of adjoints. □

When we restrict to the full subcategory $\mathbf{FinSets} \hookrightarrow \mathbf{Sets}$ of finite sets we obtain a functor $\ell^\infty = \mathbb{C}^{(-)} : \mathbf{FinSets} \to (\mathbf{FdCCstar}_{\mathrm{MIU}})^{\mathrm{op}}$. The next result is then a well-known special case of Gelfand duality (Theorem 2.3). We elaborate the proof in some detail because it is important to see where the preservation of multiplication plays a role.

Proposition 3.2. *The functor* $\mathbb{C}^{(-)} : \boldsymbol{FinSets} \to (\boldsymbol{FdCCstar}_{MIU})^{op}$ *is an equivalence of categories.*

Proof. It is easy to see that the functor $\mathbb{C}^{(-)}$ is faithful. The crucial part is to see that it is full. So assume we have two finite sets, seen as natural numbers n, m, and a MIU-homomorphism $h: \mathbb{C}^m \to \mathbb{C}^n$. For $j \in m$, let $|\,j\,\rangle \in \mathbb{C}^m$ be the standard base vector with 1 at the j-th position and 0 elsewhere. Since this $|\,j\,\rangle$ is positive, so is $h(|\,j\,\rangle)$, and thus we may write it as $h(|\,j\,\rangle) = (r_{1j}, \ldots, r_{nj})$, with $r_{ij} \in \mathbb{R}_{\geq 0}$. Because $|\,j\,\rangle \cdot |\,j\,\rangle = |\,j\,\rangle$, and h preserves multiplication, we get $h(|\,j\,\rangle) \cdot h(|\,j\,\rangle) = h(|\,j\,\rangle)$, and thus $r_{ij}^2 = r_{ij}$. This means $r_{ij} \in \{0, 1\}$, so that h is a (binary) Boolean matrix. But h is also unital, and so:

$$1 = h(1) = h(|1\rangle + \cdots + |m\rangle) = h(|1\rangle) + \cdots + h(|m\rangle). \tag{3.1}$$

For each $i \in n$ there is thus precisely one $j \in m$ with $r_{ij} = 1$ — so that h is a "functional" Boolean matrix. This yields the required function $f: n \to m$ with $\mathbb{C}^f = h$.

Corollary 2.4 says that the functor $\mathbb{C}^{(-)} : \mathbf{FinSets} \to (\mathbf{FdCCstar}_{\mathrm{MIU}})^{\mathrm{op}}$ is essentially surjective on objects, and thus an equivalence. □

This proof demonstrates that preservation of multiplication, as required for "MIU" maps, is a rather strong condition. We make this more explicit.

Corollary 3.3. *For* $n \in \mathbb{N}$ *we have* $\mathrm{MStat}(\mathbb{C}^n) \cong n$.

Proof. By identifying $n \in \mathbb{N}$ with the n-element set $n = \{0, 1, \ldots, n-1\} \in \mathbf{FinSets}$, we get by Proposition 3.2, $\mathrm{MStat}(\mathbb{C}^n) = \mathrm{Hom}_{\mathrm{MIU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathbf{FinSets}(1, n) \cong n$. □

4. Discrete probabilistic computations in C^*-algebras

We turn to probabilistic computations and will see that we remain in the world of commutative C^*-algebras, but with PU-maps (positive unital) instead of MIU-maps. Recall that the set of states $\mathrm{Stat}(A)$ of a C^*-algebra A contains the PU-maps $A \to \mathbb{C}$.

We summarize here the definition of the expectation monad given in [21]. If $[0,1]^X$ is the effect module of functions from X to $[0,1]$ with pointwise operations, $\mathcal{E}(X) = \mathbf{EMod}([0,1]^X, [0,1])$. The unit $\eta_X : X \to \mathcal{E}(X)$ is evaluation, defined as $\eta_X(x)(f) = f(x)$ for $f \in [0,1]^X$. The multiplication $\mu_X : \mathcal{E}^2(X) \to \mathcal{E}(X)$ is defined for $h \in [0,1]^{\mathcal{E}(X)} \to [0,1]$, $p \in [0,1]^X$ as

$$\mu_X(h)(p) = h\big(\lambda k \in \mathcal{E}(X).\, k(p)\big).$$

Lemma 4.1. *Sending a set X to the set of states of the C^*-algebra $\ell^\infty(X)$ yields the (underlying functor of the) expectation monad $\mathcal{E}$ from* [21]*: the mapping $X \mapsto \mathrm{Stat}(\ell^\infty(X))$ is isomorphic to the expectation monad $\mathcal{E}\colon \textbf{Sets} \to \textbf{Sets}$, defined in* [21] *via effect module homomorphisms: $\mathcal{E}(X) = \textbf{EMod}\big([0,1]^X, [0,1]\big)$.*

As a result, $\mathrm{Stat}(\mathbb{C}^n) \cong \mathcal{D}(n)$, for $n \in \mathbb{N}$, where $\mathcal{D}(n)$ is the standard n-simplex.

Proof. The predicate/effect functor $[0,1]_{(-)}\colon \mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{EMod}$ is full and faithful by Lemma 2.6, and so:

$$\begin{aligned}\mathrm{Stat}(\ell^\infty(X)) \;=\; \mathrm{Hom}_{\mathrm{PU}}\big(\ell^\infty(X), \mathbb{C}\big) &\cong \mathbf{EMod}\big([0,1]_{\ell^\infty(X)}, [0,1]_{\mathbb{C}}\big) \\ &= \mathbf{EMod}\big([0,1]^X, [0,1]\big) \;=\; \mathcal{E}(X).\end{aligned}$$

The isomorphism $\alpha\colon \mathrm{Hom}_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \xrightarrow{\cong} \mathcal{D}(n)$ follows because the expectation and distribution monad coincide on finite sets, see [21]. Explicitly, it is given by $\alpha(h) = \lambda i \in n.\, h(|i\rangle)$ and $\alpha^{-1}(\varphi)(v) = \sum_i \varphi(i) \cdot v(i)$. □

The unit η and multiplication μ structure on $\mathcal{E}(X) \cong \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C})$ is very much like for "continuation" or "double dual" monads, see [26, 31, 18], with:

$$\begin{array}{ccc} X \xrightarrow{\eta} \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C}) & \qquad & \mathrm{Hom}_{\mathrm{PU}}\Big(\ell^\infty\big(\mathrm{Hom}_{\mathrm{PU}}(\mathbb{C}^X, \mathbb{C})\big), \mathbb{C}\Big) \xrightarrow{\mu} \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C}) \\ x \longmapsto \lambda v.\, v(x) & & g \longmapsto \lambda v.\, g\big(\lambda h.\, h(v)\big). \end{array}$$

For an arbitrary monad $T = (T, \eta, \mu)$ on a category $\mathbf{B}$ we write $\mathcal{K}\ell(T)$ for the Kleisli category of T. Its objects are the same as those of $\mathbf{B}$, but its maps $X \to Y$ are the maps $X \to T(Y)$ in $\mathbf{B}$. The unit $\eta\colon X \to T(X)$ is the identity map $X \to X$ in $\mathcal{K}\ell(T)$; and composition of $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{K}\ell(T)$ is given by $g \odot f = \mu \circ T(g) \circ f$. Maps in such a Kleisli category are understood as computations with outcomes of type T, see [31]. For a monad $T\colon \mathbf{Sets} \to \mathbf{Sets}$ we write $\mathcal{K}\ell_{\mathbb{N}}(T) \hookrightarrow \mathcal{K}\ell(T)$ for the full subcategory with numbers $n \in \mathbb{N}$ as objects, considered as n-element sets.

Proposition 4.2. *The expectation monad $\mathcal{E}(X) \cong \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(X), \mathbb{C})$ gives rise to a full and faithful functor:*

$$\begin{array}{rcl} \mathcal{K}\ell(\mathcal{E}) & \xrightarrow{\quad\mathcal{C}_{\mathcal{E}}\quad} & (\textbf{CCstar}_{PU})^{op} \\ X & \longmapsto & \ell^\infty(X) \\ \big(X \xrightarrow{f} \mathcal{E}(Y)\big) & \longmapsto & \lambda v \in \ell^\infty(Y).\, \lambda x \in X.\, f(x)(v). \end{array} \tag{4.1}$$

Proof. First we need to see that $\mathcal{C}_{\mathcal{E}}(f)$ is well-defined: the function $\mathcal{C}_{\mathcal{E}}(f)(v)\colon X \to \mathbb{C}$ must be bounded. We can apply Lemma 2.2 to the function $f(x) \in \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(Y), \mathbb{C})$; it yields $\|f(x)(v)\| \le 4\|v\|$. This holds for each $x \in X$, so that $|\mathcal{C}_{\mathcal{E}}(f)(v)(x)| = |f(x)(v)|$ is bounded by $4\|v\|$. Next, the map $\mathcal{C}_{\mathcal{E}}(f)$ is a PU-map of C^*-algebras via the pointwise definitions of the relevant constructions.

We check that $\mathcal{C}_{\mathcal{E}}$ preserves (Kleisli) identities and composition:

$$\begin{aligned}
\mathcal{C}_{\mathcal{E}}(\mathrm{id})(v)(x) &= \mathcal{C}_{\mathcal{E}}(\eta)(v)(x) \\
&= \eta(x)(v) \\
&= v(x) \\
\mathcal{C}_{\mathcal{E}}(g \odot f)(v)(x) &= (g \odot f)(x)(v) \\
&= \mu\Big(\mathcal{E}(g)(f(x))\Big)(v) \\
&= \mathcal{E}(g)(f(x))\big(\lambda w.\, w(v)\big) \\
&= f(x)\big((\lambda w.\, w(v)) \circ g\big) \\
&= f(x)\big(\lambda y.\, g(y)(v)\big) \\
&= f(x)\big(\mathcal{C}_{\mathcal{E}}(g)(v)\big) \\
&= \mathcal{C}_{\mathcal{E}}(f)\big(\mathcal{C}_{\mathcal{E}}(g)(v)\big)(x) \\
&= \big(\mathcal{C}_{\mathcal{E}}(f) \circ \mathcal{C}_{\mathcal{E}}(g)\big)(v)(x).
\end{aligned}$$

Further, $\mathcal{C}_{\mathcal{E}}$ is obviously faithful, and it is full since for $h\colon \ell^\infty(Y) \to \ell^\infty(X)$ in $\mathbf{CCstar}_{\mathrm{PU}}$ we can define $f\colon X \to \mathrm{Hom}_{\mathrm{PU}}(\ell^\infty(Y), \mathbb{C})$ by $f(x)(v) = h(v)(x)$. Then each $f(x)$ is a PU-map of C^*-algebras. □

We turn to the finite case, like in the previous section. We do so by considering the Kleisli category $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{E})$ obtained by restricting to objects $n \in \mathbb{N}$. Since the expectation monad $\mathcal{E}$ and the distribution monad $\mathcal{D}$ coincide on finite sets, we have $\mathcal{K}\ell_{\mathbb{N}}(\mathcal{E}) \cong \mathcal{K}\ell_{\mathbb{N}}(\mathcal{D})$. Maps $n \to m$ in this category are probabilistic transition matrices $n \to \mathcal{D}(m)$. This category has been investigated also in [12]. The following equivalence is known, see *e.g.* [27], although possibly not in this categorical form.

Proposition 4.3. *The functor $\mathcal{C}_{\mathcal{E}}$ from* (4.1) *restricts in the finite case to an equivalence of categories:*

$$\mathcal{K}\ell_{\mathbb{N}}(\mathcal{D}) \xrightarrow[\simeq]{\mathcal{C}_{\mathcal{D}}} (\boldsymbol{FdCCstar}_{PU})^{op} \tag{4.2}$$

It is given by $\mathcal{C}_{\mathcal{D}}(n) = \mathbb{C}^n$ and $\mathcal{C}_{\mathcal{D}}\big(n \xrightarrow{f} \mathcal{D}(m)\big) = \lambda v \in \mathbb{C}^m.\, \lambda i \in n.\, \sum_{j \in m} f(i)(j) \cdot v(j)$.

This equivalence (4.2) may be read as: the category $\mathbf{FdCCstar}_{\mathrm{PU}}$ of finite-dimensional commutative C^*-algebras, with positive unital maps, is the *Lawvere theory* of the distribution monad $\mathcal{D}$.

Proof. Fullness and faithfulness of the functor $\mathcal{C}_{\mathcal{D}}$ follow from Proposition 4.2, using the isomorphism $\mathrm{Hom}_{\mathrm{PU}}(\mathbb{C}^n, \mathbb{C}) \cong \mathcal{D}(n)$ from Lemma 4.1. This functor $\mathcal{C}_{\mathcal{D}}$ is essentially surjective on objects by Corollary 2.4, using the fact that a MIU-map is a PU-map. □

5. Continuous probabilistic computations

The question arises if the full and faithful functor $\mathcal{K}\ell(\mathcal{E}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ from Proposition 4.2 can be turned into an equivalence of categories, but not just for the finite case like in Proposition 4.3. In order to make this work we have to lift the expectation monad $\mathcal{E}$ on

Sets to the category **CH** of compact Hausdorff spaces. As lifting we use what we call the *Radon* monad $\mathcal{R}$, defined on $X \in \mathbf{CH}$ as:

$$\mathcal{R}(X) = \mathrm{Stat}(C(X)) = \mathrm{Hom}_{\mathrm{PU}}\big(C(X), \mathbb{C}\big), \tag{5.1}$$

where, as usual, $C(X) = \{f\colon X \to \mathbb{C} \mid f \text{ is continuous}\}$; notice that the functions $f \in C(X)$ are automatically bounded, since X is compact. We have implicitly applied the forgetful functor from $\mathbf{CCLcvx} \to \mathbf{CH}$ to make $\mathcal{R}$ into an endofunctor of $\mathbf{CH}$. The elements of $\mathcal{R}(X)$ are related to measures in the following way. If μ is a probability measure on the Borel sets of X, integration of continuous functions with respect to μ gives a function $\int_X - d\mu \in \mathcal{R}(X)$. A Radon probability measure, or an inner regular probability measure, is one such that $\mu(S) = \sup_{K \subseteq S} \mu(K)$ where K ranges over compact sets. The map from measures to elements of $\mathcal{R}(X)$ is a bijection [33, Thm. 2.14], and accordingly we shall sometimes refer to elements of $\mathcal{R}(X)$ as measures. Therefore the Radon monad can be considered as a variant of the Giry monad. In fact there are two Giry monads, one on measurable spaces and one on Polish spaces. The Radon monad differs from the Giry monad on measurable spaces in that it uses the topology of a space, and that in the case of a space that is not a standard Borel space there can be non-Radon measures [11, 434K (d), page 192] [15, §53.10, page 231]. The Radon monad differs from the Giry monad on Polish spaces essentially only in the choice of spaces, and on compact Polish spaces they agree, as the topology Giry used is the same as the weak-* topology, and Polish spaces do not admit any non-Radon Borel probability measures.[6, Theorems 1.1 and 1.4].

This Radon monad $\mathcal{R}$ is not new: we shall see later that it occurs in [39, Theorem 3] as the monad of an adjunction ("probability measure" is used to mean "Radon probability measure" in that article). It has been used more recently in [30]. However, our duality result below — Theorem 5.1 — is not known in the literature.

From Proposition 2.5 it is immediate that $\mathcal{R}(X)$ is again a compact Hausdorff space. The unit $\eta\colon X \to \mathcal{R}(X)$ and multiplication $\mu\colon \mathcal{R}^2(X) \to \mathcal{R}(X)$ are defined as for the expectation monad, namely as $\eta(x)(v) = v(x)$ and $\mu(g)(v) = g\big(\lambda h.\, h(v)\big)$. We check that η is continuous. Recall from the proof of Proposition 2.5 that a basic open in $\mathcal{R}(X)$ is of the form $ev_s^{-1}(U) = \{h \in \mathcal{R}(X) \mid h(s) \in U\}$, where $s \in C(X)$ and $U \subseteq \mathbb{C}$ is open. Then:

$$\eta^{-1}\big(ev_s^{-1}(U)\big) = \{x \in X \mid \eta(x)(s) \in U\} = \{x \in X \mid s(x) \in U\} = s^{-1}(U).$$

The latter is an open subset of X since $s\colon X \to \mathbb{C}$ is a continuous function.

We are now ready to state our main, new duality result. It may be understood as a probabilistic version of Gelfand duality, for commutative C^*-algebras with PU maps instead of the MIU maps originally used (see Theorem 2.3).

Theorem 5.1. *The Radon monad* (5.1) *yields an equivalence of categories:*

$$\mathcal{K}\ell(\mathcal{R}) \simeq (\boldsymbol{CCstar}_{PU})^{op}.$$

Proof. We define a functor $\mathcal{C}_{\mathcal{R}}\colon \mathcal{K}\ell(\mathcal{R}) \to (\mathbf{CCstar}_{\mathrm{PU}})^{\mathrm{op}}$ like in (4.1), namely by:

$$\mathcal{C}_{\mathcal{R}}(X) = C(X) \qquad \mathcal{C}_{\mathcal{R}}(f) = \lambda v.\, \lambda x.\, f(x)(v).$$

Since $f\colon X \to \mathcal{R}(Y)$ is itself continuous, so is $f(-)(v)\colon X \to \mathbb{C}$.

The fact that $\mathcal{C}_{\mathcal{R}}$ is a full and faithful functor follows as in the proof of Proposition 4.2. This functor is essentially surjective on objects by ordinary Gelfand duality (Theorem 2.3). □

We investigate the Radon monad $\mathcal{R}$ a bit further, in particular its relation to the distribution monad $\mathcal{D}$ on **Sets**.

Lemma 5.2. *There is a map of monads* $(U, \tau)\colon \mathcal{R} \to \mathcal{D}$ *in:*

$$\mathcal{R} \circlearrowright \textbf{\textit{CH}} \xrightarrow{\;U\;} \textbf{\textit{Sets}} \circlearrowleft \mathcal{D} \qquad\qquad \mathcal{D}U \overset{\tau}{\Longrightarrow} U\mathcal{R}$$

where U is the forgetful functor and τ commutes appropriately with the units and multiplications of the monads $\mathcal{D}$ and $\mathcal{R}$. (Such a map is called a "monad functor" in [38, §1]*.)*

As a result the forgetful functor lifts to the associated categories of Eilenberg-Moore algebras:

$$\begin{array}{ccc} \mathcal{EM}(\mathcal{R}) & \longrightarrow & \mathcal{EM}(\mathcal{D}) = \textbf{\textit{Conv}} \\ \big(\mathcal{R}(X) \overset{\alpha}{\to} X\big) & \longmapsto & \big(\mathcal{D}(UX) \overset{\tau}{\to} U\mathcal{R}(X) \overset{U\alpha}{\to} UX\big) \end{array}$$

Hence the carrier of an $\mathcal{R}$-algebra is a convex compact Hausdorff space, and every algebra map is an affine function.

Proof. For $X \in \mathbf{CH}$ and $\varphi \in \mathcal{D}(UX)$, that is for $\varphi\colon UX \to [0,1]$ with finite support and $\sum_x \varphi(x) = 1$, we define $\tau(\varphi) \in U\mathcal{R}(X)$ on $h \in C(X)$ as:

$$\tau(\varphi)(h) = \textstyle\sum_x \varphi(x) \cdot h(x) \;\in\; \mathbb{C}. \tag{5.2}$$

It is easy to see that τ is a linear map $C(X) \to \mathbb{C}$ that preserves positive elements and the unit. Moreover, it commutes appropriately with the units and multiplications. For instance:

$$\big(\tau_X \circ \eta^{\mathcal{D}}_{UX}\big)(x)(h) = \tau_X(1x)(h) = h(x) = U(\eta^{\mathcal{R}}_X)(x)(h). \qquad \square$$

The continuous dual space of $C(X)$ can be ordered using (2.1), by taking the positive cone to be those linear functionals that map positive functions to positive numbers.

Definition 5.3. A state $\phi \in \mathcal{R}(X) = \mathrm{Hom}_{\mathrm{PU}}(C(X), \mathbb{C})$ is a *pure state* if for for each positive linear functional such that $\psi \leq \phi$, *i.e.* such that $\phi - \psi$ is positive, there exists an $\alpha \in [0,1]$ such that $\psi = \alpha\phi$. $\square$

Lemma 5.4. *For a compact Hausdorff space X, the subset of unit (or Dirac) measures $\{\eta(x) \mid x \in X\} \subseteq \mathcal{R}(X)$ are pure states and hence is the set of extreme points of the set of Radon measures $\mathcal{R}(X)$ — where $\eta(x) = \eta^{\mathcal{R}}(x) = \mathrm{ev}_x = \lambda h.\, h(x)$ is the unit of the monad $\mathcal{R}$.*

Proof. We rely on the basic fact, see [8, 2.5.2, page 43], that a measure is a Dirac measure iff it is a "pure" state. We prove the above lemma by showing that the pure states are precisely the extreme points of the convex set $\mathcal{R}(X)$.

- If $\phi \in \mathcal{R}(X)$ is a pure state, suppose $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$, a convex combination of two states $\phi_i \in \mathcal{R}(X)$ with $\alpha_i \in [0,1]$ satisfying $\alpha_1 + \alpha_2 = 1$, where no two elements of $\{\phi, \phi_1, \phi_2\}$ are the same. Then $\phi \geq \alpha_1\phi_1$, since for a positive function $f \in C(X)$ one has $(\phi - \alpha_1\phi_1)(f) = \alpha_2\phi_2(f) \geq 0$. Thus $\alpha_1\phi_1 = \alpha\phi$, for some $\alpha \in [0,1]$, since ϕ is pure. Then $\alpha_1 = \alpha_1\phi_1(1) = \alpha\phi(1) = \alpha$. If $\alpha_1 = 0$, then $\alpha_2 = 1$ and so $\phi = \phi_2$. If $\alpha_1 > 0$, then $\phi = \phi_1$. Hence ϕ is an extreme point.

- Suppose ϕ is an extreme point of $\mathcal{R}(X)$, *i.e.* that $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$ implies ϕ_1 or $\phi_2 = \phi$. Then if there is a positive linear functional $\psi \leq \phi$, we may take $\alpha_1 = \psi(1) \geq 0$; since $\alpha_1 = \psi(1) \leq \phi(1) = 1$, we get $\alpha_1 \in [0,1]$. If $\alpha_1 = 0$, then since $\|\psi\| = \psi(1) = 0$ we get $\psi = 0$ and $\psi = 0 \cdot \phi$. If $\alpha_1 = 1$, then $(\phi - \psi)(1) = 0$, which since $\phi - \psi$ was assumed to be positive implies $\phi - \psi = 0$ and hence $\psi = 1 \cdot \phi$. Having dealt with those cases, we have that $\alpha_1 \in (0,1)$, and so we have a state $\phi_1 = \frac{1}{\alpha_1}\psi$. We may take $\alpha_2 = 1 - \alpha_1 \in (0,1)$ and obtain a second state $\phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. By construction we have a convex decomposition of $\phi = \alpha_1\phi_1 + \alpha_2\phi_2$. Therefore either $\phi = \phi_1 = \frac{1}{\alpha_1}\psi$ or $\phi = \phi_2 = \frac{1}{\alpha_2}(\phi - \psi)$. In the first case, $\psi = \alpha_1\phi$, making ϕ pure. But also in the second case ϕ is pure, since we have $\alpha_2\phi = \phi - \psi$ and thus $\psi = (1 - \alpha_2)\phi$. □

Lemma 5.5. *Let X be a compact Hausdorff space.*

(1) *The maps $\tau_X : \mathcal{D}(UX) \to U\mathcal{R}(X)$ from* (5.2) *are injective; as a result, the unit/Dirac maps $\eta : X \to \mathcal{R}(X)$ are also injective.*
(2) *The maps $\tau_X : \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$ are dense.*

Proof. For the first point, assume $\varphi, \psi \in \mathcal{D}(UX)$ satisfying $\tau(\varphi) = \tau(\psi)$. We first show that the finite support sets are equal: $\mathit{supp}(\varphi) = \mathit{supp}(\psi)$. Since X is Hausdorff, singletons are closed, and hence finite subsets too. Suppose $\mathit{supp}(\varphi) \not\subseteq \mathit{supp}(\psi)$, so that $S = \mathit{supp}(\varphi) - \mathit{supp}(\psi)$ is non-empty. Since S and $\mathit{supp}(\psi)$ are disjoint closed subsets, there is by Urysohn's lemma a continuous function $f : X \to [0,1]$ with $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in \mathit{supp}(\psi)$. But then $\tau(\psi)(f) = 0$, whereas $\tau(\varphi)(f) \neq 0$.

Now that we know $\mathit{supp}(\varphi) = \mathit{supp}(\psi)$, assume $\varphi(x) \neq \psi(x)$, for some $x \in \mathit{supp}(\varphi)$. The closed subsets $\{x\}$ and $\mathit{supp}(\varphi) - \{x\}$ are disjoint, so there is, again by Urysohn's lemma a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in \mathit{supp}(\varphi)$. But then $\varphi(x) = \tau(\varphi)(f) = \tau(\psi)(f) = \psi(x)$, contradicting the assumption.

We can conclude that the unit $X \to \mathcal{R}(X)$ is also injective, since its underlying function can be written as the composite $U(\eta^{\mathcal{R}}) = \tau \circ \eta^{\mathcal{D}} : UX \rightarrowtail \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$, because τ is a map of monads.

To show that the image of τ_X is dense, we proceed as follows. By Lemmas 5.4 and 5.2, the extreme points of $\mathcal{R}(X)$ are

$$\{\eta^{\mathcal{R}}(x) \mid x \in X\} = \{\tau\big(\eta^{\mathcal{D}}(x)\big) \mid x \in X\}$$

and are thus in the image of $\tau : \mathcal{D}(UX) \rightarrowtail U\mathcal{R}(X)$. Since every convex combination of $\eta^{\mathcal{R}}(x)$ comes from a formal convex sum $\varphi \in \mathcal{D}(UX)$, all convex combinations of extreme points are in the image of τ_X. Using Proposition 2.5, $\mathcal{R}(X)$ can be considered an object of **CCLcvx**, i.e. a compact convex subset of a locally convex space. Accordingly, we may apply the Krein-Milman theorem [7, Proposition 7.4, page 142] to conclude the set of convex combinations of extreme points is dense. □

Lemma 5.6. *Let X and Y be compact Hausdorff spaces. Each Eilenberg-Moore algebra $\alpha : \mathcal{R}(X) \to X$ is an affine function. For each continuous map $f : X \to Y$, the function $\mathcal{R}(f) : \mathcal{R}(X) \to \mathcal{R}(Y)$ is affine.*

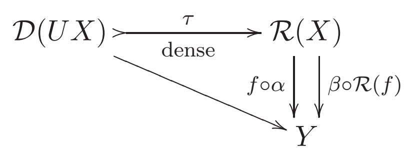
Proof. This follows from naturality of $\tau : \mathcal{D}U \Rightarrow U\mathcal{R}$. □

Proposition 5.7. *Let $\alpha : \mathcal{R}(X) \to X$ and $\beta : \mathcal{R}(Y) \to Y$ be two Eilenberg-Moore algebras of the Radon monad $\mathcal{R}$. A function $f : X \to Y$ is an algebra homomorphism if and only if f is both continuous and affine.*

As a result, the functor $\mathcal{EM}(\mathcal{R}) \to \mathcal{EM}(\mathcal{D}) = \textbf{Conv}$ from Lemma 5.2 is faithful, and an $\mathcal{EM}(\mathcal{D})$ map comes from an $\mathcal{EM}(\mathcal{R})$ map if and only if it is continuous.

We shall follow the convention of writing $\mathcal{A}(X,Y)$ for the homset of continuous and affine functions $X \to Y$.

Proof. Clearly, each algebra map is both continuous and affine. For the converse, if $f : X \to Y$ is continuous, it is a map in the category **CH** of compact Hausdorff spaces. Since it is affine, both triangles commute in:



Since Y is Hausdorff, there is at most one such map. Therefore f is an algebra map. □

The category $\mathcal{EM}(\mathcal{R})$ of Eilenberg-Moore algebras of the Radon monad may thus be understood as a suitable category of convex compact Hausdorff spaces, with affine continuous maps between them. In the next section, we see how to use a result from [39] to relate this to **CCLcvx**, which is a category of "concrete" convex sets. Using this theorem, it will be shown that "observability" conditions like in [21, top of p. 169] always hold for algebras of $\mathcal{R}$.

5.1. **Świrszcz's Theorem and Noncommutative C^*-algebras.** In this section we show that the Radon monad arises from an adjunction in [39] enabling us to use Świrszcz's theorem 3 from that paper to show that the categories **CCLcvx** and $\mathcal{EM}(\mathcal{R})$ are equivalent, which we can then apply to represent noncommutative C^*-algebras. The adjunction in question has $U : \textbf{CCLcvx} \to \textbf{CH}$ as the right adjoint, and the details of the construction of the left adjoint are not given. In order to prove that $\mathcal{R}$ is the monad arising from this adjunction, we need to know its unit and counit, so our next task is to define the left adjoint explicitly. Of course, any other left adjoint will be naturally isomorphic.

We begin as follows. We define $\acute{\mathcal{S}} : \textbf{CH} \to \textbf{CCLcvx}$ as $\acute{\mathcal{S}} = \mathrm{Stat} \circ C$. Hence $\mathcal{R} = U \circ \acute{\mathcal{S}}$. To show that $\acute{\mathcal{S}}$ is the left adjoint to U, we use the unit and counit definition of an adjunction. We already know the unit, $\eta_X : X \to U(\acute{\mathcal{S}}(X))$, as we gave it when defining the unit of $\mathcal{R}$. To define the counit we use the notion of *barycentre*.

We can understand the intuitive notion of barycentre by thinking of a Radon probability measure μ on the unit square $[0,1]^2$. If we wanted to find the centre of mass of μ, which we shall call $b \in [0,1]^2$, we would take

$$b_x = \int_{[0,1]^2} x \mathrm{d}\mu \qquad \text{and} \qquad b_y = \int_{[0,1]^2} y \mathrm{d}\mu$$

for the x and y coordinates. We can see that x and y are continuous affine functions from $[0,1]^2 \to \mathbb{R}$, assigning each point to its x and y coordinate respectively. Therefore we can rewrite the above as

$$\int_{[0,1]^2} x \mathrm{d}\mu = x(b) \qquad \text{and} \qquad \int_{[0,1]^2} y \mathrm{d}\mu = y(b).$$

In monadic terms, this means that both projections $\pi_1, \pi_2 \colon [0,1]^2 \to [0,1]$ are maps of Eilenberg-Moore algebras for the Radon monad, in the sense that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{R}([0,1]^2) & \xrightarrow{\mathcal{R}(\pi_i)} & \mathcal{R}([0,1]) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ [0,1]^2 & \xrightarrow[\pi_i]{} & [0,1] \end{array}$$

We write α for the algebra $\nu \mapsto \int \mathrm{id}\,\mathrm{d}\nu$, see also [20], and β for the product algebra structure, given by $\mu \mapsto \langle \int \pi_1 \mathrm{d}\mu, \int \pi_2 \mathrm{d}\mu \rangle = \langle \int x \mathrm{d}\mu, \int y \mathrm{d}\mu \rangle$.

If we generalize π_1 and π_2 to arbitary real-valued continuous affine functions on X, and reinterpret Radon measures as functionals (as in the start of §5), we get the idea behind the following standard definition.

Definition 5.8. If $X \in \mathbf{CCLcvx}$ and $\phi \in \acute{\mathcal{S}}(U(X))$, then a point $x \in X$ is a *barycentre* for ϕ if for all continuous affine functions f from $X \to \mathbb{R}$ we have that $\phi(f) = f(x)$. □

The theorem that every ϕ has a barycentre when X is a compact subset of a locally convex space is standard and is proven in [3, proposition I.2.1 and I.2.2].

We will require the following important lemma, one of sevaral variants of the Hahn-Banach separation lemma, and some of its corollaries, which give an affine analogue of Urysohn's lemma for objects in **CCLcvx**.

Lemma 5.9. *If V is a locally convex topological vector space, X a closed convex subset and Y a compact convex subset that is disjoint from X, then there exists a continuous linear functional $\phi \colon V \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.* □

For proof, see either [7, theorem IV.3.9] or [36, II.4.2 corollary 1].

Corollary 5.10. *Let $(K, V) \in \mathrm{Obj}(\boldsymbol{CCLcvx})$. In the following X, Y will be arbitrary closed disjoint convex subsets of K, x, y arbitrary distinct points of K.*

(i) *There is a $\phi \in \mathcal{A}(K, \mathbb{R})$ and an $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.*
(ii) *There is a $\phi \in \mathcal{A}(K, \mathbb{R})$ such that $\phi(x) \neq \phi(y)$.*
(iii) *There is a $\phi \in \boldsymbol{CCLcvx}(K, [0,1])$ and an $\alpha \in \mathbb{R}$ such that $\phi(X) \subseteq (\alpha, 1]$ and $\phi(Y) \subseteq [0, \alpha)$.*
(iv) *There is a $\phi \in \boldsymbol{CCLcvx}(K, [0,1])$ such that $\phi(x) \neq \phi(y)$.*

Proof.

(i) Apply Lemma 5.9 to obtain $\phi' \colon V \to \mathbb{R}$ separating X from Y. Since K has the subspace topology, $\phi = \phi'|_K$ is continuous, and since ϕ' is linear, ϕ is affine, hence $\phi \in \mathcal{A}(K, \mathbb{R})$. We also keep the properties that $\phi(X) \subseteq (\alpha, \infty)$ and $\phi(Y) \subseteq (-\infty, \alpha)$.
(ii) This follows directly from (i), using the fact that points are compact and convex.
(iii) We use (i) and obtain $\phi' \in \mathcal{A}(K, \mathbb{R})$ and $\alpha' \in \mathbb{R}$. Since the image of a compact space is compact, and a compact subset of $\mathbb{R}$ is closed and bounded, the numbers

$$\beta_\uparrow = \sup \phi'(K) \qquad\qquad \beta_\downarrow = \inf \phi'(K)$$

exist, and ϕ' can be considered as an affine continuous map $K \to [\beta_\downarrow, \beta_\uparrow]$. We define

$$\phi(k) = \frac{\phi(k) - \beta_\downarrow}{\beta_\uparrow - \beta_\downarrow}$$

if $\beta_\uparrow \neq \beta_\downarrow$, otherwise we define it without dividing by anything, though this can only happen if one of X or Y is empty. The image of ϕ is contained in $[0,1]$, and ϕ is affine and continuous, being the composition of affine and continuous maps. We define

$$\alpha = \frac{\alpha' - \beta_\downarrow}{\beta_\uparrow - \beta_\downarrow}$$

again not doing the division if it is zero. We have that $\phi(X) \subseteq (\alpha, \infty)$, and since the image of ϕ is contained in $[0,1]$, this implies $\phi(X) \subseteq (\alpha, 1]$. The proof that $\phi(Y) \subseteq [0, \alpha)$ is similar.

(iv) This is proven using (iii), again using the fact that points are closed, convex sets. □

Using the properties proven above, we can start to define the counit of the adjunction.

Lemma 5.11.

(i) *For every $\phi \in \acute{\mathcal{S}}(U(X))$ the barycentre is unique. The function $\varepsilon_X \colon \acute{\mathcal{S}}(U(X)) \to X$ mapping ϕ to its barycentre is well defined.*

(ii) *This ε_X is an affine map.*

Proof.

(i) We show the barycentre is unique as follows. Let (V, X) be an object of **CCLcvx**, V being the locally convex space and X the compact convex subset. Let $x, x' \in X$ be barycentres of $\phi \in \acute{\mathcal{S}}(U)$. Suppose for a contradiction that $x \neq x'$. By corollary 5.10 (ii), there is an $f \in \mathcal{A}(X, \mathbb{R})$ such that $f(x) \neq f(x')$. Since x and x' are both barycentres of ϕ,

$$f(x) = \phi(f) = f(x')$$

a contradiction. So we have $x = x'$. Therefore ε_X is well-defined, at least as a function between sets.

(ii) To show that ε_X is affine, consider two Radon measures $\phi, \psi \in \acute{\mathcal{S}}(U(X))$, such that $\varepsilon_X(\phi) = x$ and $\varepsilon_X(\psi) = y$, i.e. these are the barycentres. To show that $\varepsilon_X(\alpha\phi + (1-\alpha)\psi) = \alpha\varepsilon_X(\phi) + (1-\alpha)\varepsilon_X(\psi)$, we will show that $\alpha x + (1-\alpha)y$ is the barycentre of $\alpha\phi + (1-\alpha)\psi$. Given an continuous affine function $f \colon X \to \mathbb{R}$, we have

$$(\alpha\phi + (1-\alpha)\psi)(f) = \alpha\phi(f) + (1-\alpha)\psi(f) = \alpha x + (1-\alpha)y$$

so ε_X is affine. □

Lemma 5.12. *The barycentre map ε_X is continuous, hence a map in **CCLcvx**.*

Proof. We now show that ε_X is continuous. We use the filter-theoretic definition of continuity. Given $\phi \in \acute{\mathcal{S}}(U(X))$, with barycentre x, we want to show that for every neighbourhood V of x, there is a neighbourhood U of ϕ such that $\varepsilon_X(U) \subseteq V$. It suffices to prove this for a chosen set of basic neighbourhoods, so we choose open neighbourhoods for X and for $\acute{\mathcal{S}}(U(X))$ we choose finite intersections of elements of the following subbasis of closed neighbourhoods:

$$U_{f,\alpha,\epsilon} = \{\psi \in \acute{\mathcal{S}}(U(X)) \mid |\psi(f) - \alpha| \leq \epsilon\}$$

where $f \in C(U(X))$, $\alpha \in \mathbb{R}$ and $\epsilon \in (0, \infty)$.

We find the neighbourhood of ϕ using a compactness argument.

Consider the following subset of X.

$$\bigcap_{\substack{f \in \mathcal{A}(X,\mathbb{R}) \\ \epsilon > 0}} \overline{\varepsilon_X(U_{f,f(x),\epsilon})}$$

Since $\phi \in U_{f,f(x),\epsilon}$ for all values of f and ϵ, we have that x is in this intersection. We will show that

$$\bigcap_{\substack{f\in\mathcal{A}(X,\mathbb{R})\\ \epsilon>0}} \overline{\varepsilon_X(U_{f,f(x),\epsilon})} = \{x\} \tag{5.3}$$

As we already know x is an element of the left hand side, we will show that if $x' \in X$ and $x' \neq x$, then x' is not an element of the left hand side. So since $x \neq x'$, by Corollary 5.10(ii) there is an $f \in \mathcal{A}(X,\mathbb{R})$ such that $f(x) \neq f(x')$. We let

$$\epsilon = \frac{|f(x) - f(x')|}{3} > 0 \tag{5.4}$$

We show that $x' \notin \overline{\varepsilon_X(U_{f,f(x),\epsilon})}$ and therefore is not in (5.3) by showing there is an open set containing x' that is disjoint from $\varepsilon_X(U_{f,f(x),\epsilon})$. The open set we choose is

$$f^{-1}((f(x') - \epsilon, f(x') + \epsilon))$$

which is open because f is continuous. Assume for a contradiction that there is some $x'' \in f^{-1}((f(x') - \epsilon, f(x') + \epsilon)) \cap \varepsilon_X(U_{f,f(x),\epsilon})$. This means that

$$|f(x') - f(x'')| < \epsilon \tag{5.5}$$

and there is some $\psi \in U_{f,f(x),\epsilon}$ of which x'' is the barycentre, *i.e.* for all $g \in \mathcal{A}(X,\mathbb{R})$ $\psi(g) = g(x'')$. Therefore it must be the case that $\psi(f) = f(x'')$, and so the inequality deriving from $\psi \in U_{f,f(x),\epsilon}$, which is $|\psi(f) - f(x)| \leq \epsilon$ becomes $|f(x'') - f(x)| \leq \epsilon$. If we combine this with (5.5) and use the triangle inequality, we get $|f(x') - f(x)| \leq 2\epsilon$, which contradicts $|f(x) - f(x')| \geq 3\epsilon$ from (5.4). Therefore the assumption that x'' could exist is wrong, so x' is in an open set outside $\varepsilon_X(U_{f,f(x),\epsilon})$, and hence $x' \notin \overline{\varepsilon_X(U_{f,f(x),\epsilon})}$. This establishes that (5.3) is the case.

Now consider $X \setminus V$, which is a closed set that does not contain x, since V is an open neighbourhood of x. We therefore have

$$\emptyset = (X \setminus V) \cap \bigcap_{\substack{f\in\mathcal{A}(X,\mathbb{R})\\ \epsilon>0}} \overline{\varepsilon_X(U_{f,f(x),\epsilon})} = \bigcap_{\substack{f\in\mathcal{A}(X,\mathbb{R})\\ \epsilon>0}} (X \setminus V) \cap \overline{\varepsilon_X(U_{f,f(x),\epsilon})}$$

The right hand side is a family of closed subsets of a compact space with empty intersection. Therefore there is a finite subfamily also having empty intersection. We use the numbers $i \in \{1, \ldots, n\}$ as an index set, and take $\{\epsilon_i\}$, $\{f_i\}$ such that we have

$$\emptyset = \bigcap_{i=1}^{n} (X \setminus V) \cap \overline{\varepsilon_X(U_{f_i,f_i(x),\epsilon_i})} = (X \setminus V) \cap \bigcap_{i=1}^{n} \overline{\varepsilon_X(U_{f_i,f_i(x),\epsilon_i})}$$

Therefore we have

$$\varepsilon_X\left(\bigcap_{i=1}^{n} U_{f_i,f_i(x),\epsilon_i}\right) \subseteq \bigcap_{i=1}^{n} \varepsilon_X(U_{f_i,f_i(x),\epsilon_i}) \subseteq \bigcap_{i=1}^{n} \overline{\varepsilon_X(U_{f_i,f_i(x),\epsilon_i})} \subseteq V$$

Since V was an arbitrary open neighbourhood of $\varepsilon_X(\phi)$, we have that ε_X is continuous at ϕ. Since the choice of ϕ was arbitrary, ε_X is continuous. □

Lemma 5.13. *The family* $\{\varepsilon_X\}$ *defines a natural transformation* $\varepsilon : \acute{\mathcal{S}} \circ U \Rightarrow \mathrm{Id}$.

Proof. We must show that

$$\begin{array}{ccc} \acute{\mathcal{S}}(U(X)) & \xrightarrow{\epsilon_X} & X \\ {\scriptstyle \acute{\mathcal{S}}(U(f))}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \acute{\mathcal{S}}(U(Y)) & \xrightarrow[\epsilon_Y]{} & Y \end{array}$$

Suppose that $\phi \in \acute{\mathcal{S}}(U(X))$ and $\varepsilon_X(\phi) = x$, i.e. x is the barycentre of ϕ. It suffices to show that $f(x)$ is the barycentre of $\acute{\mathcal{S}}(U(f)(\phi)$. Let $h \in C(Y)$, and we have by definition that

$$\acute{\mathcal{S}}(U(f))(\phi)(h) = \phi(h \circ f)$$

We want to show that if h is affine, then $\acute{\mathcal{S}}(U(f))(\phi)(h) = h(f(x))$, as this would show $f(x)$ is the barycentre. Since $h \circ f$ is the composite of continuous, affine functions, it is also continuous and affine, and so, using that x is the barycentre of ϕ, we have that $\phi(h \circ f) = (h \circ f)(x) = h(f(x))$, which is what we were required to prove. □

Taken together, the preceding three lemmas define the counit. We can now move on to showing that this is actually an adjunction.

Theorem 5.14. *The functor* $\acute{\mathcal{S}} \colon \boldsymbol{CH} \to \boldsymbol{CCLcvx}$ *is the left adjoint to* $U \colon \boldsymbol{CCLcvx} \to \boldsymbol{CH}$

Proof. We show that the unit-counit diagrams commute.

First we must show that the following commutes:

$$\begin{array}{ccc} UY & \xrightarrow{\eta_{UY}} & U(\acute{\mathcal{S}}(U(Y))) \\ & {\scriptstyle \mathrm{id}_{UY}}\searrow & \big\downarrow{\scriptstyle U\varepsilon_Y} \\ & & UY \end{array}$$

In other words, we must show that for all $y \in UY$, y is the barycentre of $\eta_{UY}(y)$. Using the definition of η, we have that for any affine continuous function $f \colon X \to \mathbb{R}$ that

$$\eta_{UY}(x)(f) = f(x)$$

because that is already true for all continuous functions $f \in C(X)$. Therefore x is the barycentre of $\eta_{UY}(x)$, and so the diagram commutes.

The second diagram we must consider is the following:

$$\begin{array}{ccc} \acute{\mathcal{S}}(X) & \xrightarrow{\acute{\mathcal{S}}(\eta_X)} & \acute{\mathcal{S}}(U(\acute{\mathcal{S}}(X))) \\ & {\scriptstyle \mathrm{id}_{\acute{\mathcal{S}}(X)}}\searrow & \big\downarrow{\scriptstyle \epsilon_{\acute{\mathcal{S}}(X)}} \\ & & \acute{\mathcal{S}}(X) \end{array}$$

This time, we need to show that $\phi \in \acute{\mathcal{S}}(X)$ is the barycentre of the measure $\acute{\mathcal{S}}(\eta_X)(\phi)$. So consider an affine continuous function $k \colon \acute{\mathcal{S}}(X) \to \mathbb{R}$. We want to show that $\acute{\mathcal{S}}(\eta_X)(\phi)(k) = k(\phi)$ for all $\phi \in \acute{\mathcal{S}}(X)$. To do this, we use Lemma 5.5. We show the diagram commutes on the convex combinations of extreme points, and since this is a dense subset, the diagram commutes by continuity. So let $\{x_1, \dots x_n\}$ be a finite subset of X, and

$$\sum_{i=1}^{n} \alpha_i \eta_X(x_i)$$

a finite convex combination of extreme points of $\acute{\mathcal{S}}(X)$. Now

$$\begin{aligned}\acute{\mathcal{S}}(\eta_X)\left(\sum_{i=1}^{n} \alpha_i \eta_X(x_i)\right)(k) &= \left(\sum_{i=1}^{n} \alpha_i \eta_X(x_i)\right)(k \circ \eta_X) \\ &= \sum_{i=1}^{n} \alpha_i \eta_X(x_i)(k \circ \eta_X) \\ &= \sum_{i=1}^{n} \alpha_i k(\eta_X(x_i)) \\ &= k\left(\sum_{i=1}^{n} (\eta_X(x_i))\right)\end{aligned}$$

with the last step holding because k is an affine function.

As explained before, this shows $\acute{\mathcal{S}}(\eta_X)(\phi)(k) = k(\phi)$ for all $\phi \in \acute{\mathcal{S}}(X)$, and hence the diagram commutes. Thus we have that $\acute{\mathcal{S}}$ is the left adjoint to U. □

Now that we have defined the adjunction $\acute{\mathcal{S}} \dashv U$, we can move on to proving that $\mathcal{R}$ is not only the same functor as the monad derived from $\acute{\mathcal{S}} \dashv U$ but also the same as a monad. In order to do this, we require a few lemmas concerning the definition of μ we gave at the start of Section 5. The map μ was defined using $\lambda h.\, h(v)$. Since we need to prove certain properties about it, we give this map a name, and generalize it somewhat for later use. If A is a (possibly noncommutative) C^*-algebra, we define

$$A^{\mathrm{sa}} \xrightarrow{\zeta_A} \mathcal{A}(\mathrm{Stat}(A), \mathbb{R}) \qquad \text{as} \qquad \zeta_A(a)(\phi) = \phi(a).$$

In the special case we had earlier, we were using $\zeta_{C(X)}$ for a compact Hausdorff space X, since $C(X)^{\mathrm{sa}} = C_{\mathbb{R}}(X)$, the real-valued functions. We can see that

$$\mu_X(g)(v) = g(\zeta_{C(X)}(v)). \tag{5.6}$$

Lemma 5.15. *The map ζ_A is a bijection between A^{sa} and $\mathcal{A}(\mathrm{Stat}(A), \mathbb{R})$. $\zeta_{C(X)}$ is a bijection between $C_{\mathbb{R}}(X)$ and $\mathcal{A}(\acute{\mathcal{S}}(X), \mathbb{R})$. In fact, the bijection is an isomorphism of ordered $\mathbb{R}$-vector spaces with unit, taking these to be defined pointwise on $\mathcal{A}(\mathrm{Stat}(A), \mathbb{R})$.* □

The proof can be found in [2, Proposition 2.3]. It was originally proved by Kadison [25, Lemma 4.3, Remark 4.4] and is often stated for complete order-unit spaces (such as in [3, Theorem II.1.8]), though it was originally intended for use with C^*-algebras, as here.

Theorem 5.16. *The monad : $\boldsymbol{CH} \to \boldsymbol{CH}$ given by $\acute{\mathcal{S}} \dashv U$ is the Radon monad $\mathcal{R}$.*

Proof. We have by definition that $\mathcal{R} = U\acute{\mathcal{S}}$ and $\eta = \eta$. Therefore we only need to show that $\mu = U\varepsilon\acute{\mathcal{S}}$. What we need to show then, is that if X is a compact Hausdorff space and $\phi \in \acute{\mathcal{S}}(U(\acute{\mathcal{S}}(X)))$, then $\mu(\phi)$ is the barycentre of ϕ. That is to say, for all $f \in \mathcal{A}(\acute{\mathcal{S}}(X), \mathbb{R})$,

$\phi(f) = f(\mu_X(\phi))$. Using Lemma 5.15, we reduce to showing that for all $f \in C_{\mathbb{R}}(X)$, we have $\phi(\zeta_X(f)) = \zeta_X(f)(\mu_X(\phi))$. Using (5.6), we have

$$\zeta_X(f)(\mu_X(\phi)) = \mu_X(\phi)(f) = \phi(\zeta_X(f))$$

as required. □

Theorem 5.17 (Świrszcz's theorem)**.** *The forgetful functor* $U\colon \boldsymbol{CCLcvx} \to \boldsymbol{CH}$ *is monadic, i.e.* $\boldsymbol{CCLcvx} \simeq \mathcal{EM}(U \circ \acute{\mathcal{S}})$*. By Theorem 5.16,* $\boldsymbol{CCLcvx} \simeq \mathcal{EM}(\mathcal{R})$*.* □

This comes from [39, Theorem 3]. A proof not using any monadicity theorems can be found in [37, Proposition 7.3].

5.1.1. *Non-commutative C^*-algebras and $\mathcal{EM}(\mathcal{R})$.* In the following section we shall show that the category $\mathbf{Cstar}_{\mathrm{PU}}$ embeds fully and faithfully in $\mathcal{EM}(\mathcal{R})$. To do this, we use the fact that $\mathcal{EM}(\mathcal{R}) \simeq \mathbf{CCLcvx}$, and also the functor $\mathrm{Stat}\colon \mathbf{Cstar}_{\mathrm{PU}} \to \mathbf{CCLcvx}$.

We begin with a standard separation result from the theory of C^*-algebras.

Lemma 5.18. *If A is a C^*-algebra, and $a, b \in A$, then*

$$\phi(a) = \phi(b)$$

for all $\phi \in \mathrm{Stat}(A)$ implies $a = b$. In other words, A is separated by its states, or A has "sufficiently many states".

Proof. In [24, theorem 4.3.4 (i)] we have that if $\phi(a) = 0$ for all $\phi \in \mathrm{Stat}(A)$, then $a = 0$. We simply apply this to $a - b$. □

On the set $\mathcal{A}(X, \mathbb{C})$, for $X \in \mathrm{Obj}(\mathbf{CCLcvx})$, we can define a $\mathbb{C}$-vector space structure, a positive cone, and a distinguished unit, simply by using the fact that $\mathbb{C}$ has these things and defining them pointwise. The positive cone is $[0, \infty) \subseteq \mathbb{C}$ and the unit is 1. Given these definitions, we can prove the complexification of Lemma 5.15.

Lemma 5.19. *For each C^*-algebra A, the map $\xi_A\colon A \to \mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$, defined as*

$$\xi_A(a)(\phi) = \phi(a)$$

is an isomorphism of complex vector spaces preserving the positive cone and unit in both directions.

Proof. First we show that the map ξ_A is $\mathbb{C}$-linear and preserves *. For $\mathbb{C}$-linearity, let $z \in \mathbb{C}$, $\phi \in \mathrm{Stat}(A)$ and $a \in A$. Then

$$\xi_A(za)(\phi) = \phi(za) = z\phi(a) = z\xi_A(a)(\phi),$$

so $\xi_A(za) = z\xi_A(a)$.

To show that it preserves *, where for $f \in \mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$, f^* is calculated pointwise, we use the fact that every positive linear functional on A, and hence every state, is self-adjoint, as described in Lemma 2.2, *i.e.* $\phi(a^*) = \overline{\phi(a)}$.

Thus we have

$$\xi_A(a^*)(\phi) = \phi(a^*) = \overline{\phi(a)} = \overline{\xi_A(a)(\phi)} = \xi_A(a)^*(\phi).$$

and so $\xi_A(a^*) = \xi_A(a)^*$.

From Lemma 5.15 we have that ξ restricts to an isomorphism $\zeta\colon A^{\mathrm{sa}} \cong \mathcal{A}(\mathrm{Stat}(A), \mathbb{R})$ as an ordered vector space with unit. We extend this to complex numbers as follows. Given $a \in A$, we can define its real and imaginary parts as

$$\Re(a) = \frac{a + a^*}{2} \qquad\qquad \Im(a) = \frac{a - a^*}{2i}$$

and we see that $\Re(a) + i\Im(a) = a$. Similarly, using pointwise complex conjugation as *, we can define real and imaginary parts of an affine continuous map from $\mathrm{Stat}(A) \to \mathbb{C}$, and the self-adjoint elements are maps $\mathrm{Stat}(A) \to \mathbb{C}$. Since we know that η_X has an inverse for self-adjoint elements, we can define the inverse as

$$\xi_A^{-1}(f + ig) = \xi_A^{-1}(f) + i\xi_A^{-1}(g)$$

where f, g are self-adjoint.

We show this is the inverse of ξ_A. For one way

$$\begin{aligned} \xi_A(\xi_A^{-1}(f + ig)) &= \xi_A(\xi_A^{-1}(f) + i\xi_A^{-1}(g)) \\ &= \xi_A(\xi_A^{-1}(f)) + i\xi_A(\xi_A^{-1}(g)) \\ &= f + ig. \end{aligned}$$

For the other way, with $a, b \in A^{\mathrm{sa}}$,

$$\begin{aligned} \xi_A^{-1}(\xi_A(a + ib)) &= \xi_A^{-1}(\xi_A(a) + i\xi_A(b)) \\ &= \xi_A^{-1}(\xi_A(a)) + i\xi_A^{-1}(\xi_A(b)) \\ &= a + ib, \end{aligned}$$

where the definition of ξ_A^{-1} can be applied since ξ_A preserves * and hence preserves self-adjointness, so $\xi_A(a)$ and $\xi_A(b)$ are both self-adjoint. □

We will require the following fact in a moment.

Lemma 5.20. *If B is a C^*-algebra, $b' \in \mathcal{A}(\mathrm{Stat}(B), \mathbb{C})$, then for all $\phi \in \mathrm{Stat}(B)$*

$$\phi(\xi_B^{-1}(b')) = b'(\phi).$$

Proof. By Lemma 5.19, we have that there is some $b \in B$ such that $b' = \xi_B(b)$. Then we have

$$\phi(\xi_B^{-1}(b')) = \phi(\xi_B^{-1}(\xi_B(b))) = \phi(b) = \xi_B(b)(\phi) = b'(\phi). \qquad \square$$

We can now prove that Stat is full and faithful, and hence $(\mathbf{Cstar}_{\mathrm{PU}})^{\mathrm{op}}$ embeds fully in $\mathcal{EM}(\mathcal{R})$.

Theorem 5.21. *The state space functor $\mathrm{Stat}\colon (\boldsymbol{Cstar}_{PU})^{op} \to \boldsymbol{CCLcvx}$ is full and faithful.*

Proof.

- For faithfulness, suppose we have $f, g\colon A \to B$ in $\mathbf{Cstar}_{\mathrm{PU}}$, such that $\mathrm{Stat}(f) = \mathrm{Stat}(g)$. We have that $\mathrm{Stat}(f)(\phi) = \mathrm{Stat}(g)(\phi)$ for all $\phi \in \mathrm{Stat}(B)$, which, expanding the definitions, gives that $\phi \circ f = \phi \circ g$ for all $\phi \in \mathrm{Stat}(B)$. Now, we have that for all $a \in A$ and $\phi \in \mathrm{Stat}(B)$, that $\phi(f(a)) = \phi(g(a))$. By Lemma 5.18, we have that for all $a \in A$, $f(a) = g(a)$, and therefore $f = g$.

- For fullness, let $g\colon \mathrm{Stat}(B) \to \mathrm{Stat}(A)$ be an affine, continuous map. We must find a map $f\colon A \to B$ such that $\mathrm{Stat}(f) = g$. We take the map $f = \xi_B^{-1} \circ \mathcal{A}(g, \mathbb{C}) \circ \xi_A\colon A \to B$. First we must prove this map is positive, $\mathbb{C}$-linear, and unital. We know from Lemma 5.19 that, being isomorphisms, ξ_A and ξ_B^{-1} are $\mathbb{C}$-linear (with the pointwise structure on $\mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$) and preserve the positive cone and unit. Therefore we only need to show that $\mathcal{A}(g, \mathbb{C})$ has these properties to verify them for f. For $\mathbb{C}$-linearity, let $a_1, a_2 \in \mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$, and $z_1, z_2 \in \mathbb{C}$. Then for each $\phi \in \mathrm{Stat}(B)$

$$\begin{aligned}\mathcal{A}(g, \mathbb{C})(z_1 a_1 + z_2 a_2)(\phi) &= ((z_1 a_1 + z_2 a_2) \circ g)(\phi)\\ &= (z_1 a_1 + z_2 a_2)(g(\phi))\\ &= z_1 a_1(g(\phi)) + z_2 a_2(g(\phi))\\ &= z_1 \mathcal{A}(g, \mathbb{C})(a_1)(\phi) + z_2 \mathcal{A}(g, \mathbb{C})(a_2)(\phi)\\ &= (z_1 \mathcal{A}(g, \mathbb{C})(a_1) + z_2 \mathcal{A}(g, \mathbb{C})(a_2))(\phi),\end{aligned}$$

and so

$$\mathcal{A}(g, \mathbb{C})(z_1 a_1 + z_2 a_2) = z_1 \mathcal{A}(g, \mathbb{C})(a_1) + z_2 \mathcal{A}(g, \mathbb{C})(a_2),$$

which is to say, $\mathcal{A}(g, \mathbb{C})$ is $\mathbb{C}$-linear.

The unit of $\mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$ is given by the function $1\colon \mathrm{Stat}(A) \to \mathbb{C}$ that maps every element of $\mathrm{Stat}(A)$ to $1 \in \mathbb{C}$. We must show that $\mathcal{A}(g, \mathbb{C})$ preserves this unit. Given $\phi \in \mathrm{Stat}(B)$, we have

$$\mathcal{A}(g, \mathbb{C})(1)(\phi) = 1(g(\phi)) = 1,$$

so $\mathcal{A}(g, \mathbb{C})(1)$ takes the value $1 \in \mathbb{C}$ for all $\phi \in \mathrm{Stat}(B)$, and hence it is the unit in $\mathcal{A}(\mathrm{Stat}(B), \mathbb{C})$.

The positive elements of $\mathcal{A}(\mathrm{Stat}(A), \mathbb{C})$ are given by functions whose image is contained in the positive reals, $[0, \infty) \subseteq \mathbb{C}$. We need to show that if $a \in \mathcal{A}(\mathrm{Stat}(A), [0, \infty))$, then so is $\mathcal{A}(g, \mathbb{C})(a)$. This is easily accomplished as before. If $\phi \in \mathrm{Stat}(B)$, then

$$\mathcal{A}(g, \mathbb{C})(a)(\phi) = (a \circ g)(\phi) = a(g(\phi)).$$

Since $g(\phi) \in \mathrm{Stat}(A)$, we have that $a(g(\phi)) \in [0, \infty)$ by the assumption on a, and so $\mathcal{A}(g, \mathbb{C})(a)$ is a positive element of $\mathcal{A}(\mathrm{Stat}(B), \mathbb{C})$. All these conditions, taken together, show that f is a $\mathbf{Cstar}_{\mathrm{PU}}$ map from A to B.

Now we show that $\mathrm{Stat}(f) = g$. Let $\phi \in \mathrm{Stat}(B)$ and $a \in A$. Then

$$\begin{aligned}\mathrm{Stat}(f)(\phi)(a) &= \mathrm{Stat}(\xi_B^{-1} \circ \mathcal{A}(g, \mathbb{C}) \circ \xi_A)(\phi)(a)\\ &= (\phi \circ \xi_B^{-1} \circ \mathcal{A}(g, \mathbb{C}) \circ \xi_A)(a)\\ &= \phi(\xi_B^{-1}(\mathcal{A}(g, \mathbb{C})(\xi_A(a))))\\ &= \phi(\xi_B^{-1}(\xi_A(a) \circ g)),\end{aligned}$$

applying Lemma 5.20, we continue

$$\begin{aligned}\mathrm{Stat}(f)(\phi)(a) &= (\xi_A(a) \circ g)(\phi)\\ &= \xi_A(a)(g(\phi))\\ &= g(\phi)(a).\end{aligned}$$

Since this holds for all ϕ and a, we have the required equality $\mathrm{Stat}(f) = g$, proving Stat is full. □

Alfsen, Hanche-Olsen and Shultz have characterized the essential image of Stat [4, Corollary 8.6]. We do not give the characterization here as it involves many further definitions. Since there are PU-maps that are not completely positive, Stat is not a full functor when restricted to $\mathbf{Cstar}_{\mathrm{cPU}}$. In fact, whether a map is completely positive or not depends on the orientation (in the sense of [4]) and cannot be defined purely from the $\mathcal{EM}(\mathcal{R})$ structure of the state space. This can be seen by the fact that the transpose map, the archetypal positive but not completely positive map, is self-inverse, and hence an isomorphism as a PU map, and so by the above result defines an isomorphism in $\mathcal{EM}(\mathcal{R})$ on the state space.

6. States and effects

We start with a simple observation.

Lemma 6.1. *The unit interval* $[0,1]$ *is a compact convex subset of the locally convex space* $\mathbb{R}$*, and therefore carries a* $\mathcal{R}$*-algebra structure by Theorem 5.17. The algebra map* $\mathcal{R}([0,1]) \to [0,1]$ *maps each measure to its mean value.*

For an arbitrary $\mathcal{R}$*-algebra* X*, the homset of algebra maps:*

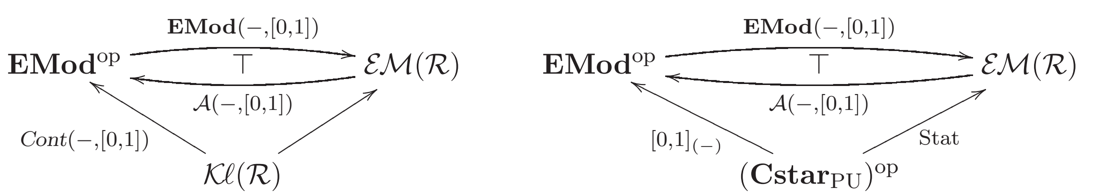
$$\mathcal{EM}(\mathcal{R})\big(X,[0,1]\big) = \mathcal{A}(X,[0,1])$$

is an effect module, with pointwise operations. Recall from Proposition 5.7 that this homset is the affine and continuous functions $X \to [0,1]$*. Taken all together, we have defined a functor* $\mathcal{A}(-,[0,1])\colon \mathcal{EM}(\mathcal{R}) \to \mathbf{EMod}^{\mathrm{op}}$*.* □

In [21] it is shown that for an effect module M, the homset $\mathbf{EMod}(M,[0,1])$ is a convex compact Hausdorff space. In fact, it carries an $\mathcal{R}$-algebra structure:

$$\begin{array}{ccc} \mathcal{R}\big(\mathbf{EMod}(M,[0,1])\big) & \xrightarrow{\alpha_M} & \mathbf{EMod}(M,[0,1]) \\ h & \longmapsto & \lambda x \in M.\, h(\mathrm{ev}_x) \end{array}$$

where $\mathrm{ev}_x = \lambda v.\, v(x)\colon C\big(\mathbf{EMod}(M,[0,1])\big) \to \mathbb{C}$. For each map of effect modules $f\colon M \to M'$ one obtains a map of $\mathcal{R}$-algebras $(-)\circ f\colon \mathbf{EMod}(M',[0,1]) \to \mathbf{EMod}(M,[0,1])$. We thus obtain the following situation:



(6.1)

Such diagrams appear in [18] as a categorical representation of the duality between states and effects, with the Schrödinger picture on the right vertex of the triangle, and the Heisenberg picture on the left vertex of the triangle (see also [20]). In these diagrams:

- The map $\mathcal{K}\ell(\mathcal{R}) \to \mathbf{EMod}^{\mathrm{op}}$ on the left is the "predicate" functor, sending a space X to the predicates on X, given by the effect module $\mathit{Cont}(X,[0,1])$ of continuous functions $X \to [0,1]$, or for C^*-algebras mapping A to the effects $[0,1]_A$. For C^*-algebras this was shown to be full and faithful in Lemma 2.6, and for $\mathcal{K}\ell(\mathcal{R})$ we combine Lemma 2.6 and

Theorem 5.1:

$$\begin{aligned}\mathbf{EMod}\big(\mathit{Cont}(Y,[0,1]),\mathit{Cont}(X,[0,1])\big) &= \mathbf{EMod}\big([0,1]_{C(Y)},[0,1]_{C(X)}\big)\\ &\cong \mathrm{Hom}_{\mathrm{PU}}\big(C(Y),C(X)\big)\\ &\cong \mathcal{K}\ell(\mathcal{R})\big(X,Y\big).\end{aligned}$$

- The "state" functor $\mathcal{K}\ell(\mathcal{R}) \to \mathcal{EM}(\mathcal{R})$ is the standard full and faithful "comparison" functor from a Kleisli category to a category of Eilenberg-Moore algebras. In the C^*-algebra case it is the functor Stat, combined with the equivalence from Theorem 5.17. It is full and faithful by Theorem 5.21.
- The diagrams in (6.1) commute (up-to-isomorphism) in one direction. For $\mathcal{K}\ell(\mathcal{R})$ we have:

$$\begin{aligned}\mathbf{EMod}\big(\mathit{Cont}(X,[0,1]),[0,1]\big) &= \mathbf{EMod}\big([0,1]_{C(X)},[0,1]_{\mathbb{C}}\big)\\ &\cong \mathrm{Hom}_{\mathrm{PU}}\big(C(X),\mathbb{C}\big) = \mathcal{R}(X),\end{aligned}$$

and similarly for $\mathbf{Cstar}_{\mathrm{PU}}$ we have

$$\begin{aligned}\mathbf{EMod}([0,1]_A,[0,1]) &\cong \mathbf{Cstar}_{\mathrm{PU}}(A,\mathbb{C}) \qquad \text{by Lemma 2.6}\\ &= \mathrm{Stat}(A)\end{aligned}$$

- The diagrams in (6.1) also commute (again, up-to-isomorphism) in the other direction, *i.e.* $\mathcal{A}(\mathcal{R}(X),[0,1]) \cong \mathit{Cont}(X,[0,1])$ and $\mathcal{A}(\mathrm{Stat}(A),[0,1]) \cong [0,1]_A$. The former follows from the latter by taking $A = C(X)$, so we reduce to the latter. By Lemma 5.19 we have that $A \cong \mathcal{A}(\mathrm{Stat}(A),\mathbb{C})$ as unital ordered vector spaces. We can then restrict both sides to their unit intervals and obtain an isomorphism $[0,1]_A \cong \mathcal{A}(\mathrm{Stat}(A),[0,1])$.

We summarise what we have just shown.

Theorem 6.2. *The diagrams* (6.1) *are commuting "state-and-effect" triangles.* □

Final remarks

The main contribution of this article lies in establishing a connection between two different worlds, namely the world of theoretical computer scientists using program language semantics (and logic) via monads, and the world of mathematicians and theoretical physicists using C^*-algebras. This connection involves the distribution monad $\mathcal{D}$ on **Sets**, which is heavily used for modeling discrete probabilistic systems (Markov chains), in the finite-dimensional case (see Proposition 4.3) and the less familiar Radon monad $\mathcal{R}$ on compact Hausdorff spaces (see Theorem 5.1). These results apply to both commutative and noncommutative C^*-algebras, but only to positive unital maps. Follow-up research will concentrate on characterizing completely positive maps in the noncommutative case.

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