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RELATING REVERSIBLE PETRI NETS AND REVERSIBLE EVENT STRUCTURES, CATEGORICALLY

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Abstract. Causal nets (CNs) are Petri nets where causal dependencies are modelled via inhibitor arcs. They play the role of occurrence nets when representing the behaviour of a concurrent and distributed system, even when reversibility is considered. In this paper we extend CNs to account also for asymmetric conflicts and study (i) how this kind of nets, and their reversible versions, can be turned into a category; and (ii) their relation with the categories of reversible asymmetric event structures.

1. Introduction

Event structures (ESs) [39] are a denotational formalism to describe the behaviour of concurrent systems as a set of event occurences and constraints on such events, regulating how such events can happen. *Prime event structures* (PESs) are among the simplest form of event structures. A Prime event structure describes a computational process as a set of events whose occurrence is constrained by two relations: *causality* and (symmetric) *conflicts*. A simple PES is depicted in Fig. 1a, where causality ($<$) is drawn with straight lines (to be read from bottom to top) and conflicts ($\#$) with curly lines. In this case, b causally depends on a (i.e., $a < b$) meaning that b cannot occur if a does not occur first; additionally, b and c are in conflict (i.e., $b \# c$) meaning that b and c are mutually exclusive and cannot occur in the same execution of the process. The behaviour of a PES can be understood in terms of a transition system defined over *configurations* (i.e., sets of events), as illustrated in Fig. 1b. For instance, the transition $\emptyset \rightarrow \{a, c\}$ indicates that the initial state $\emptyset$ (i.e., no event has been executed yet) may evolve to $\{a, c\}$ by concurrently executing a and c. Neither $\{b\}$ nor $\{a, b, c\}$ are configurations because b cannot occur without a; and b and c cannot happen in the same run.

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In order to accommodate asymmetries that may arise, e.g., in shared-memory concurrency, *Asymmetric Event Structures* (AESs) relax the notion of conflicts by considering instead *weak causality*. Intuitively, an event e weakly causes the event e' (written $e \nearrow e'$) if e' can happen after e but e cannot happen after e'. This can be considered as an asymmetric conflict because e' forbids e to take place, but not the other way round. Let us consider the following shared memory scenario:

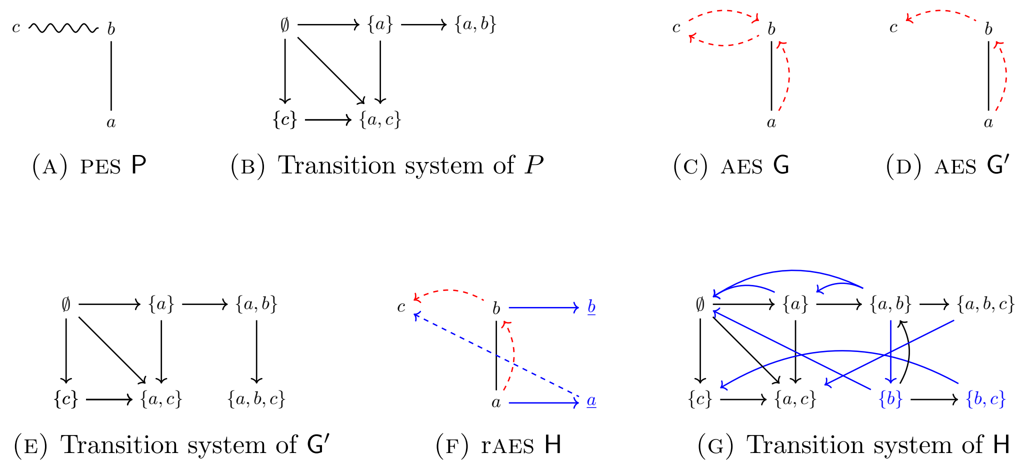
Thread 1	**Thread 2**	**Possible Values**
$1:\mathtt{lock}(\mathtt{m});$	$1:\mathtt{lock}(\mathtt{m});$	$\mathtt{x}=0;\ \mathtt{y}=1$
$2:\mathtt{if}(\mathtt{y}==0)$	$2:\mathtt{y}++;$	$\mathtt{x}=1;\ \mathtt{y}=1$
$3:\ \ \mathtt{then}\ \mathtt{x}=1;$	$1:\mathtt{unlock}(\mathtt{m});$	
$4:\mathtt{unlock}(\mathtt{m});$		

in which two variables x and y are shared among two threads. Let us suppose, for simplicity, that the two variables are initialised to 0. Assuming strong consistency, the actions of the two threads can be scheduled into two different ways leading to two different final configurations: either $x = 0$ and $y = 1$ or both variables are equal to 1. If we represent the behaviour of the above snippet as an event structure, we can identify two relevant events: the assignment of 1 to x and the increment of y. Let us call such events x_1 and y_1. We can observe that if x_1 happens then y_1 can happen, while if y_1 happens, then x_1 cannot. And this means that the x_1 weakly causes y_1, that is $x_1 \nearrow y_1$.

However, symmetric conflicts can be recovered by making a pair of conflicting events to weakly cause each other. For instance, the PES P in Fig. 1a can be rendered as the AES G in Fig. 1c, where weak causality is depicted with red, dashed arrows. Now the conflict between b and c is represented as $b \nearrow c$ and $c \nearrow b$ (the additional weak causal dependency $a \nearrow b$ is required by consistency with the causality relation). Unsurprisingly, the transition system associated with G coincides with that of P in Fig. 1b. Differently, the AES G' relaxes the conflict between b and c by making it asymmetric: we keep $b \nearrow c$ but drop $c \nearrow b$. Hence, c can be added to the configuration $\{a, b\}$ but b cannot be added to $\{a, c\}$, as rendered in the transition system in Fig. 1e.

Since their introduction [39], ESs have played a central role in the development of denotational semantics for concurrency models; in particular, for Petri nets (PNs). It is well-known that different classes of ESs correspond to different classes of PNs [39, 3, 4]. Moreover, different relations in ESs translate into different operational mechanisms of PNs. Causality and conflicts are typically modelled in nets via places *shared* among transitions. However, shared places fall short when translating other kinds of dependencies, such as weak causality, which requires contextual arcs [3].

Reversible computing has been studied in the 70's for its promise of achieving low-energy computation [15]. Lately [1, 26] is gaining interests for its application in different fields: from modelling bio-chemical reactions, to improve parallel discrete events simulation, to model fault tollerant primitives and finally to debugging. In a model of reversible computation we can distinguish two different flows of computation, i.e., in addition to the description of the standard *forward* execution, there is a *backward* flow that expresses the way in which the effects of the forward computation can be undone. If in a sequential setting it is pretty clear what is the backward flow of a computation (i.e. it is the exact inverse of the forward one), and the only challenge is on how to record computational history [14, 41]. In a concurrent setting it is more complicated. Different proposal have arisen in the last years trying to cope with different models: *causal-consistent reversibility* [7] has been shown to be the



(A) PES P (B) Transition system of P (C) AES G (D) AES G'

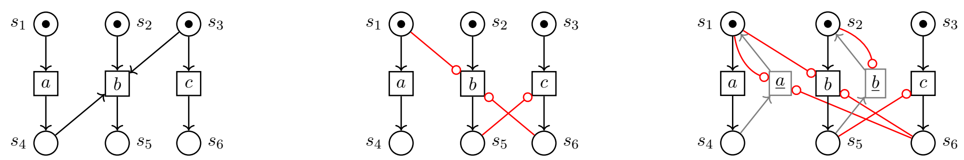
(E) Transition system of G' (F) rAES H (G) Transition system of H

FIGURE 1. A simple PES, AES, and rAES along with their associated transition systems.

right notion to model primitives for fault-handling systems such as transactions [8, 16] or roll-back recovery schemas [37, 27, 38], or to reverse-debug concurrent systems; while *out-of-causal* order reversibility [33] has been shown as being able to model bio-chemical reactions and compensations. Different models of concurrent computation [1] are available today: RCCS [7], CCSK [34], rhoπ [18], Rπ [6], reversible Petri nets [25], reversible event structures [32], to name a few. As expected, the attention has turned to the question of how these models relate to each other (see e.g. [17, 20, 21, 24, 12, 23]). This work addresses such goal by revisiting, in the context of reversibility, the connection between *Event Structures* (ESs) [39] and *Petri Nets* (PNs) established by Winskel [29].

On the event structure side, we focus on the *reversible Asymmetric Event Structures* (rAESs) introduced in [32], which are a reversible counterpart of *Asymmetric Event Structures* (AESs) [3], which in turn are a generalisation of *Prime Event Structures* (PESs) [29].

In rAESs, the backward flow is described in terms of a set of reversing events, each of them representing the undoing of some event of the forward flow. The reversal of an event e is denoted as $\underline{e}$. This notation is utilised in defining the set of reversible events and in describing the relationships associated with the undoing of events. For instance, in the AES G' in Fig. 1d, with $\{\underline{a}, \underline{b}\}$ we indicate that a and b are reversible, while c is not. Two relations, dubbed *reverse causation* ($\prec$) and *prevention* ($\lhd$), describe the backward flow and regulate the way in which reversing events occur: $\prec$ prescribes the events required for the undoing while $\lhd$ stipulates those that preclude it. The rAES H in Fig. 1f consists of the forward flow defined by G' extended with a backward flow represented by blue arrows: solid arrows correspond to reverse causation and dashed ones to prevention. Then, $a \prec \underline{a}$ says that $\underline{a}$ can be executed (meaning a can be undone) only when a has occurred (the pair $b \prec \underline{b}$ is analogous). The prevention arc $\underline{a} \lhd c$ states that a can be reversed only if c has not occurred. The transition system associated with H is in Fig. 1g: transitions corresponding to the forward flow (i.e., the ones in black mimicking those in Fig. 1e) add events to configurations; on the contrary, reversing transitions (in blue) remove events from configurations. For instance, the transition $\{a, b\} \to \{a\}$ accounts for the fact that b is reversible and can always be undone. Note that a can be reversed both in $\{a\}$ and $\{a, b\}$



$a < b$ and $b \# c$ | $a < b$, $b \nearrow c$ and $c \nearrow b$ | $\underline{a} \lhd c$, $\underline{a} \prec a$ and $\underline{b} \prec b$

(A) N_P | (B) N_R | (C) N'_R

Figure 2. An occurrence net N_P, its associated causal net N_R, and its reversible causal net N'_R.

but not in $\{a, c\}$ since the prevention relation (i.e., $\underline{a} \lhd c$) forbids a to be reversed if c has already occurred. Interestingly, a can be reversed in $\{a, b\}$ leading to the configuration $\{b\}$, which is not reachable by the forward flow (black arrows). This is known as out-of-causal order reversibility [33].

Reversible ESs have introduced further questions about the required features of their operational counterpart, suggesting that shared places are not always a suitable choice ([30, 31, 21, 23]). It has been recently shown that the operational model behind (reversible) PESs can be recovered as a subclass of contextual Petri nets, called *(reversible) Causal Nets* (rCNs), in which causality is modelled via inhibitor arcs [21] rather than relying on a shared place in which one transition produces a token to be consumed by another one. Inhibitor arcs neither produce nor consume tokens but check for the *absence* of them. This idea is rendered by the nets in Fig. 2. We recall that a PN gives an operational description of a computation in terms of *transitions* (depicted as boxes) that consume and produce *tokens* (bullets) in different *places* (circles). According to the black arrows in Fig. 2a, the transition a consumes a token from s_1 producing a token in s_4; similarly the transition b consumes from s_4, s_2 and s_3 producing in s_5. Note that b and c are in mutual exclusion (i.e., conflict) because they both compete for the shared resource (i.e., token) in s_3. The arc connecting s_4 to b indicates that b cannot be fired if s_4 does not contains any token; consequently, b can happen only after a has produced the token in s_4. The causal relation between a and b arises because of s_4. N_P in Fig. 2a is the *classical* operational counterpart of the ES P in Fig. 1a. However, we note that both the causalities and conflicts in N_P can be alternatively represented by using inhibitor arcs instead of shared places, as shown in Fig. 2b. The inhibitor arc (depicted as a red line ending with a circle) between s_1 and b models $a < b$, whereas the inhibitor arcs (s_5, c) and (s_6, b) represent the symmetric conflict between b and c. The net in Fig. 2b can be made reversible by adding a reversing transition for each reversible event, as shown in Fig. 2c (in gray). The added transitions $\underline{a}$ and $\underline{b}$ respectively reverse the effects of a and b: each of them consumes (produces) the tokens produced (resp., consumed) by the associated forward transition. Inhibitor arcs are also used for modelling reverse causation and prevention as, e.g., the inhibitor arc connecting $\underline{a}$ with s_6 stands for $\underline{a} \lhd c$.

In this paper we generalise the ideas presented in [21] so to be able to deal with rAESs (and not just with the subclass of reversible PESs). This is achieved by using inhibitor arcs to represent, not only causal dependencies, but also (symmetric) conflicts. In this way, all the dependencies between transitions are modelled *uniformly* by using inhibitor arcs. Concretely, we identify a subclass of rCNs, dubbed *reversible Asymmetric Causal Nets* (rACNs), which are the operational counterpart of rAESs. We show that the correspondence is tight by following the long tradition of comparing concurrency models in categorical terms [29, 39, 40]. To do so we first turn rAESs and rACNs into categories by providing suitable notions of morphisms, then we introduce two functors relating these categories and finally we show that the functor that associates rAESs to rACNs is the left adjoint of the one that that give rACNs out of rAESs. Besides establishing a correspondence between rAESs and rACNs, this allows us to reinterpret the results in [21] categorically.

This paper is an extended and enhanced version of [22]. In this presentation, we have refined the concept of morphisms for causal nets, specifically by relaxing the requirements on inhibitor arcs that capture causality when conflicting transitions are collapsed in the morphism's image. Additionally, we include technical details omitted in the conference paper, and we furnish complete proofs of the results. Additional examples and explanations have been added. Also, Section 5 is completely new. Finally, the whole presentation has been thoroughly refined and improved.

Structure of the paper. This paper is structured as follows: in Section 2 we recall Asymmetric Event Structures and their reversible variants, along with their corresponding categories. In Section 3 Asymmetric Causal Nets, reversible Asymmetric Causal Nets and their categories are presented. The main results about categorical connection are showns in Section 4. Section 5 discusses about some applications of our results. Finally, Section 6 concludes the paper.

2. Reversible Asymmetric Event Structures

In this section we recall the basics of *Asymmetric Event Structures* (AESs) [3] and their reversible version introduced in [32, 11].

2.1. Asymmetric Event Structures.

An AES consists of a set of events and two relations: *causality* ($<$) and *weak causality* or *precedence* ($\nearrow$). If e weakly causes e', written $e \nearrow e'$, then e cannot occur after e'; i.e., if both events occur in a computation, then e precedes e'. In this case we say that e' is in an *asymmetric* conflict with e. Events e and e' are in *(symmetric) conflict*, written $e\#e'$, iff $e \nearrow e'$ and $e' \nearrow e$; intuitively, they cannot take place in the same computation.

Definition 2.1. An *Asymmetric Event Structure* (AES) is a triple $\mathsf{G} = (E, <, \nearrow)$ where

(1) E is a set of *events*;
(2) $< \subseteq E \times E$ is an irreflexive partial order, called *causality*, defined such that $\forall e \in E$. $\lfloor e \rfloor = \{e' \in E \mid e' \leq e\}$ is finite; and
(3) $\nearrow \subseteq E \times E$, called *weak causality*, is defined such that for all $e, e' \in E$:
    (a) $e < e' \;\Rightarrow\; e \nearrow e'$;
    (b) $\nearrow \cap (\lfloor e \rfloor \times \lfloor e \rfloor)$ is acyclic; and
    (c) if $e\#e'$ and $e' < e''$ then $e\#e''$.

Each event can be attributed to a finite set of causes, as specified in Condition 2. Moreover, weak causality is consistent with causality: if e is a cause of e', then e is also a weak cause of e' (as stated in Condition 3a). Additionally, circular dependencies among the causes of an event are precluded by Condition 3b. Finally, (symmetric) conflicts ($\#$) must be inherited along causality, as required by Condition 3c.

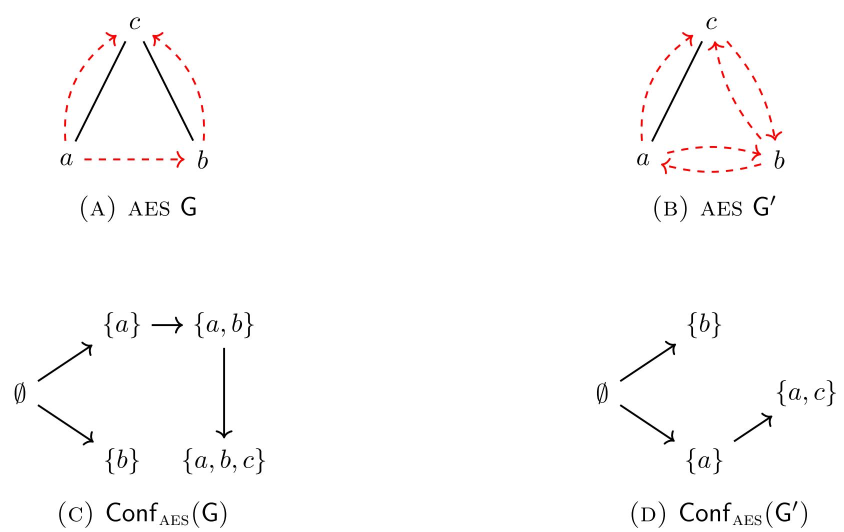


Figure 3. Two AESes and their associated configurations.

Example 2.2. Consider the AES $\mathsf{G} = (E, <, \nearrow)$ depicted in Figure 3a, where the set of events is $E = \{a, b, c\}$, the causality relation is defined such that $a < c$ and $b < c$, and the weak causality is such that $a \nearrow b$, $a \nearrow c$, and $b \nearrow c$. The set E is finite, and hence countable (Condition 1). It is straightforward to check that $<$ is an irreflexive partial order and also that each event possesses a finite set of causes (Condition 2), being $\lfloor a \rfloor = \{a\}$, $\lfloor b \rfloor = \{b\}$, and $\lfloor c \rfloor = \{a, b, c\}$. Additionally, we have that $a < c$ and $a \nearrow c$; and also $b < c$ and $b \nearrow c$, in accordance with Condition 3a. It is immediate to check that $\nearrow$ is acyclic on $\lfloor a \rfloor$, $\lfloor b \rfloor$ and $\lfloor c \rfloor$ (Condition 3b). Moreover, the conflict relation $\#$ is empty, and hence Condition 3c is trivially satisfied.

In the AES $\mathsf{G}' = (E', <', \nearrow')$ of Fig. 3b, the causality relation contains just $a <' c$ and the weak causality is $a \nearrow' b, b \nearrow' a, a \nearrow' c, b \nearrow' c, c \nearrow' b$. In this case $\nearrow'$ induces a symmetric conflict among a and b and one among b and c, hence $b \#' a$ and $b \#' c$, and also the inheritance of conflicts along the causality relation is verified (Condition 3c).

In the AES $\mathsf{G}' = (E', <', \nearrow')$ illustrated in Fig. 3b, the causality relation comprises only $a <' c$, while weak causality is defined by $a \nearrow' b$, $b \nearrow' a$, $a \nearrow' c$, $b \nearrow' c$, and $c \nearrow' b$. In this case, $\nearrow'$ results in a symmetric conflict between a and b, as well as one between b and c. As a result, $b \#' a$ and $b \#' c$, satisfying the inheritance of conflicts along causality (Condition 3c).

Definition 2.3. A *configuration* of an AES $\mathsf{G} = (E, <, \nearrow)$ is a set $X \subseteq E$ of events such that

(1) $\nearrow$ is well-founded on X;
(2) $\forall e \in X.\ \lfloor e \rfloor \subseteq X$; and
(3) $\forall e \in X$ the set $\{e' \in X \mid e' \nearrow e\}$ is finite.

The set of configurations of G is denoted by $\mathsf{Conf}_{\text{AES}}(\mathsf{G})$.

A configuration comprises a set of events representing a potential partial execution of G. Condition 1 ensures that events in X are not in conflict, as circular weak-causal dependencies are forbidden. In accordance with Condition 2, X contains all the causes of its constituent events: an event may occur only if its causes have already occurred. While a configuration may be infinite, as in a non-terminating execution, each event within a configuration has a finite set of weak causes (Condition 3).

For any pair of sets $X, Y \subseteq E$ where $X \subseteq Y$, we say that Y *extends* X if, for all $e \in X$ and $e' \in Y \setminus X$, there is no weak-causal dependency ($\neg(e' \nearrow e)$). In the context of configurations, if X and Y are configurations, then Y is considered reachable from X.

Example 2.4. The sets of configurations of the AESs G and G' introduced in Example 2.2 are shown in Figs. 3c and 3d respectively. The arrows represent the *extends* relation. Straightforwardly configurations satisfy the conditions in Definition 2.3.

A (finite) configuration $X = \{e_1, \ldots, e_n, \ldots\}$ of an AES G is *reachable* whenever the events in X can be ordered in such a way that for each $j \geq 1$ $\{e_1, \ldots, e_j\}$ extends $\{e_1, \ldots, e_{j-1}\}$ and each $\{e_1, \ldots, e_j\} \in \mathsf{Conf}_{\text{AES}}(\mathsf{G})$.

Definition 2.5. Let $\mathsf{G}_0 = (E_0, <_0, \nearrow_0)$ and $\mathsf{G}_1 = (E_1, <_1, \nearrow_1)$ be AESs. An AES-morphism, denoted as $f : \mathsf{G}_0 \to \mathsf{G}_1$, is defined as a partial function $f : E_0 \to E_1$ such that for all $e, e' \in E_0$

(1) if $f(e) \neq \bot$ then $\lfloor f(e) \rfloor \subseteq f(\lfloor e \rfloor)$; and
(2) if $f(e) \neq \bot \neq f(e')$ then
   (a) $f(e) \nearrow_1 f(e')$ implies $e \nearrow_0 e'$; and
   (b) $f(e) = f(e')$ and $e \neq e'$ imply $e \#_0 e'$.

An AES-morphism preserves the causes of each mapped event (Condition 1) and reflects its weak causes (Condition 2a). Additionally, it allows the mapping of two distinct events to the same event only when those events are in conflict (Condition 2b). The aforementioned conditions collectively guarantee that morphisms preserve computations, or configurations, as stated below.

Proposition 2.6. *Let $f : \mathsf{G}_0 \to \mathsf{G}_1$ be an AES-morphism and X a configuration of G_0, i.e., $X \in \mathsf{Conf}_{\text{AES}}(\mathsf{G}_0)$. Then, $f(X) \in \mathsf{Conf}_{\text{AES}}(\mathsf{G}_1)$.*

Proof. See [3, Lemma 3.6]. □

Since AES-morphisms compose [3], AESs and AES-morphisms turn into a category, which we denote by **AES**.

2.2. **Reversible AESs.** We now provide a summary of *reversible* AESs, introduced in [32, 11] as the reversible counterparts ofAESs. Given a set U of events and $u \in U$, we write $\underline{u}$ for the undoing of u, and $\underline{U} = \{\underline{u} \mid u \in U\}$ for the set of *undoings* of U.

Definition 2.7. A *Reversible Asymmetric Event Structure* (rAES) is a sextuple $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ where E is the set of events, $U \subseteq E$ is the set of *reversible* events, and

(1) $\nearrow \subseteq E \times E$, called *weak causality*;
(2) $\lhd \subseteq \underline{U} \times E$, called *prevention*;
(3) $< \subseteq E \times E$, called *causation*, is an irreflexive relation defined such that for all $e \in E$, $\lfloor e \rfloor_< = \{e' \in E \mid e' \leq e\}$ is finite and $(\nearrow \cup <)$ is acyclic on $\lfloor e \rfloor_<$;
(4) $\prec \subseteq E \times \underline{U}$, called *reverse causation*, is defined such that
   (a) for all $u \in U$. $u \prec \underline{u}$;
   (b) for all $u \in U$, $\lfloor \underline{u} \rfloor_\prec = \{e \in E \mid e \prec \underline{u}\}$ is finite and $(\nearrow \cup <)$ is acyclic on $\lfloor \underline{u} \rfloor_\prec$;
(5) for all $e \in E, \underline{u} \in \underline{U}$. $e \prec \underline{u} \Rightarrow \neg(\underline{u} \lhd e)$; and
(6) $(E, \prec\!\!\prec, \nearrow)$ with $\prec\!\!\prec = < \cap \{(e, e') \mid e \notin U \textit{ or } \underline{e} \lhd e'\}$ is an AES.

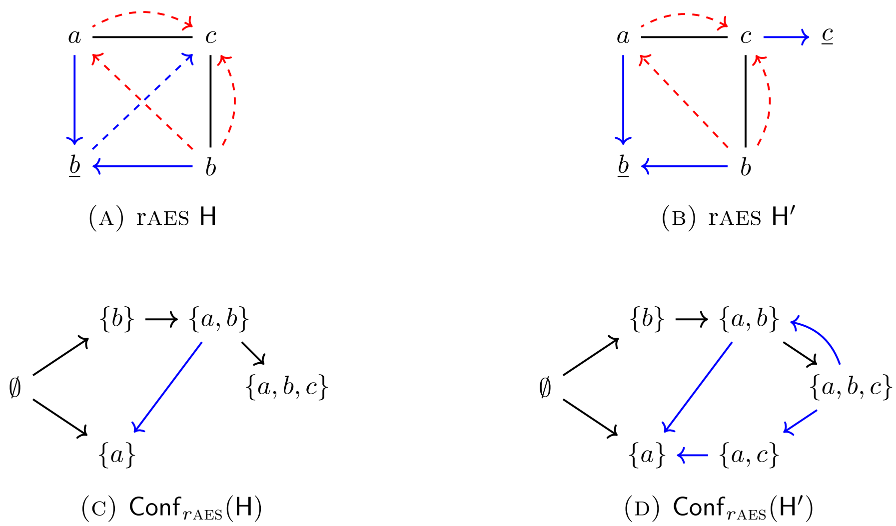


Figure 4. Two rAESes and their associated configurations.

An rAES is defined in terms of a set of events E, with those in U being reversible.

Causation ($<$) and weak causality ($\nearrow$) delineate the forward flow, while reverse causation ($\prec$, depicted as solid blue arrows) and prevention ($\lhd$, represented by dashed blue arrows) describe the backward flow. Weak causality serves the same role as in AESs, wherein $e \nearrow e'$ asserts that event e cannot occur after event e'. Prevention, on the other hand, governs the undoing of events: $\underline{e} \lhd e'$ indicates that event e cannot be undone if event e' has occurred. Similar to causality in AESs, causation indicates causal dependencies. Consistent with AESs, each event has the finite set of causes $\lfloor e \rfloor_{<}$, and these sets do not contain cyclic dependencies induced by causation and weak causality (Condition 3). The acyclicity of $\lfloor e \rfloor_{<}$ implies that for all $e, e' \in E$, if $e < e'$, then $\neg(e' \nearrow e)$. Reverse causation specifies the causes for the undoing of each event: $e \prec \underline{e'}$ states that e' can be undone only if e has occurred. Hence, condition 4a simply states that an event u can be undone only if it has occurred. Condition 4b, which is analogous to Condition 3, establishes that each undoing has a finite set of causes. Reverse causation precisely states the causes for undoing each event: $e \prec \underline{e'}$ asserts that event e' can be undone only if event e has occurred. Therefore, Condition 4a straightforwardly defines that an event u can be undone only if it has occurred. Condition 4b, akin to Condition 3, establishes that each undoing has a finite set of causes. Just as causation and weak causality demand consistency, wherein an event cannot have precedence over some of its causes (as defined in Condition 3), a similar requirement is imposed on the backward flow by Condition 5. Specifically, reverse causation must align with prevention: a cause e of the undoing of u (i.e., $e \prec \underline{u}$) cannot prevent the undoing, i.e., $\neg(\underline{u} \lhd e)$. The final condition is the most intricate. It's noteworthy that the definition of rAES does not mandate $(E, <, \nearrow)$ to be an AES, particularly because conflicts may not be inherited along causation. This flexibility is crucial for accommodating out-of-causal-order reversibility (see the example below). Condition 5 considers instead the relation $\prec\!\!\prec$, dubbed *sustained causation*. The sustained causation is derived by excluding from causation all pairs $e < e'$ where event e can be undone even when e' has occurred (i.e., $\underline{e} \lhd e'$ does not hold). It's worth noting that $\prec\!\!\prec$ coincides with $<$ when $U = \emptyset$; thus, $(E, \emptyset, <, \nearrow, \emptyset, \emptyset)$ indeed constitutes an AES.

Example 2.8. Consider $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ as depicted in Figure 4a. The (forward) events are $\{a, b, c\}$, where b is the only reversible one ($U = \{b\}$). *Causation* is defined such that $a < c$ and $b < c$, and weak causality states $a \nearrow c, b \nearrow c, b \nearrow a$. *Reverse causation* is such that $a \prec \underline{b}$ and $b \prec \underline{b}$, and prevention is $\underline{b} \lhd c$. Note that c is caused by a and b, with b weakly causing a (or a being in asymmetric conflict with b). Moreover, b can be reversed only when a is present and c has not been executed. In this case, *sustained causation* coincides with causation because the only reversible event b cannot be reversed if c (which causally depends on b) is present. It is routine to check that $(E, \prec\!\!\prec, \nearrow)$ is an AES, and H is an rAES.

Consider $\mathsf{H}' = (E, U', <, \nearrow, \prec', \lhd')$ shown in Fig. 4b, which has the same set of events, causation, and weak causality as H but also takes c as reversible (i.e., $U' = \{b, c\}$). Reverse causation is extended with the pairs $c \prec' \underline{c}$, $a \prec' \underline{b}$, and $b \prec' \underline{b}$. Observe that b can be reversed even if the event c (which depends on b) has occurred. This is known as out-of-causal-order reversibility. In this case, sustained causation consists only of $a \prec\!\!\prec' c$, i.e., the pair $b < c$ is removed because b can be reversed despite c (which causally depends on b) having occurred. It can be checked that $(E, \prec\!\!\prec', \nearrow)$ is an AES, and H' is an rAES.

Remarks 2.9. For the sake of the presentation, Definition 2.7 deviates in style from the original definition in [32], where causation and reverse causation are merged, and weak causality and prevention are combined in a single relation. Additionally, we explicitly stipulate that $(E, \prec\!\!\prec, \nearrow)$ must constitute an AES, eliminating the need to restate conditions. Further discussion is available in the Appendix.

The definition of configurations in rAESs has an operational flavor grounded in the concept of enabling. Consider $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ as an rAES, and let $X \subseteq E$ be a set of events such that $\nearrow$ is acyclic on X. For $A \subseteq E$ and $B \subseteq U$, we say that $A \cup \underline{B}$ is enabled at X if $A \cap X = \emptyset$, $B \subseteq X$, $\nearrow$ is acyclic on $A \cup X$, and

(1) for every $e \in A$, if $e' < e$ then $e' \in X \setminus B$ and if $e \nearrow e'$ then $e' \notin X \cup A$; and
(2) for every $u \in B$, if $e' \prec \underline{u}$ then $e' \in X \setminus (B \setminus \{u\})$ and if $\underline{u} \lhd e'$ then $e' \notin X \cup A$.

The first condition ensures that X contains all the causes of the events to be added (i.e., those in A) but none of their preventing events. The second condition asserts that X includes the reverse causes of the events to be undone (i.e., those in B) but none of the preventing ones. If $A \cup \underline{B}$ is *enabled* at X, then $X' = (X \setminus B) \cup A$ can be reached from X, denoted as $X \xrightarrow{A \cup \underline{B}} X'$. The initial condition above can be restated as follows: $\forall e \in A. \lfloor e \rfloor_< \subseteq X$ and $X \cup A$ extends X.

Definition 2.10. Let $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ be an rAES and $X \subseteq E$ a set of events that is well-founded with respect to $(\nearrow \cup <)$. Then, X is a *(reachable) configuration* if there exist two sequences of sets A_i and B_i, for $i = 1, \dots, n$, such that

(1) $A_i \subseteq E$ and $B_i \subseteq U$ for all i, and
(2) $X_i \xrightarrow{A_i \cup \underline{B_i}} X_{i+1}$ for all i, and $X_1 = \emptyset$ and $X_{n+1} = X$.

The set of configurations of H is denoted by $\mathsf{Conf}_{r\mathrm{AES}}(\mathsf{H})$.

Example 2.11. The configurations of the rAESs in Example 2.8 (along with the steps to reach them) are illustrated in Figures 4c and 4d. Notice that $\{a, c\}$—a configuration of H' but not of H—is reached from the configuration $\{a, b, c\}$ by undoing b.

Definition 2.12. Let $\mathsf{H}_i = (E_i, U_i, <_i, \nearrow_i, \prec_i, \lhd_i)$ with $i = 0, 1$ be two rAESs. An rAES-morphism $f : \mathsf{H}_0 \to \mathsf{H}_1$ satisfies the conditions of an AES-morphism $f : (E_0, <_0, \nearrow_0) \to (E_1, <_1, \nearrow_1)$ such that

(1) $f(U_0) \subseteq U_1$;
(2) for all $u \in U_0$, if $f(u) \neq \bot$ then $\lfloor \underline{f(u)} \rfloor_{\prec_1} \subseteq f(\lfloor \underline{u} \rfloor_{\prec_0})$; and
(3) for all $e \in E_0$ and $u \in U_0$, if $f(e) \neq \bot \neq f(u)$ then $\underline{f(u)} \lhd_1 f(e) \ \Rightarrow \underline{u} \lhd_0 e$.

Observe that $(E_0, <_0, \nearrow_0)$ and $(E_1, <_1, \nearrow_1)$ are not necessarily AESs, but here it is enough that f works on causation and weak causality as an AES-morphism. Notice that rAES-morphisms preserve the causes (and the reverse causes) of each event (resp., reversing event), and reflect preventions.

Proposition 2.13. *Let $f : \mathsf{H}_0 \to \mathsf{H}_1$ be an* rAES*-morphism and let $X \in \mathsf{Conf}_{r\text{AES}}(\mathsf{H}_0)$. Then $f(X) \in \mathsf{Conf}_{r\text{AES}}(\mathsf{H}_1)$.*

Proof. It suffices to demonstrate that, given $X \subseteq E_0$ such that $\nearrow_0$ is acyclic on X, if $A \cup \underline{B}$ is enabled at X (with $A \subseteq E_0$ and $B \subseteq U_0$), and $X \xrightarrow{A \cup \underline{B}} X'$ (with $X' = (X \setminus B) \cup A$), then $f(A) \cup \underline{f(B)}$ is enabled at $f(X)$. Observe that $f(A) \cap f(X) = \emptyset$ since $A \cap X = \emptyset$ and $f(B) \subseteq f(X)$ due to $B \subseteq X$. We now prove that $\nearrow_1$ is acyclic on $f(X \cup A) = f(X) \cup f(A)$ by contradiction. Suppose there exists a sequence of events $f(e_0), \ldots, f(e_n)$ in $f(X \cup A)$ such that $f(e_i) \nearrow_1 f(e_{i+1})$, with $0 \leq i < n$, and $f(e_n) \nearrow_1 f(e_0)$. However, as $f : (E_0, <_0, \nearrow_0) \to (E_1, < 1, \nearrow_1)$ is an AES-morphism, we have $e_i \nearrow_0 ei + 1$ for all $i \in \{0, n-1\}$, and $e_n \nearrow_0 e_0$, contradicting the acyclicity of $\nearrow_0$ on $X \cup A$.

Now consider $e \in A$ such that $f(e)$ is defined and take $e' <_0 e$ such that $f(e')$ is defined. As f is an AES-morphism, if $e' \in X \setminus B$, then $f(e') \in f(X) \setminus f(B)$, and if $f(e) \nearrow_1 f(e')$, we have $e \nearrow_0 e'$, implying that $e' \notin X \cup A$. Therefore, $f(e') \notin f(X) \cup f(A)$.

Finally, consider $u \in B$ and assume $f(u)$ is defined. Take $e' \prec_0 u$. If $f(e')$ is defined, and since $\lfloor \underline{f(u)} \rfloor_{\prec_1} \subseteq f(\lfloor \underline{u} \rfloor_{\prec_0})$, we have that $f(e') \in f(X) \setminus (f(B) \setminus \{f(u)\})$ because $e' \in X \setminus (B \setminus \{u\})$. Now, consider $\underline{f(u)} \lhd_1 f(e')$, which implies $f(e') \notin f(X) \cup f(A)$ because $\underline{f(u)} \lhd_1 f(e')$ implies $u \lhd_0 e'$ and $e' \notin X \cup A$. Hence $f(X) \xrightarrow{f(A) \cup \underline{f(B)}} f(X')$. Observing that configurations are subsets of events reachable from the empty one, we establish the thesis. ☐

As shown in [11], rAES-morphisms compose; hence rAESs and rAES-morphisms are a category, denoted with **RAES**. Moreover, **AES** is a full and faithful subcategory of **RAES**.

2.3. **Constructions.** We conclude the section showing a categorical construction for rAES. The category of rAESs has coproduct, as shown in [11]. We recall here the construction adapting it to our definition of rAES which is equivalent to the one in [32] and [11], as we discuss in the appendix.

Proposition 2.14. *Let $\mathsf{H}_0 = (E_0, U_0, <_0, \nearrow_0, \prec_0, \lhd_0)$ and $\mathsf{H}_1 = (E_1, U_1, <_1, \nearrow_1, \prec_1, \lhd_1)$ be two* rAES*s. Then $\mathsf{H}_0 + \mathsf{H}_1 = (E, U, <, \nearrow, \prec, \lhd)$ where*

- *$E = \{0\} \times E_0 \cup \{1\} \times E_1$;*
- *$U = \{0\} \times U_0 \cup \{1\} \times U_1$;*
- *$(i, e) < (j, e')$ whenever $i = j$ and $e <_i e'$;*
- *$(i, e) \nearrow (j, e')$ whenever either $i = j$ and $e \nearrow_i e'$ or $i \neq j$;*
- *$(i, e) \prec (j, u)$ whenever $i = j$ and $e \prec_i u$; and*
- *$(i, u) \lhd (j, e)$ whenever either $i = j$ and $u \lhd_i e$ or $i \neq j$,*

is their coproduct *and $\iota_i : \mathsf{H}_i \to \mathsf{H}_0 + \mathsf{H}_1$ defined as $\iota_i(e) = (i, e)$ are the injections.*

Proof. $\mathsf{H}_0 + \mathsf{H}_1$ is clearly a rAES. To show that it is indeed a coproduct, we consider any other rAES $\mathsf{H}' = (E', U', <', \nearrow', \prec', \lhd')$ together with two morphisms $f_i : \mathsf{H}_i \to \mathsf{H}'$. We now show that the mapping $g : \mathsf{H}_0 + \mathsf{H}_1 \to \mathsf{H}'$ defined as $g(i, e) = f_i(e)$ is a rAES-morphism and it is unique. Clearly $\lfloor g(i, e)\rfloor_{<'} = \lfloor f_i(e)\rfloor_{<'} \subseteq f_i(\lfloor e\rfloor_{<_i})$ as f_i is a rAES-morphism, moreover if $g(i, e)$ and $g(j, e')$ are defined and $g(i, e) \nearrow' g(j, e')$ we have two cases: either $i = j$ and then $g(i, e) = f_i(e)$, $g(j, e') = f_i(e')$ and $f_i(e) \nearrow f_i(e')$ is because $e \nearrow_i e'$, or $i \neq j$, but in this case by construction we have $g(i, e) \nearrow g(j, e')$. Assume now that $g(i, e)$ and $g(j, e')$ are defined and equal. There are two cases: if $i = j$ then $f_i(e)\#f_i(e')$ and $e\#_ie'$, and if $i \neq j$ then we have again $g(i, e) \nearrow g(j, e')$ and $g(j, e') \nearrow g(i, e)$ which implies $g(i, e)\#g(j, e')$. Turning to the reverse part, clearly $g(\{0\}\times U_0 \cup \{1\}\times U_1) = g(\{0\}\times U_0) \cup g(\{1\}\times U_1) = f_0(U_0) \cup f_1(U_1) \subseteq U'$, and $\lfloor \underline{g(i, u)}\rfloor_{\prec'} = \lfloor \underline{f_i(u)}\rfloor_{\prec'} \subseteq f_i(\lfloor \underline{u}\rfloor_{\prec})$ being f_i a rAES-morphism, and for $g(i, u) \lhd' g(j, e')$ the reasoning goes as for $\nearrow'$, hence $(i, u) \lhd (j, e')$, as required. Uniqueness follows from the fact that the events are a coproduct in the category of sets and partial mapping. □

3. Reversible Asymmetric Causal Nets

We now introduce Asymmetric Causal Nets as a subclass of nets with inhibitor arcs, wherein standard notions of causality and conflicts among transitions are modelled via inhibitor arcs.

Firstly, we revisit the notion of nets with inhibitor arcs and subsequently introduce asymmetric causal nets along with their reversible versions. We develop appropriate notions of morphisms, turning asymmetric causal nets and reversible asymmetric causal nets into categories.

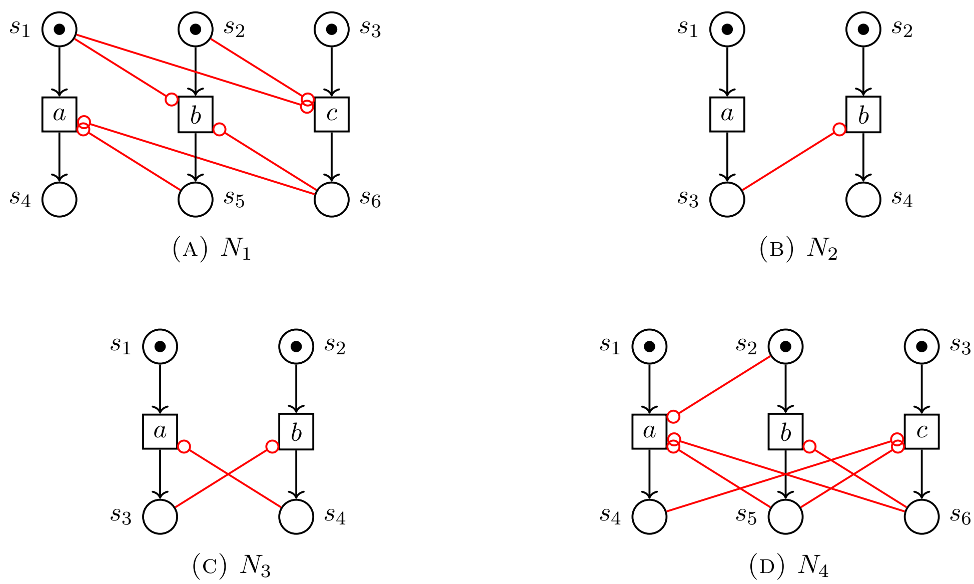
3.1. Nets with inhibitor arcs.

We summarise the basics of Petri net with inhibitor arcs along the lines of [28, 2]. We write $\mathbb{N}$ for the set of natural numbers. A *multiset* over a set A is a function $m : A \to \mathbb{N}$. We assume the usual operations of union ($+$) and difference ($-$) on multisets, and write $m \subseteq m'$ if $m(a) \leq m'(a)$ for all $a \in A$. We will use $\{\!|\ldots|\!\}$ when enumerating the elements of a multiset. The multiset $[\![m]\!]$ is defined such that $[\![m]\!](a) = 1$ if $m(a) > 0$ and $[\![m]\!](a) = 0$ otherwise. We often confuse a multiset m with the set $\{a \in A \mid m(a) \neq 0\}$ when $m = [\![m]\!]$. In such cases, $a \in m$ denotes $m(a) \neq 0$, and $m \subseteq A$ signifies that $m(a) = 1$ implies $a \in A$ for all a. The underlying set of a multiset m, namely the one formed by the elements a with $m(a)$, is precisely $[\![m]\!]$. Additionally, we will employ standard set operations like $\cap$, $\cup$, or $\setminus$. The set of all multisets over A is denoted as μA; the symbol $\mathsf{0}$ stands for the unique multiset such that $[\![\mathsf{0}]\!] = \emptyset$.

Definition 3.1. A *Petri net with inhibitor arcs* (IPT for short) is a tuple $N = \langle S, T, F, I, \mathsf{m}\rangle$, where S is a set of *places*, T is a set of *transitions* such that $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow* relation, $I \subseteq S \times T$ is the *inhibiting* relation, and $\mathsf{m} \in \mu S$ is the *initial marking*.

Given an IPT $N = \langle S, T, F, I, \mathsf{m}\rangle$ and $x \in S \cup T$, the *pre-* and *postset* of x are respectively defined as the (multi)sets ${}^\bullet x = \{y \mid (y, x) \in F\}$ and $x^\bullet = \{y \mid (x, y) \in F\}$. If $x \in S$ then ${}^\bullet x \in \mu T$ and $x^\bullet \in \mu T$; analogously, if $x \in T$ then ${}^\bullet x \in \mu S$ and $x^\bullet \in \mu S$. The *inhibitor set* of a transition t is the (multi)set ${}^\circ t = \{s \mid (s, t) \in I\}$. The definition of ${}^\bullet\cdot, \cdot^\bullet, {}^\circ\cdot$ generalise straightforwardly to multisets of transitions.

Example 3.2. Figure 5 introduces some IPTs. The IPT N_1 in Figure 5a has six places (named s_i) and three transitions a, b, and c. The initial marking is $\mathsf{m} = \{s_1, s_2, s_3\}$. For instance, the transition b consumes a token from s_2 and produces a token in s_5 and it is



(a) N_1 (b) N_2 (c) N_3 (d) N_4

Figure 5. Some IPTs.

inhibited by s_1, meaning its pre-, post-, and inhibiting sets are ${}^\bullet b = \{s_2\}$, $b^\bullet = \{s_5\}$, and ${}^\circ b = \{s_1\}$, respectively.

A (multiset of) transition(s) $A \in \mu T$ is *enabled at a marking* $m \in \mu S$, written $m\,[A\rangle$, if ${}^\bullet A \subseteq m$, ${}^\circ A \cap [\![m]\!] = \emptyset$ and $\forall t \in [\![A]\!].\ {}^\circ t \cap (A - \{t\})^\bullet = \emptyset$. Intuitively, A is enabled at m if m contains the tokens to be consumed by A (${}^\bullet A \subseteq m$) and none of the transitions in A is inhibited in m (${}^\circ A \cap [\![m]\!] = \emptyset$). The last condition avoids cases in which a transition in A produces tokens that inhibit other transition in A. Observe that the multiset $\mathsf{0}$ is enabled at every marking. If A is enabled at m, then it can *fire* and its firing produces the marking $m' = m - {}^\bullet A + A^\bullet$, which is written $m\,[A\rangle\,m'$. Hereafter, we assume that each transition t is defined such that ${}^\bullet t \neq \emptyset$, i.e., it cannot fire *spontaneously* without consuming tokens.

A marking m is *reachable* if there exists a sequence of firings $m_i\,[A_i\rangle\,m_{i+1}$ originated in the initial marking and leading to m; $\mathcal{M}_N$ stands for the set of reachable markings of N. An IPT N is *safe* if every reachable marking is a set, i.e., $\forall m \in \mathcal{M}_N.m = [\![m]\!]$. From now on, we will only consider safe IPTs.

Example 3.3. Consider the IPT N_4 in Figure 5d. Both b and c are enabled at marking $m = \{s_1, s_2, s_3\}$. On the contrary, a is not enabled because it is inhibited by the token in the place s_2. The firing of b on m produces the marking $m' = \{s_1, s_3, s_5\}$, i.e. $m\,[b\rangle\,m'$. The transition a is enabled at m' while c is disabled and cannot be fired because of the token in s_5. The firing of a on m' produces $m'' = \{s_3, s_4, s_5\}$. The reachable markings of N_4 are $\{s_1, s_2, s_3\}, \{s_1, s_3, s_5\}$ $\{s_1, s_2, s_6\}$ and $\{s_3, s_4, s_5\}$.

3.2. **Asymmetric Causal Nets.** In this section, we introduce Asymmetric Causal Nets, a class of IPTs that generalises the concept of Causal Nets introduced in [21] to accommodate asymmetric conflicts. Broadly speaking, we focus on IPTs where dependencies between transitions arise solely from *inhibitor* arcs. Similar to causal nets, the property $t^\bullet \cap {}^\bullet t' = \emptyset$ holds for all transitions t and t'. This means that if a place appears in the preset of a transition, it does not appear in the postset of any transition, and vice versa. Consequently, the flow relation induces an empty causal relation. However, causality can be recovered through inhibitor arcs. Intuitively, a transition t connected via an inhibitor arc to some place in the preset of another transition t' cannot be fired before t' if we assume that the preset of t' is marked. This is exemplified by transitions a and b in Figure 5d, where a can only be fired after b. The induced (immediate) causality relation $\lessdot$ is defined by $t \lessdot t'$ if and only if ${}^\bullet t \cap {}^\circ t' \neq \emptyset$, signifying that the firing of t consumes (at least) one of the tokens inhibiting the firing of t'.

Additionally, asymmetric causal nets impose that places should not be shared between the presets and postsets of transitions. Formally, $t^\bullet \cap t'^\bullet \neq \emptyset \ \vee \ {}^\bullet t \cap {}^\bullet t' \neq \emptyset$ implies $t = t'$ for all transitions t and t'. This ensures that the flow relation does not introduce forward or backward conflicts, which need to be recovered from inhibitor arcs. Note that a transition t inhibited by some place in the postset of another transition t' cannot be fired if t' has been fired, i.e., t' prevents t. This is exemplified by transitions b and a in Figure 5b, where b cannot be fired after a (i.e., b is prevented by a). Therefore, the induced prevention relation $\leftsquigarrow$ is defined by $t \leftsquigarrow t'$ if and only if $t^\bullet \cap {}^\circ t' \neq \emptyset$. We use $\rightsquigarrow$ to denote the inverse of $\leftsquigarrow$. It is important to note that $\rightsquigarrow$ is analogous to the weak causality of AESes: if $t' \rightsquigarrow t$, then $t^\bullet \cap {}^\circ t' \neq \emptyset$, indicating that t' cannot be fired if t has been fired; however, t can be fired after t'. Similar to AES, *symmetric* conflicts are retrieved from prevention. That is, transitions t and t' are in *symmetric* conflict, denoted as $t \natural t'$, whenever $t \leftsquigarrow t'$ and $t \rightsquigarrow t'$.

Definition 3.4. Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be an IPT. C is a *pre asymmetric causal net* (pACN) if the following conditions hold:

(1) $\forall t, t' \in T.\ t^\bullet \cap {}^\bullet t' = \emptyset$;
(2) $\forall t \in T.\ |{}^\bullet t| = |t^\bullet| = 1$; $\mathsf{m} = S \setminus T^\bullet$; and ${}^\circ T \subseteq {}^\bullet T \cup T^\bullet$;
(3) $\lessdot^+$ is a partial order;
(4) $\forall t \in T.\ \lfloor t \rfloor_\lessdot = \{t' \in T \mid t' \lessdot^* t\}$ is finite and $(\rightsquigarrow \cup \lessdot)$ is acyclic on $\lfloor t \rfloor_\lessdot$; and
(5) $\forall t, t' \in T.$ if ${}^\bullet t \cap {}^\circ t' \neq \emptyset$ then $t^\bullet \cap {}^\circ t' = \emptyset$ and if $t^\bullet \cap {}^\circ t' \neq \emptyset$ then ${}^\bullet t \cap {}^\circ t' = \emptyset$.

The first condition asserts that causal dependencies in a pACN are not established through the flow relation. The second condition stipulates that each transition has just one place in its preset and one place in its postset, and places that are either isolated or part of the presets of all transitions are initially marked. The condition on inhibitor arcs, denoted as ${}^\circ T \subseteq {}^\bullet T \cup T^\bullet$, specifies that these arcs should not link transitions to isolated places. This is because their purpose is to represent dependencies between transitions that are associated via the flow relation to those places. Since $\lessdot$ is meant to model causal dependencies, the third condition requires its transitive closure $\lessdot^+$ to form a partial order. The fourth condition mandates that each transition t has a finite set of causes, ensuring that $\lessdot^+$ is a well-founded partial order. By enforcing the acyclicity of $(\rightsquigarrow \cup \lessdot)$ on the causes of every transition t, we ensure that causes of each transition can be ordered to satisfy both causality and prevention. This constraint excludes situations where (i) prevention contradicts causality (e.g., $t \lessdot t'$ and $t' \rightsquigarrow t$), (ii) circular chains of prevention exist (i.e., $t_0 \rightsquigarrow t_1 \rightsquigarrow \ldots \rightsquigarrow t_n \rightsquigarrow t_0$), where symmetric conflicts are a particular case, and (iii) self-blocked transitions occur (i.e., $\lessdot$ needs to be irreflexive, implying ${}^\bullet t \cap {}^\circ t = \emptyset$ for all transitions t). The last condition simply says that a transition cannot depends on one that prevents it. It is worth to stress that, though in general markings are multisets, in our case they are indeed sets.

Definition 3.5. A pACN $C = \langle S, T, F, I, \mathsf{m}\rangle$ is an *asymmetric causal net* (ACN) if

(1) for all $t, t' \in T$. $t \lessdot^+ t'$ implies $t \lessdot t'$;
(2) for all $t, t' \in T$. $t \lessdot t'$ implies $t \rightsquigarrow t'$; and
(3) for all $t, t', t'' \in T$, $t \natural t' \wedge\ t' \lessdot t''$ imply $t \natural t''$.

The additional conditions imposed on ACNs entails that all dependencies among transitions must be explicitly represented in the structure of the net; in other words, causality is saturated (Condition 1) and that all conflicts must be explicitly represented in the net's structure, ensuring the presence of inhibitor arcs for all inherited symmetric conflicts (Condition 3). Condition 2 simply states that it ${}^\bullet t \cap {}^\circ t' \neq \emptyset$ then $t'^\bullet \cap {}^\circ t \neq \emptyset$. The inhibitor arc (s, t) with $\{s\} = t'^\bullet$ is somehow superflous, but it follow the intuition that in an AES causality implies weak causality.

Example 3.6. The IPTs depicted in Figure 5 satisfy the conditions of ACNs. The first two conditions of Definition 3.4 are met by all four nets, as transitions do not share places in their pre and postsets. Additionally, the places in the presets of all transitions are the only ones initially marked. For N_1 (Figure 5a), we observe that $a \lessdot b$ and $b \lessdot c$, making $\lessdot^+$ a total order, and causality is saturated with the inclusion of $a \lessdot c$. Furthermore, $\leftsquigarrow$ is empty (as no transitions have inhibitor arcs connected to the postset of other transitions), thus satisfying the fourth condition. In the case of the net N_2 (Figure 5b), the causality relation is empty, while prevention contains the sole pair $a \leftsquigarrow b$: b cannot be fired after a, but a can be executed after b. Figure 5c presents a similar net where a and b are in symmetric conflict ($a \natural b$), meaning the execution of one prevents the other. Finally, Figure 5d illustrates a net where $b \lessdot a$, $b \natural c$, and $a \natural c$, with conflicts being inherited along the causality relation $\lessdot$.

3.3. **Configurations of a (pre) Asymmetric Causal Nets.** We introduce additional concepts and findings related to (pre) ACNs, in line with our previous work in [21].

Given a function $f : A \to B$, the domain of f is defined as $dom(f) = \{a \in A \mid \exists b \in B.\ f(a) = b\}$. A sequence of elements in A is a (possibly partial) mapping $\rho : \mathbb{N} \to A$ defined such that $n \in dom(\rho)$ implies $n' \in dom(\rho)$ for all $n' < n$. Its length, denoted by $len(\rho)$, is defined as the cardinality of its domain, i.e., $len(\rho) = |dom(\rho)|$. We say that ρ is finite when its length is so (i.e., $len(\sigma) < \infty$). We often write a sequence ρ as $a_1 a_2 \cdots$ where $a_i = \rho(i)$. Note that all elements in a sequence are distinct if the the mapping is injective.

A *firing sequence* σ (abbreviated as $\mathsf{fs}\ \sigma$) of an IPT $N = \langle S, T, F, I, \mathsf{m}\rangle$ is a sequence of markings defined such that for each $i \in dom(\sigma)$ there is a multiset of transitions A_i enabled at $\sigma(i)$ and $\sigma(i)\,[A_i\rangle\,\sigma(i+1)$. A $\mathsf{fs}\ \sigma$ is written as $m_0\,[A_0\rangle\,m_1\ \cdots\ m_n\,[A_n\rangle\,m_{n+1}\ \cdots$. Additionally, $start(\sigma)$ indicates its initial marking $\sigma(0) = m_0$. If σ is finite, $lead(\sigma)$ designates its final marking $\sigma(len(\sigma))$. We say that σ starts at a marking m if $start(\sigma) = m$, and let $\mathcal{R}^N_m$ denote the set of all firing sequences of the IPT N that start at m. A marking m is *reachable* in N if there exists an $\mathsf{fs}\ \sigma \in \mathcal{R}^N_{\mathsf{m}}$ such that $m = lead(\sigma)$. The set of all reachable markings of N is $\mathcal{M}_N = \{lead(\sigma) \mid \sigma \in \mathcal{R}^N_{\mathsf{m}}\}$.

Definition 3.7. Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be a pACN. A set of transitions $X \subseteq T$ is a *configuration* of C if

(1) $\forall t \in X.\ \lfloor t\rfloor_{\lessdot} \subseteq X$ (i.e., X is *left closed* with respect to $\lessdot$); and
(2) $\rightsquigarrow \cup \lessdot$ is acyclic on X.

The set of configurations of a pACN C is denoted by $\mathsf{Conf}_{\text{pACN}}(C)$.

The conditions imposed in the definition of ACN ensure that $\leadsto = \; \leftharpoondown\!\!\leadsto$. In such case, the second condition can be alternatively expressed as $\forall t, t' \in X.\ \neg(t \natural t')$.

If $X \in \mathsf{Conf}_{p\text{ACN}}(C)$, then the transitions of X can be partially ordered with respect to $\lessdot \cup \; \leftharpoondown\!\!\leadsto$. This means that there exists a sequence $t_1, \ldots, t_n$ of the transitions in X such that $t_i \lessdot t_j$ or $t_i \leadsto t_j$ imply $i < j$.

Proposition 3.8. *Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be an p*ACN *and let $X \subseteq T$ a* configuration *of C. Then X can be partially ordered with respect to $\leadsto \cup \lessdot$.*

Proof. Take $X \in \mathsf{Conf}_{p\text{ACN}}(C)$. As X is acyclic with respect to $\leadsto \cup \lessdot$, then we have that $(\leadsto \cup \lessdot)^* \cap (X \times X)$ is a partial order and then we have the thesis. □

It is important to note a close correspondence between the configurations of a ACN and its reachable markings: any reachable marking corresponds to a configuration of the net, and conversely.

Proposition 3.9. *Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be an p*ACN*. Then,*

(1) *if $m' \in \mathcal{M}_C$ then ${}^\bullet m' \in \mathsf{Conf}_{p\text{ACN}}(C)$; and*
(2) *if $X \in \mathsf{Conf}_{p\text{ACN}}(C)$ is a reachable configuration then $\mathsf{m} - {}^\bullet X + X^\bullet \in \mathcal{M}_C$.*

Proof.

(1) If $m' \in \mathcal{M}_C$, then there exists a firing sequence σ such that $m' = \mathit{lead}(\sigma)$. We prove by induction on the length of the firing sequence that ${}^\bullet m' \in \mathsf{Conf}_{p\text{ACN}}(C)$.
   - *Base case* ($\mathit{len}(\sigma) = 0$). Then, $\mathit{lead}(\sigma) = \mathsf{m}$ and ${}^\bullet\mathsf{m} = \emptyset$, which is a configuration in $\mathsf{Conf}_{p\text{ACN}}(C)$.
   - *Inductive step* ($\mathit{len}(\sigma) = n + 1$). Then, $\sigma = \sigma' [A\rangle\, m'$ with $\mathit{len}(\sigma') = n$. By inductive hypothesis on σ', ${}^\bullet\sigma(n) \in \mathsf{Conf}_{p\text{ACN}}(C)$. Define $X' = {}^\bullet\sigma(n)$. From $\sigma(n)\, [A\rangle$, we conclude that $A \cap X' = \emptyset$, because transitions cannot be fired twice in a pACN. Also, for all $t \in A$ and $t' \lessdot t$, we have that $t' \in X'$ because ${}^\circ t \cap {}^\bullet t' \neq \emptyset$ (otherwise t would not be enabled at $\sigma(n)$). This implies that $X' \cup A$ is left-closed with respect to $\lessdot$, i.e., $\lfloor A \rfloor_\lessdot \subseteq X' \cup A$.
     It remains to show that $\leadsto \cup \lessdot$ is acyclic on $X' \cup A$. Since X' is a configuration, its transitions can be partially ordered according to $\leadsto \cup \lessdot$. Moreover, $\forall t' \in X'$, $t'^\bullet \subseteq \sigma(n)$. Hence, $t' \in X'$ implies $\neg(t' \leadsto t)$ for all $t \in A$ (otherwise, t would not be enabled at $\sigma(n)$). Since A is enabled, for all $t, t' \in A$, neither $t \leadsto t'$ nor $t \leadsto t'$, because A enabled ensures that $\forall t \in [\![A]\!].\ {}^\circ t \cap (A - \{t\})^\bullet = \emptyset$. Consequently, $\leadsto \cup \lessdot$ is also acyclic on $X' \cup A = {}^\bullet m'$.

   Therefore, we conclude that $m' \in \mathcal{M}_C$ implies ${}^\bullet m' \in \mathsf{Conf}_{p\text{ACN}}(C)$.
(2) Let X be a reachable configuration. By Proposition 3.8, the elements of X can be partially ordered. That is, there exists a sequence $t_1, \ldots, t_n$ of the elements of X such that $t_i \lessdot t_j$ or $t_i \leadsto t_j$ imply $i < j$. We prove by induction on the size n of X that $m_0\, [t_1\rangle\, m_1\, [t_2\rangle \cdots m_{n-1}\, [t_n\rangle\, m_n$ with $\mathsf{m} = m_0$ is a firing sequence and $\forall i \in \{1, \ldots n\}$, ${}^\bullet m_i = \{t_1, \ldots, t_i\}$ and $\mathsf{m} - {}^\bullet X + X^\bullet \in \mathcal{M}_C$.
   - *Base case* ($n = 0$). Hence, $X = \emptyset$. Then, the firing sequence is empty, with $m_0 = \mathsf{m}$. Moreover, ${}^\bullet m_0 = \emptyset = X$ and also $m_0 = \mathsf{m} = \mathsf{m} - {}^\bullet\emptyset + \emptyset^\bullet = \mathsf{m} - {}^\bullet X + X^\bullet$.
   - *Inductive step* ($n = k + 1$). By inductive hypothesis, there exists a firing sequence $m_0\, [t_1\rangle\, m_1\, [t_2\rangle \cdots m_{k-1}\, [t_k\rangle\, m_k$ such that ${}^\bullet m_k = X \setminus \{t_{k+1}\}$ and $m_k = \mathsf{m} - {}^\bullet(X \setminus \{t_{k+1}\}) + (X \setminus \{t_{k+1}\})^\bullet$. Since X is a configuration, $\lfloor t_{k+1} \rfloor_\lessdot \subseteq X$. This implies that ${}^\circ t_{k+1} \cap m_k = \emptyset$. Additionally, if $t' \leadsto t_{k+1}$ then $t' \notin X \setminus \{t_{k+1}\}$ (otherwise,

X would not be a configuration). Hence, ${}^\bullet m_k\,[t_{k+1}\rangle$. Then, we can conclude that $m_0\,[t_1\rangle\, m_1\,[t_2\rangle \cdots m_{k-1}\,[t_k\rangle\, m_k\,[t_{k+1}\rangle\, m_{k+1}$ is a reachable marking. Moreover, $m_{k+1} = m_k - {}^\bullet t_{k+1} + t_{k+1}{}^\bullet = \mathsf{m} - {}^\bullet(X \setminus \{t_{k+1}\}) + (X \setminus \{t_{k+1}\})^\bullet - {}^\bullet t_{k+1} + t_{k+1}{}^\bullet = \mathsf{m} - {}^\bullet X + X^\bullet$ (where the last equality holds because transitions do not share places). Also, ${}^\bullet m_k = X \setminus \{t_{k+1}\}$ implies that ${}^\bullet m_{k+1} = X$ because ${}^\bullet({}^\bullet t_{k+1}) = \emptyset$ and ${}^\bullet(t_{k+1}{}^\bullet) = t_{k+1}$. □

We emphasize that the previous results remain valid even if the nets are ACN, as the inheritance of conflicts along the $\lessdot$ relation does not play any role in the proofs above.

3.4. **Morphisms for (pre) Asymmetric Causal Nets.** We introduce a suitable notion of morphisms for pACNs, which takes into account that inhibitor arcs correspond to two distinct types of dependencies: causality and prevention. In particular, inhibitor arcs representing prevention demand a specific treatment of markings when compared to classical notions of morphisms for nets [39, 40].

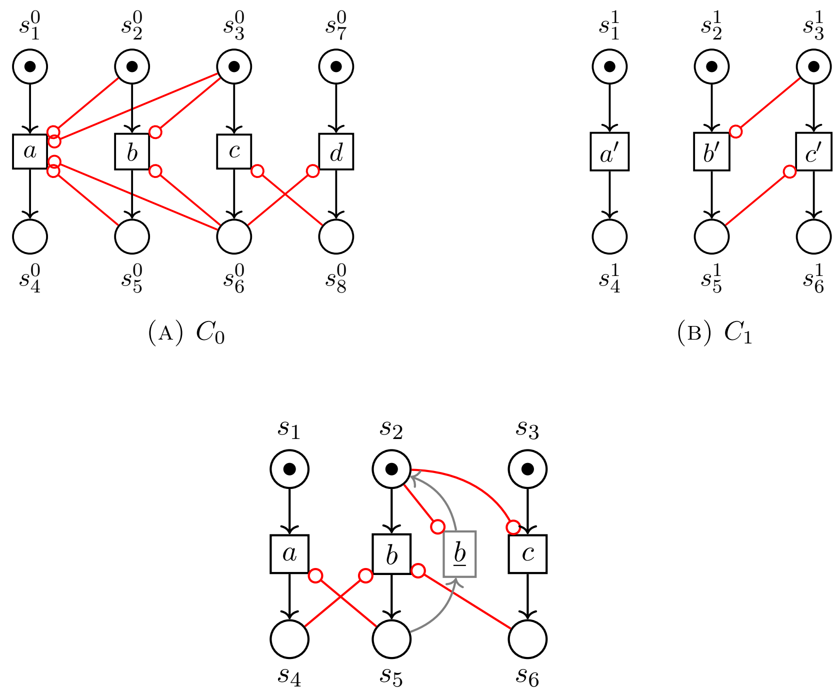
Definition 3.10. Let $C_0 = \langle S_0, T_0, F_0, I_0, \mathsf{m}_0\rangle$ and $C_1 = \langle S_1, T_1, F_1, I_1, \mathsf{m}_1\rangle$ be pACNs. An ACN-morphism is a pair (f_S, f_T) consisting of a relation $f_S \subseteq S_0 \times S_1$ and a partial function $f_T : T_0 \to T_1$ defined such that

(1) for all $t \in T_0$ if $f_T(t) \neq \bot$ then
  (a) ${}^\bullet f_T(t) = f_S({}^\bullet t)$ and $f_T(t)^\bullet = f_S(t^\bullet)$;
  (b) $\forall (s, f_T(t)) \in I_1$.
    (i) if ${}^\bullet s = \emptyset$ then $\exists s' \in f_S^{-1}(s).\ (s', t) \in I_0$;
    (ii) if ${}^\bullet s \neq \emptyset$ then $\forall s' \in f_S^{-1}(s).\ (s', t) \in I_0$.
(2) $\forall t, t' \in T_0$ if $f_T(t) \neq \bot \neq f_T(t')$ then $f_T(t) = f_T(t') \Rightarrow t\ \natural_0\ t'$.
(3) $\forall s_1 \in S_1.\ \forall s_0, s_0' \in f_S^{-1}(s_1).\ s_0 \neq s_0'$ implies $s_0{}^\bullet\ \natural_0\ s_0'{}^\bullet$ or ${}^\bullet s_0\ \natural_0\ {}^\bullet s_0'$;
(4) $[\![f_S(\mathsf{m}_0)]\!] = \mathsf{m}_1$.

The initial two conditions adhere to the standard criteria for morphisms between safe nets: the preservation of presets and postsets (Condition 1a), and the reflection of inhibitor arcs (Conditions 1(b)i and 1(b)ii). The inhibitor arcs modeling causality have to be reflected by finding a witness of the causality, whereas those representing preventions have to be reflected in every possible instance. Notably, Condition 1b implies that ${}^\circ f_T(t) \subseteq [\![f_S({}^\circ t)]\!]$. Furthermore, only conflicting transitions can be identified, as stipulated by Condition 2.

In contrast to conventional requirements, Condition 3 permits f_S to actively *identify* places in the preset of different transitions. This might raise concerns, as a place in the target of the morphism could represent different tokens in the source, potentially evolving independently. However, under the constraints of Condition 3, f_S is only allowed to identify places connected to transitions that are in (symmetric) conflict. Moreover, the initial markings corresponds via the relation f_S (Condition 4). It is worth noting that the notion of morphisms remains unaffected by the inheritance of symmetric conflicts along the causality relation, making these conditions applicable to ACNs as well.

Example 3.11. Examine the ACNs C_0 and C_1 illustrated in Figure 6. The morphism $(f_S, f_T) : C_0 \to C_1$ is defined as follows. For transitions, the mapping f_T is defined by $f_T(a) = a'$, $f_T(b) = b'$, and $f_T(c) = c' = f_T(d)$, thereby identifying the conflicting transitions c and d. The relationship on places is established as expected, specifically $f_S(s_i^0, s_i^1)$ for $1 \leq i \leq 6$, $f_S(s_7^0, s_3^1)$, and $f_S(s_8^0, s_6^1)$. The inhibitor arc (s_3^1, b') in C_1 is reflected by the arc (s_3^0, b) in C_0. It is noteworthy that the mapping does not preserve the remaining arcs of



(A) C_0 (B) C_1

(C) V

Figure 6

C_0. We also stress that if flattening is not applied we would have two tokens in s_3^1, one *originated* from s_3^0 and the other from s_7^0.

The following result is instrumental for the following developments, and ensures that any place in the preset of a transition (in the target of an ACN-morphisms) possesses a corresponding pre-image within the morphism.

Proposition 3.12. *Let $C_0 = \langle S_0, T_0, F_0, I_0, \mathsf{m}_0 \rangle$ and $C_1 = \langle S_1, T_1, F_1, I_1, \mathsf{m}_1 \rangle$ be pACNs, and $(f_S, f_T) : C_0 \to C_1$ an ACN-morphism. Then, for all $s_1 \in S_1$, if ${}^\bullet s_1 = \emptyset$, then $f_S^{-1}(s_1) \neq \emptyset$.*

Proof. By contradiction. Assume $f_S^{-1}(s_1) = \emptyset$. Hence, there does does not exists $s_0 \in S_0$ such that $f_S(s_0) = s_1$. Hence, $s_1 \notin f_S(\mathsf{m}_0)$; and consequently, $[\![f_S(\mathsf{m}_0)]\!] \neq \mathsf{m}_1$, which contradicts the hypothesis that (f_S, f_T) is an ACN-morphism (Definition 3.10(4)). □

In connection with a previous property, it is noteworthy that this property does not hold for transitions, which do not have a guaranteed pre-image. Additionally, places in the postset of transitions that are outside of the image of the morphism, may not necessarily be within the image of the morphism.

Now, our focus is on demonstrating that ACN-morphisms preserve the token game; specifically, a firing $(m\,[A\rangle\, m')$ in the source net is correspondingly translated to a firing in the target net. It is noteworthy that not every marking is meaningful in a ACN. For instance, we anticipate that a marking should not simultaneously contain tokens for both the pre- and postset of the same transition. Similarly, a valid marking is not expected to place tokens in the postset of conflicting transitions. The upcoming definition introduces the concept of markings that are coherent for ACNs.

Definition 3.13. Given a ACN C, we say m is a *coherent marking* for C if $[\![m]\!] = m$,

(1) for all t, ${}^\bullet t \in m$ iff $t^\bullet \notin m$.
(2) for all t, t', $t \natural t'$ and $t^\bullet \in m$ implies $t'^\bullet \notin m$.

Proposition 3.14. *Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be a* ACN*. Then* m *is coherent*

Proof. Since C is a ACN, $[\![\mathsf{m}]\!] = \mathsf{m}$. Moreover, $\mathsf{m} = S \setminus T^\bullet$. Hence, the two conditions are trivially satisfied. □

Another important consideration is that an ACN-morphism may collapse different conflicting transitions in the source into a single transition in the target. In other words, two conflicting transitions t and t' may be mapped to the same transition in the target, denoted as $f_T(t) = f_T(t')$. Consequently, $f_S({}^\bullet t) = f_S({}^\bullet t')$ and $f_S(t^\bullet) = f_S(t'^\bullet)$. Therefore, a coherent marking in the source that includes $t^\bullet$ and ${}^\bullet t'$ corresponds to an execution where the conflict has been resolved—t has been fired, and subsequently, t' is precluded. When mapping such a marking to the target, the conventional approach in morphisms for Petri nets, applying f_S to the marking, would include both $f_S(t^\bullet)$ and $f_S({}^\bullet t')$ in the target marking. Since $f_T(t) = f_T(t')$, this would result in a marking that is not coherent for the net in the target of the morphism. Therefore, when mapping a marking, it is imperative to account for conflicts that have already been resolved. Specifically, tokens in the preset of transitions that were in conflict with others that have been fired must be disregarded. This is formally addressed by the next definition, which defines the relevant information of a marking.

The next results establish the correspondence among firing between ACNs related via a morphism.

Definition 3.15. Let $(f_S, f_T) : C_0 \to C_1$ be an ACN-morphism, and m a coherent marking of C_0. The *relevant information of* m is $\underline{m} = m - \{\!\{s \mid s, s' \in m \wedge s \in {}^\bullet t \wedge s' \in t'^\bullet \wedge t \natural t'\}\!\}$

Proposition 3.16. *Let $(f_S, f_T) : C_0 \to C_1$ be an* ACN*-morphism, and m a coherent marking of C_0. Then, $m\,[A\rangle\, m'$ implies m' coherent and $[\![f_S(\underline{m})]\!]\,[f_T(A)\rangle\,[\![f_S(\underline{m'})]\!]$.*

Proof. Assume $m\,[A\rangle$. Since A is enabled at m, we have (i) ${}^\bullet A \subseteq m$, (ii) ${}^\circ A \cap\ [\![m]\!] = \emptyset$, and (iii) $\forall t \in [\![A]\!].\ {}^\circ t \cap (A - \{t\})^\bullet = \emptyset$. Firstly, we show that $f_S({}^\bullet A) = [\![f_S({}^\bullet A)]\!]$. Suppose the contrary, i.e., $f_S({}^\bullet A) \neq [\![f_S({}^\bullet A)]\!]$. Since, m is a set, then all transitions in A are different, otherwise A would not be enabled at A. Since C_0 is an ACNs, each of its transitions has just one place in its preset; i.e., for all $t \in C_0$, ${}^\bullet t$ is a singleton. Consequently, there should be two different transitions $t \neq t' \in A$ whose preset is mapped to the same place, i.e., $f_S({}^\bullet t) = f_S({}^\bullet t')$. Since (f_S, f_T) is a pACN morphism, $t \natural_0 t'$ by Definition 3.10(2), which contradicts the assumption that A is enabled at m. Hence, $f_S({}^\bullet A) = [\![f_S({}^\bullet A)]\!]$. From (i), we conclude that $[\![f_S({}^\bullet A)]\!] \subseteq [\![f_S(m)]\!]$. By contradiction, we show that $[\![f_S({}^\bullet A)]\!] \subseteq [\![f_S(m)]\!]$ implies $[\![f_S({}^\bullet A)]\!] \subseteq [\![f_S(\underline{m})]\!]$. Suppose there is $s \in f_S({}^\bullet A)$, $s \in f_S(m)$, and $s \notin f_S(\underline{m})$. Hence, there exist s' and t' such that $s' \in m$, $s' \in t'^\bullet$ and $t' \natural t$. But this implies t is not enabled at m, contradicting (ii). Hence, $[\![f_S({}^\bullet A)]\!] \subseteq [\![f_S(\underline{m})]\!]$. Since (f_S, f_T) is a pACN morphism, ${}^\bullet f_T(A) = f_S({}^\bullet A) = [\![f_S({}^\bullet A)]\!]$; and consequently, ${}^\bullet f_T(A) \subseteq [\![f_S(\underline{m})]\!]$.

Secondly, we show that ${}^\circ f_T(A) \cap\ [\![f_S(\underline{m})]\!] = \emptyset$. We proceed by contradiction. Assume that there exist $t \in A$ and $s \in [\![f_S(\underline{m})]\!]$ such that s inhibits $f_T(t)$, i.e., $s \in {}^\circ f_T(t)$. We have two cases:

- ${}^\bullet s = \emptyset$: Since $s \in [\![f_S(\underline{m})]\!]$, we conclude $f_S^{-1}(s) \neq \emptyset$. Hence, there exists $s' \in S_0$ such that $(s', s) \in f_S$ and $s' \in {}^\circ t$, because inhibitor arcs are reflected by Definition 3.10(1(b)i). Since

C_0 is an ACN, there exists exactly one transition $t' \in T_0$ such that $s' \in {}^\bullet t'$. There are two cases:
- $s' \in \underline{m}$: Hence, we have $s' \in m$ and $s' \in {}^\circ t$, $t \in A$; which is in contradiction with (ii).
- $s' \notin \underline{m}$. Since $f_S(s') \in [\![f_S(\underline{m})]\!]$, there should exist $s'' \in \underline{m}$ such that $f(s'') = f(s)' = s$. Let t'' be the transition such that $s'' \in {}^\bullet t''$. Since (f_S, f_T) is an ACN-morphism, $s'^\bullet \natural s''^\bullet$ (Definition 3.10(3)). Hence, $t' \natural t''$. Since m is a coherent marking, $s' \notin m$ implies $t'^\bullet \in m$. Since $t'^\bullet \in m$ and $t' \natural t''$, by definition of relevant information of a marking, we conclude that ${}^\bullet t'' \notin \underline{m}$, which contradicts the assumption $s'' \in \underline{m}$.

- ${}^\bullet s \neq \emptyset$. Since $s \in [\![f_S(\underline{m})]\!]$, then there exist $s' \in \underline{m} \subseteq m$ such that $f_S(s') = s$. By Definition 3.10(1(b)ii), inhibitor arcs are reflected, and hence $s' \in {}^\circ t$, which contradicts (ii).

Finally, we show that $\forall t \in [\![f_T(A)]\!].\ {}^\circ t \cap (f_T(A) - \{t\})^\bullet = \emptyset$. If this were not the case, then there would exist $t_1 \in f_T(A)$ such that $t_1 \neq t$ and ${}^\circ t \cap t_1{}^\bullet \neq \emptyset$. Since (f_S, f_T) is a pACN morphism, there would exists $t', t'_1 \in A$ such that $f_S(t') = t$ and $f_S(t'_1) = t_1$ and ${}^\circ t' \cap t_1'^\bullet$, which contradicts (iii). Consequently, $f_T(A)$ is enabled at $[\![f_S(\underline{m})]\!]$. Moreover, $[\![f_S(\underline{m})]\!]\,[f_T(A)\rangle\,[\![f_S(\underline{m'})]\!]$ as $[\![f_S(\underline{m'})]\!] = [\![f_S(\underline{m})]\!] - [\![f_S({}^\bullet A)]\!] + [\![f_S(A^\bullet)]\!]$. □

The next results shows that ACN-morphisms are closed under composition.

Proposition 3.17. *Let $(f_S, f_T) : C_0 \to C_1$ and $(g_S, g_T) : C_1 \to C_2$ be two ACN-morphisms. Then $(f_S; g_S, f_T; g_T) : C_0 \to C_2$ is a ACN-morphism as well.*

Proof. We check the conditions of Definition 3.10.

(1) Take $t \in T_0$ if $f_T; g_T(t) \neq \bot$ then
  (a) Since f_T and g_T define morphisms, we have that ${}^\bullet(f_T; g_T)(t) = {}^\bullet g_T(f_T({}^\bullet t)) = g_S(f_S({}^\bullet t)) = (f_S; g_S)({}^\bullet t)$. By analogous reasoning, we deduce that $(f_T; g_T)(t)^\bullet = (f_S; g_S)(t^\bullet)$.
  (b) Consider $(s, f_T; g_T(t)) \in I_2$.
    (i) Assume $s^\bullet \neq \emptyset$, and $f_S^{-1}(g_S^{-1}(s)) \neq \emptyset$. By the definition of (g_S, g_T), there exists $\tilde{s} \in g_S^{-1}(s)$. $(\tilde{s}, f_T(t)) \in I_1$. Furthermore $\tilde{s}^\bullet \neq \emptyset$. Again by definition of (f_S, f_T), we have again that there is an $s' \in g_T^{-1}(g_S^{-1}(s))$ such that $(s', t) \in I_0$. We can conclude that if $(s, f_T; g_T(t)) \in I_2$ and if $f_S^{-1}(g_S^{-1}(s)) \neq \emptyset$ then we have that there exists $s' \in (f_T; g_T)^{-1}(s))$, $(s', t) \in I_0$.
    (ii) Assume $s^\bullet = \emptyset$. By the definition of (g_S, g_T), for all $\tilde{s} \in g_S^{-1}(s)$. $(\tilde{s}, f_T(t)) \in I_1$. Furthermore $\tilde{s}^\bullet = \emptyset$. Again by definition of (f_S, f_T), we have that for all $s' \in g_T^{-1}(g_S^{-1}(s))$ we have $(s', t) \in I_0$. We can conclude that if $(s, f_T; g_T(t)) \in I_2$, then for all $s' \in (f_T; g_T)^{-1}(s))$ we have $(s', t) \in I_0$.

(2) Assume $f_T; g_T(t) = f_T; g_T(t')$. There are two cases:
  - $f_T(t) = f_T(t')$. Then, $t \natural_0 t'$ because (f_S, f_T) is an ACN morphism.
  - $f_T(t) \neq f_T(t')$ and $g_T(t) = g_T(t')$. Since, (g_S, g_T) is an ACN morphism, $f_S(t) \natural_1 f_s(t')$. By definition of symmetric conflicts, $f_S(t)^\bullet \cap {}^\circ f_S(t') \neq \emptyset$ and $f_S(t')^\bullet \cap {}^\circ f_S(t) \neq \emptyset$. Since pACN are defined such that the postset of every transition is a singleton, we have that $f_S(t)^\bullet \subseteq {}^\circ f_S(t')$ and $f_S(t')^\bullet \subseteq {}^\circ f_S(t)$. Since (f_S, f_T) is a pACN-morphism, we have that $t'^\bullet \subseteq {}^\circ t$ and $t^\bullet \subseteq {}^\circ t'$, by Definition 3.10(1b). Consequently, $t \natural_0 t'$.

(3) We check that $\forall s_2 \in S_2, \forall s_0, s'_0 \in f_S^{-1}(g_S^{-1}(s_2))$ either $s_0{}^\bullet \natural_0 s_0'^\bullet$ or ${}^\bullet s_0 \natural_0 {}^\bullet s'_0$. Therefore, either there exists a place $s_1 \in g_S^{-1}(s_2)$ such that both s_0 and s'_0 belong to $f_S^{-1}(s_1)$, or there are two places s_1 and s'_1 in $g_S^{-1}(s_2)$ and s_0 and s'_0 are both in $f_S^{-1}(\{s_1, s'_1\})$. In the first case $s_0{}^\bullet \natural_0 s_0'^\bullet$ or ${}^\bullet s_0 \natural_0 {}^\bullet s'_0$ holds as (f_S, f_T) is an ACN-morphism; in the second

case we have ${s_1}^\bullet \natural_1 {s'_1}^\bullet$ or ${}^\bullet s_1 \natural_1 {}^\bullet s'_1$ as (g_S, g_T) is a ACN-morphism. Since, conflicts are reflected, also ${s_0}^\bullet \natural_0 {s'_0}^\bullet$ or ${}^\bullet s_0 \natural_0 {}^\bullet s'_0$ holds.

(4) Condition $[\![f_S; g_S(m_0)]\!] = m_2$ straightforwardly follows by the definitions of f_S and g_S that ensure $[\![f_S(m_0)]\!] = m_1$ and $[\![g_S(m_1)]\!] = m_2$. □

We designate the category of pACNs and ACN-morphisms as **pACN**. Within this category, there exists a full and faithful subcategory denoted as **ACN**, wherein the objects are ACNs. This subcategory, **ACN**, is the specific category of interest.

3.4.1. *Morphisms and configurations:* We conclude this part by noticing that morphisms preserve configurations, a fact established through the following proposition.

Proposition 3.18. *Let $(f_S, f_T) : C_0 \to C_1$ be an ACN-morphism. Then*

(1) *for all $t \in T_0$, if $f_T(t) \neq \perp$ then $\lfloor f_T(t) \rfloor_{\lessdot_1} \subseteq f_T(\lfloor t \rfloor_{\lessdot_0})$; and*

(2) *for all $t_0, t'_0 \in T_0$ such that $f_T(t_0) \neq \perp \neq f_T(t'_0)$, if $f_T(t_0) \leftsquigarrow_1 f_T(t'_0)$ then $t_0 \leftsquigarrow_0 t'_0$.*

Proof. (1) Take $t \in T_0$ such that $f_T(t) \neq \perp$. For each $t_0 \in \lfloor f_T(t) \rfloor_{\lessdot_1}$ we have that either ${}^\bullet t_0 \cap {}^\circ f_T(t) \neq \emptyset$ or there exists transitions $t_1, \ldots, t_n \in \lfloor f_T(t_0) \rfloor_{\lessdot_1}$ such that $({}^\bullet t_0, t_1), ({}^\bullet t_1, t_2), \ldots, ({}^\bullet t_n, f_T(t)) \in I_1$. Since (f_S, f_T) is a ACN-morphism, this leads to either a place $s_0 \in S_0$ with ${s_0}^\bullet = t'_0$, $f_T(t) = t'_0$ and $(s_0, t'_0) \in I_0$, or the existence of $s_0, s_1, \ldots, s_n$ such that ${s_i}^\bullet = t'_i$, $f_T(t'_i) = t_i$ and $(s_i, t'_i) \in I_0$. This implies $f_T(t'_0) = t_0 \in f_T(\lfloor t_0 \rfloor_{\lessdot_1})$. Thus, the inclusion is established.

(2) Assume $f_T(t_0) \leftsquigarrow_1 f_T(t'_0)$, then $(s, f_T(t_0)) \in I_1$ and $s \in f_T(t'_0)^\bullet$, but then for all $s' \in f_S^{-1}(s)$ we have $(s', t_0) \in I_0$ and there must be an $s' \in {t'_0}^\bullet$. But then $t_0 \leftsquigarrow_0 t'_0$. □

Proposition 3.19. *Let $(f_S, f_T) : C_0 \to C_1$ be an ACN-morphism. If $X \in \mathsf{Conf}_{p\text{ACN}}(C_0)$, then $f_T(X) \in \mathsf{Conf}_{p\text{ACN}}(C_1)$.*

Proof. Take $X \in \mathsf{Conf}_{p\text{ACN}}(C_0)$. Hence, for every $t_1 \in f_T(X)$, there exists $t_0 \in X$ such that $f_T(t_0) = t_1$. By Proposition 3.18 $\lfloor f_T(t_0) \rfloor_{\lessdot_1} \subseteq f_T(\lfloor t_0 \rfloor_{\lessdot_0})$. Then, it follows that $\lfloor t_1 \rfloor_{\lessdot_1} \subseteq f_T(X)$. It remains to show that $\rightsquigarrow_1 \cup \lessdot_1$ is acyclic on $f_T(X)$. We proceed by contradiction. Assume that $\rightsquigarrow_1 \cup \lessdot_1$ has a cycle on $f_T(X)$. Since both relations are induced by inhibitor arcs, which are reflected by ACN-morphisms, this implies that $\rightsquigarrow_0 \cup \lessdot_0$ has a cycle on X, which contradicts the assumption that X is a configuration. Therefore, $\rightsquigarrow_1 \cup \lessdot_1$ is acyclic on $f_T(X)$ and $f_T(X) \in \mathsf{Conf}_{p\text{ACN}}(C_1)$. □

Corollary 3.20. *Let $(f_S, f_T) : C_0 \to C_1$ be an ACN-morphism and C_0, C_1 be two ACNs. If $X \in \mathsf{Conf}_{\text{ACN}}(C_0)$ then $f_T(X) \in \mathsf{Conf}_{\text{ACN}}(C_1)$.*

3.5. **Reversible Asymmetric Causal Nets.** In this section, we introduce the concept of Reversible Asymmetric Causal Nets, following the approach outlined in [21]. We extend ACNs by incorporating *backward* transitions, responsible for undoing or reversing the effects of previously executed *forward* transitions, i.e., ordinary transitions. We assume that the set T of transitions in a net is divided into two sets: $\overline{T}$ for forward transitions and $\underline{T}$ for backward transitions. Furthermore, each backward transition $\underline{t} \in \underline{T}$ is designed to undo the effect of precisely one forward transition $t \in \overline{T}$. Nonetheless, there may be forward transitions that are irreversible. For simplicity, we will use t to refer to the forward transition and $\underline{t}$ to denote its associated reversing transition, when applicable.

Definition 3.21. An IPT $V = \langle S, T, F, I, \mathsf{m}\rangle$ is a *reversible Asymmetric Causal Net* (rACN) if there exists a partition $\{\overline{T}, \underline{T}\}$ of T (where $\overline{T}$ represents the forward transitions, and $\underline{T}$ denotes the backward transitions) satisfying the following conditions:

(1) $V_{\overline{T}} = \langle S, \overline{T}, F_{|\overline{T}\times\overline{T}}, I_{|\overline{T}\times\overline{T}}, \mathsf{m}\rangle$ is a pACN net;
(2) $\forall \underline{t} \in \underline{T}.\ \exists!\ t \in \overline{T}$ such that $t^\bullet = {}^\bullet\underline{t}$, ${}^\bullet t = \underline{t}^\bullet$, and ${}^\bullet t \subseteq {}^\circ\underline{t}$;
(3) $\forall \underline{t} \in \underline{T}.\ K_{\underline{t}} = \{t' \in \overline{T} \mid {}^\circ\underline{t} \cap {}^\bullet t' \neq \emptyset\}$ is finite and $\rightsquigarrow$ acyclic on $K_{\underline{t}}$;
(4) $\forall \underline{t} \in \underline{T}.\ \forall t \in \overline{T}$ if ${}^\bullet t \cap {}^\circ\underline{t} \neq \emptyset$ then $t^\bullet \cap {}^\circ\underline{t} = \emptyset$;
(5) $\forall t, t', t'' \in \overline{T}.\ t \natural t' \ \wedge\ t' \lll t'' \ \Rightarrow\ t \natural t''$ with $\lll$ being the transitive closure of $\lessdot \cap \{(t, t') \mid \underline{t} \notin \underline{T}$ *or* ${}^\circ\underline{t} \cap t'^\bullet \neq \emptyset\}$.

Sometimes we use $V^{\underline{T}}$ to represent a rACN V with the set $\underline{T}$ of backward transitions.

In accordance with Condition 1, the underlying net $V_{\overline{T}}$, encompassing solely the forward transitions of V, is a pACN. The insistence on it being a pACN rather than an ACN stems from the fact that the conflict relation is not always inherited along $\lessdot$, which serves as the causation relationship. This deviation is due to the fact that reversing transitions may allow potentially conflicting transitions to be executed. Condition 2 stipulates that each backward transition $\underline{t}$ unequivocally reverses one and only one forward transition t. Consequently, $\underline{t}$ consumes the tokens produced by t ($t^\bullet = {}^\bullet\underline{t}$) and generates the tokens consumed by t ($\underline{t}^\bullet = {}^\bullet t$). The requirement ${}^\bullet t \subseteq {}^\circ\underline{t}$ in Condition 2 signifies that the reversal of t (i.e., $\underline{t}$) can only occur if t has been executed. In other words, a transition can only be reversed if it has been fired. Condition 3 requires a finite set of causes for undoing each transition; in essence, ${}^\bullet K_{\underline{t}}$ encompasses all the forward transitions t' that enable the execution of $\underline{t}$. Condition 4 asserts that if a backward transition $\underline{t}'$ causally depends on the forward transition t (i.e., ${}^\bullet t \cap {}^\circ\underline{t}' \neq \emptyset$), then $\underline{t}'$ cannot be prevented by the same transition t ($t^\bullet \cap {}^\circ\underline{t}' = \emptyset$), as otherwise, it would be blocked. Condition 5 introduces the relation $\lll$, which is analogous to the *sustained causation* in rAES. This relation coincides with causality, except in cases in which a cause can be reversed even after a causally-dependent transition has been fired. Note that conflicts should be inherited along $\lll$ rather than along the $\lessdot$ relation.

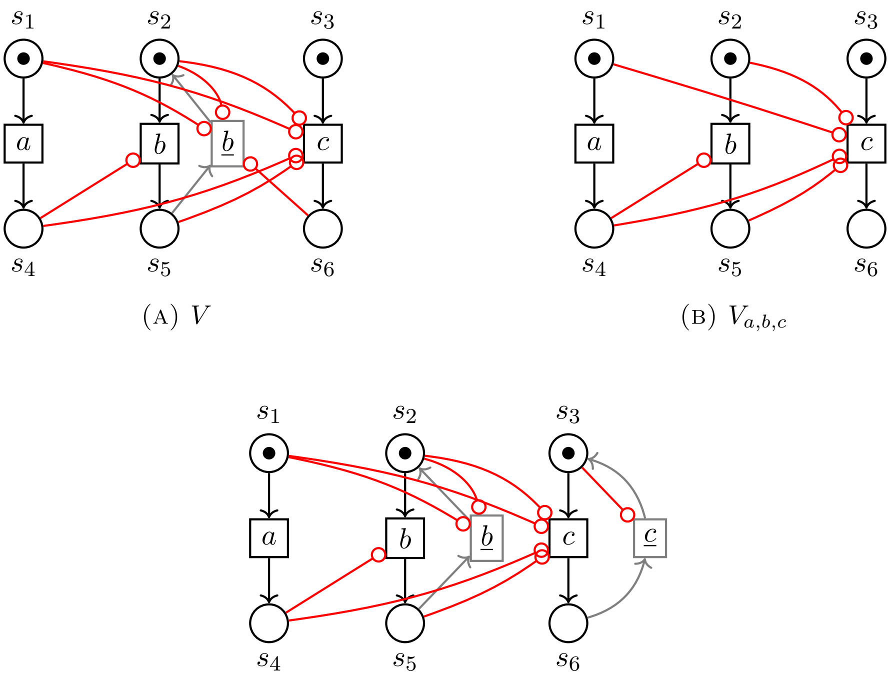
The inhibitor arcs of an rACN V induce four distinct relations. Two of these pertain to the forward flow, defined on $\overline{T} \times \overline{T}$, and correspond to those found in pACNs, namely $\lessdot$ and $\rightsquigarrow$. Additionally, two relations concern the backward flow. These are *reverse causation* $\prec\ \subseteq \overline{T} \times \underline{T}$, characterized by $t \prec \underline{t'}$ if and only if ${}^\bullet t \cap {}^\circ\underline{t}' \neq \emptyset$, and *prevention* $\lhd\ \subseteq \underline{T} \times \overline{T}$, defined by $\underline{t'} \lhd t$ if and only if $t^\bullet \cap {}^\circ\underline{t}' \neq \emptyset$.

Example 3.22. Verifying that the IPT V in Figure 7a is an rACN is straightforward. Note that c causally depends on a and b due to the inhibitor arcs connecting c with the presets of a and b. Moreover, a and b are in asymmetric conflict: the inhibitor arc linking s_4 and b indicates that b is prevented by a. The sole reversible transition is b, and its associated undoing $\underline{b}$ causally depends on both a and b and is prevented by c. In other words, the reversal of b can only occur after the firing of a and b, but only if c has not taken place.

The net $V_{a,b,c}$ in Figure 7b is obtained by eliminating the transition $\underline{b}$ along with the connected inhibitor arcs from V. It is immediate that $V_{a,b,c}$ is a ACN.

The net V' in Figure 7c is obtained by extending V_1 with the reversal of c, i.e., $\underline{c}$. If $\underline{b}$ and $\underline{c}$ along with their associated inhibitor arcs are removed from V', we obtain the ACN depicted in Fig. 7b.

Example 3.23. Consider the net V in Fig. 6. Note that the removal of the reversing transition $\underline{b}$ does not yield a ACN, as the conflict between a and b is not inherited along



(a) V

(b) $V_{a,b,c}$

(c) V'

Figure 7

$b \lessdot c$. It is noteworthy that the following represents a valid execution of V: initiating with b, followed by c, and then reversing b (performing $\underline{b}$), enabling the subsequent execution of a. The ability to execute a after the firing and reversal of b would be forbidden if the conflict between a and b were inherited along $b \lessdot c$.

3.6. **Configurations of rACNs.** We now introduce the concept of configuration for rACNs. In contrast to the configuration for standard nets, which is typically defined in terms of causality, a configuration of rACN is simply required reachable via a firing sequence.

Definition 3.24. Let $V^{\underline{T}} = \langle S, T, F, I, \mathsf{m}\rangle$ be a rACN. A *configuration* of $V^{\underline{T}}$ is any subset $X \subseteq \overline{T}$ of forward transitions such that there exists a firing sequence $\mathsf{m}\,[A_1\rangle \ldots m'$ with $X = {}^{\bullet}m' \cap \overline{T}$.

Example 3.25. Consider the net V_2 in Fig. 7c; one of its configuration is $\{a, c\}$, obtained by executing b first, followed by a, then c and subsequently undoing b (executing $\underline{b}$). This configuration is only reachable by undoing b since, given that b is a cause of c, the presence of b is necessary to execute c.

3.7. **Morphisms for Reversible Asymmetric Causal Nets.** The notion of morphisms for rACNs is given below.

Definition 3.26. Let $V^{\underline{T_0}} = \langle S_0, T_0, F_0, I_0, \mathsf{m}_0\rangle$ and $V^{\underline{T_1}} = \langle S_1, T_1, F_1, I_1, \mathsf{m}_1\rangle$ be two rACNs. A rACN-morphism is a pair (f_S, f_T) consisting of a relation $f_S \subseteq S_0 \times S_1$ and a partial function $f_T : T_0 \to T_1$ satisfying the following conditions

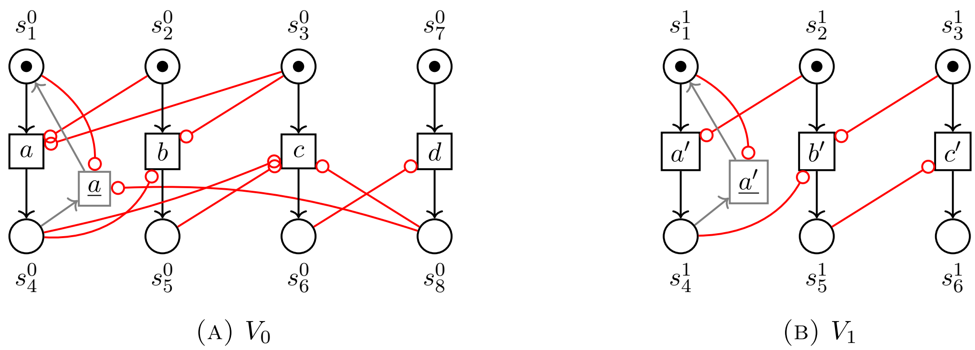


FIGURE 8. Two rACN

(1) $f_T(\overline{T_0}) \subseteq \overline{T_1}$ and $f_T(\underline{T_0}) \subseteq \underline{T_1}$;
(2) $(f_S, f_T|_{\overline{T_0}}) : V_{\overline{T_0}} \to V_{\overline{T_1}}$ is an ACN-morphism,
(3) $\forall \underline{t} \in \underline{T_0}$. if $f_T(t) \neq \bot$ then
 (a) $f_T(\underline{t}) \neq \bot$ and $f_T(\underline{t}) = \underline{f_T(t)}$; and
 (b) $\forall (s, f_T(\underline{t})) \in I_1$.
  (i) if $s^\bullet \cap \overline{T_1} \neq \emptyset$, $f_S^{-1}(s) \neq \emptyset$ implies $\exists\ s' \in f_S^{-1}(s)$. $(s', \underline{t}) \in I_0$; and
  (ii) if $s^\bullet \cap \overline{T_1} = \emptyset$, $\forall\ s' \in f_S^{-1}(s)$. $(s', \underline{t}) \in I_0$.

The first condition stipulates that forward and backward transitions are respectively mapped to forward and backward transitions. According to Condition 2, when considering forward transitions only (i.e., the restriction of f_T to $\overline{T_0}$) we have an ACN-morphism on the underlying $V_{\overline{T_0}}$ and $V_{\overline{T_1}}$ (i.e., the nets consisting only of forward transitions). Condition 3a ensures that the transition $\underline{t}$, responsible for reversing t, must be mapped to the transition $\underline{f_T(t)}$, which, in turn, reverses $f_T(t)$. Finally, Condition 3(b)i prevents the merging of inhibitor arcs, meaning that the distinction between various causes that might prevent the reversal of a transition is preserved, whereas Condition 3(b)ii assures that is a prevention is present in the target then all the reverse images of this prevention must be present in the origin. We have to restrict our attention to $s^\bullet \cap \overline{T_1}$ as we have to consider the relations induced by the inhibitor arcs in this case that are either reverse causation or reverse prevention, and these are defined among forward transitions and backward ones.

Example 3.27. An rACN-morphism $(f_S, f_T) : V_0 \to V_1$ for the rACNs V_0 and V_1 in Fig. 8 is as follows. Consider the multirelation f_S on places, as detailed in Example 3.11, and define the mapping on transitions f_T such that it mirrors the one in Example 3.11 for forward transitions. On the reversing transition $\underline{a}$, set $f_T(\underline{a}) = \underline{a}'$. It is evident that $\underline{f_T(a)} = f_T(\underline{a})$. The other conditions are straightforwardly satisfied.

The following two results state that rACN-morphisms preserve behaviours and that they are closed under composition.

We first need to specialize the notion of coherent marking for rACN. The idea is that we have to focus on forward transitions only. In fact, a coherent marking does not place tokens simultaneously in both the preset and postset of a transition, and does not place tokens in the postsets of two conflicting transitions. Note that this can guaranteed by just looking at forward transitions. Therefore given a marking m of the rACN $V = V^{\underline{T}}$, m is coherent if it is coherent in $V|_{\overline{T}}$ (recall that the places in V and $V|_{\overline{T}}$ are the same). Similarly let

$(f_S, f_T) : V^{\underline{T_0}} \to V^{\underline{T_1}}$ be an rACN-morphism, and let m be a coherent marking of $V^{\underline{T_0}}$. The relevant information of m are the relevant information of the marking m in $V|_{\overline{T_0}}$.

Proposition 3.28. *Let $(f_S, f_T) : V^{\underline{T_0}} \to V^{\underline{T_1}}$ be a rACN-morphism. Let m be a coherent marking of $V^{\underline{T_0}}$. Then, $m\,[A\rangle\, m'$ implies m' coherent and $[\![f_S(\underline{m})]\!]\,[f_T(A)\rangle\,[\![f_S(\underline{m'})]\!]$.*

Proof. We notice that conditions 3a and 3b of Definition 3.26 are essentially the same of a ACN-morphism, thus without distinguish among reverse transitions and forward ones, we have that a rACN-morphism is a ACN-morphism. Consequently, the tokens game is preserved. □

Proposition 3.29. *Let $(f_S, f_T) : V_0 \to V_1$ and $(g_S, g_T) : V_1 \to V_2$ be two rACN-morphisms. Then $(f_S; g_S, f_T; g_T) : V_0 \to V_2$ is a rACN-morphism as well.*

Proof. The proof follows the same lines as the one in Proposition 3.17. □

Therefore, rACNs and rACN-morphisms constitute a category, which is denoted by **RACN**.

3.7.1. *Configurations and morphisms:* We show that morphisms preserve configurations.

Proposition 3.30. *Let $(f_S, f_T) : V^{\underline{T_0}} \to V^{\underline{T_1}}$ be a rACN-morphism and X a configuration, i.e., $X \in \mathsf{Conf}_{r\text{ACN}}(V^{\underline{T_0}})$. Then, $f_T(X) \in \mathsf{Conf}_{r\text{ACN}}(V^{\underline{T_1}})$.*

Proof. It is sufficient to observe that configurations in rACN are characterized by reachable markings, and rACN-morphisms preserve them. □

3.8. Constructions.

Similarly to what we have done for rAES, we have a coproduct also in the category of rACN.

Proposition 3.31. *Let $V_0^{\underline{T_0}} = (S_0, T_0, F_0, I_0, \mathsf{m}_0)$ and $V_1^{\underline{T_1}} = (S_1, T_1, F_1, I_1, \mathsf{m}_1)$ be two rACNs. Then $V_0^{\underline{T_0}} + V_1^{\underline{T_1}} = (S, T, F, I, \mathsf{m})$ where*

- $S = \{0\} \times S_0 \cup \{1\} \times S_1$;
- $T = \{0\} \times T_0 \cup \{1\} \times T_1$;
- $\forall (i, a) \in S \cup T,\ \forall (j, b) \in S \cup T.\ ((i, a), (j, b)) \in F$ *whenever* $i = j$ *and* $(a, b) \in F_i$;
- $\forall (i, s) \in S,\ \forall (j, t) \in T.\ ((i, s), (j, t)) \in I$ *whenever either* $i = j$ *and* $(s, t) \in I_i$ *or* $i \neq j$; *and*
- $\forall (i, s) \in S.\ \mathsf{m}(i, s) = \mathsf{m}_i(s)$

is their coproduct *and* $(in_S^i, in_T^i) : V_i^{\underline{T_i}} \to V_0^{\underline{T_0}} + V_1^{\underline{T_1}}$ *defined as* $(s, (i, s)) \in in_S^i$ *and* $in_T^i(t) = (i, t)$ *are the injections.*

Proof. $V_0^{\underline{T_0}} + V_1^{\underline{T_1}}$ is clearly an rACN. It remains to prove that is indeed a coproduct in the category **RACN**. Consider an rACN $V_2^{\underline{T_2}}$ and two morphisms $(f_S, f_T) : V_0^{\underline{T_0}} \to V_2^{\underline{T_2}}$ and $(g_S, g_T) : V_1^{\underline{T_1}} \to V_2^{\underline{T_2}}$, we show that there exists a unique morphisms $(h_S, h_T) : V_0^{\underline{T_0}} + V_1^{\underline{T_1}} \to V_2^{\underline{T_2}}$. Define h_S as the relation comprising the pairs $((i, s), s')$ if either $(s, s') \in f_S$ or $(s, s') \in g_S$ and the mapping on transitions $h_T(i, t)$ equal to $f_T(t)$ if $i = 0$ and to $g_T(t)$ if $i = 1$. We check first that this is indeed an rACN-morphism. Consider $(i, t) \in T$ such that $h_T(i, t)$ is defined. Then we have that ${}^{\bullet}h_T(i, t) = {}^{\bullet}f_T(t)$ if $i = 0$ and ${}^{\bullet}h_T(i, t) = {}^{\bullet}g_T(t)$ if $i = 1$ and as (f_S, f_T) and (g_S, g_T) are morphisms we have that either ${}^{\bullet}h_T(i, t) = \mu f_S({}^{\bullet}t)$

or ${}^{\bullet}h_T(i,t) = \mu g_S({}^{\bullet}t)$ and similarly for $h_T(i,t)^{\bullet} = \mu f_S(t^{\bullet})$ or $h_T(i,t)^{\bullet} = \mu g_S(t^{\bullet})$. Consider now $((i,s), h_T(j,t)) \in I_2$ and $s' \in f_S^{-1}(i,s) \cup g_S^{-1}(i,s)$. If $s' \in f_S^{-1}(i,s)$ and $j = 0 = i$ then $((i,s'),(i,t)) \in I$ and if $i \neq j$ then $((i,s'),(j,t)) \in I$ by construction. The case $s' \in g_S^{-1}(i,s)$ is the same. Assume now $h_T(i,t)$ and $h_T(j,t')$ are both defined and equal. If $i = j$ then $(i,t) \natural (i,t')$ and if $i \neq j$ we have $(i,t) \natural (j,t')$ as well as each transition of the first one is prevented by the happening of a transition of the second one and vice versa. Condition 3 of Definition 3.10 is proven similarly to the previous one, and the same argument is used to show that fact that $[\![h_S]\!](\mathsf{m}) = \mathsf{m}_2$. Observing that forward transitions are mapped to forward ones and backward transitions are mapped to backward ones, and if $(i,\underline{t}) \in \underline{T}$ and $h_T(i,t)$ is defined we have that $h_T(i,\underline{t})$ is defined and $h_T(i,\underline{t}) = \underline{h_T(i,t)}$ as (f_S, f_T) and (g_S, g_T) are morphisms. The rACN-morphism (h_S, h_T) is unique as h_T is unique and there is just one h_S satisfying the requirements. ☐

Also the category **RACN** does not have products.

4. Relating models

In this section, we explore the interplay between the categories of (reversible) asymmetric causal nets and (reversible) asymmetric event structures. We establish functors and demonstrate the emergence of an adjunction.

We delve into the connection between rAESs and rACNs. The following definition outlines the process of recovering an rAES from the flow and inhibitor arcs of an rACN.

As expected, the relations induced by an rAES —–namely $\lessdot$ (causality), $\rightsquigarrow$ (weak causality), $\prec$ (reverse causality), $\lhd$ (prevention), and $\lll$ (sustained causation)—play a crucial role in defining the relations of the corresponding rAES.

Definition 4.1. Let $V^{\underline{T}} = \langle S, T, F, I, \mathsf{m}\rangle$ be an rACN, and let $\lessdot$, $\rightsquigarrow$, $\prec$, $\lhd$ and $\lll$ denote the causation, weak causality, reverse causality, prevention and sustained causation induced by F and I. Then, the corresponding structure $\mathcal{E}_r(V^{\underline{T}})$ is the tuple $(\overline{T}, U, \lessdot, \lll \cup \rightsquigarrow, \prec, \lhd)$, where $U = \{t \in \overline{T} \mid \underline{t} \in \underline{T}\}$ represents the reversible events.

The following result ensures that the above construction generates an rAES.

Theorem 4.2. *Let $V^{\underline{T}}$ be an rACN. Then $\mathcal{E}_r(V^{\underline{T}})$ is an rAES.*

Proof. We show that the conditions in Definition 2.7 are satisfied:

(1) Weak causality is set to $\lll \cup \rightsquigarrow$. By definition, both $\lll$ and $\rightsquigarrow$ are relations over $\overline{T} \times \overline{T}$. Hence, $(\lll \cup \rightsquigarrow) \subseteq \overline{T} \times \overline{T}$.
(2) Prevention corresponds to $\lhd$, which, by definition satisfies $\lhd \subseteq \underline{T} \times \overline{T}$.
(3) Firstly, causation is set to $\lessdot$, which, by definition, satisfies $\lessdot \subseteq \overline{\overline{T}} \times \overline{T}$. Moreover, since $V^{\underline{T}}$ is an rACN, $V_{\overline{T}}$ is an ACN by Definition 3.21(1). Hence, $\lessdot$ is irreflexive because of Definition 3.4(4). By Definition 3.4(4), $\forall t \in \overline{T}.\ \lfloor t \rfloor_{\lessdot} = \{t' \in \overline{T} \mid t' \lessdot^* t\}$ is finite and $(\rightsquigarrow \cup \lessdot)$ is acyclic on $\lfloor t \rfloor_{\lessdot}$. By Definition 3.21(5), $\lll \subset \lessdot$, hence $(\lll \cup \rightsquigarrow \cup \lessdot) = (\rightsquigarrow \cup \lessdot)$. Therefore, $(\rightsquigarrow \cup \lessdot)$ acyclic on $\lfloor t \rfloor_{\lessdot}$ implies $(\lll \cup \rightsquigarrow \cup \lessdot)$ acyclic on $\lfloor t \rfloor_{\lessdot}$.
(4) Reverse causation is set to $\prec$, which is by definition $\prec \subseteq \overline{T} \times \underline{T}$.
    (a) By Definition 3.21(2), $\forall \underline{t} \in \underline{T}$, ${}^{\bullet}t \subseteq {}^{\circ}\underline{t}$. Hence, $t \prec \underline{t}$.
    (b) By Definition 3.21(3), for all $\underline{t} \in \underline{T}$, $K_{\underline{t}} = \{t' \in \overline{T} \mid {}^{\circ}\underline{t} \cap {}^{\bullet}t' \neq \emptyset\}$ is finite and $\rightsquigarrow$ acyclic on $K_{\underline{t}}$. Note that $K_{\underline{t}} = \{t' \in \overline{T} \mid t \prec \underline{t'}\} = \lfloor \underline{t} \rfloor_{\prec}$. Hence, $\lfloor \underline{t} \rfloor_{\prec}$ is finite. It remains to show that $\lll \cup \rightsquigarrow \cup \lessdot$ is acyclic on $K_{\underline{t}}$. As in the previous item,

$(\lll \cup \leadsto \cup \lessdot) = (\leadsto \cup \lessdot)$. Hence, we need to show that $(\leadsto \cup \lessdot)$ is acyclic on $K_{\underline{t}}$. We know that $\leadsto$ is acyclic on $K_{\underline{t}}$, and $\lessdot^+$ is a partial order since $\mathcal{E}_r(V^{\underline{T}})$ is a rACN. Hence, acyclicity on $K_{\underline{t}}$ can only be violated if, for any two $t', t'' \in \lfloor \underline{t} \rfloor_{\prec}$, we have $t' \leadsto t''$ and $t'' \lessdot t'$. However, this implies that $t''^{\bullet} \cap {}^{\circ}t' \neq \emptyset$ and ${}^{\bullet}t'' \cap {}^{\circ}t' \neq \emptyset$, which contradicts the fact that $\mathcal{E}_r(V^{\underline{T}})$ is an rACN.

(5) for all $t \in \overline{T}, \underline{t'} \in \underline{T}$. $t \prec \underline{t'} \ \Rightarrow \neg(\underline{t'} \lhd t)$. Note that $t \prec \underline{t'}$ implies that ${}^{\bullet}t \cap {}^{\circ}\underline{t'} \neq \emptyset$. Consequently, $t^{\bullet} \cap {}^{\circ}\underline{t'} = \emptyset$, indicating that $\neg(\underline{t'} \lhd t)$.

(6) We need to show that $(\overline{T}, \lessdot\!\!\!\lessdot, \lll \cup \leadsto)$ with $\lessdot\!\!\!\lessdot = \lessdot \cap \{(t, t') \mid t \notin U \ \mathit{or} \ \underline{t} \lhd t'\}$ is an AES. We now check the conditions of Definition 2.1. First note that $\lessdot\!\!\!\lessdot = \lll$ by the definition of $\lll$.

  (a) We need to show that $\lessdot\!\!\!\lessdot \subseteq \overline{T} \times \overline{T}$ is an irreflexive partial order defined such that $\forall t \in \overline{T}$. $\lfloor t \rfloor_{\lessdot\!\!\!\lessdot}$ is finite. The fact that $\lessdot\!\!\!\lessdot$ irreflexive and antisymmetric because $\lll \subseteq \lessdot +$ and $\lessdot^+$ is an irreflexive partial order. Transitivity follows from the definition of $\lll$ (Definition 3.21(5)).
  $\lfloor t \rfloor_{\lessdot\!\!\!\lessdot}$ is finite because $\lessdot\!\!\!\lessdot = \lll \subseteq \lessdot$ and $\lfloor t \rfloor_{\lessdot}$ is finite (Definition 3.4(4)).

  (b) $(\lll \cup \leadsto) \subseteq \overline{T} \times \overline{T}$ by definition. We need to show that for all $t, t' \in \overline{T}$:
    (i) $t \lll t' \ \Rightarrow \ t \ (\lll \cup \leadsto) \ t'$; which immediately follows because $\lll \subseteq (\lll \cup \leadsto)$.
    (ii) $(\lll \cup \leadsto) \cap (\lfloor t \rfloor_{\lessdot\!\!\!\lessdot} \times \lfloor t \rfloor_{\lessdot\!\!\!\lessdot})$ is acyclic. Note that $(\lll \cup \leadsto) \cap (\lfloor t \rfloor_{\lessdot\!\!\!\lessdot} \times \lfloor t \rfloor_{\lessdot\!\!\!\lessdot}) = (\lll \cup \leadsto) \cap (\lfloor t \rfloor_{\lll} \times \lfloor t \rfloor_{\lll}) = (\lfloor t \rfloor_{\lll} \times \lfloor t \rfloor_{\lll}) \subseteq (\lfloor t \rfloor_{\lessdot} \times \lfloor t \rfloor_{\lessdot})$, which is acyclic because $V_{\overline{T}}$ is an ACN (Definition 3.4(4)).
    (iii) if $t \# t'$ and $t' \lessdot\!\!\!\lessdot t''$ then $t \# t''$. It follows from Definition 3.21(5).

We can conclude that $\mathcal{E}_r(V)$ is indeed an rAES. □

Example 4.3. Consider the rACN V_1 Figure 7. The corresponding rAES $\mathcal{E}_r(V_1)$ is defined as H_1 in Example 2.8. In fact, within V_1, causality is characterized by $b \lessdot c$ and $a \lessdot c$ due to the conditions ${}^{\bullet}a \cap {}^{\circ}c \neq \emptyset$ and ${}^{\bullet}b \cap {}^{\circ}c \neq \emptyset$. Moreover, weak causality includes $b \nearrow a$ as a consequence of $b \leadsto a$, and also encompasses $a \nearrow c$ and $b \nearrow c$ induced by the sustained causation, which coincides with $\lessdot$. On the other hand, reverse causality encompasses $b \prec \underline{b}$ and $a \prec \underline{b}$, while prevention is limited to $\underline{b} \lhd c$.

Recall that an rACN lacking reversing transitions (i.e., $V^{\emptyset}$) is essentially an ACN. Consequently, the construction outlined in Definition 4.1 is applicable to ACNs as well.

Corollary 4.4. *Let $C = \langle S, T, F, I, \mathsf{m} \rangle$ be an* ACN. *Then $\mathcal{E}_r(C)$ is an* AES.

$\mathcal{E}_r$ extends to a functor by observing that an rACN-morphism $(f_S, f_T) : V_0 \to V_1$ induces an rAES-morphism $\mathcal{E}_r(f_S, f_T) = f_T$.

Proposition 4.5. $\mathcal{E}_r : \mathbf{RACN} \to \mathbf{RAES}$ *is a well-defined functor.*

The construction linking a net to an event structure adheres to the conventional intuition, where places represent appropriate subsets of events, and transitions are connected to places based on the relations specified in the event structure. In our context, the event subsets have a cardinality of at most one, meaning they are either the empty set or a singleton. To streamline notation, we will use the symbol e to represent the singleton set $\{e\}$ and, consequently, avoid the use of braces.

Definition 4.6. Let $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ be an rAES. Then, the associated net $\mathcal{N}_r(\mathsf{H})$ is defined as $\langle S, E \cup \{\underline{u} \mid u \in U\}, F, I, \mathsf{m} \rangle$ where

$$
\begin{aligned}
S &= \{\ (\emptyset, e)\ \mid\ e \in E\} \qquad \cup \{\ (e, e)\ \mid\ e \in E\} \qquad \cup \{\ (\emptyset, \emptyset)\ \}\\
F &= \{\ ((\emptyset, e), e)\ \mid\ e \in E\} \cup \{\ (e, (e, e))\ \mid\ e \in E\} \quad \cup\\
&\quad\ \{\ ((u, u), \underline{u})\ \mid\ u \in U\} \cup \{\ (\underline{u}, (\emptyset, u))\mid\ u \in U\}\\
I &= \{\ ((\emptyset, e'), e)\ \mid\ e' < e\} \cup \{\ ((e', e'), e)\ \mid\ e \nearrow e'\} \cup\\
&\quad\ \{\ ((\emptyset, e), \underline{u})\ \mid\ e \prec \underline{u}\ \} \cup \{\ ((e, e), \underline{u})\mid\ \underline{u} \lhd e\}\\
\mathsf{m} &= \{\ (\emptyset, e)\ \mid\ e \in E\} \qquad \cup \{\ (\emptyset, \emptyset)\ \}
\end{aligned}
$$

Events in E are mapped to the forward transitions of the net. For each event e, two associated places are considered: $(\emptyset, e)$ signifies that the event e has not occurred, while (e, e) indicates its execution. As a result, the preset of the forward transition e is $(\emptyset, e)$, and its postset is (e, e). A transition $\underline{u}$ corresponds to the undoing of the forward transition u. This transition consumes tokens from (u, u) and produces them in $(\emptyset, u)$. The inclusion of the special place $(\emptyset, \emptyset)$ serves no operational purpose; its motivation is purely technical. Its significance will become clearer when demonstrating that the mapping extends to a functor, as explained in the subsequent discussion. The inhibitor arcs in I model both causality (forward or backward) and precedence (forward or backward). Additionally, all places not appearing in the postset of forward transitions are initially marked.

Theorem 4.7. *If $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ is a* rAES*, then $\mathcal{N}_r(\mathsf{H})$ is an* rACN*.*

Proof. Consider the rAES $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ and $\mathcal{N}_r(\mathsf{H}) = \langle S, E \cup \{\underline{u} \mid u \in U\}, F, I, \mathsf{m}\rangle$ as defined in Definition 4.6. Initially, we show that the net $\langle S, E, F', I', \mathsf{m}\rangle$, where F' and I' are the restrictions of F and I to the transitions in E, is a pACN. The first three conditions of Definition 3.4 follow through a straightforward examination of the definition of $\mathcal{N}_r(\mathsf{H})$. The fourth condition is a consequence of H being an rAES, satisfying Definition 2.7(3), which translates to Condition 4 in Definition 3.4. The final condition is implied by the fact that $(E, \prec\!\!\prec, \nearrow)$ constitutes an AES with $\prec\!\!\prec \subseteq <$.

For each transition in $\{\underline{u} \mid u \in U\}$, there is precisely one corresponding transition in E, represented by u, as dictated by H being an rAES. Condition 3 in Definition 3.21—asserting that $K_{\underline{u}}$ is finite—stems from $\lfloor \underline{u} \rfloor_{\prec}$ being finite in an rAES. For the same reason, causation and weak causality are acyclic on such sets. Hence, Condition 4 is also satisfied by construction. The condition 5 of Definition 3.21 is verified as it mimic the requirement of sustained causality of the rAES H. The final condition depends again on the analogous one in rAES. We can conclude that indeed $\mathcal{N}_r(\mathsf{H})$ is an rACN. □

Example 4.8. Consider the rAES H' from Example 2.8. The places are $(\emptyset, \emptyset)$, $(\emptyset, a)$, $(\emptyset, b)$, $(\emptyset, c)$, (a, a), (b, b) and (c, c). The corresponding transitions are a, b, c (forward ones) and $\underline{b}, \underline{c}$ (reversing ones). The flow arcs adhere to the specifications outlined in Definition 4.6, while the inhibitor arcs are determined by the relations within H'. The resulting net bears resemblance to the one depicted in Figure 7c, with distinct place labels and the exclusion of the isolated place $(\emptyset, \emptyset)$.

Corollary 4.9. *If $\mathsf{G} = (E, <, \nearrow)$ is an* AES*, then $\mathcal{N}_r(E, \emptyset, <, \nearrow, \emptyset, \emptyset)$ is an* ACN*.*

We establish the extension of the mapping $\mathcal{N}_r$ to a functor by demonstrating that any rAES-morphism $f : \mathsf{H}_0 \to \mathsf{H}_1$ induces a rACN-morphism $\mathcal{N}_r(f) : \mathcal{N}_r(\mathsf{H}_0) \to \mathcal{N}_r(\mathsf{H}_1)$.

Definition 4.10. Let $f : \mathsf{H}_0 \to \mathsf{H}_1$ be rAES-morphism. Assume $\mathcal{N}_r(\mathsf{H}_i) = \langle S_i, E_i \cup \{\underline{u} \mid u \in U_i\}, F_i, I_i, \mathsf{m}_i\rangle$ for $i = 0, 1$. We define $\mathcal{N}_r(f) = (f_S, f_T)$ as follows:

(1) $f_S \subseteq S_0 \times S_1$ where

$$(s_0, s_1) \in f_S \iff \begin{cases} s_0 = s_1 = (\emptyset, \emptyset);\ or \\ s_0 = (\emptyset, \emptyset)\ \wedge\ s_1 = (\emptyset, e)\ \wedge\ f^{-1}(e) = \bot;\ or \\ s_0 = (\emptyset, e)\ \wedge\ f(e) \neq \bot\ \wedge\ s_1 = (\emptyset, f(e));\ or \\ s_0 = (e, e)\ \wedge\ f(e) \neq \bot\ \wedge\ s_1 = (f(e), f(e)). \end{cases}$$

(2) $f_T : E_0 \cup \{\underline{u} \mid u \in U_0\} \to E_1 \cup \{\underline{u} \mid u \in U_1\}$ is defined as follows:

$$f_T(t) = \begin{cases} f(t) & \text{if } t \in E_0,\ \wedge\ f(e) \neq \bot \\ \underline{f(u)} & \text{if } t = \underline{u} \text{ and } f(u) \neq \bot. \end{cases}$$

According to Condition (1), the relation f_S is established to associate the place $(\emptyset, \emptyset)$ in the source with its counterpart $(\emptyset, \emptyset)$ in the target. Additionally, it is linked to all the places $(\emptyset, e)$ in the target, where e is not in the image of the morphism f. This connection is crucial to guarantee that the definition satisfies the condition regarding the initial markings of ACN-morphisms (Definition 3.10(4)). Each place in the source, denoted as $(\emptyset, e)$, which corresponds to the preset of the event e, is paired with the place in the target representing the preset of the corresponding event $f(e)$; denoted as $(\emptyset, f(e))$. Similarly, each place in the source, denoted as (e, e), indicating the postset of the event e, is associated with its counterpart in the target representing the postset of the corresponding event $f(e)$; designated as $(f(e), f(e))$. The mapping for transitions is straightforward. A transition representing a forward event e is mapped to a transition representing the corresponding event $f(e)$. Reversing transitions are similarly mapped to reversing transitions.

The following result ensures that the construction above actually gives an rACN-morphism.

Proposition 4.11. *If $f : \mathsf{H}_0 \to \mathsf{H}_1$ is an rAES-morphism, then $\mathcal{N}_r(f) : \mathcal{N}_r(\mathsf{H}_0) \to \mathcal{N}_r(\mathsf{H}_1)$ is a well-defined rACN-morphism.*

Proof. We verify that $\mathcal{N}_r(f)$ constitutes an rACN-morphism from $\mathcal{N}_r(\mathsf{H}_0)$ to $\mathcal{N}_r(\mathsf{H}_1)$ by checking the conditions of Definition 3.26. Assume $\mathcal{N}_r(\mathsf{H}_i) = \langle S_i, T_i, F_i, I_i, \mathsf{m}_i \rangle$ for $i = 0, 1$. By definition of $\mathcal{N}_r(\mathsf{H}_i)$, note that $\overline{T_i} = E_i$ and $\underline{T_i} = \{\underline{u} \mid u \in U_i\}$.

(1) By definition of $\mathcal{N}_r(f)$, $f_T(\overline{T_0}) = f_T(E_0) \subseteq f_T(E_1) = \overline{T_1}$. Similarly, $f_T(\underline{T_0}) = f_T(\{\underline{u} \mid u \in U_0\}) \subseteq f_T(\{\underline{u} \mid u \in U_1\}) = \underline{T_1}$.
(2) We need to show that $(f_S, f_T|_{E_0}) : V_{E_0} \to V_{E_1}$ is an ACN-morphism. Hence, we check the conditions of Definition 3.10.
   (a) for all $e \in E_0$ if $f_T(e) \neq \bot$ then
      (i) We show that ${}^\bullet f_T(e) = f_S({}^\bullet e)$ and $f_T(e)^\bullet = f_S(e^\bullet)$

$$\begin{aligned} {}^\bullet f_T(e) &= {}^\bullet f(e) && \text{By def. of } f_T(e). \\ &= \{(\emptyset, f(e))\} && \text{By def. of } F_0. \\ &= f_S(\{(\emptyset, e)\}) && \text{By def. of } f_S. \\ &= f_S({}^\bullet e) && \text{By def. of } F_0. \end{aligned}$$

Analogously, $f_T(t)^\bullet = f(e)^\bullet = \{(f(e), f(e))\} = f_S(\{(e, e)\}) = f_S(t^\bullet)$.
      (ii) Assume now $(s, f_T(e)) \in I_1$. We distinguish two cases, either $s^\bullet \neq \emptyset$ or $s^\bullet = \emptyset$. In the first case, if $f_S^{-1}(s) = \emptyset$ we have nothing to prove, thus assume that $f_S^{-1}(s) \neq \emptyset$ By Definition 4.6, if $(s, f_T(e)) \in I_1$, then there are two cases:

- $s = (\emptyset, e')$ and $e' <_1 f_T(e)$: Since f is an rAES-morphism, it is also an AES-morphism. Consequently, it preserves causes (Definition 2.5(1)). Therefore, it should hold that $\lfloor f(e) \rfloor \subseteq f(\lfloor e \rfloor)$. Since $e' \in \lfloor f(e) \rfloor$, there must be $e_0 \in \lfloor e \rfloor$ such that $f(e_0) = e'$. Then, it suffices to consider $s' = (\emptyset, e_0)$. It is immediate that $s' \in f_S^{-1}(s)$ because $f(e_0) = e'$. Moreover, $((\emptyset, e_0), e) \in I_0$ because $e_0 <_0 e$ (Definition 4.6).
- $s = (e', e')$ and $f_T(e) \nearrow e'$. As $f_S^{-1}(s) \neq \emptyset$, we can consider all $e_0 \in E_0$ such that $f(e_0) = e'$. Since f is an rAES-morphism, it reflects weak causality (Definition 2.5(2a)). Hence, for all e_0 such that $f(e_0) = e'$, we have that $e \nearrow e_0$. Hence, $((e_0, e_0), e) \in I_0$ by Definition 4.6.

(b) We show that $\forall e, e' \in E_0$ if $f_T(e) \neq \bot \neq f_T(e')$ then $f_T(e) = f_T(e') \;\Rightarrow\; e \natural_0 e'$. Since f is an rAES-morphism, $f_T(e) = f_T(e')$ implies $e \#_0 e'$, i.e., $e \nearrow e'$ and $e' \nearrow e$. Then, by Definition 4.6, $((e, e), e') \in I$ and $((e', e'), e) \in I$. Hence, $e \natural e'$.

(c) We show that $\forall s_1 \in S_1.\ \forall s_0, s_0' \in f_S^{-1}(s_1).\ s_0 \neq s_0'$ implies ${s_0}^\bullet \natural_0 {s_0'}^\bullet$ or ${}^\bullet s_0 \natural_0 {}^\bullet s_0'$. We proceed by case analysis on the shape of s_1.

- $s_1 = (\emptyset, \emptyset)$ or $s_1 = (\emptyset, e)$ and $f^{-1}(e) = \bot$. Then, $f_S^{-1}(s_1)$ is a singleton by Definition 4.10; hence, the thesis follows vacuously.
- $s_1 = (\emptyset, e_1)$ with $e_1 = f(e_0)$ for some $e_0 \in E_0$. Then, there exist $e_0, e_0' \in E_0$ such that $f(e_0) = f(e_0') = e_1$ and $s_0 = (\emptyset, e_0)$ and $s_0' = (\emptyset, e_0')$. Since f is an rAES-morphism, $e_0 \#_0 e_0'$. Therefore, $e_0 \nearrow_0 e_0'$ and $e_0' \nearrow_0 e_0$. By Definition 4.6, both $((e_0, e_0), e_0')$ and $((e_0', e_0'), e_0)$ are in I_0. Hence, $e_0 \natural_0 e_0'$. Therefore, ${s_0}^\bullet \natural_0 {s_0'}^\bullet$.
- $s_1 = (e_1, e_1)$ with $e_1 = f(e_0)$ for some $e_0 \in E_0$. By reasoning analogously to the previous case, it is shown that ${}^\bullet s_0 \natural_0 {}^\bullet s_0'$.

(d) We show that $[\![f_S(\mathsf{m}_0)]\!] = \mathsf{m}_1$. By Definition 3.4(2), $\mathsf{m}_i = S_i \setminus {T_i}^\bullet$. By Definition 4.6,

$$\mathsf{m}_i = \{(\emptyset, e) \mid e \in E_i\} \;\cup\; \{(\emptyset, \emptyset)\}$$

Consequently,

$$f_S(\mathsf{m}_0) = f_S(\{(\emptyset, e) \mid e \in E_i\}) \cup f_S(\{(\emptyset, \emptyset)\})$$

By Definition 4.10(1),

$$f_S(\mathsf{m}_0) = \{\!|(\emptyset, f(e)) \mid e \in E_0, f(e) \neq \bot|\!\} + \{\!|(\emptyset, e) \mid e \in E_1, f^{-1}(e) \neq \bot|\!\} + \{\!|(\emptyset, \emptyset)|\!\}$$

Then,

$$[\![f_S(\mathsf{m}_0)]\!] = \{(\emptyset, f(e)) \mid e \in E_0, f(e) \neq \bot\} \cup \{(\emptyset, e) \mid e \in E_1, f^{-1}(e) \neq \bot\} \cup \{(\emptyset, \emptyset)\}$$

Note that the first set on the right-hand-side contains all places $(\emptyset, e)$ with $e \in E_1$ that are in the image of f, while the second one contains all places $(\emptyset, e')$ with $e' \in E_1$ that are not in the image in of f. Hence, they cover all the elements of E_1. Hence,

$$[\![f_S(\mathsf{m}_0)]\!] = \{(\emptyset, e')) \mid e' \in E_1\} \;\cup\; \{(\emptyset, \emptyset)\} \;=\; \mathsf{m}_1$$

(3) We check that $\forall \underline{t} \in \underline{T_0}$. If $\underline{t} \in \underline{T_0}$, then $\underline{t} = \underline{u}$ with $u \in U_0$. Then, the corresponding forward transition is $t = u$ with $u \in U_0 \subseteq E_0$. If $f_T(t) = f_T(u) \neq \bot$ then $f(t) = f(u) \neq \bot$ by Definition 4.10(2).

(a) By Definition 4.10(2), $f_T(\underline{t}) = \underline{f(u)} \neq \bot$. Since $f_T(t) = f(u)$, $\underline{f_T(t)} = \underline{f(u)}$ by Definition 4.6. Hence, $f_T(\underline{t}) = \underline{f_T(t)}$.

(b) Consider now $(s, f_T(\underline{t})) \in I_1$, we have again to distinguish two cases:

- $s = (\emptyset, e)$ and $e \prec f_T(\underline{u})$. Since f is an rAES-morphism, by Definition 2.12(2), $\lfloor \underline{f(u)} \rfloor_{\prec_1} \subseteq f(\lfloor \underline{u} \rfloor_{\prec_0})$. Consequently, there exists $e_0 \in \lfloor \underline{u} \rfloor_{\prec_0}$ such that $f(e_0) = e$. Then, take $s' = (\emptyset, e_0)$ and note that $((\emptyset, e_0), \underline{u}) \in I_0$ by Definition 4.6.
- $s = (f(e_0), f(e_0))$ and $f(\underline{u}) \lhd f(e)$. By Definition 4.10, $(e, e) \in f_S^{-1}(s)$. Since f is an rAES-morphism , $\underline{u} \lhd e$, by Definition 2.12(3). Then, $((e, e), \underline{u}) \in I_0$, by Definition 4.10. □

Proposition 4.12. $\mathcal{N}_r : \mathbf{RAES} \to \mathbf{RACN}$ *is a well-defined functor.*

Proposition 4.13. *Let* H *be an* rAES*, then* $\mathcal{E}_r(\mathcal{N}_r(\mathsf{H})) = \mathsf{H}$.

Proof. Consider the $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ and the associated rACN $\mathcal{N}_r(\mathsf{H}) = \langle S, T, F, I, \mathsf{m} \rangle$ as defined in Definition 4.6. Let $\mathcal{E}_r(\mathcal{N}_r(\mathsf{H})) = (E', U', <', \nearrow', \prec', \lhd')$. We now show that the elements coincides:

- E': By Definition 4.6, $T = E \cup \{\underline{u} \mid u \in U\}$ with $\overline{T} = E$. By Definition 4.1, $E' = \overline{T} = E$.
- U': By Definition 4.6, $\underline{T} = \{\underline{u} \mid u \in U\}$. By Definition 4.1, $U' = \{t \in \overline{T} \mid \underline{t} \in \underline{T}\}$. By substituting $\overline{T}$ and $\underline{T}$, $U' = \{t \in E \mid \underline{t} \in \{\underline{u} \mid u \in U\}\}$. Hence, $U' = U$.
- $<'$: By Definition 4.1, $<' = \lessdot$. Recall that $t \lessdot t'$ iff ${}^{\bullet}t \cap {}^{\circ}t' \neq \emptyset$. By inspecting the definition of F and I, ${}^{\bullet}t \cap {}^{\circ}t' \neq \emptyset$ implies $t = e$, $t = e'$ with $e, e' \in E$ and $e < e'$. Hence, $<' = <$.
- $\nearrow'$: By Definition 4.1, $\nearrow' = \lll \cup \rightsquigarrow$. We now show that $e \nearrow e'$ iff $e \nearrow' e'$. Assume that $e \nearrow e'$. Then, $((e', e'), e) \in I$, by Definition 4.6. Hence, $e \rightsquigarrow e'$. Therefore, $e \nearrow' e'$. On the contrary, assume $e \nearrow' e'$. Then, either $e \lll e'$ or $e \rightsquigarrow e'$. If $e \rightsquigarrow e'$, then $((e', e'), e) \in I$. By Definition 4.6, $e \nearrow e'$. In case $e \lll e'$, we recall that $\lll$ is the transitive closure of $\lessdot \cap \{(t, t') \mid \underline{t} \notin \underline{T} \ \mathit{or} \ {}^{\circ}\underline{t} \cap t'^{\bullet} \neq \emptyset\}$. Hence, $e \lll e'$ implies $e \lessdot e'$. Therefore, ${}^{\bullet}e \cap {}^{\circ}e' \neq \emptyset$. Then, $((\emptyset, e), e') \in I$. By inspecting Definition 4.6, $e < e'$. Since H is an rAES, $e < e'$ implies $e \nearrow e'$ (Definition 2.1(3a)).
- $\prec'$: Then, $\prec' = \prec_{\mathcal{N}_r(\mathsf{H})}$. Therefore, $e \prec' \underline{u}$ implies ${}^{\bullet}e \cap {}^{\circ}\underline{u} \neq \emptyset$. Then, $((\emptyset, e), \underline{u}) \in I$. By inspecting Definition 4.6, we conclude that $e \prec \underline{u}$.
- $\lhd'$: Then, $\lhd' = \lhd_{\mathcal{N}_r(\mathsf{H})}$. Therefore, $\underline{u} \lhd' e$ implies $e^{\bullet} \cap {}^{\circ}\underline{u} \neq \emptyset$. Then, $((e, e), \underline{u}) \in I$. By inspecting Definition 4.6, we conclude that $\underline{u} \lhd e$. □

The main result is that there is a precise relation between **RACN** and **RAES**, namely the functor $\mathcal{N}_r$ is the left adjoint of $\mathcal{E}_r$.

Theorem 4.14. *The functor* $\mathcal{N}_r : \mathbf{RAES} \to \mathbf{RACN}$ *is the left adjoint of the functor* $\mathcal{E}_r : \mathbf{RACN} \to \mathbf{RAES}$.

Proof. Let $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ be an rAES and $\mathcal{N}_r(\mathsf{H}) = \langle S, T, F, I, \mathsf{m} \rangle$ the associated rACN. By Proposition 4.13, we have that $\mathsf{H} = \mathcal{E}_r(\mathcal{N}_r(\mathsf{H}))$ and we have the obvious identity mapping $id : \mathsf{H} \to \mathcal{E}_r(\mathcal{N}_r(\mathsf{H}))$. To prove the result, we show that for any rACN $V = \langle S_V, T_V, F_V, I_V, \mathsf{m}_V \rangle$, and any rAES-morphism $g : \mathsf{H} \to \mathcal{E}_r(V)$, there exists a unique morphism $h = (h_S, h_T) : \mathcal{N}_r(\mathsf{H}) \to V$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathsf{H} \arrow[r, "id"] \arrow[dr, "g"'] & \mathcal{E}_r(\mathcal{N}_r(\mathsf{H})) \arrow[d, dashed, "\mathcal{E}_r(h)"] \\
 & \mathcal{E}_r(V)
\end{tikzcd}$$

We begin by establishing the existence of the morphism. Let $h = (h_S, h_T)$ be defined as follows:

$$h_S(s_0, s_1) \text{ iff } \begin{cases} s_0 = (\emptyset, \emptyset) \wedge s_1 \in \mathsf{m}_V \wedge\ g({s_1}^\bullet) = \bot \\ s_0 = (\emptyset, \{e\}) \wedge s_1 \in \mathsf{m}_V \wedge\ g(e) \in g({s_1}^\bullet) \\ s_0 = (\{e\}, \{e\}) \wedge s_1 \in g(e)^\bullet \end{cases}$$

and $h_T(e) = g(e)$ and $h_T(\underline{u}) = \underline{g(u)}$ if $u \in U$.

We verify that h is an rACN-morphism from $\mathcal{N}_r(\mathsf{H})$ to V by checking the conditions of Definition 3.26. By definition of $\mathcal{N}_r(\mathsf{H})$, note that $\overline{T} = E$ and $\underline{T} = \{\underline{u} \mid u \in U\}$.

(1) By definition of h, $h_T(\overline{T}) = g(\overline{T}) = g(E)$. Since g is an rAES-morphism, $g(E) \subseteq \overline{T_V}$. Similarly, $h_T(\underline{T}) = g(\{\underline{u} \mid u \in U\})$. Since g is an rAES-morphism, $g(\{\underline{u} \mid u \in U\}) \subseteq \underline{T_V}$.

(2) We need to show that $(h_S, h_T|_E) : \mathcal{N}_r(\mathsf{H})_E \to V_{\overline{T_V}}$ is an ACN-morphism. Hence, we check the conditions of Definition 3.10.

  (a) for all $e \in E$ if $h_T(e) \neq \bot$ then

   (i) We show that ${}^\bullet h_T(e) = h_S({}^\bullet e)$ and $h_T(e)^\bullet = h_S(e^\bullet)$

$$\begin{aligned} {}^\bullet h_T(e) &= {}^\bullet g(e) && \text{By def. of } h_T(e). \\ &= \{(\emptyset, g(e))\} && \text{By def. of } F. \\ &= h_S(\{(\emptyset, e)\}) && \text{By def. of } h_S. \\ &= h_S({}^\bullet e) && \text{By def. of } F. \end{aligned}$$

   Analogously, $h_T(t)^\bullet = g(e)^\bullet = \{(g(e), g(e))\} = h_S(\{(e, e)\}) = h_S(t^\bullet)$.

   (ii) We now check that inhibitor arcs are reflected. Consider now $e \in E$ and $s \in S_V$ such that $(s, h_T(e)) \in I_V$.

    (A) Case ${}^\bullet s = \emptyset$. By Proposition 3.12, $h_S^{-1}(s) \neq \emptyset$. Hence, there exists $s' \in S$ such that $h_S(s') = s$. Moreover, $s' = (\emptyset, e')$ and $g(e') = s^\bullet$. As $(s, h_T(e)) \in I_V$ implies that $s^\bullet \lessdot h_T(e)$, i.e., $s^\bullet = g(e') < g(e)$. As g is an rAES-morphisms, $e' < e$. Consequently, $(s', e) \in I$;

    (B) Case ${}^\bullet s \neq \emptyset$. For any $s' \in S$ such that $h_S(s') = s$, it should be $s' = (e', e')$ and $g(e') = {}^\bullet s$. As $(s, h_T(e)) \in I_V$ implies that $s^\bullet \nearrow h_T(e)$; hence $s^\bullet = g(e') \nearrow g(e)$. As g is an rAES-morphisms, $e' \nearrow e$. Consequently, $(s', e) \in I$;

  (b) We show that $\forall e, e' \in E$ if $h_T(e) \neq \bot \neq h_T(e')$ then $h_T(e) = h_T(e') \;\Rightarrow\; e \natural e'$. Note that $h_T(e) = h_T(e')$ implies $g(e) = g(e')$, by the definition of h_T. Since g is an rAES-morphism, $g(e) = g(e')$ implies $e \# e'$, i.e., $e \nearrow e'$ and $e' \nearrow e$. Then, by Definition 4.6, $\mathcal{N}_r(\mathsf{H})$ is defined such that $((e, e), e') \in I$ and $((e', e'), e) \in I$. Hence, $e \natural e'$.

  (c) We show that $\forall s_1 \in S_V.\ \forall s_0, s_0' \in h_S^{-1}(s_1).\ s_0 \neq s_0'$ implies ${s_0}^\bullet \natural {s_0'}^\bullet$ or ${}^\bullet s_0 \natural {}^\bullet s_0'$. We proceed by case analysis on s_1 (according to the definition of h_S):

  - $s_1 \in \mathsf{m}_V \wedge\ g({s_1}^\bullet) = \bot$: Hence, $h_S^{-1}(s_1)$ is the singleton $\{(\emptyset, \emptyset)\}$. Therefore, thesis follows vacuously.
  - $s_1 \in \mathsf{m}_V \wedge\ g({s_1}^\bullet) \neq \bot$. Then, there exist $e, e' \in E$ such that $g(e) = g(e')$ and $s_0 = (\emptyset, e)$ and $s_0' = (\emptyset, e')$. Since g is an rAES-morphism, $e \# e'$. Therefore, $e \nearrow e$ and $e' \nearrow e$. By Definition 4.6, both $((e, e), e')$ and $((e', e'), e)$ are in I. Hence, $e \natural e'$. Therefore, ${s_0}^\bullet \natural {s_0'}^\bullet$.
  - $s_1 \in (g(e))^\bullet)$. As in the previous case, there exist $e, e' \in E$ such that $g(e) = g(e')$ and $s_0 = (e, e)$ and $s_0' = (e', e')$. Since g is an rAES-morphism, $e \# e'$. Therefore,

$e \nearrow e$ and $e' \nearrow e$. By Definition 4.6, both $((e, e), e')$ and $((e', e'), e)$ are in I. Hence, $e \natural e'$. Therefore, ${}^\bullet s_0 \natural {}^\bullet s'_0$.

(d) We prove that $[\![h_S(\mathsf{m})]\!] = \mathsf{m}_V$. By Definition 4.6,

$$\mathsf{m} = \{(\emptyset, e) \mid e \in E\} \cup \{(\emptyset, \emptyset)\}$$

Consequently,

$$h_S(\mathsf{m}) = h_S(\{(\emptyset, e) \mid e \in E\}) \cup h_S(\{(\emptyset, \emptyset)\})$$

By the definition of h_S,

$$h_S(\mathsf{m}) = \{\!\!\{s_1 \mid s_1 \in \mathsf{m}_V \wedge \ g(s_1{}^\bullet) = \bot\}\!\!\} + \{\!\!\{s_1 \mid s_1 \in \mathsf{m}_V \wedge \ g(s_1{}^\bullet) \neq \bot\}\!\!\}$$

Note that both multisets on the right-hand-side are actually sets. Moreover, their union covers all places in m_V. Hence,

$$[\![h_S(\mathsf{m}_0)]\!] = \mathsf{m}_V$$

(3) $\forall \underline{t} \in \underline{T}$. If $\underline{t} \in \underline{T}$, then $\underline{t} = \underline{u}$ with $u \in U$. Then, the corresponding forward transition is $t = u$ with $u \in U \subseteq E$. If $h_T(t) = h_T(u) \neq \bot$ then $g(t) = g(u) \neq \bot$ by the definition of h.

(a) By the definition of h_T, $h_T(\underline{t}) = h_T(\underline{u}) = \underline{g(u)} = \underline{h_T(u)}$.

(b) Consider $(s, h_T(\underline{t})) \in I_V$.

(i) Case $s^\bullet \cap \overline{T_V} \neq \emptyset$. If $h_T^{-1}(s) \neq \emptyset$ then there exists a $s' = (\emptyset, e')$ and $g(e') = s^\bullet$. Since, $(s, \underline{t}) \in I_V$, it implies $posts = g(e') \prec g(\underline{t})$. Since g is a rAES-morphism, $e \prec \underline{t}$. Therefore, $(s', \underline{t}) \in I$

(ii) Case ${}^\bullet s \cap \overline{T_V} = \emptyset$. Take $s' \in h_T^{-1}(s)$. Hence, $s' = (e, e)$ and $g(e') = {}^\bullet s$. Since, $(s, \underline{t}) \in I_V$, it implies ${}^\bullet s = g(\underline{t}) \lhd g(e')$. Since g is a rAES-morphism, $\underline{t} \lhd e$. Therefore, $(s', \underline{t}) \in I$.

The uniqueness of the mapping is established by assuming the existence of another $h' = (h'_S, h'_T)$. As $\mathcal{E}_r(h')$ makes the diagram commutative, we can conclude $g = h'_T$. Consequently, h'_S must be the relation h_S, concluding the proof. □

Denoting with $\mathcal{E}$ and $\mathcal{N}$ the functors defined as $\mathcal{E}_r$ and $\mathcal{N}_r$ but acting on objects that are either ACNs or AESs, i.e. if we restrict our attentions to the two full and faithful subcategories of **ACN** and **AES**, we have that the same relation the between them exists.

Theorem 4.15. *The functor* $\mathcal{N}$: **AES** $\to$ **ACN** *is the left adjoint of the functor* $\mathcal{E}$: **ACN** $\to$ **AES**.

Corollary 4.16. *The adjunctions* $\mathcal{E}_r \vdash \mathcal{N}_r$ *and* $\mathcal{E} \vdash \mathcal{N}$ *are coreflections.*

Proof. It is suffices to observe that the units of the adjunctions are natural isomorphisms. □

5. Applications

Reversible Debugging. One of the successful applications of reversibility is causal-consistent *reversible debugging* [9, 19]. Reversible debuggers extend the classical ones with the possibility to get back in the execution of a program, so to find the source of a bug in an effective way. Causal-consistent reversible debuggers improve on reversible ones by exploiting causal information while undoing computations. So far this technique has been applied to message passing concurrent systems (e.g., actor-like languages), but not to shared-memory based concurrency. Consider the next code snippet consisting of a two-threaded program that accesses dynamically allocated memory.

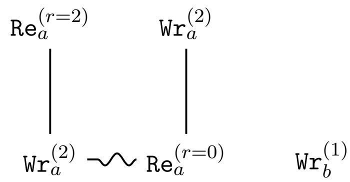
```
1  pthread_mutex_t m = PTHREAD_MUTEX_INITIALIZER;
2  int *x;
3
4  void thread(void *arg)
5  {
6      pthread_mutex_lock(&m);
7      if(x != NULL)
8          doSomething(x);
9      pthread_mutex_unlock(&m);
10 }

11 int main()
12 {
13     x= malloc(sizeof(int));
14     pthread_t t;
15     pthread_create(&t, NULL, thread, NULL);
16     pthread_mutex_lock(&m);
17         free(x);
18     pthread_mutex_unlock(&m);
19     return 0;
20 }
```

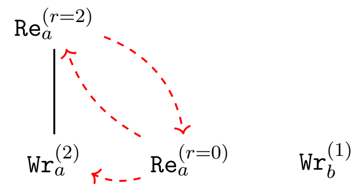
The behaviour of the program can be thought in terms of events. Take a as the event corresponding to the initialisation of x at line 13, b for the instruction at line 8 and c for the one at line 17. It is clear that both b and c causally depend on a ($a < b$ and $a < c$); while c can happen after b, b cannot happen after c, that is $b \nearrow c$. Moreover, the reversal of a complete execution of the program should ensure that c is reversed (i.e., the memory is allocated) before b is reversed, hence $\underline{b} \lhd c$. Consider instead a version of the program in which c is executed outside a critical section (e.g., without acquiring and releasing the lock). In this case, the execution may raise a *segmentation fault* error. When debugging such a faulty execution, the programmer would observe that the execution violates $b \nearrow c$ because b happened after c. On the hand, an execution can be visualised in terms of events, i.e., the programmer can be provided with a high-level description of the current state of the system (a configuration of the event structure) along with the relevant dependencies. On the other hand, the instrumented execution of the program and of its reversal can be handled by the underlying operational model (i.e., a reversible causal net). Also, one could think of the undoing of an event as a backward breakpoint. That is, one could trigger the undoing of an event from the net and then the debugger will execute all the necessary backward steps in the code, to undo such event. The seamless integration of rAES and rACN with causal-consistent reversible debuggers can be a nice exploitation of our results, which we will consider in future work.

Another way to exploit our results is to generate *directly* a reversible semantics for languages for shared memory [5, 13]. In these works, weak memory models are interpreted as event structures. By exploiting our results, we could use *r*PES to interpret such models and give them a reversible debugging tool in terms of corresponding Petri net. For example, let us consider a simple snippet (taken from [5])

$$r \leftarrow a \,||\, a := 2 \,||\, b := 1$$



(A) ES (from [5])



(B) AES

where r is a thread-local variable (e.g., a register) while a and b are global shared variables, and all the variables are initialised to 0. The interpretation of such a snippet in terms of event structures is depicted in Figure 9a, where the assignment of 2 to a (event $\mathtt{Wr}_a^{(2)}$), causes the first thread to read the value 2 (event $\mathtt{Re}_a^{(r=2)}$). On the other hand, if the first thread reads the value 0, then the assignment of 2 to a happens. The event of the third thread (i.e., the assignment of 1 to b) is independent of the others, hence it has no conflict neither causes. This interpretation works fine when considering just the normal forward flow of computation, but it turns out to be *too strict* if one wants to reverse debug such a snippet, since sequencing is seen as causation. If we want to reverse debug such a snippet, we could reason as follows: the event $\mathtt{Wr}_b^{(1)}$ can be reversed at any time since it has no causes, neither it is caused by another event. Also, the undoing of $\mathtt{Wr}_a^{(2)}$ should cause the undoing of the event $\mathtt{Re}_a^{(r=2)}$. But then, the undoing of $\mathtt{Re}_a^{(r=0)}$ should not cause the undoing of $\mathtt{Wr}_a^{(2)}$, since the second event is not a consequence of the first one. Hence, the right interpretation for reversible debugging should be the AES depicted in Figure 9b. Again, we leave as future work the adaptation of the results in [5, 13] for reversible systems.

A speculative scenario. We now show how our framework can be used to model a *speculative scenario* borrowed from [35]. In this scenario, value speculation is used as a mechanism to boost parallelism by predicting values of data dependencies between tasks. Whenever a value prediction is incorrect, corrective actions must be taken in order to re-execute the data consumer code with the correct data value. In this regard, as shown in [10] for a shared-memory setting, reversible execution can permit to relieve programmers from the burden of properly undoing the actions subsequent to an incorrect prediction. For simplicity, our scenario will involve a *producer* and a *consumer*. The producer produces a value on which the consumer has to perform some calculation. Since the production of such a value may require some time, the consumer can try to guess (e.g. speculate) the value and launch a computation on the predicted value. At the end of the computation, the consumer compares the predicted value with the real one, and if it has done a wrong guess then it reverts its computation and redoes the calculation with the actual value. If the guess was right, then nothing has to be reverted. Hence we can model such a scenario with the *r*AES

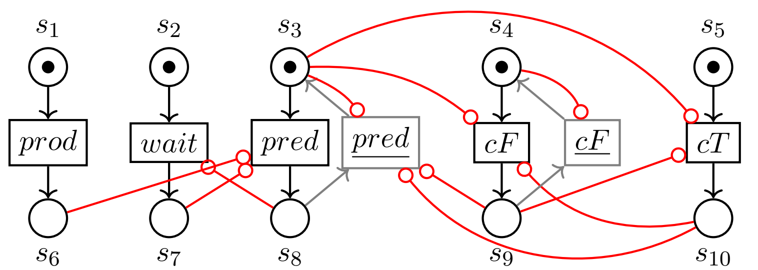


FIGURE 10. Speculative scenario *r*ACN

$\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ where:

$$
\begin{aligned}
E &= \{prod, wait, pred, cT, cF\}\\
U &= \{cF, pred\}\\
< &= \{(pred, cF), (pred, cT)\}\\
\nearrow &= \{(wait, pred), (pred, wait), (cT, cF), (cF, cT), (prod, pred)\}\\
\prec &= \{(cF, \underline{cF}), (pred, \underline{pred})\}\\
\lhd &= \{(\underline{pred}, cT), (\underline{pred}, cF)\}
\end{aligned}
$$

The sef of *forward* events E includes the following events: *prod* (indicating the production of the value), *wait* (indicating that the consumer decided to wait for the real value), *pred* (indicating that the consumer predicts the value and hence speculates), cT (the predicted value coincides with the one generated by the producer) and cF (the predicted value is wrong). The only *reversible* events (set U) are cF and *pred*, as the computation has to be reverted just in case of a wrong prediction. The causality relation $<$ is as expected: predicting the value causes its comparison with the real one. About conflicts, we have that $pred\#wait$ and $cT\#cF$. Also we have that the consumer may decide to predict the value and afterwards the producer produces the value. The opposite cannon happen, as it is worthless to speculate if the value has been already generated. Hence, $pred \nearrow prod$. Reverse causation $\prec$ is as expected. The prevention relation $\lhd$ allows for preventing the undoing of the prediction if the prediction was right, hence $\underline{pred} \lhd cT$, and in case the prediction was wrong to revert first the comparison and then the prediction ($\underline{pred} \lhd cF$). The net corresponding to the *r*AES H is depicted in Figure 10, where some inhibitor arcs (those induced by the weak causality of forward events) are omitted.

6. Conclusions

In this paper we complete previous efforts [30, 31, 21] aimed at relating classes of reversible event structures with classes of Petri nets: Firstly, we account for the full class of *r*AESs instead of proper subclasses (being *r*AESs the most general reversible event structures considered in the literature). Secondly, the correspondence is established according to the standard technique of exhibiting a coreflection between suitable categories. Regarding the philosophy behind our nets, it may be asked whether we adhere to the *individual token philosophy* or the *collective token philosophy*, as defined in [36]. In the individual token philosophy, each place can be uniquely marked, whereas in the collective token philosophy, there can be multiple ways to place a token in a given place. If we only consider the forward flow of our nets, they adhere to the individual token philosophy. However, asymmetric conflicts suggest that the *past* of the transition putting a token in a given place may not be unique. Moreover, when considering the reversing flow, things become more complex, particularly with out-of-causal order reversibility, as a forward transition may only be executed because some reversing transition has been executed. To settle this question, it may be necessary to seek a different definition for the individual token philosophy, perhaps by focusing on the forward dependencies of the pre-asymmetric causal net in the rACN, rather than considering the entire rACN.

Besides the theoretical relevance of establishing a correspondence between these two different models, such connection may be exploited in concrete scenarios as discussed in the previous section.

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Appendix A. Appendix

In this appendix we prove that the notions of rAES and ACN we used are consistent to what is established in literature ([21], [32] and [11])

We start fixing notation and introducing auxiliary definitions concerning nets that will be useful in the following. Given an fs σ, $\mathcal{X}_\sigma$ is the set of all sequences of multisets of transitions that *agree* with σ, namely the set $\{\theta \mid len(\theta) = len(\sigma) \ \wedge\ (\sigma(i)\,[\theta(i)\rangle\,\sigma(i+1)\ \mathit{with}\ i < len(\sigma))\}$, and $X_\sigma = \{\sum_{i=1}^{len(\theta)} \theta(i) \mid \theta \in \mathcal{X}_\sigma\}$ for the set of multisets of transitions associated to an fs. Each multiset of X_σ is a *state* of the net and write $\mathsf{St}(N) = \bigcup_{\sigma \in \mathcal{R}^N_{\mathsf{m}}} X_\sigma$ for the set of states of the net N, and $\theta \in \mathcal{X}_\sigma$ is an *execution* of the net.

A.1. Causal Nets and Asymmetric Causal Nets.

In [21] we introduced the notion of *causal net* to show that it was the proper kind of net corresponding to reversible prime event structures of [32]. In that paper we proved that each occurrence nets, the classical counterpart of prime event structures in net terms, could be seen as a causal net. Here we show that each causal net can be seen as an asymmetric causal nets, and this implies that the tight correspondence between causal nets and reversible prime event structures can be transferred to rACN. In that paper we did not considered causal nets in categorical terms.

We first recall the definition of causal net from [21].

Definition A.1. Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be an IPT. C is a *causal* net (CN) if the following conditions are satisfied:

(1) $\forall t, t' \in T.\ t^\bullet \cap {}^\bullet t' = \emptyset$;
(2) $[\![T^\bullet]\!] = T^\bullet$;
(3) $\forall t \neq t' \in T.\ {}^\bullet t \cap {}^\bullet t' \cap\ {}^\circ T = \emptyset$,
(4) $\forall t \in T.\ {}^\circ t$ is finite;
(5) $\lessdot$ is an irreflexive partial order;
(6) $\forall t', t'' \in \lfloor t \rfloor_\lessdot.\ t' \natural t'' \Rightarrow\ t' = t''$;
(7) $\forall t, t', t'' \in T.\ t \natural t' \wedge\ t' \lessdot t''\ \Rightarrow\ t \natural t''$; and
(8) $\mathsf{m} = {}^\bullet T$ and ${}^\circ T \subseteq \mathsf{m}$.

The requirement that $\forall t \in T.\ {}^{\circ}t$ is finite implies that $\lfloor t \rfloor_{\lessdot}$ is finite and $\forall t', t'' \in \lfloor t \rfloor_{\lessdot}.\ t' \natural t'' \Rightarrow t' = t''$ is analogous to the requirement that the inverse prevention relation and the causality one are acyclic on $\lfloor t \rfloor_{\lessdot}$.

Proposition A.2. *Let $C = \langle S, T, F, I, \mathsf{m}\rangle$ be a* CN *and let $S_\# \subseteq S$ be the shared places, i.e. $S_\# = \{s \in S \mid |s^\bullet| > 1\}$. Define $\mathcal{W}(C) = \langle S', T, F', I', \mathsf{m}'\rangle$ be the* IPT *where*
(1) $S' = S \setminus S_\#$;
(2) $F' = F \cap ((S' \times T) \cup (T' \times S'))$;
(3) $I' = I \cup \{({}^\bullet t, t') \mid \exists s \in S_\#. t \neq t' \ \wedge\ t, t' \in s^\bullet\}$; *and*
(4) $\mathsf{m}' = \mathsf{m} \cap S'$.
Then $\mathcal{W}(C)$ is an ACN *and $\mathsf{St}(C) = \mathsf{St}(\mathcal{W}(C))$.*

Proof. $\mathcal{W}(C)$ is a ACN as $\forall t \in T.\ {}^{\circ}t$ finite means that $\lfloor t \rfloor_{\lessdot}$ is finite and requiring that $\lfloor t \rfloor_{\lessdot}$ does not contain conflicting transitions account to prescribe that the reverse prevention and the causality are acyclic on the set of transitions $\lfloor t \rfloor_{\lessdot}$. The other requirements are trivial.

To show that $\mathsf{St}(C) = \mathsf{St}(\mathcal{W}(C))$ it is enough to observe that each transition can be executed just once and that, given any marking $m \in \mathcal{M}_C$ and any multiset of transitions $A \in \mu T$ such that $m\,[A\rangle$ there exists a marking $\widetilde{m} \in \mathcal{M}_{\mathcal{W}(C)}$ such that $\widetilde{m}\,[A\rangle$. Given a marking $m \in \mathcal{M}_C$, the associated marking $\widetilde{m}$ is $m \cap S'$. Now consider $A \in \mu T$ such that $m\,[A\rangle$, we have to show $\widetilde{m}\,[A\rangle$. Clearly ${}^\bullet A \subseteq m$ and this implies that ${}^\bullet A \subseteq \widetilde{m}$, now assume that there is a transition $t \in A$ such that there exists a place $s \in S_\#$ such that $s \in {}^\bullet t$. This implies that for all $t' \in s^\bullet \setminus \{t\}$ $t' \not\in A$ and $t'^\bullet \not\in m$, hence $\widetilde{m}\,[A\rangle$. It is straightforward to observe that if $m\,[A\rangle\, m'$ then $\widetilde{m}\,[A\rangle\, \widetilde{m'}$. Consider now a marking $m \in \mathcal{M}_{\mathcal{W}(C)}$ and a step $A \in \mu T$ such that $m\,[A\rangle$, we show that there exists a marking $\widehat{m} \in \mathcal{M}_C$ such that $\widehat{m}\,[A\rangle$. Consider $m \in \mathcal{M}_{\mathcal{W}(C)}$, the transitions executed to reach that marking are $T' \subseteq T$ such that for each $t \in T'.\ t^\bullet \subseteq m$. Define $\widehat{m} = m \cup \widehat{S_\#}$ where $\widehat{S_\#} = \{s \in S_\# \mid s \not\in T'^\bullet\}$. It is easy to see that if $m\,[A\rangle$ then $\widehat{m}\,[A\rangle$ reasoning as above, and observing that $\mathsf{m} = \widehat{\mathsf{m}'}$ we have that $\widehat{m} \in \mathcal{M}_C$. Observing that all the steps in a CN (and in a ACN) are indeed execution of set of transitions and that each transition is fired just once, we can conclude that $\mathsf{St}(C) = \mathsf{St}(\mathcal{W}(C))$. $\square$

As we have a precise correspondence between reachable markings of a CN and of the associated ACN and the states coincide, we can conclude that ACNs are a proper generalisation of CNs.

A.2. **Equivalence of the definitions of raess.**

The definition of reversible asymmetric event structure given in [32] and [11] has just two relations which, as we already said, comprises both the forward and the reverse causality and prevention. To avoid confusion we call them *standard reversible asymmetric event structures.*

Definition A.3. A *standard reversible asymmetric event structures* (srAES) is the quadruple $\mathsf{K} = (E, U, \prec, \lhd)$ where E is the set of events and
(1) $U \subseteq E$ is the set of *reversible* events;
(2) $\prec \subseteq E \times (E \cup \underline{U})$ is an irreflexive causation relation;
(3) $\lhd \subseteq (E \cup \underline{U}) \times E$ is an irreflexive *precedence* relation such that for all $\alpha \in E \cup \underline{U}$. $\{e \in E \mid e \prec \alpha\}$ is finite and acyclic with respect to $\lhd \cup \prec$;
(4) $\forall u \in U.\ u \prec \underline{u}$;
(5) for all $e \in E$ and $\alpha \in E \cup \underline{U}$, if $e \prec \alpha$ then not $\alpha \lhd e$; and

(6) the relation $\prec\!\!\prec$, defined as $e \prec\!\!\prec e'$ when $e \prec e'$ and if $e = u$, with $u \in U$, then $\underline{u} \lhd e'$, is such that
- $e \prec\!\!\prec e'$ implies $e \lhd e'$;
- it is a transitive relation; and
- if $e \# e'$ and $e \prec\!\!\prec e''$ then $e' \# e''$, where $\# = \lhd \cap \rhd$.

In this definition $\prec$ comprises the forward causality (which we called *causation*) and the reverse causality, and $\lhd$ comprises the weak causality (forward relation) and the prevention involving the undoing of events. Just observing that the relations on subsets of events reduces always to the part concerning *forward* relations, and that the last condition of the previous definition can be rewritten just requiring that, when restricted to forward events only, sustained causation and the restriction of the prevention to these events is an AES, it is straightforward to see that the two proposition below hold.

Proposition A.4. *Let* $\mathsf{K} = (E, U, \prec, \lhd)$ *be an sr*AES*, then* $\mathsf{H} = (E, U, <_{\mathsf{H}}, \nearrow_{\mathsf{H}}, \prec_{\mathsf{H}}, \lhd_{\mathsf{H}})$ *is an r*AES*, where* $<_{\mathsf{H}} = \prec \cap (E \times E)$*,* $\nearrow_{\mathsf{H}} = \lhd \cap (E \times E)$*,* $\prec_{\mathsf{H}} = \prec \cap (E \times \underline{U})$ *and* $\lhd_{\mathsf{H}} = \lhd \cap (\underline{U} \times E)$*.*

Proposition A.5. *Let* $\mathsf{H} = (E, U, <, \nearrow, \prec, \lhd)$ *be an r*AES*, then* $\mathsf{K} = (E, U, < \cup \prec, \nearrow \cup \lhd)$ *is an sr*AES*.*

The proofs of both propositions are trivial and omitted.