PROBLEMS IN NUMBER THEORY
FROM BUSY BEAVER COMPETITION

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Abstract. By introducing the busy beaver competition of Turing machines, in 1962, Rado defined noncomputable functions on positive integers. The study of these functions and variants leads to many mathematical challenges. This article takes up the following one: How can a small Turing machine manage to produce very big numbers? It provides the following answer: mostly by simulating Collatz-like functions, that are generalizations of the famous $3x+1$ function. These functions, like the $3x+1$ function, lead to new unsolved problems in number theory.

1. Introduction

1.1. A well defined noncomputable function. It is easy to define a noncomputable function on nonnegative integers. Indeed, given a programming language, you produce a systematic list of the programs for functions, and, by diagonalization, you define a function whose output, on input $n$, is different from the output of the $n$th program. This simple definition raises many problems: Which programming language? How to list the programs? How to choose the output?

In 1962, Rado [Ra62] gave a practical solution by defining the busy beaver game, also called now the busy beaver competition. Consider all Turing machines on one infinite tape, with $n$ states (plus a special halting state), and two symbols (1, and the blank symbol 0), and launch all of them on a blank tape. Define $S(n)$ as the maximum number of computation steps made by such a machine before it stops, and define $\Sigma(n)$ as the maximum number of symbols 1 left on the tape by a machine when it stops. Then functions $S$ and $\Sigma$ are noncomputable, and, moreover, grow faster than any computable function, that is, for any computable function $f$, there exists an integer $N$ such that, for any $n \geq N$, $S(n) > \Sigma(n) > f(n)$.


Key words and phrases: busy beaver, Collatz-like functions.

Corresponding address: 59 rue du Cardinal Lemoine, 75005 Paris, France.
More than fifty years later, no better choice has been found for a practical noncomputable function. Only variants of Rado’s definition have been proposed. So, in 1988, Brady \[Br88\] defined similar functions $S(n, m)$ and $\Sigma(n, m)$ for $n \times m$ Turing machines, that is Turing machines with $n$ states and $m$ symbols. He also introduced analogous functions for two-dimensional Turing machines and “TurNing machines”, later resumed and expanded by Tim Hutton \[HuW\]. Bátfai \[Ba09a, Ba09b\] relaxed the rule about head moving, by allowing the head to stand still.

In this article, we will consider functions $S(n, m)$ and $\Sigma(n, m)$. Recall that $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$.

1.2. Computing the values of noncomputable functions. The busy beaver functions $S$ and $\Sigma$ are explicitly defined, and it is possible to compute $S(n, m)$ and $\Sigma(n, m)$ for small $n$ and $m$. In the first article on busy beavers, Rado \[Ra62\] gave $\Sigma(2) = 2$ and $\Sigma(3) \geq 6$.

These results show that two problems are at stake:

- **Problem 1**: To give lower bounds on $S(n, m)$ and $\Sigma(n, m)$ by finding Turing machines with high scores.
- **Problem 2**: To compute $S(n, m)$ and $\Sigma(n, m)$ by proving that no Turing machines do better than the known best ones.

Problem 1 can be tackled either by hand search, as did, for example, Green \[Gr64\] and Lynn \[Ly72\], or by computer search, using acceleration techniques of computation and, for example, simulated annealing, as did T. and S. Ligocki \[Li05\].

Solving Problem 2 requires more work to be done: clever enumeration of $n \times m$ Turing machines, simulation of computation with acceleration techniques, proofs of non-halting for the machines that do not halt.

Currently, the following results are known (see Michel \[Mi09, MiWa\] for a historical survey):

- $S(2) = 6$ and $\Sigma(2) = 4$ (Rado \[Ra62\]),
- $S(3) = 21$ and $\Sigma(3) = 6$ (Lin and Rado \[LR65\]),
- $S(4) = 107$ and $\Sigma(4) = 13$ (Brady \[Br66, Br83\], Machlin and Stout \[MS90\]),
- $S(5) \geq 47, 176, 870$ and $\Sigma(5) \geq 4098$ (Marxen and Buntrock \[MB90\]),
- $S(6) > 7.4 \times 10^{36534}$ and $\Sigma(6) > 3.5 \times 10^{18267}$ (P. Kropitz in 2010),
- $S(2, 3) = 38$ and $\Sigma(2, 3) = 9$ (Lafitte and Papazian \[LP07\]),
- $S(3, 3) > 1.1 \times 10^{147}$ and $\Sigma(3, 3) \geq 347, 676, 383$ (T. and S. Ligocki in 2007),
- $S(4, 3) > 1.0 \times 10^{4072}$ and $\Sigma(4, 3) > 1.3 \times 10^{7036}$ (T. and S. Ligocki in 2008),
- $S(2, 4) \geq 3, 932, 964$ and $\Sigma(2, 4) \geq 2050$ (T. and S. Ligocki in 2005),
- $S(3, 4) > 5.2 \times 10^{13036}$ and $\Sigma(3, 4) > 3.7 \times 10^{6518}$ (T. and S. Ligocki in 2007),
- $S(2, 5) \geq 7.9 \times 10^{704}$ and $\Sigma(2, 5) > 1.7 \times 10^{352}$ (T. and S. Ligocki in 2007),
- $S(2, 6) > 2.4 \times 10^{9866}$ and $\Sigma(2, 6) > 1.9 \times 10^{4933}$ (T. and S. Ligocki in 2008).

In order to achieve these results, many computational and mathematical challenges had to be taken up.

**A**: Computational challenges.

- **A1.** To generate all $n \times m$ Turing machines, or rather, to treat all cases without having to generate all $n \times m$ Turing machines.
- **A2.** To simulate the computation of a machine by using acceleration techniques (see Marxen and Buntrock \[MB90\], Marxen \[MaW\]).
A3. To gave automatic proofs that non-halting machines do not halt (see Brady [Br83], Marxen and Buntrock [MB90], Machlin and Stout [MS90], Hertel [He09], Lafitte and Papazian [LP07]).

B: Mathematical challenges.

B1. To prove by hand that a non-halting machine that resists the computational proof does not halt.

B2. To understand how the Turing machines that reach high scores manage to do it.

1.3. Facing open problems in number theory. Let us come back to mathematical challenge B1. For example, the computational study of 5 × 2 Turing machines by Marxen and Buntrock [MB90], Skelet [GeW] and Hertel [He09] left holdouts that needed to be analyzed by hand. Marxen and Buntrock [MB90] gave an unsettled 5 × 2 Turing machine, named #4, that turned out to never halt, by an intricate analysis.

Actually, the halting problem for Turing machines launched on a blank tape is m-complete, and this implies that this problem is as hard as the problem of the provability of a mathematical statement in a logical theory such as ZFC (Zermelo Fraenkel set theory with axiom of choice). So, when Turing machines with more and more states and symbols are studied, potentially all theorems of mathematics will be met. When more and more non-halting Turing machines are studied to be proved non-halting, one has to expect to face hard open problems in mathematics, that is problems that current mathematical knowledge can’t settle.

Consider now mathematical challenge B2, which is the very subject of this article.

From 1983 to 1989, several 5 × 2 Turing machines with high scores were discovered by Uwe Schult, by George Uhing, and by Heiner Marxen and Jürgen Buntrock. Michel [Mi92, Mi93] analyzed some of these machines and found that their behavior is Collatz-like, which implies that the halting problems on general inputs for these machines are open problems in number theory (see Table 1).

From 2005 and 2007, many 3 × 3, 2 × 4 and 2 × 5 Turing machines with high scores were discovered, mainly by two teams: the French one of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Collatz-like behavior of these champions seems to be the rule (see Tables 3, 4 and 5).

However, the behaviors of 6 × 2 Turing machines display some variety. Many machines were discovered, from 1990 to 2010, by Heiner Marxen and Jürgen Buntrock, by Terry and Shawn Ligocki and by Pavel Kropitz. The analyses of some of these machines, by Robert Munafo, Clive Tooth, Shawn Ligocki and the author, show that the behaviors can be Collatz-like, exponential Collatz-like, loosely Collatz-like, or definitely not Collatz-like. Almost all of them raise open problems (see Table 2).

Note: The Turing machines listed in Tables 1-5 are those for which an analysis is known by the author. The machines without references for the study of behavior were analyzed by the author [Mi09, MiWb]. Many other machines are waiting for their analyses.

1.4. Collatz functions, Collatz-like functions and other functions. The 3x+1 function, or Collatz function, is the function $T$ on positive integers defined by

$$T(n) = \begin{cases} 
  n/2 & \text{if } n \text{ is even} \\
  (3n+1)/2 & \text{if } n \text{ is odd}
\end{cases}$$
This function is famous because, when it is iterated on a positive integer, it seems to lead to the loop 2, 1, 2, 1, . . . . Is it always true? This is an open problem. See Lagarias [La85, La03, La06, La10] for more information.

It is natural to generalize the definition of the 3x+1 function by replacing \( n \) even, \( n \) odd by \( n \equiv 0, \ldots, d - 1 \pmod{d} \), and by replacing \( n/2, (3n + 1)/2 \) by \( an + b \) for rational numbers \( a, b \). Unfortunately, no name for such functions is currently taken for granted. Formal definitions were given by Rawsthorne [Ra85], who proposed Collatz-type iteration functions, by Buttsworth and Matthews [BM90], who proposed generalized Collatz mappings, by Kašcák [Ka92], who proposed one-state linear operator algorithms (OLOA), and by Kohl [Ko07], who proposed residue-class-wise affine functions (RCWA). Without giving a formal definition, Lagarias [La85] proposed periodically linear functions, and Wagon [Wa85] proposed Collatz-like functions.

We will choose the following definitions.

**Definition 1.1.** A mapping \( f : \mathbb{Z} \to \mathbb{Z} \) is a generalized Collatz mapping if there exists an integer \( d \geq 2 \) such that the following three equivalent conditions are satisfied:

(i) (see [W98, p.14]) There exist rational numbers \( q_0, \ldots, q_{d-1}, r_0, \ldots, r_{d-1} \), such that, for all \( i, 0 \leq i \leq d - 1 \), we have \( q_i d \in \mathbb{Z}, q_i r + r_i \in \mathbb{Z} \), and, for all \( n \in \mathbb{Z} \), \( f(n) = q_i n + r_i \) if \( n \equiv i \pmod{d} \).

(ii) (see [BM90]) There exist integers \( m_0, \ldots, m_{d-1}, p_0, \ldots, p_{d-1} \), such that, for all \( i, 0 \leq i \leq d - 1 \), we have \( p_i \equiv i m_i \pmod{d} \) and, for all \( n \in \mathbb{Z} \), \( f(n) = (m_i n - p_i)/d \) if \( n \equiv i \pmod{d} \).

(iii) There exist integers \( a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1} \), such that we have, for all \( i, 0 \leq i \leq d - 1 \), for all \( n \in \mathbb{Z} \), \( f(n + i) = a_i n + b_i \).

These definitions are easily seen to be equivalent: we have \( a_i = m_i = q_i d \) and \( b_i = (im_i - p_i)/d = q_i r + r_i \).

The definitions above concern total functions, but, in this article, we always deal with partial functions and functions with parameters, so we introduce the following definitions.

**Definition 1.2.** A partial function \( f : \mathbb{Z} \to \mathbb{Z} \) is a generalized Collatz function, or a pure Collatz-like function (without parameter) if, in the previous definition, \( f(dn + i) \) can be undefined for one or many \( i \), \( 0 \leq i \leq d - 1 \).

**Definition 1.3.** A partial function \( f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \) is a pure Collatz-like function with parameter if there exist an integer \( d \geq 2 \), integers \( a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1} \), a set \( S \) of integers and a function \( p : \{0, \ldots, d - 1\} \times S \to S \) such that, for all \( i, 0 \leq i \leq d - 1 \), for all \( n \in \mathbb{Z} \), for all \( s \in S \), \( f(dn + i, s) = (a_i n + b_i, p(i, s)) \) or is undefined.

**Definition 1.4.** If, in the definitions above, \( a_i = a \) for all \( i, 0 \leq i \leq d - 1 \), we say that \( f \) is pure Collatz-like of type \( d \to a \).

We also need to define a new type of function, as follows.

**Definition 1.5.** A partial function \( f : \mathbb{Z} \to \mathbb{Z} \) is an exponential Collatz-like function if there exist integers \( d, p \geq 2 \), integers \( a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1}, c_0, \ldots, c_{d-1} \), such that, for all \( i, 0 \leq i \leq d - 1 \), all \( n \in \mathbb{Z} \), \( f(dn + i) = (a_i p^n + b_i)/c_i \) or is undefined. In this definition, integers \( p, a_i, b_i, c_i \) are chosen such that \( (a_i p^n + b_i)/c_i \) is an integer for all \( n \in \mathbb{Z} \).

Currently, no study of this type of function is known. Note that iterates \( f(n), f^2(n), \ldots \) grow much faster for exponential Collatz-like functions than for pure Collatz-like functions.
1.5. From Collatz-like functions to high scores. The Turing machines studied in this article have behaviors modeled on iterations of functions, where halting configurations correspond to undefined values of functions.

How do Turing machines simulate Collatz-like functions?

First note that Baiocchi [Ba98], Margenstern [Ma00] and Michel [Mi14] found Turing machines that simulate the $3x+1$ function with a very small number of states and symbols. In these articles, clever tricks were designed to minimize the size of the Turing machines.

On the other hand, the Turing machines in the present article were frontally attacked, and it is with hindsight that they were found to simulate Collatz-like functions.

Note also that, while the theorems giving the behavior of Turing machines below are hard to be formally proven, they can be easily verified for the small values of the parameters, by using programs simulating the Turing machines.

In Section 3, we present a $3 \times 3$ Turing machine $M_1$ whose behavior is pure Collatz-like, of type $8 \rightarrow 14$. In Section 4, we present a $2 \times 4$ Turing machine $M_2$ whose behavior is pure Collatz-like with parameter, of type $3 \rightarrow 5$. In Section 5, we present a $2 \times 5$ Turing machine $M_3$ whose behavior is pure Collatz-like with parameter, of type $2 \rightarrow 3$. Thus, the halting problem for machines $M_1$, $M_2$ and $M_3$ depends on an open problem about iterating Collatz-like functions.

In Section 6, we present a $6 \times 2$ Turing machine $M_4$ whose behavior is exponential Collatz-like.

In Section 7, we present a $6 \times 2$ Turing machine $M_5$ whose behavior depends on iterating a partial function $g_5(n, p)$. Without being Collatz-like, this function seems to share some properties with Collatz-like functions.

In Section 8, we present a $6 \times 2$ Turing machine $M_6$ whose behavior looks like a loosely Collatz-like behavior with parameter, of type $2 \rightarrow 5$. The novelty is that a potentially infinite set of rules seems to be necessary to completely describe the behavior of the machine on inputs $00x$, $x \in \{0, 1\}^*$. A string $x \in \{0, 1\}^*$ ending with symbol 1 can be taken as the binary representation of a number $p$, read in the opposite direction, so $x = R(\text{bin}(p))$, where $\text{bin}(p)$ is the usual binary representation of $p$, and $R(w)$ is the reverse of string $w$, that is $R(w_1 \ldots w_n) = w_n \ldots w_1$. In Table 2 we write “$R(\text{bin}(p))$” to indicate the machines with a behavior involving a potentially infinite set of rules. Of course, only a finite subset of these rules are used when the machine is launched on a blank tape.

In Section 9, we present a $6 \times 2$ Turing machine $M_7$ whose behavior on the blank tape depends on configurations $C(n)$ all of them provably leading to a halting configuration. We present such a machine to show how a Turing machine can take a long time to stop without calling for Collatz-like functions.

2. Preliminaries

A Turing machine involved in the busy beaver competition is defined as follows. It has a tape made of cells, infinite in both directions. Each cell contains a symbol, and one head can move on the tape and read and write a symbol on a cell. The Turing machine can be in a finite number of states. A computation of the Turing machine is a sequence of steps. In a step of computation, according to the current state and the symbol read by the head on the current cell, the head writes a symbol on the cell, moves to the next cell on the right side or on the left side, and the machine enters a new state.
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 1983</td>
<td>Pure Collatz-like (4 → 9)</td>
<td>Robinson, cited in [De84], Michel [Mi92]</td>
</tr>
<tr>
<td>Uwe Schult, σ = 501, s = 134,467</td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 transitions</td>
<td></td>
</tr>
<tr>
<td>December 1984</td>
<td>Pure Collatz-like (3 → 8)</td>
<td>Michel [Mi92]</td>
</tr>
<tr>
<td>George Uthing, σ = 1915, s = 2,133,492</td>
<td>with parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9 transitions</td>
<td></td>
</tr>
<tr>
<td>February 1986</td>
<td>Pure Collatz-like (8 → 15)</td>
<td></td>
</tr>
<tr>
<td>George Uthing, σ = 1471, s = 2,358,064</td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11 transitions</td>
<td></td>
</tr>
<tr>
<td>August 1989</td>
<td>Pure Collatz-like (3 → 5)</td>
<td>Michel [Mi92]</td>
</tr>
<tr>
<td>Marxen, Buntrock, σ = 4098, s = 11,798,826</td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15 transitions</td>
<td></td>
</tr>
<tr>
<td>September 1989</td>
<td>Pure Collatz-like (3 → 5)</td>
<td>Michel [Mi92]</td>
</tr>
<tr>
<td>Marxen, Buntrock, σ = 4097, s = 23,554,764</td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15 transitions</td>
<td></td>
</tr>
<tr>
<td>September 1989</td>
<td>Pure Collatz-like (3 → 5)</td>
<td>Michel [Mi93]</td>
</tr>
<tr>
<td>Marxen, Buntrock, σ = 4098, s = 47,176,870</td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15 transitions</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Study of behavior of 5 × 2 machines. For each machine, in the first column, one can find when it was discovered, by whom, the number σ of non-blank symbols left on the tape when the machine halts, and the number s of steps of the computation. In the second column, the behavior of the machine is given, and we refer to Def. 1.2–1.4 for the precise definition of pure Collatz-like function, with and without parameter, of type $d \rightarrow a$. The number of rules gives roughly the length of the definition of the function. The number of transitions gives the number of times that the rules are used during the computation of the machine on a blank tape. In the last column, machines without references were analyzed by the author [Mi09, MiWb].

Formally, a Turing machine $M = (Q, \Gamma, \delta)$ has a finite set of states $Q = \{q_0, q_1, \ldots, q_{n-1}\}$, a finite set of symbols $\Gamma = \{0, 1, \ldots, m-1\}$, and a transition function (or next move function) $\delta$, which is a mapping

$$\delta : Q \times \Gamma \rightarrow (\Gamma \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$  

If $\delta(q, a) = (b, D, q')$, then the Turing machine, when it is in state $q$ reading symbol $a$ on the current cell, writes symbol $b$ instead of $a$ on this cell, moves one cell left if $D = L$, one cell right if $D = R$, and enters state $q'$. The transition function is usually given by a transition table.
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>September 1997, Marxen and Buntrock, $s &gt; 8.6 \times 10^{15}$</td>
<td>Pure Collatz-like ($4 \rightarrow 10$) without parameter 5 rules 21 transitions</td>
<td>Munafo [MuWa]</td>
</tr>
<tr>
<td>October 2000, Marxen and Buntrock, (machine o), $s &gt; 6.1 \times 10^{119}$</td>
<td>R(bin(p)) ($2 \rightarrow 3$) 9 rules 337 transitions</td>
<td></td>
</tr>
<tr>
<td>October 2000, Marxen and Buntrock, (machine q), $s &gt; 6.1 \times 10^{925}$</td>
<td>All $C(n)$ stop 4 rules 5 transitions</td>
<td>Munafo [MuWb] Section 9</td>
</tr>
<tr>
<td>March 2001, Marxen and Buntrock, $s &gt; 3.0 \times 10^{1730}$</td>
<td>R(bin(p)) ($2 \rightarrow 3$) 20 rules 4911 transitions</td>
<td>Tooth [To02]</td>
</tr>
<tr>
<td>November 2007, T. and S. Ligocki, $s &gt; 8.9 \times 10^{1762}$</td>
<td>R(bin(p)) ($2 \rightarrow 5$) 12 rules 3346 transitions</td>
<td>Section 8</td>
</tr>
<tr>
<td>December 2007, T. and S. Ligocki, $s &gt; 2.5 \times 10^{2879}$</td>
<td>R(bin(p)) ($4 \rightarrow 6$) 18 rules 11026 transitions</td>
<td></td>
</tr>
<tr>
<td>May 2010, Pavel Kropitz, $s &gt; 3.8 \times 10^{21132}$</td>
<td>Unclassifiable 6 rules 22158 transitions</td>
<td>S. Ligocki [Li10] Section 7</td>
</tr>
<tr>
<td>June 2010, Pavel Kropitz, $s &gt; 7.4 \times 10^{36534}$</td>
<td>Exponential Collatz-like without parameter 4 rules 5 transitions</td>
<td>Section 6</td>
</tr>
</tbody>
</table>

Table 2: Study of behavior of $6 \times 2$ machines. See Def. 1.5 for the definition of exponential Collatz-like function. We write R(bin(p)) when a potentially infinite set of rules would be needed for a complete analysis of the machine.

For greater convenience, the states of the Turing machines will be denoted by capital letters: $A, B, \ldots$. There is a special state $A$, called the initial state, and a special symbol 0, called the blank symbol. In the busy beaver competition, at the beginning of a computation, the Turing machine is in state $A$, and the tape is blank, that is all the cells of the tape contain the blank symbol. There is another state $H$, the halting state, not in the set $Q$ of states. When a Turing machine enters this state, the computation stops. We impose that, at the last step, the machine writes 1, moves right, and enters state $H$. The machines are normalized in the following way: When they are launched on a blank tape, they enter new states in the order $B, C, \ldots$, and they write new symbols in the order 1, 2, \ldots.
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>December 2004 Brady</td>
<td>Pure Collatz-like (2 → 5) with parameter 5 rules</td>
<td></td>
</tr>
<tr>
<td>$s = 92,649,163$</td>
<td>11 transitions</td>
<td></td>
</tr>
<tr>
<td>July 2005 Souris</td>
<td>Pure Collatz-like (2 → 5) with parameter 5 rules</td>
<td></td>
</tr>
<tr>
<td>$\sigma = 36089$</td>
<td>12 transitions</td>
<td></td>
</tr>
<tr>
<td>$s = 310,341,163$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>July 2005 Souris</td>
<td>Pure Collatz-like (3 → 7) with parameter 7 rules</td>
<td></td>
</tr>
<tr>
<td>$s = 544,884,219$</td>
<td>12 transitions</td>
<td></td>
</tr>
<tr>
<td>August 2005 Lafitte and Papazian</td>
<td>Pure colatz-like (4 → 7) with parameter 8 rules</td>
<td></td>
</tr>
<tr>
<td>$s &gt; 4.9 \times 10^9$</td>
<td>21 transitions</td>
<td></td>
</tr>
<tr>
<td>September 2005 Lafitte and Papazian</td>
<td>Pure Collatz-like (4 → 7) with parameter 7 rules</td>
<td></td>
</tr>
<tr>
<td>$s &gt; 9.8 \times 10^{11}$</td>
<td>24 transitions</td>
<td></td>
</tr>
<tr>
<td>April 2006 Lafitte and Papazian</td>
<td>Pure Collatz-like (2 → 5) with parameter 5 rules</td>
<td></td>
</tr>
<tr>
<td>$s &gt; 4.1 \times 10^{12}$</td>
<td>16 transitions</td>
<td></td>
</tr>
<tr>
<td>August 2006 T. and S. Ligocki</td>
<td>Pure Collatz-like (2 → 5) with parameter 4 rules</td>
<td>S. Ligocki [Li06b]</td>
</tr>
<tr>
<td>$s &gt; 4.3 \times 10^{15}$</td>
<td>20 transitions</td>
<td></td>
</tr>
<tr>
<td>November 2007 T. and S. Ligocki</td>
<td>Pure Collatz-like (8 → 14) without parameter 9 rules</td>
<td>Section 3</td>
</tr>
<tr>
<td>$s &gt; 1.1 \times 10^{17}$</td>
<td>34 transitions</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Study of behavior of 3 × 3 machines

A *word* is a finite string of symbols. The set of words with symbols in the set $\Gamma$ is denoted by $\Gamma^*$. The number of symbols in a word $x \in \Gamma^*$ is called the *length* of $x$ and is denoted by $|x|$. The *empty word* is the word of length zero, denoted by $\lambda$. If $x \in \Gamma^*$, and $n \geq 0$, $x^n$ is the word $xx \ldots x$, where $x$ is repeated $n$ times, that is, formally: $x^0 = \lambda$, $x^1 = x$ and $x^{n+1} = x^n x$. An infinite to the left string of 0 is denoted by $0^\omega$, and an infinite to the right string of 0 is denoted by $0^\omega$.

A *configuration* is a way to encode the symbols on the tape, the state, and the cell currently read by the head. The Turing machine is in configuration $0x(Sa)y0^\omega$, with $S \in Q \cup \{H\}$, $a \in \Gamma$, $x, y \in \Gamma^*$, if the word $xay$ is written on the tape, the state is $S$, and the head is reading symbol $a$. Since, at the beginning of the computation, the state is $A$ and the tape is blank, the initial configuration is $0^\omega(A0)0^\omega$. If the state is $H$, the configuration
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988 Brady</td>
<td>Pure Collatz-like (3 → 5) with parameter 6 rules 7 transitions</td>
<td></td>
</tr>
<tr>
<td>February 2005 T. and S. Ligocki</td>
<td>Pure Collatz-like (3 → 5) with parameter 7 rules 14 transitions</td>
<td>Section 4</td>
</tr>
</tbody>
</table>

Table 4: Study of behavior of 2 × 4 machines

<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 2005 Lafitte, Papazian</td>
<td>Pure Collatz-like (2 → 5) with parameter 7 rules 15 transitions</td>
<td></td>
</tr>
<tr>
<td>December 2005 Lafitte, Papazian</td>
<td>Pure Collatz-like (2 → 5) with parameter 5 rules 14 transitions</td>
<td></td>
</tr>
<tr>
<td>May 2006 Lafitte, Papazian</td>
<td>Pure Collatz-like (3 → 4) with parameter 7 rules 45 transitions</td>
<td></td>
</tr>
<tr>
<td>June 2006 Lafitte, Papazian</td>
<td>Pure Collatz-like (2 → 3) with parameter 9 rules 36 transitions</td>
<td></td>
</tr>
<tr>
<td>August 2006 T. and S. Ligocki</td>
<td>Pure Collatz-like (2 → 5) with parameter 9 rules 30 transitions</td>
<td>S. Ligocki [Li06a]</td>
</tr>
<tr>
<td>November 2007 T. and S. Ligocki</td>
<td>Pure Collatz-like (2 → 3) with parameter 17 rules 2002 transitions</td>
<td>Section 5</td>
</tr>
</tbody>
</table>

Table 5: Study of behavior of 2 × 5 machines

is halting. We also consider configurations $x(Sa)y$ with finite length. If the computation from configuration $C_1$ to configuration $C_2$ takes $t$ steps, we write $C_1 \vdash (t) C_2$, and $t$ is said to be the time taken by the machine to go from $C_1$ to $C_2$. If $C_2$ is a halting configuration, we also write $C_1 \vdash (t) \text{END}$. We write $C_1 \vdash ( ) C_2$ if the time is not specified. If $C_1$ and $C_2$ are configurations with finite length, then they refer to the same part of the tape. For example, $(A0)0 \vdash (1) 1(B0)$ if $\delta(A,0) = (1,R,B)$. 
Table 6: Machine $M_1$ discovered in November 2007 by T. and S. Ligocki. Such tables are read as in the following example. When machine $M_1$ is in state $A$ and reads symbol 0, then it writes symbol 1 instead of symbol 0, moves one cell to the right, and enters state $B$.

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1RB</td>
<td>2LA</td>
<td>1LC</td>
</tr>
<tr>
<td>$B$</td>
<td>0LA</td>
<td>2RB</td>
<td>1LB</td>
</tr>
<tr>
<td>$C$</td>
<td>1RH</td>
<td>1RA</td>
<td>1RC</td>
</tr>
</tbody>
</table>

A Turing machine $M$ computes a partial function $f_M : \Gamma^* \rightarrow \Gamma^*$ as follows. Let $x = x_1 \ldots x_n \in \Gamma^*$. Then $x$ becomes an input for $M$ by considering the computation of $M$ on initial configuration $\omega 0(Ax_1)x_2 \ldots x_n 0^\omega$. If $M$ never stops on this configuration, then $f_M(x)$ is undefined. If $M$ stops, in configuration $\omega 0y(Ha)z0^\omega$, with $a \in \Gamma$, $y, z \in \Gamma^*$, then the output $f_M(x)$ is defined from this configuration by a suitable convention. The halting set is $\{ x \in \Gamma^* : f_M(x) \text{ is defined} \}$. The halting problem for machine $M$ is the problem consisting in determining the halting set. Note that the Turing machines with two symbols 0 and 1 are powerful enough to compute any computable function, and their halting sets can be any computably enumerable (also called recursively enumerable) set.

A Turing machine with $n$ states and $m$ symbols is called a $n \times m$ machine. The set of $n \times m$ machines is denoted by $\text{TM}(n,m)$. With our definition of the transition function, there are $(2nm + 1)^{nm}$ machines in the set $\text{TM}(n,m)$. In the busy beaver competition, for fixed numbers of states $n$ and symbols $m$, all the $(2nm + 1)^{nm}$ Turing machines in $\text{TM}(n,m)$ are launched on the blank tape. Some of them never stop. Those which stop are called busy beaver. Each busy beaver takes some time to stop, and leaves some non-blank symbols on the tape, so busy beavers are involved in two competitions: to take the longest time before stopping, and to leave the greatest number of non-blank symbols on the tape when stopping. The time taken by Turing machine $M$ to stop is denoted by $s(M)$, and the number of non-blank symbols left by $M$ when it stops is denoted by $\sigma(M)$. The busy beaver functions are defined by

$$S(n, m) = \max \{ s(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols} \}$$

$$\Sigma(n, m) = \max \{ \sigma(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols} \}$$

Rado [Ra62] initially defined functions $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$ for Turing machines with $n$ states and two symbols.

3. Pure Collatz-like behavior

Let $M_1$ be the $3 \times 3$ Turing machine defined by Table 6. We have $s(M_1) = 119,112,334,170,342,540$ and $\sigma(M_1) = 374,676,383$.

This machine is the current champion for the busy beaver competition for $3 \times 3$ machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on November, 9th) that they enumerated all the $3 \times 3$ machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine $M_1$ to reach a halting configuration from a blank tape.
Theorem 3.1. We have the following transitions between configurations of Turing machine $M_1$. Let $C(n) = \omega(A0)2^n\omega^\omega$. Then

(a) $\omega(A0)0^\omega \vdash (3) \ C(1)$,

and, for all $k \geq 0$,

(b) $C(8k + 1) \vdash (112k^2 + 116k + 13) \ C(14k + 3)$,

(c) $C(8k + 2) \vdash (112k^2 + 144k + 38) \ C(14k + 7)$,

(d) $C(8k + 3) \vdash (112k^2 + 172k + 54) \ C(14k + 8)$,

(e) $C(8k + 4) \vdash (112k^2 + 200k + 74) \ C(14k + 9)$,

(f) $C(8k + 5) \vdash (112k^2 + 228k + 97) \ C(01(H1)2^{14k+9}0\omega)$,

(g) $C(8k + 6) \vdash (112k^2 + 256k + 139) \ C(14k + 14)$,

(h) $C(8k + 7) \vdash (112k^2 + 284k + 169) \ C(14k + 15)$,

(i) $C(8k + 8) \vdash (112k^2 + 312k + 203) \ C(14k + 16)$.

Proof. A direct inspection of the transition table gives

(1) $0(A0)0 \vdash (3) \ (A0)20$,

(2) $0^3(A0)2^5 \vdash (53) \ (B1)1^8$,

(3) $0(A1) \vdash (1) \ (A0)2$,

(4) $1(A1) \vdash (1) \ (A1)2$,

(5) $02(A1) \vdash (3) \ 1(H1)2$,

(6) $12(A1) \vdash (4) \ (A1)22$,

(7) $22(A1) \vdash (8) \ (A1)22$,

(8) $2(B1)0^2 \vdash (7) \ 11(A1)0$,

(9) $B11 \vdash (1) \ 2(B1)$,

(10) $B12 \vdash (1) \ 2(B2)$,

(11) $0^3(B2) \vdash (14) \ 1^3(B1)$,

(12) $B21 \vdash (1) \ (B2)1$,

(13) $B22 \vdash (1) \ (B2)1$.

From this point, $k$ will be an integer, $k \geq 0$.

Iterating, respectively, (4), (7), (9) and (13) gives

(14) $1^k(A1) \vdash (k) \ (A1)2^k$,

(15) $2^{2k}(A1) \vdash (8k) \ (A1)2^{2k}$,

(16) $(B1)1^k \vdash (k) \ 2^k(B1)$,

(17) $2^k(B2) \vdash (k) \ (B2)1^k$.

Using consecutively (16), (10), (17) and (12), we get

(18) $1(B1)1^k2 \vdash (2k + 3) \ (B1)1^{k+2}$.

Using (16), (10), (17) and (11), we get

(19) $0^3(B1)1^{k2} \vdash (2k + 16) \ 1^3(B1)1^{k+1}$.

Using (19) and three times (18), we get

(20) $0^3(B1)1^{k2}4 \vdash (8k + 43) \ (B1)1^{k+7}$.

For any $n \geq 0$, by induction on $k$, using (20), we get

(21) $0^{3k}(B1)1^{n2^{4k}} \vdash (28k^2 + (8n + 15)k) \ (B1)1^{7k+n}$.

By taking $n = 8$ in (21), we get

(22) $0^{3k}(B1)1^82^{4k} \vdash (28k^2 + 79k) \ (B1)1^{7k+8}$.

Using (8), (14) and (15), we get

(23) $2^{2k+1}(B1)0^2 \vdash (8k + 9) \ (A1)2^{2k+2}$.

We are now ready to prove the results of the theorem.
Using (2), (22) and (16), we get
\[ 0^{4k+3}(A_0)2^{4k+5} \vdash (28k^2 + 86k + 61) 2^{7k+8} (B1). \]
Using (24), (23) and (5) we get
\[ 0^{6k+4}(A_0)2^{8k+5}0^2 \vdash (112k^2 + 228k + 97) 1(H1)2^{14k+9}. \]
Using (24), (23) and (3) we get
\[ 0^{6k+7}(A_0)2^{8k+9}0^2 \vdash (112k^2 + 340k + 214) (A_0)2^{14k+17}. \]
and this result is still true for \( k = -1 \), so we have
\[ 0^{6k+1}(A_0)2^{8k+1}0^2 \vdash (112k^2 + 116k + 13) (A_0)2^{14k+3}. \]
Using (2), (22), (19) and (16) we get
\[ 0^{4k+6}(A_0)2^{4k+6} \vdash (28k^2 + 100k + 94) 1^{3}2^{7k+9} (B1). \]
Using (27), (23), (14) and (3) we get
\[ 0^{6k+7}(A_0)2^{8k+6}0^2 \vdash (112k^2 + 256k + 139) (A_0)2^{14k+14}. \]
Using (27), (23), (6), (14) and (3) we get
\[ 0^{6k+10}(A_0)2^{8k+10}0^2 \vdash (112k^2 + 368k + 294) (A_0)2^{14k+21}. \]
and this result is still true for \( k = -1 \), so we have
\[ 0^{6k+4}(A_0)2^{8k+2}0^2 \vdash (112k^2 + 144k + 38) (A_0)2^{14k+7}. \]
Using (2), (22), (19), (18), (16), (8) and (14) we get
\[ 0^{4k+6}(A_0)2^{4k+7}0^2 \vdash (28k^2 + 114k + 126) 1^22^{2k+10}(A1)220. \]
Using (30), (15), (14) and (3) we get
\[ 0^{4k+7}(A_0)2^{8k+7}0^2 \vdash (112k^2 + 284k + 169) (A_0)2^{14k+15}. \]
Using (30), (15), (6), (4) and (3) we get
\[ 0^{6k+10}(A_0)2^{8k+11}0^2 \vdash (112k^2 + 396k + 338) (A_0)2^{14k+22}. \]
and this result is still true for \( k = -1 \), so we have
\[ 0^{6k+4}(A_0)2^{8k+3}0^2 \vdash (112k^2 + 172k + 54) (A_0)2^{14k+8}. \]
Using (2), (22), (19), (18), (16), (8) and (14) we get
\[ 0^{4k+6}(A_0)2^{4k+8}0^2 \vdash (28k^2 + 128k + 153) 12^{7k+12}(A1)220. \]
Using (33), (15), (4) and (3) we get
\[ 0^{4k+7}(A_0)2^{8k+8}0^2 \vdash (112k^2 + 312k + 203) (A_0)2^{14k+16}. \]
Using (33), (15), (6) and (3) we get
\[ 0^{6k+10}(A_0)2^{8k+12}0^2 \vdash (112k^2 + 424k + 386) (A_0)2^{14k+23}. \]
and this result is still true for \( k = -1 \), so we have
\[ 0^{6k+4}(A_0)2^{8k+4}0^2 \vdash (112k^2 + 200k + 74) (A_0)2^{14k+9}. \]
The results (1), (26), (29), (32), (35), (25), (28), (31) and (34) give, respectively, the results
(a)–(i) of the theorem.

Using the rules of this theorem, we have, in 34 transitions,
\[ ω^0(A_0)0^ω \vdash (3) C(1) \vdash (13) C(3) \vdash (\cdots) \vdash (\cdots) ω^01(H_1)2^{374,676,381}0^ω. \]
Let us try to give an informal explanation of how Turing machine \( M_1 \) works. Note that this explanation will also apply to the machines \( M_2 \) and \( M_3 \) below. The two main ideas are the following ones. First, a string of symbols 2 is nibbled four by four, and each time a string of four symbols 2 is nibbled, a string of seven symbols 2 is potentially created (see transitions (20) and (21) in the proof of Theorem[3,1]). This explains the type 8 \( \rightarrow \) 14 of the simulated Collatz-like function. Second, there are edge effects when the ends of a string of symbols 2 are reached, which explain the multiplicity of cases.
Let $g_1$ be the pure Collatz-like function defined by: for $k \geq 0$,

$$
g_1(8k + 1) = 14k + 3,
g_1(8k + 2) = 14k + 7,
g_1(8k + 3) = 14k + 8,
g_1(8k + 4) = 14k + 9,
g_1(8k + 5) \text{ undefined},
g_1(8k + 6) = 14k + 14,
g_1(8k + 7) = 14k + 15,
g_1(8k + 8) = 14k + 16.
$$

Then $g_1^{33}(1)$ is undefined.

The theorem gives immediately the following proposition.

**Proposition 3.2.** The behavior of Turing machine $M_1$, on inputs $02^n$, $n \geq 1$, depends on the behavior of iterated $g_1^k(n)$, $k \geq 1$.

Since the behavior of iterated $g_1^k(n)$ is an open problem in mathematics, this is also the case for the halting problem for Turing machine $M_1$.

Let $h_1(n) = \min\{k : g_1^k(n) \text{ is undefined}\}$. We have seen that $h_1(1) = 33$. We also have $h_1(144) = 41$, $h_1(270) = 51$.

4. Collatz-like with parameter: first example

Let $M_2$ be the $2 \times 4$ Turing machine defined by Table 7.

<table>
<thead>
<tr>
<th>$M_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1RB</td>
<td>2LA</td>
<td>1RA</td>
<td>1RA</td>
</tr>
<tr>
<td>$B$</td>
<td>1LB</td>
<td>1LA</td>
<td>3RB</td>
<td>1RH</td>
</tr>
</tbody>
</table>

Table 7: Machine $M_2$ discovered in February 2005 by T. and S. Ligocki

We have $s(M_2) = 3,932,964$ and $\sigma(M_2) = 2050$.

This machine is the current champion for the busy beaver competition for $2 \times 4$ machines. It was discovered in February 2005 by Terry and Shawn Ligocki, who wrote (email on February, 13th) that they found this machine using simulated annealing.

The following theorem gives the rules that enable Turing machine $M_2$ to reach a halting configuration from a blank tape.

**Theorem 4.1.** We have the following transitions between configurations of Turing machine $M_2$. Let

$$
C(n, 1) = \omega0(A0)2^n10^\omega,
C(n, 2) = \omega0(A0)2^n110^\omega.
$$

Then

(a) $\omega0(A0)0^\omega \vdash (6) \ C(1, 2),$
and, for all $k \geq 0$,
Proof. A direct inspection of the transition table gives

Using (3), (15) and (11), (6), (9) and (7) we get

Using (2), (13), (6), (14), (8), (13) and (7) we get

Using (2), (13), (6), (14), (8), (13), (6) and (7) we get

By induction on \(k\), from (22), we get

so we have, for \(n = 2\) and \(n = 4\)

We are now ready to prove the theorem.
Using (20), (24) and (17) we get
\[0(A0)^{2^{3k+2}10^{2k+2}} \vdash (15k^2 + 39k + 27) \] \((A0)^{2^{5k+6}1},
and the result is still true for \(k = -1\), so we have
(26) \[0(A0)^{2^{3k}10^{2k}} \vdash (15k^2 + 9k + 3) \] \((A0)^{2^{5k+1}}.
Using (20), (24), (18) and (19) we get
\[0(A0)^{2^{3k+4}10^{2k+3}} \vdash (15k^2 + 54k + 52) \] \(13^{5k+7}1(H1),
and the result is still true for \(k = -1\), so we have
(27) \[0(A0)^{2^{3k+1}10^{2k+1}} \vdash (15k^2 + 24k + 13) \] \(13^{5k+2}1(H1).
Using (20), (24) and (16) we get
\[0(A0)^{2^{3k+2}2^{10k+2}} \vdash (15k^2 + 29k + 17) \] \((A0)^{2^{5k+4}11}.
Using (21), (25) and (16) we get
\[0(A0)^{2^{3k+3}110^{2k+2}} \vdash (15k^2 + 41k + 29) \] \((A0)^{2^{5k+6}11},
and the result is still true for \(k = -1\), so we have
(29) \[0(A0)^{2^{3k}110^{2k}} \vdash (15k^2 + 11k + 3) \] \((A0)^{2^{5k+1}11}.
Using (21), (25), (18) and (19) we get
\[0(A0)^{2^{3k+4}110^{2k+2}} \vdash (15k^2 + 51k + 43) \] \((A0)^{2^{5k+8}1},
and the result is still true for \(k = -1\), so we have
(30) \[0(A0)^{2^{3k+1}110^{2k}} \vdash (15k^2 + 21k + 7) \] \((A0)^{2^{5k+3}1}.
Using (21), (25), (16) and (17) we get
\[0(A0)^{2^{3k+5}110^{2k+3}} \vdash (15k^2 + 66k + 74) \] \(13^{5k+9}1(H1),
and the result is still true for \(k = -1\), so we have
(31) \[0(A0)^{2^{3k+2}110^{2k+1}} \vdash (15k^2 + 36k + 23) \] \(13^{5k+4}1(H1).
The theorem comes from results (1) and (26)–(31).

Using the rules of this theorem, we have, in 14 transitions,
\[\omega0(A0)^{0\omega} \vdash (6) \quad C(1, 2) \vdash (7) \quad C(3, 1) \vdash (\cdots) \vdash (\omega013^{2047}1(H1)0\omega).
Informally, Turing machine \(M_2\) works according to the same ideas as Turing machine \(M_1\) does, but, in addition, edge effects depend also on the parameter.

Let \(g_2\) be the pure Collatz-like function with parameter defined by: for \(k \geq 0,\)
\[
g_2(3k, 1) = (5k + 1, 1), \quad g_2(3k + 1, 1) \text{ undefined}, \quad g_2(3k + 2, 1) = (5k + 4, 2), \quad g_2(3k, 2) = (5k + 1, 2), \quad g_2(3k + 1, 2) = (5k + 3, 1), \quad g_2(3k + 2, 2) \text{ undefined}.
Then \(g_2^{13}(1, 2)\) is undefined.

The theorem gives immediately the following proposition.

**Proposition 4.2.** The behavior of Turing machine \(M_2\), on inputs \(02^n1^i\), \(n \geq 1, i \in \{1, 2\}\), depends on the behavior of iterated \(g_2^k(n, i)\), \(k \geq 1\).

Since the behavior of iterated \(g_2^k(n, i)\) is an open problem in mathematics, this is also the case for the halting problem for Turing machine \(M_2\).

Let \(h_2(n, i) = \min\{k : g_2^k(n, i)\ \text{is undefined}\}\). We have seen that \(h_2(1, 2) = 13\). We also have \(h_2(137, 1) = 16, h_2(210, 2) = 20\).
Then

\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
   & 0   & 1   & 2   & 3   & 4   \\
\hline
A & 1RB & 2LA & 1RA & 2LB & 2LA \\
\hline
B & 0LA & 2RB & 3RB & 4RA & 1RH \\
\hline
\end{tabular}
\end{table}

Table 8: Machine $M_3$ discovered in November 2007 by T. and S. Ligocki

5. **Collatz-like with parameter: second example**

Let $M_3$ be the $2 \times 5$ Turing machine defined by Table ~8. We have $s(M_3) > 1.9 \times 10^{704}$ and $\sigma(M_3) > 1.7 \times 10^{352}$.

This machine is the current champion for the busy beaver competition for $2 \times 5$ machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on November, 9th) that, as they did for $3 \times 3$ machine $M_1$, they enumerated all the $2 \times 5$ machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine $M_3$ to reach a halting configuration from a blank tape.

**Theorem 5.1.** We have the following transitions between configurations of Turing machine $M_3$. Let

\[
C(n, 1) = \omega 013^n (B0) 0^\omega,
\]

\[
C(n, 2) = \omega 023^n (B0) 0^\omega,
\]

\[
C(n, 3) = \omega 03^n (B0) 0^\omega,
\]

\[
C(n, 4) = \omega 04113^n (B0) 0^\omega,
\]

\[
C(n, 5) = \omega 04123^n (B0) 0^\omega,
\]

\[
C(n, 6) = \omega 0413^n (B0) 0^\omega,
\]

\[
C(n, 7) = \omega 0423^n (B0) 0^\omega,
\]

\[
C(n, 8) = \omega 043^n (B0) 0^\omega.
\]

Then

\begin{enumerate}
\item \(\omega 0(A0) 0^\omega \vdash (1) C(0, 1)\),
\item and, for all \(k \geq 0\),
\item \(C(2k, 1) \vdash (3k^2 + 8k + 4) C(3k + 1, 1)\),
\item \(C(2k, 2) \vdash (3k^2 + 14k + 9) C(3k + 2, 1)\),
\item \(C(2k, 3) \vdash (3k^2 + 8k + 2) C(3k, 1)\),
\item \(C(2k, 4) \vdash (3k^2 + 8k + 8) C(3k + 3, 1)\),
\item \(C(2k, 5) \vdash (3k^2 + 14k + 13) C(3k + 4, 1)\),
\item \(C(2k, 6) \vdash (3k^2 + 8k + 6) C(3k + 2, 1)\),
\item \(C(2k, 7) \vdash (3k^2 + 14k + 11) C(3k + 3, 1)\),
\item \(C(2k, 8) \vdash (3k^2 + 8k + 4) C(3k + 1, 1)\),
\item \(C(2k + 1, 1) \vdash (3k^2 + 8k + 4) C(3k + 1, 2)\),
\item \(C(2k + 1, 2) \vdash (3k^2 + 8k + 4) C(3k + 2, 3)\),
\item \(C(2k + 1, 3) \vdash (3k^2 + 8k + 22) C(3k + 1, 4)\),
\item \(C(2k + 1, 4) \vdash (3k^2 + 8k + 4) C(3k + 1, 5)\),
\item \(C(2k + 1, 5) \vdash (3k^2 + 8k + 4) C(3k + 2, 6)\),
\item \(C(2k + 1, 6) \vdash (3k^2 + 8k + 4) C(3k + 1, 7)\),
\item \(C(2k + 1, 7) \vdash (3k^2 + 8k + 4) C(3k + 2, 8)\),
\item \(C(2k + 1, 8) \vdash (3k^2 + 5k + 3) \omega 01(H2)^{2k} 0^\omega\).
\end{enumerate}

**Proof.** A direct inspection of the transition table gives
(1) \((A_0)0 \vdash (1) 1(B_0)\),
(2) \(0(A_0)0 \vdash (17) 41(A_0)\),
(3) \((A_0)2 \vdash (1) 1(B_2)\),
(4) \(0(A_1) \vdash (1) (A_0)2\),
(5) \(1(A_1) \vdash (1) (A_1)2\),
(6) \(4(A_1) \vdash (1) (A_4)2\),
(7) \((A_2)0^2 \vdash (2) 1^2(B_0)\),
(8) \((A_2)2 \vdash (1) 1(A_2)\),
(9) \(1(B_0) \vdash (1) (A_1)0\),
(10) \(3^2(B_0)0 \vdash (5) 41^2(B_0)\),
(11) \((B_2)0 \vdash (1) 3(B_0)\),
(12) \((B_2)2 \vdash (1) 3(B_2)\),
(13) \(0(A_4) \vdash (1) (A_0)2\),
(14) \(1(A_4) \vdash (1) (A_1)2\),
(15) \(2(A_4) \vdash (1) (A_2)2\),
(16) \(0^23(A_4) \vdash (3) (A_0)02^2\),
(17) \(13(A_4) \vdash (4) 23(B_2)\),
(18) \(23(A_4) \vdash (4) 33(B_2)\),
(19) \(3^2(A_4) \vdash (4) 41(A_2)\),
(20) \(43(A_4) \vdash (3) 1(H_2)2\),
(21) \(04(A_4) \vdash (2) (A_0)2^2\).

Iterating, respectively, (5), (8) and (12) gives

(22) \(1^k(A_1) \vdash (k) (A_1)2^k\),
(23) \((A_2)2^k \vdash (k) 1^k(A_2)\),
(24) \((B_2)2^k \vdash (k) 3^k(B_2)\).

Using (9), (22), (6), (19), (23) and (7) we get
\[3^241^{k+1}(B_0)0 \vdash (2k + 9) 41^{k+1}(B_0),\]
and the result is still true for \(k = -1\), so we have
\[3^241^k(B_0)0 \vdash (2k + 7) 41^{k+1}(B_0).\]

For any \(n \geq 0\), by induction on \(k\), using (25), we get
\[3^241^n(B_0)0^k \vdash (3k^2 + (2n + 4)k) 41^{k+n}(B_0),\]
so we have, for \(n = 2\) in (26),
\[3^241^2(B_0)0^k \vdash (3k^2 + 8k) 41^{k+2}(B_0).\]

Using (10), (27), (9), (22) and (6), we get
\[3^{2k+2}(B_0)0^{k+1} \vdash (3k^2 + 11k + 8) (A_4)2^{3k+2}0.\]

Using (3), (24) and (11) we get
\[(A_0)2^{k+1}0 \vdash (k + 2) 13^{k+1}(B_0),\]
and the result is still true for \(k = -1\), so we have
\[(A_0)2^{k}0 \vdash (k + 1) 13^{k}(B_0).\]

Using (15), (23), (7), (9) and (22), we get
\[(2(A_4)2^k0^2 \vdash (2k + 7) (A_1)2^{k+2}0.\]

We are now ready to prove the theorem.

Using (28), (14), (4) and (29) we get
\[013^{2k+2}(B_0)0^{k+1} \vdash (3k^2 + 14k + 15) 13^{3k+4}(B_0),\]
and the result is still true for \(k = -1\), so we have
\[013^2k(B_0)0^k \vdash (3k^2 + 8k + 4) 13^{3k+1}(B_0).\]
Using (28), (30), (4) and (29) we get
\[0^{3k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 26) \ 13^{3k+5}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{3k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 9) \ 13^{3k+2}(B0).\]

Using (28), (13) and (29) we get
\[0^{3k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 13) \ 13^{3k+3}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{3k}(B0)0^{k} \vdash (3k^2 + 8k + 2) \ 13^{3k}(B0).\]

Using (28), (14), (5), (6), (13) and (29) we get
\[0^{4113^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 19) \ 13^{3k+6}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{4113^{3k}(B0)0^{k} \vdash (3k^2 + 8k + 8) \ 13^{3k+3}(B0).\]

Using (28), (30), (4) and (29) we get
\[0^{4123^{2k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 30) \ 13^{3k+7}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{4123^{2k}(B0)0^{k+1} \vdash (3k^2 + 14k + 13) \ 13^{3k+4}(B0).\]

Using (28), (14), (6), (13) and (29) we get
\[0^{413^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 17) \ 13^{3k+5}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{413^{3k}(B0)0^{k} \vdash (3k^2 + 8k + 6) \ 13^{3k+2}(B0).\]

Using (28), (30), (4) and (29) we get
\[0^{420^{3k+2}(B0)0^{k+2} \vdash (3k^2 + 20k + 28) \ 13^{3k+6}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{420^{3k}(B0)0^{k+1} \vdash (3k^2 + 14k + 11) \ 13^{3k+3}(B0).\]

Using (28), (21) and (29) we get
\[0^{43^{2k+2}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \ 13^{3k+4}(B0),\]
and the result is still true for \(k = -1\), so we have
\[0^{43^{3k}(B0)0^{k} \vdash (3k^2 + 8k + 4) \ 13^{3k+1}(B0).\]

Using (28), (17), (24) and (11) we get
\[13^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \ 23^{3k+4}(B0),\]
and the result is still true for \(k = -1\), so we have
\[13^{2k+1}(B0)0^{k} \vdash (3k^2 + 8k + 4) \ 23^{3k+1}(B0).\]

Using (28), (18), (24) and (11) we get
\[23^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \ 3^{3k+5}(B0),\]
and the result is still true for \(k = -1\), so we have
\[23^{2k+1}(B0)0^{k} \vdash (3k^2 + 8k + 4) \ 3^{3k+2}(B0).\]

Using (28), (16), (2) and (29) we get
\[0^{3^{2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 33) \ 41^{23^{3k+4}(B0)},\]
and the result is still true for \(k = -1\), so we have
\[0^{3^{2k+1}(B0)0^{k} \vdash (3k^2 + 8k + 22) \ 41^{23^{3k+1}(B0).}\]

Using (39) we get
\[0^{4113^{2k+1}(B0)0^{k} \vdash (3k^2 + 8k + 4) \ 04123^{3k+1}(B0).\]

Using (40) we get
\[0^{4123^{2k+1}(B0)0^{k} \vdash (3k^2 + 8k + 4) \ 0413^{3k+2}(B0).\]
Theorem 6.1.

Let \( M \) be the pure Collatz-like function with parameter defined by: for \( k \geq 0 \),

\[
\begin{align*}
g_3(2k,1) &= (3k + 1,1) \\
g_3(2k,2) &= (3k + 2,1) \\
g_3(2k,3) &= (3k,1) \\
g_3(2k,4) &= (3k + 3,1) \\
g_3(2k,5) &= (3k + 4,1) \\
g_3(2k,6) &= (3k + 2,1) \\
g_3(2k,7) &= (3k + 3,1) \\
g_3(2k,8) &= (3k + 1,1)
\end{align*}
\]

Then \( g_3^{2001}(0,1) \) is undefined.

**Proposition 5.2.** The behavior of Turing machine \( M_3 \), on inputs \( 02^n, n \geq 1 \), depends on the behavior of iterated \( g_3^k(n,1), k \geq 1 \).

**Proof.** We have \( \omega \cdot 0(A0)2^n0^\omega \vdash (n + 1) \quad \omega \cdot 013^n(B0)0^\omega = C(n,1) \).

Since the behavior of iterated \( g_3^k(n,1) \) is an open problem in mathematics, this si also the case for the halting problem for Turing machine \( M_3 \).

Note that the way by which a high score is obtained is particularly clear for machine \( M_3 \). The parameter \( p, 1 \leq p \leq 8 \), can be seen as a state. If \( n \) is odd, \( g_3(n,p) = (n',p + 1) \), the state goes from \( p \) to \( p + 1 \), and the computation stops when state 8 is reached. If \( n \) is even, \( g_3(n,p) = (n',1) \), and the state goes back to 1.

6. **Exponential Collatz-like**

Let \( M_4 \) be the \( 6 \times 2 \) Turing machine defined by Table 3.

We have \( s(M_4) > 7.4 \times 10^{36534} \) and \( \sigma(M_4) > 3.5 \times 10^{18267} \).

This machine is the current champion for the busy beaver competition for \( 6 \times 2 \) machines.

It was discovered in June 2010 by Pavel Kropitz.

The following theorem gives the rules observed by Turing machine \( M_4 \).

**Theorem 6.1.** We have the following transitions between configurations of Turing machine \( M_4 \). Let \( C(n) = \omega \cdot 0(A0)1^n0^\omega \). Then
and Proof. A direct inspection of the transition table gives

<table>
<thead>
<tr>
<th>$M_4$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>1RB</td>
<td>1LE</td>
</tr>
<tr>
<td>$B$</td>
<td>1RC</td>
<td>1RF</td>
</tr>
<tr>
<td>$C$</td>
<td>1LD</td>
<td>0RB</td>
</tr>
<tr>
<td>$D$</td>
<td>1RE</td>
<td>0LC</td>
</tr>
<tr>
<td>$E$</td>
<td>1LA</td>
<td>0RD</td>
</tr>
<tr>
<td>$F$</td>
<td>1RH</td>
<td>1RC</td>
</tr>
</tbody>
</table>

Table 9: Machine $M_4$ discovered in June 2010 by P. Kropitz

(a) $C(0) \vdash (29) C(9)$,
(b) $C(2) \vdash (36) C(11)$,
(c) $C(3) \vdash (48) C(13)$,
and, for all $k \geq 0$,
(d) $C(3k + 1) \vdash (3k + 3)$ \quad \text{and} \quad \omega_{0111(011)^k(H0)0^\omega}$,
(e) $C(9k + 5) \vdash \left((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 25362)/270\right) C((98 \times 4^k - 11)/3)$,
(f) $C(9k + 6) \vdash \left((125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27\right) C((50 \times 4^{k+1} - 59)/3)$,
(g) $C(9k + 8) \vdash \left((4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270\right) C((98 \times 4^k + 1)/3)$,
(h) $C(9k + 9) \vdash \left((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27\right) C((50 \times 4^{k+1} - 11)/3)$,
(i) $C(9k + 11) \vdash \left((4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270\right) C((98 \times 4^{k+1} - 59)/3)$,
(j) $C(9k + 12) \vdash \left((125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27\right) C((50 \times 4^{k+1} + 1)/3)$.

Note that the behavior of this Turing machine on the blank tape involves only items (a), (d), (h) and (j).

Proof. A direct inspection of the transition table gives

- $0^3(A0)0^6 \vdash (29) (A0)1^9$,
- $0^4(A0)1^20^5 \vdash (36) (A0)1^{11}$,
- $0^4(A0)1^30^6 \vdash (48) (A0)1^{13}$,
- $(A0)1 \vdash (1) 1(B1)$,
- $01(E0) \vdash (2) (E0)11$,
- $0(E0) \vdash (1) (A0)1$,
- $11(E0) \vdash (2) (E1)11$,
- $01(C0) \vdash (2) (C0)01$,
- $11(C0) \vdash (2) (C1)01$,
- $0(C0) \vdash (2) 1(E1)$,
- $(E1)01 \vdash (2) 01(E1)$,
- $(E1)00 \vdash (2) 01(E0)$,
- $(E1)1 \vdash (2) (C0)0$,
- $(B1)1^3 \vdash (3) 110(B1)$,
- $(B1)00 \vdash (2) 11(H0)$,
- $(B1)10 \vdash (6) 01(C1)$,
- $(B1)1100 \vdash (12) (01)^2(C1)$,
- $(C1)01 \vdash (2) 01(C1)$,
- $(C1)00 \vdash (2) 01(C0)$,
- $(C1)1^60^6 \vdash (44) 01(E1)1^{10}$,
- $(C1)1^80^{11} \vdash (113) 1(01)^5(E1)1^8$.

Iterating, respectively, (5), (8), (11), (14) and (18) gives
Using (19), (23), (10) and (24), we get
(27) \( 0(01)^k(C1)00 \vdash (4k + 8)(1)0(1)^{k+1}(E1). \)

Using (12), (22) and (6), we get
(28) \( 0(01)^k(E1)00 \vdash (2k + 5)(A0)1^{2k+3}. \)

Using (12), (22) and (7), we get
(29) \( 11(01)^k(E1)00 \vdash (2k + 6)(E1)1^{2k+4}. \)

Using (13), (23), (10) and (24), we get
(30) \( 0(01)^k(E1)1 \vdash (4k + 4)(1)0(1)^k(E1)0. \)

Using (13), (23), (9) and (26), we get
(31) \( 11(01)^k(E1)1 \vdash (4k + 6)(1)0(1)^{k+1}(C1)0. \)

Using (20), (30), (11), (31) and (18), we get
(32) \( 10(01)^k(C1)1^{6}0^{b} \vdash (8k + 70)(1)0(1)^{k+4}(C1)1^{6}. \)

By induction on \( n \), using (32), we get
(33) \( (10)^n(01)^k(C1)1^{6}0^{bn} \vdash (16n^2 + 8kn + 54n)(1)0(1)^{kn}+k(C1)1^{6}. \)

Using (30) and (11), we get
(34) \( 00(01)^k(E1)11 \vdash (4k + 6)(1)0(1)^{k+2}(E1). \)

By induction on \( n \), using (34), we get
(35) \( 0^{2n}(01)^k(E1)1^{2n} \vdash (4n^2 + 4kn + 2n)(1)0(1)^{2n+k}(E1). \)

Using (20), (31), (18), (21), (31), (18) and (33), we get
(36) \( 11(01)^k(C1)1^{6}0^{6k+29} \vdash (16k^2 + 178k + 481)(1)0(1)^{4k+15}(C1)1^{6}. \)

Using (20), (35) and (28), we get
(37) \( 0(01)^k(C1)1^{6}0^{b} \vdash (22k + 201)(A0)1^{2k+25}. \)

Using (20), (35), (31), (18), (32) and (36), we get
(38) \( (110)^30(01)^k(C1)1^{6}0^{6k+101} \vdash (16k^2 + 514k + 4045)(1)0(1)^{4k+55}(C1)1^{6}. \)

By induction on \( k \), using (38) we get
(39) \( (110)^30(01)^k(C1)1^{6}0^{a} \vdash (T)(1)0(1)^{6}(C1)1^{6}, \)

with \( a = (2(3n+55)4^k - 6n - 27k - 110)/3 \), \( b = ((3n+55)4^k - 55)/3 \), and \( T = \frac{16(3n+55)^2}{135}16k - \frac{218(3n+55)^2}{27}4^k - \frac{5}{9}k - \frac{16(3n+55)^2}{135} + \frac{218(3n+55)}{27}. \)

Using (4), (25), (17), (27), (29), (31), (18), (21), (31), (18) and (33), we get
(40) \( (A0)1^{3k+9}0^{29} \vdash (3k + 484)(1)0(1)^{6}(C1)1^{6}. \)

Using (40), (39), (32) and (37), we get
(41) \( 0^{11}(A0)1^{9k+9}0^{a} \vdash (T)(A0)1^{b}, \)

with \( a = (50 \times 4^{k+1} - 11)/3 - 9k - 20 \), \( b = (50 \times 4^{k+1} - 11)/3 \), and \( T = (125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 2289)/27. \)

Using (40), (39), (20), (35), (31), (18) and (37), we get
(42) \( 0^{12}(A0)1^{9k+12}0^{a} \vdash (T)(A0)1^{b}, \)

with \( a = (50 \times 4^{k+1} + 1)/3 - 9k - 24 \), \( b = (50 \times 4^{k+1} + 1)/3 \), and \( T = (125 \times 16^{k+2} + 325 \times 4^{k+2} + 228k - 912)/27. \)
Using (4), (25) and (15), we get
\[(43) \quad (A0)^{1\,3k+1/100} \vdash (3k + 3) \quad 111(011)^k(H0).\]

Results (1), (43), (41) and (42) are results (a), (d), (h) and (j) of the theorem. They are sufficient to analyze the behavior of the Turing machine on a blank tape. The following gives its behavior from configurations *^0(A0)1^n0^o^t*, where \( n = 9k + m, \ m \in \{5, 6, 8, 11\} \).

Using (4), (25), (16), (27), (29), (31), (18), (32) and (36), we get
\[(44) \quad (A0)^{1^{3k+14}/0^8} \vdash (3k + 3076) \quad 1(110)^x0(01)^y(C1)^z1^b.\]

Using (44), (39), (32) and (37), we get
\[0^{11}(A0)^{1^{9k+14}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k+1} - 11)/3 \) and \( T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k + 1) - 25362)/270, \)
and this result is still true for \( k = -1 \), since
\[0^{11}(A0)^{1^{5}/0^{13}} \vdash (285) \quad (A0)^1^{29},\]
so we get
\[(45) \quad 0^{11}(A0)^{1^{9k+14}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k} - 11)/3 \) and \( T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 25362)/270. \)

Using (20), (35), (31), (18), (32), (36) and (37), we get
\[(46) \quad 0^{10}(110)^20(01)^k(C1)^160^b+103 \vdash (16k^2 + 570k + 4886) \quad (A0)^1^{8k+127}.\]

Using (40), (39) and (46), we get
\[0^{10}(A0)^{1^{9k+16}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (50 \times 4^{k+2} - 59)/3 \) and \( T = (125 \times 16^{k+3} - 575 \times 4^{k+3} + 228(k + 1) - 2226)/27, \)
and this result is still true for \( k = -1 \), since
\[0^{10}(A0)^{1^{6}/0^{31}} \vdash (762) \quad (A0)^1^{47},\]
so we get
\[(47) \quad 0^{10}(A0)^{1^{9k+16}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (50 \times 4^{k+1} - 59)/3 \) and \( T = (125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27. \)

Using (20), (30), (11), (31), (18) and (37), we get
\[(48) \quad 0^{12}11100(01)^k(C1)^160^{14} \vdash (30k + 407) \quad (A0)^1^{2k+37}.\]

Using (44), (39) and (48), we get
\[0^{12}(A0)^{1^{9k+17}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k+1} + 1)/3 \) and \( T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k + 1) - 11592)/270, \)
and this result is still true for \( k = -1 \), since
\[0^{12}(A0)^{1^{8}/0^{13}} \vdash (336) \quad (A0)^1^{33},\]
so we get
\[(49) \quad 0^{12}(A0)^{1^{9k+8}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k} + 1)/3 \) and \( T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270. \)

Using (44), (39) and (46), we get
\[0^{10}(A0)^{1^{9k+20}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k+2} - 59)/3 \) and \( T = (4802 \times 16^{k+3} - 11270 \times 4^{k+3} + 2280(k + 1) - 22452)/270, \)
and this result is still true for \( k = -1 \), since
\[0^{10}(A0)^{1^{11}/0^{90}} \vdash (3802) \quad (A0)^1^{111},\]
so we get
\[(50) \quad 0^{10}(A0)^{1^{9k+11}/0^a} \vdash (T) \quad (A0)^1b,\]
with \( b = (98 \times 4^{k+1} - 59)/3 \) and \( T = (4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270. \)

Results (45), (47), (49) and (50) are results (e), (f), (g) and (i) of the theorem.
Results (1), (2) and (3) give special cases (a), (b) and (c) of the theorem. Using the rules of this theorem, we have,

\[ \omega_0(A_0)0 \vdash (29) \quad C(9) \vdash (1293) \quad C(63) \vdash (19, 884, 896, 677) \]

\[ C(273063) \vdash (T_1) \quad C((50 \times 4^{30340} + 1)/3) \vdash (T_2) \quad \omega_0111(011)^K(H0)0^\omega, \]

with \( T_1 = (125 \times 16^{30341} + 325 \times 4^{30341} + 6916380)/27, \)
\( T_2 = (50 \times 4^{30340} + 7)/3, \)
\( K = (50 \times 4^{30340} - 2)/9. \)

The total time is \( s(M_4) = (125 \times 16^{30341} + 1750 \times 4^{30340} + 15)/27 + 19, 885, 154, 163, \)
and the final number of symbols 1 is \( \sigma(M_4) = (25 \times 4^{30341} + 23)/9. \)

Let us give some informal comments on how Turing machine \( M_4 \) works. As in the case of machine \( M_1 \), edge effects give the multiplicity of cases. Also as in the case of machine \( M_1 \), machine \( M_4 \) nibbles strings \((01)^n\), transforming them into strings four times longer (see transitions (32) and (33) in the proof of Theorem 6.1). A new idea allows machine \( M_4 \) to reach exponential scores. Machine \( M_4 \) nibbles strings \((110)^{3k}\) and, each time a string \((110)^{3k}\) is nibbled, the length of a string \((01)^n\) becomes four times longer (see transitions (38) and (39) in the proof of Theorem 6.1). This explains why powers of four are obtained in the transitions of Theorem 6.1.

Let \( g_4 \) be the exponential Collatz-like function defined by: for \( k \geq 0, \)

\[
\begin{align*}
g_4(0) &= 9, \\
g_4(2) &= 11, \\
g_4(3) &= 13, \\
g_4(3k + 1) &= \text{undefined}, \\
g_4(9k + 5) &= (98 \times 4^k - 11)/3, \\
g_4(9k + 6) &= (50 \times 4^{k+1} - 59)/3, \\
g_4(9k + 8) &= (98 \times 4^k + 1)/3, \\
g_4(9k + 9) &= (50 \times 4^{k+1} - 11)/3, \\
g_4(9k + 11) &= (98 \times 4^{k+1} - 59)/3, \\
g_4(9k + 12) &= (50 \times 4^{k+1} + 1)/3.
\end{align*}
\]

Then \( g_5^0(0) \) is undefined.

The theorem gives immediately the following proposition.

**Proposition 6.2.** The behavior of Turing machine \( M_4 \), on inputs \((01)^n, n \geq 0\), depends on the behavior of iterated \( g_4^k(n) \), \( k \geq 1. \)

Since the behavior of iterated \( g_4^k(n) \) is an open problem in mathematics, this is also the case for the halting problem for Turing machine \( M_4 \).

Let \( h_4(n) = \min\{k : g_4^k(n) \text{ is undefined}\} \). We have seen that \( h_4(0) = 5. \) We also have \( h_4(2) = 8, \) and \( C(2) \vdash (T) \text{ END with } T > 10^{10^{10^{10^{10^{10^{10^{10^{10^{10}}}}}}}}}. \) We also have \( h_4(36) = 15. \)

### 7. Unclassifiable Machine

Let \( M_5 \) be the \( 6 \times 2 \) Turing machine defined by Table 10.

We have \( s(M_5) > 3.8 \times 10^{21132} \) and \( \sigma(M_5) > 3.1 \times 10^{10566} \).

This machine was discovered in May 2010 by Pavel Kropitz. It was the champion for the busy beaver competition for \( 6 \times 2 \) machines from May to June 2010.
Table 10: Machine $M_5$ discovered in May 2010 by P. Kropitz

The following theorem is adapted from an analysis of S. Ligocki [Li10]. It gives the rules that enable Turing machine $M_5$ to reach a halting configuration from a blank tape.

**Theorem 7.1.** We have the following transitions between configurations of Turing machine $M_5$. Let $C(k,n) = \omega 01^a 1(C1)1^{3k}0^b$. Then

(a) $\omega 0(A0)0^a \vdash (47) C(2,5)$,

and, for all $k \geq 0$,

(b) $C(k,0) \vdash (3) \omega 1(H0)1^{3k+1}0^a$,

(c) $C(k,1) \vdash (3k + 37) C(2,2k + 2)$,

(d) $C(k,2) \vdash (12k + 44) C(k + 2,4)$,

(e) $C(k,3) \vdash (3k + 57) C(2,2k + 8)$,

(f) $C(k,n + 4) \vdash (27k^2 + 105k + 112) C(3k + 5,n)$.

**Proof.** A direct inspection of the transition table gives

(1) $0^k(A0)0^k \vdash (47) 10^k 1(C1)1^6$,

(2) $0(C0) \vdash (1) (C0)1$,

(3) $1(C0) \vdash (1) (C1)1$,

(4) $(B1)00 \vdash (2) 0^2(C0)$,

(5) $(B1)01 \vdash (4) 01(B1)$,

(6) $(B1)10 \vdash (4) 01(B1)$,

(7) $(B1)10^3 \vdash (14) 10^3(B1)$,

(8) $(B1)1^20^2 \vdash (7) 0^21(B1)1$,

(9) $(B1)1^3 \vdash (3) 0^3(B1)$,

(10) $0(C1) \vdash (2) 1(B1)$,

(11) $11(C1) \vdash (3) 1(H0)1$,

(12) $0^31(C1) \vdash (10) (C1)1^4$,

(13) $0^310^21(C1) \vdash (8) 1(B1)0^210^21$,

(14) $101(C1) \vdash (8) 1(B1)1^2$.

Iterating, respectively, (2) and (9) gives

(15) $0^k(C0) \vdash (k) (C0)1^k$,

(16) $(B1)1^3k \vdash (3k) 0^3k(B1)$.

Using (4), (15) and (3), we get

(17) $10^k(B1)00 \vdash (k + 5) (C1)1^{k+3}$.

Using (16), (17) and (10), we get

(18) $01(B1)1^3k00 \vdash (6k + 7) 1(B1)1^{3k+3}$.

By induction on $k$, using (18), we get

(19) $0^k1(B1)0^2k \vdash (3k^2 + 4k) 1(B1)1^{3k}$. 

Using (12), (10), (16), (6), (19), (16) and (17), we get
\[ (20) \quad 0^4(C1)^{13k}0^{6k+11} \vdash (27k^2 + 105k + 112) \quad 1(C1)^{19k+15}. \]

Using (14), (16), (8), (7) and (17), we get
\[ (21) \quad 101(C1)^{13k}0^7 \vdash (3k + 37) \quad 10^{3k+2}1(C1)^{16}. \]

Using (13), (17), (10), (9), (6), (5), (16), (17), (10), (16) and (17), we get
\[ (22) \quad 0^410^21(C1)^{13k}0^7 \vdash (12k + 44) \quad 10^{3k+4}1(C1)^{16}. \]

Using (12), (12), (10), (16), (8), (6), (17), (10), (9) and (17), we get
\[ (23) \quad 0^410^31(C1)^{13k}0^7 \vdash (3k + 57) \quad 10^{3k+81}(C1)^{16}. \]

Results (1), (11), (21), (22), (23) and (20) give results (a)–(f) of the theorem.

Using the rules of this theorem, we have, in 22158 transitions,
\[ \omega 0(A0)0^\omega \vdash (47) \quad C(2, 5) \vdash (430) \quad C(11, 1) \vdash ( ) \cdots \vdash ( ) \quad END. \]

Let us give some informal comments on how Turing machine \( M_5 \) works. The computation of machine \( M_5 \) on a blank tape is a succession of two types of phases. In the first one, the second parameter \( p \) of configuration \( C(n, p) \) is smaller than four, and transitions (b)–(e) of Theorem 7.1 immediately give a greater value to this second parameter. In the second type of phase, the second parameter is greater than four, and slowly decreases, while the first parameter rapidly increases. In this second type of phase, machine \( M_5 \) nibbles strings of symbols 0 two by two, and each time a string of two symbols 0 is nibbled, a string of three symbols 1 is produced (see transitions (18) and (19) in the proof of Theorem 7.1). In the computation of Turing machine \( M_5 \) on a blank tape, phases of the second type occur four times:

- from \( C(2, 5) \) to \( C(11, 1) \) (1 transition),
- from \( C(2, 35) \) to \( C(29522, 3) \) (8 transitions),
- from \( C(2, 88574) \) to \( C(m, 2) \) (22143 transitions),
- from \( C(m + 2, 4) \) to \( C(3(m + 2) + 5, 0) \) (1 transition),

where \( m \) is a big number.

Let \( g_5 \) be the partial function defined by: for \( k, n \geq 0 \),
\[
\begin{align*}
g_5(k, 0) &= \text{undefined}, \\
g_5(k, 1) &= (2, 3k + 2), \\
g_5(k, 2) &= (k + 2, 4), \\
g_5(k, 3) &= (2, 3k + 8), \\
g_5(k, n + 4) &= (3k + 5, n).
\end{align*}
\]

Then \( g_5^{22157}(2, 5) \) is undefined.

**Proposition 7.2.** The behavior of Turing machine \( M_5 \), on inputs \( 01^{3n+3} \), \( n \geq 0 \), depends on the behavior of iterated \( g_5^k(2, 3n + 2) \).

**Proof.** We have \( \omega 0(A0)1^{3n+3}0^\omega \vdash (3n + 30) \quad \omega 01^{3n+2}1(C1)^{16}0^\omega = C(2, 3n + 2) \).

Since the behavior of iterated \( g_5^k(n, p) \) is an open problem in mathematics, this is also the case for the halting problem for Turing machine \( M_5 \).

The following proposition shows that some configurations take a long time to halt.

**Proposition 7.3.** For Turing machine \( M_5 \), we have \( C(9, 1) \vdash (T) \quad END \) with \( T > 10^{10^610^{10^{3520}}} \).
Proof. By induction on \( n \), using Theorem 7.1 (f), it is easy to prove that, if \( n \geq 0 \), \( 0 \leq r \leq 3 \), we have

\[
C(2, 4n + r) \vdash (t_n) \quad C(u_n, r),
\]

with \( u_n = (3^{n+2} - 5)/2 \) and \( t_n = (3 \times 9^{n+3} - 80 \times 3^{n+3} + 584n - 27)/32 \).

By induction on \( k \), it is easy to prove that, if \( k \geq 2 \), we have

\[
3^{2^{k-1}} \equiv 2^{k+1} + 1 \pmod{2^{k+2}}
\]

so the multiplicative order of 3 modulo \( 2^{k+2} \) is \( 2^k \) for \( k \geq 1 \). Thus we can prove that, for \( k \geq 1 \), \( n, m \geq 0 \), we have

\[
n \equiv m \pmod{2^k} \iff u_n \equiv u_m \pmod{2^{k+1}}.
\]

Now, suppose that, for \( a \in \{1, 3\} \), \( n, n' \geq 1 \), \( q, q' \geq 1 \), \( 0 \leq r, r' \leq 3 \), we have

\[
C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3n + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),
\]

and

\[
C(n', a) \vdash (3n' + 27 + 10a) \quad C(2, 3n' + 3a - 1) = C(2, 4q' + r') \vdash (t_{q'}) \quad C(u_{q'}, r'),
\]

and let \( k \geq 2 \) such that \( n \equiv n' \pmod{2^{k+1}} \). Then it is easy to prove that \( r = r' \) and \( u_q \equiv u_{q'} \pmod{2^k} \). So the behavior of configurations \( C(n, a) \) is mirrored by the behavior of configurations \( C(n', a) \) with \( n' \leq 2^k \) for suitable \( k \).

In the following computation on \( C(9, 1) \):

\[
C(9, 1) \vdash (t_1) \quad C(2, 4 \times 7 + 1) \vdash (t_7)
\]

\[
C(9839, 1) \vdash (t_1) \quad C(2, 4 \times 7379 + 3) \vdash (t_{7379})
\]

\[
C(u_{7379}, 3) \vdash (t_1) \quad C(2, 4 \times q_3 + r_3) \vdash (t_{q_3})
\]

\[
C(u_{q_3}, r_3) \vdash (t_1) \quad C(2, 4 \times q_4 + r_4) \vdash (t_{q_4})
\]

\[
C(u_{q_4}, r_4) \vdash (t_1) \quad C(2, 4 \times q_5 + r_5) \vdash (t_{q_5})
\]

\[
C(u_{q_5}, r_5) \vdash (t_1) \quad C(2, 4 \times q_6 + r_6) \vdash (t_{q_6})
\]

\[
C(u_{q_6}, r_6) \vdash (t_1) \quad C(2, 4 \times q_7 + r_7) \vdash (t_{q_7})
\]

we know that \( r_6 = 0 \) because we have

\[
\begin{align*}
\quad u_{q_1} &= u_7 \equiv 47 \pmod{64}, \quad (3 \times 47) + 2 = (4 \times 35) + 3, \quad q'_2 = 35, \quad r_2 = 3, \\
\quad u_{q_2} &= u_{q_2} \equiv 23 \pmod{32}, \quad (3 \times 23) + 8 = (4 \times 19) + 1, \quad q'_3 = 19, \quad r_3 = 1, \\
\quad u_{q_3} &= u_{q_3} \equiv 7 \pmod{16}, \quad (3 \times 7) + 2 = (4 \times 5) + 3, \quad q'_4 = 5, \quad r_4 = 3, \\
\quad u_{q_4} &= u_{q_4} \equiv 3 \pmod{8}, \quad (3 \times 3) + 8 = (4 \times 4) + 1, \quad q'_5 = 4, \quad r_5 = 1, \\
\quad u_{q_5} &= u_{q_5} \equiv 2 \pmod{4}, \quad (3 \times 2) + 2 = (4 \times 2) + 0, \quad q'_6 = 2, \quad r_6 = 0.
\end{align*}
\]

It is easy to see that, if \( a \in \{1, 3\} \), \( n \geq 0 \), and

\[
C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3n + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),
\]

then \( q \geq (3n - 1)/4 \) and \( u_q > (3^{3/4})^n > 2^n \).

And we also have \( n \geq 5 \Rightarrow t_n > 68 \times 9^n \), so, if \( C(9, 1) \vdash (T) \) END, we have

\[
T > t_{q_6} > 9^{q_6} > 9^{3u_{q_6}/4} > 5^{u_{q_5}},
\]

and \( u_{q_5} > 2^{u_{q_4}}, u_{q_4} > 2^{u_{q_3}}, u_{q_3} > 2^{u_{q_2}} = 2^{u_{7379}}, \) so \( T > 5^{2^{2^{u_{7379}}}} \).

Using \( u_{7379} > 10^{2521} \), and, for \( x \geq 1 \), \( 2^{10^x} > 10^{10^{10^{x-0.53}}}, 2^{10^{10^x}} > 10^{10^{10^{10^{x-0.03}}}}, 2^{10^{10^{10}}}>10^{10^{10^{10^{10^{x-0.03}}}}, \) and \( 5^{10^{10^{10^{10}}}}, \) we are done. \( \square \)
8. A potentially infinite set of rules

Let $M_6$ be the $6 \times 2$ Turing machine defined by Table 11.

We have $s(M_6) > 8.9 \times 10^{1762}$ and $\sigma(M_6) > 2.5 \times 10^{881}$.

This machine was discovered in November 2007 by Terry and Shawn Ligocki. It was the champion for the busy beaver competition for $6 \times 2$ machines from November to December 2007.

The complete analysis of Turing machine $M_6$ seems to need an infinite set of rules, but proving this assertion could be difficult. Of course, only a finite subset of these rules are needed when the machine is launched on a blank tape.

The following theorem gives the rules that enable Turing machine $M_6$ to reach a halting configuration from a blank tape.

Recall that $\text{bin}(p)$ is the usual binary representation of number $p$, and $R(w_1 \ldots w_n) = w_n \ldots w_1$.

**Theorem 8.1.** We have the following transitions between configurations of Turing machine $M_6$. Let $C(n, p) = \omega_0(F0)(10)^n R(\text{bin}(p))0^{\omega}$, so that $C(k, 4m + 1) = C(k + 1, m)$. Then

(a) $\omega_0(A0)0^{\omega} \vdash (6) C(0, 15)$, and, for all $k, m \geq 0$,

(b) $C(k, 4m + 3) \vdash (4k + 6) C(k + 2, m)$,

(c) $C(2k, 4m) \vdash (30k^2 + 20k + 15) C(5k + 2, 2m + 1)$,

(d) $C(2k + 1, 4m) \vdash (30k^2 + 40k + 25) C(5k + 2, 32m + 20)$,

(e) $C(k, 8m + 2) \vdash (8k + 20) C(k + 3, 2m + 1)$,

(f) $C(2k, 16m + 6) \vdash (30k^2 + 40k + 23) C(5k + 2, 32m + 20)$,

(g) $C(2k + 1, 16m + 6) \vdash (30k^2 + 80k + 63) C(5k + 7, 2m + 1)$,

(h) $C(k, 32m + 14) \vdash (4k + 18) C(k + 3, 2m + 1)$,

(i) $C(2k, 128m + 94) \vdash (30k^2 + 40k + 39) C(5k + 2, 256m + 84)$,

(j) $C(2k + 1, 128m + 94) \vdash (30k^2 + 80k + 79) C(5k + 9, m)$,

(k) $C(k, 256m + 190) \vdash (4k + 34) C(k + 5, m)$,

(l) $C(k, 512m + 30) \vdash (2k + 43) \omega_0(10)^k(10)^2(01)^2 R(\text{bin}(m))0^{\omega}$.

**Proof.** A direct inspection of the transition table gives...
Using (16), (20), (14), (4), (11), (18), (8) and (21), we get

Using (25) and (22), we get

and the result is still true for

Using (16), (2), (4), (11), (18), (8) and (21), we get

By induction on

Using (24), with

Using (22) and (21), we get

Iterating, respectively, (3), (10), (6) and (12) gives

Using (5), (19) and (13), we get

Using (16), (4), (17) and (9), we get

Using (16), (20), (15), (17), (9) and (21), we get

Using (16), (2), (10), (6) and (12) gives

Using (16), (20), (15), (17), (9) and (21), we get

Using (22) and (21), we get

Using (22) and (21), we get

Using (22) and (21), we get

Using (22) and (21), we get

Using (22) and (21), we get

Using (22) and (21), we get

From (16), (17), (18), (26) and (27), we get

Using (25) and (22), we get

Using (25) and (22), we get

Using (25) and (22), we get

Using (25) and (22), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get

From (16), (20), (14), (4), (11), (18), (8) and (21), we get
Using (30) and (25), we get
\[ (31) \quad 0^{6k+5}(F0)(10)^{2k+2}01^20 \vdash (30k^2 + 40k + 23) \quad (F0)(10)^{5k+2}00101. \]

Using (30) and (28), we get
\[ (32) \quad 0^{6k+9}(F0)(10)^{2k+1}01^20 \vdash (30k^2 + 80k + 63) \quad (F0)(10)^{5k+7}. \]

Using (16), (20), (14), (4), (17) and (9), we get
\[ (33) \quad 0^2(F0)(10)^k01^30 \vdash (4k + 18) \quad (F0)(10)^{k+3}. \]

Using (34), (4), (17), (11), (18), (8) and (21), we get
\[ (35) \quad 0^5(F0)(10)^k01^401 \vdash (4k + 39) \quad (F0)(10)^{2k+1}00(10)^3. \]

Using (35) and (25), we get
\[ (36) \quad 0^{6k+5}(F0)(10)^{2k+1}01^401 \vdash (30k^2 + 40k + 39) \quad (F0)(10)^{5k+2}02(10)^3. \]

Using (35), (25) and (22), we get
\[ (37) \quad 0^{6k+9}(F0)(10)^{2k+1}01^401 \vdash (30k^2 + 80k + 79) \quad (F0)(10)^{5k+9}. \]

Using (16), (20), (14), (4), (9) and (16), we get
\[ (38) \quad (F0)(01)^k01^50 \vdash (2k + 22) \quad (10)^{k+3}(F0)1. \]

Using (38), (4), (17) and (9), we get
\[ (39) \quad 0^2(F0)(10)^k01^501 \vdash (4k + 34) \quad (F0)(10)^{k+5}. \]

Using (34), (3), (2), (4), (11), (18) and (7), we get
\[ (40) \quad (F0)(10)^k01^404 \vdash (2k + 43) \quad (10)^k1(H0)(10)^2(01)^2. \]

Results (1), (21), (29), (27), (22), (31), (32), (33), (36), (37), (39) and (40) give results (a)–(l) of the theorem.

Note that the rules (a)–(l) are written in their order of occurrence in the computation of Turing machine \( M_6 \) on the blank tape.

Using the rules of this theorem, we have, in 3346 transitions,
\[
\omega 0(A0)0^\omega \vdash (6) \quad C(0, 15) \vdash (6) \quad C(2, 3) \vdash (6) \cdots \vdash (6) \quad \text{END}
\]

We have
\[
\omega 0(A0)0R(bin(p))0^\omega \vdash (6) \quad \omega 0(F0)1^40R(bin(p))0^\omega
\]
\[
= C(0, 32p + 15) \vdash (6) \quad C(2, 8p + 3) \vdash (14) \quad C(4, 2p),
\]
so the behavior of Turing machine \( M_6 \) on inputs \( 00x, \ x \in \{0, 1\}^* \), depends on the behavior of configurations \( C(n, p) \), and the halting problem for Turing machine \( M_6 \) depends on this behavior.

Let us give some informal comments on how Turing machine \( M_6 \) works. First, as in the case of the previous machines, machine \( M_6 \) nibbles strings \( (01)^{2n} \) and transforms them into strings \( (01)^{5n} \) (see transitions (23) and (24) in the proof of Theorem 8.1). This explains why some transitions of Theorem 8.1 looks like a Collatz-like behavior of type \( 2 \to 5 \). Second, unlike the previous machines, edge effects are more chaotic. A study of Turing machine \( M_6 \) beyond the rules stated in Theorem 8.1 seems to lead to an infinite set of rules, but we could not prove this assertion.
Table 12: Machine $M_7$ discovered in October 2000 by Marxen and Buntrock

<table>
<thead>
<tr>
<th>$M_7$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1RB</td>
<td>0LB</td>
</tr>
<tr>
<td>B</td>
<td>0RC</td>
<td>1LB</td>
</tr>
<tr>
<td>C</td>
<td>1RD</td>
<td>0LA</td>
</tr>
<tr>
<td>D</td>
<td>1LE</td>
<td>1LF</td>
</tr>
<tr>
<td>E</td>
<td>1LA</td>
<td>0LD</td>
</tr>
<tr>
<td>F</td>
<td>1RH</td>
<td>1LE</td>
</tr>
</tbody>
</table>

Let $M_7$ be the $6 \times 2$ Turing machine defined by Table 12.

We have $s(M_7) > 6.1 \times 10^{925}$ and $\sigma(M_7) > 6.4 \times 10^{462}$.

This machine was discovered in October 2000 by Heiner Marxen and Jürgen Buntrock.

It was the champion for the busy beaver competition for $6 \times 2$ machines from October 2000 to March 2001.

The following theorem was initially obtained by Munafo [MuWb]. It gives the rules observed by Turing machine $M_7$.

**Theorem 9.1.** We have the following transitions between configurations of Turing machine $M_7$. Let $C(n) = \omega 01^n(B0)\omega^\omega$. Then

(a) $\omega 0(A0)0^\omega \vdash (1) C(1),$ and, for all $k \geq 0$,

(b) $C(3k) \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86)$, $C(9 \times 2^{k+1} - 8),$ $C(2^{k+5} - 8),$ $\omega 01(H1)(011)^k01010^\omega$.

(c) $C(3k + 1) \vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792)$, $C(2^{k+5} - 8),$ $\omega 01(H1)(011)^k01010^\omega$.

(d) $C(3k + 2) \vdash (3k + 8)$, $C(9 \times 2^{k+1} - 8),$ $C(2^{k+5} - 8),$ $\omega 01(H1)(011)^k01010^\omega$.

**Proof.** A direct inspection of the transition table gives

(1) $11(B0)00 \vdash (6) (D1)0101,$
(2) $00(B0)01 \vdash (8) (B0)1^4,$
(3) $11(B0)01 \vdash (6) (B1)01^3,$
(4) $(B0)1 \vdash (3) 1(B0),$
(5) $0(B1) \vdash (1) (B0)1,$
(6) $1(B1) \vdash (1) (B1)1,$
(7) $0(D1) \vdash (2) 1(H1),$
(8) $0^31(D1) \vdash (6) (B0)1^4,$
(9) $0^41^2(D1) \vdash (8) (B0)1^301^2,$
(10) $1^3(D1) \vdash (3) (D1)011.$

Iterating, respectively, (4), (6) and (10) gives

(11) $(B0)1^k \vdash (3k) 1^k(B0),$  
(12) $1^k(B1) \vdash (k) (B1)1^k,$  
(13) $1^{3k}(D1) \vdash (3k) (D1)(011)^k.$

Using (3), (12), (5) and (11), we get

$01^{k+2}(B0)01 \vdash (4k + 10) 1^{k+1}(B0)01^3$, and the result is still true for $k = -1$, so we have

(14) $01^{k+1}(B0)01 \vdash (4k + 6) 1^k(B0)01^3$.

By induction on $k$, using (14), we get
Using (1), (13) and (7), we get
\[ 01^{3k+2}(B0)0^2 \vdash (3k + 8) (H1)(011)^k0101. \]

Using (1), (13) and (9), we get
\[ 0^41^{3k+4}(B0)0^2 \vdash (3k + 14) (B0)1^3(011)^{k+1}0101, \]
and the result is still true for \( k = -1 \), so we have
\[ 0^41^{3k+4}(B0)0^2 \vdash (3k + 11) (B0)1^3(011)^k0101. \]

Using (1), (13) and (8), we get
\[ 0^41^{3k+3}(B0)0^2 \vdash (3k + 12) (B0)1^4(011)^k0101. \]

By induction on \( k \), using (19), we get
\[ 0^4k(n+5)−5−n−3k(B0)1^n(011)^k \vdash (T) (B0)1^2k(n+5)−5, \]
with \( T = 2(n + 5)^2(4^k − 1)/3 − 13(n + 5)(2^k − 1) + 23k. \)

Using (20), for \( n = 3 \) and \( n = 4 \), we get respectively
\[ 0^4k+3−3k−8(B0)1^3(011)^k \vdash 128(4^k − 1)/3 − 13 \times 2^{k+3} + 23k + 104 \]
\[ (B0)1^{2k+3}−5, \]
\[ 0^4k+3−3k−9(B0)1^4(011)^k \vdash (54 \times 4^k − 117 \times 2^k + 23k + 63) \]
\[ (B0)1^{9×2^k−5}. \]

Using (11), (15), (2), (11), (15), (2) and (11), we get
\[ 0^4k+8(B0)1^k0101 \vdash 10k^2 + 65k + 112 \]
\[ 1^{4k+12}(B0). \]

Using (17), (21) and (23), we get
\[ 0^4k+5−3k−111^3k+1(B0)0^2 \vdash (2048 \times (4^k − 1)/3 − 3 \times 2^{k+7} + 26k + 792) \]
\[ 1^{2k+5}−8(B0). \]

Using (18), (22) and (23), we get
\[ 0^4k+2−3k−131^3k+3(B0)0^2 \vdash (54 \times 4^{k+2} − 27 \times 2^{k+4} + 26k + 112) \]
\[ 1^{9×2^{k+2}}−8(B0), \]
and the result is still true for \( k = -1 \), so we have
\[ 0^4k+3−3k−101^3k(B0)0^2 \vdash (54 \times 4^{k+1} − 27 \times 2^{k+3} + 26k + 86) \]
\[ 1^{9×2^{k+1}}−8(B0). \]

Results (25), (24) and (16) give results (b)–(d) of the theorem.

Using the rules of this theorem, we have
\[ \vdash 0^40(A0)0^\omega \vdash (1) C(1) \vdash (408) C(24) \vdash (14100774) \]
\[ C(4600) \vdash (T) C(2)^{1538} \vdash (2^{1538} − 2) \vdash 01(H1)(011)^p0101\omega, \]
with \( T = 2048 \times (4^{1533} − 1)/3 − 3 \times 2^{1540} + 40650 \) and \( p = (2^{1538} − 10)/3 \).

So the total time is \( s(M_7) = 2048 \times (4^{1533} − 1)/3 − 11 \times 2^{1538} + 1414831 \), and the final number of symbols 1 is \( \sigma(M_7) = 2 \times (2^{1538} − 10)/3 + 4 \).

As in the case of Turing machine \( M_4 \), we find two ideas that explain how Turing machine \( M_7 \) works. First, machine \( M_7 \) nibbles strings of symbols 0 one by one, transforming them into strings two times longer (see transitions (14) and (15) in the proof of Theorem \[ \text{[0.1]} \]. Second, machine \( M_7 \) nibbles strings (011)\(^k\) and, each time a string 011 is nibbled, the length of a string of symbols 1 is doubled (see transitions (19) and (20) in the proof of Theorem \[ \text{[0.1]} \]). This explains why powers of two are obtained.

Note that
\[ C(6k + 1) \vdash ( ) C(3m) \vdash ( ) C(6\rho + 4) \vdash ( ) C(3q + 2) \vdash ( ) \text{END}, \]
with \( m = \frac{(2^{2k+5} - 8)}{3} \), \( p = 3 \times 2^m - 2 \), \( q = \frac{(2^{2p+6} - 10)}{3} \). So all configurations \( C(n) \) lead to a halting configuration. Those taking the most time are \( C(6k + 1) \). For example, \( C(7) \vdash (t) \) \( \mathrm{END} \) with \( t > 10^{8.9 \times 10^{12}} \). More generally, \( C(6k + 1) \vdash (t(k)) \mathrm{END} \) with \( t(k) > 10^{10^{(3k+2)/5}} \).

10. Conclusion

We discuss two questions as a conclusion to this article.

A. How simulating Collatz-like functions allows Turing machines to achieve high scores?

Lagarias [La85] noted that the successive iterates of the \( 3x+1 \) function \( T \) have an irregular behavior. For example, 7 iterations of function \( T \) on \( n = 26 \) lead to the value 1, but 70 iterations are necessary on \( n = 27 \). It seems that many Collatz-like functions have the same irregular behavior. Iterating them on small numbers may produce very long runs before stopping.

Adding parameters may increase the number of iterations by allowing the iterated values to range the set of parameters before stopping. The pure Collatz-like function with parameter \( g_3(n, p) \) presented in Section 5 is particularly illustrative.

Another way to high scores is given by exponential Collatz-like functions such as function \( g_4 \) in Section 6. Only five iterations are performed on a blank tape, but exponential growth ensures a high score.

Irregular behavior is a condition for a Collatz-like function to be eligible to the busy beaver competition. Another condition is, of course, being computable by a very small Turing machine.

B. Are some universal devices more natural than others?

Conway [Co72] proved that there is no algorithm that, given as inputs a Collatz-like function \( g \) and two integers \( n, p \), outputs an answer yes or no to the question: Does there exist a positive integer \( k \) such that \( g^k(n) = p? \) Conway [Co72, Co87] also proved that Collatz-like functions can be used to simulate all computable (also called recursive) functions. These properties can be summed up by writing that Collatz-like functions provide a universal model of computation with a \( m \)-complete decision problem.

Many universal models of computation are known: Turing machines, tag-systems, cellular automata, Diophantine equations, etc. (see [MM10]. Of course, any universal model can simulate and be simulated by any other universal model. But it is Collatz-like functions, and not another model, that appear naturally in this study. Their unexpectedly pervasive presence leads to wonder about the significance of their status among mathematical beings.

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References


