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ACZEL-MENDLER BISIMULATIONS IN A REGULAR CATEGORY

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Abstract. Aczel-Mendler bisimulations are a coalgebraic extension of a variety of computational relations between systems. It is usual to assume that the underlying category satisfies some form of the axiom of choice, so that the collection of bisimulations enjoys desirable properties, such as closure under composition. In this paper, we accommodate the definition in general regular categories and toposes. We show that this general definition: 1) is closed under composition without using the axiom of choice, 2) coincides with other types of coalgebraic formulations under milder conditions, 3) coincides with the usual definition when the category satisfies the regular axiom of choice. In particular, the case of toposes heavily relies on power-objects, for which we recover some favourable properties along the way. Finally, we describe several examples in Stone spaces, toposes for name-passing, and modules over a ring.

Introduction

Bisimilarity is a way to describe that two states of two systems behave in the same way. It formalises the fact that one can mimic any execution starting from one state with an execution from the other state, and vice versa. In contrast to language equivalence, which requires one to consider entire (possibly infinite) executions, bisimilarity is a local notion, focusing only on the next step of the execution. As such, bisimilarity is often far more tractable than the comparison of trace languages.

Since the seminal work by Park [Par81] on labelled transition systems, a plethora of different notions of bisimilarity has arisen in various contexts: for probabilistic [LS91], timed [Wan90], hybrid [GP05], and truly concurrent [vG91] systems, among others. Although they deal with very different types of systems, these notions share common ground: connections with logic, games, fixpoints, or even some form of decidability that exhibits a similar flavour. This has suggested that these theories could be abstracted into a meta-theory that captures the essence of these shared foundations.

Categorical modelling is one such effort to abstract concrete theories into purely mathematical ones, expressed in the language of category theory. If an earlier success in computer science lies in the denotational semantics of programming languages (see, for example, the Curry-Howard-Lambek correspondence, first published in [LS88]), a more recent achievement

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is the categorical modelling of bisimulations and computational systems using coalgebras. In this modelling, systems are represented as coalgebras—that is, morphisms of the form $X \longrightarrow FX$, where X is an object in some category representing the state space of the system, and F is an endofunctor on this category, representing the type of allowed transitions. By varying the underlying category and the functor F, one can capture various (known and novel) types of systems. In this abstract view, morphisms between coalgebras play an important role: they encompass the intuition of bisimulation maps, that is, transformations of systems that induce bisimulations. Building on this intuition, several abstract notions of bisimilarity can be defined, all more or less equivalent (see [Jac16, Sta11] for an overview).

In the present paper, we are particularly interested in Aczel-Mendler bisimilarity [AM89], which defines a bisimulation as an abstract relation (that is, a subobject of a product) which itself carries a coalgebra structure, from which the coalgebra structures of the systems being compared can be recovered via projections. This abstract notion has the advantage of being very close to the usual notions of bisimulation in terms of relations, but this comes at the cost of being overly set-flavoured. For instance, some basic properties (such as closure under composition, or their relation to bisimulation maps) only hold when the underlying category satisfies some form of the axiom of choice.

These issues hinder the use of Aczel-Mendler bisimulations in certain interesting categories. Regular categories-and in particular, toposes- form a class of categories that enjoy very desirable properties, notably a convenient theory of relations, which is crucial for abstract bisimulations. However, they do not satisfy the axiom of choice. This is the case, for example, with the effective topos [Hyl82], which internalises concepts such as decidable sets and computable functions, or the topos of nominal sets [Law89], which models name-passing and, more generally, infinite systems possessing some form of decidability. Being able to abstract bisimulations in such categories thus becomes essential, offering a potential route to general decidability results.

Outline. The remainder of the paper is organised as follows. In Section 1, we recall some necessary background on relations in a general category and allegories, with a particular focus on maps. This includes the definition of relations, their basic constructions (diagonal, composition, converse, and intersection), the definition of an allegory, maps and tabulations, and finally, the characterisations of relation maps and the tabularity of the allegory of relations. In Section 2, we recall the definition of Aczel-Mendler bisimulations and some of their properties that only hold under certain forms of the axiom of choice. We then extend them to regular AM-bisimulations, which behave well in any regular category. Section 3 explores the power-object monad and some of its well-known properties that illuminate its role in AM-bisimulations. We recover these properties in a purely relational way by observing that Kleisli composition corresponds to the composition of relations. In Section 4, we present a more elegant reformulation of regular AM-bisimulations in toposes, enabled by the power-object monad. Section 5 extends this refined formulation to simulations. Finally, in Section 6, we explore examples of regular AM-bisimulations for Stone spaces, toposes modelling name-passing, and linear weighted systems.

Contributions. Our contributions may be summarised as follows:

(1) An extension of the theory of Aczel-Mendler bisimulations that works in any regular category, without relying on the axiom of choice. In particular, we prove that closure

under composition (Proposition 2.16) and coincidence with other notions of coalgebraic bisimulations (Theorem 2.18) do not require the axiom of choice.
(2) An elementary and relational account of folklore properties of power-objects, including the fact that they yield a commutative monad whose Kleisli category is isomorphic to the category of relations (Theorem 3.3), and that there are simple conditions for the existence of (weak) distributive laws with respect to it (Corollary 3.12).
(3) A more elegant formulation of regular AM-bisimulations in the case of toposes, enabled by the power-object monad, with a connection to tabulations of coalgebra homomorphisms that can be established (Corollary 4.6), again without assuming the axiom of choice.
(4) An extension of this more refined formulation to simulations in a topos (Section 5).

Related work. Section 1 provides a summary of the material required from the textbook [FS90] on allegories, with a particular focus on allegories of relations. Applications of allegories, and their extensions, to computer science include fuzzy logic [Win07], logic programme compilation [AL12], and generic programming [BH99]. Topos theory has a well-established literature covering a variety of aspects. For a comprehensive reference on the subject, we recommend [Joh02]. Coalgebra theory—-particularly bisimulations for coalgebras—-has also seen substantial recent development. Most of the results in this paper concerning bisimulations are grounded in concepts discussed in the textbook [Jac16]. A detailed comparison of various notions of coalgebraic bisimilarity can be found in [Sta11]. Aczel-Mendler bisimulations originate from [AM89]. Their connection to bisimulation and simulation maps within a categorical framework lies at the heart of the theory of open maps [JNW96, WDKH19].

Comparison with the CALCO 2023 paper. In addition to the numerous complete proofs, this version adds Section 3 about the power-object monad.

Notations. Given two morphisms $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ in a category with binary product, we denote the pairing by $\langle f, g\rangle : X \longrightarrow Y \times Y'$ (if $X = X'$), and the product by $f \times g : X \times X' \longrightarrow Y \times Y'$.

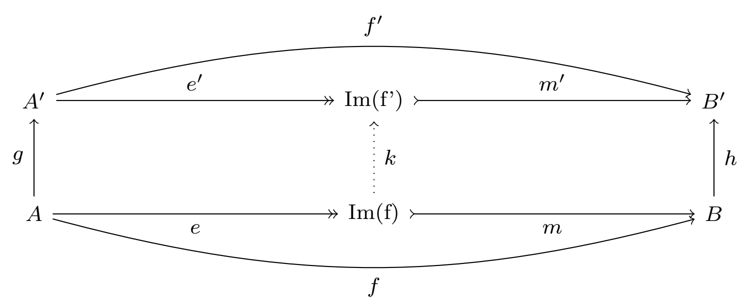
1. Allegory of Relations

In this section, we present the general notion of relations in a category, focusing in particular on the fact that they form a tabular allegory. Definitions, propositions, and proofs may be found in [FS90]. Our main motivations for introducing allegories in this paper are: 1) to highlight that regular categories provide the appropriate level of abstraction for studying bisimulations; and 2) to introduce maps—that is, left adjoints in allegories—which we aim to relate to coalgebra homomorphisms, in order to provide an abstract justification for the idea that "coalgebra homomorphisms are bisimulation maps".

1.1. **Subobjects and Factorisations.** In this paper, subobjects will play a crucial role throughout. Let us then spend some time on their definition. Fix an object A of $\mathcal{C}$. There is a preorder on the class of monos of the form $m : X \rightarrowtail A$ defined by $m : X \rightarrowtail A \sqsubseteq m' : X' \rightarrowtail A$ if and only if there is a morphism $u : X \longrightarrow X'$ such that $m' \cdot u = m$. In this case, u is unique and is a mono. A *subobject of* A is then an equivalence class of monos with $m : X \rightarrowtail A \equiv m' : X' \rightarrowtail A$ if $m \sqsubseteq m'$ and $m \sqsupseteq m'$, that is, there are u and u' such that $m' \cdot u = m$ and $m \cdot u' = m'$. In this case, u and u' are inverses of each other. The preorder on the monos becomes a partial order on subobjects, also denoted by $\sqsubseteq$. Throughout the paper, when reasoning about subobjects, we will instead reason using a representing mono. This is harmless when dealing with notions such as pullbacks and factorisations that are unique only up to isomorphism.

Example 1.1. In **Set**, since monos are injective functions, subobjects of a set are in bijection with its subsets. The order $\sqsubseteq$ then corresponds to the usual inclusion $\subseteq$ of sets.

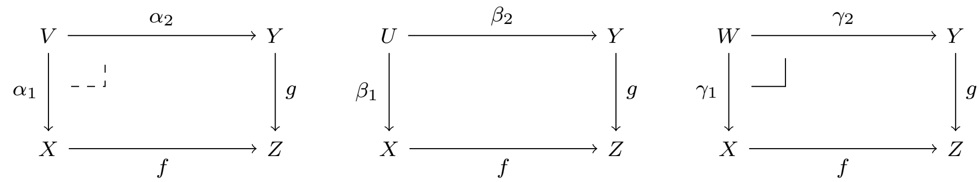
Given a morphism $f : A \longrightarrow B$, there is a particular subobject of B called the *image of* f. In general, it is defined as the smallest (for $\sqsubseteq$) subobject $\mathrm{Im}(f)$ of B such that f can be factorised as $m \cdot e$, where m is any representing mono. The existence of the image is not guaranteed in general. It is, however, when the category $\mathcal{C}$ has a nice (epi, mono)-factorisation system, as is the case for regular categories (and so for toposes). In a regular category, every morphism f can be uniquely (up to unique isomorphism) factorised as $m \cdot e$, where m is a mono and e is a regular epi, and furthermore, this factorisation is the image factorisation. In addition, this factorisation is functorial and is preserved by pullbacks, meaning that if we have a commutative diagram of the following form (outer rectangle):



there is a (dotted) morphism that makes the two squares commute, and if the outer rectangle is a pullback, then the rightmost square is also a pullback. For a gentle overview of regular categories, an interested reader can look into [But98].

Example 1.2. In **Set**, the image of a function is the usual notion of image, that is, the subset $f(a) \mid a \in A$ of B. Since **Set** is regular, and regular epis are surjective functions, the image factorisation is given by the (surjection, injection)-factorisation of the function f.

Remark 1.3 (Pullbacks vs. weak pullbacks)**.** In many places in this paper, where pullbacks would naturally play a role, they can be replaced by weak pullbacks, leading to laxer conditions. A *weak pullback* of a cospan $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is given by a commutative square (as on the left):



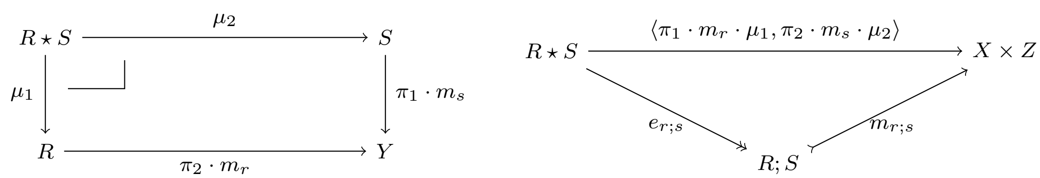
such that, for every other commutative square as in the middle, there is (not necessarily a unique) morphism $\phi : U \longrightarrow V$ with $\alpha_i \cdot \phi = \beta_i$. We denote them by a dashed corner (while proper pullbacks are denoted by plain corners). If we are in a category where pullbacks exist, weak pullbacks can be equivalently reformulated as the commutative squares as on the left, such that, if the pullback of f and g is given as on the right, then the unique morphism $\psi : V \longrightarrow W$ with $\gamma_i \cdot \psi = \alpha_i$ is a split epi.

As a first example of replacement of pullbacks by weak pullbacks, the preservation of images by pullbacks and the functoriality also imply the preservation of images by weak pullbacks, in the sense that, if the outer rectangle is a weak pullback, then the rightmost square is also a weak pullback.

1.2. **Relations in a Regular Category.** From now on, let us assume that the category $\mathcal{C}$ is regular, that is, it has finite limits and a pullback-stable (regular epi, mono)-factorisation as described in the previous section. Everything in this section can be done in a locally regular category, but less conveniently. In general:

Definition 1.4. A *relation from X to Y* is a subobject of $X \times Y$.

Objects of $\mathcal{C}$ and relations between them form a category, denoted by $\mathbf{Rel}(\mathcal{C})$. The composition is defined as follows. Let $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail Y \times Z$ be two monos, representing two relations, r from X to Y and s from Y to Z. Form the following pullback and (regular epi, mono)-factorisation:



The composition $r; s$ from X to Z is then the subobject represented by the mono part $m_{r;s}$.

Remark 1.5 (Pullbacks vs. weak pullbacks, continued)**.** In the definition of the composition, we chose to form a pullback, because we know it exists. However, the definition is unchanged if we take any weak pullback instead.

The *identity relation* Δ_X is represented by the diagonal $\langle \mathrm{id}, \mathrm{id} \rangle : X \rightarrowtail X \times X$.

Proposition 1.6. $\mathbf{Rel}$*($\mathcal{C}$) is a category.*

Example 1.7. In **Set**, the composition of relations is the usual one:

$$R; S = \{(x, z) \in X \times Z \mid \exists y \in Y. \, (x, y) \in R \wedge (y, z) \in S\},$$

while the identity relation is the usual diagonal $\Delta_X = \{(x, x) \mid x \in X\}$.

Of course, $\mathbf{Rel}(\mathcal{C})$ has much more structure. First, since subobjects are naturally ordered by $\sqsubseteq$, and since this order is compatible with the composition, $\mathbf{Rel}(\mathcal{C})$ has a structure of a locally ordered 2-category. Furthermore, it comes equipped with an anti-involution $(\_)^\dagger : \mathbf{Rel}(\mathcal{C})^{op} \longrightarrow \mathbf{Rel}(\mathcal{C})$ which makes it an I-category in the sense of [FS90]. This involution is given by the converse of a relation, as follows. If the relation r is represented by the mono $m_r : R \rightarrowtail X \times Y$, then $r^\dagger$ is represented by $m_{r^\dagger} = \langle \pi_2, \pi_1 \rangle \cdot m_r : R \rightarrowtail Y \times X$. Finally, the meet of two relations for the partial order $\sqsubseteq$ is defined and is called the intersection. Given $m_r : R \rightarrowtail X \times Y$ and $m_s : S \rightarrowtail X \times Y$ representing r and s respectively, the intersection $r \cap s$ is then represented by the pullback of m_r and m_s. Altogether:

Theorem 1.8. $\mathbf{Rel}(\mathcal{C})$ *is an allegory, meaning that all this data satisfies the modular law:*

$$(R;S) \cap T \sqsubseteq (R \cap (T;S^\dagger));S.$$

Example 1.9. In **Set**, $R^\dagger$ is the usual converse of the relation R: $R^\dagger = \{(y,x) \mid (x,y) \in R\}$. The intersection $\cap$ is the intersection of relations as sets. Let us show what the modular law means in $\mathbf{Rel}(\mathbf{Set})$. The relation $(R;S) \cap T$ is given by the set

$$\{(x,z) \in T \mid \exists y.\, (x,y) \in R \wedge (y,z) \in S\}.$$

Let (x,z) be in this set and fix a witness y as in the definition above. This means in particular that $(y,z) \in S$ and so $(z,y) \in S^\dagger$. Since $(x,z) \in T$, then $(x,y) \in T;S^\dagger$. In summary, $(x,y) \in R \cap (T;S^\dagger)$ and $(x,z) \in (R \cap (T;S^\dagger));S$. Intuitively, the modular law is an algebraic law expressing how composition preserves intersection in a weak way. More generally, this law is crucial to make adjoints in an allegory behave like direct/inverse images, (see the next section, and the Frobenius reciprocity [Law70]).

1.3. Maps in Allegories.

From an allegory (intuitively of relations), it is possible to recover the morphisms of the original category through the notion of *maps*. In a general allegory $\mathcal{A}$, a map is a morphism which is a left adjoint (in the 2-categorical sense). Maps form a subcategory of $\mathcal{A}$ denoted by $\mathbf{Map}(\mathcal{A})$. In the case of an allegory of relations:

Theorem 1.10. $\mathbf{Map}(\mathbf{Rel}(\mathcal{C}))$ *is isomorphic to* $\mathcal{C}$.

The reason for it is that maps (left adjoints) in $\mathbf{Rel}(\mathcal{C})$ are precisely the relations represented by a mono of the form $\langle \mathrm{id}, f \rangle$ for some morphism f of $\mathcal{C}$, justifying the remark from Example 1.9 that left adjoints in an allegory behave like direct images. Similarly, their right adjoints are relations represented by $\langle f, \mathrm{id} \rangle$, corresponding to inverse images. This also implies that $\mathbf{Rel}(\mathcal{C})$ is *tabular*, that is, it is generated by maps in the following sense. A *tabulation* of a morphism $\phi : X \longrightarrow Y$ in an allegory is a pair of maps $\psi : Z \longrightarrow X$ and $\xi : Z \longrightarrow Y$ such that $\phi = \xi \cdot \psi^\dagger$ and $\psi^\dagger \cdot \psi \cap \xi^\dagger \cdot \xi = \mathrm{id}_Z$.

Theorem 1.11. *In an allegory of relations, the tabulations of a relation R are exactly those pairs of relations (S,T) represented by monos of the form $\langle id, f \rangle$ and $\langle id, g \rangle$ respectively, with f and g jointly monic, and such that $R = T^\dagger;S$. In particular, every relation has a tabulation, that is,* $\mathbf{Rel}(\mathcal{C})$ *is tabular.*

The intuition of this theorem is that relations are precisely jointly monic spans.

Example 1.12. In **Set**, maps are graphs of functions, that is, relations of the form $\{(x, f(x)) \mid x \in X\}$ for some function $f : X \longrightarrow Y$. Consequently, every relation R is the same as the span of $f : R \longrightarrow X$ $(x, y) \mapsto x$ and $g : R \longrightarrow Y$ $(x, y) \mapsto y$, that is, $R = \{(f(r), g(r)) \mid r \in R\}$.

2. Aczel-Mendler Bisimulations, in Regular Categories

We now start investigating our original problem: a nice general theory of bisimulations in terms of relations. The development of this section will start with the notion of Aczel-Mendler bisimulations [AM89], where systems are described as coalgebras. We will witness that one bottleneck of this theory is the role of the axiom of choice that is necessary to prove even some basic properties of this notion of bisimulations. This prevents the use of this notion in most regular categories. We will then show that we can fix this issue by a careful usage of relations.

2.1. Systems as Coalgebras.

In this section, we will briefly recall coalgebras, and how to model systems with them. For a more complete introduction, see for example [Jac16].

Coalgebras require two ingredients:

- a category $\mathcal{C}$ that describes the type of state spaces of our systems; and
- an endofunctor F on $\mathcal{C}$ that describes the type of allowed transitions.

A *coalgebra* is then a morphism of type $\alpha : X \longrightarrow FX$. Intuitively, X is the state space of the system and α maps a state to the collection of transitions from this state.

Example 2.1. For example, deterministic transition systems labelled in the alphabet Σ can be modelled with the **Set**-functor $X \mapsto \Sigma \Rightarrow X$. A coalgebra for this functor is a function $X \to \Sigma \Rightarrow X$. It maps a state to a function from Σ to X, describing what the next state is after reading a particular letter. Non-deterministic labelled transition systems can be described using the functor $X \mapsto \mathcal{P}(\Sigma \times X)$. A coalgebra then maps a state to a set of transitions, given by a letter and a state, describing the states we can reach from another state reading a particular letter. Another typical example is a probabilistic system, that can be described using the distribution functor $\mathcal{D}$. A transition for those systems is then a distribution on the states, describing what is the probability of reaching a given state in the next step.

A *morphism of coalgebras* from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ is a morphism $f : X \longrightarrow Y$ of $\mathcal{C}$ such that the following diagram commutes:

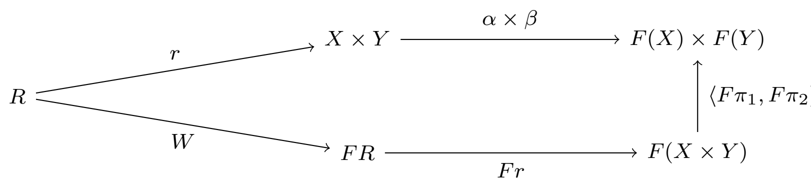
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ FX & \xrightarrow[\quad Ff \quad]{} & FY \end{array}$$

Coalgebras on F and homomorphisms of coalgebras form a category, which we denote by $\mathbf{Coal}(F)$.

2.2. **Aczel-Mendler Bisimulations of Coalgebras.** In this section, we follow closely the development of [Jac16]. We recall the definition of Aczel-Mendler bisimulations and give some of their properties.

2.2.1. *AM-Bisimulations.*

Definition 2.2. We say that a relation is an *Aczel-Mendler bisimulation* (AM-bisimulation for short) from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow FR$, *a witness*, such that:



Example 2.3. In the case of non-deterministic labelled transition systems, AM-bisimulations correspond to the usual strong bisimulations. The function W maps a pair (x, y) of states of α and β to a subset of triples (a, x', y') such that $(x', y') \in R$. The commutation condition means that the set $c(x)$ of transitions from x corresponds exactly to the set $\{(a, x') \mid \exists y'. (a, x', y') \in W(x, y)\}$, and similarly for y. This implies the characteristic property of a bisimulation: if there is a transition (a, x') from x, then there exists a transition (a, y') from y such that $(x', y') \in R$; and vice versa.

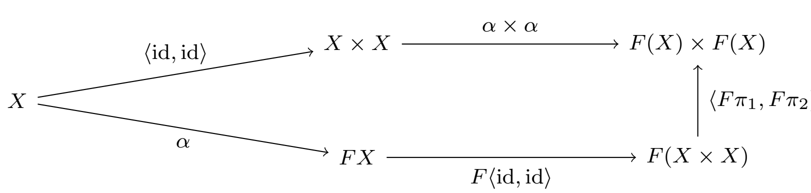
2.2.2. *I-Category of Bisimulations, under the Axiom of Choice.* We show now that AM-bisimulations behave well under the regular axiom of choice.

Definition 2.4. A category has the regular axiom of choice if every regular epi is split.

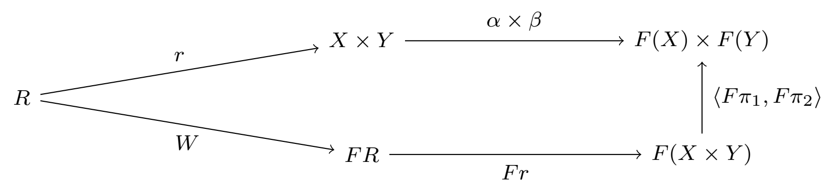
Proposition 2.5. *Assume that $\mathcal{C}$ has the regular axiom of choice and that F preserves weak pullbacks. Then the following is an I-category in the sense of* [FS90]*, denoted by* **Bis***(F):*
- *objects are coalgebras on F,*
- *morphisms are AM-bisimulations,*
- *$\sqsubseteq$, identities, composition, and $(\_)^\dagger$ are defined as in* **Rel***($\mathcal{C}$).*

That is, diagonals are AM-bisimulations, and AM-bisimulations are closed under composition and converse.

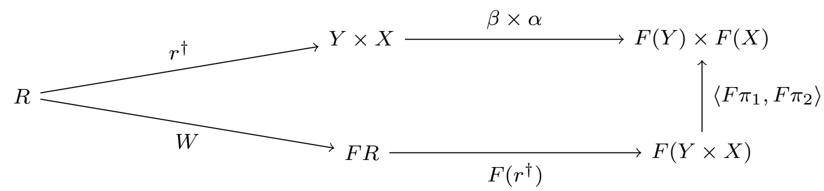
Proof. It boils down to proving the following three facts:
- **Diagonals are Aczel-Mendler bisimulations:** We have the following commutative diagram:



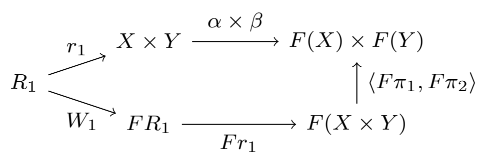
- **Aczel-Mendler bisimulations are closed under converse:** Assume given a witness for r:

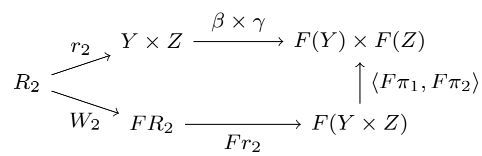


Then it is also a witness for $r^\dagger = \langle \pi_2, \pi_1 \rangle \cdot r$:

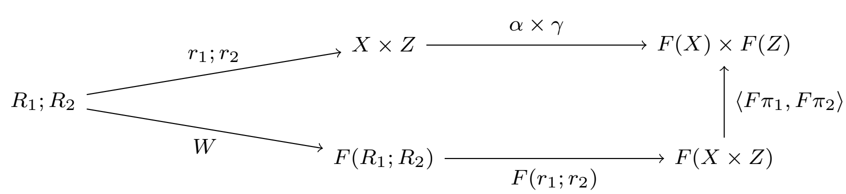


- **Aczel-Mendler bisimulations are closed under composition:** We then have two witnesses:

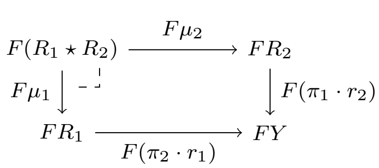


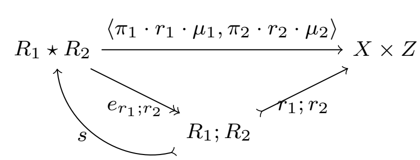


We then want to construct a morphism $W : R_1; R_2 \longrightarrow F(R_1; R_2)$ such that



Since F preserves weak pullbacks and by definition of the composition, we have the following weak pullback and (regular epi, mono)-factorisation:

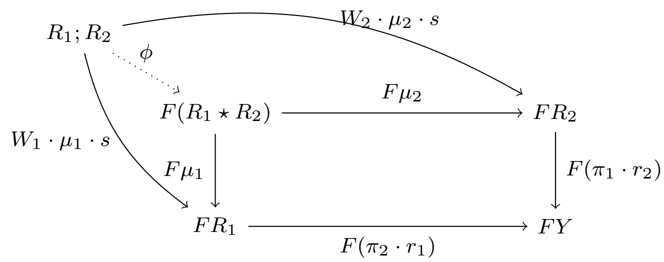




Denote by s a section of $e_{r_1;r_2}$, which exists by the regular axiom of choice. Then we have the following:

$$\begin{array}{rcll} F(\pi_1 \cdot r_2) \cdot W_2 \cdot \mu_2 \cdot s & = & \beta \cdot \pi_1 \cdot r_2 \cdot \mu_2 \cdot s & (r_2 \text{ is AM-bisimulation}) \\ & = & \beta \cdot \pi_2 \cdot r_1 \cdot \mu_1 \cdot s & (\text{definition of } \mu_i) \\ & = & F(\pi_2 \cdot r_1) \cdot W_1 \cdot \mu_1 \cdot s & (r_1 \text{ is AM-bisimulation}) \end{array}$$

By the universal property of weak pullbacks, we have $\phi : R_1; R_2 \longrightarrow F(R_1 \star R_2)$, such that



Now $W = Fe_{r_1;r_2} \cdot \phi$ is the expected witness:

$$
\begin{aligned}
& \langle F\pi_1, F\pi_2\rangle \cdot F(r_1; r_2) \cdot W \\
= \;& \langle F\pi_1, F\pi_2\rangle \cdot F(r_1; r_2) \cdot Fe_{r_1;r_2} \cdot \phi && \text{(definition of } W\text{)} \\
= \;& \langle F\pi_1, F\pi_2\rangle \cdot F\langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2\rangle \cdot \phi && \text{(definition of } r_1; r_2\text{)} \\
= \;& \langle F(\pi_1 \cdot r_1 \cdot \mu_1), F(\pi_2 \cdot r_2 \cdot \mu_2)\rangle \cdot \phi && \text{(computation on products)} \\
= \;& F(\pi_1 \cdot r_1) \times F(\pi_2 \cdot r_2) \cdot \langle F(\mu_1) \cdot \phi, F(\mu_2) \cdot \phi\rangle && \text{(computation on products)} \\
= \;& F(\pi_1 \cdot r_1) \times F(\pi_2 \cdot r_2) \cdot \langle W_1 \cdot \mu_1 \cdot s, W_2 \cdot \mu_2 \cdot s\rangle && \text{(definition of } \phi\text{)} \\
= \;& \langle \alpha \cdot \pi_1 \cdot r_1 \cdot \mu_1 \cdot s, \gamma \cdot \pi_2 \cdot r_2 \cdot \mu_2 \cdot s\rangle && \text{(definition of the } W_i\text{)} \\
= \;& \alpha \times \gamma \cdot \langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2\rangle \cdot s && \text{(computation on products)} \\
= \;& \alpha \times \gamma \cdot (r_1; r_2) && \text{(definition of } s\text{)} \quad \square
\end{aligned}
$$

Remark 2.6. As we have already seen, the preservation of weak pullbacks is a crucial property for a functor related to relations. More surprisingly, the reliance on the axiom of choice is necessary to prove closure under composition. This was already noted in [Jac16, Sta11]. Sometimes, this proposition is stated under the assumption that F preserves pullbacks. When pullbacks exist, since any functor preserves split epis, it follows from Remark 1.3 that if a functor preserves pullbacks, then it also preserves weak pullbacks.

In the proof, we rely on the regular axiom of choice in the following way: we require that the epi part $e_{r_1;r_2} : R_1 \star R_2 \twoheadrightarrow R_1; R_2$ of a (regular epi, mono)-factorisation be split, that is, that there exists a section $s : R_1; R_2 \rightarrowtail R_1 \star R_2$. In **Set**, $R_1 \star R_2$ consists of triples (x, y, z) such that $(x, y) \in R_1$ and $(y, z) \in R_2$. The section then corresponds to making a choice of such an intermediate y for every pair (x, z) in the composite relation. This kind of choice is common, for instance, in the proof that strong bisimulations are closed under composition: given a transition (a, x') from x, to show that a similar transition exists from z, one picks an intermediate y, uses the assumption that R_1 is a bisimulation to obtain a transition from y, and finally uses that R_2 is a bisimulation to conclude the argument.

2.2.3. *Bisimulation Maps are Coalgebra Homomorphisms.* In this I-category of bisimulations, we can also discuss maps and tabulations, as we did in the context of relations. Moreover, since the 2-categorical structure of $\mathbf{Bis}(F)$ is inherited from that of $\mathbf{Rel}(\mathcal{C})$—specifically, because the local posets of bisimulations embed into the corresponding local posets of relations—we may apply results from Section 1.3 within this setting. In particular, we can establish the following:

Theorem 2.7. *Under the assumptions of Proposition 2.5,* $\mathbf{Map}(\mathbf{Bis}(F))$ *is isomorphic to* $\mathbf{Coal}(F)$*.*

Using results from Section 1.3, proving this theorem boils down to proving that bisimulations that are maps are precisely graphs of coalgebra homomorphisms:

Proposition 2.8. *A morphism $h : X \longrightarrow Y$ of $\mathcal{C}$ is a coalgebra homomorphism from α to β if and only if the mono $\langle id, h\rangle : X \rightarrowtail X \times Y$ represents an AM-bisimulation from α to β.*

Proof. Let us prove both implications:

$\Rightarrow$ Assume given a coalgebra homomorphism $h : X \longrightarrow Y$ from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, that is, with

$$Fh \cdot \alpha = \beta \cdot h.$$

We then want $W : X \longrightarrow FX$ such that

$$\alpha \times \beta \cdot \langle \mathrm{id}, h\rangle = \langle F\pi_1, F\pi_2\rangle \cdot F\langle \mathrm{id}, h\rangle \cdot W.$$

Using $W = \alpha$ does the job:

$$\begin{aligned}
\alpha \times \beta \cdot \langle \mathrm{id}, h\rangle &= \langle \alpha, \beta \cdot h\rangle && \text{(computation on products)}\\
&= \langle \alpha, Fh \cdot \alpha\rangle && \text{(h homomorphism)}\\
&= \langle F\pi_1, F\pi_2\rangle \cdot F\langle \mathrm{id}, h\rangle \cdot \alpha && \text{(computation on products)}
\end{aligned}$$

$\Leftarrow$ Let us assume that we have a morphism $W : X \longrightarrow FX$ such that

$$\langle F\pi_1, F\pi_2\rangle \cdot F\langle \mathrm{id}, h\rangle \cdot W = \alpha \times \beta \cdot \langle \mathrm{id}, h\rangle.$$

Then:

$$\begin{aligned}
\alpha &= \pi_1 \cdot \alpha \times \beta \cdot \langle \mathrm{id}, h\rangle && \text{(computation on products)}\\
&= F\pi_1 \cdot F\langle \mathrm{id}, h\rangle \cdot W && \text{(definition of W)}\\
&= W && \text{(computation on products)}\\
\beta \cdot h &= \pi_2 \cdot \alpha \times \beta \cdot \langle \mathrm{id}, h\rangle && \text{(computation on products)}\\
&= F\pi_2 \cdot F\langle \mathrm{id}, h\rangle \cdot W && \text{(definition of W)}\\
&= Fh \cdot W && \text{(computation on products)}
\end{aligned}$$

Consequently,

$$Fh \cdot \alpha = \beta \cdot h$$

and h is a coalgebra homomorphism. □

Using this characterisation of maps for AM-bisimulations, and using the tabularity of the allegory of relations, we can prove that an AM-bisimulation can be described as a span of homomorphisms of coalgebras, under some form of the axiom of choice (see [Jac16]). We can formulate this in terms of tabulations:

Proposition 2.9. *If U is an AM-bisimulation from α to β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U, then there is a coalgebra structure γ on Z such that f is a coalgebra homomorphism from γ to α and g is a coalgebra homomorphism from γ to β.*

Proof. The fact that f, g is a tabulation of U means that $\langle f, g\rangle : Z \longrightarrow X \times Y$ is a mono and represents U. The fact that U is a AM-bisimulation gives a witness which is a F-coalgebra structure on Z. The commutativity of the diagram defining this witness implies that f and g are coalgebra homomorphisms. □

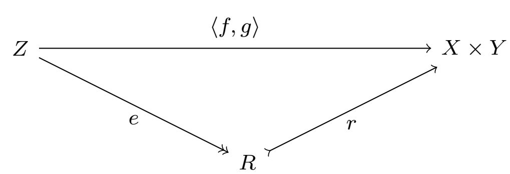
Corollary 2.10. *Assume $\mathcal{C}$ has the regular axiom of choice. Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. Then the following two statements are equivalent:*

(1) *There is an AM-bisimulation* $r : R \rightarrowtail X \times Y$ *from* α *to* β*, and a point* $c : * \longrightarrow R$ *such that* $r \cdot c = \langle p, q\rangle$*.*

(2) *There is a span* $X \xleftarrow{f} Z \xrightarrow{g} Y$ *, an* F*-coalgebra structure* γ *on* Z *such that* f *is a coalgebra homomorphism from* γ *to* α *and* g *from* γ *to* β*, and a point* $w : * \longrightarrow Z$ *such that* $f \cdot w = p$ *and* $g \cdot w = q$*.*

Proof. Let us prove both implications:

- $1 \Rightarrow 2$) By Proposition 2.9, we obtain a tabulation $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ together with $\gamma : Z \longrightarrow FZ$ that makes f and g coalgebra homomorphisms. In particular, U is represented by $\langle f, g\rangle$. Since r also represents U, there is an iso ϕ such that $\langle f, g\rangle \cdot \phi = r$. By taking $w = \phi \cdot c$, we have
$$f \cdot w = \pi_1 \cdot \langle f, g\rangle \cdot \phi \cdot c = \pi_1 \cdot r \cdot c = \pi_1 \cdot \langle p, q\rangle = p,$$
and similarly $g \cdot w = q$.
- $2 \Rightarrow 1$) Let us assume that we have a span of homomorphisms. Then, since f and g are coalgebra homomorphisms, $\langle \mathrm{id}, f\rangle$ and $\langle \mathrm{id}, g\rangle$ represent bisimulations by Proposition 2.8. Since bisimulations are closed under converse, $\langle \mathrm{id}, f\rangle^\dagger$ also represents a bisimulation. To conclude, we would like to prove that $\langle \mathrm{id}, f\rangle^\dagger; \langle \mathrm{id}, g\rangle$ is a bisimulation by using the closure under composition. However, the general closure under composition requires both the regular axiom of choice and that F preserves weak pullbacks. But since the relations we are composing are of special forms, namely that $\langle \mathrm{id}, f\rangle^\dagger = \langle f, \mathrm{id}\rangle$ is a right adjoint and $\langle \mathrm{id}, g\rangle$ is a map, the construction in the proof of Proposition 2.5 does not need the preservation of weak pullbacks, and we can conclude with just the regular axiom of choice that $\langle \mathrm{id}, f\rangle^\dagger; \langle \mathrm{id}, g\rangle$ is an AM-bisimulation. By definition of the composition, this bisimulation is represented by the mono part r of the (regular epi, mono)-factorisation:

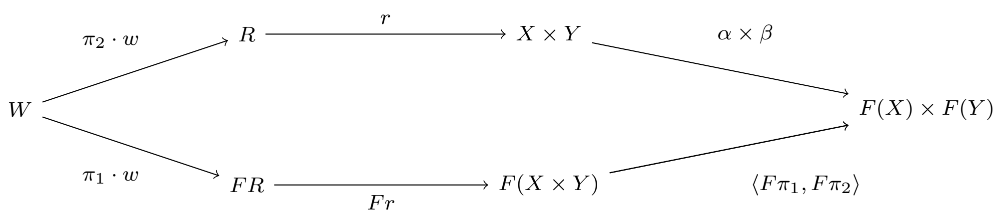


Now, if we have w as in 2), define $c = e \cdot w$. We have
$$r \cdot c = r \cdot e \cdot w = \langle f, g\rangle \cdot w = \langle p, q\rangle.$$ □

Remark 2.11. Here $*$ is usually the terminal object (since we are talking about points), but it can really be any object.

2.3. **Picking vs. Collecting: AM-Bisimulations for Regular Categories.** We have seen that several results about AM-bisimulations depend on the regular axiom of choice, preventing its usage in more exotic toposes and regular categories. Actually, the only occurrences are of similar flavour: one wants to prove some property of elements (x, z) in a composition of relations, and for that, one has to pick a witness y in between. The main idea of our proposal is that, instead of picking a witness (which would require the axiom of choice), it is enough to collect all the witnesses, prove properties about all of them, and make sure that there is enough of them. This can be done in any regular category as follows:

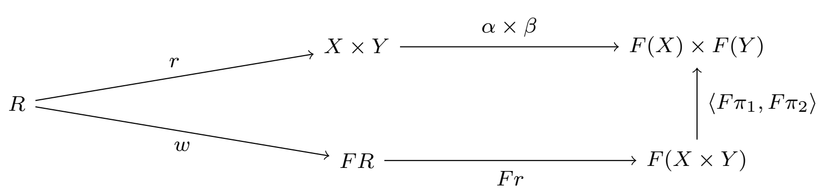
Definition 2.12. We say that a relation R is a *regular AM-bisimulation* from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is another relation represented by $w : W \rightarrowtail FR \times R$ such that $\pi_2 \cdot w$ is a regular epi and:



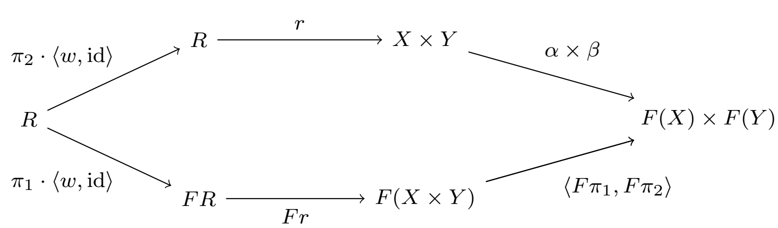
The intuition is as follows: W collects witnesses that R is a bisimulation. In particular, for a given pair (x, y) in R, there might be several witnesses. The fact $\pi_2 \cdot w$ is a regular epi guarantees that every pair in R has at least one witness. Of course, we have to prove that this extends plain AM-bisimulations:

Proposition 2.13. *If $\mathcal{C}$ is a regular category, then a AM-bisimulation is a regular AM-bisimulation. If additionally $\mathcal{C}$ satisfies the regular axiom of choice, then a regular AM-bisimulation is a AM-bisimulation.*

Proof. • Assume that we have a AM-bisimulation

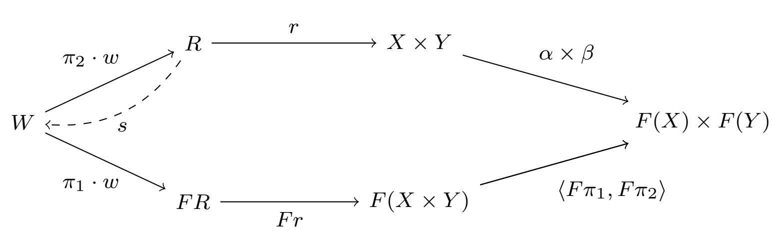


Then

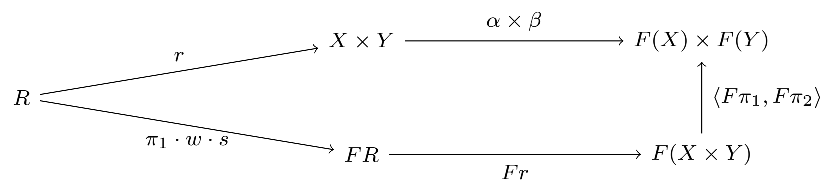


witnesses r as a regular AM-bisimulation.

• Assume that $\mathcal{C}$ has the regular axiom of choice and that we have a regular AM-bisimulation



Since $\pi_2 \cdot w$ is regular epi so is a split epi by the regular axiom of choice, there is $s : R \longrightarrow W$ such that $\pi_2 \cdot w \cdot s = \mathrm{id}$. Now let us prove that

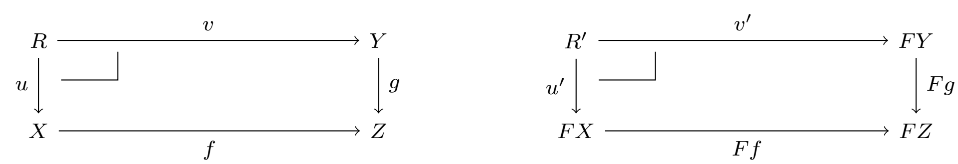


witnesses r as an AM-bisimulation.

$$\begin{array}{rcll} \langle F\pi_1, F\pi_2\rangle \cdot Fr \cdot \pi_1 \cdot w \cdot s & = & (\alpha \times \beta) \cdot r \cdot \pi_2 \cdot w \cdot s & (r \text{ regular AM-bisimulation}) \\ & = & (\alpha \times \beta) \cdot r & (\text{definition of } s) \end{array}$$ □

Also, regular bisimulations are closed under composition. This requires a mild condition on F as already observed in [Sta11].

Definition 2.14. We say that F *covers pullbacks* if for every pair of pullbacks:

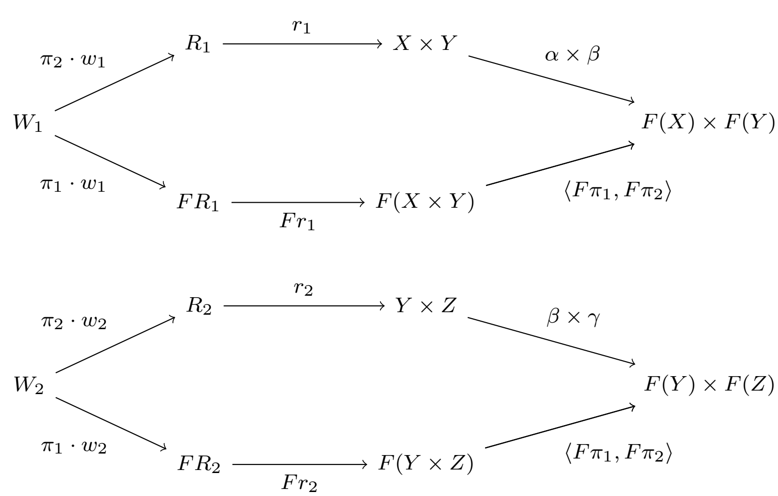


the unique morphism $\gamma : FR \longrightarrow R'$ such that $u' \cdot \gamma = Fu$ and $v' \cdot \gamma = Fv$ is a regular epi.

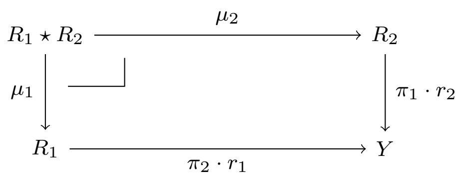
Remark 2.15. When F preserves weak pullbacks, then F covers pullbacks. When $\mathcal{C}$ has the regular axiom of choice, then both notions coincide.

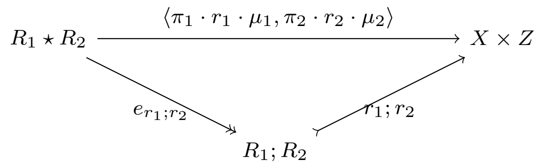
Proposition 2.16. *When F covers pullbacks, regular AM-bisimulations are closed under compositions.*

Proof. Assume that we have two regular AM-bisimulations

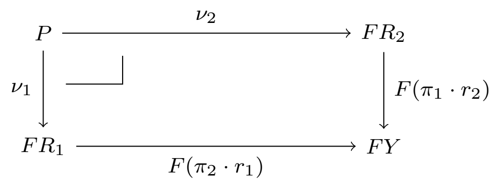


and we want to prove that the composition $r_1; r_2$ is also a regular AM-bisimulation. This composition is defined by the following pullback and (regular epi, mono)-factorisation:





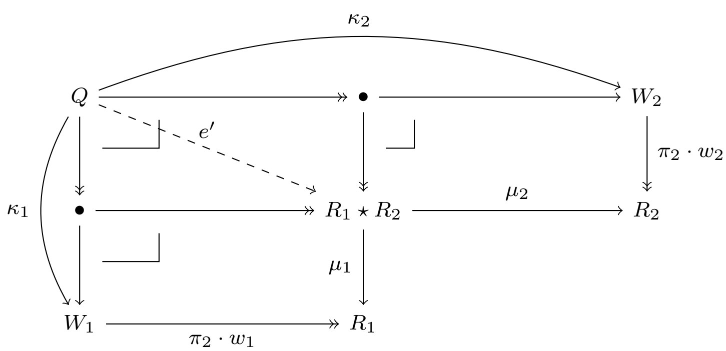
Let us form the following pullback



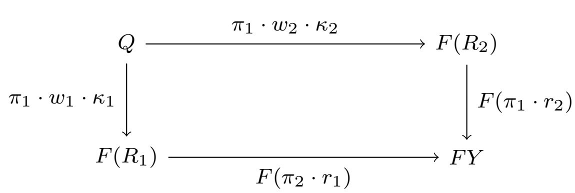
and since F covers pullbacks, there is a regular epi $e : F(R_1 \star R_2) \twoheadrightarrow P$ such that

$$\nu_1 \cdot e = F\mu_1 \quad \text{and} \quad \nu_2 \cdot e = F\mu_2.$$

Now, form the following three pullbacks:



Since $\pi_2 \cdot w_1$ and $\pi_2 \cdot w_2$ are regular epis, and regular epis are closed under pullbacks in a regular category, the four morphisms forming the top-left pullback are regular epis. Let us call $e' : Q \twoheadrightarrow R_1 \star R_2$ the composition of those regular epis, so that e' is also a regular epi. Now, the following square commutes:



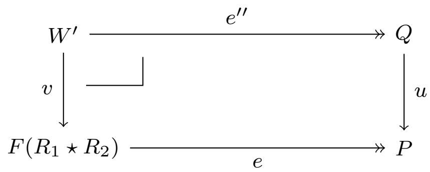
Indeed,

$$\begin{aligned}
F(\pi_2 \cdot r_1) \cdot \pi_1 \cdot w_1 \cdot \kappa_1 &= \beta \cdot \pi_2 \cdot r_1 \cdot \pi_2 \cdot w_1 \cdot \kappa_1 && (r_1 \text{ regular AM-bisimulation}) \\
&= \beta \cdot \pi_1 \cdot r_2 \cdot \pi_2 \cdot w_2 \cdot \kappa_2 && (\text{various pullbacks}) \\
&= F(\pi_1 \cdot r_2) \cdot \pi_1 \cdot w_2 \cdot \kappa_2 && (r_2 \text{ regular AM-bisimulation})
\end{aligned}$$

Then, by universality of P, there is a unique morphism $u : Q \longrightarrow P$ such that

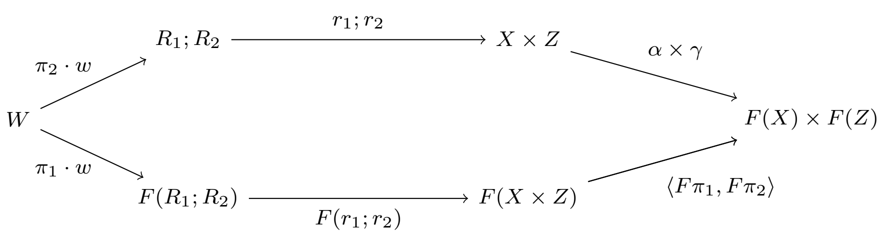
$$\pi_1 \cdot w_1 \cdot \kappa_1 = \nu_1 \cdot u \quad \text{and} \quad \pi_1 \cdot w_2 \cdot \kappa_2 = \nu_2 \cdot u.$$

Finally, form the following pullback



Since e is a regular epi and regular epi are closed under pullbacks in a regular category, then e'' is also a regular epi.

Now define $w' = \langle F(e_{r_1;r_2}) \cdot v, e_{r_1;r_2} \cdot e' \cdot e'' \rangle : W' \longrightarrow F(R_1; R_2) \times (R_1; R_2)$. Observe in particular that $\pi_2 \cdot w'$ is a regular epi as the composition of regular epis. Now, take the (regular epi, mono)-factorisation of w', that is, we have $\rho : W' \longrightarrow W$ regular epi and $w : W \longrightarrow F(R_1; R_2) \times (R_1; R_2)$ mono, such that, $w \cdot \rho = w'$. Observe that $\pi_2 \cdot w \cdot \rho = \pi_2 \cdot w'$, and since $\pi_2 \cdot w'$ is regular epi, then $\pi_2 \cdot w$ is regular epi (this is a usual property of regular epis). It remains to prove that the following diagram commutes:



Let us prove

$$\alpha \cdot \pi_1 \cdot r_1; r_2 \cdot \pi_2 \cdot w = F(\pi_1 \cdot r_1; r_2) \cdot \pi_1 \cdot w,$$

the other side is similar. Since ρ is epi, it is then enough to prove that

$$\alpha \cdot \pi_1 \cdot r_1; r_2 \cdot \pi_2 \cdot w' = \alpha \cdot \pi_1 \cdot r_1; r_2 \cdot \pi_2 \cdot w \cdot \rho = F(\pi_1 \cdot r_1; r_2) \cdot \pi_1 \cdot w \cdot \rho = F(\pi_1 \cdot r_1; r_2) \cdot \pi_1 \cdot w'.$$

Indeed,

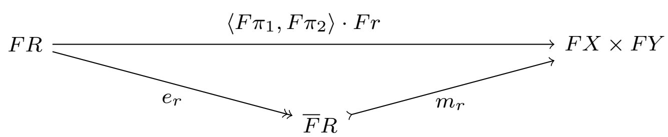
$$\begin{array}{rlr}
& \alpha \cdot \pi_1 \cdot r_1; r_2 \cdot \pi_2 \cdot w' & \\
= & \alpha \cdot \pi_1 \cdot r_1; r_2 \cdot e_{r_1;r_2} \cdot e' \cdot e'' & \text{(definition of } w'\text{)} \\
= & \alpha \cdot \pi_1 \cdot \langle \pi_1 \cdot r_1 \cdot \mu_1, \pi_2 \cdot r_2 \cdot \mu_2 \rangle \cdot e' \cdot e'' & \text{(definition of } r_1; r_2\text{)} \\
= & \alpha \cdot \pi_1 \cdot r_1 \cdot \mu_1 \cdot e' \cdot e'' & \text{(computation)} \\
= & \alpha \cdot \pi_1 \cdot r_1 \cdot \pi_2 \cdot w_1 \cdot \kappa_1 \cdot e'' & \text{(definition of } e' \text{ and } \kappa_1\text{)} \\
= & F(\pi_1 \cdot r_1) \cdot \pi_1 \cdot w_1 \cdot \kappa_1 \cdot e'' & (r_1 \text{ is regular AM-bisimulation)} \\
= & F(\pi_1 \cdot r_1) \cdot \nu_1 \cdot u \cdot e'' & \text{(definition of } u\text{)} \\
= & F(\pi_1 \cdot r_1) \cdot \nu_1 \cdot e \cdot v & \text{(definition of } e'' \text{ and } v\text{)} \\
= & F(\pi_1 \cdot r_1 \cdot \mu_1) \cdot v & \text{(definition of } e\text{)} \\
= & F(\pi_1 \cdot r_1; r_2 \cdot e_{r_1;r_2}) \cdot v & \text{(definition of } r_1; r_2\text{)} \\
= & F(\pi_1 \cdot r_1; r_2) \cdot \pi_1 \cdot w' & \text{(definition of } w'\text{)}
\end{array}$$

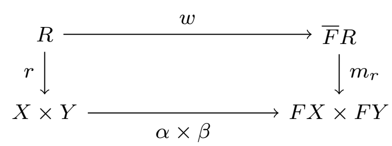
□

In [Sta11], Staton described conditions for several coalgebraic notions of bisimulations to coincide. In this picture, AM-bisimulations were quite weak, as they would coincide with other notions only under some form of the axiom of choice (again). Here, we will show that the picture is much nicer with regular AM-bisimulations.

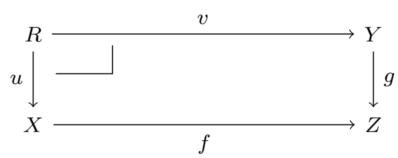
Let us recall two notions with which we will compare regular AM-bisimulations.

Definition 2.17. A relation from X to Y is a *Hermida-Jacobs bisimulation* (HJ-bisimulation for short) from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if if there is a mono $r : R \rightarrowtail X \times Y$ representing it and a morphism $w : R \longrightarrow \overline{F}R$ where $\overline{F}R$ is obtained by the (epi, mono)-factorisation on the left, and such that the square on the right commutes:



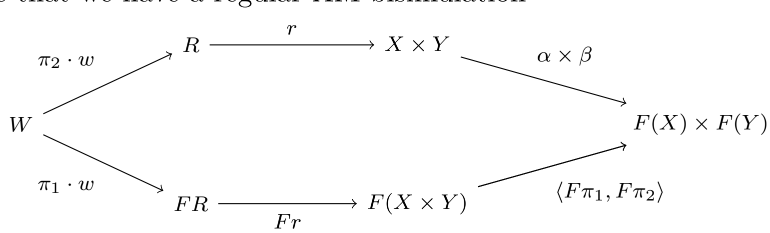


A relation is a *behavioural equivalence* from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ if it is represented by a pullback of coalgebra homomorphisms, that is, if there are a coalgebra $\gamma : Z \longrightarrow FZ$ and two coalgebra homomorphisms $f : \alpha \longrightarrow \gamma$ and $g : \beta \longrightarrow \gamma$ such that the mono $\langle u, v\rangle : R \rightarrowtail X \times Y$ obtained from their pullback in $\mathcal{C}$ represents it.

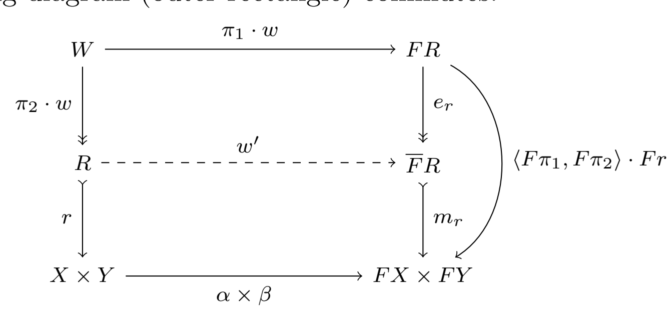


Theorem 2.18. *Assume that $\mathcal{C}$ is a regular category. Then:*
- *a relation is a regular AM-bisimulation if and only if it is a HJ-bisimulation,*
- *if $\mathcal{C}$ has pushouts, then a regular AM-bisimulation is included in a behavioural equivalence,*
- *if F covers pullbacks, then a behavioural equivalence is a regular AM-bisimulation.*

Proof. Let us prove that regular AM-bisimulations coincide with HJ-bisimulations.
- Let us assume that we have a regular AM-bisimulation

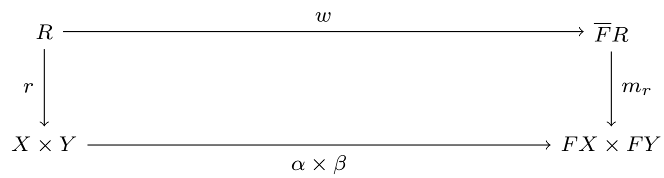


Then the following diagram (outer rectangle) commutes:

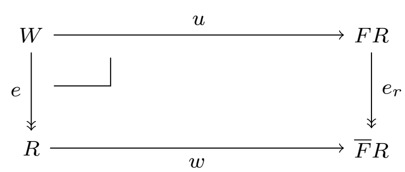


since r is a regular AM-bisimulation. Furthermore, by definition $\pi_2 \cdot w$ and e_r are regular epis, and r and m_r are monos. So by functoriality of the (regular epi, mono)-factorisation, there is $w' : R \longrightarrow \overline{F}R$ as above (dashed). The lower square witnesses that r is an HJ-bisimulation.

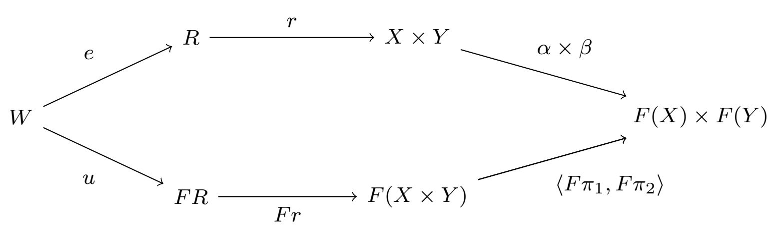
- Assume that r is an HJ-bisimulation



Form the following pullback:



Since e_r is a regular epi and regular epis are closed under pullbacks in a regular category then e is a regular epi. If we define $w' = \langle u, e\rangle : W \longrightarrow FR \times R$, then w' is a mono. Indeed, if we fix $\phi, \psi : Z \longrightarrow W$, such that $w' \cdot \phi = w' \cdot \psi$, then ϕ and ψ are morphisms of cones from $(Z, u \cdot \phi = u \cdot \psi, e \cdot \phi = e \cdot \psi)$ to (W, u, e). By universality of the pullback, such a morphism of cones is unique, so $\phi = \psi$. It remains to prove that the following diagram commutes

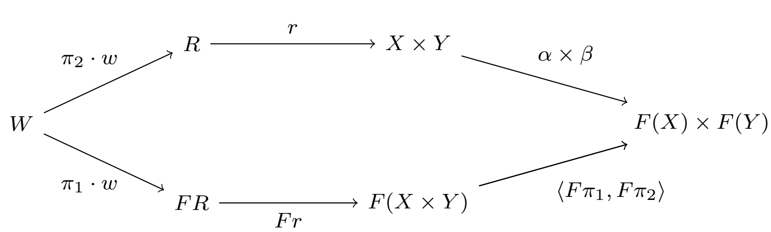


Let us do it for α,

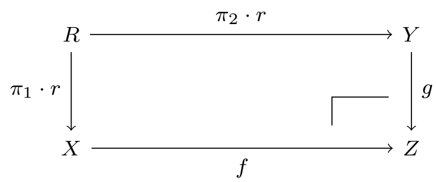
$$\begin{aligned}
\alpha \cdot \pi_1 \cdot r \cdot e &= \pi_1 \cdot m_r \cdot w \cdot e && (r \text{ is HJ bisimulation})\\
&= \pi_1 \cdot m_r \cdot e_r \cdot u && (\text{definition of } e \text{ and } u)\\
&= \pi_1 \cdot \langle F\pi_1, F\pi_2\rangle \cdot Fr \cdot u && (\text{definition of } e_r \text{ and } m_r)\\
&= F(\pi_1 \cdot r) \cdot Fr \cdot u && (\text{computation})
\end{aligned}$$

At this point we could just invoke [Sta11] to conclude, but we provide dedicated proofs here.

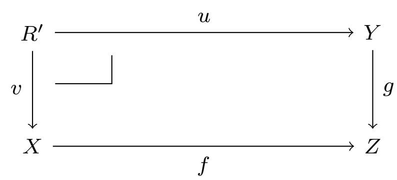
Let us assume that $\mathcal{C}$ has pushouts and assume that we have a regular AM-bisimulation



Form the following pushout:



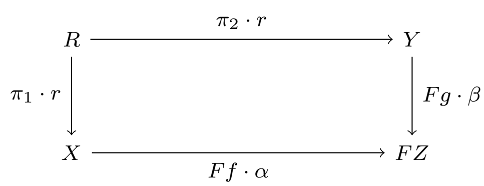
Now, forming the pullback



by universality of this pullback, there is a unique morphism $\kappa : R \longrightarrow R'$ such that

$$r = \langle u, v \rangle \cdot \kappa,$$

witnessing that $r \leq \langle u, v \rangle$ as monos, that is, the relation represented by r is included in the relation represented by $\langle u, v \rangle$. To conclude, it remains to prove that $\langle u, v \rangle$ represents a behavioural equivalence, that is, there exists a coalgebra structure $\gamma : Z \longrightarrow FZ$ making f and g coalgebra homomorphisms. Let us prove that the following square commutes



Since $\pi_2 \cdot w$ is epi it is enough to prove that

$$Ff \cdot \alpha \cdot \pi_1 \cdot r \cdot \pi_2 \cdot w = Fg \cdot \beta \cdot \pi_2 \cdot r \cdot \pi_2 \cdot w.$$

Indeed,

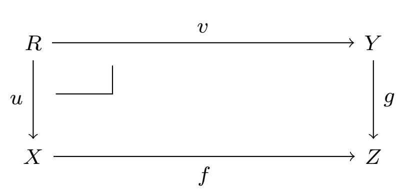
$$\begin{aligned} Ff \cdot \alpha \cdot \pi_1 \cdot r \cdot \pi_2 \cdot w &= F(f \cdot \pi_1 \cdot r) \cdot \pi_1 \cdot w && (r \text{ is AM-bisimulation}) \\ &= F(g \cdot \pi_2 \cdot r) \cdot \pi_1 \cdot w && (\text{definition of } f \text{ and } g) \\ &= Fg \cdot \beta \cdot \pi_2 \cdot r \cdot \pi_2 \cdot w && (r \text{ is AM-bisimulation}) \end{aligned}$$

By universality of Z as a pushout, there is a unique $\gamma : Z \longrightarrow FZ$ such that

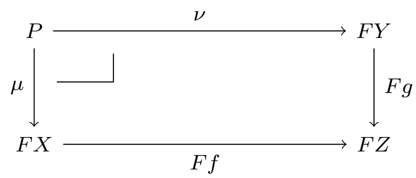
$$\gamma \cdot f = Ff \cdot \alpha \quad \text{and} \quad \gamma \cdot g = Fg \cdot \beta,$$

that is f and g are coalgebra homomorphisms.

Finally, let us assume that F covers pullbacks and that we have a behavioural equivalence



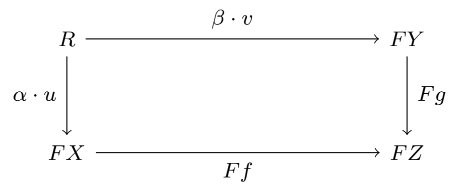
with $f : \alpha \longrightarrow \gamma$ and $g : \beta \longrightarrow \gamma$ coalgebra homomorphisms. Form the following pullback



Since F covers pullbacks, there is a regular epi $e : FR \twoheadrightarrow P$ such that

$$Fu = \mu \cdot e \quad \text{and} \quad Fv = \nu \cdot e.$$

Now the following square commutes



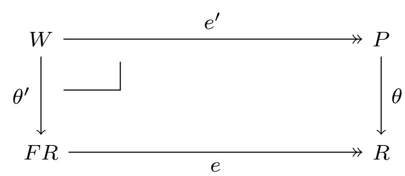
Indeed,

$$\begin{array}{rcll} Ff \cdot \alpha \cdot u & = & \gamma \cdot f \cdot u & (f \text{ is coalgebra homomorphism}) \\ & = & \gamma \cdot g \cdot v & (\text{definition of } u \text{ and } v) \\ & = & Fg \cdot \beta \cdot v & (g \text{ is coalgebra homomorphism}) \end{array}$$

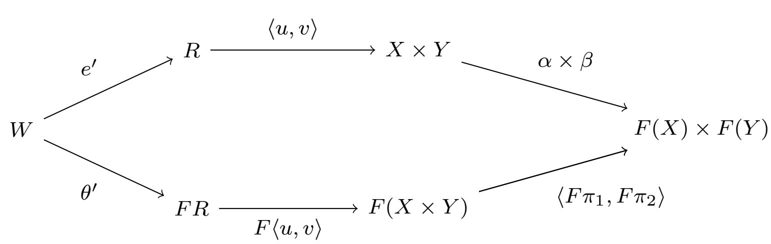
By universality of P as a pullback, there is a unique $\theta : R \longrightarrow P$ such that

$$\alpha \cdot u = \mu \cdot \theta \quad \text{and} \quad \beta \cdot v = \nu \cdot \theta.$$

Then form the following pullback:



Since e is a regular epi, and regular epis are closed under pullbacks in a regular category, e' is also a regular epi. So it remains to prove that the following diagram commutes



Let us prove it for α (the other side is similar):

$$
\begin{aligned}
\alpha \cdot u \cdot e' &= \mu \cdot \theta \cdot e' && \text{(definition of } \theta) \\
&= \mu \cdot e \cdot \theta' && \text{(definition of } \theta' \text{ and } e') \\
&= Fu \cdot \theta' && \text{(definition of } e)
\end{aligned}
$$
□

In Section 2.2, we described that AM-bisimilarity coincides with the existence of a span of coalgebra homomorphisms. This can also be formulated in the context of regular AM-bisimulations. The witness $w : W \rightarrowtail FR \times R$ can be seen as a coalgebra in $\mathbf{Rel}(\mathcal{C})$ (although F is technically not a functor on it). The coalgebra $\alpha : X \longrightarrow FX$ can also be seen as a coalgebra in $\mathbf{Rel}(\mathcal{C})$ as $\langle \alpha, \mathrm{id} \rangle : X \rightarrowtail FX \times X$. Then $\pi_1 \cdot r$ can be seen as a coalgebra homomorphism from w to α, since the following diagram commutes

$$
\begin{array}{ccc}
W & \overset{w}{\rightarrowtail} & FR \times R \\
{\scriptstyle \pi_1 \cdot r \cdot \pi_2 \cdot w}\downarrow & & \downarrow{\scriptstyle F(\pi_1 \cdot r) \times \pi_1 \cdot r} \\
X & \underset{\langle \alpha, \mathrm{id} \rangle}{\rightarrowtail} & FX \times X
\end{array}
$$

Regular AM-bisimulations can be interpreted as spans of coalgebra homomorphisms in $\mathbf{Rel}(\mathcal{C})$.

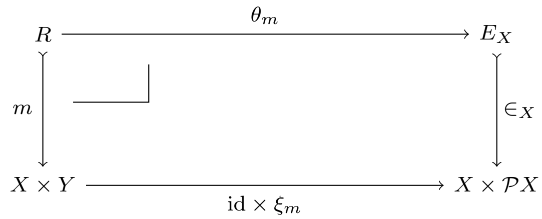
3. The Relational Essence of Power-Objects in a Topos

In this section, we investigate toposes and their power-objects in a purely relational way. The gain is that some ingredients of the proof, particularly the precise correspondence between composition of relations and Kleisli composition, will be used later on. From this observation, we (re)prove that 1) power-objects form a commutative monad whose Kleisli category is isomorphic to the category of relations, 2) power-objects behave well with epis, 3) under some mild conditions on a monad in terms of weak pullbacks and epis, there is a (weak) distributive law with respect to the power-object monad. During the proofs, we will denote by mono(f) the mono part of the (epi, mono)-factorisation of f.

This section is mostly directed at coalgebraists who are not very familiar with toposes. The results here are known (sometimes folklore) but scattered in the rich literature. However, the proofs of the statements as presented in this section, which we call "relational" as they only rely on properties of relations, could not be found anywhere. In total, this section should be seen an an advertisement that 1) many things that are done in coalgebra in **Set** with the powerset functor can be done automatically in any topos with the power-object functor, and 2) anyone intersted in toposes should invest in learning about the internal logic of a topos, as this makes the rather technical relational proofs much more concise.

3.1. **Toposes, as Relation Classifiers.**

Definition 3.1. A topos is a finitely complete category with power-objects. The latter condition means that for every object X, there is a mono $\in_X : E_X \rightarrowtail X \times \mathcal{P}X$ such that for every mono of the form $m : R \rightarrowtail X \times Y$ there is a unique morphism $\xi_m : Y \longrightarrow \mathcal{P}X$ such that there is a pullback diagram of the form:



Here θ_m is not required to be unique, only ξ_m is. This formulation passes to relations since $\xi_m = \xi_{m'}$ if and only if m and m' represent the same relation r. In that case, we will write ξ_r for $\xi_m = \xi_{m'}$. Another formulation of toposes uses sub-object classifiers which can be recovered as $\mathbb{T} = \in_{\mathbf{1}}\colon \mathbf{1} \simeq E_{\mathbf{1}} \to \mathbf{1} \times \mathcal{P}\mathbf{1} \simeq \mathcal{P}\mathbf{1}$. The formulation by power-objects implies that a topos is closed, which is not the case for the one by sub-object classifiers. Conversely, $\mathcal{P}X$ is equal to Ω^X and $\in_X$ is any mono corresponding to the evaluation morphism $X \times \Omega^X \to \Omega$ of the cartesian-closed structure.

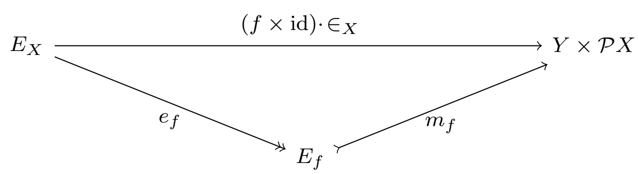
Example 3.2. In **Set**, $\mathcal{P}X$ is given by the usual power-set and E_X is the subset of $X \times \mathcal{P}X$ consisting of pairs (x, U) such that $x \in U$. In **Scha**-the Schanuel topos **Scha** [Law89], equivalent to the category of nominal sets and equivariant functions-$\mathcal{P}X$ is the nominal set of finitely supported subsets of X. In **Eff**-the effective topos [Hyl82], intuitively, the category of effective sets and computable functions-$\mathcal{P}X$ is intuitively given by the set of decidable subsets of X (although the formal description is much more abstract).

3.2. **The Power-Object Monad.** The following is a folklore result about power-objects that can be proved, for example, by noticing that the proof in **Set** does not use either the law of excluded-middle nor the axiom of choice, and the fact that any such statement is true in any topos:

Theorem 3.3. *In a topos $\mathcal{C}$, $\mathcal{P}$ extends to a commutative monad whose Kleisli category is isomorphic to the category of relations* **Rel**$(\mathcal{C})$.

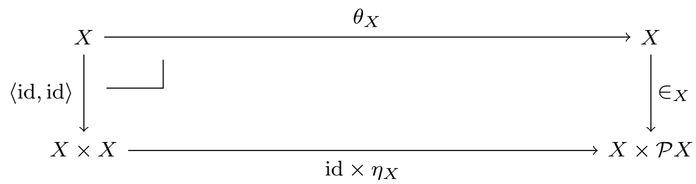
During the course of this section, we will give an elementary and relational proof of this statement.

Let us describe some parts of this statement that will be useful in the following discussion. First, the structure of a *covariant* functor (not to be confused with the contravariant structure that is also sometimes used) is given as follows. Given a morphism $f : X \longrightarrow Y$, $\mathcal{P}f : \mathcal{P}X \longrightarrow \mathcal{P}Y$ is defined as follows. Consider first the following (epi, mono)-factorisation:

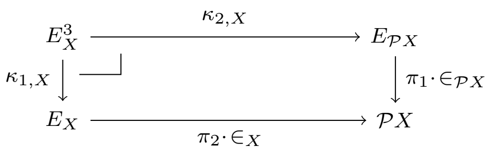


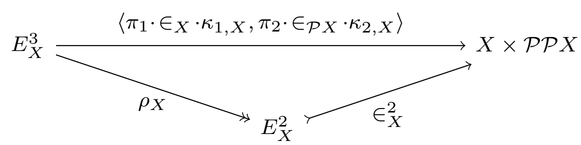
Then $\mathcal{P}f : \mathcal{P}X \longrightarrow \mathcal{P}Y$ is the unique morphism corresponding to m_f.

The unit $\eta_X : X \longrightarrow \mathcal{P}X$ is defined as ξ_{Δ_X}, that is, the unique morphism such that there is a pullback of the form:



for some θ_X. The multiplication $\mu_X : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X$ is defined as the unique morphism associated with the composition of relations $\in_X;\in_{\mathcal{P}X}$. In diagrams, this means that we form a similar pattern of pullback followed by (epi, mono)-factorisation:





and define μ_X as the unique morphism $\xi_{\in^2_X}$.

3.3. The Kleisli Category is the Allegory of Relations.

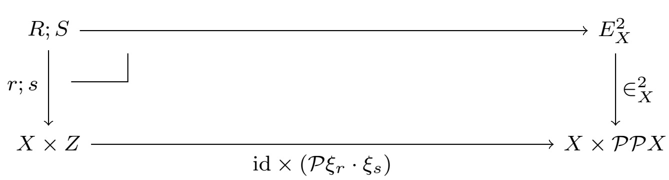
The operator ξ obtained from the definition connects a topos with the opposite of its category of relations. It maps a relation from X to Y to a morphism of the form $Y \to \mathcal{P}X$, that is, a Kleisli morphism for $\mathcal{P}$. The definition of a topos means that this is a one-to-one correspondence. To show that the Kleisli category and the opposite of the category of relations coincide, it is then enough that the composition and the identities are preserved by the operator ξ. For the identities, it is by design: the identities of the Kleisli category are given by the units, which are *defined* as ξ_{Δ_X}, and the diagonals are the identity relations.

The only remaining part is then about compositions. This is the main technical result of this section. In plain words, the following proposition means that ξ maps the opposite of the composition of relations to the Kleisli composition:

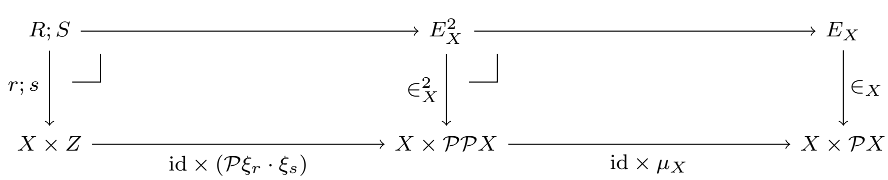
Proposition 3.4. *Given two relations, r from X to Y and s from Y to Z, $\xi_{r;s} = \mu_X \cdot \mathcal{P}\xi_r \cdot \xi_s$.*

The proof is quite technical and relies on a lot of diagram chasing.

Proof. The main trick is to prove that we have a pullback of the form

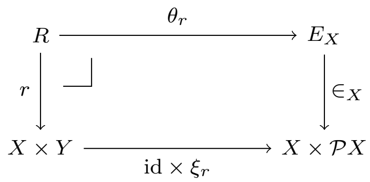


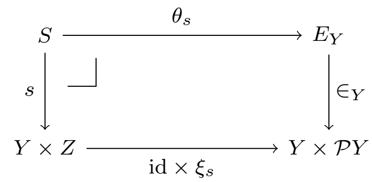
by using the preservation of the image by pullback on a suitable pullback. Then considering the following composition of pullbacks



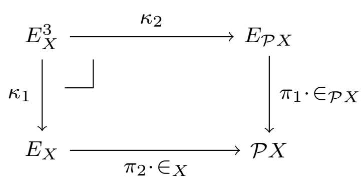
does the job.

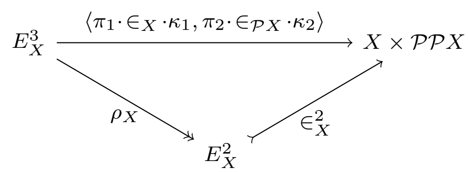
First, let us describe the pullbacks and the factorisations we have by assumption, to introduce notations. By definition of ξ_r and ξ_s, we have the following two pullbacks:



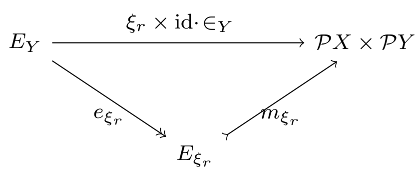


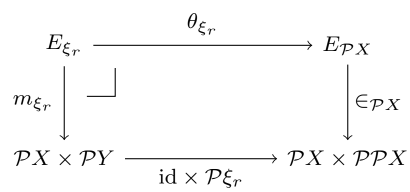
By definition of $\in^2_X$ we have the following pullback and factorisation:



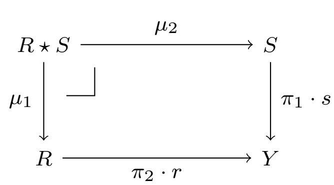


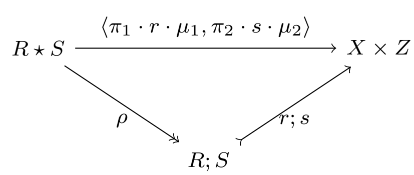
By definition of $\mathcal{P}\xi_r$, we have the following factorisation and the pullback:



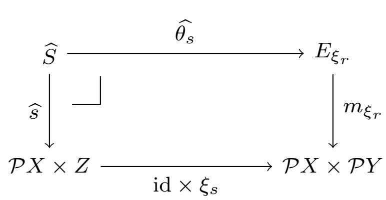


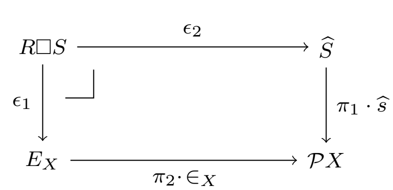
Finally, by definition of $r; s$ we have the following pullback and factorisation:





Now, let us describe the suitable pullback we want to look at. It is defined in several steps. First, form the following two pullbacks:





Our suitable pullback will have the following form:

$$\begin{array}{ccc} R\Box S & \xrightarrow{\quad w \quad} & E_3^X \\ {\scriptstyle \langle \pi_1 \cdot \in_X \cdot \epsilon_1, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \rangle} \downarrow \quad \lrcorner & & \downarrow {\scriptstyle \langle \pi_1 \cdot \in_X \cdot \kappa_1, \pi_2 \cdot \in_{\mathcal{P}X} \cdot \kappa_2 \rangle} \\ X \times Z & \xrightarrow[\mathrm{id} \times (\mathcal{P}\xi_r \cdot \xi_s)]{} & X \times \mathcal{P}\mathcal{P}X \end{array}$$

for some w we describe now. We have the following commutative diagram:

$$\begin{array}{ccc} R\Box S & \xrightarrow{\theta_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2} & E_{\mathcal{P}X} \\ {\scriptstyle \epsilon_1} \downarrow & & \downarrow {\scriptstyle \pi_1 \cdot \in_{\mathcal{P}X}} \\ E_X & \xrightarrow[\pi_2 \cdot \in_X]{} & \mathcal{P}X \end{array}$$

Indeed,

$$\begin{array}{rcll} \pi_1 \cdot \in_{\mathcal{P}X} \cdot \theta_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 & = & \pi_1 \cdot m_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 & \text{(definition of } \mathcal{P}\xi_r) \\ & = & \pi_1 \cdot \widehat{s} \cdot \epsilon_2 & \text{(definition of } \widehat{S}) \\ & = & \pi_2 \cdot \in_X \cdot \epsilon_1 & \text{(definition of } R\Box S) \end{array}$$

So by the universal property of E_X^3, there is a unique morphism $w : R\Box S \longrightarrow E_X^3$ such that

$$\kappa_1 \cdot w = \epsilon_1 \quad \text{and} \quad \kappa_2 \cdot w = \theta_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2.$$

Let us prove that the suitable pullback is indeed a pullback. First it is a commutative diagram:

$$\begin{array}{rcll} \pi_1 \cdot \in_X \cdot \kappa_1 \cdot w & = & \pi_1 \cdot \in_X \cdot \epsilon_1 & \text{(definition of } w) \\ \pi_2 \cdot \in_{\mathcal{P}X} \cdot \kappa_2 \cdot w & = & \pi_2 \cdot \in_{\mathcal{P}X} \cdot \theta_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 & \text{(definition of } w) \\ & = & \mathcal{P}\xi_r \cdot \pi_2 \cdot m_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 & \text{(definition of } \mathcal{P}\xi_r) \\ & = & \mathcal{P}\xi_r \cdot \xi_s \cdot \pi_2 \cdot \widehat{s} \cdot \epsilon_2 & \text{(definition of } \widehat{S}) \end{array}$$

Now, assume given another commutative diagram of the form:

$$\begin{array}{ccc} W & \xrightarrow{\quad \phi \quad} & E_3^X \\ {\scriptstyle \psi} \downarrow & & \downarrow {\scriptstyle \langle \pi_1 \cdot \in_X \cdot \kappa_1, \pi_2 \cdot \in_{\mathcal{P}X} \cdot \kappa_2 \rangle} \\ X \times Z & \xrightarrow[\mathrm{id} \times (\mathcal{P}\xi_r \cdot \xi_s)]{} & X \times \mathcal{P}\mathcal{P}X \end{array}$$

We construct a morphism $\gamma : W \longrightarrow R\Box S$ using three universal properties of pullbacks as follows. First we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\quad \kappa_2 \cdot \phi \quad} & E_{\mathcal{P}X} \\ {\scriptstyle \langle \pi_2 \cdot \in_X \cdot \kappa_1 \cdot \phi, \xi_s \cdot \pi_2 \cdot \psi \rangle} \downarrow & & \downarrow {\scriptstyle \in_{\mathcal{P}X}} \\ \mathcal{P}X \times \mathcal{P}Y & \xrightarrow[\mathrm{id} \times \mathcal{P}\xi_r]{} & \mathcal{P}X \times \mathcal{P}\mathcal{P}X \end{array}$$

Indeed,

$$\begin{array}{rcll}\pi_1\cdot\in_{\mathcal{P}X}\cdot\kappa_2\cdot\phi & = & \pi_2\cdot\in_X\cdot\kappa_1\cdot\phi & \text{(definition of } E^3_X\text{)}\\ \pi_2\cdot\in_{\mathcal{P}X}\cdot\kappa_2\cdot\phi & = & \mathcal{P}\xi_r\cdot\xi_s\cdot\pi_2\cdot\psi & \text{(assumption on } W\text{)}\end{array}$$

So by the universal property of E_{ξ_r}, there is a unique morphism $\alpha : W \longrightarrow E_{\xi_r}$ such that

$$m_{\xi_r}\cdot\alpha = \langle\pi_2\cdot\in_X\cdot\kappa_1\cdot\phi, \xi_s\cdot\pi_2\cdot\psi\rangle$$

and

$$\theta_{\xi_r}\cdot\alpha = \kappa_2\cdot\phi.$$

Secondly, we have the following commutative diagram, by definition of α:

$$\begin{array}{ccc} W & \xrightarrow{\quad\alpha\quad} & E_{\xi_r}\\ {\scriptstyle\langle\pi_2\cdot\in_X\cdot\kappa_1\cdot\phi,\pi_2\cdot\psi\rangle}\downarrow & & \downarrow{\scriptstyle m_{\xi_r}}\\ \mathcal{P}X\times Z & \xrightarrow[\mathrm{id}\times\xi_s]{} & \mathcal{P}X\times\mathcal{P}Y\end{array}$$

So by the universal property of $\widehat{S}$, there is a unique morphism $\beta : W \longrightarrow \widehat{S}$ such that

$$\widehat{s}\cdot\beta = \langle\pi_2\cdot\in_X\cdot\kappa_1\cdot\phi, \pi_2\cdot\psi\rangle$$

and

$$\widehat{\theta}_s\cdot\beta = \alpha.$$

Finally, we have the following commutative diagram, by definition of β:

$$\begin{array}{ccc} W & \xrightarrow{\quad\beta\quad} & \widehat{S}\\ {\scriptstyle\kappa_1\cdot\phi}\downarrow & & \downarrow{\scriptstyle\pi_1\cdot\widehat{s}}\\ E_X & \xrightarrow[\pi_2\cdot\in_X]{} & \mathcal{P}X\end{array}$$

So by the universal property of $R\Box S$, there is a unique morphism $\gamma : W \longrightarrow R\Box S$ such that

$$\epsilon_1\cdot\gamma = \kappa_1\cdot\phi \quad\text{and}\quad \epsilon_2\cdot\gamma = \beta.$$

Let us prove that γ is the unique morphism from W to $R\Box S$ such that

$$w\cdot\gamma = \phi \quad\text{and}\quad \langle\pi_1\cdot\in_X\cdot\epsilon_1, \pi_2\cdot\widehat{s}\cdot\epsilon_2\rangle\cdot\gamma = \psi.$$

First, it satisfies those conditions. For the first one, by the unicity of the pullback property of E^3_X, it is enough to prove the following

$$\begin{array}{rcll}\kappa_1\cdot w\cdot\gamma & = & \epsilon_1\cdot\gamma & \text{(definition of } w\text{)}\\ & = & \kappa_1\cdot\phi & \text{(definition of } \gamma\text{)}\\ \kappa_2\cdot w\cdot\gamma & = & \theta_{\xi_r}\cdot\widehat{\theta}_s\cdot\epsilon_2\cdot\gamma & \text{(definition of } w\text{)}\\ & = & \theta_{\xi_r}\cdot\widehat{\theta}_s\cdot\beta & \text{(definition of } \gamma\text{)}\\ & = & \theta_{\xi_r}\cdot\alpha & \text{(definition of } \beta\text{)}\\ & = & \kappa_2\cdot\phi & \text{(definition of } \alpha\text{)}\end{array}$$

For the second one:

$$\begin{array}{rcll}\pi_1\cdot\in_X\cdot\epsilon_1\cdot\gamma & = & \pi_1\cdot\in_X\cdot\kappa_1\cdot\phi & \text{(definition of } \gamma\text{)}\\ & = & \pi_1\cdot\psi & \text{(assumption on } W\text{)}\\ \pi_2\cdot\widehat{s}\cdot\epsilon_2\cdot\gamma & = & \pi_2\cdot\widehat{s}\cdot\beta & \text{(definition of } \gamma\text{)}\\ & = & \pi_2\cdot\psi & \text{(definition of } \beta\text{)}\end{array}$$

Now assume that there is another γ' from W to $R\Box S$ such that

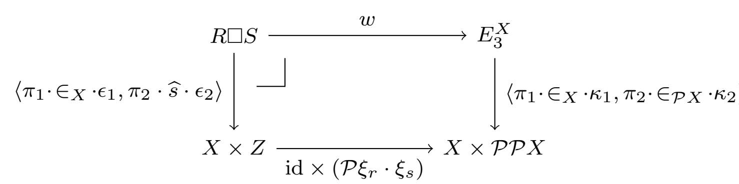
$$w \cdot \gamma' = \phi \quad \text{and} \quad \langle \pi_1 \cdot \in_X \cdot \epsilon_1, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \rangle \cdot \gamma' = \psi.$$

By the unicity properties of α, β and γ, it is enough to prove the following five equations:

$$\begin{array}{rcll}
\kappa_1 \cdot \phi & = & \kappa_1 \cdot w \cdot \gamma' & \text{(assumption on } \gamma') \\
 & = & \epsilon_1 \cdot \gamma' & \text{(definition of } w) \\
\pi_1 \cdot \widehat{s} \cdot \epsilon_2 \cdot \gamma' & = & \pi_2 \cdot \in_X \cdot \epsilon_1 \cdot \gamma' & \text{(definition of } R\Box S) \\
 & = & \pi_2 \cdot \in_X \cdot \kappa_1 \cdot \phi & \text{(assumption on } \gamma') \\
\pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \gamma' & = & \pi_2 \cdot \psi & \text{(assumption on } \gamma') \\
\kappa_2 \cdot \phi & = & \kappa_2 \cdot w \cdot \gamma' & \text{(assumption on } \gamma') \\
 & = & \theta_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 \cdot \gamma' & \text{(definition of } w) \\
m_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 \cdot \gamma' & = & \mathrm{id} \times \xi_s \cdot \widehat{s} \cdot \epsilon_2 \cdot \gamma' & \text{(definition of } \widehat{S}) \\
 & = & \langle pi_1 \cdot \widehat{s} \cdot \epsilon_2 \cdot \gamma', \xi_s \cdot \pi_2 \cdot \psi \rangle & \text{(assumption on } \gamma') \\
 & = & \langle \pi_2 \cdot \in_X \cdot \epsilon_1 \cdot \gamma', \xi_s \cdot \pi_2 \cdot \psi \rangle & \text{(definition of } R\Box S) \\
 & = & \langle \pi_2 \cdot \in_X \cdot \kappa_1 \cdot w \cdot \gamma', \xi_s \cdot \pi_2 \cdot \psi \rangle & \text{(definition of } w) \\
 & = & \langle \pi_2 \cdot \in_X \cdot \kappa_1 \cdot \phi, \xi_s \cdot \pi_2 \cdot \psi \rangle & \text{(assumption on } \gamma')
\end{array}$$

from which we deduce that $\alpha = \widehat{\theta_s} \cdot \epsilon_2 \cdot \gamma'$, then $\beta = \epsilon_2 \cdot \gamma'$, and finally $\gamma = \gamma'$.

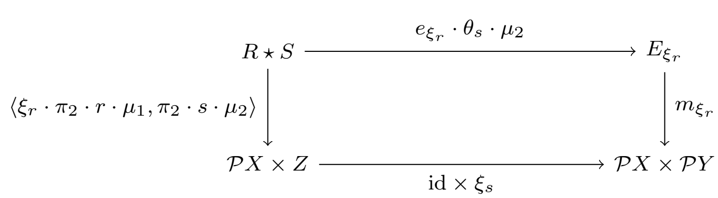
So we have our suitable pullback:



To conclude with the preservation of the image by pullback, we have to prove that we have the correct (epi, mono)-factorisations, that is:

- $\mathrm{mono}(\langle \pi_1 \cdot \in_X \cdot \kappa_1, \pi_2 \cdot \in_{\mathcal{P}X} \cdot \kappa_2 \rangle) \equiv \in_X^2$: this is the case by definition of $\in_X^2$.
- $\mathrm{mono}(\langle \pi_1 \cdot \in_X \cdot \epsilon_1, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \rangle) \equiv r; s$: this part is much more complicated. We know, by construction, that $r; s \equiv \mathrm{mono}(\langle \pi_1 \cdot r \cdot \mu_1, \pi_2 \cdot s \cdot \mu_2 \rangle)$, so we need to compare those two morphisms. We start by constructing a morphism $v : R \star S \longrightarrow R\Box S$, by using two pullbacks properties as follows.

  First we have the following commutative diagram:



  Indeed,

$$\begin{array}{rcll}
m_{\xi_r} \cdot e_{\xi_r} \cdot \theta_s \cdot \mu_2 & = & \xi_r \times \mathrm{id} \cdot \in_Y \cdot \theta_s \cdot \mu_2 & \text{(definition of } E_{\xi_r}) \\
 & = & \xi_r \times \xi_s \cdot s \cdot \mu_2 & \text{(definition of } \xi_s) \\
 & = & \mathrm{id} \times \xi_s \cdot \langle \xi_r \cdot \pi_1 \cdot s \cdot \mu_2, \pi_2 \cdot s \cdot \mu_2 \rangle & \text{(computation on products)} \\
 & = & \mathrm{id} \times \xi_s \cdot \langle \xi_r \cdot \pi_2 \cdot r \cdot \mu_1, \pi_2 \cdot s \cdot \mu_2 \rangle & \text{(definition of } R \star S)
\end{array}$$

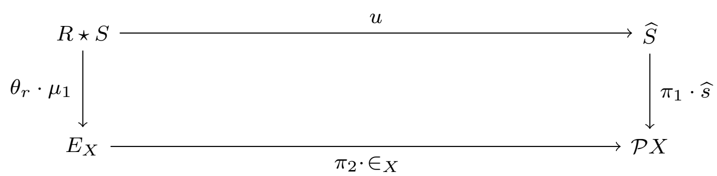
So by the universal property of $\widehat{S}$, there is a unique morphism $u : R \star S \longrightarrow \widehat{S}$ such that

$$\widehat{\theta}_s \cdot u = e_{\xi_r} \cdot \theta_s \cdot \mu_2$$

and

$$\widehat{s} \cdot u = \langle \xi_r \cdot \pi_2 \cdot r \cdot \mu_1, \pi_2 \cdot s \cdot \mu_2 \rangle.$$

Next we have the following commutative diagram:



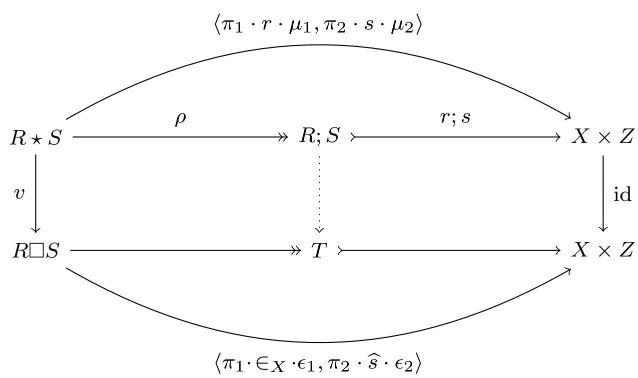
Indeed,

$$\begin{array}{rcll}
\pi_2 \cdot \in_X \cdot \theta_r \cdot \mu_1 & = & \xi_r \cdot \pi_2 \cdot r \cdot \mu_1 & \text{(definition of } \xi_r) \\
& = & \xi_r \cdot \pi_1 \cdot s \cdot \mu_2 & \text{(definition of } R \star S) \\
& = & \xi_r \cdot \pi_1 \cdot \in_Y \cdot \theta_s \cdot \mu_2 & \text{(definition of } \xi_s) \\
& = & \pi_1 \cdot m_{\xi_r} \cdot e_{\xi_r} \cdot \theta_s \cdot \mu_2 & \text{(definition of } E_{\xi_r}) \\
& = & \pi_1 \cdot m_{\xi_r} \cdot \widehat{\theta}_s \cdot u & \text{(definition of } u) \\
& = & \pi_1 \cdot \widehat{s} \cdot u & \text{(definition of } \widehat{S})
\end{array}$$

So by the universal property of $R\Box S$, there is a unique morphism $v : R \star S \longrightarrow R\Box S$ such that

$$\epsilon_1 \cdot v = \theta_r \cdot \mu_1 \quad \text{and} \quad \epsilon_2 \cdot v = u.$$

Now, we can compare the two morphisms and their (epi, mono)-factorisations, since we have the following commutative diagram:



Indeed,

$$\begin{array}{rcll}
\pi_1 \cdot \in_X \cdot \epsilon_1 \cdot v & = & \pi_1 \cdot \in_X \cdot \theta_r \cdot \mu_1 & \text{(definition of } v) \\
& = & \pi_1 \cdot r \cdot \mu_1 & \text{(definition of } \xi_r) \\
\pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot v & = & \pi_2 \cdot \widehat{s} \cdot u & \text{(definition of } v) \\
& = & \pi_2 \cdot s \cdot \mu_2 & \text{(definition of } u)
\end{array}$$

So by functoriality of the (epi, mono)-factorisation, we have the dotted morphism as above. To conclude, we need to prove that this is an iso. The right square tells us this is a mono. If we can prove that v is an epi, then this dotted morphism would also be an epi, and since since we are in a topos, this would be an iso.

To prove that v is an epi, we will use the fact that epis are closed under pullback in a topos. To this end, let us prove that the following square is a pullback:

$$\begin{array}{ccc} R\star S & \xrightarrow{\theta_s\cdot\mu_2} & E_Y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle e_{\xi_r}} \\ R\Box S & \xrightarrow[\widehat{\theta}_s\cdot\epsilon_2]{} & E_{\xi_r}\end{array}$$

First, it is a commutative square:

$$\begin{array}{rcll} \widehat{\theta}_s\cdot\epsilon_2\cdot v & = & \widehat{\theta}_s\cdot u & \text{(definition of } v\text{)} \\ & = & e_{\xi_r}\cdot\theta_s\cdot\mu_2 & \text{(definition of } u\text{)} \end{array}$$

Now assume given another commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\phi} & E_Y \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle e_{\xi_r}} \\ R\Box S & \xrightarrow[\widehat{\theta}_s\cdot\epsilon_2]{} & E_{\xi_r}\end{array}$$

We want to construct a morphism $\gamma : W \longrightarrow R\star S$. This is done by using three pullback properties as follows. First we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\epsilon_1\cdot\psi} & E_X \\ {\scriptstyle \langle\pi_1\cdot\in_X\cdot\epsilon_1\cdot\psi,\,\pi_1\cdot\in_Y\cdot\phi\rangle}\downarrow & & \downarrow{\scriptstyle \in_X} \\ X\times Y & \xrightarrow[\mathrm{id}\times\xi_r]{} & X\times\mathcal{P}X\end{array}$$

Indeed,

$$\begin{array}{rcll} \pi_2\cdot\in_X\cdot\epsilon_1\cdot\psi & = & \pi_1\cdot\widehat{s}\cdot\epsilon_2\cdot\psi & \text{(definition of } R\Box S\text{)} \\ & = & \pi_1\cdot m_{\xi_r}\cdot\widehat{\theta}_s\cdot\epsilon_2\cdot\phi & \text{(definition of } \widehat{S}\text{)} \\ & = & \pi_1\cdot m_{\xi_r}\cdot e_{\xi_r}\cdot\phi & \text{(assumption on } W\text{)} \\ & = & \xi_r\cdot\pi_1\cdot\in_Y\cdot\phi & \text{(definition of } E_{\xi_r}\text{)} \end{array}$$

So by the universal property of R, there is a unique morphism $\alpha : W \longrightarrow R$ such that

$$r\cdot\alpha = \langle\pi_1\cdot\in_X\cdot\epsilon_1\cdot\psi,\pi_1\cdot\in_Y\cdot\phi\rangle$$

and

$$\theta_r\cdot\alpha = \epsilon_1\cdot\psi.$$

Next we have the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\phi} & E_Y \\ {\scriptstyle \langle\pi_1\cdot\in_Y\cdot\phi,\,\pi_2\cdot\widehat{s}\cdot\epsilon_2\cdot\psi\rangle}\downarrow & & \downarrow{\scriptstyle \in_Y} \\ Y\times Z & \xrightarrow[\mathrm{id}\times\xi_s]{} & Y\times\mathcal{P}Y\end{array}$$

Indeed,

$$\begin{array}{rcll} \xi_s\cdot\pi_2\cdot\widehat{s}\cdot\epsilon_2\cdot\psi & = & \pi_2\cdot m_{\xi_r}\cdot\widehat{\theta}_s\cdot\epsilon_2\cdot\psi & \text{(definition of } \widehat{S}\text{)} \\ & = & \pi_2\cdot m_{\xi_r}\cdot e_{\xi_r}\cdot\phi & \text{(assumption on } W\text{)} \\ & = & \pi_2\cdot\in_Y\cdot\phi & \text{(definition of } E_{\xi_r}\text{)} \end{array}$$

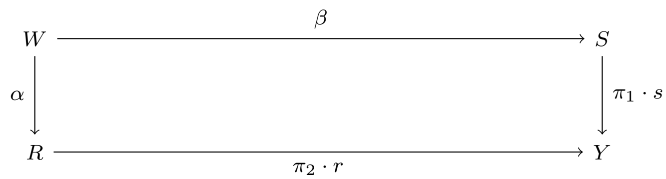
So by the universal property of S, there is a unique morphism $\beta : W \longrightarrow S$ such that

$$s \cdot \beta = \langle \pi_1 \cdot \in_Y \cdot \phi, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi \rangle$$

and

$$\theta_s \cdot \beta = \phi.$$

Finally we have the following commutative diagram:



Indeed,

$$\begin{aligned} \pi_2 \cdot r \cdot \alpha &= \pi_1 \cdot \in_Y \cdot \phi && \text{(definition of } \alpha) \\ &= \pi_1 \cdot s \cdot \beta && \text{(definition of } \beta) \end{aligned}$$

So by the universal property of $R \star S$, there is a unique morphism $\gamma : W \longrightarrow R \star S$ such that

$$\mu_1 \cdot \gamma = \alpha \quad \text{and} \quad \mu_2 \cdot \gamma = \beta.$$

Let us prove that γ is the unique morphism from W to $R \star S$ such that

$$v \cdot \gamma = \psi \quad \text{and} \quad \theta_s \cdot \mu_2 \cdot \gamma = \phi.$$

First, it satisfies those properties. For the first one, by unicity in the pullback property of $R \Box S$ and the fact that $\widehat{s}$ is a mono, it is enough to prove:

$$\begin{aligned} \epsilon_1 \cdot v \cdot \gamma &= \theta_r \cdot \mu_1 \cdot \gamma && \text{(definition of } v) \\ &= \theta_r \cdot \alpha && \text{(definition of } \gamma) \\ &= \epsilon_1 \cdot \psi && \text{(definition of } \alpha) \\ \widehat{s} \cdot \epsilon_2 \cdot v \cdot \gamma &= \widehat{s} \cdot u \cdot \gamma && \text{(definition of } v) \\ &= \langle \xi_r \cdot \pi_2 \cdot r \cdot \mu_1 \cdot \gamma, \pi_2 \cdot s \cdot \mu_2 \cdot \gamma \rangle && \text{(definition of } u) \\ &= \langle \xi_r \cdot \pi_2 \cdot r \cdot \alpha, \pi_2 \cdot s \cdot \beta \rangle && \text{(definition of } \gamma) \\ &= \langle \xi_r \cdot \pi_1 \cdot \in_Y \cdot \phi, \pi_2 \cdot s \cdot \beta \rangle && \text{(definition of } \alpha) \\ &= \langle \xi_r \cdot \pi_1 \cdot \in_Y \cdot \phi, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi \rangle && \text{(definition of } \beta) \\ &= \langle \pi_1 \cdot m_{\xi_r} \cdot e_{\xi_r} \cdot \phi, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi \rangle && \text{(definition of } E_{\xi_r}) \\ &= \langle \pi_1 \cdot m_{\xi_r} \cdot \widehat{\theta_s} \cdot \epsilon_2 \cdot \psi, \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi \rangle && \text{(assumption on } W) \\ &= \pi_1 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi && \text{(definition of } \widehat{S}) \\ &= \widehat{s} \cdot \epsilon_2 \cdot \psi && \text{(easy)} \end{aligned}$$

For the second one,

$$\theta_s \cdot \mu_2 \cdot \gamma = \theta_s \cdot \beta = \phi.$$

Now assume that there is another γ' from W to $R \star S$ such that

$$v \cdot \gamma' = \psi \quad \text{and} \quad \theta_s \cdot \mu_2 \cdot \gamma' = \phi.$$

Using the unicity of γ it is enough to prove that $\mu_1 \cdot \gamma' = \alpha$ and $\mu_2 \cdot \gamma' = \beta$. For the first one, by unicity of α it enough to prove the following:

$$
\begin{array}{rcll}
\theta_r \cdot \mu_1 \cdot \gamma' & = & \epsilon_1 \cdot v \cdot \gamma' & \text{(definition of } v\text{)} \\
& = & \epsilon_1 \cdot \psi & \text{(assumption on } \gamma'\text{)} \\
\pi_1 \cdot r \cdot \mu_1 \cdot \gamma' & = & \pi_1 \cdot \in_X \cdot \theta_r \cdot \mu_1 \cdot \gamma' & \text{(definition of } \xi_r\text{)} \\
& = & \pi_1 \cdot \in_X \cdot \epsilon_1 \cdot \psi & \text{(similar to the previous case)} \\
\pi_2 \cdot r \cdot \mu_1 \cdot \gamma' & = & \pi_1 \cdot s \cdot \mu_2 \cdot \gamma' & \text{(definition of } R \star S\text{)} \\
& = & \pi_1 \cdot \in_Y \cdot \theta_s \cdot \mu_2 \cdot \gamma' & \text{(definition of } \xi_s\text{)} \\
& = & \pi_1 \cdot \in_Y \cdot \phi & \text{(assumption on } \gamma'\text{)}
\end{array}
$$

For the second one, by unicity of β, it is enough to prove the following:

$$
\begin{array}{rcll}
\theta_s \cdot \mu_2 \cdot \gamma' & = & \phi & \text{(assumption on } \gamma'\text{)} \\
\pi_1 \cdot s \cdot \mu_2 \cdot \gamma' & = & \pi_1 \cdot \in_Y \cdot \theta_s \cdot \mu_2 \cdot \gamma' & \text{(definition of } \xi_s\text{)} \\
& = & \pi_1 \cdot \in_Y \cdot \phi & \text{(assumption on } \gamma'\text{)} \\
\pi_2 \cdot s \cdot \mu_2 \cdot \gamma' & = & \pi_2 \cdot \widehat{s} \cdot u \cdot \gamma' & \text{(definition of } u\text{)} \\
& = & \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot v \cdot \gamma' & \text{(definition of } v\text{)} \\
& = & \pi_2 \cdot \widehat{s} \cdot \epsilon_2 \cdot \psi & \text{(assumption on } \gamma'\text{)}
\end{array}
$$
□

In addition, given a morphism $f : X \longrightarrow Y$ of the topos, we have a corresponding morphism in the Kleisli category with $\eta_Y \cdot f$. Through ξ, this morphism corresponds to the right adjoint $\langle f, \mathrm{id}\rangle$. Using Theorem 1.10, we obtain that this functor from the topos to the Kleisli category is in reality an embedding. In particular, this means:

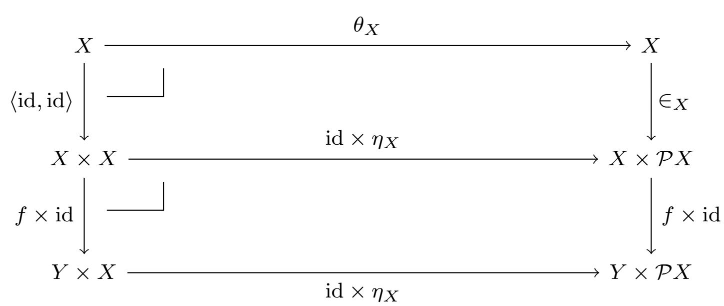
Lemma 3.5. *For all X, η_X is a mono.*

In this explanation, we can get rid of the "opposite", since an allegory is self-dual.

3.4. **Naturality and Coherence Axioms.** We are now all set to prove the first part of Theorem 3.3: Both naturalities are also easy or consequence of Proposition 3.4:

Lemma 3.6. *η and μ are natural. Furthermore, we have $\mu_X \cdot \eta_{\mathcal{P}X} = id_{\mathcal{P}X}$, $\mu_X \cdot \mathcal{P}\eta_X = id_{\mathcal{P}X}$, and $\mu_X \cdot \mathcal{P}\mu_X = \mu_X \cdot \mu_{\mathcal{P}X}$. Consequently, $\mathcal{P}$ is a monad whose Kleisli category is the allegory of relations.*

Proof of Lemma 3.6. • η **is natural:** Let $f : X \longrightarrow Y$. We have seen that $\eta_Y \cdot f = \xi_{\langle f,\mathrm{id}\rangle}$, and by unicity, it is enough to prove that $\mathcal{P}f \cdot \eta_X = \xi_{\langle f,\mathrm{id}\rangle}$. We have the following composition of pullbacks:



Since $f \times \mathrm{id} \cdot \langle \mathrm{id}, \mathrm{id}\rangle = \langle f, \mathrm{id}\rangle$ is a mono, and the (epi, mono)-factorisation of $f \times \mathrm{id} \cdot \in_X$ is given by $m_f \cdot e_f$, then by preservation of the image by pullback, we have the following composition of pullbacks:

$$\begin{array}{ccccc} X & \xrightarrow{\theta} & E_f & \xrightarrow{\theta_f} & Y \\ \downarrow{\scriptstyle\langle f,\mathrm{id}\rangle} & & \downarrow{\scriptstyle m_f} & & \downarrow{\scriptstyle\in_Y} \\ Y\times X & \xrightarrow[\mathrm{id}\times\eta_X]{} & Y\times\mathcal{P}X & \xrightarrow[\mathrm{id}\times\mathcal{P}f]{} & Y\times\mathcal{P}Y \end{array}$$

for some θ.

- μ **is natural:** Let $f : X \longrightarrow Y$. Observe that we have the following monos representing the same relations:
  $$\langle f,\mathrm{id}\rangle;\in_X^2 \equiv \langle f,\mathrm{id}\rangle;\in_X;\in_{\mathcal{P}X} \equiv m_f;\in_{\mathcal{P}X}.$$
  Then, by Proposition 3.4, we have:
  - $\xi_{\langle f,\mathrm{id}\rangle;\in_X^2} = \mu_Y \cdot \mathcal{P}\xi_{\langle f,\mathrm{id}\rangle} \cdot \xi_{\in_X^2} = \mu_Y \cdot \mathcal{P}(\eta_Y \cdot f) \cdot \mu_X = \mathcal{P}f \cdot \mu_X$, by a coherence axiom that we prove next.
  - $\xi_{m_f;\in_{\mathcal{P}X}} = \mu_Y \cdot \mathcal{P}\xi_{m_f} \cdot \xi_{\in_{\mathcal{P}X}} = \mu_Y \cdot \mathcal{P}\mathcal{P}f$, which uses the fact that $\xi_{\in_X} = \mathrm{id}_{\mathcal{P}X}$.
- **coherence axioms:**
  - $\mathrm{id}_{\mathcal{P}X} = \xi_{\in_X} = \xi_{\in_X;\Delta_{\mathcal{P}X}} = \mu_X \cdot \mathcal{P}\xi_{\in_X} \cdot \xi_{\Delta_{\mathcal{P}X}} = \mu_X \cdot \eta_{\mathcal{P}X}$.
  - $\mathrm{id}_{\mathcal{P}X} = \xi_{\in_X} = \xi_{\Delta_X;\in_X} = \mu_X \cdot \mathcal{P}\xi_{\Delta_X} \cdot \xi_{\in_X} = \mu_X \cdot \mathcal{P}\eta_X$.
  - $\xi_{\in_X;\in_{\mathcal{P}X};\in_{\mathcal{P}\mathcal{P}X}} = \mu_X \cdot \mathcal{P}\xi_{\in_X;\in_{\mathcal{P}X}} \cdot \xi_{\in_{\mathcal{P}\mathcal{P}X}} = \mu_X \cdot \mathcal{P}\xi_{\in_X^2} = \mu_X \cdot \mathcal{P}\mu_X$ and $\xi_{\in_X;\in_{\mathcal{P}X};\in_{\mathcal{P}\mathcal{P}X}} = \mu_X \cdot \mathcal{P}\xi_{\in_X} \cdot \xi_{\in_{\mathcal{P}X};\in_{\mathcal{P}\mathcal{P}X}} = \mu_X \cdot \xi_{\in_{\mathcal{P}X}^2} = \mu_X \cdot \mu_{\mathcal{P}X}$. □

The remaining part of Theorem 3.3 is about the strength of the monad. This will be explained as a particular case of proto-distributive laws later on.

3.5. **Pseudo-Inverse of a Morphism.** Let us continue this section with another useful consequence of Lemma 3.4. A morphism $f : X \longrightarrow Y$ induces a Kleisli morphism $\eta_Y \cdot f : X \longrightarrow \mathcal{P}Y$, or a relation as the right adjoint $\langle f,\mathrm{id}\rangle$. This relation has a converse which is given by its left adjoint (or map) $\langle \mathrm{id}, f\rangle$. This relation then corresponds to a unique morphism $f^\dagger = \xi_{\langle\mathrm{id},f\rangle} : Y \longrightarrow \mathcal{P}X$. This pseudo-inverse has nice properties when f is a mono or an epi:

Proposition 3.7. *When f is a mono, $f^\dagger \cdot f = \eta_X$ and $\mathcal{P}f$ is a split mono. When f is an epi, $\mathcal{P}f \cdot f^\dagger = \eta_Y$ and $\mathcal{P}f$ is a split epi.*

Corollary 3.8. *$\mathcal{P}$ preserves epis and monos, and so (epi, mono)-factorisations.*

When translating the proof of Proposition 2.16 to toposes, the main argument becomes the fact that $\mathcal{P}$ maps epis to split epis.

Proof of Proposition 3.7. We have the following composition of pullbacks:

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}} & X & \xrightarrow{\theta_X} & E_X \\ \downarrow{\scriptstyle\langle \mathrm{id},\mathrm{id}\rangle} & & \downarrow{\scriptstyle\langle \mathrm{id},f\rangle} & & \downarrow{\scriptstyle\in_X} \\ X\times X & \xrightarrow[\mathrm{id}\times f]{} & X\times Y & \xrightarrow[\mathrm{id}\times f^\dagger]{} & X\times\mathcal{P}X \end{array}$$

Observe that the left one is a pullback only when f is a mono. Then the equality holds by unicity of $\xi_{\langle\mathrm{id},\mathrm{id}\rangle} = \eta_X$. From this equality, we deduce that

$$(\mu_X \cdot \mathcal{P}f^\dagger) \cdot \mathcal{P}f = \mu_X \cdot \mathcal{P}\eta_X = \mathrm{id},$$

and so that $\mathcal{P}f$ is a split mono.

When f is an epi,

$$\langle f, \mathrm{id}\rangle ; \langle \mathrm{id}, f\rangle = \langle \mathrm{id}, \mathrm{id}\rangle.$$

Indeed, $\langle f, \mathrm{id}\rangle ; \langle \mathrm{id}, f\rangle$ is reprsented by the mono part of the (epi, mono)-factorisation of $\langle f, f\rangle$, which is given by $\langle \mathrm{id}, \mathrm{id}\rangle \cdot f$ when f is epi. Consequently, from Proposition 3.4:

$$\begin{aligned}\eta_Y = \xi_{\langle \mathrm{id},\mathrm{id}\rangle} = \xi_{\langle f,\mathrm{id}\rangle;\langle \mathrm{id},f\rangle} &= \mu_Y \cdot \mathcal{P}\xi_{\langle f,\mathrm{id}\rangle} \cdot \xi_{\langle \mathrm{id},f\rangle} \\ &= \mu_Y \cdot \mathcal{P}\eta_Y \cdot \mathcal{P}f \cdot f^\dagger = \mathcal{P}f \cdot f^\dagger.\end{aligned}$$

From this equality, we deduce that

$$\mathcal{P}f \cdot (\mu_X \cdot \mathcal{P}f^\dagger) = \mu_Y \cdot \mathcal{P}(\mathcal{P}f \cdot f^\dagger) = \mu_Y \cdot \mathcal{P}\eta_Y = \mathrm{id},$$

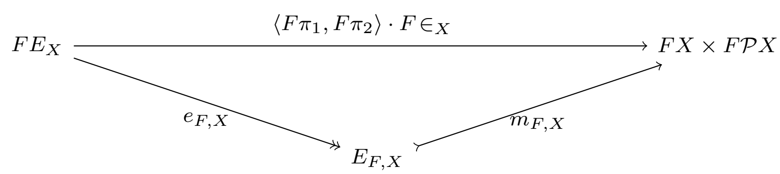
and so that $\mathcal{P}f$ is a split epi. □

3.6. **Proto-Distributive Laws.** As a side remark, we can easily derive some candidates for (weak) distributive laws for every functor, also called cross-operator in [dM94]. Everything written here already appears in some form in [GPA21], but proved in a purely relational way.

The power-set monad (and more generally, the power-object monad) is often combined with other functors to model the non-determinism of a system. Having weak distributive laws then allows to simplify the analysis by transferring it from the original category to the Kleisli category (which we know well in the case of the power-object monad). See for example [UH18].

The interesting observation behind the definition of power-objects is that there is a canonical way to define a candidate for a distributive law of $\mathcal{P}$ over *any functor* F. We will see that these canonical candidates give rise to well-known (weak) distributive laws in the literature.

Given a functor F on the topos and any object X of the topos, we define $\sigma_{F,X} : F\mathcal{P}X \longrightarrow \mathcal{P}FX$ as the usual pattern (epi, mono)-factorisation followed by unique morphism from the definition. In this case, we consider the (epi, mono)-factorisation of $\langle F\pi_1, F\pi_2\rangle \cdot F\in_X$:



and $\sigma_{F,X}$ is defined as $\xi_{m_{F,X}}$.

Those proto-distributive laws are related to liftings of functors to the Kleisli category, here to the category of relations. In the case of **Set**, several papers [GP20, Gar20] investigate this connection, and particularly, some conditions are given for the existence of (weak) distributive laws. We can prove a similar theorem in any topos, as already stated in [GPA21]:

Proposition 3.9. *If F preserves weak pullbacks and epis, then $\sigma_{F,X}$ is natural in X. Furthermore, we have: $\sigma_{F,X} \cdot F\eta_X = \eta_{FX}$ and $\sigma_{F,X} \cdot F\mu_X = \mu_{FX} \cdot \mathcal{P}\sigma_{F,X} \cdot \sigma_{F,\mathcal{P}X}$.*

Remark 3.10. In **Set**, there is no need for the second condition, as any functor preserves epis: every epi is split in **Set** by the axiom of choice.

Proof. Let $f : X \longrightarrow Y$. We want to prove that the following square commutes:

$$\begin{array}{ccc} F\mathcal{P}X & \xrightarrow{\sigma_{F,X}} & \mathcal{P}FX \\ {\scriptstyle F\mathcal{P}f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}Ff} \\ F\mathcal{P}Y & \xrightarrow[\sigma_{F,Y}]{} & \mathcal{P}FY \end{array}$$

On one side, we have the following composition of pullbacks:

$$\begin{array}{ccc} E_{F,X} & \xrightarrow{\theta_{F,X}} & E_{FX} \\ {\scriptstyle m_{F,X}}\downarrow & \lrcorner & \downarrow{\scriptstyle \in_{FX}} \\ FX \times F\mathcal{P}X & \xrightarrow{\mathrm{id} \times \sigma_{F,X}} & FX \times \mathcal{P}FX \\ {\scriptstyle Ff \times \mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle Ff \times \mathrm{id}} \\ FY \times F\mathcal{P}X & \xrightarrow[\mathrm{id} \times \sigma_{F,X}]{} & FY \times \mathcal{P}FX \end{array}$$

Since $\mathrm{mono}((Ff \times \mathrm{id}) \cdot \in_{FX}) = m_{Ff}$, then by preservation of image by pullbacks, we have a pullback of the following shape:

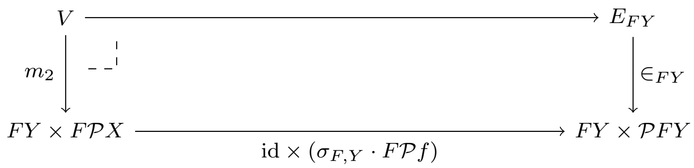
$$\begin{array}{ccc} U & \longrightarrow & E_{FY} \\ {\scriptstyle m_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \in_{FY}} \\ FY \times F\mathcal{P}X & \xrightarrow[\mathrm{id} \times (\mathcal{P}Ff \cdot \sigma_{F,X})]{} & FY \times \mathcal{P}FY \end{array}$$

where $m_1 = \mathrm{mono}((Ff \times \mathrm{id}) \cdot m_{F,X})$. So, $\mathcal{P}Ff \cdot \sigma_{F,X}$ is the unique morphism associated to m_1, and it is enough to prove that $\sigma_{F,Y} \cdot F\mathcal{P}f$ is also associated to m_1 to conclude.

On the other side, we have the following composition of weak pullbacks:

$$\begin{array}{ccc} FE_f & \xrightarrow{F\theta_f} & FE_Y \\ {\scriptstyle Fm_f}\downarrow & \text{weak} \lrcorner & \downarrow{\scriptstyle F\in_Y} \\ F(Y \times \mathcal{P}X) & \xrightarrow{F(\mathrm{id} \times \mathcal{P}f)} & F(Y \times \mathcal{P}Y) \\ {\scriptstyle \langle F\pi_1, \mathrm{id}\rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle F\pi_1, \mathrm{id}\rangle} \\ FY \times F(Y \times \mathcal{P}X) & \xrightarrow[\mathrm{id} \times F(\mathrm{id} \times \mathcal{P}f)]{} & FY \times F(Y \times \mathcal{P}Y) \\ {\scriptstyle \mathrm{id} \times F\pi_2}\downarrow & \text{weak} \lrcorner & \downarrow{\scriptstyle \mathrm{id} \times F\pi_2} \\ FY \times F\mathcal{P}X & \xrightarrow[\mathrm{id} \times F\mathcal{P}f]{} & FY \times F\mathcal{P}Y \end{array}$$

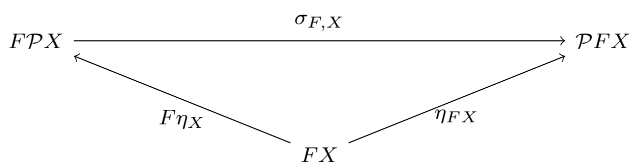
The upper weak pullback comes from the preservation of weak pullbacks by F and by the definition of $\mathcal{P}f$. The middle pullback is easy. The lower weak pullback comes from the preservation of weak pullbacks by F and product functors. Then by preservation of images by weak pullbacks, there is a weak pullback of the following shape:



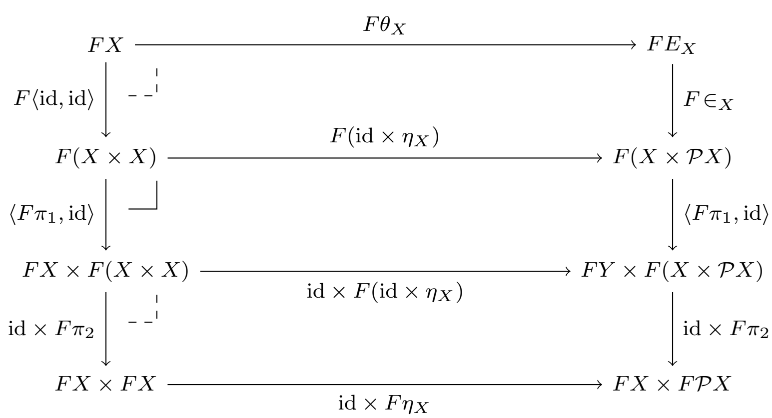
where $m_2 = \mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot Fm_f)$. To conclude, it is enough to prove that $m_1 \equiv m_2$:

$$
\begin{array}{rcll}
m_2 & \equiv & \mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot Fm_f) & \text{(definition)} \\
& \equiv & \mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot Fm_f \cdot Fe_f) & \text{(F preserves epis)} \\
& \equiv & \mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot F(f \times \mathrm{id}) \cdot F\in_X) & \text{(definition)} \\
& \equiv & \mathrm{mono}((Ff \times \mathrm{id}) \cdot \langle F\pi_1, F\pi_2\rangle \cdot F\in_X) & \text{(calculation)} \\
& \equiv & \mathrm{mono}((Ff \times \mathrm{id}) \cdot m_{F,X} \cdot e_{F,X}) & \text{(definition)} \\
& \equiv & \mathrm{mono}((Ff \times \mathrm{id}) \cdot m_{F,X}) & \text{($e_{F,X}$ epi)} \\
& \equiv & m_1 & \text{(definition)}
\end{array}
$$

Now, let us prove the first coherence axiom:



Similarly to the above proof, we have the following composition of weak pullbacks:

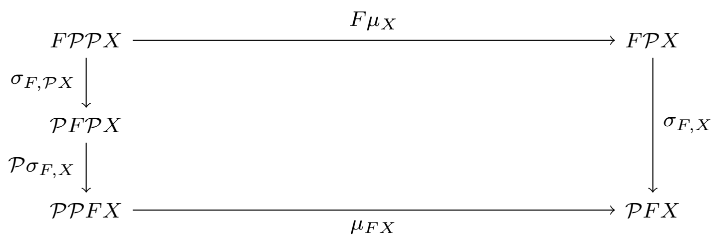


By preservation of images by weak pullbacks, this implies that $\sigma_{F,X} \cdot F\eta_X$ is the unique morphism associated to

$$\mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot F\langle \mathrm{id}, \mathrm{id}\rangle) = \langle \mathrm{id}_{FX}, \mathrm{id}_{FX}\rangle,$$

and so is η_{FX}.

Finally, let us prove the second coherence axiom:



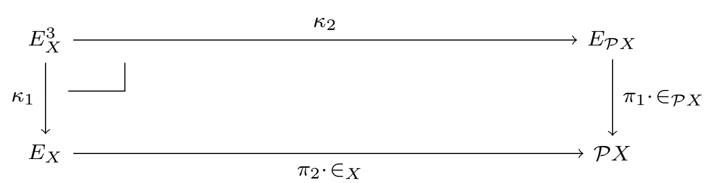
First, by Proposition 3.4, $\mu_{FX} \cdot \mathcal{P}\sigma_{F,X} \cdot \sigma_{F,\mathcal{P}X}$ is the unique morphism associated with $m_{F,X}; m_{F,\mathcal{P}X}$. On the other side, with the same kind of composition of weak pullbacks, we have that $\sigma_{F,X} \cdot F\mu_X$ is the unique morphism associated to

$$\mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot F\in_X^2),$$

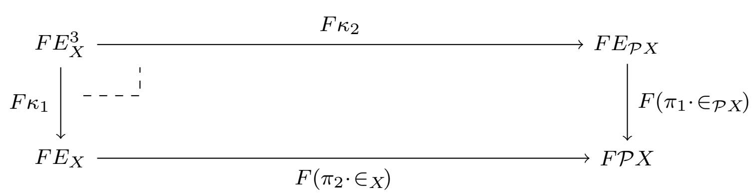
so it is enough to prove that both monos are the same. Using the fact that F preserves epis, we can observe that

$$\mathrm{mono}(\langle F\pi_1, F\pi_2\rangle \cdot F\in_X^2) \equiv \mathrm{mono}(\langle F(\pi_1\cdot \in_X \cdot\kappa_1), F(\pi_2\cdot \in_{\mathcal{P}X} \cdot\kappa_2)\rangle), \tag{3.1}$$

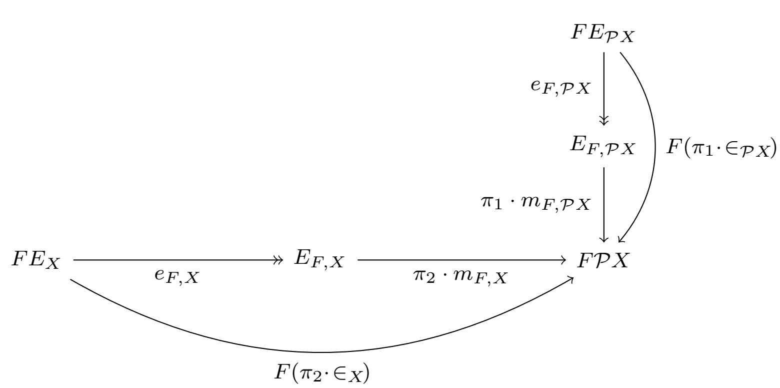
where κ_1 and κ_2 are obtained with the following pullback:



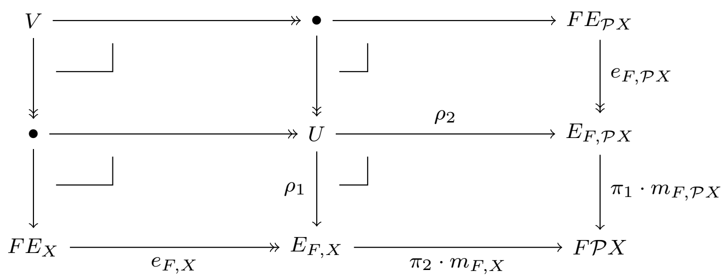
By preservation of weak pullbacks by F, the following is then a weak pullback:



If we analyse the strict pullback of the same diagram, then we realise that it is also the limit of the following cospan:



Now, we can compute this pullback by computing four smaller pullbacks, which gives us the following situation, using the preservation of epis by pullbacks:



Since FE^3_X is a weak pullback of this cospan, the unique morphism of cones from FE^3_X to V is a split epi. In total, this means that there is an epi $u : FE^3_X \twoheadrightarrow U$ such that

$$\rho_1 \cdot u = e_{F,X} \cdot F\kappa_1 \quad \text{and} \quad \rho_2 \cdot u = e_{F,\mathcal{P}X} \cdot F\kappa_2. \tag{3.2}$$

Now, the lower-right pullback is the one used to define the composition $m_{F,X}; m_{F,\mathcal{P}X}$, which means that:

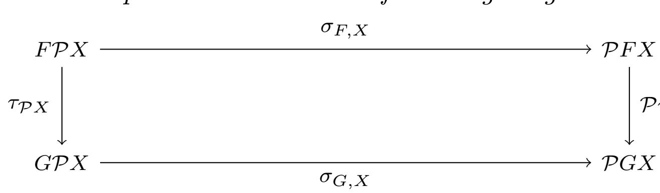
$$m_{F,X}; m_{F,\mathcal{P}X} = \text{mono}(\langle \pi_1 \cdot m_{F,X} \cdot \rho_1, \pi_2 \cdot m_{F,\mathcal{P}X} \cdot \rho_2 \rangle).$$

To conclude, it is enough to observe:

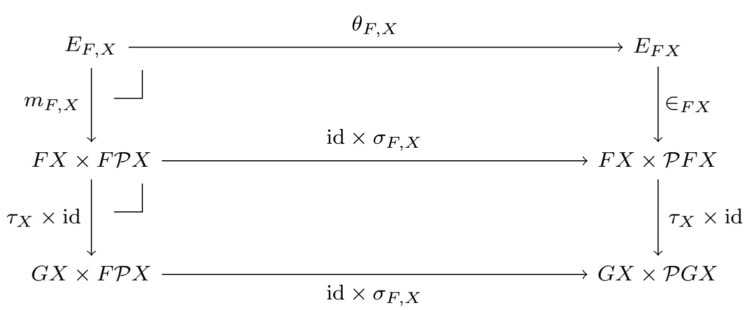
$$\begin{array}{lll} & \text{mono}(\langle \pi_1 \cdot m_{F,X} \cdot \rho_1, \pi_2 \cdot m_{F,\mathcal{P}X} \cdot \rho_2 \rangle) & \\ \equiv & \text{mono}(\langle \pi_1 \cdot m_{F,X} \cdot \rho_1, \pi_2 \cdot m_{F,\mathcal{P}X} \cdot \rho_2 \rangle \cdot u) & (u \text{ is epi}) \\ \equiv & \text{mono}(\langle \pi_1 \cdot m_{F,X} \cdot e_{F,X} \cdot F\kappa_1, \pi_2 \cdot m_{F,\mathcal{P}X} \cdot e_{F,\mathcal{P}X} \cdot F\kappa_2 \rangle) & (\text{by } (3.2)) \\ \equiv & \text{mono}(\langle F(\pi_1 \cdot \in_X) \cdot F\kappa_1, F(\pi_2 \cdot \in_{\mathcal{P}X}) \cdot F\kappa_2 \rangle) & (\text{definition}) \\ \equiv & \text{mono}(\langle F\pi_1, F\pi_2 \rangle \cdot F \in^2_X) & (\text{by } (3.1)) \end{array}$$ □

In [Gar20], some conditions are also given to get (weak) distributive laws. Those results can be encompassed in a result about naturality of $\sigma_{F,X}$ with respect to F in the following sense:

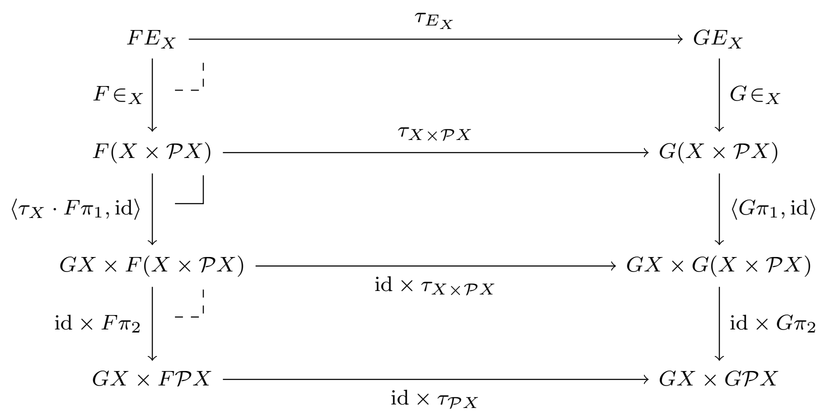
Proposition 3.11. *Assume given a natural transformation $\tau : F \Longrightarrow G$ such that its naturality squares are weak pullbacks. Then the following diagram commutes for any X:*



Proof. On one side, we have the following composition of pullbacks:



which implies that $\mathcal{P}\tau_X \cdot \sigma_{F,X}$ is the unique morphism corresponding to mono$((\tau_X \times \mathrm{id}) \cdot m_{F,X})$. On the other side, we have the following composition of weak pullbacks:



Indeed, the upper and lower weak pullbacks come from the naturality squares of τ, and the middle pullback is easy. This means that $\sigma_{G,X} \cdot \tau_{\mathcal{P}X}$ is the unique morphism corresponding to mono$(\mathrm{id} \times F\pi_2 \cdot \langle \tau_X \cdot F\pi_1, \mathrm{id}\rangle \cdot F\in_X)$. It is easy to check that both monos are the same. □

As stated in [GPA21]:

Corollary 3.12. *If (T, μ^T, η^T) is a monad which preserves weak pullbacks and epis, and for which the naturality squares of μ^T are weak pullbacks, then $\sigma_{T,X}$ is a weak distributive law. If the naturality squares of η^T are also weak pullbacks, then $\sigma_{T,X}$ is a distributive law.*

Before proving Corollary 3.12, let us prove an easy lemma about $\sigma_{F,X}$:

Lemma 3.13. *We have the following equalities:*

- $\sigma_{Id,X} = id_{\mathcal{P}X}$,
- *if G preserves weak pullbacks and epis, then $\sigma_{G\cdot F,X} = \sigma_{G,FX} \cdot G\sigma_{F,X}$.*

Proof. • By definition, $\sigma_{\mathrm{Id},X}$ correspond to the mono $\in_X$, which is also the case of $\mathrm{id}_{\mathcal{P}X}$.

- By definition, $\sigma_{G\cdot F,X}$ corresponds to the mono $m_{G\cdot F,X}$. Also, by the same kind of composition of weak pullbacks as previous proofs, $\sigma_{G,FX} \cdot G\sigma_{F,X}$ corresponds to the mono mono$(\langle G\pi_1, G\pi_2\rangle \cdot Gm_{F,X})$. Using the fact that G preserves epi and the definition of $m_{F,X}$, it is easy to check that both monos are the same. □

Proof of Corollary 3.12. For the first part, we can apply Proposition 3.11 with μ^T which is a natural transformation from TT to T. We then obtain:

$$\sigma_{T,X} \cdot \mu^T_{\mathcal{P}X} = \mathcal{P}\mu^T_X \cdot \sigma_{TT,X}.$$

Then by Lemma 3.13:

$$\sigma_{T,X} \cdot \mu^T_{\mathcal{P}X} = \mathcal{P}\mu^T_X \cdot \sigma_{T,TX} \cdot T\sigma_{T,X}$$

which is the coherence axiom to prove. Similarly, the second part consists in using Proposition 3.11 with η^T and then the first point of Lemma 3.13. □

Example 3.14. In **Set**, when F is $\mathcal{P}$ itself, $\sigma_{F,X}$ is a weak distributive law, as described in [GP20], namely, $\sigma_{\mathcal{P},X} : U \in \mathcal{P}\mathcal{P}X \mapsto \{V \subseteq X \mid V \subseteq \bigcup U \wedge \forall W \in U.\, W \cap V \neq \emptyset\}$. A similar, analysis can be done for the distribution monad $\mathcal{D}$. More generally (see [GPA21]), in any topos, $\mathcal{P}$ satisfies the assumptions of Proposition 3.9 and the first part of Corollary 3.12, meaning that $\sigma_{\mathcal{P},X}$ is a weak distributive law. However, it satisfies the second part only when the topos is trivial.

Proof of Example 3.14. Here, we want to prove that in any topos $\sigma_{\mathcal{P},X}$ satisfies the assumptions of Proposition 3.9. We already know that $\mathcal{P}$ preserves epis. Let us prove that it preserves weak pullbacks.

Assume given a weak pullback of the form:

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_2} & Y_2 \\ {\scriptstyle \epsilon_1}\big\downarrow & & \big\downarrow{\scriptstyle \mu_2} \\ Y_1 & \xrightarrow[\mu_1]{} & Z \end{array}$$

We want to prove that we have the following weak pullback:

$$\begin{array}{ccc} \mathcal{P}X & \xrightarrow{\mathcal{P}\epsilon_2} & \mathcal{P}Y_2 \\ {\scriptstyle \mathcal{P}\epsilon_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}\mu_2} \\ \mathcal{P}Y_1 & \xrightarrow[\mathcal{P}\mu_1]{} & \mathcal{P}Z \end{array}$$

So we assume given another commutative square of the form:

$$\begin{array}{ccc} W & \xrightarrow{\phi_2} & \mathcal{P}Y_2 \\ {\scriptstyle \phi_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{P}\mu_2} \\ \mathcal{P}Y_1 & \xrightarrow[\mathcal{P}\mu_1]{} & \mathcal{P}Z \end{array}$$

We want to construct a morphism $\phi : W \longrightarrow \mathcal{P}X$, and the trick is to play with the correspondence with relations. First, let us form the following pullbacks:

$$\begin{array}{ccc} R_i & \xrightarrow{\theta_i} & E_{Y_i} \\ {\scriptstyle r_i}\big\downarrow & & \big\downarrow{\scriptstyle \in_{Y_i}} \\ Y_i \times W & \xrightarrow[\mathrm{id} \times \phi_i]{} & Y_i \times \mathcal{P}Y_i \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\rho_2} & R_2 \\ {\scriptstyle \rho_1}\big\downarrow & & \big\downarrow{\scriptstyle (\mu_2 \times \mathrm{id}) \cdot r_2} \\ R_1 & \xrightarrow[(\mu_1 \times \mathrm{id}) \cdot r_1]{} & Z \times W \end{array}$$

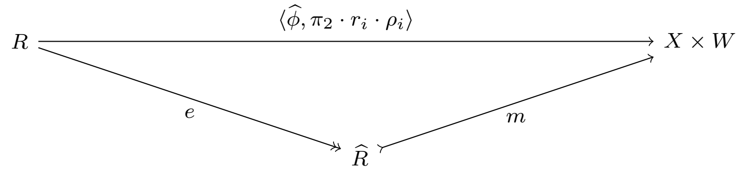
So by construction, we have the following commutative square:

$$\begin{array}{ccc} R & \xrightarrow{\pi_1 \cdot r_2 \cdot \rho_2} & Y_2 \\ {\scriptstyle \pi_1 \cdot r_1 \cdot \rho_1}\big\downarrow & & \big\downarrow{\scriptstyle \mu_2} \\ Y_1 & \xrightarrow[\mu_1]{} & Z \end{array}$$

and by the universal property of X, there is a (non necessarily unique) morphism $\widehat{\phi} : R \longrightarrow X$ such that

$$\epsilon_i \cdot \widehat{\phi} = \pi_1 \cdot r_i \cdot \rho_i.$$

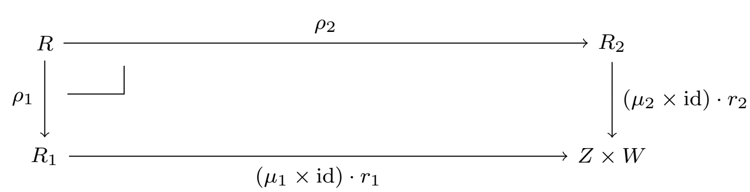
Since we have $\pi_2 \cdot r_1 \cdot \rho_1 = \pi_2 \cdot r_2 \cdot \rho_2$, we have the following unique (epi, mono)-factorisation:



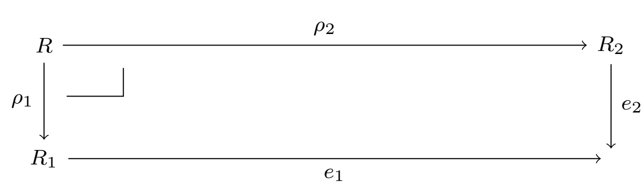
Define then $\phi = \xi_m$. To conclude, we need to prove that $\phi_i = \mathcal{P}\epsilon_i \cdot \phi$. But we know that:

$$
\begin{array}{rcll}
\phi_i & = & \xi_{r_i} & \text{(definition of } \phi_i\text{)} \\
\mathcal{P}\epsilon_i \cdot \phi & = & \mu_{Y_i} \cdot \mathcal{P}\eta_{Y_i} \cdot \mathcal{P}\epsilon_i \cdot \phi & \text{(unit coherence axiom)} \\
& = & \mu_{Y_i} \cdot \mathcal{P}\xi_{\langle \epsilon_i, \mathrm{id} \rangle} \cdot \phi & \text{(calculation)} \\
& = & \mu_{Y_i} \cdot \mathcal{P}\xi_{\langle \epsilon_i, \mathrm{id} \rangle} \cdot \xi_m & \text{(definition of } m\text{)} \\
& = & \xi_{\langle \epsilon_i, \mathrm{id} \rangle ; m} & \text{(Proposition 3.4)}
\end{array}
$$

So we need to prove that $r_i \equiv \langle \epsilon_i, \mathrm{id} \rangle ; m$. We know by definition of composition that $\langle \epsilon_i, \mathrm{id} \rangle ; m$ is $\mathrm{mono}(\langle \epsilon_i \cdot \pi_1 \cdot m, \pi_2 \cdot m \rangle)$. Since e is an epi, this is also $\mathrm{mono}(\langle \epsilon_i \cdot \pi_1 \cdot m, \pi_2 \cdot m \rangle \cdot e) \equiv \mathrm{mono}(r_i \cdot \rho_i)$. So to conclude, it is enough to prove that ρ_i is an epi. By assumption, we know that $\mathcal{P}\mu_1 \cdot \phi_1 = \mathcal{P}\mu_2 \cdot \phi_2$. By using again the same trick, this implies that $\mathrm{mono}(\langle \mu_1, \mathrm{id} \rangle ; r_1) = \mathrm{mono}(\langle \mu_2, \mathrm{id} \rangle ; r_2)$. Let us write e_1 and e_2 their corresponding epic parts. But we also know that we have the following pullback:



which means we have the following pullback:



Since e_i is epi, ρ_i is epi by preservation of epis by pullbacks. □

As a consequence, let us look at the strength and costrength of $\mathcal{P}$. Indeed, define the strength as: $t_{X,Y} = \sigma_{X \times \_, Y} : X \times \mathcal{P}Y \longrightarrow \mathcal{P}(X \times Y)$.

Proposition 3.15. *$t_{X,Y}$ is the strength of $\mathcal{P}$.*

Proof. Naturality in X comes from Proposition 3.9. Naturality in Y comes from Proposition 3.11. The coherence axioms are also consequences of either Proposition. □

Dually, the costrength can be defined as $t'_{X,Y} = \sigma_{\_ \times Y, X} : \mathcal{P}X \times Y \longrightarrow \mathcal{P}(X \times Y)$. By the naturality of Proposition 3.11, we indeed have the expected equality $t'_{X,Y} = \mathcal{P}\lambda_{Y,X} \cdot t_{Y,X} \cdot \lambda_{\mathcal{P}X,Y}$, where $\lambda_{X,Y} : X \times Y \longrightarrow Y \times X$ is the symmetry of the product.

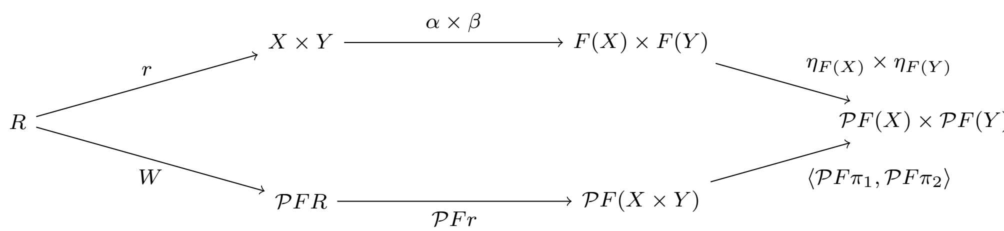
Theorem 3.16. *$\mathcal{P}$ is a commutative strong monad.*

Proof. The commutation axiom is a consequence of Proposition 3.4. □

4. AM-Bisimulations in a Topos

Since toposes are regular categories, the notion of regular AM-bisimulations makes sense. We show here that they can be reformulated as follows.

Definition 4.1. We say that a relation is a *toposal AM-bisimulation* from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow \mathcal{P}FR$ such that:



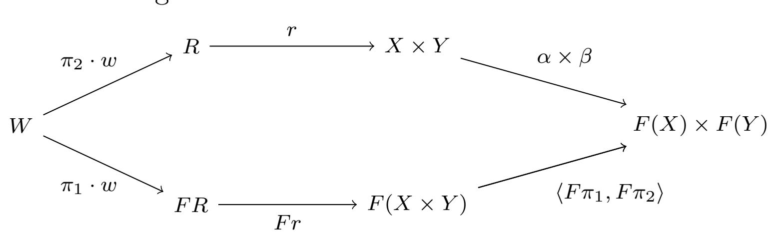
In other words, an F-toposal AM-bisimulation between α and β is a $\mathcal{P}F$-AM-bisimulation between $\eta \cdot \alpha$ and $\eta \cdot \beta$. Intuitively, this means that toposal bisimulations look at systems as non-deterministic. This allows us to *collect* witnesses as a morphism $W : R \longrightarrow \mathcal{P}FR$ instead of picking some, very much like regular AM-bisimulations.

We have to make sure that toposal and regular AM-bisimulations coincide.

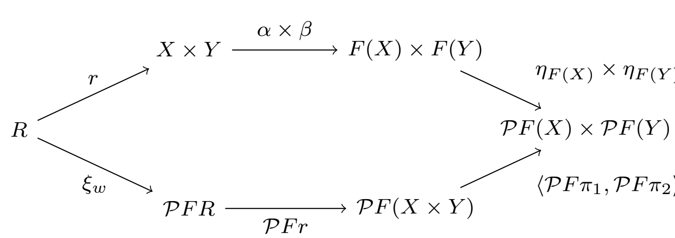
Proposition 4.2. *Assume that $\mathcal{C}$ is a topos. Then for every relation U from X to Y, every coalgebra $\alpha : X \longrightarrow FX$ and $\beta : Y \longrightarrow FY$, U is a toposal AM-bisimulation from α to β if and only if it is a regular AM-bisimulation between them.*

Proof. Assume that $\mathcal{C}$ is a topos.

- Assume that we have a regular AM-bisimulation



The relation $w : W \rightarrowtail FR \times R$ uniquely corresponds to a morphism $\xi_w : R \longrightarrow \mathcal{P}FR$. Let us prove that this witnesses r as a toposal bisimulation



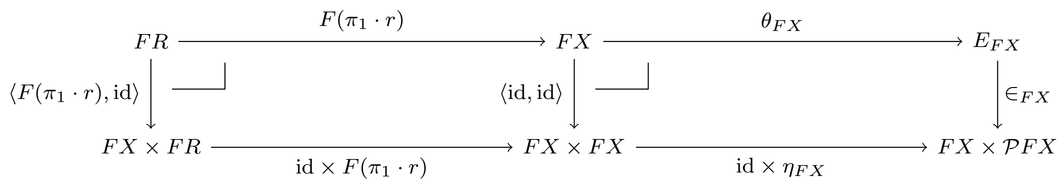
Let us then prove that

$$\eta_{FX} \cdot \alpha \cdot \pi_1 \cdot r = \mathcal{P}F(\pi_1 \cdot r) \cdot \xi_w,$$

the statement for Y and β being similar. To prove this equality, since they are both morphisms from R to $\mathcal{P}FX$, it is enough to prove they correspond to the same relation on $FX \times R$. First,

$$\begin{array}{rcll} \mathcal{P}F(\pi_1 \cdot r) \cdot \xi_w & = & \mu_{FX} \cdot \mathcal{P}(\eta_{FX} \cdot F(\pi_1 \cdot r)) \cdot \xi_w & \text{(coherence axiom)} \\ & = & \mu_{FX} \cdot \mathcal{P}\xi_{\langle F(\pi_1 \cdot r), \mathrm{id}\rangle} \cdot \xi_w & (*) \\ & = & \xi_{\langle F(\pi_1 \cdot r), \mathrm{id}\rangle ; w} & \text{(Lemma 3.4)} \end{array}$$

Here $(*)$ comes from the fact we have the following composition of pullbacks:

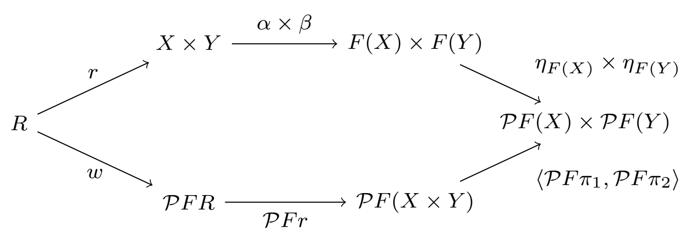


where the left pullback is by simple computation and the right one is by definition of η_{FX}. Now, by definition, the composition of relations $\langle F(\pi_1 \cdot r), \mathrm{id}\rangle ; w$ is given by the monic part of the (epi, mono)-factorisation of

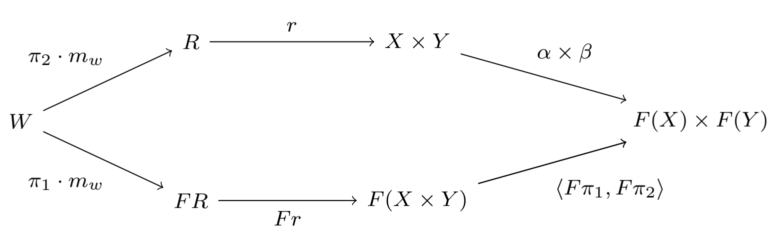
$$\langle F(\pi_1 \cdot r) \cdot \pi_1 \cdot w, \pi_2 \cdot w\rangle = \langle \alpha \cdot \pi_1 \cdot r \cdot \pi_2 \cdot w, \pi_2 \cdot w\rangle.$$

Since $\pi_2 \cdot w$ is epi, and $\langle \alpha \cdot \pi_1 \cdot r, \mathrm{id}\rangle$ is mono, then the monic part of $\langle F(\pi_1 \cdot r) \cdot \pi_1 \cdot w, \pi_2 \cdot w\rangle$ is $\langle \alpha \cdot \pi_1 \cdot r, \mathrm{id}\rangle$, which corresponds to the morphism $\eta_{FX} \cdot \alpha \cdot \pi_1 \cdot r$ (similarly to $*$).

- Now assume we have a toposal bisimulation



Then w corresponds to a unique relation represented by a mono $m_w : W \rightarrowtail FR \times R$. Let us prove that this witnesses r as a regular AM-bisimulation, that is, that the following diagram commutes



and that $\pi_2 \cdot m_w$ is epi. Using the same calculation as the previous point, the diagram of r being a toposal bisimulation can be translated in terms of relations as

$$\langle F(\pi_1 \cdot r), \mathrm{id}\rangle ; m_w = \langle \alpha \cdot \pi_1 \cdot r, \mathrm{id}\rangle \quad \text{and} \quad \langle F(\pi_2 \cdot r), \mathrm{id}\rangle ; m_w = \langle \beta \cdot \pi_2 \cdot r, \mathrm{id}\rangle.$$

Let's concentrate on α (β will be similar). The composition $\langle F(\pi_1 \cdot r), \mathrm{id}\rangle ; m_w$ is again given by the monic part of $\langle F(\pi_1 \cdot r) \cdot \pi_1 \cdot m_w, \pi_2 \cdot m_w\rangle$, which is equal to $\langle \alpha \cdot \pi_1 \cdot r, \mathrm{id}\rangle$.

This means that there is an epi e such that

$$\langle F(\pi_1 \cdot r) \cdot \pi_1 \cdot m_w, \pi_2 \cdot m_w\rangle = \langle \alpha \cdot \pi_1 \cdot r, \mathrm{id}\rangle \cdot e.$$

Consequently, $\pi_2 \cdot m_w = e$ and $\pi_2 \cdot m_w$ is an epi. Furthermore,

$$F(\pi_1 \cdot r) \cdot \pi_1 \cdot m_w = \alpha \cdot \pi_1 \cdot r \cdot e = \alpha \cdot \pi_1 \cdot r \cdot \pi_2 \cdot m_w. \qquad \square$$

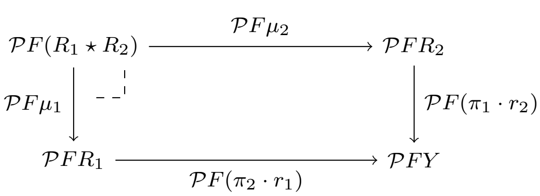
This nicer formulation allows us to prove a much nicer tabularity property, which could only be informally described for regular AM-bisimulations:

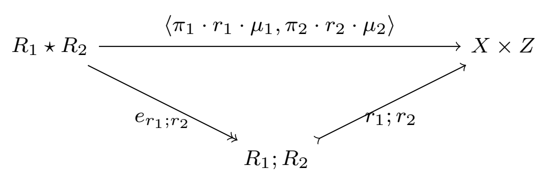
Proposition 4.3. *Assume that $\mathcal{C}$ is a topos and that F covers pullbacks. Then the following is an I-category: objects are coalgebras on F, morphisms are toposal AM-bisimulations, $\sqsubseteq$, identities, composition, and $(\_)^\dagger$ are defined as in $\mathbf{Rel}(\mathcal{C})$.*

Remark 4.4. Remark that this Proposition is similar to Proposition 2.5, without the axiom of choice and assuming only that F covers pullbacks, but by replacing plain AM-bisimulations by toposal AM-bisimulations.

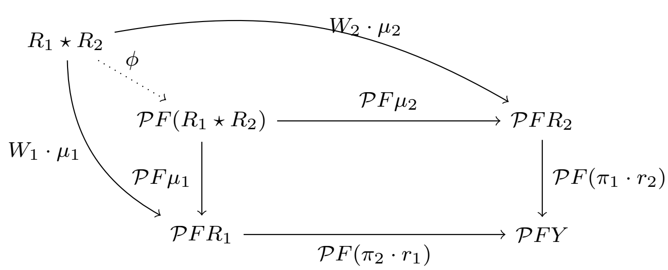
Proof. We could directly conclude this from Propositions 2.16 and 4.2, but let us show that the proof of Proposition 2.5 can be adapted more easily in the case when F preserves weak pullbacks.

The only thing to prove is that toposal bisimulations are closed under composition, without using the regular axiom of choice. The proof starts the same way as Proposition 2.5. We have two witnesses $W_i : R_i \longrightarrow \mathcal{P}FR_i$ and we want to construct a witness $W : R_1;R_2 \longrightarrow \mathcal{P}F(R_1;R_2)$. Since F and $\mathcal{P}$ preserve weak pullbacks and by definition of composition, we have the following weak pullback and (epi, mono)-factorisation:





By the universal property of weak pullbacks, we have $\phi : R_1 \star R_2 \longrightarrow \mathcal{P}F(R_1 \star R_2)$, such that



Now $W = \mathcal{P}F(e_{r_1;r_2}) \cdot \mu_{F(R_1 \star R_2)} \cdot \mathcal{P}\phi \cdot (e_{r_1;r_2})^\dagger$ is the expected witness:

$$
\begin{array}{lll}
& \mathcal{P}F\pi_1 \cdot \mathcal{P}F(r_1;r_2) \cdot W & \\
= & \mathcal{P}F(\pi_1 \cdot r_1; r_2 \cdot e_{r_1;r_2}) \cdot \mu_{F(R_1\star R_2)} \cdot \mathcal{P}\phi \cdot (e_{r_1;r_2})^\dagger & \text{(definition of } W\text{)} \\
= & \mu_{FX} \cdot \mathcal{P}\mathcal{P}F(\pi_1 \cdot r_1; r_2 \cdot e_{r_1;r_2}) \cdot \mathcal{P}\phi \cdot (e_{r_1;r_2})^\dagger & \text{(naturality of } \mu\text{)} \\
= & \mu_{FX} \cdot \mathcal{P}\mathcal{P}F(\pi_1 \cdot r_1 \cdot \mu_1) \cdot \mathcal{P}\phi \cdot (e_{r_1;r_2})^\dagger & \text{(definition of } r_1;r_2\text{)} \\
= & \mu_{FX} \cdot \mathcal{P}(\mathcal{P}F(\pi_1 \cdot r) \cdot W_1 \cdot \mu_1) \cdot (e_{r_1;r_2})^\dagger & \text{(definition of } \phi\text{)} \\
= & \mu_{FX} \cdot \mathcal{P}(\eta_{FX} \cdot \alpha \cdot \pi_1 \cdot r_1 \cdot \mu_1) \cdot (e_{r_1;r_2})^\dagger & \text{(assumption on } W_1\text{)} \\
= & \mathcal{P}(\alpha \cdot \pi_1 \cdot r_1 \cdot \mu_1) \cdot (e_{r_1;r_2})^\dagger & \text{(unit coherence axiom)} \\
= & \mathcal{P}(\alpha \cdot \pi_1 \cdot r_1; r_2 \cdot e_{r_1;r_2}) \cdot (e_{r_1;r_2})^\dagger & \text{(definition of } r_1;r_2\text{)} \\
= & \mathcal{P}(\alpha \cdot \pi_1 \cdot r_1; r_2) \cdot \eta_{R_1;R_2} & (e_{r_1;r_2} \text{ is epi)} \\
= & \eta_{FX} \cdot \alpha \cdot \pi_1 \cdot r_1; r_2 & \text{(naturality of } \eta\text{)}
\end{array}
$$

Similarly, we can prove that $\mathcal{P}F\pi_2 \cdot \mathcal{P}F(r_1;r_2) \cdot W = \eta_{FZ} \cdot \gamma \cdot \pi_2 \cdot r_1; r_2$, which completes the proof. □

Obviously, the category of maps of the I-category of toposal bisimulations is then not isomorphic to $\mathbf{Coal}(F)$, but to the category of F-coalgebras with $\mathcal{P}F$-coalgebra homomorphisms between them. Then tabularity can be formulated as follows:

Proposition 4.5. *If U is a toposal bisimulation from the F-coalgebra α to the F-coalgebra β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U, then there is a $\mathcal{P}F$-coalgebra structure γ on Z such that f is a $\mathcal{P}F$-coalgebra homomorphism from γ to $\eta_X \cdot \alpha$ and g is a $\mathcal{P}F$-coalgebra homomorphism from γ to $\eta_Y \cdot \beta$.*

Corollary 4.6. *Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. the following two statements are equivalent:*

(1) *There is a toposal bisimulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ such that $r \cdot c = \langle p, q\rangle$ if and only if*
(2) *there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, a $\mathcal{P}F$-coalgebra structure γ on Z, and a point $w : * \longrightarrow Z$ such that f is a $\mathcal{P}F$-coalgebra homomorphism from γ to $\eta_X \cdot \alpha$, g from γ to $\eta_Y \cdot \beta$, $f \cdot w = p$, and $g \cdot w = q$.*

5. From Bisimulations to Simulations

In this section, we would like to extend the analysis of the previous sections to deal with *simulations*. Classically, simulations for coalgebras require a notion of order on morphisms of the form $X \longrightarrow FY$, to allow one to define that there is fewer transitions coming out of a state than another. This allows one to easily modify the definition of AM-bisimulations to obtain *AM-simulations*. We will show that toposal bisimulations can also be extended to simulations in a nice way to mitigate these issues. The only reason we chose to stay in a topos and not in a general regular category is because theorems have a nicer formulation there, but most of the discussion here can be done in a regular category.

5.1. Order-Structure on Functors, and Lax Coalgebra Homomorphisms.

We want to be able to compare two morphisms of the form $X \longrightarrow FY$. So, assuming a preorder $\leq$ on each Hom-set $\mathcal{C}(X, FY)$, we can define *lax homomorphisms of coalgebras*, as follows:

Definition 5.1. A lax homomorphism of coalgebras from $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$ is a morphism $f : X \longrightarrow Y$ of $\mathcal{C}$ such that the following diagram laxly commutes,

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\alpha}\downarrow & \leq & \downarrow{\scriptstyle\beta} \\ FX & \xrightarrow[\quad Ff \quad]{} & FY \end{array}$$

meaning that $Ff \cdot \alpha \leq \beta \cdot f$ in $\mathcal{C}(X, FY)$.

Unfortunately, coalgebras and lax homomorphisms of coalgebras do not form a category in general, and some axioms are required for the interaction of $\leq$ with the composition.

Definition 5.2. A *good order structure on* F is a preorder $\leq$ on each Hom-set of the form $\mathcal{C}(X, FY)$ such that:

(1) if $\alpha \leq \beta$ in $\mathcal{C}(X, FY)$, $f : X' \longrightarrow X$, and $g : Y \longrightarrow Y'$, then $Fg \cdot \alpha \cdot f \leq Fg \cdot \beta \cdot f$ in $\mathcal{C}(X', FY')$;
(2) if $h : X \longrightarrow FZ$, $k : X \longrightarrow FY$, $g : Y \longrightarrow Z$, and $h \leq Fg \cdot k$ in $\mathcal{C}(X, FZ)$, then there is $k' : X \longrightarrow FY$ such that $k' \leq k$ in $\mathcal{C}(X, FY)$ and $h = Fg \cdot k'$.

Lemma 5.3. *When $\leq$ is a good order structure on F, then coalgebras and lax homomorphisms of coalgebras form a category, denoted by* $\mathbf{Coal}_{\boldsymbol{lax}}(F)$*.*

Example 5.4. When F is the functor modelling non-deterministic labelled transition systems and $\leq$ is given by point-wise inclusion, lax homomorphisms of coalgebras are exactly morphisms of systems in the sense of [JNW96]. Those morphisms are intuitively morphisms whose graphs are simulations. More generally, we will see that lax homomorphisms are simulation maps. In this picture, it can be proved in some cases that coalgebra homomorphisms are precisely open maps, that is, lax homomorphisms with some lifting properties (see [WDKH19], from which the notion of good order is adapted).

5.2. **AM-Simulations.**

Definition 5.5. We say that a relation is an *AM-simulation* from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow FR$ such that:

$$\begin{array}{ccccc} & & X \times Y & \xrightarrow{\quad \alpha \times \beta \quad} & F(X) \times F(Y) \\ & \nearrow^{r} & & & \uparrow{\scriptstyle \langle F\pi_1, F\pi_2 \rangle} \\ R & & & \leq \times \geq & \\ & \searrow_{W} & & & \\ & & FR & \xrightarrow[\quad Fr \quad]{} & F(X \times Y) \end{array}$$

meaning that $\alpha \cdot \pi_1 \cdot r \leq F\pi_1 \cdot Fr \cdot W$ and $\beta \cdot \pi_2 \cdot r \geq F\pi_2 \cdot Fr \cdot W$.

The definition can be simplified:

Proposition 5.6. *When $\leq$ is a good order structure, it is equivalent to require that the left inequality is actually an equality $\alpha \cdot \pi_1 \cdot r = F\pi_1 \cdot Fr \cdot W$.*

Proof. We start with W such that

$$\alpha \cdot \pi_1 \cdot r \leq F\pi_1 \cdot Fr \cdot W \quad \text{and} \quad \beta \cdot \pi_2 \cdot r \geq F\pi_2 \cdot Fr \cdot W.$$

Use the second assumption of a good order structure with $h = \alpha \cdot \pi_1 \cdot r$, $g = \pi_1 \cdot r$ and $k = W$. We then obtain $W' \leq W$ with

$$\alpha \cdot \pi_1 \cdot r = F\pi_1 \cdot Fr \cdot W'.$$

Then since composition is monotone,

$$\beta \cdot \pi_2 \cdot r \geq F\pi_2 \cdot Fr \cdot W \geq F\pi_2 \cdot Fr \cdot W'. \qquad \square$$

Example 5.7. When $F : X \mapsto \mathcal{P}(\Sigma \times X)$, AM-simulations correspond to strong simulations. The left part of the commutativity means that for every $(x, y) \in R$ and $(a, x') \in \alpha(x)$, there is y' such that $(a, (x', y')) \in W(x, y)$. The right part then implies that necessarily $(a, y') \in \beta(y)$.

Much as in the case of AM-bisimulations, diagonals (and actually all AM-bisimulations) are AM-simulations, and AM-simulations are closed under composition only under some conditions. However, they are not closed under converse. These observations can be encompassed as follows:

Proposition 5.8. *When $\mathcal{C}$ has the regular axiom of choice and F preserves weak pullbacks, then the following is a locally ordered 2-category:*

- *objects are F-coalgebras,*
- *morphisms are AM-simulations,*
- *identitites, compositions, and $\sqsubseteq$ are given by $\mathbf{Rel}(\mathcal{C})$.*

We denote this category by $\mathbf{Sim}(F)$.

We can formalise the relationship between lax coalgebra homomorphisms and simulation maps:

Theorem 5.9. *Maps in $\mathbf{Rel}(\mathcal{C})$ that are AM-simulations are precisely lax homomorphisms of coalgebras.*

Note that this theorem cannot have a form as nice as Theorem 2.7 because AM-simulations are not closed under converse, and the right adjoint of a map has to be its converse. At this point, we can also describe the tabulations of AM-simulations:

Proposition 5.10. *If U is an AM-simulation from α to β, and if $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$ is a tabulation of U then, there is a coalgebra structure γ on Z such that f is a coalgebra homomorphism from γ to α and g is a lax coalgebra homomorphism from γ to β.*

Corollary 5.11. *Assume $\mathcal{C}$ has the regular axiom of choice. Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. The following two statements are equivalent:*

(1) *There is an AM-simulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ with $r \cdot c = \langle p, q \rangle$.*
(2) *There is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, an F-coalgebra structure γ on Z such that f is a coalgebra homomorphism from γ to α and g is a lax coalgebra homomorphism from γ to β, and a point $w : * \longrightarrow Z$ such that $f \cdot w = p$ and $g \cdot w = q$.*

This formalises some observations that simulations are spans of a bisimulations map and a simulation map (see [Tab04] for examples of this fact in the context of open maps).

5.3. **Extending the Order-Structure.** In Section 5.1, we started by assuming a relation $\leq$ on the Hom-sets of the form $\mathcal{C}(X, FY)$ satisfying some properties. This good order structure was necessary to prove the properties of Section 5.2. In the coming section, we will pass again from plain to toposal, by considering F-coalgebras as $\mathcal{P}F$-coalgebras. It is then necessary to extend good order structures on F to good order structures on $\mathcal{P}F$.

Assume a relation $\leq$ is given on all Hom-sets of the form $\mathcal{C}(X, FY)$. We define $\leq_\mathcal{P}$ on $\mathcal{C}(X, \mathcal{P}FY)$ as follows. A morphism $f : X \longrightarrow \mathcal{P}FY$ uniquely (up to isos) corresponds to a mono of the form $m_f : U_f \longrightarrow FY \times X$ by definition of $\mathcal{P}$. Then, given two morphisms $f, g : X \longrightarrow \mathcal{P}FY$, $f \leq_\mathcal{P} g$ if there exist a morphism $u : Z \longrightarrow U_g$ and an epi $e : Z \twoheadrightarrow U_f$ such that: $\pi_1 \cdot m_f \cdot e \leq \pi_1 \cdot m_g \cdot u$ and $\pi_2 \cdot m_f \cdot e = \pi_2 \cdot m_g \cdot u$.

Example 5.12. The order $\leq_\mathcal{P}$ might appear complicated, but it can be interpreted easily in **Set**, especially when the order structure on $\mathcal{C}(X, FY)$ is a point-wise order, assuming that FY itself is preordered. Indeed, given two functions $f, g : X \longrightarrow \mathcal{P}FY$, $f \leq_\mathcal{P} g$ if and only if for every $x \in X$, and every $a \in f(x) \subseteq FY$ there is $b \in g(x)$ such that $a \leq b$ in $F(Y)$.

To make it consistent with the previous section, we show that this preserves goodness:

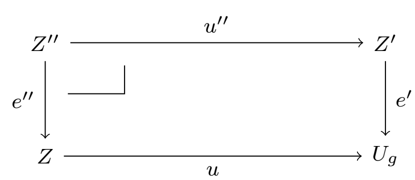
Proposition 5.13. *$\leq_\mathcal{P}$ is a good order structure if $\leq$ is.*

Proof. Let us prove that $\leq_\mathcal{P}$ is a good order structure on $\mathcal{P}F$.

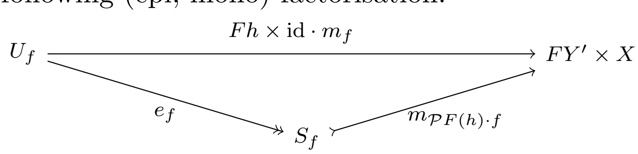
- **$\leq_\mathcal{P}$ is a preorder.**
  - **reflexivity.** To prove $f \leq_\mathcal{P} f$, take $u = e = \mathrm{id}$.
  - **transitivity**: Assume $f \leq_\mathcal{P} g \leq_\mathcal{P} h$. So there are a morphism $u : Z \longrightarrow U_g$, and an epi $e : Z \twoheadrightarrow U_f$ such that:
    - $\pi_1 \cdot m_f \cdot e \leq \pi_1 \cdot m_g \cdot u$,
    - $\pi_2 \cdot m_f \cdot e = \pi_2 \cdot m_g \cdot u$.

    and there a morphism $u' : Z' \longrightarrow U_h$ and an epi $e' : Z' \twoheadrightarrow U_g$ such that:
    - $\pi_1 \cdot m_g \cdot e' \leq \pi_1 \cdot m_h \cdot u'$,
    - $\pi_2 \cdot m_g \cdot e' = \pi_2 \cdot m_h \cdot u'$.

    Form the following pullback:

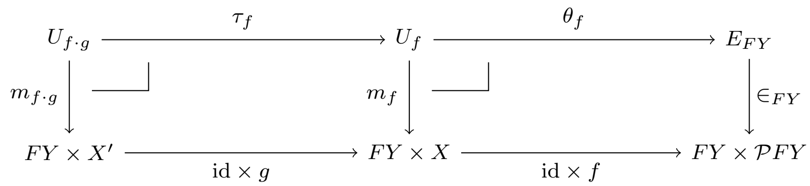
    

    Since in a topos, epis are closed under pullbacks, e'' is an epi, and so is $e \cdot e''$. So then, Z'', $u' \cdot u''$ and $e \cdot e''$ witness the fact that $f \leq_\mathcal{P} h$.
- **Composition is monotone.** Assume $f \leq_\mathcal{P} f' : X \longrightarrow \mathcal{P}F(Y)$, with witnesses Z, u and e
  - **composition to the left**: Assume $h : Y \longrightarrow Y'$. Then $\mathcal{P}F(h) \cdot f$ corresponds to the mono part of the following (epi, mono)-factorisation:

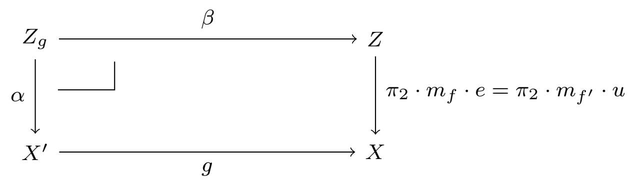
    

    Same for $\mathcal{P}F(h) \cdot f'$. Then Z, $e_{f'} \cdot u$ and $e_f \cdot e$ is a witness of the fact that $\mathcal{P}F(h) \cdot f \leq_\mathcal{P} \mathcal{P}F(h) \cdot f'$.

– **composition to the right**: Assume given $g : X' \longrightarrow X$. Then $f \cdot g$ corresponds to the relation represented by $m_{f \cdot g}$, given by the following composition of pullbacks:



for some τ_f, θ_f. Same for $m_{f' \cdot g}$. Form the following pullback:



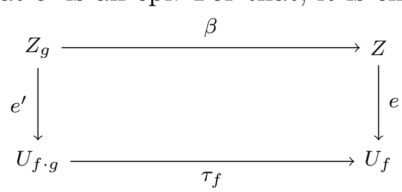
So then, we have that $m_f \cdot e \cdot \beta = \mathrm{id} \times g \cdot \langle \pi_1 \cdot m_f \cdot e \cdot \beta, \alpha\rangle$, and by the universal property of $U_{f \cdot g}$, there is a unique morphism $e' : Z_g \longrightarrow U_{f \cdot g}$ such that

$$\tau_f \cdot e' = e \cdot \beta \quad \text{and} \quad m_{f\cdot g} \cdot e' = \langle \pi_1 \cdot m_f \cdot e \cdot \beta, \alpha\rangle.$$

Similarly, there is $u' : Z_g \longrightarrow U_{f' \cdot g}$ such that

$$\tau_{f'} \cdot u' = u \cdot \beta \quad \text{and} \quad m_{f\cdot g} \cdot u' = \langle \pi_1 \cdot m_{f'} \cdot u \cdot \beta, \alpha\rangle.$$

The only interesting part in proving that Z_g, u' and e' is a witness of the fact that $f \cdot g \leq_{\mathcal{P}} f' \cdot g$ is the fact that e' is an epi. For that, it is enough to observe that:

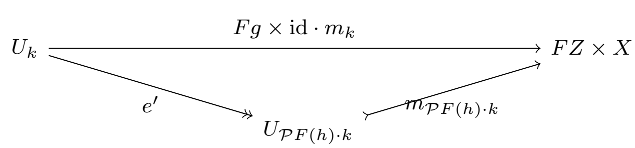


is a pullback square, and to use the fact that in a topos, epis are closed under pullback.

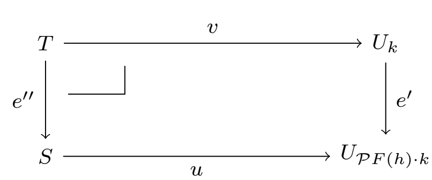
- **Last axiom of good order structure.** Assume $h \leq_{\mathcal{P}} \mathcal{P}F(g) \cdot k$, with $h : X \longrightarrow \mathcal{P}FZ$, $k : X \longrightarrow \mathcal{P}FY$ and $g : Y \longrightarrow Z$. So we have a morphism $u : S \longrightarrow U_{\mathcal{P}F(h)\cdot k}$ and an epi $e : S \longrightarrow U_h$ such that

$$\pi_1 \cdot m_h \cdot e \leq \pi_1 \cdot m_{\mathcal{P}F(h)\cdot k} \cdot u \quad \text{and} \quad \pi_2 \cdot m_h \cdot e = \pi_2 \cdot m_{\mathcal{P}F(h)\cdot k} \cdot u.$$

$m_{\mathcal{P}F(h)\cdot k}$ is obtained by the following factorisation:



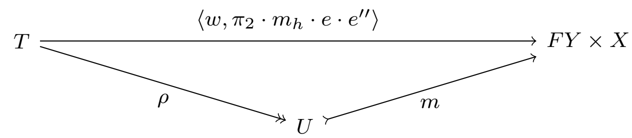
Form the following pullback:



Since epis are closed under pullbacks in a topos, e'' is an epi, and we have

$$\pi_1 \cdot m_h \cdot e \cdot e'' \leq Fg \cdot \pi_1 \cdot m_k \cdot v.$$

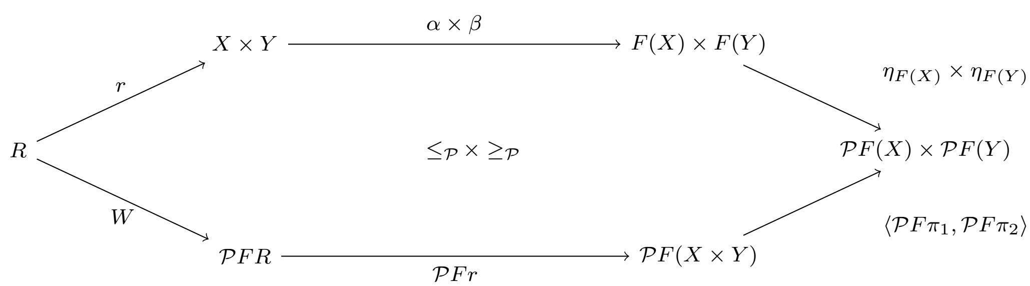
By using the fact that $\leq$ is a good order structure, we obtain $w \leq \pi_1 \cdot m_k \cdot v$, such that $Fg \cdot w = \pi_1 \cdot m_h \cdot e \cdot e''$, and then $m_h \cdot e \cdot e' = Fg \times \mathrm{id} \cdot \langle w, \pi_2 \cdot m_h \cdot e \cdot e'' \rangle$. Consider the following factorisation:



and define k' as ξ_m. Then m_h, which is the monic part of $m_h \cdot \rho \cdot \rho'$, is also the monic part of $Fg \times \mathrm{id} \cdot m$. But $m_{\mathcal{P}Fg \cdot k'}$ is also the monic part of $Fg \times \mathrm{id} \cdot m$, so by unicity of the (epi, mono)-factorisation, $m_{\mathcal{P}Fg \cdot k'} \equiv m_h$, which means that $\mathcal{P}Fg \cdot k' = h$. Furthermore, T, v and ρ is a witness of $k' \leq_{\mathcal{P}} k$. □

5.4. **Toposal AM-Simulations.** With all those ingredients, we can easily deduce the right notion of *AM toposal-simulations*:

Definition 5.14. We say that a relation is a *toposal AM-simulation* from the coalgebra $\alpha : X \longrightarrow FX$ to $\beta : Y \longrightarrow FY$, if for any mono $r : R \rightarrowtail X \times Y$ representing it, there is a morphism $W : R \longrightarrow \mathcal{P}FR$ such that:

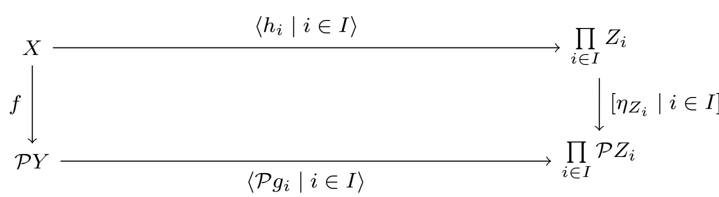


Plain and toposal AM-simulations also coincide under the axiom of choice:

Proposition 5.15. *Assume that $\mathcal{C}$ has the regular axiom of choice. Then for every relation U from X to Y, every coalgebra $\alpha : X \longrightarrow FX$ and $\beta : Y \longrightarrow FY$, U is an AM-simulation from α to β if and only if it is a toposal AM-simulation between them.*

The proof of this Proposition relies on the following lemma, relating the regular axiom of choice and picking elements in a power-object:

Lemma 5.16. *Assume that every epi is split and assume given F with a good order structure $\leq$. Assume also given a square:*



such that I is finite and:

- *for all $i \in I$, either:*
  - $\mathcal{P}g_i \cdot f = \eta_{Z_i} \cdot h_i$*, or*
  - g_i *is of the form* Fg'_i *and* $\mathcal{P}g_i \cdot f \leq_{\mathcal{P}} \eta_{Z_i} \cdot h_i$*,*
- *there is $i_0 \in I$ satisfying the first case of the first point.*

Then there is $f' : X \longrightarrow Y$ such that:

- *if $m : R \rightarrowtail Y \times X$ is the relation corresponding to f, then $\langle f', id\rangle \sqsubseteq m$,*
- *for all i satisfying the first case, $g_i \cdot f' = h_i$, and*
- *for all i satisfying the second case, $g_i \cdot f' \leq h_i$.*

Proof. The conclusion $\langle f', \mathrm{id}\rangle \sqsubseteq m$ means that we are looking for f' such that there is $u : X \longrightarrow R$ such that $\langle f', \mathrm{id}\rangle = m \cdot u$, that is:

$$f' = \pi_1 \cdot m \cdot u \quad \text{and} \quad \mathrm{id} = \pi_2 \cdot m \cdot u.$$

To obtain this u is then enough to prove that $\pi_2 \cdot m$ is an epi and using the regular axiom of choice. Observe that we have the following:

$$\begin{array}{rcll}
\xi_{\langle h_i, \mathrm{id}\rangle} & = & \eta_{Z_i} \cdot h_i & \text{(definition of } \xi_{\langle h_i, \mathrm{id}\rangle}) \\
& \simeq_i & \mathcal{P}g_i \cdot f & \text{(assumption)} \\
& = & \mu_{Z_i} \cdot \mathcal{P}\eta_{Z_i} \cdot \mathcal{P}g_i \cdot f & \text{(unit coherence axiom)} \\
& = & \mu_{Z_i} \cdot \mathcal{P}\eta_{Z_i} \cdot g_i \cdot f & \text{(Pow is a functor)} \\
& = & \mu_{Z_i} \cdot \mathcal{P}\xi_{\langle g_i, \mathrm{id}\rangle} \cdot \xi_m & \text{(definition of } m \text{ and } \xi_{\langle g_i, \mathrm{id}\rangle}) \\
& = & \xi_{\langle g_i, \mathrm{id}\rangle ; m} & \text{(Proposition 3.4)}
\end{array}$$

where $\simeq_i$ is either $=$ if i satisfies the first case, or $\geq_{\mathcal{P}}$ otherwise.

This means that for all i satisfying the first case (and there is at least one), $\langle h_i, \mathrm{id}\rangle \equiv \langle g_i, \mathrm{id}\rangle; m$, which means that there is an iso v_i such that $\langle g_i, \mathrm{id}\rangle; m = \langle h_i, \mathrm{id}\rangle \cdot v_i$. Unfolding the definition of the composition, there is an epi e_i such that:

$$\langle g_i, \mathrm{id}\rangle; m \cdot e_i = \langle g \cdot \pi_1 \cdot m, \pi_2 \cdot r\rangle.$$

In total, $\pi_2 \cdot m = v_i \cdot e_i$ which is an epi.

It then remains to prove that $f' = \pi_1 \cdot m \cdot u$ satisfies all the statements in the conclusion. The first one is by construction. Now, for i satisfying the first case,

$$\begin{array}{rcll}
g_i \cdot \pi_1 \cdot m \cdot u & = & \pi_1 \cdot (\langle g_i, \mathrm{id}\rangle; m) \cdot e_i \cdot u & \text{(definition of } \langle g_i, \mathrm{id}\rangle; m) \\
& = & \pi_1 \cdot \langle h_i, \mathrm{id}\rangle \cdot v_i \cdot e_i \cdot u & \text{(definition of } v_i) \\
& = & h_i \cdot v_i \cdot e_i \cdot u & \text{(computation on products)} \\
& = & h_i \cdot \pi_2 \cdot m \cdot u & \text{(see previously)} \\
& = & h_i & \text{(definition of } u)
\end{array}$$

Now, for i in the second case, we have proved that $\xi_{\langle g_i, \mathrm{id}\rangle ; m} \leq_{\mathcal{P}} \xi_{\langle h_i, \mathrm{id}\rangle}$. By definition, this means that there is an epi ρ_i and a morphism u_i such that:

$$\pi_1 \cdot \langle g_i, \mathrm{id}\rangle; m \cdot \rho \leq h_i \cdot u_i \quad \text{and} \quad \pi_2 \cdot \langle g_i, \mathrm{id}\rangle; m \cdot \rho = u_i.$$

Since every is split and $\leq$ is a good order structure, this implies that:

$$\pi_1 \cdot \langle g_i, \mathrm{id}\rangle; m \leq h_i \cdot \pi_2 \cdot \langle g_i, \mathrm{id}\rangle; m,$$

from which it is easy to deduce that $g_i \cdot \pi_1 \cdot m \cdot u \leq h_i$. □

Remark 5.17. Actually, the converse of the previous lemma also holds. Assume given an epi $e : X \twoheadrightarrow Y$. Since e is an epi, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\quad !_Y \quad} & * \\ {\scriptstyle e^\dagger}\downarrow & & \downarrow{\scriptstyle \eta_*} \\ \mathcal{P}X & \xrightarrow[\quad \mathcal{P}! \quad]{} & \mathcal{P}* \end{array}$$

Then using the conclusion of the previous lemma, we obtain $s : Y \longrightarrow X$ such that $\langle s, \mathrm{id}\rangle \sqsubseteq \langle \mathrm{id}, e\rangle$. This means that there is a mono u such that $s = u$ and $\mathrm{id} = e \cdot u$, and e is split.

An additional argument is needed, namely:

Lemma 5.18. *If $f \leq g$ then $\eta \cdot f \leq_{\mathcal{P}} \eta \cdot g$.*

Proof. $\eta \cdot f$ (resp. $\eta \cdot g$) corresponds to the mono $\langle f, \mathrm{id}\rangle$ (resp. $\langle g, \mathrm{id}\rangle$). So we have:

$$\pi_1 \cdot \langle f, \mathrm{id}\rangle = f \leq g = \pi_1 \cdot \langle g, \mathrm{id}\rangle$$

and

$$\pi_2 \cdot \langle f, \mathrm{id}\rangle = \mathrm{id} = \pi_2 \cdot \langle g, \mathrm{id}\rangle,$$

which means that $\eta \cdot f \leq_{\mathcal{P}} \eta \cdot g$. □

Proof of Proposition 5.15. Let us prove both implications:

- If U is AM-simulation and $W : R \longrightarrow FR$ is a witness, then $\eta_{FR} \cdot W : R \longrightarrow FR$ is a toposal witness by naturality of η and the previous Lemma.
- If U is a toposal AM-simulation, then we obtain a witness to prove it is a plain AM-simulation directly by Lemma 5.16. □

Finally, we can prove the closure under composition and the characterisation with spans without the axiom of choice:

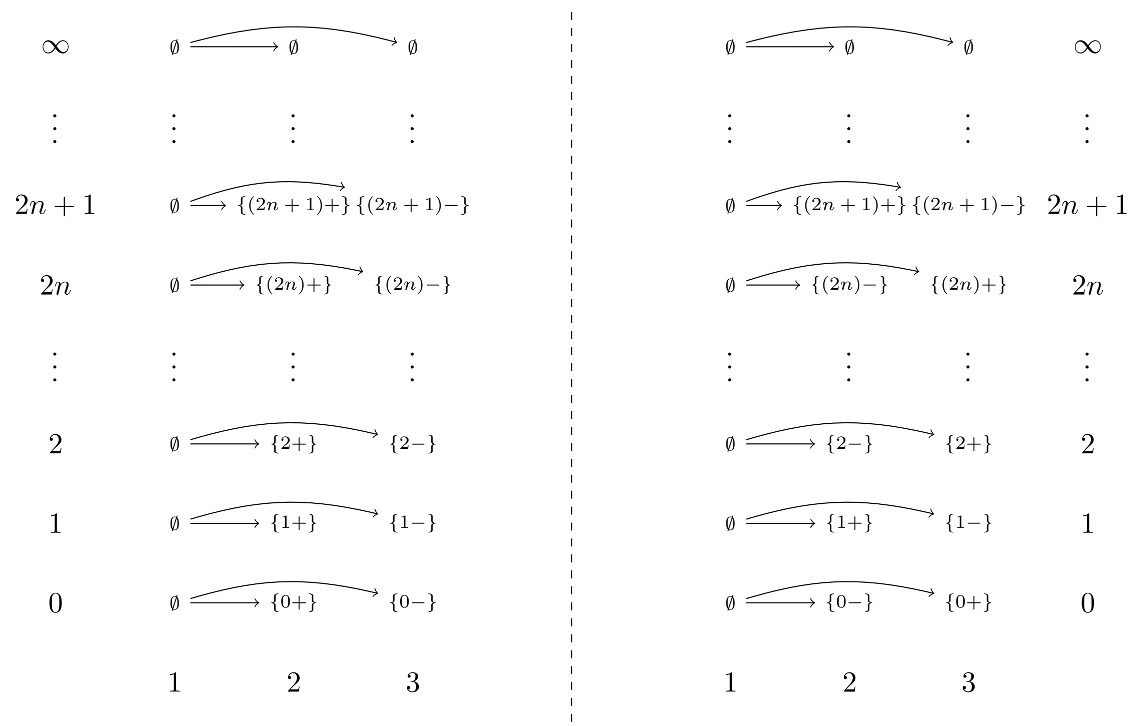
Proposition 5.19. *Proposition 5.8 holds without regular axiom of choice when replacing AM-simulations by toposal AM-simulations.*

Theorem 5.20. *Assume given two coalgebras $\alpha : X \longrightarrow F(X)$ and $\beta : Y \longrightarrow F(Y)$, and two points $p : * \longrightarrow X$ and $q : * \longrightarrow Y$. There is a toposal AM-simulation $r : R \rightarrowtail X \times Y$ from α to β, and a point $c : * \longrightarrow R$ such that $r \cdot c = \langle p, q\rangle$ if and only if there is a span $X \xleftarrow{f} Z \xrightarrow{g} Y$, a $\mathcal{P}F$-coalgebra structure γ on Z such that f is a $\mathcal{P}F$-coalgebra homomorphism from γ to $\eta_X \cdot \alpha$ and g a lax $\mathcal{P}F$-coalgebra homomorphism from γ to $\eta_Y \cdot \beta$, and a point $w : * \longrightarrow Z$ such that $f \cdot w = p$ and $g \cdot w = q$.*

6. Examples

In this section, let us develop some examples in different regular categories.

6.1. **Vietoris Bisimulations.** In [BFV10], the authors study bisimulations for the Vietoris functor, which maps a topological space to its set of closed subspaces equipped with a suitable topology, in the category **Stone** of Stone spaces and continuous functions. More specifically, they show that so-called descriptive models coincide with coalgebras of the form $X \to \mathcal{V}(X) \times A$ where $\mathcal{V}$ is the Vietoris functor and A is some fixed Stone space (such as a finite set of sets of propositions equipped with the discrete topology). They are interested in describing relation liftings (similar to those defining HJ-bisimulations) that coincide with behavioural equivalences. They actually proved that in this case AM-bisimilarity does not coincide with behavioural equivalence. The main reason for discrepancy is that the Vietoris functor does not preserves weak pullbacks. In [Sta11], Staton proved that the Vietoris functor is a so-called $\mathcal{S}$-powerset functor, and that it, in particular, covers pullbacks. Combining this with the (well-known) fact that the category of Stone spaces is regular and has pushouts, Theorem 2.18 holds in this case, and all three notions-regular AM-bisimulations, HJ-bisimulations, and behavioural equivalences-coincide.



We now develop the counter-examples described in [BFV10]. Consider the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, which is obtained as the Alexandroff-compactification of $\mathbb{N}$ equipped with the discrete topology. Specifically, the open sets of $\overline{\mathbb{N}}$ are $\{U \subseteq \mathbb{N}\} \cup \{U \cup \{\infty\} \mid U \subseteq \mathbb{N} \wedge \exists n \in U. \forall m \geq n. m \in u\}$. Denote $\overline{\mathbb{N}} \oplus \overline{\mathbb{N}} \oplus \overline{\mathbb{N}}$, the coproduct of three copies of $\overline{\mathbb{N}}$, by $3\overline{\mathbb{N}}$. Let us also consider $A = \mathcal{P}(\mathbb{N} \times \{+, -\})$ (with the product topology on $2^{\mathbb{N}\times\{+,-\}}$, which is compact by Tychonoff's theorem). Define the continuous function $\tau : 3\overline{\mathbb{N}} \longrightarrow \mathcal{V}(3\overline{\mathbb{N}})$ as follows: $\tau(i_1) = \{i_2, i_3\}$ and $\tau(i_2) = \tau(i_3) = \emptyset$, where i_j denotes the j-th copy of $i \in \overline{\mathbb{N}}$. Define two continuous functions $\lambda, \lambda' : 3\overline{\mathbb{N}} \longrightarrow A$ $\lambda(i_1) = \lambda'(i_1) = \emptyset$ for all $i \in \overline{\mathbb{N}}$; $\lambda(\infty_j) = \lambda'(\infty_j) = \emptyset$ for $j \in \{2, 3\}$; $\lambda(i_2) = \lambda'(i_2) = \{i+\}$, $\lambda(i_3) = \lambda'(i_3) = \{i-\}$, for i odd; $\lambda(i_2) = \lambda'(i_3) = \{i+\}$, $\lambda(i_2) = \lambda'(i_3) = \{i-\}$ for i even. Altogether, this defines two coalgebras $\alpha = \langle \tau, \lambda \rangle$ and $\beta = \langle \tau, \lambda' \rangle$.

In [BFV10], they proved that the following relation (for Stone spaces, relations coincide with closed subspaces of a product):

$$R = \{(i_1, i_1) \mid i \in \overline{\mathbb{N}}\} \cup \{(i_2, i_2), (i_3, i_3) \mid i \in \mathbb{N} \text{ odd}\} \cup \{(i_2, i_3), (i_3, i_2) \mid i \in \mathbb{N} \text{ even}\} \\ \cup \{(\infty_j, \infty_k) \mid j, k \in \{2, 3\}\}$$

is a Vietoris bisimulation but not an AM-bisimulation. We can reformulate this as:

Theorem 6.1. *R is a regular AM-bisimulation but not an AM-bisimulation.*

For the second part of this statement, this means that there is no continuous function $W : R \longrightarrow \mathcal{V}(R) \times A$ satisfying the requirement of an AM-bisimulation. However, there is a relation $W \subseteq (\mathcal{V}(R) \times A) \times R$ that satisfies the requirement of a regular AM-bisimulation as:

$$\begin{aligned} W = \{&((\{(i_2, i_2), (i_3, i_3)\}, \emptyset), (i_1, i_1)) \mid i \in \mathbb{N} \text{ odd}\} \\ &\cup \{((\{(i_2, i_3), (i_3, i_2)\}, \emptyset), (i_1, i_1)) \mid i \in \mathbb{N} \text{ even}\} \\ &\cup \{((\{(\infty_2, \infty_2), (\infty_3, \infty_3)\}, \emptyset), (\infty_1, \infty_1)), ((\{(\infty_2, \infty_3), (\infty_3, \infty_2)\}, \emptyset), (\infty_1, \infty_1))\} \\ &\cup \{((\emptyset, \lambda(i_j)), (i_j, i_k)) \mid i \in \overline{\mathbb{N}} \wedge (i_j, i_k) \in R\} \end{aligned}$$

The interesting part is that (∞_1, ∞_1) is related to two elements, and that if one of them is removed, then W is not closed anymore, and so not a relation in **Stone**. This explains why this relation cannot be restricted to the graph of a continuous function.

6.2. **Toposes for Name-Passing.** In [Sta11], the author studies models of name-passing and their bisimulations. Three toposes and functors are presented to model different parts of the theory. The first topos is the category of name substitution, which is the category of presheaves over non-empty finite subsets of a fixed countable set, together with all functions between them. It comes with a functor combining non-determinism and name-binding. This functor satisfies strong properties: in particular, AM-bisimulations coincide with HJ-bisimulations, and the largest AM-bisimulation coincide with the largest behavioural equivalence. This framework is already nice as AM-bisimulations describe precisely open bisimulations [San96].

The second topos is a refinement of the first one, as the category of functors over all finite subsets of the given countable set, together with injections. The proposed functor in this case is not as nice: it does not preserve weak pullbacks, and AM-bisimulations no longer coincide with HJ-bisimulations anymore. However, it is sufficiently well-behaved in our theory: it covers pullbacks, and the category is a topos, thus regular and with pushouts. Consequently, HJ-bisimulations coincide with regular AM-bisimulations, and their existence coincides with the existence of a behavioural equivalence.

For this topos, it is noted in [Sta11] that if a relation is a HJ-bisimulation (so a regular/toposal AM-bisimulation), then its $\neg\neg$-completion is an AM-bisimulation. This means, in particular, that this framework for name-passing behaves much more nicely when restricted to $\neg\neg$-sheaves. One main reason for this is that the sheaf topos for the $\neg\neg$-topology satisfies the axiom of choice when the base topos is a presheaf topos over a poset [MM92], which is the case here.

6.3. **Weighted Linear Systems.** In [BBB+12], the authors study linear weighted systems, that is, coalgebras for the endofunctor $X \mapsto K \times X^A$ on $K\mathbf{Vect}$, in the category of K-vector spaces, with K a field, and A a set. The following discussion can also be carried out in the category of modules over a ring. The category $K\mathbf{Vect}$ is abelian, and thus regular and with pushouts. The endofunctor in question actually preserves pullbacks, so the three notions of bisimilarity coincide by Theorem 2.18. In this paper, the focus is on linear bisimulations, which coincide with behavioural equivalence, and so to the other two notions of bisimilarities.

In perspective, usual weighted systems are described in the category **Set**, with the functor $X \mapsto A \Rightarrow K^{(X)}$ where $K^{(X)}$ is the set of functions from X to K that take finitely many non-zero values. In this context, this functor does not even cover pullbacks in general, and they actually prove that AM-bisimilarity (and so regular AM-bisimilarity since **Set** has the regular axiom of choice) does not coincide with behavioural equivalence.

7. Conclusion

This paper introduces some foundations for the theory of bisimulations and simulations in a general regular category, mitigating some known issues with Aczel-Mendler bisimulations. Relations and power-objects are the key ingredients in this mitigation: while the axiom of choice allows for picking some witnesses of bisimilarity, relations and power-objects enable us to collect them without the need to choose. This paves the way for studying such bisimulations in more exotic regular categories and toposes.

One direction for future work is to investigate regular AM-bisimulations for probabilistic systems, in comparison to what is done in [DEP02, DDLP06] for behavioural equivalences. The main challenge lies in identifying a suitable *regular* category of "probabilistic space" and a "probabilistic distribution functor" that *covers pullbacks*. For the first property, the work on Quasi-Borel spaces [HKSY17], which yields a quasi-topos, is of interest. For the second, one possible avenue is to explore categories of σ-frames (see for example [Sim12]), in which pullbacks do not coincide with those in the category of measurable spaces–a solution under investigation.

Another avenue would be to explore other general properties of bisimulations, for instance those related to the largest bisimulation or to up-to techniques [SM92]. These approaches require considering (finite or infinite) unions of bisimulations, and hence of relations, which necessitates working within coherent categories.

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