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MULTI-REPRESENTATION ASSOCIATED TO THE NUMBERING OF A SUBBASIS AND FORMAL INCLUSION RELATIONS

EMMANUEL RAUZY

Universität der Bundeswehr München, München, Germany

Abstract. We revisit Dieter Spreen's notion of a representation associated to a numbered basis equipped with a strong inclusion relation. We show that by relaxing his requirements, we obtain different classically considered representations as subcases, including representations considered by Grubba, Weihrauch and Schröder. We show that the use of an appropriate strong inclusion relation guarantees that the representation associated to a computable metric space seen as a topological space always coincides with the Cauchy representation. We also show how the use of a formal inclusion relation guarantees that when defining multi-representations on a set and on one of its subsets, the obtained multi-representations will be compatible, i.e. inclusion will be a computable map. The proposed definitions are also more robust under change of equivalent bases.

1. Introduction

In order to study computability in areas of mathematics where mathematician freely define very abstract objects, one has first to answer the question: how can a machine manipulate an abstract object?

Turing, in his seminal paper [Tur37], gave a first approach to this problem, and he was able to define computable functions of a computable real variable.

Kleene's general solution is known as *realizability*: to represent abstract objects by concrete descriptions, thanks to the use of semantic functions which give meaning to a priori inert symbols. A function between abstract objects is then called *computable* if it can be realized by a computable function on concrete objects.

In the Type Two theory of Effectivity (TTE), the considered set of concrete objects is the Baire space $\mathbb{N}^{\mathbb{N}}$, and the notion of computability is given by Turing machines that work with infinite tapes. The semantic functions that map elements of the Baire space to abstract objects are called *representations*, they were introduced by Kreitz and Weihrauch in [KW85]. This notion was extended by Schröder to *multi-representations* in his dissertation [Sch03].

One of the defining features of TTE is that the study of computable functions is always related to the study of continuity, because, on the Baire space, a function is continuous if and only if it is computable with respect to some oracle. Thanks to Schröder's generalization of Weihrauch's notion of *admissible representation* [Sch02], this phenomenon extends to other

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topological spaces. One can then investigate in parallel computability and continuity, and reductions between problems, or translations between representations, exist both in terms of computable functions and in terms of continuous functions.

A celebrated theorem of Schröder [Sch03] characterizes those topological spaces that admit admissible multi-representations as those whose sequentialization is the quotient of countably based spaces (qcb-spaces). On such a space, there is a unique admissible multi-representation, up to continuous translation.

However, when studying computability, representations are considered up to computable translation.

One could of course ask to distinguish amongst admissible representations those that are appropriate to study computability -and possibly call them "computably admissible".

However, because qcb-spaces can have continuously many auto-homeomorphisms, there is no hope of distinguishing a single equivalence class of representation as *the* correct one to study computability. Indeed, if $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is an admissible representation of a topological space X, which we know to be appropriate for studying computability, then for any auto-homeomorphism $\Theta : X \to X$ of X, $\Theta \circ \rho$ will be another representation of X which will have exactly the same properties as ρ. The representation $\Theta \circ \rho$ is computably equivalent to ρ exactly when Θ and its inverse are (ρ, ρ)-computable, and thus there can only be countably many of these representations that are computably equivalent.

In practice, there is often a single correct choice of a representation on a set, but this comes from the fact that we consider sets that have more intrinsic structure than just a topology. For example, any permutation of $\mathbb{N}$ is an auto-homeomorphism of $\mathbb{N}$ for the discrete topology, and thus there is no hope of fixing *the* correct representation of $\mathbb{N}$ using only its topology. However, if we ask that addition should be computable, or that the order relation should be decidable, etc, we may end up distinguishing a single representation as the appropriate one. Such questions are linked to the study of computable model theory.

A possible way to equip a topological space X with an additional structure that will permit to distinguish a single class of representations as "computably admissible", and that has been used in computable analysis since early work of Weihrauch [KW85, Wei87], is to consider a numbered basis $(\mathfrak{B}, \beta)$ associated to X, i.e. a countable basis $\mathfrak{B}$ for the topology of X together with a partial surjection $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$. Note that the crucial point here is that by fixing a numbering of a basis we are already choosing the desired notion of computability. Fixing an abstract basis, as a set and not as a numbered set, would not be sufficient for this purpose. This is a very natural way to proceed because, in practice, when working with explicit topological spaces, there is often an obvious numbering of a basis that stands out as the correct one to study computability.

In fact, one can easily remark that in many settings where authors have used numbered bases, one might as well have considered only numbered subbases and obtained the same results. Additionally, because it is important for us to be able to study computability on non-T_0 spaces (for instance to include at least all finite topological spaces in our field of study), we will allow the use of multi-representations.

This gives the setting of the present study: a topological space equipped with a numbered subbasis, on which we want to define a multi-representation.

However, even once a numbered subbasis has been fixed on a space, there still are several, all seemingly natural, but sometimes non-equivalent, ways to define a multi-representation associated to this numbered subbasis.

In the present article, we consider a family of representations introduced by Spreen in [Spr01], defined thanks to a *strong inclusion relation*.

In [Spr01], these representations are used in a context in which they are computably equivalent to the more commonly used “standard representation” of Weihrauch.

Our purpose here is to show that in more general settings, representations defined thanks to strong inclusion relations do not have to be equivalent to the standard representation of Weihrauch, and that they can in some cases be *better behaved* than this standard representation.

Throughout, we fix a set X, and denote by $(\mathfrak{B}, \beta)$ a numbered subbasis for X. This is simply a countable subset of $\mathcal{P}(X)$ equipped with a partial surjection $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$.

There are two main approaches that authors have used to define a multi-representation $\rho :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ associated to the numbered subbasis $(\mathfrak{B}, \beta)$.

In each case the ρ-name of a point x of X is a sequence $(u_n)_{n\in\mathbb{N}}$ of β-names of basic sets which form a neighborhood basis of x. But with this idea, there are two possible approaches:

- The sequence $(u_n)_{n\in\mathbb{N}}$ is asked to contain β-names for *sufficiently many basic sets* so as to define a neighborhood basis of x. This representation was first used by Weihrauch in [Wei87]. It is particularly important since it was used by Schröder to prove the characterization theorem of topological spaces that admit admissible multi-representations. See [Sch01, Sch02]. It also appears in [KP22, Example 2.2].
- Or the sequence $(u_n)_{n\in\mathbb{N}}$ is asked to contain *all the β-names for basic sets that contain x*. This was first used in [KW85] under the name *standard representation*. This is also the definition of Weihrauch and Grubba [WG09]. This is now the common approach: see [HR16, Sch21].

In Schröder’s dissertation [Sch03], both approaches are used, depending on whether the focus is solely continuity (first approach, see Section 3.1.2 in [Sch03], which deals with limit spaces and limit bases), or also computability (second approach, see Section 4.3.6 in [Sch03]).

We thus define two multi-representations associated to the numbered subbasis $(\mathcal{B}, \beta)$. They are denoted by $\rho_\beta^{\min} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ and $\rho_\beta^{\max} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$, and are defined by:

$$\rho_\beta^{\min}(f) \ni x \Leftrightarrow \begin{cases} \forall n \in \mathrm{Im}(f),\, n \in \mathrm{dom}(\beta)\,\&\, x \in \beta(n), \\ \forall B \in \mathfrak{B},\, x \in B \Rightarrow \exists n_1, ..., n_k \in \mathbb{N},\, \beta(f(n_1)) \cap ... \cap \beta(f(n_k)) \subseteq B; \end{cases}$$

$$\rho_\beta^{\max}(f) \ni x \iff \mathrm{Im}(f) = \{n \in \mathrm{dom}(\beta),\, x \in \beta(n)\}.$$

Note that there always exists a translation $\rho_\beta^{\max} \leq \rho_\beta^{\min}$, witnessed by the identity realizer on Baire space.

Both approaches highlighted above are particular cases of a more general definition based on a strong inclusion relation, corresponding respectively to the coarsest and finest reflexive strong inclusion relations.

The idea of replacing set inclusion by a formal relation which need not be extensional can be traced back to Schreiber and Weihrauch in [WS81]. It is now commonly used in the theory of domain representation [SHT08].

The use of a formal inclusion relation in relation with numbered bases was initiated by Dieter Spreen in [Spr98], in terms of *strong inclusion relation*s.

Definition 1.1 [Spr98, Definition 2.3]**.** Let $\mathfrak{B}$ be a subset of $P(X)$, and $\beta :\subseteq \mathbb{N} \to \mathfrak{B}$ a numbering of $\mathfrak{B}$. Let $\mathring{\subseteq}$ be a binary relation on $\mathrm{dom}(\beta)$. We say that $\mathring{\subseteq}$ is a *strong inclusion relation for* $(\mathfrak{B}, \beta)$ if the following hold:

(1) The relation $\mathring{\subseteq}$ is transitive;
(2) $\forall b_1, b_2 \in \mathrm{dom}(\beta)$, $b_1 \mathring{\subseteq} b_2 \implies \beta(b_1) \subseteq \beta(b_2)$.

The general definition of the multi-representation associated to a numbered subbasis based on a strong inclusion is the following:

- A sequence $(u_n)_{n\in\mathbb{N}}$ of β-names constitutes a ρ-name of x if it contains sufficiently many basic sets, but *sufficiently many with respect to the strong inclusion.*

More precisely, consider a numbered subbasis $(\mathfrak{B}, \beta)$.

We denote by $(\hat{\mathfrak{B}}, \hat{\beta})$ the numbered basis induced by $(\mathfrak{B}, \beta)$: $\hat{\mathfrak{B}}$ it the set of all finite intersections of elements of $\mathfrak{B}$, and $\hat{\beta}$ is the naturally associated numbering: the $\hat{\beta}$-name of an intersection $B_1 \cap ... \cap B_n$ encodes β-names for B_1, ...,B_n.

When the basis $(\hat{\mathfrak{B}}, \hat{\beta})$ is equipped with a strong inclusion relation $\mathring{\subseteq}$, we define a multi-representation $\rho_{\beta}^{\mathring{\subseteq}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ by:

$$\rho_{\beta}^{\mathring{\subseteq}}(f) \ni x \iff \begin{cases} \forall b_1 \in \mathrm{Im}(f),\, b_1 \in \mathrm{dom}(\hat{\beta}) \,\&\, x \in \hat{\beta}(b_1), \\ \forall b_1 \in \mathrm{dom}(\hat{\beta}),\, x \in \hat{\beta}(b_1) \implies \exists b_2 \in \mathrm{Im}(f),\, b_2 \mathring{\subseteq} b_1 \end{cases}.$$

Let X be a set equipped with a numbered subbasis $(\mathcal{B}, \beta)$, and suppose that the induced numbered basis $(\hat{\mathfrak{B}}, \hat{\beta})$ admits a strong inclusion relation $\mathring{\subseteq}$. When x is a point of X, a subset $A \subseteq \mathrm{dom}(\hat{\beta})$ is called a *strong neighborhood basis for* x if and only if

$$\forall b \in A,\, x \in \hat{\beta}(b);$$
$$\forall b_1 \in \mathrm{dom}(\hat{\beta}),\, x \in \hat{\beta}(b_1) \implies \exists b_2 \in A,\, b_2 \mathring{\subseteq} b_1.$$

Thus the $\rho_{\beta}^{\mathring{\subseteq}}$-name of a point x is a list of $\hat{\beta}$-names that forms a strong neighborhood basis of x.

Note that in general, not all points of X need to have a strong neighborhood basis with respect to a strong inclusion relation $\mathring{\subseteq}$. The formula given above for $\rho_{\beta}^{\mathring{\subseteq}}$ correctly defines a multi-representation if and only if every point admits a strong neighborhood basis, otherwise it is not surjective. Thus in the present article, we will always suppose that all points admit strong neighborhood bases. Two natural conditions are sufficient for all points to admit strong neighborhood bases:

- If $\mathring{\subseteq}$ is a reflexive relation. Reflexive strong inclusions induce equivalence relations, this is often useful, see [Rau23].
- If the basis is a *strong basis* in the sense of Spreen [Spr98]: if for every two $\hat{\beta}$-names a and b of basic sets that intersect, and any x in their intersection, there is a $\hat{\beta}$-name c of a set that contain x and for which $c \mathring{\subseteq} a$ and $c \mathring{\subseteq} b$.

The definition of $\rho_{\beta}^{\mathring{\subseteq}}$ generalizes the two previous definitions:

- When we take the strong inclusion to be the actual inclusion relation, i.e. $b_1 \mathring{\subseteq} b_2 \iff \hat{\beta}(b_1) \subseteq \hat{\beta}(b_2)$, $\rho_{\beta}^{\mathring{\subseteq}}$ is exactly $\rho_{\beta}^{\min}$. Note that this is the coarsest strong inclusion relation.
- When we take the strong inclusion to be equality, i.e. $b_1 \mathring{\subseteq} b_2 \iff b_1 = b_2$, $\rho_{\beta}^{\mathring{\subseteq}}$ is exactly $\rho_{\beta}^{\max}$. Note that equality is the finest reflexive strong inclusion relation.

Note that for every strong inclusion relation $\mathring{\subseteq}$ with respect to which all points admit strong neighborhood bases, we have $\rho_{\beta}^{\max} \le \rho_{\beta}^{\mathring{\subseteq}} \le \rho_{\beta}^{\min}$, the identity on Baire space being a realizer for both translations.

The central idea of the present article can be summarized as follows.

Let $(\mathcal{B}, \beta)$ be a numbered basis for a set X.

The correct definition of the representation associated to β will often involve giving more information about points than simply listing names for a neighborhood basis. For instance in a metric space, we want the name of a point x to give us access to open balls with small radii that contain a x -if x is isolated, a neighborhood basis could contain only balls that are explicitly given with big radii, and we would have to be able to guess that x is isolated to make use of this name.

There is a certain amount of "additional information" required to define the correct representation. It is precisely determined by an appropriate choice of a strong inclusion relation.

The solution devised by Kreitz and Weihrauch in [KW85] is to systematically provide the maximal amount of information that the basis can provide, giving *all* names of balls that contain a given point x. This corresponds to systematically using the finest reflexive strong inclusion relation.

Using a strong inclusion relation different from the finest one gives a precise quantification of the information that the basis should provide, offering a better understanding of the representation at hand. Again, in metric spaces, it is clear that the correct amount of information required to manipulate a point x is to have balls with arbitrarily small radii that contain x be given. Listing *all* the balls with rational radii that contain x obviously involves listing much more than what is actually necessary.

And the main problem that arises with the Kreitz-Weihrauch method, where the maximal amount of information is systematically given, is that as soon as one tries to use non computably enumerable bases, this amount of information becomes too important, and no point has a computable name. For instance, when trying to use as a basis of the topology of $\mathbb{R}$ the set of open intervals with computable reals as endpoints, the representation $\rho_\beta^{\max}$ has no computable point. But for every reasonable notion of "computable equivalence" of bases, the basis that uses rational intervals is equivalent to the one that uses intervals with computable reals as endpoints -just like classically the basis consisting of all open intervals is equivalent to the one where rational intervals are used.

Quantifying more precisely the amount of information a basis is supposed to provide, thanks to a strong inclusion relation, allows to avoid this caveat.

This article is organized around five main topics, which correspond respectively to Sections 3, 4, 5, 6 and 7.

Firstly, we show that in any situation, all multi-representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are admissible for the topology generated by the basis $\mathfrak{B}$. They can thus be used interchangeably when focusing on continuity.

Then, we discuss what happens when considering the multi-representations associated to a set X and to a subset A of X, and investigate whether we can guarantee that the embedding $A \hookrightarrow X$ will be computable.

We then focus on metric spaces, and on the problem of defining, thanks to a numbering of open balls, a representation that is equivalent to the Cauchy representation.

We then consider semi-decidable strong inclusions. We render explicit the benefit of using a c.e. strong inclusion relation. We also note that when a basis $\mathcal{B}$ is equipped with a partial numbering β, it is always possible to totalize β while preserving the representations $\rho_\beta^{\min}$,

$\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subsetneq}}$, but it might not be possible to preserve the fact that $\mathrm{dom}(\beta)$ was equipped with a semi-decidable strong inclusion.

Finally, we study different notions of equivalence of bases. We give examples of bases that one would naturally expect to be equivalent, but which yield different representations $\rho^{\max}$. We describe a notion of *representation-equivalent bases* which guarantees that two bases yield equivalent representations.

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2. Preliminaries

2.1. Multi-representations: translation and admissibility.

Definition 2.1. A *multi-representation* [Sch03] of a set X is a partial multi-function: $\rho :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ such that every point of X is the image of some point in $\mathrm{dom}(\rho)$.

Note that, extentionally, a partial multi-function: $\rho :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ is nothing but a total function to the power-set of X: $\rho : \mathbb{N}^\mathbb{N} \to \mathcal{P}(X)$. The domain of ρ in this case is $\{f \in \mathbb{N}^\mathbb{N}, \rho(f) \neq \emptyset\}$. But in terms of the way one uses a multi-representation, it should not be seen as a function to $\mathcal{P}(X)$. For instance, the preimage of a subset $A \subseteq X$ by ρ is not defined as $\{f \in \mathbb{N}^\mathbb{N}, \rho(f) = A\}$, but as $\rho^{-1}(A) = \{f \in \mathbb{N}^\mathbb{N}, \rho(f) \cap A \neq \emptyset\}$.

For $f \in \mathbb{N}^\mathbb{N}$, if $x \in \rho(f)$, then f is a *ρ-name* of x.

Definition 2.2. A partial function $H :\subseteq X \rightrightarrows Y$ between multi-represented sets (X, ρ_1) and (Y, ρ_2) is called (ρ_1, ρ_2)-computable if there exists a computable function[1] $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ defined at least on all $f \in \mathrm{dom}(\rho_1)$ for which $\rho_1(f) \cap \mathrm{dom}(H) \neq \emptyset$ such that

$$\forall x \in \mathrm{dom}(H), \forall f \in \mathrm{dom}(\rho_1), x \in \rho_1(f) \implies H(x) \in \rho_2(F(f)).$$

This definition has a very simple interpretation: the multi-function H is (ρ_1, ρ_2)-computable if there is a computable map which, when given a name for a point x, produces a name for the image of x by H.

When $H :\subseteq X \to Y$ is a partial function between represented spaces (X, ρ_1) and (Y, ρ_2), any partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ which satisfies the condition written in the above definition is called a *realizer* for H. A function is computable if and only if it has a computable realizer.

Definition 2.3. If ρ_1 and ρ_2 are multi-representations of a set X, we say that ρ_1 *translates to* ρ_2, denoted by $\rho_1 \leq \rho_2$, if the identity id_X of X is (ρ_1, ρ_2)-computable. The multi-representations ρ_1 and ρ_2 are called equivalent if each one translates to the other, this is denoted by $\rho_1 \equiv \rho_2$.

The fact that ρ_1 translates to ρ_2 can be interpreted as meaning that the ρ_1-name of a point in X provides more information on this point than a ρ_2-name would. Note that for representations, $\rho_1 \leq \rho_2$ if and only if there exists a computable $h :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ so that

[1] Or "computable functional" in the sense of Kleene [Kle52] or Grzegorczyk [Grz55], see [Wei00].

$\rho_2 \circ h = \rho_1$, while for multi-representations, $\rho_1 \leq \rho_2$ if and only if there exists a computable $h :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ so that $\rho_1 \sqsubseteq \rho_2 \circ h$, i.e. for all $x \in \mathrm{dom}(\rho_1)$, $\rho_1(x) \subseteq \rho_2 \circ h(x)$.

We also have a notion of continuous reduction:

Definition 2.4. If ρ_1 and ρ_2 are multi-representations of a set X, we say that ρ_1 *continuously translates to* ρ_2 if the identity id_X of X admits a continuous realizer. This is denoted by $\rho_1 \leq_t \rho_2$, where the subscript t stands for *topological*.

The multi-representations ρ_1 and ρ_2 are *continuously equivalent* when each one continuously translates to the other. This is denoted by $\rho_1 \equiv_t \rho_2$.

When X is equipped with a multi-representation $\rho :\subseteq \mathbb{N}^\mathbb{N} \to X$, we can naturally equip X with a topology, the final topology $\mathcal{T}_\rho$ of the multi-representation. The topology $\mathcal{T}_\rho$ is defined as follows [Sch03, p. 53]: a set A is open in X if and only if $A = \rho(\rho^{-1}(A))$ and $\rho^{-1}(A)$ is open in the Baire space topology (more precisely: open in the topology on $\mathrm{dom}(\rho)$ induced by the topology of Baire space).

Note that when ρ is a function, $A = \rho(\rho^{-1}(A))$ is automatically satisfied, since ρ is surjective. Generalizing this to multi-functions, we have render explicit this condition.

In this paper, as we focus on second countable topological spaces, the topology of each space we consider is determined by converging sequences, and thus by the *limit space* induced by the topology. However, when working with multi-representations, we need the notion of *sequentially continuous multi-representation* even when focusing solely on second countable spaces.

Definition 2.5 [Sch03, Section 2.4.4]**.** A multi-function $F : X \rightrightarrows Y$ is called *sequentially continuous* if for every sequence $(x_n) \in X^\mathbb{N}$ that converges to a point x_∞, every sequence $(y_n) \in Y^\mathbb{N}$ with $y_n \in F(x_n)$ for all n, and every $y_\infty \in F(x_\infty)$, (y_n) converges to y_∞.

The reason why we need this definition even in a sequential setting is the following. Consider the degenerate multi-function $F : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ which maps every point to $\mathbb{N}^\mathbb{N}$. The preimage of any open set by F is open in Baire space, and the preimage of any closed set is closed (thus F is both *lower* and *upper semi–continuous*). However, there is no continuous translation between F (seen as a multi-representation) to the usual identity representation of Baire space. This comes from the fact that while F is in a sense continuous, the final topology of F is still coarser than the Baire space topology. This cannot happens for single valued functions: a function $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is continuous if and only if the final topology of F is finer than the topology of Baire space.

Definition 2.6 [Sch03]**.** Let $(X, \mathcal{T})$ be a sequential topological space. A multi-representation $\rho :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ of $(X, \mathcal{T})$ is called *admissible* if it is a maximal sequentially continuous multi-representation: ρ is sequentially continuous, and for any sequentially continuous multi-representation $\phi :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ of X we have $\phi \leq_t \rho$.

Note that the following lemma guarantees that when we consider a T_0-space, the above supremum can be taken only over continuous representations.

Lemma 2.7 ([Sch03, Lemma 2.4.6])**.** *If a multi-function $F : X \rightrightarrows Y$ towards a T_0 space is sequentially continuous, it is in fact a single valued function.*

2.2. **Sierpiński representation of open sets.**

Associated to every multi-represented space (X, ρ) is a representation of the open sets of the final topology of ρ, called the Sierpiński representation [Sch03], denoted by $[\rho \to \rho_{\mathfrak{Si}}]$ as in [Sch03]. See also [Pau16]. Note that even when ρ is a multi-representation, $[\rho \to \rho_{\mathfrak{Si}}]$ remains an actual representation.

This representation is defined as follows: $[\rho \to \rho_{\mathfrak{Si}}](p) = O$ if and only if p encodes a pair (n, q), where $n \in \mathbb{N}$ and $q \in \mathbb{N}^{\mathbb{N}}$, and n is the code of a Type 2 Turing machine which, when run on input $f \in \mathrm{dom}(\rho)$ using q as an oracle, will stop if and only if $\rho(f) \in O$. For more details, see [Sch03].

A subset of X is called *c.e. open* if it is a computable point of the Sierpiński representation. In other words, it is simply a ρ-semi-decidable set.

2.3. **Numberings.** A numbering of a set X is a partial surjection $\nu :\subseteq \mathbb{N} \to X$. This can equivalently be seen as a representation whose domain is a subset of the set of constant sequences, and thus the notion of computable function and the order $\leq$ defined in the previous section can also be applied to numberings.

We say that X is *ν-computably enumerable* if there is a c.e. subset A of $\mathbb{N}$ such that $X = \nu(A)$. This is one of several possible notions that formalize the idea of being "effectively countable", and probably the more commonly considered one.

Supposing that a numbering ν has domain $\mathbb{N}$, or has a c.e. domain, in particular implies that the numbered set is ν-computably enumerable. Applied to numbered bases, this gives a notion of effective second countability.

2.4. **Basis induced by a subbasis and induced strong inclusion.** Fix a set X, and denote by $(\mathfrak{B}, \beta)$ a numbered subbasis for X. The *induced numbered basis* is a pair $(\hat{\mathfrak{B}}, \hat{\beta})$: $\hat{\mathfrak{B}}$ is the set of finite intersections of elements of $\mathfrak{B}$, and $\hat{\beta}$ is a numbering defined as follows. Denote by Δ the standard numbering of finite subsets of $\mathbb{N}$. Then put:

$$\mathrm{dom}(\hat{\beta}) = \{n \in \mathbb{N},\, \Delta_n \subseteq \mathrm{dom}(\beta)\};$$
$$\forall n \in \mathrm{dom}(\hat{\beta}),\, \hat{\beta}(n) = \bigcap_{B \in \beta(\Delta_n)} B.$$

When $(\mathfrak{B}, \beta)$ is equipped with a strong inclusion $\mathring{\subseteq}$, this strong inclusion can naturally be extended to $(\hat{\mathfrak{B}}, \hat{\beta})$, as follows:

$$\forall n_1, n_2 \in \mathrm{dom}(\hat{\beta}), n_1 \mathring{\subseteq} n_2 \iff \forall k \in \Delta_{n_2},\, \exists p \in \Delta_{n_1},\, p \mathring{\subseteq} k.$$

One easily check that this extension preserves the conditions of being a strong inclusion relation, and that if $\mathring{\subseteq}$ was reflexive, the extension remains reflexive.

Remark 2.8. In the present article, we often consider a numbered subbasis $(\mathfrak{B}, \beta)$ which induces a basis $(\hat{\mathfrak{B}}, \hat{\beta})$, and a strong inclusion relation $\mathring{\subseteq}$ for $(\hat{\mathfrak{B}}, \hat{\beta})$ -and not for $(\mathfrak{B}, \beta)$. The reason for this is as follows. As shown above, every strong inclusion relation for $(\mathfrak{B}, \beta)$ induces a strong inclusion relation on $(\hat{\mathfrak{B}}, \hat{\beta})$. However, not every strong inclusion for $(\hat{\mathfrak{B}}, \hat{\beta})$ needs to come from a strong inclusion of $(\mathfrak{B}, \beta)$. In particular, the "set inclusion strong relation" $n_1 \mathring{\subseteq} n_2 \iff \beta(n_1) \subseteq \beta(n_2)$ on $(\mathfrak{B}, \beta)$ does not induce the set inclusion strong relation on $(\hat{\mathfrak{B}}, \hat{\beta})$. And the relation $n_1 \mathring{\subseteq} n_2 \iff \hat{\beta}(n_1) \subseteq \hat{\beta}(n_2)$ for $(\hat{\mathfrak{B}}, \hat{\beta})$ does not have to be induced by a strong inclusion of $(\mathfrak{B}, \beta)$.

Our setting is thus made more general by allowing any strong inclusion relation on $(\hat{\mathfrak{B}}, \hat{\beta})$, and not only those that come from strong inclusions of the subbasis.

2.5. **Computable topological spaces.** We now introduce here the notion of "computable topological space" that was introduced by Weihrauch and Grubba in [WG09]. It is a special case of a definition of Bauer: [Bau00, Definition 5.4.2].

We want to note here that the term "computable topological space" is not an appropriate name for this notion, since it is a notion of *computable basis*, and not the definition of a computable topological space. Furthermore, it is only one amongst several possible notions of computable basis, and not the most general one one can think of. For instance it does not apply to all non-computably separable computable metric spaces.

Denote by $W_i = \mathrm{dom}(\varphi_i)$ the usual numbering of r.e. subsets of $\mathbb{N}$.

Definition 2.9 [WG09]**.** A *"computable topological space"* is a triple $(X, \mathcal{B}, \beta)$, where X is a set, $\mathcal{B}$ is a topological basis on X that makes of it a T_0 space, and $\beta : \mathbb{N} \to \mathcal{B}$ is a total surjective numbering of $\mathcal{B}$, for which there exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ of such that for any i, j in $\mathbb{N}$:

$$\beta(i) \cap \beta(j) = \bigcup_{k \in W_{f(i,j)}} \beta(k).$$

Note that the requirement that β be total can be read as imposing that $(X, \mathcal{B}, \beta)$ be *computably second countable*. This is a requirement one might want to do away with.

A possible better name for the notion above would be that of a *Lacombe basis*. In particular, if we do not ask β to be total, and if we do not suppose that X will be T_0, then the conditions imposed on the basis are exactly *the necessary and sufficient conditions in order for the Lacombe sets*[2] *to form a computable topology*: so that finite intersection and computable unions be computable.

Associated to a "computable topological space" is a representation of open sets:

$$\rho_{(\mathfrak{B},\beta)}(f) = \bigcup_{\{n,\, \exists p \in \mathbb{N},\, f(p) = n+1\}} \beta(n).$$

(In the above, if $f = 0^\omega$, then it is a name of the empty set.)

The condition of Definition 2.9 are sufficient in order for finite intersections and countable unions to be computable for $\rho_{(\mathfrak{B},\beta)}$, and, again, removing the condition that X be T_0 and that β be total, we obtain necessary and sufficient conditions.

The representation $\rho_\beta^{\max}$ has been up to now often associated to the above definition of a computable basis. One of the purposes of this article is to show that the notion of "computable topological space" described above, while very relevant to the study of the associated representation $\rho_{(\mathfrak{B},\beta)}$ of open sets, is not relevant to the study of the representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ associated to a numbered basis. In particular, none of the conditions of Definition 2.9 are useful in showing that $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are admissible representations (or admissible multi-representations, if we allow non T_0-spaces).

2.6. **Computable metric spaces.** The following is the common definition for computable metric spaces.

[2]Lacombe sets are computable union of basic open sets, this name goes back to Lachlan [Lac64] and Moschovakis [Mos64].

Definition 2.10 [Wei03, BP03]**.** A *computable metric space* (CMS) is a quadruple (X, A, ν, d), where (X, d) is a metric space, A is a countable and dense subset of X, $\nu : \mathbb{N} \to A$ is a total numbering of A, and such that the metric $d : A \times A \to \mathbb{R}_c$ is $(\nu \times \nu, c_{\mathbb{R}})$-computable.

The following older definition is in fact more general. It can be seen as the generalization of Moschovakis' notion of a *recursive metric space,* that comes from [Mos64] and which concerns only countable spaces, to a Type 2 setting. (Indeed, not every recursive metric space in the sense of [Mos64] is a computable metric space.)

Definition 2.11 [Her96]**.** A *non-necessarily effectively separable computable metric space* is a quadruple (X, A, ν, d), where (X, d) is a metric space, A is a countable and dense subset of X, $\nu :\subseteq \mathbb{N} \to A$ is a partial numbering of A, and such that the metric $d : A \times A \to \mathbb{R}_c$ is $(\nu \times \nu, c_{\mathbb{R}})$-computable.

The existence of constructively non separable metric spaces has been known for a long time. In [Sli72], Slisenko constructs, in the context of countable numbered sets, a non-computably separable metric space which cannot be embedded in a computably separable one. A more conceptual proof of this result was given by Weihrauch in [Wei13]: any CMS satisfies a strong form of effective regularity, known as SCT_3, and non-separable computable metric spaces, while always computably regular (CT_3), do not have to be strongly regular. But because SCT_3 is a property inherited by subsets, the example of a non-computably separable metric space that satisfies CT_3 but not SCT_3 [Wei13, Example 5.4] is a non-computably separable metric space that does not embed into a CMS.

Non-computably separable spaces naturally arise, see the author's paper [Rau21], which motivated the present note. Note also that, when applying the Schröder Metrization theorem [Sch98, Wei13] to a represented space, one builds a computable metric, but there is no guarantee that the resulting space will indeed be computably separable, and an actual computable metric space.

Denote by $c_{\mathbb{Q}}$ the usual numbering of $\mathbb{Q}$, which is total. Associated to a (non-necessarily effectively separable) computable metric space (X, A, ν, d) is a numbering β of open balls centered at points of A:

$$\mathrm{dom}(\beta) = \{\langle n, m\rangle \in \mathbb{N},\ n \in \mathrm{dom}(\nu),\ c_{\mathbb{Q}}(m) > 0\},$$

$$\forall \langle n, m\rangle \in \mathrm{dom}(\beta),\ \beta(\langle n, m\rangle) = B(\nu(n), c_{\mathbb{Q}}(m)).$$

Note that when ν is total, β has a recursive domain, we can then suppose that it is total.

The are two natural strong inclusion relations on metric spaces:

$$\langle n_1, m_1\rangle \mathring{\subseteq} \langle n_2, m_2\rangle \iff d(\nu(n_1), \nu(n_2)) + c_{\mathbb{Q}}(m_1) < c_{\mathbb{Q}}(m_2);$$

$$\langle n_1, m_1\rangle \mathring{\subseteq} \langle n_2, m_2\rangle \iff d(\nu(n_1), \nu(n_2)) + c_{\mathbb{Q}}(m_1) \leq c_{\mathbb{Q}}(m_2).$$

The first one has the advantage of being semi-decidable, the second one of being reflexive, both have found use in the literature [WS81, Spr98]. But in fact, these two different relations induce the same notion of strong neighborhood basis, and thus they can be studied interchangeably when studying the representation $\rho_{\beta}^{\mathring{\subseteq}}$.

Metric spaces come equipped with the Cauchy representation ρ_{Cau}:

$$\mathrm{dom}(\rho_{Cau}) = \{p \in \mathrm{dom}(\nu)^{\mathbb{N}},\ \forall i > j,\ d(\nu(p(i)), \nu(p(j))) < 2^{-j},\ \exists x \in X,\ x = \lim_{i \to +\infty} \nu(p(i))\}$$

$$\forall p \in \text{dom}(\rho_{Cau}),\ \rho_{Cau}(p) = \lim_{i\to+\infty} \nu(p(i)).$$

3. Admissibility theorem

In this section, we prove that all multi-representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are admissible.

Proposition 3.1. *For any numbered set $(\mathcal{B}, \beta)$ of subsets of a set X, and any strong inclusion relation $\mathring{\subseteq}$ for $(\mathcal{B}, \beta)$, the three multi-representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are equivalent modulo continuous translations.*

Proof. As remarked in the introduction, we always have $\rho_\beta^{\max} \le \rho_\beta^{\mathring{\subseteq}} \le \rho_\beta^{\min}$.We thus prove that $\rho_\beta^{\min} \le_t \rho_\beta^{\max}$.

With a powerful enough oracle, the inclusion can be decided on dom(β) (i.e. the relation R defined by $nRm \iff \beta(n) \subseteq \beta(m)$). Also, a powerful enough oracle can enumerate dom(β). With such an oracle, we can, given the $\rho_\beta^{\min}$-name of a point x, enumerate in parallel all names of balls that contain a ball that contains x, this will precisely give a $\rho_\beta^{\max}$-name of x. □

Theorem 3.2. *For any numbered set $(\mathcal{B}, \beta)$ of subsets of a set X, and any strong inclusion relation $\mathring{\subseteq}$ for $(\hat{\mathcal{B}}, \hat{\beta})$ for which points have strong neighborhood bases, the three multi-representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are admissible with respect to the topology of X generated by $\mathcal{B}$ (as a subbasis).*

Proof. This is an immediate corollary of the previous result, together with the theorem of Schröder which states that $\rho_\beta^{\min}$ is always admissible. See in particular in [Sch03]: Proposition 3.1.6, for the case of limit spaces, and Lemma 3.1.10 for the transfer of this result to topological spaces. □

Note that in cases where the topology generated by $(\mathcal{B}, \beta)$ satisfies the T_0 axiom, and $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ are representations instead of multi-representations, the above result follows from classical results of Weihrauch [WK87, Wei00] who shows that $\rho_\beta^{\max}$ is admissible. And in fact, one can also deduce Theorem 3.2 from Weihrauch's result, together with the following lemma, suggested to us by an anonymous referee. Recall that on a topological space $(X, \mathcal{T})$, we define an equivalence relation $\sim$ by:

$$a \sim b \iff \forall O \in \mathcal{T},\, a \in O \iff b \in O.$$

The quotient $X/\sim$, that comes equipped with the quotient topology, is called the Kolmogorov quotient of $(X, \mathcal{T})$.

Lemma 3.3. *Let (X, ρ) be a multi-represented space with final topology $\mathcal{T}$. Consider the Kolmogorov quotient $X/\sim$ of X, and the quotient representation $\tilde{\rho}$. Then $\tilde{\rho}$ is admissible if and only if ρ is.*

Proof. Suppose first that $\tilde{\rho}$ is admissible.

Let τ be any sequentially continuous multi-representation of X. Because the quotient map $X \xrightarrow{q} X/\sim$ is sequentially continuous, $q \circ \tau$ is a sequentially continuous multi-function whose codomain is T_0, by Lemma 2.7, $q \circ \tau$ is actually single valued. It is a continuous representation of $X/\sim$, and thus $q \circ \tau \le_t \tilde{\rho}$, i.e. there is a continuous function h so that $q \circ \tau \circ h = \tilde{\rho}$ on dom(ρ).

This indicates that for every $x \in \mathrm{dom}(\rho)$, every $y \in \tau \circ h(x)$ and every z in $\rho(x)$, $y \sim z$. But by definition of the final topology of ρ, if $y \sim z$ and $z \in \rho(x)$, then $y \in \rho(x)$. And thus $\tau \circ h(x) \subseteq \rho(x)$ for every x, which indicates that $\tau \circ h \leq_t \rho$.

Conversely, if ρ is admissible, $\tilde{\rho}$ is also admissible. Indeed, let $\tilde{\tau} :\subseteq \mathbb{N}^{\mathbb{N}} \to X/\sim$ be a continuous representation of $X/\sim$. Then $q^{-1} \circ \tilde{\tau}$ is a sequentially continuous multi-representation of X. By admissibility of ρ, $q^{-1} \circ \tilde{\tau} \leq_t \rho$. And $q \circ q^{-1} \circ \tilde{\tau} \leq_t q \circ \rho$, i.e. $\tilde{\tau} \leq_t \tilde{\rho}$. □

4. Compatibility of the multi-representations of a set and of a subset

If (X, ρ) is a multi-represented set, and if A is a subset of X, we naturally define a multi-representation $\rho_{|A}$ of A, *the restriction of ρ to A,* by the following:

$$\mathrm{dom}(\rho_{|A}) = \{f \in \mathrm{dom}(\rho),\, \rho(f) \cap A \neq \emptyset\};$$

$$\forall f \in \mathrm{dom}(\rho_{|A}),\, \rho_{|A}(f) = \rho(f) \cap A.$$

Proposition 4.1. *If A is a subset of X and ρ is a multi-representation of X, then the embedding $A \hookrightarrow X$ is always $(\rho_{|A}, \rho)$-computable, and the identity on Baire space is a realizer.*

Proof. Immediate. □

We now consider a numbered subbasis $(\mathfrak{B}, \beta)$ for X equipped with a strong inclusion relation $\mathring{\subseteq}$. We thus have three multi-representations $\rho_\beta^{\min}$, $\rho_\beta^{\max}$ and $\rho_\beta^{\mathring{\subseteq}}$ of X.

We can now consider a numbered subbasis $(\mathfrak{A}, \alpha)$ for A, defined by restriction of $(\mathfrak{B}, \beta)$:

$$\mathrm{dom}(\alpha) = \mathrm{dom}(\beta);$$

$$\forall n \in \mathrm{dom}(\beta),\, \alpha(n) = A \cap \beta(n).$$

In fact, $(\mathfrak{A}, \alpha)$ is naturally equipped with the same strong inclusion relation as β. Indeed, the condition $n \mathring{\subseteq} m \implies \alpha(n) \subseteq \alpha(m)$ is always valid, since

$$\forall n, m \in \mathrm{dom}(\beta),\, n \mathring{\subseteq} m \implies \beta(n) \subseteq \beta(m) \implies A \cap \beta(n) \subseteq A \cap \beta(m).$$

(Note that the new $\mathring{\subseteq}$ remains reflexive, and points still admit strong neighborhood bases, if this was the case for the original $\mathring{\subseteq}$.)

We then have three multi-representations of A: $\rho_\alpha^{\min}$, $\rho_\alpha^{\max}$ and $\rho_\alpha^{\mathring{\subseteq}}$.

Both $\rho_\alpha^{\max}$ and $\rho_\alpha^{\mathring{\subseteq}}$ are well behaved with respect to the inclusion $A \hookrightarrow X$, but $\rho_\alpha^{\min}$ is not. This is what we show now.

Proposition 4.2. *We have:* $\rho_\alpha^{\max} \equiv (\rho_\beta^{\max})_{|A}$ and $\rho_\alpha^{\mathring{\subseteq}} \equiv (\rho_\beta^{\mathring{\subseteq}})_{|A}$.

Proof. Left to the reader. □

Corollary 4.3. *The embedding $A \hookrightarrow X$* is always $(\rho_\alpha^{\max}, \rho_\beta^{\max})$-computable and $(\rho_\alpha^{\mathring{\subseteq}}, \rho_\beta^{\mathring{\subseteq}})$-computable.

Proof. By Proposition 4.1. □

Proposition 4.4. *The embedding $A \hookrightarrow X$* is not always $(\rho_\alpha^{\min}, \rho_\beta^{\min})$-computable.

The proof of this proposition uses a construction that appears in Section 5.1 and is postponed to this section.

Corollary 4.5. *It is not always the case that* $\rho_\alpha^{\min} \equiv (\rho_\beta^{\min})_{|A}$.

Proof. This follows by Proposition 4.1. □

5. Metric spaces and the Cauchy representation

Recall from Section 2.6 that on a non-computably separable computable metric space, we have four representations: $\rho_\beta^{\max}$, $\rho_\beta^{\min}$, $\rho_\beta^{\mathring{\subseteq}}$ (where $\mathring{\subseteq}$ is the strong inclusion that comes from the metric) and ρ_{Cau}, the Cauchy representation.

In this section, we establish the following result:

Theorem 5.1. *The following reductions hold on any computable metric space:*

$$\rho_\beta^{\max} \equiv \rho_{Cau} \equiv \rho_\beta^{\mathring{\subseteq}} \leq \rho_\beta^{\min},$$

and $\leq$ *can sometimes be strict.*

The following reductions hold on any non-computably separable computable metric space:

$$\rho_\beta^{\max} \leq \rho_{Cau} \equiv \rho_\beta^{\mathring{\subseteq}} \leq \rho_\beta^{\min},$$

and each $\leq$ *can sometimes be strict.*

Note that the positive statements about computable metric spaces in the above can be gathered from [KW85, Wei87, Wei00, Spr01].

5.1. **The "sufficiently many basic sets" approach and metric spaces.**

Let (X, A, ν, d) be a non-necessarily computably separable computable metric space, and denote by β the numbering of open balls with rational radii associated to (X, A, ν, d).

Proposition 5.2. *We have* $\rho_{Cau} \leq \rho_\beta^{\min}$.

Proof. The ρ_{Cau} name of a point x is a sequence (of names) of points $(u_n)_{n\in\mathbb{N}}$ that converges towards x at exponential speed. A sequence of names for the balls $(B(u_n, 2^{-n}))_{n\in\mathbb{N}}$ is then a $\rho_\beta^{\min}$-name of x. □

It was already remarked by Weihrauch in [Wei87, Section 3.4, Therorem 18] that the existence of isolated point could be a problem for the representation $\rho_\beta^{\min}$. We show:

Proposition 5.3. *The representation* $\rho_\beta^{\min}$ *does not have to be equivalent to the Cauchy representation of* X*, even if* X *is computably separable.*

The idea is the following. We build a computable metric space which is discrete in some places and not discrete in others. For a point x which is isolated, if n is a name of a ball $B(x, r)$ which contains only x, i.e. $B(x, r) = \{x\}$, then $(n, n, n, n, ...)$ is a $\rho_\beta^{\min}$-name for x. However, to construct a Cauchy name for x starting from this $\rho_\beta^{\min}$-name, one has to be able to computably understand that x is isolated, and that the ball $B(x, r)$ determines x uniquely. We choose a space where this is not possible.

Note that this example is precisely based upon a case where the strong inclusion relation is different from the actual inclusion relation. Indeed, with the notations above, we have:

- For any m such that $x \in \beta(m)$, then $\beta(n) \subseteq \beta(m)$.
- It is not true that for any m such that $x \in \beta(m)$, then $n \mathring{\subseteq} m$. Indeed, this fails if m is a name for $B(x, r/2)$ (i.e. a name where the radius *explicitly given* is $r/2$).

Proof. We take a certain subset of $\mathbb{R}$. Denote by K the halting set: $K = \{n \in \mathbb{N},\, \varphi_n(n)\downarrow\}$. Consider the union

$$A = \bigcup_{n\in K} [n-1/2, n+1/2] \cup \bigcup_{n\notin K} \{n\}.$$

Thus A is discrete in some places and not discrete in others. A admits a dense and computable sequence: the set of natural numbers, together with the set of rationals in $[n-1/2, n+1/2]$, for each n in K, which is a r.e. set because K is.

The usual metric of $\mathbb{R}$ remains computable when restricted to the set of rationals in A. Thus A is a computable metric space. Denote by β the numbering of open intervals of A induced by that of $\mathbb{R}$: if γ is a numbering of open intervals of $\mathbb{R}$ that have rational endpoints, put

$$\beta(n) = \gamma(n) \cap A.$$

In this setting, for $n \notin K$, denote by m a γ-name of the basic set $]n-1/4, n+1/4[$. Then the constant sequence $(m, m, m, m...)$ constitutes a valid $\rho_\beta^{\min}$-name of n. Suppose there is a Type-2 machine T that on input the $\rho_\beta^{\min}$-name of a point of A transforms it in a ρ_{Cau}-name of this point. Then we can enumerate numbers n for which the ρ_{Cau}-name produced by T when given as input a constant sequence as above gives a precision better than 1/4. This gives precisely an enumeration of K^c. This is a contradiction. □

We finally use the construction above to prove Proposition 4.4.

Proof of Proposition 4.4. In the construction that appears above in the proof of Proposition 5.3, we build a metric space A equipped with a numbered basis with numbering β and where $\rho_\beta^{\min} \not\leq \rho_{Cau}$. The constructed set is a subset of $\mathbb{R}$, and it is easy to see that in this case the Cauchy numbering ρ_{Cau} on A is the restriction of the Cauchy numbering on $\mathbb{R}$, which is itself equivalent to $\rho_\gamma^{\min}$, where γ denotes the numbering of open intervals of $\mathbb{R}$ with rational endpoints. And thus we have: $\rho_\beta^{\min} \not\leq \rho_{Cau}$, $\rho_{Cau} \equiv (\rho_\gamma^{\min})_{|A}$, and thus $\rho_\beta^{\min} \not\leq (\rho_\gamma^{\min})_{|A}$. Finally this directly implies that the embedding $A \hookrightarrow \mathbb{R}$ is not $(\rho_\beta^{\min}, \rho_\gamma^{\min})$-computable. □

5.2. **The representation associated to "all names of basic sets" and non-computably separable metric spaces.** Let (X, A, ν, d) denote again a non-necessarily computably separable computable metric space, and denote by β the numbering of open balls with rational radii associated to (X, A, ν, d).

Proposition 5.4. *We have $\rho_\beta^{\max} \leq \rho_{Cau}$.*

Proof. This is very simple: a $\rho_\beta^{\max}$-name contains names of balls of arbitrarily small radius. Given a $\rho_\beta^{\max}$-name of a point x, a blind search in this name for a 2^{-n} good-approximation of x will always terminate. □

The following is well known: it shows that the definition of *computable topological space* as introduced in [WG09] (see Definition 2.9) is coherent with the definition of computable metric space as used since [Wei03]. It is for instance recalled in [Sch21, Example 9.4.16].

Proposition 5.5. *As soon as (X, A, ν, d) has a dense and computable sequence, for the numbering β of open balls given by rational radii and centers in the dense sequence, one also has $\rho_{Cau} \equiv \rho_\beta^{\max}$.*

Note however, as we will discuss in Section 7, that the above proposition holds only for certain choices of a numbering of open balls.

Finally, if (X, A, ν, d) does not have a dense and computable sequence, it does not have to have a computably enumerable basis of open balls. In this case, we have:

Proposition 5.6. *If β is the natural numbering of open balls with rational radii in a non-computably separable computable metric space, it is possible that $\rho_{Cau} \not\leq \rho_\beta^{\max}$. And it is possible that $\rho_\beta^{\max}$ defines no computable point.*

Proof. We give a very simple example. Consider the set K^c, the complement of the halting set, which is not r.e.. The numbering of K^c is the numbering induced by the identity on $\mathbb{N}$. Take the usual metric of $\mathbb{N}$, $d(i,j) = |i - j|$, and the associated numbering β of balls with rational radii and centers in K^c.

In this setting, no point of K^c has admits a computable $\rho_\beta^{\max}$-name.

Indeed, a computable enumeration of *all balls* centered at points in K^c that contain a given point x gives in particular an enumeration of the set of all their centers, which is exactly K^c. □

5.3. **The strong inclusion approach in metric spaces.** Let (X, A, ν, d) be a non-necessarily computably separable computable metric space. Let β be the numbering of balls with rational radii induced by ν. Denote by $\mathring{\subseteq}$ the strong inclusion of β induced by the metric, which comes from the relation on balls parametrized by pair (point-radius) defined by

$$(x, r_1) \mathring{\subseteq} (y, r_2) \iff d(x, y) + r_1 < r_2.$$

We have already defined the Cauchy representation on X and the representation $\rho_\beta^{\mathring{\subseteq}}$ induced by the numbering of the basis and the strong inclusion relation. As noted before, we can equivalently replace $<$ by $\leq$ in the definition of $\mathring{\subseteq}$ above.

Proposition 5.7. *The equivalence $\rho_{Cau} \equiv \rho_\beta^{\mathring{\subseteq}}$ holds.*

Proof. Denote by t_n a $c_\mathbb{Q}$-name of 2^{-n}. The map $n \mapsto t_n$ can be supposed computable. If p is a ρ_{Cau}-name for a point x, then q defined by

$$q(n) = \langle p(n), t_n \rangle$$

defines a $\rho_\beta^{\mathring{\subseteq}}$-name of x. This name is given as a computable function of p.

Conversely, suppose that q is a $\rho_\beta^{\mathring{\subseteq}}$-name of x. Denote by fst and snd the two halves of the inverse of the pairing function used to define the numbering β of balls in (X, A, ν, d). Then the following p gives a ρ_{Cau}-name for point x:

$$p(n) = \text{fst}(q(\mu i,\ \text{snd}(q(i)) < 2^{-n-1}))).$$

In words: $p(n)$ is defined as the center of the first ball of radius less than 2^{-n} found in the name q of x. The fact that this application of the μ-operator produces a total function comes exactly from the hypothesis that $\rho_\beta^{\mathring{\subseteq}}$-names give arbitrarily precise information *with respect to the strong inclusion:* in the $\rho_\beta^{\mathring{\subseteq}}$-name of x appear balls given by arbitrarily small radii. □

6. Semi-decidable strong inclusion

In this section we consider semi-decidable strong inclusion relations.

6.1. Semi-decidable strong inclusion and c.e. open sets.

One of the purposes of using strong inclusion relations is to be able to replace set inclusion by a semi-decidable relation. This plays a crucial role in many applications: [Her96, Spr98, DM23].

Here we highlight what seems to us to be the most important benefit of having access to a semi-decidable strong inclusion relation $\mathring{\subseteq}$: basic open sets are c.e. open with respect to the representation $\rho_\beta^{\mathring{\subseteq}}$. The following is essentially contained in [Spr01, Lemma 8], we are simply extending it to non-T_0 spaces.

Proposition 6.1. *Let $(\mathcal{B}, \beta)$ be a numbered subbasis for X, and let $\mathring{\subseteq}$ be a strong inclusion for $(\hat{\mathcal{B}}, \hat{\beta})$. Suppose that $\mathring{\subseteq}$ is semi-decidable. Suppose also that every point of X admits a strong neighborhood basis.*

Then the basic open sets are uniformly c.e. open sets with respect to the multi-representation $\rho_\beta^{\mathring{\subseteq}}$, i.e.:

$$\beta \leq [\rho_\beta^{\mathring{\subseteq}} \to \rho_{\mathfrak{Si}}].$$

Proof. All we have to show is that there is a Type-2 Turing machine that, on input the ρ-name of a point x and the β-name b_0 of a basic set B, will stop if and only if $x \in B$. But notice that, by definition of $\rho_\beta^{\mathring{\subseteq}}$, $x \in B$ if and only if any $\rho_\beta^{\mathring{\subseteq}}$-name of x contains a name $b_1 \in \mathrm{dom}(\beta)$ with $b_1 \mathring{\subseteq} b_0$. Since we suppose $\mathring{\subseteq}$ semi-decidable, the condition "containing a name $b_1 \in \mathrm{dom}(\beta)$ with $b_1 \mathring{\subseteq} b_0$" is also semi-decidable. □

We will now show that the result of this proposition is not true in general: we give an example of a numbered basis $(\mathcal{B}, \beta)$, which defines a representation $\rho_\beta^{\min}$, and yet the elements of $\mathcal{B}$ are not c.e. open for $\rho_\beta^{\min}$.

Example 6.2. Let (X, ν) be a numbered set on which equality is not semi-decidable. (For example: the set of c.e. subsets of $\mathbb{N}$ with numbering W, the computable reals with their numbering.) Consider a numbering $\beta :\subseteq \mathbb{N} \to \mathcal{P}(X)$ given by $\mathrm{dom}(\beta) = \mathrm{dom}(\nu)$ and $\forall n \in \mathrm{dom}(\beta)$, $\beta(n) = \{\nu(n)\}$. Then we get a representation $\rho_\beta^{\min}$ of X. The $\rho_\beta^{\min}$-name of a point x of X is a sequence that contains one or more ν-names for x. But since equality is not semi-decidable for ν, membership in singletons cannot be semi-decidable.

6.2. Semi-decidability of the strong inclusion and "totalization of a numbered basis".

In this section, we show that the obvious "totalization" of the numbering of a basis cannot always be achieved while preserving semi-decidability of a strong inclusion relation.

Suppose that we start with a partially numbered basis $(\mathcal{B}, \beta :\subseteq \mathbb{N} \to \mathcal{B})$. We can totalize the numbering β: we define a numbering $\tilde{\beta}$, with $\mathrm{dom}(\tilde{\beta}) = \mathbb{N}$, by

$$\forall n \in \mathrm{dom}(\beta),\ \tilde{\beta}(n) = \beta(n);$$

$$\forall n \notin \mathrm{dom}(\beta),\ \tilde{\beta}(n) = \emptyset.$$

If $(\mathcal{B}, \beta)$ was equipped with a strong inclusion relation $\mathring{\subseteq}$, we can naturally extend it, and define a strong inclusion $\mathring{\subseteq}'$ for $\tilde{\beta}$ as follows:

$$n \mathring{\subseteq}' m \iff (n, m \in \mathrm{dom}(\beta) \,\&\, n \mathring{\subseteq} m) \text{ or } n = m.$$

Note that if $\mathring{\subseteq}$ was reflexive, $\mathring{\subseteq}'$ remains so.

There are several other possibilities to define $\mathring{\subseteq}'$, one could for instance say that $n \mathring{\subseteq}' m$ when $n \notin \mathrm{dom}(\beta)$ and $m \in \mathrm{dom}(\beta)$, this also yields a strong inclusion relation.

It is immediate to see that, whatever the chosen extension $\mathring{\subseteq}'$ of $\mathring{\subseteq}$, we get $\rho_{\beta}^{\mathring{\subseteq}} \equiv \rho_{\tilde{\beta}}^{\mathring{\subseteq}'}$. Indeed, names of the empty set never play a role in the definition of the representation generated by a basis.

Consider as a basis for the topology of $\mathbb{R}$ the set $\mathcal{B}$ of open balls with computable reals as center and rational radii. It is equipped with the numbering β given by

$$\mathrm{dom}(\beta) = \{\langle n, m \rangle,\, n \in \mathrm{dom}(c_{\mathbb{R}}),\, c_{\mathbb{Q}}(m) > 0\},$$

$$\forall \langle n, m \rangle \in \mathrm{dom}(\beta),\, \beta(\langle n, m \rangle) =]c_{\mathbb{R}}(n) - c_{\mathbb{Q}}(m), c_{\mathbb{R}}(n) + c_{\mathbb{Q}}(m)[$$

where $c_{\mathbb{R}}$ is the usual numbering of computable reals and $c_{\mathbb{Q}}$ is any usual total numbering of $\mathbb{Q}$. We have the semi-decidable strong inclusion given by

$$\langle n_1, m_1 \rangle \mathring{\subseteq} \langle n_2, m_2 \rangle \iff |c_{\mathbb{R}}(n_1) - c_{\mathbb{R}}(n_2)| + c_{\mathbb{Q}}(m_1) < c_{\mathbb{Q}}(m_2).$$

Proposition 6.3. *For any of the totalizations of β described above, the resulting numbered basis cannot be equipped with a semi-decidable strong inclusion that extends $\mathring{\subseteq}$.*

Proof. Consider a β-name n for $B(0, 1) =]-1, 1[$. Suppose that $\mathring{\subseteq}'$ is a semi-decidable extension of $\mathring{\subseteq}$ to $\mathbb{N}$. For $i \in \mathbb{N}$, let b_i be the code of a Turing machine defined as follows: on input k, run $\varphi_i(k)$, if it stops output 0. Thus if $i \in \mathrm{Tot}$, b_i is a $c_{\mathbb{R}}$-name of 0, otherwise $b_i \notin \mathrm{dom}(c_{\mathbb{R}})$. Denote by n_2 a $c_{\mathbb{Q}}$-name of 2. Then $\langle b_i, n_2 \rangle$ is a $\tilde{\beta}$-name of $B(0, 2)$ when $i \in \mathrm{Tot}$, and a name of the empty set otherwise. And then a program semi-deciding for $\mathring{\subseteq}'$ would semi-decide membership in Tot. ◻

7. Equivalence of bases

In this section, we restrict our attention to numbered bases, instead of numbered subbases. But subbases are equivalent when the bases they generate are equivalent, and the same relation holds for notion of effective equivalence.

7.1. Representation $\rho_\beta^{\max}$ and equivalence of bases.

We first note that the representation $\rho_\beta^{\max}$ is badly behaved with respect to equivalence of bases: bases that "should be" equivalent can give non-equivalent representations. We first show this by an example that uses a non-computably enumerable basis, and then modify it so that it uses only computably enumerable bases.

The example used here is that of open balls of $\mathbb{R}$ given either by rational radii/center or by computable reals for their radii and center, and a totalized version of this last basis, which fills every gap with the empty set. In the following section, we present several notions of equivalence of bases, the first two bases considered here are equivalent according to all these definitions, and the "totalized basis" is also representation-equivalent and Nogina equivalent to the other two, but it fails to be Lacombe equivalent to them.

Denote by $c_{\mathbb{Q}}$ the usual numbering of $\mathbb{Q}$, which is total.

Denote by $c_{\mathbb{R}}$ the Cauchy numbering of $\mathbb{R}_c$.

Denote by $\mathfrak{B}_{\mathbb{Q}}$ the set of open intervals of $\mathbb{R}$ with rational endpoints.

Define $\beta_{\mathbb{Q}} :\subseteq \mathbb{N} \to \mathfrak{B}_{\mathbb{Q}}$ by

$$\mathrm{dom}(\beta_{\mathbb{Q}}) = \{\langle n, m\rangle,\, c_{\mathbb{Q}}(m) > 0\};$$
$$\beta_{\mathbb{Q}}(\langle n, m\rangle) = B(c_{\mathbb{Q}}(n), c_{\mathbb{Q}}(m)).$$

The domain of $\beta_{\mathbb{Q}}$ is easily seen to be recursive, and we can thus in fact suppose that $\beta_{\mathbb{Q}}$ is defined on all of $\mathbb{N}$.

Denote by $\mathfrak{B}_{\mathbb{R}}$ the set of open intervals of $\mathbb{R}$ with computable reals as endpoints.

Define $\beta_{\mathbb{R}} :\subseteq \mathbb{N} \to \mathfrak{B}_{\mathbb{R}}$ by

$$\mathrm{dom}(\beta_{\mathbb{R}}) = \{\langle n, m\rangle,\, c_{\mathbb{R}}(m) > 0\};$$
$$\beta_{\mathbb{R}}(\langle n, m\rangle) = B(c_{\mathbb{R}}(n), c_{\mathbb{R}}(m)).$$

Finally, we use a totalization of $\beta_{\mathbb{R}}$, denoted by $\tilde{\beta}_{\mathbb{R}}$, by adding the empty set to $\mathfrak{B}_{\mathbb{R}}$ and changing the numbering as follows:

$$\mathrm{dom}(\tilde{\beta}_{\mathbb{R}}) = \mathbb{N};$$
$$\forall n \in \mathrm{dom}(\beta_{\mathbb{R}}),\, \tilde{\beta}_{\mathbb{R}}(n) = \beta_{\mathbb{R}}(n),$$
$$\forall n \notin \mathrm{dom}(\beta_{\mathbb{R}}),\, \tilde{\beta}_{\mathbb{R}}(n) = \emptyset.$$

We then have:

Proposition 7.1. *The representations $\rho^{\max}_{\beta_{\mathbb{R}}}$ and $\rho^{\max}_{\beta_{\mathbb{Q}}}$ are not equivalent.*

Proof. This follows from the following strong fact: there is no $\rho^{\max}_{\beta_{\mathbb{R}}}$-computable point. This follows immediately from the fact that there does not exist a computable enumeration of all computable reals. □

Proposition 7.2. *We have $\rho^{\max}_{\beta_{\mathbb{R}}} \equiv \rho^{\max}_{\hat{\beta}_{\mathbb{R}}}$, and thus $\rho^{\max}_{\tilde{\beta}_{\mathbb{R}}}$ and $\rho^{\max}_{\beta_{\mathbb{Q}}}$ are not equivalent, even though $\tilde{\beta}_{\mathbb{R}}$ is a total numbering.*

Proof. The identity of Baire space is a realizer for both directions. □

However, it is very easy to check that for the natural strong inclusion $\mathring{\subseteq}$ on $\mathbb{R}$, that comes from the metric of $\mathbb{R}$, we have:

Proposition 7.3. *The representations $\rho^{\mathring{\subseteq}}_{\beta_{\mathbb{R}}}$, $\rho^{\mathring{\subseteq}}_{\tilde{\beta}_{\mathbb{R}}}$ and $\rho^{\mathring{\subseteq}}_{\beta_{\mathbb{Q}}}$ are equivalent.*

7.2. **Representation-equivalent subbases.** The notion of topological space of Definition 2.9 naturally comes with a notion of equivalence of bases:

Definition 7.4. Two numbered bases $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ are called *Lacombe equivalent* if there is a program that takes as input the β_1-name of a basic open set B_1 and outputs the name of a β_2-computable sequence $(B_n)_{n\geq 2}$ of basic open sets such that

$$B_1 = \bigcup_{n\geq 2} B_n,$$

and a program that does the converse operation, with the roles of $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ reversed.

Recall that associated to a Lacombe basis $(\mathfrak{B}_1, \beta_1)$ is a representation of open sets: the name of an open set O is a sequence $(b_i)_{i\in\mathbb{N}} \in \mathrm{dom}(\beta_1)$ such that

$$O = \bigcup_{i\geq 0} \beta_1(b_i).$$

Definition 7.4 gives the correct notion of equivalence of basis with respect to this representation by the following easy proposition:

Proposition 7.5. *The numbered bases $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ induce equivalent representations of open sets if and only if $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ are Lacombe equivalent.*

Other notions of equivalence of bases can be appropriate, we quote one to illustrate the variety of possible definitions of equivalence of bases. The following notion is appropriate to a notion of effective basis that was first described by Nogina [Nog66] for numbered sets and used in the context of Type 2 computability in [GKP16].

Definition 7.6. Suppose that (X, ρ) is a second countable represented space. Two numbered bases $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ are called *Nogina equivalent* if there is a program that takes as input the β_1-name of a basic open set B_1 and the ρ-name of a point x in B_1 and outputs the β_2-name of a basic open sets B_2 such that $x \in B_2 \subseteq B_1$, and a program that does the converse operation, with the roles of $(\mathfrak{B}_1, \beta_1)$ and $(\mathfrak{B}_2, \beta_2)$ reversed.

In this case, one can also define a certain naturally associated representation of open sets, and show that two numbered bases define equivalent representations of open sets exactly when they are Nogina equivalent [Rau23].

However, neither of these notions of equivalence of bases is appropriate to the study of the multi-representations $\rho^{\max}_{\beta_1}$, $\rho^{\min}_{\beta_1}$ and $\rho^{\mathring{\subseteq}}_{\beta_1}$ associated to a numbered subbasis $(B_1, \beta_1, \mathring{\subseteq})$. In particular, Lacombe equivalent bases can yield non-equivalent representations of points.

We now describe the notion of equivalence of bases appropriate to the study of the representations $\rho^{\mathring{\subseteq}}_{\beta_1}$.

When A is a set, denote by A^* the set of (empty or) finite sequences of elements of A.

If $\mathring{\subseteq}_1$ is a strong inclusion relation on $\mathrm{dom}(\beta_1)$, we extend $\mathring{\subseteq}_1$ to $\mathrm{dom}(\beta_1)^*$ by

$$(b_1, ..., b_n)\mathring{\subseteq}_1(b'_1, ..., b'_m) \iff \forall i \leq m, \exists j \leq n,\, b_j \mathring{\subseteq}_1 b'_i.$$

Definition 7.7. Consider two numbered bases $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ of a set X equipped with strong inclusion relations. We say that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ is *representation coarser* than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ if there exists a computable function $f :\subseteq \mathrm{dom}(\beta_2)^* \to \mathrm{dom}(\beta_1)^*$ defined at least on all sequences $(b_1, ..., b_n) \in \mathrm{dom}(\beta_2)^*$ such that $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \neq \emptyset$, and such that:

- For all sequence $(b_1, ..., b_n) \in \mathrm{dom}(\beta_2)^*$, if $f((b_1, ..., b_n)) = (d_1, ..., d_m)$, then $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \subseteq \beta_1(d_1) \cap ... \cap \beta_1(d_m)$[3].
- For all x in X and d in $\mathrm{dom}(\beta_1)$ with $x \in \beta_1(d)$, for any sequence $(b_i)_{i\in\mathbb{N}} \in \mathrm{dom}(\beta_2)$ that defines a strong basis of neighborhood of x, there exists $k \in \mathbb{N}$ such that for all $n \geq k$,

$$f((b_1, ..., b_n))\mathring{\subseteq}_1 d.$$

We say that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ are *representation-equivalent* if each one is representation coarser than the other.

[3]In case $f((b_1, ..., b_n))$ is just the empty sequence, we use the convention that an empty intersection gives X.

The second condition written above says that as the sequence $(b_1, b_2, ...)$ closes in on a point, the sequence of images by f should also produce a strong neighborhood basis of this point.

We also introduce a more restrictive notion of equivalence of bases, which implies equivalence of the associated representations, and which is more natural to work with:

Definition 7.8. Consider two numbered bases $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ of a set X equipped with strong inclusion relations. We say that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ is *uniformly representation coarser* than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ if there exists a computable function $f :\subseteq \mathrm{dom}(\beta_2)^* \to \mathrm{dom}(\beta_1)^*$ defined at least on all sequences $(b_1, ..., b_n) \in \mathrm{dom}(\beta_2)^*$ such that $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \neq \emptyset$, and such that:

- For all sequence $(b_1, ..., b_n) \in \mathrm{dom}(\beta_2)^*$, if $f((b_1, ..., b_n)) = (d_1, ..., d_m)$, then $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \subseteq \beta_1(d_1) \cap ... \cap \beta_1(d_m)$.
- For all x in X and d in $\mathrm{dom}(\beta_1)$ with $x \in \beta_1(d)$, there exists $(b_1, ..., b_n)$ in $\mathrm{dom}(\beta_2)$, with $x \in \beta_2(b_1) \cap ... \cap \beta_2(b_n)$, and such that

$$\forall (b'_1, ..., b'_k) \in \mathrm{dom}(\beta_2)^*,\ (b'_1, ..., b'_k) \mathring{\subseteq}_2 (b_1, ..., b_n) \implies f((b'_1, ..., b'_k)) \mathring{\subseteq}_1 d.$$

We say that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ are *uniformly representation-equivalent* if each one is uniformly representation coarser than the other.

One checks that the uniform version is more restrictive that the general notion of representation-equivalence.

Lemma 7.9. *Let $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ be two numbered bases of a set X. If $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ is uniformly representation coarser than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$, then it is also representation coarser than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$. If $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ are uniformly representation-equivalent, then they are also representation-equivalent.*

It is easy to see that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ being representation coarser than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ does imply that the topology generated by $\mathfrak{B}_1$ as subbasis is coarser than the topology generated by the subbasis $\mathfrak{B}_2$ (in terms of classical mathematics).

The following lemma shows why we have the correct notion of equivalence of bases.

Lemma 7.10. *Let $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ be two numbered bases of a set X. Then*

$$\rho_{\beta_2}^{\mathring{\subseteq}_2} \leq \rho_{\beta_1}^{\mathring{\subseteq}_1} \iff (\mathfrak{B}_1,\beta_1,\mathring{\subseteq}_1) \text{ is effectively coarser than } (\mathfrak{B}_2,\beta_2,\mathring{\subseteq}_2);$$

$$\rho_{\beta_2}^{\mathring{\subseteq}_2} \equiv \rho_{\beta_1}^{\mathring{\subseteq}_1} \iff (\mathfrak{B}_1,\beta_1,\mathring{\subseteq}_1) \text{ and } (\mathfrak{B}_2,\beta_2,\mathring{\subseteq}_2) \text{ are representation-equivalent.}$$

Proof. The second equivalence is a direct consequence of the first one, we thus focus on the first one.

Suppose first that $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ is effectively coarser than $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$, and thus that we have a function f as in Definition 7.7.

Given the $\rho_{\beta_2}^{\mathring{\subseteq}_2}$-name of a point x, we show how to compute a $\rho_{\beta_1}^{\mathring{\subseteq}_1}$-name of it.

Simply apply the function f along all initial segments of the $\rho_{\beta_2}^{\mathring{\subseteq}_2}$-name of x, and output the concatenation of all the results. The fact that f produces oversets implies that x does belong to all produced basic open sets. The second condition on f guarantees that we indeed construct a strong neighborhood basis of x.

Suppose now that $\rho_{\beta_2}^{\mathring{\subseteq}_2} \leq \rho_{\beta_1}^{\mathring{\subseteq}_1}$.

By a classical characterization of Type 2 computable functions in terms of isotone[4] functions [Sch02], this implies that there is a computable isotone function $f :\subseteq \text{dom}(\beta_2)^* \to \text{dom}(\beta_1)^*$ that testifies for the relation $\rho_{\beta_2}^{\mathring{\subseteq}_2} \leq \rho_{\beta_1}^{\mathring{\subseteq}_1}$. This function f is defined at least on all sequences $(b_1, ..., b_n) \in \text{dom}(\beta_2)^*$ such that $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \neq \emptyset$, because any such sequence is the beginning of the $\rho_{\beta_2}^{\mathring{\subseteq}}$-name of some point.

Note first that for any sequence $(b_1, ..., b_n) \in \text{dom}(\beta_2)^*$, if $f((b_1, ..., b_n)) = (d_1, ..., d_m)$, then $\beta_2(b_1) \cap ... \cap \beta_2(b_n) \subseteq \beta_1(d_1) \cap ... \cap \beta_1(d_m)$. Indeed suppose that this is not the case. It means that there is a point $x \in \beta_2(b_1) \cap ... \cap \beta_2(b_n) \setminus (\beta_1(d_1) \cap ... \cap \beta_1(d_m))$. The sequence $(b_1, ..., b_n)$ could be completed to a $\rho_{\beta_2}^{\mathring{\subseteq}}$-name of x, however f cannot map a sequence that starts with $(b_1, ..., b_n)$ to a $\rho_{\beta_1}^{\mathring{\subseteq}}$-name of x. This is a contradiction, and thus the desired inclusion holds.

Finally, f applied along a strong neighborhood basis (with respect to $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$) of a point x will always produce a strong neighborhood basis of x (with respect to $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$). This guarantees that the last condition of Definition 7.7 is satisfied. □

The reason why we introduce the uniform version of representation-equivalence is twofold:

- It has a nice interpretation in metric spaces, and it is in general easier to understand, see the example below.
- It is unclear whether two bases can be representation-equivalent while not being uniformly representation-equivalent. We thus ask:

Problem 7.11. Can two subbases $(\mathfrak{B}_1, \beta_1, \mathring{\subseteq}_1)$ and $(\mathfrak{B}_2, \beta_2, \mathring{\subseteq}_2)$ of a set X be representation-equivalent while not being uniformly representation-equivalent?

Example 7.12. Suppose we are set in a separable metric space (X, d). There are many possible numberings of open balls: for any numbering ν of a dense subset $A \subseteq X$, and any numbering $c :\subseteq \mathbb{N} \to T \subseteq \mathbb{R}$ of a set of positive real numbers that has 0 as an accumulation point, the numbering

$$\beta(\langle n, m\rangle) = B(\nu(n), c(m))$$

is a numbering of a basis for the topology of X.

For two such numberings β_1 and β_2, the condition of uniform representation-equivalence with respect to the strong inclusion of metric spaces says that there is an algorithm that, given a finite intersection $B_1 \cap ... \cap B_n$ of balls given by β_1-names, covers it by a finite intersection $B'_1 \cap ... \cap B'_m \supseteq B_1 \cap ... \cap B_n$ of balls given by β_2-names, and, additionally, that when the minimal radius appearing in the first intersection $B_1 \cap ... \cap B_n$ goes to 0, then the minimal radius appearing in $B'_1 \cap ... \cap B'_m$ should go to 0 as well.

The condition of representation-equivalence only asks of this algorithm that along each sequence $(b_i)_{i\in\mathbb{N}}$ of β_1-names which defines a sequence of balls with arbitrarily small radii, the corresponding sequence of β_2-names should also encode small radii, but the dependence does not have to be uniform in the radii anymore.

[4]A function $f : A^* \to B^*$ is called *isotone* if it is increasing for the prefix relation: if u is a prefix of v, then $f(u)$ is a prefix of $f(v)$.

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