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EXTENDED STONE DUALITY VIA MONOIDAL ADJUNCTIONS

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Abstract. Extensions of Stone-type dualities have a long history in algebraic logic and have also been instrumental in proving results in algebraic language theory. We show how to extend abstract categorical dualities via monoidal adjunctions, subsuming various incarnations of classical extended Stone and Priestley duality as special cases, and providing the foundation for two new concrete dualities: First, we investigate residuation algebras, which are lattices with additional residual operators modeling language derivatives algebraically. We show that the subcategory of derivation algebras is dually equivalent to the category of profinite ordered monoids, restricting to a duality between Boolean residuation algebras and profinite monoids. We further refine this duality to capture relational morphisms of profinite ordered monoids, which dualize to natural morphisms of residuation algebras. Second, we apply the categorical extended duality to the discrete setting of sets and complete atomic Boolean algebras to obtain a concrete description for the dual of the category of all small categories.

1. Introduction

Marshall H. Stone's representation theorem for Boolean algebras [Sto36], the foundation for the so called *Stone duality* between Boolean algebras and Stone spaces, manifests a tight connection between logic and topology. It has thus become an ubiquitous tool in various areas of theoretical computer science, not only in logic, but also, for example, in domain theory [Abr91] and automata theory [Pip97, GGP08].

From algebraic logic arose the need for extending Stone duality to capture Boolean algebras equipped with additional operators (modelling quantifiers or modalities). Originating from Jónsson and Tarski's representation theorem for Boolean algebras with operators [JT51, JT52], a representation in the spirit of Stone was proven by Halmos [Hal58]; the general categorical picture of the duality of Kripke frames and modal algebras is based on an adjunction between operators and continuous relations developed by Sambin and Vaccaro [SV88].

In the context of automata theory, the need for extensions of Stone duality was only unveiled in this millennium: Using ordinary Stone duality, Pippenger [Pip97] (see also Almeida [Alm94]) has already shown that the Boolean algebra of regular languages on an alphabet Σ corresponds to the Stone space $\widehat{\Sigma^*}$ of profinite words. This result and the theory surrounding it was placed in the bigger picture by categorical frameworks that have identified Stone-type dualities to be one of the cornerstones of algebraic language

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theory [UACM17, Sal17, Blu21]. On the other hand, Gehrke et al. [GGP08] discovered that, under Goldblatt's form of extended Priestley duality [Gol89], the *residuals* of language concatenation dualize to *multiplication* of profinite words, but so far this result could not yet be placed in the categorical big picture. One reason might be that, despite some progress in recent years [BKR07, HN15], extended (Stone) dualities for (co-)algebras are themselves not fully understood as instances of a crisp categorical idea.

Contributions. In the present paper, we introduce in Section 3 a simple categorical framework for extending any categorical duality $\mathbf{C} \simeq^{\mathrm{op}} \widehat{\mathbf{C}}$ via *monoidal adjunctions*: for a given adjunction on $\mathbf{C}$ with a strong monoidal right adjoint U, we establish a dual equivalence between U-operators on $\mathbf{C}$ and operators in a Kleisli category on $\widehat{\mathbf{C}}$. This framework is compositional both in its parameter – the strong monoidal right adjoint – and the object level of U-operators, which yields a simple categorical version of correspondence theory.

We demonstrate the power of this framework by working out several examples. On one hand, we show how to recover existing dualities and applications thereof, but we also indicate how to extend existing dualities including completely new instances of dualities.

To this end, we show in Section 4 how to use our framework to recover, and mildly extend, extended Priestley duality for distributive lattices with operators [Gol89] and relational morphisms. We also show how to use the compositionality of the abstract extended duality to recover results from modal correspondence theory for free.

Guided by our categorical foundations for extended Stone duality, we subsequently investigate in Section 5 the correspondence between language derivatives and multiplication of profinite words in the setting of *residuation algebras*. The key observation is that, on complete and completely distributive lattices, the residuals are equivalent to a *coalgebraic* operator on the lattice. This equivalence can then be composed with an extended duality based on the discrete duality between complete atomic Boolean algebras (CABAs) and sets to obtain a duality between certain complete residuation algebras and ordered monoids. We then proceed to lift this correspondence to locally finite structures, i.e. structures built up from finite substructures. By identifying suitable non-full subcategories – derivation algebras and (lattice) comonoids, respectively – and an appropriate definition of morphism for residuation algebras, we augment Gehrke's characterization of Stone-topological algebras in terms of residuation algebras to a duality between the categories of derivation algebras and that of profinite ordered monoids:

$$\mathbf{Der} \;\cong\; \mathbf{Comon} \;\simeq^{\mathrm{op}}\; \mathbf{ProfOrdMon}. \tag{1.1}$$

The abstract theory of extended duality now suggests that the dual equivalence between profinite ordered monoids on the one side and comonoids as well as derivation algebras on the other side extends to a more general duality capturing morphisms of *relational* type of profinite ordered monoids. To this end, we identify a natural notion of relational morphism for residuation algebras and comonoids, and we use our abstract extended duality theorem to obtain the dual equivalence

$$\mathbf{RelDer} \;\cong\; \mathbf{RelComon} \;\simeq^{\mathrm{op}}\; \mathbf{RelProfOrdMon}$$

which extends (1.1) to relational morphisms.

Finally, we combine the ideas underlying these results to derive a novel duality between the category of all small categories and the category of *categorical residuation algebras*:

$$\mathbf{Cat} \;\simeq^{\mathrm{op}}\; \mathbf{CatResCABA}. \tag{1.2}$$

To achieve this, we use the duality between monoids and residuation CABAs and combine it with two observations: first, small categories are equivalent to certain *relational monoids*; second, the discrete version of the duality (1.1) admits an extension to a duality between relational monoids and *residuation CABAs*. We prove that the composite of these equivalences restricts to (1.2).

Related Work.

This paper is a completely revised and extended version of our conference paper [BMU24] presented at FoSSaCS 2024. Besides providing detailed proofs of all results, we have included additional material: Proposition 3.12 simplifies dualization of composite operators, and it is used in Section 4.3 to show how to derive results from modal correspondence theory in a categorical way, by encoding modal formulas as morphisms. Section 5.3 has been extended to complete lattices, instead of just finite ones, to set the base duality for the material in Section 6. We have added Proposition 5.40 to complete the picture relating profinite ordered monoids and Priestley monoids similarly to the unordered case. Finally, the dual characterization of the category **Cat** of small categories (Section 6) is a new application of our abstract methods.

Extended Stone Duality. Duality for (complete) Boolean algebras with operators goes back to Jónsson and Tarski [JT51, JT52]. This duality was refined by the topological approach via Stone spaces taken by Halmos [Hal58], which allowed to characterize the relations arising as the duals of operators, namely *Boolean relations*. Halmos' duality was extended to distributive lattices with (n-ary) operators by Goldblatt [Gol89] and Cignoli [CLP91]. Kupke et al. [KKV04] recognized that Boolean relations elegantly describe descriptive frames as coalgebras for the (underlying functor of) the Vietoris monad on the category of Stone spaces; notions of bisimulation for these coalgebras were investigated by Bezhanishvili et al. [BFV10]. Bonsangue et al. [BKR07] introduced a framework for dualities over distributive lattices equipped with a theory of operators for a signature, which are dual to certain coalgebras. Hofmann and Nora [HN15] have taken a categorical approach to extend natural dualities to algebras for a signature equipped with unary operators preserving only some of the operations prescribed by the signature; they relate these to coalgebras for (the underlying functor of) a suitable monad T. Recent work by Bezhanishvili et al. [BHM23] clarifies the relation between free constructions on distributive lattices and different versions of the Vietoris monad to derive several dualities for distributive lattices with different types of operators and their corresponding Priestley relations.

Similarly to Hofmann and Nora's work [HN15] we have tried to fit most of these developments in a single, categorical framework. Our approach differs in two ways: first, incorporating monoidal structures into our setting immediately allows us to dualize operators with multiple in- and outputs. Second, besides a "nice" base duality, our framework depends only on some monoidal right adjoint. In contrast, op. cit. requires, in addition to the right adjoint (represented by a subsignature), a candidate monad T and gives a criterion whether an extended duality between operators for the subsignature and T-coalgebras exists.

Residuation Algebras. The original work by Jónsson and Tarski [JT51] already used duality for residual operators (also called *conjugates*) on Boolean algebras. Residuated Boolean algebras, i.e. Boolean algebras with a residuated binary operator, were explicitly studied by Jónsson and Tsinakis [JT93] to highlight the role of the residuals in relation algebra. Gehrke et al. [GGP08] exposed the connection between the residuals of the concatenation of regular languages and the multiplication on profinite words and investigated applications to automata theory, most notably a duality-theoretic proof of Eilenberg's variety theorem [Eil76]. The duality theory behind the correspondence of general residuation algebras and Priestley-topological algebras developed by Gehrke [Geh16b] is based on Goldblatt's extension of Stone duality [Gol89] for distributive lattices. This duality was further investigated via the theory of canonical extensions [GJ94, Geh09, GP07] to show that certain crucial parts are not entirely "algebraic": While Gehrke [Geh16b] provides a condition under which the dual relation of the residuals is functional, Fussner and Palmigiano [FP19] showed that it cannot be equationally defined in the language of residuation algebras.

Our duality for Priestley monoids is a non-trivial restriction of Gehrke's duality [Geh16b], and, to our knowledge, the first duality result for relational morphisms of profinite monoids, which are a ubiquitous tool in algebraic language theory [Pin88] and semigroup theory [RS09]. We also note that, while our results are closely related to Gehrke's [Geh16b], our methods are fundamentally different: while most of the proofs in op. cit. are of topological nature, we only work on the side of ordered structures, and outsource any topology to already existing dualities. In our opinion, this not only simplifies the theory, but it also clarifies the relation between Gehrke's results and the duality by Rhodes and Steinberg [RS09] between profinite monoids and counital Boolean bialgebras: on the algebraic side, derivation algebras correspond to Boolean comonoids, which are precisely counital Boolean bialgebras.

2. Preliminaries

Readers are assumed to be familiar with basic category theory, such as functors, natural transformations, adjunctions and monoidal categories, see Mac Lane [ML98] for an introduction.

We briefly recall the foundations of Stone duality [Sto36] and Priestley duality [Pri70]. By the latter we mean the dual equivalence $\mathbf{DL} \simeq^{\mathrm{op}} \mathbf{Priest}$ between the category $\mathbf{DL}$ of bounded distributive lattices and lattice homomorphisms, and the category $\mathbf{Priest}$ of Priestley spaces (ordered compact topological spaces in which for every $x \not\leq y$ there exists a clopen upset containing x but not y) and continuous monotone maps. The duality maps a distributive lattice D to the pointwise-ordered space $\mathbf{DL}(D, 2)$ of homomorphisms into the two-element lattice (equivalently, prime filters, ordered by inclusion), and topologized via pointwise convergence. In the reverse direction, it maps a Priestley space X to the distributive lattice $\mathbf{Priest}(X, 2)$ of continuous maps into the two-element poset $2 = \{0 \leq 1\}$ with discrete topology (equivalently, clopen upsets), with the pointwise lattice structure. Priestley duality restricts to Stone duality $\mathbf{BA} \simeq^{\mathrm{op}} \mathbf{Stone}$ between the full subcategories $\mathbf{BA}$ of Boolean algebras and $\mathbf{Stone}$ of Stone spaces (discretely ordered Priestley spaces).

This duality also has a "topologically discrete" version: First recall that an element k in a lattice D is *compact* if for every subset $A \subseteq D$ with $k \leq \bigvee A$ there exists a finite subset $F \subseteq A$ with $k \leq F$. A lattice is *algebraic* if every element is the join of the compact elements below it. There exists a duality $\mathbf{Pos} \simeq^{\mathrm{op}} \mathbf{ACDL}$ between the category $\mathbf{Pos}$ of posets with order-preserving maps and the category $\mathbf{ACDL}$ of algebraic completely distributive lattices

(ACDLs) with maps preserving all joins and meets. Under this duality a poset P is mapped to the set $\mathbf{Pos}(P, 2) \cong \mathcal{D}P$, which corresponds to the set of downsets of P. In the reverse direction, an ACDL D is mapped to the pointwise-ordered poset $\mathbf{ACDL}(D, 2) \cong \mathcal{J}P$, which corresponds to the poset of completely join-prime elements of D, i.e. those elements of D satisfying $p \leq \bigvee A \Rightarrow p \leq a$ for some $a \in A$. This duality restricts to the well-known duality $\mathbf{Set} \simeq^{\mathrm{op}} \mathbf{CABA}$ between the category $\mathbf{Set}$ and the category $\mathbf{CABA}$ of complete atomic boolean algebras.

Both the topological and the discrete duality restrict to Birkhoff duality [Bir37] $\mathbf{DL}_{\mathrm{f}} \simeq^{\mathrm{op}} \mathbf{Pos}_{\mathrm{f}}$ between finite distributive lattices and finite posets. For a comprehensive introduction to ordered structures and their dualities, see the first two chapters of the classic textbook by Johnstone [Joh82].

3. Extending Dualities

Our first contribution is a general categorical framework for extending Stone-type dualities via monoidal adjunctions. It is motivated by the extension of Priestley duality to operators due to Goldblatt [Gol89] (which is recovered in Section 4) and serves as the basis for several concrete duality results derived subsequently.

We start this chapter by setting up some notation for the two ingredients involved: adjunctions and monoidal categories.

Notation 3.1.

(1) Given functors $U\colon \mathbf{C} \to \mathbf{D}$ and $F\colon \mathbf{D} \to \mathbf{C}$, we write

$$F\colon \mathbf{D} \dashv \mathbf{C} :U$$

(or simply $F \dashv U$) if F is left adjoint to U. We denote the unit and counit by

$$\eta\colon \mathrm{Id} \to UF \qquad \text{and} \qquad \varepsilon\colon FU \to \mathrm{Id}; \tag{3.1}$$

the transposing isomorphisms are denoted by

$$(-)^+\colon \mathbf{D}(C, UD) \rightleftarrows \mathbf{C}(FC, D) :(-)^-$$

Hence, for every $f\colon C \to UD$ and $g\colon FC \to D$ we have

$$f^+ = \varepsilon_D \cdot Ff, \qquad \text{and} \qquad g^- = Ug \cdot \eta_C.$$

(2) For a category $\mathbf{C}$ with dual $\widehat{\mathbf{C}}$ we denote both contravariant functors witnessing the dual equivalence by

$$\widehat{(-)}\colon \mathbf{C} \xrightarrow{\simeq} \widehat{\mathbf{C}} \qquad \text{and} \qquad \widehat{(-)}\colon \widehat{\mathbf{C}} \xrightarrow{\simeq} \mathbf{C}.$$

Moreover, if $F\colon \mathbf{C} \to \mathbf{D}$ is a functor and $\widehat{\mathbf{D}}$ is dual to $\mathbf{D}$, then we denote the dual of F by

$$\widehat{F} = \widehat{(-)} \cdot F \cdot \widehat{(-)}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{D}}.$$

(3) The Kleisli category of a monad (T, η, μ) on the category $\mathbf{C}$ is denoted by $\mathbf{C}_T$. It has the same objects as $\mathbf{C}$ and $\mathbf{C}_T(X, Y) = \mathbf{C}(X, TY)$; the composition of *Kleisli morphisms* $f\colon C \to TD$ and $g\colon D \to TE$ is defined by

$$g \cdot f = \big(C \xrightarrow{f} TD \xrightarrow{Tg} TTE \xrightarrow{\mu_E} TE\big).$$

A Kleisli morphism $f\colon C \to TD$ is *pure* if $f = \eta_D \cdot f'$ for some $f'\colon C \to D$ in $\mathbf{C}$. We write $J_T\colon \mathbf{C} \to \mathbf{C}_T$ for the usual identity-on-objects functor mapping $f\colon A \to B$ to $\eta_B \cdot f\colon A \to TB$.

Convention 3.2. To lighten notation, we omit subscripts indicating components of natural transformations when they are clear from the context, e.g. we write $\eta\colon A \to TA$ for η_A.

Definition 3.3.
(1) A *monoidal category* is a category $\mathbf{C}$ with a bifunctor $\otimes\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called *tensor*, an object $I \in \mathbf{C}$ called *unit* together with natural isomorphisms

$$\upsilon\colon I \otimes \mathrm{Id}_{\mathbf{C}} \cong \mathrm{Id}_{\mathbf{C}} \qquad \text{and} \qquad \alpha\colon (\mathrm{Id}_{\mathbf{C}} \otimes \mathrm{Id}_{\mathbf{C}}) \otimes \mathrm{Id}_{\mathbf{C}} \cong \mathrm{Id}_{\mathbf{C}} \otimes (\mathrm{Id}_{\mathbf{C}} \otimes \mathrm{Id}_{\mathbf{C}}),$$

which are subject to natural *coherence axioms* (see e.g. [ML98, Section VII]). A monoidal category is *strict* if υ and α are identities.
(2) A functor $U\colon \mathbf{C} \to \mathbf{D}$ between monoidal categories that is equipped with a morphism $\epsilon\colon I_{\mathbf{D}} \to UI_{\mathbf{C}}$ and a transformation

$$\lambda\colon U(-) \otimes U(-) \to U(- \otimes -)$$

which is natural in both components is *lax monoidal* if it satisfies appropriate associativity and unitarity axioms with respect to υ and α. If both ϵ and λ are isomorphisms (identities) then we say U is *strong* (*strict*) monoidal.

Notation 3.4. Given an object X of a monoidal category, the nth tensor power is denoted

$$X^{\otimes n} = \bigotimes_{i=1}^{n} X,$$

and for a functor G the expression $GX^{\otimes n}$ is parsed as $G(X^{\otimes n})$, as usual.

We now introduce the setting in which our framework for extending dualities applies. In the simplest sense, the only ingredient is a strong monoidal functor:

Assumption 3.5. We fix strict monoidal categories $\mathbf{C}, \mathbf{D}$ with dually equivalent categories $\widehat{\mathbf{C}}, \widehat{\mathbf{D}}$; we regard $\widehat{\mathbf{C}}, \widehat{\mathbf{D}}$ as monoidal categories with tensor products $\widehat{\otimes}$ dual to the tensor products $\otimes$ of $\mathbf{C}, \mathbf{D}$. Moreover, we fix an adjunction $F\colon \mathbf{D} \dashv \mathbf{C} :U$ with unit η and counit ε (3.1), and we assume that U is a strong monoidal functor with associated natural isomorphisms

$$\lambda\colon UX \otimes UY \cong U(X \otimes Y) \qquad \text{and} \qquad \epsilon\colon I_{\mathbf{D}} \cong UI_{\mathbf{C}}.$$

We denote the monad dual to the comonad FU by $T = \widehat{F}\widehat{U}$, with unit and multiplication

$$e = \hat{\varepsilon}\colon \mathrm{Id} \to T \qquad \text{and} \qquad m = \widehat{F}\hat{\eta}\widehat{U}\colon TT \to T.$$

Overall, we have the following situation:

$$\begin{tikzcd} \mathbf{D} \arrow[d, bend right, "F"'] \arrow[r, phantom, "\simeq^{\mathrm{op}}"] \arrow[d, phantom, "\dashv"] & \widehat{\mathbf{D}} \arrow[d, bend right, "\widehat{F}"'] \arrow[d, phantom, "\vdash"] \\ \mathbf{C} \arrow[u, bend right, "U"'] \arrow[r, phantom, "\simeq^{\mathrm{op}}"] & \widehat{\mathbf{C}} \arrow[u, bend right, "\widehat{U}"'] \arrow[loop right, "T"] \end{tikzcd} \tag{3.2}$$

Remark 3.6.
(1) By Mac Lane's *Coherence Theorem* [ML98, Section VII.2], every monoidal category is equivalent to a strict monoidal category, hence the strictness requirement on $\mathbf{C}$ in Assumption 3.5 is without loss of generality.

(2) The isomorphism λ can be extended to an isomorphism

$$\lambda\colon UX_1 \otimes \cdots \otimes UX_n \cong U(X_1 \otimes \cdots \otimes X_n)$$

for all finite families of objects $X_1 \ldots, X_n$ in $\mathbf{C}$.

(3) The functor $\widehat{U}\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{D}}$ dual to U is a strong monoidal *left* adjoint of $\widehat{F}$, and the unit and counit of the adjunction $\widehat{U} \dashv \widehat{F}$ are given by $\hat{\varepsilon}$ and $\hat{\eta}$, respectively. Since $\widehat{U}$ is strong monoidal with respect to the isomorphisms

$$\hat{\epsilon}\colon \widehat{I}_{\mathbf{D}} \cong \widehat{U}\widehat{I}_{\mathbf{C}} \qquad \text{and} \qquad \hat{\lambda}\colon \widehat{U}X \mathbin{\widehat{\otimes}} \widehat{U}Y \cong \widehat{U}(X \mathbin{\widehat{\otimes}} Y) : \hat{\lambda}^{-1},$$

its right adjoint $\widehat{F}$ is lax monoidal (see e.g. [SS02, p. 17]) for the natural transformations

$$(\hat{\epsilon}^{-1})^{-}\colon \widehat{I}_{\mathbf{C}} \to \widehat{F}\widehat{I}_{\mathbf{D}} \qquad \text{and} \qquad ((\hat{\eta} \mathbin{\widehat{\otimes}} \hat{\eta}) \cdot \hat{\lambda}^{-1})^{-}\colon \widehat{F}X \mathbin{\widehat{\otimes}} \widehat{F}Y \to \widehat{F}(X \mathbin{\widehat{\otimes}} Y).$$

This makes $\widehat{U} \dashv \widehat{F}$ a monoidal adjunction and $T = \widehat{F}\widehat{U}$ a *monoidal* or *commutative* monad on $\widehat{\mathbf{C}}$ with monoidal structure $\hat{\delta}\colon TX \mathbin{\widehat{\otimes}} TY \to T(X \mathbin{\widehat{\otimes}} Y)$ given as the appropriate composite of the monoidal structures of $\widehat{F}$ and $\widehat{U}$. Note that $\hat{\delta}$ also extends to any arity, that is, for every n-tuple of objects $X_1, \ldots X_n$, we obtain a natural transformation

$$\hat{\delta}\colon TX_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} TX_n \to T(X_1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} X_n).$$

(4) The tensor product $\widehat{\otimes}$ of $\widehat{\mathbf{C}}$ lifts to the Kleisli category $\widehat{\mathbf{C}}_T$; the lifting maps a pair $(f\colon X \to TY, g\colon X' \to TY')$ of $\widehat{\mathbf{C}}_T$-morphisms to the $\widehat{\mathbf{C}}_T$-morphism

$$\hat{\delta} \cdot (f \widehat{\otimes} g)\colon X \widehat{\otimes} X' \to TY \widehat{\otimes} TY' \to T(Y \widehat{\otimes} Y').$$

This makes $\widehat{\mathbf{C}}_T$ itself a monoidal category (see e.g. [Sea53, Prop. 1.2.2]) with tensor $\widehat{\otimes}$, and the canonical left adjoint $J_T\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}_T$ a strict monoidal functor.

We now show how to extend the duality $\mathbf{C} \simeq^{\mathrm{op}} \widehat{\mathbf{C}}$ along the adjunction $F \dashv U$ to yield a duality between operators.

Definition 3.7.

(1) Let $G\colon \mathbf{A} \to \mathbf{B}$ be a functor between monoidal categories, and let $k, n \in \mathbb{N}$. An *(k,n)-ary G-operator* consists of an object $A \in \mathbf{A}$ and a morphism $a\colon (GA)^{\otimes k} \to (GA)^{\otimes n}$ of $\mathbf{B}$. A *(k,n)-ary G-operator morphism* from (A,a) to (B,b) is a morphism $h\colon GA \to GB$ of $\mathbf{B}$ such that the following square commutes:

$$\begin{tikzcd}
(GA)^{\otimes k} \arrow[r, "a"] \arrow[d, "h^{\otimes k}"'] & (GA)^{\otimes n} \arrow[d, "h^{\otimes n}"] \\
(GB)^{\otimes k} \arrow[r, "b"] & (GB)^{\otimes n}
\end{tikzcd}$$

The category of (k,n)-ary G-operators and their morphisms is denoted by $\mathrm{Op}_G^{k,n}(\mathbf{A})$.

(2) A *G-operator* is a (k,n)-ary G-operator for some k and n.

(3) A *G-algebra* is a $(k,1)$-ary G-operator, and a *G-coalgebra* is a $(1,n)$-ary one.

(4) If G is strong monoidal, then a (k,n)-ary G-operator (A,a) is *pure* if there exists a morphism $a'\colon A^{\otimes k} \to A^{\otimes n}$ such that

$$a = \big((GA)^{\otimes k} \xrightarrow{\lambda} G(A^{\otimes k}) \xrightarrow{Ga'} G(A^{\otimes n}) \xrightarrow{\lambda^{-1}} (GA)^{\otimes n}\big),$$

where λ is given analogously to Assumption 3.5. An operator morphism $h\colon (A,a) \to (B,b)$ is *pure* if there exists a morphism $h'\colon A \to B$ such that $h = Gh'\colon GA \to GB$.

Two cases will be important for intuition: (1) if G is a forgetful functor between varieties of algebraic structures, then a G-operator on an object A is a map preserving only part of the structure on A; (2) if T is a powerset-like monad on a category of spaces, then an operator for the embedding $J_T\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}_T$ on a space $\widehat{A}$ is a (continuous) *relation* $\widehat{A} \to T\widehat{A}$.

Theorem 3.8 (Abstract Extended Duality)**.** *The category of (k,n)-ary U-operators is dually equivalent to the category of (n,k)-ary J_T-operators:*

$$\mathrm{Op}_U^{k,n}(\mathbf{C}) \ \simeq^{\mathrm{op}} \ \mathrm{Op}_{J_T}^{n,k}(\widehat{\mathbf{C}}).$$

Proof. The desired equivalence of categories is given by the functor

$$\Phi\colon\ \mathrm{Op}_{J_T}^{n,k}(\widehat{\mathbf{C}}) \to \mathrm{Op}_U^{k,n}(\mathbf{C}) \tag{3.3}$$

which maps a (n,k)-ary J_T-operator

$$\widehat{a}\colon \widehat{A}^{\widehat{\otimes} n} \to T\widehat{A}^{\widehat{\otimes} k} = \widehat{F}\widehat{U}\widehat{A}^{\widehat{\otimes} k}$$

to the (k,n)-ary U-operator

$$(UA)^{\otimes k} \xrightarrow{\ \lambda\ } UA^{\otimes k} \xrightarrow{\ a^-\ } UA^{\otimes n} \xrightarrow{\ \lambda^{-1}\ } (UA)^{\otimes n}, \tag{3.4}$$

and an operator morphism $\widehat{f}\colon (\widehat{A},\widehat{a}) \to (\widehat{B},\widehat{b})$ to $f^-\colon UB \to UA$.

Let us first verify that the definition of Φ makes sense on morphisms, i.e. that f^- is an operator morphism from $(B, \lambda^{-1}\cdot b^- \cdot \lambda)$ to $(A, \lambda^{-1}\cdot a^- \cdot \lambda)$. Unfolding the definitions of composition and monoidal structure in the Kleisli category, we see that $\widehat{f}$ is a morphism $\widehat{A} \to T\widehat{B} = \widehat{F}\widehat{U}\widehat{B}$ such that the following diagram in $\widehat{\mathbf{C}}$ commutes:

$$\begin{tikzcd}
\widehat{A}^{\widehat{\otimes} n} \arrow[r, "\widehat{a}"] \arrow[d, "\widehat{f}^{\widehat{\otimes} n}"] & T\widehat{A}^{\widehat{\otimes} k} \arrow[r, "T\widehat{f}^{\widehat{\otimes} k}"] & T(T\widehat{B})^{\widehat{\otimes} k} \arrow[r, "T\delta"] & TT\widehat{B}^{\widehat{\otimes} k} \arrow[d, "m"] \\
(T\widehat{B})^{\widehat{\otimes} n} \arrow[r, "\delta"] & T\widehat{B}^{\widehat{\otimes} n} \arrow[r, "T\widehat{b}"] & TT\widehat{B}^{\widehat{\otimes} k} \arrow[r, "m"] & T\widehat{B}^{\widehat{\otimes} k}
\end{tikzcd} \tag{3.5}$$

The multiplication m of the monad is the dual of $F\eta U$, so the dual diagram of (3.5) is precisely the outside of the following diagram in $\mathbf{C}$, in which all parts except (#) commute by naturality of η, ε and λ, and the triangle equations.

$$\begin{tikzcd}
FUFUB^{\otimes k} \arrow[d, "FUF\lambda^{-1}"] \arrow[dddd, bend right, "FU\delta"'] & FUB^{\otimes k} \arrow[l, "F\eta U"'] \arrow[r, "F\eta U"] \arrow[d, "F\lambda^{-1}"] \arrow[rr, bend right, "Fb^-"'] & FUFUB^{\otimes k} \arrow[r, "FUb"] & FUB^{\otimes n} \arrow[rr, "\delta"] \arrow[d, "F\lambda^{-1}"] & & (FUB)^{\otimes n} \arrow[dddd, bend left, "f^{\otimes n}"] \\
FUF(UB)^{\otimes k} \arrow[d, "FUF(\eta U)^{\otimes k}"] & F(UB)^{\otimes k} \arrow[l, "F\eta"'] \arrow[d, "F(\eta U)^{\otimes k}"] \arrow[dr, "F(f^-)^{\otimes k}"] & & F(UB)^{\otimes n} \arrow[r, "F\eta^{\otimes n}"] \arrow[dr, "F(f^-)^{\otimes n}"'] & F(UFUB)^{\otimes n} \arrow[r, "F\lambda"] \arrow[d, "F(Uf)^{\otimes n}"] & FU(FUB)^{\otimes n} \arrow[u, "\varepsilon"'] \arrow[d, "FUf^{\otimes n}"'] \\
FUF(UFUB)^{\otimes k} \arrow[d, "FUF\lambda"] \arrow[dr, "FUF(Uf)^{\otimes k}"] & F(UFUB)^{\otimes k} \arrow[l, "F\eta"'] \arrow[r, "F(Uf)^{\otimes k}"] & F(UA)^{\otimes k} \arrow[dl, "F\eta"'] \arrow[d, "F\lambda"] & & F(UA)^{\otimes n} \arrow[r, "F\lambda"] & FUA^{\otimes n} \arrow[dd, "\varepsilon"] \\
FUFU(FUB)^{\otimes k} \arrow[d, "FU\varepsilon"] \arrow[dr, "FUFUf^{\otimes k}"] & FUF(UA)^{\otimes k} \arrow[d, "FUF\lambda"] & FUA^{\otimes k} \arrow[dl, "F\eta U"'] \arrow[d, equal] & (\#) & & \\
FU(FUB)^{\otimes k} \arrow[rr, bend right, "FUf^{\otimes k}"'] & FUFUA^{\otimes k} \arrow[r, "FU\varepsilon"] & FUA^{\otimes k} \arrow[rrr, "a"] & & & A^{\otimes n}
\end{tikzcd}$$

To see that (#) also commutes, note that the counit $\varepsilon = \mathrm{id}^+$ is the adjoint transpose of the identity, and transposition is natural. So transposing the two paths from $FUB^{\otimes k}$ to $A^{\otimes n}$

that form part (#) yields the inner square of the following commutative diagram in $\mathbf{D}$:

$$\begin{tikzcd}
(UB)^{\otimes k} \arrow[r, "\lambda"] \arrow[dr, equals] & UB^{\otimes k} \arrow[r, "b^-"] \arrow[d, "\lambda^{-1}"] & UB^{\otimes n} \arrow[r, "\lambda^{-1}"] & (UB)^{\otimes n} \arrow[r, "(f^-)^{\otimes n}"] & (UA)^{\otimes n} \arrow[d, "\lambda"] \arrow[dr, equals] & \\
& (UB)^{\otimes k} \arrow[r, "(f^-)^{\otimes k}"] & (UA)^{\otimes k} \arrow[r, "\lambda"] & UA^{\otimes k} \arrow[r, "a^-"] & UA^{\otimes n} \arrow[r, "\lambda^{-1}"] & U(A^{\otimes n})
\end{tikzcd}$$

By pre- and postcomposition of this square with λ and λ^{-1}, respectively, and replacing a^- and b^- by their respective conjugates $\alpha = \lambda^{-1} \cdot a^- \cdot \lambda$ and $\beta = \lambda^{-1} \cdot b^- \cdot \lambda$ this diagram simply becomes the square

$$\begin{tikzcd}
(UB)^{\otimes k} \arrow[r, "\beta"] \arrow[d, "(f^-)^{\otimes k}"'] & (UB)^{\otimes n} \arrow[d, "(f^-)^{\otimes n}"] \\
(UA)^{\otimes k} \arrow[r, "\alpha"] & (UA)^{\otimes n}
\end{tikzcd}$$

in $\mathbf{D}$, which is a homomorphism diagram of (k, n)-ary U-operators.

We have shown that Φ is indeed a functor. Now we verify that it is an equivalence.

The (natural) isomorphisms

$$\mathbf{D}(UA, UB) \cong \mathbf{C}(FUA, B) \cong \widehat{\mathbf{C}}(\widehat{B}, \widehat{F}\widehat{U}\widehat{A}) = \widehat{\mathbf{C}}(\widehat{B}, T\widehat{A})$$

given by the duality and the adjunction $F \dashv U$ ensure that this functor is fully faithful.

To see that it is essentially surjective, pick any U-operator $c\colon (UC)^{\otimes k} \to (UC)^{\otimes n}$. We have to show that it is isomorphic to an operator of the form (3.4). Since the original duality $\mathbf{C} \simeq^{\mathrm{op}} \widehat{\mathbf{C}}$ is essentially surjective there exists an isomorphism $h\colon C \xrightarrow{\sim} \widehat{X}$ for some $X \in \widehat{\mathbf{C}}$. Let $a\colon FU\widehat{X}^{\otimes k} \to \widehat{X}^{\otimes n}$ be the adjoint transpose of the $\mathbf{D}$-morphism

$$U\widehat{X}^{\otimes k} \xrightarrow{\lambda^{-1}} (U\widehat{X})^{\otimes k} \xrightarrow{(Uh^{-1})^{\otimes k}} (UC)^{\otimes k} \xrightarrow{c} (UC)^{\otimes n} \xrightarrow{(Uh)^{\otimes n}} (U\widehat{X})^{\otimes n} \xrightarrow{\lambda} U\widehat{X}^{\otimes n}.$$

Then $Uh\colon (UC, c) \to (U\hat{X}, \lambda^{-1} \cdot a^- \cdot \lambda)$ is a pure operator isomorphism. □

Remark 3.9. The definition of Φ (3.3) can be slightly generalized to yield a dual correspondence between morphisms $a\colon (UA)^{\otimes k} \to (UB)^{\otimes n}$ of $\mathbf{D}$ and morphisms $\rho\colon \hat{B}^{\widehat{\otimes} n} \to \hat{T}A^{\widehat{\otimes} k}$ of $\widehat{\mathbf{D}}$: the dual of a is given by $\rho = \widehat{h^+}$, where $h = \lambda \cdot a \cdot \lambda^{-1}\colon UA^{\otimes k} \to UB^{\otimes n}$.

Our approach to extending dualities is *compositional* on two levels: first, compositionality of adjunctions allows us to dualize certain operators more precisely; second, on the object level, U-operators themselves admit a compositional structure under morphism composition and tensor product, leading to *simpler* calculations of dual operators. We will use the first and second version of compositionality in Sections 4.2 and 4.3, respectively.

We elaborate on the first point: Let $\mathbf{E}$ be a monoidal category with monoidal adjunctions

$$F_1\colon \mathbf{E} \dashv \mathbf{C} : U_1 \qquad \text{and} \qquad F_2\colon \mathbf{D} \dashv \mathbf{E} : U_2$$

which *split* $F \dashv U$ (i.e. $F = F_1F_2$, $U = U_2U_1$), and suppose that the monoidal structure of U is given by $\lambda = U_2\lambda_1 \cdot \lambda_2 U_1$. The compositionality of adjunctions leads to the following lifting property, applying to both operators and operator morphisms by setting $A = B$ and $k = n = 1$, respectively, in the following proposition. Here we say that, for a monoidal functor G, a morphism $f\colon (GX)^{\otimes i} \to (GY)^{\otimes j}$ *lifts along* G, if there exists a morphism $g\colon X^{\otimes i} \to Y^{\otimes j}$ with $f = \lambda^{-1} \cdot Gg \cdot \lambda$. The morphism g is called a *lifting* of f.

Proposition 3.10. *Let* $a\colon (UA)^{\otimes k} \to (UB)^{\otimes n}$ *be a morphism in* $\mathbf{D}$ *dual to* $\rho\colon \widehat{B}^{\widehat{\otimes} n} \to T\widehat{A}^{\widehat{\otimes} k}$ *in* $\widehat{\mathbf{C}}$ *as in Remark 3.9. Then the following are equivalent:*

(1) *a lifts along U_2, that is,*

$$a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2 \qquad \text{for some } b\colon (U_1 A)^{\otimes k} \to (U_1 B)^{\otimes n};$$

(2) *ρ factorizes through the monad morphism $\widehat{F}_1 \hat{\varepsilon}_2 \widehat{U}_1 \colon T_1 \to T$ (where $T_1 = \widehat{F}_1 \widehat{U}_1$), that is,*

$$\rho = \widehat{F}_1 \hat{\varepsilon}_2 \widehat{U}_1 \cdot \sigma \qquad \text{for some } \sigma\colon \widehat{B}^{\widehat{\otimes} n} \to T_1 \widehat{A}^{\widehat{\otimes} k}.$$

Proof. The dual of the morphism $a\colon (U_2 U_1 A)^{\otimes k} \to (U_2 U_1 B)^{\otimes n}$ under the abstract extended duality is given by

$$\rho = \widehat{h^+}\colon \widehat{B}^{\widehat{\otimes} n} \to T\widehat{A}^{\widehat{\otimes} k} = \widehat{F}_1 \widehat{F}_2 \widehat{U}_2 \widehat{U}_1 \widehat{A}^{\widehat{\otimes} k},$$

where h is the unique morphism making the outside of the following diagram commute:

$$\begin{tikzcd}
U_2 U_1 A^{\otimes k} \arrow[r, "h"] & U_2 U_1 B^{\otimes n} \arrow[d, "U_2 \lambda_1^{-1}"] \arrow[dd, bend left=60, "\lambda^{-1}"] \\
U_2 (U_1 A)^{\otimes k} \arrow[u, "U_2 \lambda_1"] & U_2 (U_1 B)^{\otimes n} \arrow[d, "\lambda_2^{-1}"] \\
(U_2 U_1 A)^{\otimes k} \arrow[u, "\lambda_2"] \arrow[r, "a"] \arrow[uu, bend left=60, "\lambda"] & (U_2 U_1 B)^{\otimes n}
\end{tikzcd}$$

We denote the transposition isomorphisms of the adjunctions $F_i \dashv U_i$ by

$$(-)^{\flat_i}\colon \mathbf{C}(F_i X, Y) \rightleftarrows \mathbf{D}(X, U_i Y) : (-)^{\sharp_i}.$$

Now assume that $\widehat{g^{\sharp_1}}\colon \widehat{B}^{\widehat{\otimes} n} \to T_1 \widehat{A}^{\widehat{\otimes} k} = \widehat{F}_1 \widehat{U}_1 \widehat{A}^{\widehat{\otimes} k}$ is a morphism in $\widehat{\mathbf{C}}$ such that we have a factorization $\widehat{h^+} = \widehat{F}_1 \hat{\varepsilon}_2 \widehat{U}_1 \cdot \widehat{g^{\sharp_1}}$. Under duality this is equivalent to

$$h^+ = g^{\sharp_1} \cdot F_1 \varepsilon_2 U_1 \iff h^{\sharp_2} = (h^+)^{\flat_1} = g \cdot \varepsilon_2 U_1 \iff h = (h^{\sharp_2})^{\flat_2} = U_2 g \cdot (\varepsilon_2 U_1)^{\flat_2} = U_2 g$$

using naturality of the isomorphisms $\flat_1, \flat_2$. The dual of the J_{T_1}-operator $\widehat{g^{\sharp_1}}$ under the abstract extended duality along the adjunction $F_1 \dashv U_1$ is the U_1-operator $b = \lambda_1^{-1} \cdot g \cdot \lambda_1$. Therefore the following diagram commutes:

$$\begin{tikzcd}
U_2 U_1 A^{\otimes k} \arrow[r, "U_2 g = h"] & U_2 U_1 B^{\otimes n} \arrow[d, "U_2 \lambda_1^{-1}"] \arrow[dd, bend left=60, "\lambda^{-1}"] \\
U_2 (U_1 A)^{\otimes k} \arrow[u, "U_2 \lambda_1"] \arrow[r, "U_2 b"] & U_2 (U_1 B)^{\otimes n} \arrow[d, "\lambda_2^{-1}"] \\
(U_2 U_1 A^{\otimes k}) \arrow[u, "\lambda_2"] \arrow[r, "a"] \arrow[uu, bend left=60, "\lambda"] & (U_2 U_1) B^{\otimes n}
\end{tikzcd}$$

This is equivalent to a admitting the lifting b, that is, $a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2$. $\square$

Remark 3.11.

(1) A special case of Proposition 3.10 proves that extended dualities preserve purity: splitting $F \dashv U$ into $F_1 = \mathrm{Id} \dashv \mathrm{Id} = U_1$ and $F_2 = F \dashv U = U_2$, we see that a U-operator (or operator morphism) a is pure iff its dual f is pure as a Kleisli morphism, that is, it factorizes through the unit $\hat{\varepsilon}$ of the monad T.

(2) The right adjoint U_2 often is faithful and in this case $\widehat{F}_1\hat{\varepsilon}_2\widehat{U}_1$ is monic, that is, T_1 is a submonad of T: indeed, faithfulness of U_2 is equivalent to having an epic counit ε_2, hence $\hat{\varepsilon}_2\widehat{U}_1$ is monic, and the right adjoint $\hat{F}_1$ preserves monos. In particular, if T is 'powerset-like', then $\widehat{\mathbf{C}}_T$ is a category of relations, and we think of U-operators (or operator morphisms) of the form $a = \lambda_2^{-1} \cdot U_2 b \cdot \lambda_2$ as dualizing to 'more functional' relations. This idea is illustrated by the examples in Section 4.2.

The compositionality on the level of U-operators manifests itself as follows:

Proposition 3.12. *Let $h, g \colon UA \to UA$ be U-operators with respective duals $\rho, \sigma \colon \widehat{A} \to T\widehat{A}$. On objects the abstract extended duality of Theorem 3.8 preserves*

(1) Tensor products of operators:

$$h \otimes g \colon UA \otimes UA \to UA \otimes UA \quad \textit{is dual to} \quad \hat{\delta} \cdot (\rho \mathbin{\widehat{\otimes}} \sigma) \colon \widehat{A} \mathbin{\widehat{\otimes}} \widehat{A} \to T\widehat{A} \mathbin{\widehat{\otimes}} T\widehat{A} \to T(\widehat{A} \mathbin{\widehat{\otimes}} \widehat{A}).$$

(2) Composition of operators:

$$g \cdot h \colon UA \to UA \to UA \quad \textit{is dual to} \quad m \cdot T\rho \cdot \sigma \colon \widehat{A} \to T\widehat{A} \to TT\widehat{A} \to T\widehat{A}.$$

(3) Identity operators:

$$\mathrm{id}_{UA} \colon UA \to UA \quad \textit{is dual to} \quad e_{\widehat{A}} \colon \widehat{A} \to T\widehat{A}.$$

Proof.

(1) Under the extended duality the operator $h \otimes g$ is mapped to $\widehat{\alpha^+}$, the dual of the adjoint transpose α^+ of the conjugate α of $h \otimes g$:

$$\alpha = \big(U(A \otimes A) \xrightarrow{\lambda^{-1}} UA \otimes UA \xrightarrow{h \otimes g} UA \otimes UA \xrightarrow{\lambda} U(A \otimes A)\big).$$

We now prove that the following diagram commutes, where δ is the comonoidal structure of FU dual to the monoidal structure $\hat{\delta}$ of T from Remark 3.6.

$$\begin{tikzcd}
FU(A \otimes A) \arrow[r, "F\lambda^{-1}"] \arrow[d, "\alpha^+"] \arrow[dr, "\delta"] & F(UA \otimes UA) \arrow[r, "F(\eta \otimes \eta)"] \arrow[rr, bend left, "F(h \otimes g)"] & F(UFUA \otimes UFUA) \arrow[r, "F(Uh^+ \otimes Ug^+)"] \arrow[d, "F\lambda"] & F(UA \otimes UA) \arrow[d, "F\lambda"] \\
A \otimes A & FUA \otimes FUA \arrow[l, "h^+ \otimes g^+"'] & FU(FUA \otimes FUA) \arrow[l, "\varepsilon"'] \arrow[r, "FU(h^+ \otimes g^+)"] & FU(A \otimes A) \arrow[lll, bend left, "\varepsilon"]
\end{tikzcd}$$

Its outside commutes using the definition of the adjoint transpose α^+, and the upper part also commutes by adjoint transposition. The right-hand and lower parts commute by naturality of λ and ε, respectively. The middle part commutes trivially by the definition of δ. Thus, the left-hand triangle commutes. This shows that, under the duality $\mathbf{C} \simeq^{\mathrm{op}} \widehat{\mathbf{C}}$, the dual of α^+ is equal to the dual of $(h^+ \otimes g^+) \cdot \delta$, which in turn is given by $\hat{\delta} \cdot (\rho \mathbin{\widehat{\otimes}} \sigma)$.

(2) Similarly, the adjoint transpose of $g \cdot h$ is equal to $g^+ \cdot FUh^+ \cdot F\eta U$ whose dual is given by $m \cdot T\rho \cdot \sigma$.

(3) The adjoint transpose of id_{UA} is ε_A, whose dual is the unit $\hat{\varepsilon}_{\hat{A}} = e_{\hat{A}}$ of T. $\square$

4. Example: Extended Priestley Duality

As a first application of our adjoint framework, we investigate the classical Priestley duality (Section 2) and derive a generalized version of Goldblatt's duality [Gol89] between distributive lattices with operators and relational Priestley spaces. We instantiate (3.2) to the following categories and functors, which we will subsequently explain in detail:

$$\begin{tikzcd} \mathbf{D} \arrow[r, phantom, "\simeq^{\mathrm{op}}"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \widehat{\mathbf{D}} \arrow[d, bend right, "\widehat{F}"'] \arrow[d, phantom, "\vdash"] \\ \mathbf{C} \arrow[r, phantom, "\simeq^{\mathrm{op}}"] \arrow[u, bend right, "U"'] & \widehat{\mathbf{C}} \arrow[u, bend right, "\widehat{U}"'] \arrow[loop right, "T"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{JSL} \arrow[r, phantom, "\simeq^{\mathrm{op}}"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \mathbf{StoneJSL} \arrow[d, bend right, "\widehat{F}"'] \arrow[d, phantom, "\vdash"] \\ \mathbf{DL} \arrow[r, phantom, "\simeq^{\mathrm{op}}"] \arrow[u, bend right, "U"'] & \mathbf{Priest} \arrow[u, bend right, "\mathbb{V}_{\downarrow}"'] \arrow[loop right, "\mathbb{V}_{\downarrow}"] \end{tikzcd} \tag{4.1}$$

Categories. The upper duality is Hofman-Mislove-Stralka duality [HMS74] between the category of join-semilattices with a bottom element and the category of Stone semilattices (i.e. topological join-semilattices with a bottom element whose underlying topological space is a Stone space) and continuous semilattice homomorphisms. The duality maps a join-semilattice J to the Stone semilattice $\mathbf{JSL}(J,2)$ of semilattice homomorphisms into the two-element semilattice, topologized by pointwise convergence. Equivalently, $\mathbf{JSL}(J,2)$ is the space $\mathrm{Idl}(J)$ of *ideals* (downward closed and upward directed subsets) of J, ordered by reverse inclusion, with topology generated by the subbasic open sets $\sigma(j) = \{I \in \mathrm{Idl}(J) \mid j \in I\}$ and their complements for $j \in J$. In the other direction, a Stone semilattice X is mapped to its semilattice $\mathbf{StoneJSL}(X,2)$ of clopen ideals, ordered by inclusion.

Functors. The functor $U\colon \mathbf{DL} \to \mathbf{JSL}$ is the obvious forgetful functor. Its left adjoint $F\colon \mathbf{JSL} \to \mathbf{DL}$ maps a join-semilattice J to the set $\mathcal{U}^{\partial}_{\mathrm{fg}}(J)$ of finitely generated upsets of J, ordered by reverse inclusion. The dual right adjoint $\widehat{F}$ of F is the forgetful functor mapping a Stone semilattice to its underlying Priestley space. Indeed, since $F \dashv U$, we compute for the carrier set $|X|$ of a Stone semilattice X that

$$|\widehat{F}X| = |\widehat{F\widehat{X}}| \cong |\mathbf{DL}(F\widehat{X},2)| \cong |\mathbf{JSL}(\widehat{X},U2)| = |\mathbf{JSL}(\widehat{X},2)| \cong |X|,$$

and this bijection is a homeomorphism.

The left adjoint $\widehat{U}\colon \mathbf{Priest} \to \mathbf{StoneJSL}$ maps a Priestley space X to the Stone join-semilattice

$$\widehat{U\widehat{X}} = \mathbf{JSL}(U(\mathbf{Priest}(X,2)),2) \cong \mathrm{Idl}(\mathrm{Cl}_{\uparrow} X)$$

of ideals of clopen upsets of X. This space is isomorphic to the *(downset) Vietoris hyperspace* $\mathbb{V}_{\downarrow}X$ of X carried by the set of closed downsets of X. The isomorphism is given by

$$\begin{array}{rcl} \mathrm{Idl}(\mathrm{Cl}_{\uparrow} X) & \cong & \mathbb{V}_{\downarrow}X \\ I & \mapsto & \bigcap_{U \in I} X \setminus U \\ \{U \in \mathrm{Cl}_{\uparrow} X \mid C \subseteq X \setminus U\} & \leftarrow\!\shortmid & C. \end{array}$$

The topology of pointwise convergence on $\mathbf{JSL}(U(\mathbf{Priest}(X,2)),2)$ translates to the *hit-or-miss topology* on $\mathbb{V}_{\downarrow}X$ generated by the subbasic open sets

$$\{A \subseteq X \text{ closed } \mid A \cap U \neq \emptyset\} \quad \text{for} \quad U \in \mathrm{Cl}_{\uparrow} X$$

and their complements. Note that for a Stone space X, the Stone join-semilattice $\mathbb{V}_{\downarrow}X$ is the *free Stone join-semilattice* on X, as observed by Johnstone [Joh82, Sec. 4.8]. The monad induced by the adjunction is the *(downset) Vietoris monad*; its unit $e\colon X \to \mathbb{V}_{\downarrow}X$ is given by $x \mapsto {\downarrow}x$ and multiplication is given by union [HN15]. The monad $\mathbb{V}_{\downarrow}$ restricts to the Vietoris monad $\mathbb{V}$ on the category **Stone** of Stone spaces.

The duality of modal algebras and coalgebras for the Vietoris construction, going back to Esakia [Esa74], has often been rediscovered and extended since, see e.g. [KKV04, VV14]; for recent accounts with detailed computations regarding the dualities of FU and $\mathbb{V}_{\downarrow}$ see Bezhanishvili et al. [BHM23] or the textbook by Gehrke and van Gool [GvG24, Section 6.4].

Remark 4.1 (Continuous Relations)**.** Continuous maps in **Priest** of the form

$$\rho\colon X \to \mathbb{V}_{\downarrow}Y$$

are known in the literature under a variety of names; we call them as *Priestley relations* as in [CLP91, Gol89], or *Stone relations* if X, Y are Stone spaces. We write $x\,\rho\,y$ for $y \in \rho(x)$, and sometimes identify ρ with a subset of $X \times Y$. Let us note that some authors (e.g. [RS09]) call a relation $R \subseteq X \times Y$ between topological spaces *continuous* if it is closed as a subspace of $X \times Y$. Every Priestley relation is continuous. The converse generally fails: for instance, consider any non-discrete Stone space X and let $C \subseteq X$ be a subset that is closed but not open. The relation $C \times 1 \subseteq X \times 1$ is closed, but the corresponding map $\rho\colon X \to \mathbb{V}1$ (given by $\rho(x) = 1$ if $x \in C$ and $\rho(x) = \emptyset$ otherwise) is not continuous because $\rho^{-1}[1] = C$ is not open.

Monoidal Structure. The category **JSL** of join-semilattices has a tensor product $\otimes$ that represents *join-bilinear* maps, that is, maps $J \times J' \to K$ between join-semilattices preserving finite joins in each argument:

$$\mathrm{Bilin}(J \times J', K) \;\cong\; \mathbf{JSL}(J \otimes J', K).$$

Join-bilinear maps $J \times J' \to K$ and their corresponding **JSL**-morphisms $J \otimes J' \to K$ are often tacitly identified. The tensor product $\otimes$ makes **JSL** a monoidal category with unit 2, i.e. $2 \otimes J \cong J$. The standard presentation of $J \otimes J'$ as a join-semilattice is given by the generators $\{j \otimes j' \mid j \in J, j' \in J'\}$ with equations

$$j_1 \otimes 0 = 0 \otimes j_1' = 0, \quad (j_1 \vee j_2) \otimes j' = j_1 \otimes j' \vee j_2 \otimes j' \quad \text{and} \quad j \otimes (j_1' \vee j_2') = j \otimes j_1' \vee j \otimes j_2'$$

ranging over $j_1, j_2 \in J$ and $j_1', j_2' \in J'$. We call (the equivalence class of) a generating element $j \otimes j'$ a *pure tensor*. If D, D' are bounded distributive lattices then so is $UD \otimes UD'$ [Fra76], with meet given on pure tensors by $(d \otimes d') \wedge (e \otimes e') = (d \wedge e) \otimes (d' \wedge e')$. Moreover, the lattice $UD \otimes UD'$ is the coproduct of D, D' in **DL**: the coproduct injections are given by

$$\iota(d) = d \otimes 1' \qquad \text{and} \qquad \iota'(d') = 1 \otimes d'$$

for $d \in D, d' \in D'$, and the copairing of lattice homomorphisms $f\colon D \to E, f'\colon D' \to E$ is given by the extension of the join-bilinear map

$$\wedge \cdot (f \times f')\colon D \times D' \to E, \qquad (d, d') \mapsto f(d) \wedge f(d').$$

Taking coproducts yields a monoidal structure on **DL**, and since $U(D + D') = UD \otimes UD'$, the functor U is strict monoidal.

Lemma 4.2. *The dual monoidal structure $\hat{\delta}$ of $\mathbb{V}_{\downarrow}$ is given by product:*

$$\hat{\delta}\colon \mathbb{V}_{\downarrow}X \times \mathbb{V}_{\downarrow}Y \to \mathbb{V}_{\downarrow}(X \times Y), \qquad (C, D) \mapsto C \times D.$$

Proof. Let $C \in \mathbb{V}_{\downarrow}X, D \in \mathbb{V}_{\downarrow}Y$ be closed downsets. We first represent them by their respective ideals I_C, I_D of $\mathrm{Cl}_{\uparrow} X$ and $\mathrm{Cl}_{\uparrow} Y$, which are equivalently **JSL**-morphisms

$$c\colon U\,\mathrm{Cl}_{\uparrow} X \to U2,\ d\colon U\,\mathrm{Cl}_{\uparrow} Y \to U2.$$

The map $\hat{\delta}$ is the given as the dual

$$\mathbf{DL}(FU\,\mathrm{Cl}_{\uparrow} X + FU\,\mathrm{Cl}_{\uparrow} Y, 2) \to \mathbf{DL}(FU(\mathrm{Cl}_{\uparrow} X + \mathrm{Cl}_{\uparrow} Y), 2)$$

of the comonoidal structure $FU(A+B) \to FUA + FUB$ mapping $[c^+, d^+]$ to the prime filter that is the transpose of

$$\begin{aligned} U[c^+, d^+] \cdot (\eta \otimes \eta)\colon U(&\mathrm{Cl}_{\uparrow} X + \mathrm{Cl}_{\uparrow} Y) \\ &\cong U(\mathrm{Cl}_{\uparrow} X) \otimes U(\mathrm{Cl}_{\uparrow} Y) \to UFU\,\mathrm{Cl}_{\uparrow} X \otimes UFU\,\mathrm{Cl}_{\uparrow} Y \\ &\cong U(FU\,\mathrm{Cl}_{\uparrow} X + FU\,\mathrm{Cl}_{\uparrow} Y) \to U2. \end{aligned}$$

The latter map sends a pure tensor $A \otimes B \in U(\mathrm{Cl}_{\uparrow} X + \mathrm{Cl}_{\uparrow} Y)$ to its 'product' $c(A) \wedge d(B)$. Therefore the closed set $\hat{\delta}(C, D)$ corresponding to $U[c^+, d^+] \cdot (\eta \otimes \eta)$ contains a pair (x, y) iff $x \in C$ and $y \in D$. □

Expanding Definition 3.7, the category $\mathrm{Op}^{n,k}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest})$ is given as follows:

Definition 4.3. A (*(n,k)-ary*) *relational Priestley space* consists of a carrier Priestley space X and a Priestley relation $\rho\colon X^n \to \mathbb{V}_{\downarrow}X^k$. A *relational morphism* from a relational Priestley space (X, ρ) to a relational Priestley space (X', ρ') is given by a Priestley relation $\beta\colon X \to \mathbb{V}_{\downarrow}Y$ such that, for all $\mathbf{x} \in X^n$, $\mathbf{y} \in X^k$, and $\mathbf{y}' \in X'^k$,

$$\mathbf{x}\ \rho\ \mathbf{y} \wedge (\forall i\colon y_i\ \beta\ y'_i) \implies \exists \mathbf{x}'\colon (\forall i\colon x_i\ \beta\ x'_i) \wedge \mathbf{x}'\ \rho'\ \mathbf{y}',$$

and, for all $\mathbf{x} \in X^n$, $\mathbf{x}' \in X'^m$, and $\mathbf{y}' \in X'^k$,

$$(\forall i\colon x_i\ \beta\ x'_i) \wedge \mathbf{x}'\ \rho'\ \mathbf{y}' \implies \exists \mathbf{y}\colon \mathbf{x}\ \rho\ \mathbf{y} \wedge (\forall i\colon y_i\ \beta\ y'_i).$$

We denote by $\mathrm{Op}^{n,k}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest})$ the category of (n,k)-ary relational Priestley spaces and relational morphisms.

Then Theorem 3.8 instantiates to the following result:

Theorem 4.4 (Extended Priestley duality)**.** *The category of (k,n)-ary U-operators of distributive lattices is dually equivalent to the category of (n,k)-ary relational Priestley spaces and relational morphisms:*

$$\mathrm{Op}^{k,n}_{U}(\mathbf{DL}) \ \simeq^{\mathrm{op}}\ \mathrm{Op}^{n,k}_{J_{\mathbb{V}_{\downarrow}}}(\mathbf{Priest}).$$

By setting $n = 1$ and restricting the operator morphisms on both sides to be pure (Remark 3.11(1)), we recover Goldblatt's duality [Gol89]. In the latter work, pure relational morphisms are called *bounded morphisms* and n-ary U-algebras $(UD)^{\otimes n} \to UD$ in **JSL** are called n-ary *join-hemimorphisms.*

Corollary 4.5 (Goldblatt duality)**.** *The category of distributive lattices with n-ary join-hemimorphisms, and pure morphisms between them, is dually equivalent to the category of $(1,n)$-relational Priestley spaces and bounded morphisms.*

4.1. **Deriving Concrete Formulas.** We proceed to show how an order-enriched extension of our adjoint framework can be used to methodically derive concrete (i.e. element-based) formulas for the dual join operator of a continuous relation and vice versa. Let us first observe that all involved categories are *order-enriched* if we equip the homsets with the usual pointwise order on functions. Moreover, from the definitions it is clear that the transposing isomorphisms of the adjunction $F \dashv U$ and the duality $\mathbf{DL} \simeq^{\mathrm{op}} \mathbf{Priest}$ are order-isomorphisms.

Second, we can represent an element $\hat{x}$ of a Priestley space $\widehat{X}$ as a continuous function $\hat{x} \in \mathbf{Priest}(1, \widehat{X})$; on the lattice side, elements j of a join-semilattice J correspond bijectively to $\mathbf{JSL}$-morphisms $j \in \mathbf{JSL}(2, J)$, using that 2 is the free semilattice on a single generator.

For the rest of this subsection let us fix a U-algebra h and its dual Priestley relation ρ:

$$h\colon (UX)^{\otimes n} \to UX \qquad \text{and} \qquad \rho\colon \widehat{X} \to \mathbb{V}_{\downarrow}\widehat{X}^n.$$

We first show how to express ρ in terms of h. Viewing ρ as a relation from $\widehat{X}$ to $\widehat{X}^n$ (Remark 4.1), two elements $\hat{x} \in \widehat{X}, \hat{\mathbf{x}} \in \widehat{X}^n$ are related by ρ (i.e. $\hat{x}\ \rho\ \hat{\mathbf{x}}$) iff the inequality $e(\hat{\mathbf{x}}) = {\downarrow}\hat{\mathbf{x}} \leq \rho(\hat{x})$ holds, equivalently, iff the left diagram below commutes laxly as indicated:

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\rho"] & \mathbb{V}_{\downarrow}\widehat{X}^n & \\
1 \arrow[u, "\hat{x}"] \arrow[r, "\hat{\mathbf{x}}"'] \arrow[ur, phantom, "\leq"] \arrow[rr, bend right, "\Delta"'] & \widehat{X}^n \arrow[u, "e"'] & 1^n \arrow[l, "\prod_i \hat{x}_i"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
UX \arrow[d, "Ux"'] \arrow[dr, phantom, "\leq"] & (UX)^{\otimes n} \arrow[l, "h"'] \arrow[d, "\bigotimes_i Ux_i"] \\
U2 & (U2)^{\otimes n} \arrow[l, "\nabla"]
\end{tikzcd}$$

The duals of $\hat{x}, \hat{x}_i$ are $\mathbf{DL}$ morphisms $x, x_i\colon X \to 2$. Under duality and transposition the left diagram corresponds to the right diagram where ∇ is the codiagonal given by n-fold conjunction, i.e. it maps $\bigotimes_{i=1}^n x_i$ to $\bigwedge_{i=1}^n x_i$. Writing $F_z = z^{-1}(1)$ for the prime filter corresponding to a morphism $z \in \mathbf{DL}(X, 2)$ the right diagram yields Goldblatt's formula [Gol89, p. 186] for the dual Priestley relation of an algebra h: we have

$$\hat{x}\ \rho\ \hat{\mathbf{x}} \qquad \text{iff} \qquad h[\prod_i F_{x_i}] \subseteq F_x.$$

Conversely, to express h in terms of ρ, it suffices to describe $h(\mathbf{x})$ for a pure tensor $\mathbf{x} \in (UX)^{\otimes n}$ by the universal property of the tensor product. We factorize

$$\mathbf{x} = \bigotimes_i x_i \cdot \nabla^{-1}\colon U2 \cong (U2)^{\otimes n} \to (UX)^{\otimes n}$$

to see that the element $h(\mathbf{x})$ corresponds to the following morphism representing an element of the join-semilattice UX:

$$h \cdot \bigotimes_i x_i \cdot \nabla^{-1}\colon U2 \cong (U2)^{\otimes n} \to (UX)^{\otimes n} \to UX.$$

Its dual is the characteristic function

$$\widehat{X} \xrightarrow{\rho} \mathbb{V}_{\downarrow}\widehat{X}^n \xrightarrow{\mathbb{V}_{\downarrow}(\prod_i C_i)} \mathbb{V}_{\downarrow}(\mathbb{V}_{\downarrow}1)^n \xrightarrow{\mathbb{V}_{\downarrow}\hat{\delta}} \mathbb{V}_{\downarrow}\mathbb{V}_{\downarrow}1^n \xrightarrow{m} \mathbb{V}_{\downarrow}1^n \xrightarrow{\mathbb{V}_{\downarrow}\Delta^{-1}} \mathbb{V}_{\downarrow}1 = 2,$$

where $C_i = \widehat{x_i^+}$ is the clopen upset of $\widehat{X}$ dual to

$$x_i \in \mathbf{JSL}(U2, UX) \;\cong\; \mathbf{DL}(FU2, X) \;\cong\; \mathbf{Priest}(\widehat{X}, \mathbb{V}_{\downarrow}1) \;\cong\; \mathbf{Priest}(\widehat{X}, 2).$$

This shows that $h(\mathbf{x}) \in X \;\cong\; \mathrm{Cl}_{\uparrow} \widehat{X}$ corresponds to the clopen upset

$$h(\mathbf{x}) = \{a \in \widehat{X} \mid \exists (b_1, \dots, b_n) \in \rho(a)\colon \forall i\colon b_i \in C_i = \widehat{x_i^+}\} \in \mathrm{Cl}_{\uparrow}(\widehat{X}),$$

so we derived Goldblatt's formula [Gol89, p. 184] for the dual algebra of a relation ρ.

4.2. **Functional Properties of Priestley Relations.** As a further application of the adjoint framework to extended Stone duality, we show how to recover the characterization of those operators on distributive lattices whose dual Priestley relation is a partial function or a total relation, respectively. As outlined in Remark 3.11, we achieve this by considering suitable splittings of the adjunction $F\colon \mathbf{JSL} \dashv \mathbf{DL} : U$ to obtain submonads of $\mathbb{V}_{\downarrow}$ on which we instantiate Proposition 3.10.

Partial Functions. We split the adjunction $F \dashv U$ into

$$F_1\colon \mathbf{DL}_0 \dashv \mathbf{DL} : U_1 \qquad \text{and} \qquad F_2\colon \mathbf{JSL} \dashv \mathbf{DL}_0 : U_2,$$

where $\mathbf{DL}_0$ is the category of distributive lattices that are only bounded from below, and U_1, U_2 are forgetful functors. The tensor product of objects of $\mathbf{DL}_0$ is given by the tensor product of their underlying join-semilattices. The left adjoint F_1 adds a top element to a lattice in $\mathbf{DL}_0$.

Lemma 4.6. *The submonad T_1 on* **Priest** *dual to the adjunction $F_1 \dashv U_1$ is given by the partial function monad*

$$T_1 X = X + \{\emptyset\}.$$

Proof. The submonad $T_1 = \widehat{F}_1 \widehat{U}_1 \hookrightarrow \mathbb{V}_{\downarrow}$ on **Priest** is given by

$$\widehat{F}_1 \widehat{U}_1 \widehat{D} \;\cong\; \widehat{F_1 U_1 D} \;\cong\; \mathbf{DL}(F_1 U_1 D, 2) \;\cong\; \mathbf{DL}_0(U_1 D, U_1 2).$$

Every morphism $f \in \mathbf{DL}_0(U_1 D, U_1 2)$ either satisfies $f(\top) = \top$, in which case $f \in \widehat{D}$ is prime; or $f(\top) = \bot$, but then f is the constant zero map $\bot!\colon U_1 D \to U_1 2$. The map $\bot!$ is the bottom element in the pointwise ordering of $\mathbf{DL}_0(U_1 D, U_1 2)$, so the monad $\widehat{F}_1 \widehat{U}_1$ just freely adjoins a bottom element to a Priestley space. □

In particular, the dual category of $\mathbf{DL}_0$ is readily seen to be equivalent to $\mathbf{Priest}_0$, the category of Priestley spaces with a bottom element, and bottom-preserving continuous monotone maps. A Kleisli morphism $\rho\colon X \to T_1 X \hookrightarrow \mathbb{V}_{\downarrow} X$ is a *partial continuous function*, and a U-operator lifts along U_2 iff it preserves non-empty meets. Proposition 3.10 thus recovers the following result (for the unary version see [Hal58, HN15]).

Corollary 4.7. *The dual Priestley relation of a U-operator is a partial function iff the operator preserves non-empty meets.*

Total Relations. We split the adjunction $F \dashv U$ into

$$F_1'\colon \mathbf{JSL}_1 \dashv \mathbf{DL} : U_1' \qquad \text{and} \qquad F_2'\colon \mathbf{JSL} \dashv \mathbf{JSL}_1 : U_2',$$

where $\mathbf{JSL}_1$ is the category of join-semilattices with both a bottom and top element with morphisms preserving joins, bottom, and top. The right adjoints U_1', U_2' are forgetful functors, and the tensor product of objects of $\mathbf{JSL}_1$ is given by the tensor product of their underlying join-semilattices. The left adjoint F_1' maps $J \in \mathbf{JSL}_1$ to the distributive lattice $\mathcal{U}^{\partial}_{\mathrm{fg}+} J$ of *non-empty* finitely generated upsets of J, ordered by reverse inclusion.

Lemma 4.8. *The submonad T_1' on* **Priest** *dual to the adjunction $F_1' \dashv U_1'$ is given by the non-empty Vietoris monad*

$$T_1' X = \mathbb{V}_{\downarrow}^{+} X.$$

Proof. We have

$$\widehat{F_1'}\widehat{U_1'}\widehat{D} \;\cong\; \mathbf{DL}(F_1'U_1'D, 2) \;\cong\; \mathbf{JSL}_1(U_1'D, U_1'2).$$

The only ideal $f \in \mathbf{JSL}(UD, U2)$ that is not an element of $\mathbf{JSL}_1(U_1'D, U_1'2)$ is the trivial ideal $\bot!\colon d \longmapsto \bot$: if $f(\top) \neq \top$ then $f(\top) = \bot$, so f maps all elements to $\bot$ by monotonicity.. The trivial ideal corresponds to the empty set $\emptyset \in \mathbb{V}_{\downarrow}\widehat{D}$. Thus $T_1'X = \mathbb{V}_{\downarrow}X \setminus \{\emptyset\}$. □

Kleisli morphisms $X \to T_1'Y$ therefore are simply *total* Priestley relations. Moreover, a U-operator lifts along U_2' iff it preserves the top element. Proposition 3.10 then yields the following result (for the unary version see [Hal58, HN15]).

Corollary 4.9. *The dual Priestley relation of a U-operator is a total relation iff the operator preserves the top element.*

4.3. **Equational Properties of Operators.** In Section 4.2 we have shown how to use factorizations of the adjunction $F \dashv U$ to obtain more precise dualities for operators with additional properties. However, spelling out and computing the factorizing adjunctions and their dual monads for the desired operator property individually is tedious, and there often is a simpler method: If an operator property can be described *equationally*, we can use the fact that inequations of operators translate to inequations of their dual relations to obtain additional information under dualization.

The results we recover in this section are usually associated with modal correspondence theory [vB77, GT75, BdRV01]. Note that our proofs neither use first-order logic nor canonical frames: after having found a suitable encoding of order-theoretic properties as operator (in-)equations they are mere instantiations of the abstract results from Section 3.

Proposition 4.10. *Let $h\colon UA \to UA$ be an operator on a bounded distributive lattice A with dual Priestley relation $\rho\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}$. The following correspondences hold:*

(1) *ρ is reflexive iff $\forall a \in A\colon a \leq h(a)$.*
(2) *ρ is symmetric iff $\forall a, b \in A\colon a \wedge h(b) \leq h(h(a) \wedge b)$.*
(3) *ρ is euclidean iff $\forall a, b \in A\colon h(a) \wedge h(b) \leq h(a \wedge h(b))$.*
(4) *ρ is transitive iff $\forall a, b \in A\colon h(a) \wedge h(b) \geq h(a \wedge h(b))$.*
(5) *ρ is total iff $h(\top) = \top$.*
(6) *ρ is empty iff $h(\top) = \bot$.*

The proof follows a simple pattern: (a) translate the inequation into an inequation between operators, (b) dualize the inequation between operators to an inequation between Priestley relations and (c) convert the relational inequality into the corresponding first-order property.

Proof. First we dualize some canonical operators whose combinations we can then dualize using Proposition 3.12. Recall from Proposition 3.12(3) that the identity operator $\mathrm{id}_{UA}\colon UA \to UA$ dualizes to the unit $e\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}$. The conjunction operator $\wedge\colon UA \otimes UA \to UA$ is equal to the conjugate

$$\lambda \cdot U\nabla \cdot \lambda^{-1}\colon UA \otimes UA \cong U(A+A) \to UA \cong UA$$

and thus dualizes to the diagonal function $\Delta\colon \widehat{A} \to \widehat{A} \times \widehat{A}$.

Now let $h\colon UA \to UA$ be a unary operator with dual Priestley relation $\rho\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}$.

(1) The condition $\forall a \in A\colon a \le h(a)$ is equivalent to the operator inequality $\mathrm{id}_{UA} \le h$. The dual inequality is given by $e \le \rho\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}$, that is, $\forall x \in \widehat{A}\colon \{x\} \subseteq \rho(x)$, which states precisely that ρ is reflexive.

(2) The condition $\forall a, b \in A\colon a \wedge h(b) \le h(h(a) \wedge b)$ is equivalent to $\wedge \cdot (\mathrm{id} \otimes h) \le h \cdot \wedge \cdot (h \otimes \mathrm{id})$. The left side dualizes to

$$\hat{\delta} \cdot (e \times \rho) \cdot \Delta\colon \widehat{A} \to \widehat{A} \times \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A} \times \mathbb{V}_{\downarrow}\widehat{A} \to \mathbb{V}_{\downarrow}(\widehat{A} \times \widehat{A})$$

which is given by $x \mapsto \{(x, y) \mid y \in \rho(x)\}$, while the right side dualizes to

$$m \cdot \hat{\delta} \cdot \mathbb{V}_{\downarrow}(\rho \times e) \cdot \mathbb{V}_{\downarrow}\Delta \cdot \rho\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A} \to \mathbb{V}_{\downarrow}(\widehat{A} \times \widehat{A}) \to \mathbb{V}_{\downarrow}(\mathbb{V}_{\downarrow}\widehat{A} \times \mathbb{V}_{\downarrow}\widehat{A}) \to \mathbb{V}_{\downarrow}\mathbb{V}_{\downarrow}(\widehat{A} \times \widehat{A}) \to \mathbb{V}_{\downarrow}(\widehat{A} \times \widehat{A})$$

which is given by $x \mapsto \{(z, y) \mid y \in \rho(x), z \in \rho(y)\}$. On elements this inequality of relations reads

$$\forall x, y, z \in \widehat{A}\colon y \in \rho(x) \Rightarrow x \in \rho(y),$$

which states precisely that ρ is symmetric.

(3), (4) The proof is analogous to part (2) and left as an exercise to the reader.

(5) Here we have to use two auxiliary operators that exist on every Boolean algebra: the 'bottom' operator

$$z\colon UA \to UA, \quad x \mapsto \bot \qquad \text{whose dual is the empty relation} \qquad \zeta\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}, \quad x \mapsto \emptyset.$$

and the 'top' operator

$$t\colon UA \to UA, \quad x \mapsto \begin{cases} \bot & \text{if } x = \bot \\ \top & \text{otherwise,} \end{cases} \qquad \text{whose dual is} \qquad \tau\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}, \quad x \mapsto \widehat{A}.$$

The equation $h(\top) = \top$ holds iff the equation $h \cdot t = t\colon UA \to UA$ of operators holds. The left side dualizes to

$$m \cdot \mathbb{V}_{\downarrow}\tau \cdot \rho\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A} \to \mathbb{V}_{\downarrow}\mathbb{V}_{\downarrow}\widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}, \quad x \mapsto \bigcup_{y \in \rho(x)} \tau(y) = \begin{cases} \widehat{A} & \text{if } \rho(x) \text{ is non-empty,} \\ \emptyset & \text{else.} \end{cases}$$

The right side is simply $\tau\colon \widehat{A} \to \mathbb{V}_{\downarrow}\widehat{A}, x \mapsto \widehat{A}$, and these relations are equal iff ρ is total, that is, $\rho(x)$ is non-empty for every x.

(3) One has $h(\top) = \bot$ iff the operator equation $h = z$ holds. Its dual equation $\rho = \zeta$ simply states that $\rho(x)$ is empty for every x. ∎

Remark 4.11. The axioms (1)–(4) considered in Proposition 4.10 correspond to the classical modal axioms (T), (B), (D) and (5). Usually, axioms (2)–(4) are phrased differently by using *Boolean* modal logic: (2) is equivalent to $\forall a\colon a \leq \neg h(\neg h(a))$, (3) to $\forall a\colon h(a) \leq \neg h(\neg h(a))$ and (4) to $h(h(a)) \leq h(a)$. We show that axiom (2) is equivalent to (B) and leave the rest as an easy exercise for the reader.

Suppose that (2) holds in a Boolean algebra B, that is, it satisfies $\forall a, b \in B\colon a \wedge h(b) \leq h(h(a) \wedge b)$. We prove that it satisfies $\forall a \in B\colon a \leq \neg h(\neg h(a))$. Note that in a Boolean algebra $p \leq q$ is equivalent to $p \wedge \neg q \leq \bot$. Hence, we have

$$a \wedge \neg\neg h(\neg h(a)) = a \wedge h(\neg h(a)) \leq h(h(a) \wedge \neg h(a)) \leq h(\bot) = \bot,$$

where we use (2) instantiated with $b = \neg h(a)$ in the middle inequality. So B satisfies (B). For the converse, suppose that B satisfies $\forall a\colon a \leq \neg h(\neg h(a))$. We show that it satisfies $\forall a, b\colon a \wedge h(b) \leq h(h(a) \wedge b)$:

$$\begin{aligned}
&(a \wedge h(b)) \wedge \neg h(h(a) \wedge b) \leq \neg h(\neg h(a)) \wedge h(b) \wedge \neg h(h(a) \wedge b)\\
=\,& h(b) \wedge \neg(h(\neg h(a)) \vee h(h(a) \wedge b)) = h(b) \wedge \neg h(\neg h(a) \vee (h(a) \wedge b))\\
=\,& h(b) \wedge \neg h(\neg h(a) \vee b) = h(b) \wedge \neg h(\neg h(a)) \wedge \neg h(b) = \bot,
\end{aligned}$$

where we used (B) in the first step. Thus, B satisfies (2).

Our phrasings of (B) and (D) as (2) and (3) have the clear benefit of not using negation, and therefore being applicable to all (and not just Boolean) bounded distributive lattices, which are models of *positive* modal logic. As for the phrasing of transitivity as (4), we simply found the symmetry to the euclidean property appealing.

Proposition 4.10 allows us, for example, to generalize a classic result of algebraic (Boolean) modal logic due to Halmos [Hal58] to the positive setting: an operator $h\colon UA \to UA$ is a *quantifier* if it satisfies the inequations from (1), (3) and (5):

Corollary 4.12 (Halmos)**.** *An operator on a bounded distributive lattice algebra is a quantifier if and only if its dual relation is an equivalence relation.*

5. Monoids, Comonoids and Residuation Algebras

In this section we investigate *residuation algebras*, as introduced by Gehrke [Geh16b], which are ordered structures with residual operators similar to language derivatives. After recalling some foundations, we divide our study into two steps: First, we start with the simpler case of *complete* ordered structures, for which we prove a duality between certain complete residuation algebras and ordered monoids. This result will then serve as the foundation for both major applications in Sections 5.5 and 6: the duality for complete structures in particular restricts to finite structures, since finite lattices are complete, which can then be extended to a duality for more general structures by forming appropriate completions. On the other hand, the discrete duality for monoids is also the basis for the duality of the category of all categories.

5.1. **Foundations: Discrete Duality.** We recall some facts and notation for distributive lattices and their complete counterparts. In particular, we describe the instantiation of the following discrete version of the extended duality setting from Diagram (4.1):

$$\begin{tikzcd} \mathbf{D} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \simeq^{\mathrm{op}} & \widehat{\mathbf{D}} \arrow[d, bend right, "\widehat{F}"'] \arrow[d, phantom, "\vdash"] \\ \mathbf{C} \arrow[u, bend right, "U"'] & \simeq^{\mathrm{op}} & \widehat{\mathbf{C}} \arrow[u, bend right, "\widehat{U}"'] \arrow[loop right, "T"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{CSL}_{\bigwedge} \arrow[d, bend right, "\mathcal{D}"'] \arrow[d, phantom, "\dashv"] & \simeq^{\mathrm{op}} & \mathbf{CSL}_{\bigvee} \arrow[d, bend right, "|-|"'] \arrow[d, phantom, "\vdash"] \\ \mathbf{ACDL} \arrow[u, bend right, "V_{\wedge}"'] & \simeq^{\mathrm{op}} & \mathbf{Pos} \arrow[u, bend right, "\mathcal{D}"'] \arrow[loop right, "\mathcal{D}"] \end{tikzcd} \tag{5.1}$$

(Complete) Semilattices. We denote the category of meet-semilattices with a top element, which is isomorphic to **JSL**, by **MSL**. It has a monoidal structure given by tensor product $\boxtimes$ of meet-semilattices. The category **MSL** is dual to the category of Stone meet-semilattices [HMS74]. Henceforth, we denote the forgetful functors from **DL** to **JSL** and **MSL** by $U_\vee$ and $U_\wedge$, respectively, to avoid ambiguity. The monad on **Priest** induced by the dual of $F_\wedge \dashv U_\wedge$ maps a Priestley space X to its hyperspace $\mathbb{V}_\uparrow X$ of closed *upsets* [BHM23]. The respective comonads on **DL** for the adjunctions $F_\wedge \dashv U_\wedge$ and $F_\vee \dashv U_\vee$ are not isomorphic but *conjugate*: $F_\wedge U_\wedge \cong (F_\vee U_\vee(-)^\partial)^\partial$, where X^∂ denotes the *order-dual* of X. Their restrictions to the category of Boolean algebras are isomorphic since their dual monads on **Stone** satisfy $\mathbb{V}_\downarrow = \mathbb{V} = \mathbb{V}_\uparrow$.

Similarly, the category $\mathbf{CSL}_{\bigvee}$ ($\mathbf{CSL}_{\bigwedge}$) consists of complete join- (meet-) semilattices with morphisms preserving all joins (meets). Note that every complete join- or meet-semilattice X also has all meets and joins, respectively, given by

$$\bigwedge A = \bigvee\{x \mid \forall a \in A\colon x \leq a\}, \qquad \bigvee A = \bigwedge\{x \mid \forall a \in A\colon x \geq a\}.$$

The categories $\mathbf{CSL}_{\bigvee}$ and $\mathbf{CSL}_{\bigwedge}$ are easily seen to be dual to each other by swapping joins for meets and taking right, respectively left adjoints of morphisms. This duality is often stated equivalently as $\mathbf{CSL}_{\bigvee} \simeq^{\mathrm{op}} \mathbf{CSL}_{\bigvee}$, if the duality also reverses the order on objects. Analogously to its finitary counterpart, $\mathbf{CSL}_{\bigvee}$ has a monoidal structure given by the complete tensor product $\otimes_{\mathrm{C}}$ representing $\bigvee$-bilinear maps, making the left adjoints $\mathcal{D}\colon (\mathbf{Pos}, \times) \to (\mathbf{CSL}_{\bigvee}, \otimes_{\mathrm{C}})$ and $\mathcal{P}\colon (\mathbf{Set}, \times) \to (\mathbf{CSL}_{\bigvee}, \otimes_{\mathrm{C}})$ strong monoidal (the anlogous structure for $\mathbf{CSL}_{\bigwedge}$ is denoted $\boxtimes_{\mathrm{C}}$). Analogously to the finitary setting, the tensor product of complete join-semilattices J, J' is a quotient of $\mathcal{D}(|J| \times |J'|)$, that is, it is presented by generators $(j, j') \in |J| \times |J'$, which we also write as $j \otimes j'$, modulo the equations

$$(\bigvee_{j\in A}) \otimes (\bigvee_{j'\in A'}) = \bigvee_{j\in A, j'\in A'} j \otimes j' \qquad \text{for all } A \subseteq J, A' \subseteq J'.$$

The monad on **Pos** induced by this adjunction is the *downset monad* $\mathcal{D}$. It maps a set X to the set of all downward closed subsets of X. Its unit and multiplication are given by $\downarrow(-)\colon X \to \mathcal{D}X$ and union, respectively, and its $\times$-monoidal structure simply takes products of downsets:

$$\hat{\delta}\colon \mathcal{D}X \times \mathcal{D}Y \to \mathcal{D}(X \times Y), \quad (A, B) \mapsto A \times B.$$

The Kleisli category of $\mathcal{D}$ is the category **OrdRel** of posets with *order relations* as morphisms, that is, those relations $R \subseteq X \times Y$ between posets satisfying $x' \geq xRy \geq y' \implies x'Ry'$. We denote the lifting of the cartesian structure of **Pos** to **OrdRel** by $\bar{\times}$: on objects of

OrdRel we have $X \bar{\times} Y = X \times Y$, and the tensor $r \bar{\times} r'$ of order relations $r\colon X \to \mathcal{D}Y$ and $r'\colon X' \to \mathcal{D}Y'$ is defined as

$$\hat{\delta} \cdot (r \times r')\colon X \times X' \to \mathcal{D}Y \times \mathcal{D}Y' \to \mathcal{D}(Y \times Y'), \quad (x, x') \mapsto \{(y, y') \mid y \in r(x), y' \in r(y)\}.$$

The order-discrete restriction $\mathcal{P}\colon \mathbf{Set} \dashv \mathbf{CSL}_{\bigvee} :|-|$ induces the classical product monoidal structure on the powerset monad, and its Kleisli category $\mathbf{Set}_{\mathcal{P}}$ is well-known to be the category of sets and relations.

Completely Distributive Lattices. We denote the completely join-prime elements of an ACDL D by $\mathcal{J}D \cong \mathbf{ACDL}(D, 2)$ and the atoms of a CABA B by $\mathcal{A}B \cong \mathbf{CABA}(B, 2)$. The left adjoint to the forgetful functor $V_\wedge\colon \mathbf{ACDL} \to \mathbf{CSL}_{\bigwedge}$ takes downsets, i.e. $\mathcal{D} \dashv V_\wedge$, and its unit $M \to \mathcal{D}V_\wedge$ is given by

$$\eta_M\colon M \to \mathcal{D}M, \quad x \mapsto \{y \mid y \leq x\}.$$

Note that the left adjoint of the forgetful functor $\mathbf{CABA} \to \mathbf{CSL}_{\bigwedge}$ is given by $\mathcal{P}$, with the same unit η_M. If we respectively equip $\mathbf{ACDL}$ and $\mathbf{CSL}_{\bigwedge}$ with coproduct $+$ and tensor product $\boxtimes_{\mathrm{C}}$ of complete meet-semilattices as monoidal structures, then the forgetful functor $(\mathbf{ACDL}, +) \to (\mathbf{CSL}_{\bigwedge}, \boxtimes_{\mathrm{C}})$ is strong monoidal, analogously to the forgetful functor $(\mathbf{DL}, +) \to (\mathbf{CSL}_{\bigwedge}, \boxtimes)$; the same holds for the forgetful functor $(\mathbf{CABA}, +) \to (\mathbf{MSL}, \boxtimes_{\mathrm{C}})$. A straightforward verification shows that the adjunctions $\mathcal{D} \dashv V_\wedge$ and $\mathcal{D} \dashv |-|$ are dual with respect to the dualities $\mathbf{ACDL} \simeq^{\mathrm{op}} \mathbf{Pos}$ and $\mathbf{CSL}_{\bigwedge} \simeq^{\mathrm{op}} \mathbf{CSL}_{\bigvee}$. For detailed verifications on the order-discrete setting we refer the reader to Bezhanishvili et al. [BCM22].

Remark 5.1. Restricting to finite carriers, the diagrams (4.1) and (5.1) coincide, since

$$\mathbf{ACDL}_{\mathrm{f}} = \mathbf{DL}_{\mathrm{f}}, \qquad \mathbf{JSL}_{\mathrm{f}} \cong \mathbf{CSL}_{\bigvee,\mathrm{f}}, \qquad \mathbf{Priest}_{\mathrm{f}} \cong \mathbf{Pos}_{\mathrm{f}},$$

and so the results we establish in the following for complete structures will in particular hold for finite ones.

Notation 5.2. We tacitly omit the forgetful functors $U_\wedge$ and $U_\vee$ for notational brevity, whenever they are clear from the context, and just write the join- and meet-semilattice tensor products of the underlying semilattices of distributive lattices lattices D, D' as $D \otimes D'$ and $D \boxtimes D'$, respectively. The same holds for the complete versions: we omit the functors $V_\wedge, V_\vee$ and write $D \boxtimes_{\mathrm{C}} D'$ and $D \otimes_{\mathrm{C}} D'$ for the complete tensor products of ACDLs D and D'.

Remark 5.3 (Adjunctions on Lattices)**.** It is well known that a monotone map $f\colon D \to D'$ between complete lattices preserves all joins if and only if it has a right adjoint $f_*\colon D' \to D$, which is then given by $f_*(d') = \bigvee_{f(d) \leq d'} d$; dually, it preserves all meets iff it has a left adjoint $f^*\colon D' \to D$, given by $f^*(d') = \bigwedge_{d' \leq f(d)} d$. The join-primes $\mathcal{J}D$ of a bounded distributive lattice D are precisely those elements $p \in D$ whose characteristic function $\chi_p\colon D \to 2$ (mapping $x \in D$ to 1 iff $p \leq x$) is a homomorphism. The left adjoint of χ_p, denoted $p\colon 2 \to D$, is the join-semilattice morphism defined by $1 \mapsto p$.

Lemma 5.4.

(1) *The join- and meet-semilattice tensor products of bounded distributive lattices D, E yield isomorphic lattices, i.e. there exists an isomorphism*

$$\omega\colon U_\vee D \otimes U_\vee E \xrightarrow{\cong} U_\wedge D \boxtimes U_\wedge E$$

satisfying $\omega(d \otimes 1) = d \boxtimes 0$ and $\omega(1 \otimes e) = 0 \boxtimes e$.

(2) *Adjunctions on bounded distributive lattices 'compose horizontally': Given adjunctions $f\colon D \dashv E : g$ and $f'\colon D' \dashv E' : g'$ between distributive lattices we get adjunctions:*

$$
\begin{tikzcd}
E \boxtimes E' \arrow[r, shift left, "g\boxtimes g'"] \arrow[r, phantom, "\top"] & D \boxtimes D' \arrow[l, shift left] \arrow[d, "\omega^{-1}"] \\
E \otimes E' \arrow[u, "\omega"] & D \otimes D' \arrow[l, "f\otimes f'"]
\end{tikzcd}
\quad
\begin{tikzcd}
E \boxtimes E' \arrow[r, "g\boxtimes g'"] \arrow[dr, shift left] \arrow[dr, phantom, "\top"] & D \boxtimes D' \arrow[d, "\omega^{-1}"] \\
E \otimes E' \arrow[u, "\omega"] & D \otimes D' \arrow[l, "f\otimes f'"] \arrow[ul, shift left]
\end{tikzcd}
\quad
\begin{tikzcd}
E \boxtimes E' \arrow[r, "g\boxtimes g'"] & D \boxtimes D' \arrow[d, "\omega^{-1}"] \arrow[dl, shift left] \\
E \otimes E' \arrow[u, "\omega"] \arrow[ur, shift left] \arrow[ur, phantom, "\top"] & D \otimes D' \arrow[l, "f\otimes f'"]
\end{tikzcd}
\quad
\begin{tikzcd}
E \boxtimes E' \arrow[r, "g\boxtimes g'"] & D \boxtimes D' \arrow[d, "\omega^{-1}"] \\
E \otimes E' \arrow[u, "\omega"] \arrow[r, shift left] \arrow[r, phantom, "\top"] & D \otimes D' \arrow[l, shift left, "f\otimes f'"]
\end{tikzcd}
$$

If the right adjoints g and g' preserve finite joins, then this simplifies to

$$f \otimes f' \dashv g \otimes g' = \omega^{-1}(g \boxtimes g')\omega.$$

Dually, $\omega(f \otimes f')\omega^{-1} = f \boxtimes f'$, if f and f' preserve finite meets.

Proof.

(1) We have already seen that for bounded distributive lattices D, E their **JSL** tensor product $U_\vee D \otimes U_\vee E$ is their coproduct in **DL**. But by order-duality, the meet-semilattice tensor product $U_\wedge D \boxtimes U_\wedge E$ also gives a representation of the coproduct $D + E$ in **DL**: its inclusions $\hat\iota_1, \hat\iota_2$ map $d \in D, e \in E$ to $\hat\iota_1(d) = d \boxtimes 0$ and $\hat\iota_2(e) = 0 \boxtimes e$, respectively. By the universal property of the coproduct we obtain a unique isomorphism

$$\omega\colon U_\vee D \otimes U_\vee E \to U_\wedge D \boxtimes U_\wedge E \qquad \text{such that} \qquad \omega \cdot \iota_i = \hat\iota_i \text{ for } i = 1, 2.$$

We now show that ω is given by the following concrete formula:

$$\bigvee\nolimits_{i\in I} d_i \otimes e_i \mapsto \bigwedge\nolimits_{A \in \mathcal{P}I} \big(\bigvee\nolimits_{i\in A} d_i\big) \boxtimes \big(\bigvee\nolimits_{i\notin A} e_i\big).$$

By definition, the canonical isomorphism ω is the copairing $\omega = [\hat\iota_1, \hat\iota_2]$ of the inclusions of the meet-semilattice tensor product. Therefore on pure tensors ω maps $d \otimes e \mapsto d \boxtimes 0 \wedge 0 \boxtimes e$, which extends to general elements of $D \otimes E$ via distributivity as

$$
\begin{aligned}
\omega\big(\bigvee\nolimits_{i\in I} d_i \otimes e_i\big) &= \bigvee\nolimits_{i\in I} \omega(d_i \otimes e_i) \\
&= \bigvee\nolimits_{i\in I} d_i \boxtimes 0 \wedge 0 \boxtimes e_i \\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} \bigvee\nolimits_{i\in A} d_i \boxtimes 0 \vee \bigvee\nolimits_{i\notin A} 0 \boxtimes e_i \\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} \big(\bigvee\nolimits_{i\in A} d_i\big) \boxtimes 0 \vee 0 \boxtimes \big(\bigvee\nolimits_{i\notin A} e_i\big) \\
&= \bigwedge\nolimits_{A\in\mathcal{P}I} \big(\bigvee\nolimits_{i\in A} d_i\big) \boxtimes \big(\bigvee\nolimits_{i\notin A} e_i\big),
\end{aligned}
$$

where we use in the last two steps that joins in $D \boxtimes E$ satisfy the equation

$$(a \boxtimes b) \vee (c \boxtimes d) = (a \vee c) \boxtimes (b \vee d).$$

Note that by order-duality the inverse ω^{-1} is given by

$$\bigwedge\nolimits_{i\in I} d_i \boxtimes e_i \mapsto \bigvee\nolimits_{A\in\mathcal{P}I} \big(\bigwedge\nolimits_{i\in A} d_i\big) \otimes \big(\bigwedge\nolimits_{i\notin A} e_i\big).$$

(2) We only need to prove that one of the squares is an adjunction, since we obtain all others by suitable composition with ω and its inverse. We show this for the third diagram, that is, we show that there is an adjunction

$$(f \otimes f') \cdot \omega^{-1} \dashv (g \boxtimes g') \cdot \omega$$

by verifying the unit and counit inequalities

$$\mathrm{id} \leq (g \boxtimes g') \cdot \omega \cdot (f \otimes f') \cdot \omega^{-1} \qquad \text{and} \qquad (f \otimes f') \cdot \omega^{-1} \cdot (g \boxtimes g') \cdot \omega \leq \mathrm{id}.$$

We only prove the counit inequality; the proof of the unit inequality is dual. Recall that the right adjoint g preserves meets. Given a finite index set A we write $x_A = \bigvee_{i \in A} x_i$ and compute for every element $\bigvee_i x_i \otimes y_i \in D \otimes E$

$$\begin{aligned}
&(f \otimes f')\omega^{-1}(g \boxtimes g')\omega(\textstyle\bigvee_i x_i \otimes y_i) \\
&= (f \otimes f')\omega^{-1}(g \boxtimes g')(\textstyle\bigwedge_{A \in \mathcal{P}I} x_A \boxtimes y_{A^c}) && \text{def. } \omega \\
&= (f \otimes f')\omega^{-1}(\textstyle\bigwedge_{A \in \mathcal{P}I} g(x_A) \boxtimes g'(y_{A^c})) && \text{def. } g \boxtimes g' \\
&= (f \otimes f')(\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} (\bigwedge_{A \in B} g(x_A)) \otimes (\bigwedge_{A \in B^c} g'(y_{A^c}))) && \text{def. } \omega^{-1} \\
&= (f \otimes f')(\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} g(\bigwedge_{A \in B} x_A) \otimes g'(\bigwedge_{A \in B^c} y_{A^c})) && g \text{ preserves meets} \\
&= (\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} fg(\bigwedge_{A \in B} x_A) \otimes f'g'(\bigwedge_{A \in B^c} y_{A^c})) && \text{def. } f \otimes f' \\
&\leq (\textstyle\bigvee_{B \in \mathcal{P}\mathcal{P}I} \mathrm{id}(\bigwedge_{A \in B} x_A) \otimes \mathrm{id}(\bigwedge_{A \in B^c} y_{A^c})) && \text{counits } f \dashv g, f' \dashv g' \\
&= \omega^{-1}\omega(\textstyle\bigvee_i x_i \otimes y_i) = \bigvee_i x_i \otimes y_i.
\end{aligned}$$

As for the last statement, if g and g' preserve finite joins, then $g \otimes g'$ is defined (otherwise it would not be!), and it is clear that $f \otimes f' \dashv g \otimes g'$. By uniqueness of adjoints this implies $g \otimes g' = \omega^{-1}(g \boxtimes g')\omega$. □

Remark 5.5. Lemma 5.4 holds analogously for complete tensor products of ACDLs: there exists a unique isomorphism $\omega \colon V_\vee D \otimes_{\mathrm{C}} V_\vee E \simeq V_\wedge D \boxtimes_{\mathrm{C}} V_\wedge E$ commuting with the coproduct injections.

Proposition 5.6. *Let D be an ACDL.*

(1) *For every $x \in D$ the $\mathbf{CSL}_\bigvee$-morphism*

$$x \otimes (-) \colon D \to D \otimes_{\mathrm{C}} D, \qquad y \mapsto x \otimes y,$$

has a right adjoint

$$\begin{aligned} x \multimap (-) \colon D \otimes_{\mathrm{C}} D &\to D \\ T &\mapsto \textstyle\bigvee_{x \otimes y \leq T} y \end{aligned}$$

called tensor implication. *It can be extended to a binary function*

$$(-) \multimap (-) \colon D^\partial \boxtimes_{\mathrm{C}} (D \otimes_{\mathrm{C}} D) \to D.$$

Analogously, $(-) \otimes x \colon D \to D \otimes_{\mathrm{C}} D$ has a right adjoint $(-) \multimapinv x \colon D \otimes_{\mathrm{C}} D \to D$.

(2) *If $p \in \mathcal{J}D$ is completely join-prime then $p \multimap (-)$ is a lattice homomorphism given by*

$$\lambda \cdot (\chi_p + \mathrm{id}) \colon D + D \to 2 + D \cong D, \qquad \textstyle\bigvee_{i \in I} p_i \otimes q_i \mapsto \bigvee_{p \leq p_i} q_i.$$

(3) *Every adjunction $l \colon E \dashv D : r$ between ACDLs satisfies*

$$x \multimap \omega^{-1}(r \boxtimes r)\omega(T) = r(l(x) \multimap T)$$

as well as

$$l(x \multimap T) \leq l(x) \multimap (l \otimes l)(T) \qquad \text{and} \qquad r(x \multimap T) \leq r(x) \multimap \omega^{-1}(r \boxtimes r)\omega(T),$$

where the latter inequality holds with equality if r is order-reflecting.

Proof.
(1) The function $x \otimes (-)$ preserves joins by definition, so its right adjoint $x \multimap (-)$ exists and is given by $T \mapsto \bigvee_{x \otimes y \leq T} y$. We can write $x \otimes (-)$ as

$$x \otimes (-) = \lambda^{-1} \cdot (x \otimes \mathrm{id})\colon D \cong 2 \otimes D \to D \otimes D.$$

By Lemma 5.4(2), this has the right adjoint $\lambda \cdot \omega^{-1} \cdot (\chi_x \boxtimes \mathrm{id}) \cdot \omega$. The extension to a binary function $(-) \multimap (-)\colon D^\partial \boxtimes (D \otimes D) \to D$ is an instance of an *adjunction with a parameter* (cf. [ML98, Ch. IV.7]).
(2) If $x = p \in \mathcal{J}D$ is join-prime, then χ_p preserves joins. Hence, $p \multimap (-)$ simplifies to

$$\lambda^{-1} \cdot (\chi_p \otimes \mathrm{id})\colon D \otimes D \to 2 \otimes D \cong 2, \quad \bigvee_i p_i \otimes q_i \mapsto \bigvee_{p \leq p_i} q_i,$$

which is an ACDL morphism since λ^{-}, χ_p and id are.
(3) Let $T \in D \otimes D$ and $x \in E$. Then for all $y \in E$ we have

$$\begin{aligned} y \leq x \multimap \omega^{-1}(r \boxtimes r)\omega(T) &\iff x \otimes y \leq \omega^{-1}(r \boxtimes r)\omega(T) \\ &\iff l(x) \otimes l(y) \leq T \\ &\iff l(y) \leq l(x) \multimap T \\ &\iff y \leq r(l(x) \multimap T). \end{aligned}$$

So the first statement follows. Using this we compute

$$x \multimap T \leq x \multimap \omega^{-1}(r \boxtimes r)\omega(l \otimes l)(T) = r(l(x) \multimap (l \otimes l)(T)),$$

which by adjoint transposition is equivalent to

$$l(x \multimap T) \leq l(x) \multimap (l \otimes l)(T).$$

Similarly,

$$r(x \multimap T) \leq r(l(r(x)) \multimap T) = r(x) \multimap \omega^{-1}(r \boxtimes r)\omega(T),$$

and the first step is an equality if $l \cdot r = \mathrm{id}$, which is equivalent to r being order-reflecting. □

5.2. **Residuation Algebras.** We proceed with recalling the definition of residuation algebras [Geh16b], which are distributive lattices equipped with two additional operations to be thought of as abstractions of language derivatives (Example 5.9.(4)). We furthermore introduce an obvious extension of residuation algebras for the complete setting.

Definition 5.7.
(1) A *(Boolean) residuation algebra* consists of a (Boolean) lattice $R \in \mathbf{DL}$ equipped with $\mathbf{MSL}$-morphisms $\backslash\colon R^\partial \boxtimes R \to R$ and $/\colon R \boxtimes R^\partial \to R$, the *left* and *right residual*, satisfying the *residuation property*: $b \leq a \backslash c \iff a \leq c \,/\, b$.
(2) A *residuation ACDL (CABA)* R is an ACDL (CABA) whose residuals are complete morphisms $R^\partial \boxtimes_{\mathrm{C}} R \to R \leftarrow R \boxtimes_{\mathrm{C}} R^\partial$.

(3) A residuation algebra R is *associative* if it satisfies

$$x \setminus (z \,/\, y) = (x \setminus z) \,/\, y \qquad \text{for all } x, y, z \in R,$$

and it is *(prime-)unital* if there exists a (join-prime) element $e \in R$ which is a *unit*:

$$e \setminus z = z = z \,/\, e.$$

Remark 5.8. Units are unique: If e, e' are units, then $e = e' \setminus e$ since e' is a unit, so by the residuation property $e' \leq e \,/\, e = e$. Analogously we have $e \leq e'$, so $e = e'$.

As indicated above, residuals serve as algebraic generalizations of language derivatives, but as the following examples indicate they are not limited to this interpretation.

Examples 5.9.

(1) Every Heyting algebra is an associative residuation algebra with residuals $a \setminus c = a \to c$ and $c \,/\, b = b \to c$.

(2) Every Boolean algebra B is a non-associative residuation algebra with $x \setminus 1 = 1$ and $x \setminus z = \neg x$ for $z \neq 1$. If B is non-trivial, then it is not prime-unital by the first equality.

(3) Every continuous binary function $f\colon X \times X \to X$ on a Stone space X induces a residuation algebra on its dual Boolean algebra $\widehat{X}$ of clopens: given $A, B, C \in \widehat{X}$, put

$$A \setminus C = \{x \in X \mid \forall a \in A\colon f(a, x) \in C\},$$
$$C \,/\, B = \{x \in X \mid \forall b \in B\colon f(x, b) \in C\}.$$

(4) The set $\operatorname{Reg}\Sigma$ of all regular languages over a finite alphabet Σ forms an associative Boolean residuation algebra with residuals given by left and right *extended derivatives*:

$$K \setminus L = \{v \in \Sigma^* \mid Kv \subseteq L\}, \qquad L \,/\, K = \{v \in \Sigma^* \mid vK \subseteq L\}.$$

The unit is the singleton $\{\varepsilon\}$, where ε is the empty word. This example is a special case of (3): take as (X, f) the *free profinite monoid* on Σ, which is the Stone dual of $\operatorname{Reg}\Sigma$.

We now introduce the notion of a *residuation morphism* between residuation algebras and also its *relational* generalization.

Definition 5.10.

(1) A lattice morphism $f\colon R \to S$ between prime-unital residuation algebras is a *(pure) residuation morphism* if it satisfies the conditions

$$\forall x, z \in R\colon f(x \setminus z) \leq f(x) \setminus f(z) \text{ and } f(z \,/\, x) \leq f(z) \,/\, f(x) \tag{Forth}$$
$$\forall (y, z) \in S \times R\colon \exists x_{y,z} \in R\colon y \leq f(x_{y,z}) \text{ and } y \setminus f(z) = f(x_{y,z} \setminus z) \tag{Back}$$
$$\forall (y, z) \in S \times R\colon \exists x_{y,z} \in R\colon y \leq f(x_{y,z}) \text{ and } f(z) \,/\, y = f(z \,/\, x_{y,z}) \tag{Back'}$$
$$\forall x \in R\colon e \leq x \Leftrightarrow e' \leq f(x) \tag{Unit}$$

where e and e' are the units of R and S. The morphism f is *open* if, additionally, it has a left adjoint. Prime-unital residuation algebras and residuation morphisms form a category **Res**.

(2) A *corelational residuation morphism* from a prime-unital residuation algebra R to a prime-unital residuation algebra S is a morphism $\rho \in \mathbf{JSL}_1(R, S)$ satisfying

$$\rho(x \setminus z) \leq \rho(x) \setminus \rho(z) \quad \text{and} \quad e' \leq \rho(e).$$

Prime-unital residuation algebras with relational morphisms form a category **RelRes**.

Notation 5.11.

(1) We use the convention that for a subcategory $\mathbf{C}$ of $\mathbf{Res}$ or $\mathbf{RelRes}$ we denote the full subcategory of $\mathbf{C}$ with Boolean carriers by $\mathbf{BC}$.

(2) All categories above have obvious counterparts for residuation ACDLs and residuation CABAs with complete morphisms, which we denote by $\mathbf{ResACDL}, \mathbf{RelResACDL}$, etc.

Remark 5.12. Let us provide some intuition behind Definition 5.10.

(1) The notion of residuation morphism is derived from a result by Gehrke [Geh16b, Thm. 3.19], where it is shown to capture precisely the conditions satisfied by the duals of morphisms of binary Stone algebras.

(2) We speak about *corelational* morphisms of residuation algebras since for these will dualize precisely to relational morphisms of monoids. Recall that a *relational morphism* a monoid M to a monoid N is a total relation $\rho\colon M \to \mathcal{P}^+N$ satisfying

$$\rho(x)\rho(y) \subseteq \rho(xy) \qquad \text{and} \qquad 1_N \in \rho(1_M). \tag{5.2}$$

Relational morphisms represent inverses of surjective monoid homomorphisms [RS09, p. 38]. More precisely, the inverse relation h^{-1} of a surjective monoid homomorphism $h\colon N \twoheadrightarrow M$ is a relational morphism; conversely, if a relational morphism $h^{-1}\colon M \to \mathcal{P}^+N$ is the inverse of a function $h\colon N \to M$, then h is a surjective monoid homomorphism.

Categorically, we can consider an inverse relation $e^{-1}\colon M \to \mathcal{P}N$ of a surjective map $e\colon N \twoheadrightarrow M$ as is its *right adjoint* in the order-enriched category $\mathbf{Rel} \simeq \mathbf{Set}_{\mathcal{P}}$ of sets and relations: as relations they satisfy $\mathrm{id}_N \leq e^{-1} \cdot e$ and $e \cdot e^{-1} \leq \mathrm{id}_M$. Under duality the composition is reversed, so an inverse relation e^{-1} dualizes to a *left adjoint* $\widehat{e^{-1}} \dashv \widehat{e}$. Since left adjoints between lattices are precisely the join-preserving functions, this justifies our choice that corelational morphisms of residuation algebras preserve (finite) joins (and not necessarily meets). Note also that totality of e^{-1} is equivalent to surjectivity of e, which by Corollary 4.9 dualizes to the property that $\widehat{e^{-1}}$ preserves the top element.

(3) This is also the rationale behind the naming for *open* residuation morphisms: if $e\colon M \twoheadrightarrow N$ is a continuous surjection between Stone monoids then $e^{-1}\colon N \to \mathbb{V}M$ is continuous precisely iff e is an open map.

For open residuation morphisms the three conditions (Back), (Forth), (Unit) can be replaced by two equivalent, yet much simpler conditions. Over complete residuation algebras this is particularly convenient, since every residuation morphism is open.

Lemma 5.13. *Let R, S be prime-unital residuation algebras. A lattice morphism $f\colon R \to S$ with a left adjoint $f^*\colon S \to R$ is an open residuation morphism iff it satisfies the equations*

$$f^*(e') = e, \qquad\qquad \forall y \in S, z \in R\colon y \setminus f(z) = f(f^*(y) \setminus z), \tag{Open}$$

and the same equation for the right residual $/$.

In the proof below we omit mentioning the right residual because the arguments for it are completely analogous. This will be the case in most of the subsequent proofs involving properties of residuals.

Proof. We first show that if f satisfies (Open), then it is an open residuation morphism. The (Forth) condition follows from $f^* \cdot f \leq \mathrm{id}$ and contravariance of $\setminus$ in the first argument:

$$f(x \setminus z) \leq f(f^*(f(x)) \setminus z) = f(x) \setminus f(z).$$

For the (Back) condition, given $y \in S$ and $z \in R$ we choose the element $x_{y,z} = f^*(y) \in R$ (independently of z). The unit of the adjunction yields $y \leq f(f^*(y)) = f(x_{y,z})$, and using (Open) we obtain

$$y \setminus f(z) = f(f^*(y) \setminus z) = f(x_{y,z} \setminus z).$$

For the other direction, we prove that every open residuation morphism satisfies the condition (Open). Let $(y, z) \in S \times R$. By the (Back) condition, there exists $x_{y,z} \in R$ such that $y \leq f(x_{y,z})$ and $y \setminus f(z) = f(x_{y,z} \setminus z)$. This implies $f^*(y) \leq x_{y,z}$, and using (Back) and contravariance of $\setminus$ in the first argument, we obtain

$$y \setminus f(z) = f(x_{y,z} \setminus z) \leq f(f^*(y) \setminus z).$$

On the other hand, the adjunction unit $y \leq f(f^*(y))$, (Forth) and contravariance of $\setminus$ yield

$$f(f^*(y) \setminus z) \leq f(f^*(y)) \setminus f(z) \leq y \setminus f(z).$$

This proves that f indeed satisfies (Open).

For the respective unitality conditions we have by $f^* \dashv f$ that

$$\forall x \colon e \leq x \Leftrightarrow e' \leq f(x) \Leftrightarrow f^*(e') \leq x,$$

which is equivalent to $e = f^*(e')$. □

Example 5.14. Let Σ and Δ be finite alphabets. Every substitution $f_0 \colon \Sigma \to \Delta^*$ can be extended to a monoid homomorphism $f \colon \Sigma^* \to \Delta^*$, and for regular languages $L \in \operatorname{Reg}\Sigma$ and $K \in \operatorname{Reg}\Delta$, both $f[L]$ and $f^{-1}[K]$ are also regular. Then $f^{-1} \colon \operatorname{Reg}\Delta \to \operatorname{Reg}\Sigma$ is an open residuation morphism. Indeed, its left adjoint is given by the direct image map $f[-] \colon \operatorname{Reg}\Sigma \to \operatorname{Reg}\Delta$, satisfying $f[\{\varepsilon\}] = \{f(\varepsilon)\} = \{\varepsilon\}$ and

$$K \setminus f^{-1}[L] = \{w \mid Kw \subseteq f^{-1}[L]\} = \{w \mid f[K]f(w) \subseteq L\} = f^{-1}(f[K] \setminus L).$$

5.3. **Residuation ACDLs.** We start by investigating complete residuation algebras, whose characterization (Theorem 5.23) in terms of coalgebras forms not only the backbone of the classification and duality theory of *locally finite* residuation algebras in Sections 5.4 and 5.5, but also of the duality for the category of small categories in Section 6. Concretely, we use the tensor implication operator introduced in the last section to associate a *comultiplication* to the residuals and investigate its properties.

Construction 5.15.

(1) Every $V_\vee$-algebra $V_\vee D \otimes_{\mathrm{C}} V_\vee D \to V_\vee D$ on an ACDL D has a right adjoint $\gamma \colon V_\wedge D \to V_\wedge(D \otimes_{\mathrm{C}} D)$ that can be extended, by using the isomorphism ω from Lemma 5.4, to a $V_\wedge$-coalgebra

$$\bar{\gamma} = V_\wedge\omega \cdot \gamma \colon V_\wedge D \to V_\wedge(D \otimes_{\mathrm{C}} D) \cong V_\wedge(D \boxtimes_{\mathrm{C}} D) = V_\wedge D \boxtimes_{\mathrm{C}} V_\wedge D.$$

We refer to both versions γ and $\bar{\gamma}$ as *comultiplication* or *coalgebra structure*. Conversely, we obtain a $V_\vee$-algebra from a comultiplication $\gamma \colon V_\wedge D \to V_\wedge(D \otimes_{\mathrm{C}} D)$ by taking its left adjoint.

(2) In a residuation ACDL R the partially applied residuals have respective left adjoints $\mu(x, -) \dashv (x \setminus -)$ and $\mu(-, y) \dashv (- \mathbin{/} y)$ that can be combined into a $V_\vee$-algebra structure $\mu \colon V_\vee R \otimes_{\mathrm{C}} V_\vee R \to V_\vee R$ that we call *multiplication*. By part (1), the multiplication μ induces the comultiplication $\gamma \colon V_\wedge R \to V_\wedge(R \otimes_{\mathrm{C}} R)$, or $\bar{\gamma} \colon V_\wedge R \to V_\wedge R \boxtimes_{\mathrm{C}} V_\wedge R$.

Each of the operators $/$, $\backslash$, μ, γ determines the others uniquely due to the equivalences

$$x \leq z \,/\, y \iff y \leq x \setminus z \iff \mu(x \otimes y) \leq z \iff x \otimes y \leq \gamma(z).$$

The following lemma provides the concrete formulas.

Lemma 5.16. *Let R be a residuation ACDL.*
(1) *The residuals can be calculated from the comultiplication:*

$$x \setminus z = x \multimap \gamma(z) \quad \text{and} \quad z \,/\, y = \gamma(z) \multimapinv y,$$

where $\multimap$ and $\multimapinv$ are the tensor implications given by Proposition 5.6.
(2) *The comultiplication can be calculated from the residuals:*

$$\gamma(z) = \bigvee_{x \in R} x \otimes (x \setminus z) = \bigvee_{p \in \mathcal{J}R} p \otimes (p \setminus z).$$

Proof.
(1) For all $x, y, z \in R$ we have

$$y \leq x \setminus z \iff \mu(x \otimes y) \leq z \iff x \otimes y \leq \gamma(z) \iff y \leq x \multimap \gamma(z),$$

and analogously $x \leq z \,/\, y = \gamma(z) \multimapinv y$.
(2) We compute

$$\begin{aligned}
\gamma(z) &= \bigvee\{\bigvee_i x_i \otimes y_i \mid x_i, y_i \in R, \mu(\bigvee_i x_i \otimes y_i) \leq z\} && \text{formula for right adjoint}\\
&= \bigvee\{x \otimes y \mid x, y \in R, \mu(x \otimes y) \leq z\} && \mu \text{ preserves joins}\\
&= \bigvee\{x \otimes y \mid x, y \in R, y \leq x \setminus z\} && \mu(x \otimes -) \dashv (x \setminus -)\\
&= \bigvee_{x \in R} x \otimes x \setminus z && \text{simplification.}
\end{aligned}$$

It is clear that $\bigvee_{p \in \mathcal{J}R} p \otimes p \setminus z \leq \bigvee_{x \in R} x \otimes x \setminus z$, for the reverse inclusion we compute

$$x \otimes x \setminus z = (\bigvee_{p \leq x} p) \otimes x \setminus z = \bigvee_{p \leq x} p \otimes x \setminus z \leq \bigvee_{p \leq x} p \otimes p \setminus z,$$

where p ranges over $\mathcal{J}R$ and we use contravariance of $(- \setminus z)$ in the last step. ∎

We now investigate when the comultiplication is *pure*, that is, lifts to a complete lattice morphism $R \to R + R$ and thus corresponds to a pure multiplication.

Lemma 5.17. *For a residuation ACDL R, the following are equivalent:*
(1) *The comultiplication is pure: for all $A \subseteq R$ we have $\gamma(\bigvee_{x \in A} x) = \bigvee_{a \in A} \gamma(x)$.*
(2) *For all $p \in \mathcal{J}R$ and $A \subseteq R$ we have*

$$p \setminus (\bigvee_{x \in A} x) = \bigvee_{x \in A} p \setminus x \quad \text{and} \quad (\bigvee_{x \in A}) \,/\, p = \bigvee_{x \in A} x \,/\, p$$

(3) *For all $x, y \in R$: $\mu(x \otimes y) = 0 \iff x = 0 \vee y = 0$, and $\mu[\mathcal{J}(R + R)] \subseteq \mathcal{J}R$.*

Proof. (1) $\iff$ (3). First, we have

$$\begin{aligned}\gamma(0)=0 &\iff \gamma(0)\le 0\\ &\iff \forall T\colon T\le\gamma(0)\Rightarrow T\le 0\\ &\iff \forall T=\bigvee\nolimits_i x_i\otimes y_i\colon \mu(T)=\bigvee\nolimits_i \mu(x_i\otimes y_i)\le 0\Rightarrow \forall i\colon x_i\otimes y_i\le 0\\ &\iff \forall x,y\colon \mu(x\otimes y)=0\Rightarrow x\otimes y=0\\ &\iff \forall x,y\colon \mu(x\otimes y)=0\Leftrightarrow x=0\vee y=0,\end{aligned}$$

where we use in the penultimate equivalence that $\mu(0)=0$, and in the last equivalence that $x\otimes y=0$ iff $x=0$ or $y=0$. To show that γ preserves joins, note that the join-primes of $R\otimes R$ are given by pure tensors of $p\otimes q$ of join-primes $p,q\in\mathcal{J}R$ and that in a distributive lattice an element j is join-prime iff it is is *join-prime*: for $A\subseteq R$ if $j\le\bigvee A$ then $j\le x$ for some $x\in R$. Given $A\subseteq R$, we compute:

$$\begin{aligned}&\gamma(\bigvee_{x\in A}x)=\bigvee_{x\in A}\gamma(x)\\ \iff&\forall a,b\in\mathcal{J}R: a\otimes b\le\bigvee_{x\in A}\gamma(x)\Rightarrow a\otimes b\le\bigvee_{x\in A}\gamma(x)\\ \iff&\forall a,b\in\mathcal{J}R: a\otimes b\le\gamma(\bigvee_{x\in A}x)\Rightarrow[\exists x\in A\colon a\otimes b\le\gamma(x)]\\ \iff&\forall a,b\in\mathcal{J}R: \mu(a\otimes b)\le\bigvee_{x\in A}x\Rightarrow[\exists x\in A\colon \mu(a\otimes b)\le x]\\ \iff&\forall a\otimes b\in\mathcal{J}[R\otimes R]: \mu(a\otimes b)\in\mathcal{J}R.\end{aligned}$$

For the equivalence (1) $\iff$ (2), we combine Lemma 5.16 with the preservation properties of $x\multimap(-)$ from Proposition 5.6(1): If γ is pure, then it preserves joins and so does $(p\setminus -)=p\multimap\gamma(-)$ for $p\in\mathcal{J}R$; and if every $(p\setminus -)$ preserves joins, then so does $\gamma=\bigvee_{p\in\mathcal{J}R}p\otimes p\setminus(-)$. ☐

Next we show how structural identities like (co-)associativity or unitality translate between γ, μ and the residuals. Note that while the statements are to be expected, the proof is non-trivial due to the complication introduced by the seemingly innocent isomorphism $\omega\colon R\otimes_{\mathrm{C}}R\cong R\boxtimes_{\mathrm{C}}R$. Recall that a coalgebra $c\colon R\to R\boxtimes_{\mathrm{C}}R$ is *coassociative* if $(c\boxtimes\mathrm{id})\cdot c=(\mathrm{id}\boxtimes c)\cdot c$ and *(prime-)counital* if it is equipped with a (prime) *counit* $\varepsilon\in\mathbf{CSL}_{\bigwedge}(R,2)$ ($\varepsilon\in\mathbf{ACDL}(R,2)$) satisfying $(\varepsilon\boxtimes\mathrm{id})\cdot c=\mathrm{id}=(\mathrm{id}\boxtimes\varepsilon)\cdot c$. Diagrammatically, these equations are dual to the well-known monoid equations:

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "c"] & C\boxtimes_{\mathrm{C}}C \arrow[d, "\mathrm{id}\boxtimes_{\mathrm{C}}c"] \\ C\boxtimes_{\mathrm{C}}C \arrow[r, "c\boxtimes_{\mathrm{C}}\mathrm{id}"] & C\boxtimes_{\mathrm{C}}C\boxtimes_{\mathrm{C}}C \end{tikzcd} \qquad \begin{tikzcd} C\boxtimes_{\mathrm{C}}C \arrow[d, "\mathrm{id}\boxtimes_{\mathrm{C}}\varepsilon"] & C \arrow[l, "c"'] \arrow[r, "c"] & C\boxtimes_{\mathrm{C}}C \arrow[d, "\varepsilon\boxtimes_{\mathrm{C}}\mathrm{id}"] \\ C\boxtimes_{\mathrm{C}}2 \arrow[r, phantom, "\cong"] & C \arrow[r, phantom, "\cong"] & 2\boxtimes_{\mathrm{C}}C \end{tikzcd}$$

Lemma 5.18. *The following are equivalent for a residuation ACDL R:*

(1) *The comultiplication on R is coassociative and has a (prime) counit ε.*

(2) *The residuals are associative and R has a (prime) unit $e\in R$.*

(3) *The multiplication μ is associative and has a (prime) unit, that is, there exists $e\in R$ satisfying $\mu(e\otimes -)=\mathrm{id}=\mu(-\otimes e)$.*

Proof. For the proof we only use 'adjunctional calculus'. The equivalence (3) $\iff$ (2) follows from uniqueness of adjoints; to see this, we write associativity of μ as

$$\forall x, y \colon \mu(- \otimes y) \cdot \mu(x \otimes -) = \mu(x \otimes -) \cdot \mu(- \otimes y)$$

and associativity of the residuals as

$$\forall x, y \colon (x \setminus -) \cdot (- \mathbin{/} y) = (- \mathbin{/} y) \cdot (x \setminus -).$$

Since the respective left and right sides of the above equalities are adjoint, and adjoints are unique, it is clear that one of the equations holds iff the other one does. The unit of the residuals is the left adjoint of the comultiplication γ, which implies the equivalence of the (co-)unit properties.

The equivalence (1) $\iff$ (3) is shown similarly, but we have to be careful, since μ and $\bar{\gamma}$ are adjoint only up to composition with the isomorphism $\omega \colon R \otimes_{\mathrm{C}} R \to R \boxtimes_{\mathrm{C}} R$. By Lemma 5.4(2), we have the following diagram of adjunctions:

$$\begin{tikzcd}
 & & & & R \arrow[dll, shift right, "\gamma"'] \arrow[drr, shift left, "\gamma"] \arrow[dllll, bend right=40, "\bar{\gamma}"'] \arrow[drrrr, bend left=40, "\bar{\gamma}"] & & & & \\
 & & R \otimes_{\mathrm{C}} R \arrow[urr, shift right, "\mu"'] \arrow[urr, phantom, "\top"] \arrow[dll, "\omega"] \arrow[drr, shift right, "\mathrm{id} \otimes_{\mathrm{C}} \gamma"'] & & & & R \otimes_{\mathrm{C}} R \arrow[ull, shift left, "\mu"] \arrow[ull, phantom, "\top"] \arrow[dll, shift left, "\gamma \otimes_{\mathrm{C}} \mathrm{id}"] \arrow[drr, "\omega"] & & \\
R \boxtimes_{\mathrm{C}} R \arrow[drr, "\mathrm{id} \boxtimes_{\mathrm{C}} \gamma"] \arrow[drrrr, bend right=40, "\mathrm{id} \boxtimes_{\mathrm{C}} \bar{\gamma}"'] & & & & R \otimes_{\mathrm{C}} R \otimes_{\mathrm{C}} R \arrow[ull, shift right, "\mathrm{id} \otimes_{\mathrm{C}} \mu"'] \arrow[ull, phantom, "\perp"] \arrow[urr, shift left, "\mu \otimes_{\mathrm{C}} \mathrm{id}"] \arrow[urr, phantom, "\perp"] \arrow[dll, shift left, "\omega"] \arrow[drr, shift right, "\omega"'] & & & & R \boxtimes_{\mathrm{C}} R \arrow[dll, "\gamma \boxtimes_{\mathrm{C}} \mathrm{id}"'] \arrow[dllll, bend left=40, "\bar{\gamma} \boxtimes_{\mathrm{C}} \mathrm{id}"] \\
 & & R \boxtimes_{\mathrm{C}} (R \otimes_{\mathrm{C}} R) \arrow[urr, shift left, "\omega^{-1}"] \arrow[drr, "\mathrm{id} \boxtimes_{\mathrm{C}} \omega"] & & & & (R \otimes_{\mathrm{C}} R) \boxtimes_{\mathrm{C}} R \arrow[ull, shift right, "\omega^{-1}"'] \arrow[dll, "\omega \boxtimes_{\mathrm{C}} \mathrm{id}"'] & & \\
 & & & & R \boxtimes_{\mathrm{C}} R \boxtimes_{\mathrm{C}} R & & & &
\end{tikzcd}$$

The left and right diamonds come from the horizontal composition of adjunctions under the respective tensor products. The bottom diamond is easily seen to commute. If μ is associative the top inner diamond commutes, and so by uniqueness the outer big diamond commutes by uniqueness of adjoints, proving $\bar{\gamma}$ coassociative. Dually, if $\bar{\gamma}$ is coassociative, then μ is associative. The unit of μ is the left adjoint of the counit of γ, so one is prime iff the other one is. $\square$

These lemmas suggest the following definitions:

Definition 5.19.

(1) A residuation ACDL R is *pure* if it satisfies one of the equivalent conditions of Lemma 5.17.

(2) A *derivation ACDL* is a pure residuation ACDL that satisfies the equivalent conditions of Lemma 5.18. We denote the respective full subcategories by

$$\mathbf{DerACDL} \hookrightarrow \mathbf{ResACDL} \qquad \text{and} \qquad \mathbf{RelDerACDL} \hookrightarrow \mathbf{RelResACDL}.$$

(3) A $V_\wedge$-coalgebra $\bar\gamma\colon V_\wedge C \to V_\wedge C \boxtimes_{\mathrm{C}} V_\wedge C$ is a $V_\wedge$-*comonoid* if it is coassociative and prime-counital, and a *comonoid* if $\bar\gamma$ is pure. We analogously define $U_\wedge$-*comonoids* and (pure) *comonoids* in **DL**.

In order to extend the correspondence of residuation ACDLs and coalgebras to a categorical equivalence we introduce appropriate morphisms for coalgebras, which we also define for the general case of $U_\wedge$-coalgebras.

Definition 5.20.
(1) A *pure morphism* from a prime-counital $V_\wedge$-coalgebra $(C, \bar\gamma, \epsilon)$ to $(C', \bar\gamma', \epsilon')$ is a morphism $f \in \mathbf{ACDL}(C, D)$ satisfying $(f \boxtimes_{\mathrm{C}} f) \cdot \bar\gamma = \bar\gamma' \cdot f$ and $\epsilon = \epsilon' \cdot f$.

$$
\begin{tikzcd}
V_\wedge C \arrow[r, "V_\wedge f"] \arrow[d, "\bar\gamma"] & V_\wedge C' \arrow[d, "\bar\gamma'"] \\
V_\wedge C \boxtimes_{\mathrm{C}} V_\wedge C \arrow[r, "V_\wedge f \boxtimes_{\mathrm{C}} V_\wedge f"] & V_\wedge C' \boxtimes_{\mathrm{C}} V_\wedge C'
\end{tikzcd}
\qquad
\begin{tikzcd}
V_\wedge C \arrow[r, "V_\wedge f"] \arrow[dr, "V_\wedge \epsilon"'] & V_\wedge C' \arrow[d, "V_\wedge \epsilon'"] \\
& V_\wedge 2
\end{tikzcd}
$$

The category of prime-counital $V_\wedge$-coalgebras with pure morphisms is denoted by $\mathbf{Coalg}(V_\wedge)$ and its full subcategory of $V_\wedge$-comonoids by $\mathbf{Comon}(V_\wedge)$, again with the full subcategory **Comon** of comonoids.
(2) Let C and C' be comonoids in **ACDL**. A *corelational morphism* from C to C' is a morphism $\rho \in \mathbf{CSL}_\bigvee(C, C')$ satisfying $\rho(1) = 1$, $(\rho \otimes_{\mathrm{C}} \rho) \cdot \gamma \leq \gamma' \cdot \rho$ and $\epsilon \leq \epsilon' \cdot \rho$, that is, the following diagrams in $\mathbf{CSL}_\bigvee$ commute laxly as indicated:

$$
\begin{tikzcd}
V_\vee C \arrow[r, "\rho"] \arrow[d, "V_\vee \gamma"] \arrow[dr, phantom, "\leq"] & V_\vee C' \arrow[d, "V_\vee \gamma'"] \\
V_\vee C \otimes_{\mathrm{C}} V_\vee C \arrow[r, "\rho \otimes_{\mathrm{C}} \rho"] & V_\vee C' \otimes_{\mathrm{C}} V_\vee C'
\end{tikzcd}
\qquad
\begin{tikzcd}
V_\vee C \arrow[r, "\rho"] \arrow[dr, "V_\vee \epsilon"'] & V_\vee C' \arrow[d, "V_\vee \epsilon'"] \arrow[dl, phantom, "\leq"] \\
& V_\vee 2
\end{tikzcd}
$$

Comonoids with corelational morphisms form a category **RelComon**.
(3) Analogously, we define the category $\mathbf{Coalg}(U_\wedge)$ with its subcategories $\mathbf{Comon}(U_\wedge)$ and **Comon** of (pure) $U_\wedge$-comonoids. We also denote by **RelComon** the category of comonoids with corelational morphisms. So we overload notation; whether we mean comonoids in **MSL** or $\mathbf{CSL}_\bigwedge$ will be clear from context.

Recall from Lemma 5.13 that every morphism of residuation ACDLs is open.

Proposition 5.21. *Let R and R' be prime-unital residuation ACDLs.*
(1) *A morphism $f \in \mathbf{ACDL}(R, R')$ is a pure coalgebra morphism iff it is a residuation morphism.*
(2) *If R and R' are comonoids, then a morphism $\rho \in \mathbf{CSL}_\bigvee(R, R')$ preserving the top element is a corelational comonoid morphism iff it is a corelational residuation morphism.*

Proof.

(1) First, let f be a pure coalgebra morphism. Then

$$\begin{aligned}
x \setminus f(z) &= x \multimap \gamma' f(z) && \text{Lemma 5.16(1)} \\
&= x \multimap \omega^{-1} \bar{\gamma}' f(z) && \bar{\gamma}' = \omega \cdot \gamma' \\
&= x \multimap \omega^{-1} (f \boxtimes_{\mathrm{C}} f) \omega \gamma(z) && f \text{ is a coalgebra morphism} \\
&= f(f^*(x) \multimap \gamma(z)) && \text{Proposition 5.6(3)} \\
&= f(f^*(x) \setminus z) && \text{Lemma 5.16,}
\end{aligned}$$

which by Lemma 5.13 shows that f is a residuation morphism.

Conversely, if f is a residuation morphism, then for every $z \in R$ we compute

$$\begin{aligned}
\gamma' f(z) &= \bigvee\nolimits_{x' \in R'} x' \otimes x' \setminus f(z) && \text{Lemma 5.16(2)} \\
&= \bigvee\nolimits_{x' \in R'} x' \otimes f(f^*(x') \setminus z) && f \text{ residuation morphism} \\
&\leq \bigvee\nolimits_{x' \in R'} f f^*(x') \otimes f(f^*(x') \setminus z) && \mathrm{id} \leq f \cdot f^* \\
&= (f \otimes_{\mathrm{C}} f)(\bigvee\nolimits_{x' \in R'} f^*(x') \otimes f^*(x') \setminus z) && f \otimes f \text{ preserves joins} \\
&\leq (f \otimes_{\mathrm{C}} f)(\bigvee\nolimits_{x \in R} x \otimes x \setminus z) && f^*[R'] \subseteq R \\
&= (f \otimes_{\mathrm{C}} f)\gamma(z) && \text{Lemma 5.16(2).}
\end{aligned}$$

Now (order-isomorphic) postcomposition with ω gives

$$\bar{\gamma}' f = \omega \gamma' f \leq \omega (f \otimes_{\mathrm{C}} f) \gamma = (f \boxtimes_{\mathrm{C}} f) \omega \gamma = (f \boxtimes_{\mathrm{C}} f) \bar{\gamma}.$$

Conversely,

$$\begin{aligned}
(f \otimes_{\mathrm{C}} f)\gamma(z) &= \bigvee\nolimits_{x \in R} f(x) \otimes f(x \setminus z) && \text{Lemma 5.16(2)} \\
&\leq \bigvee\nolimits_{x \in R} f(x) \otimes f(f^* f(x) \setminus z) && f^* f \leq \mathrm{id} \text{ and contravariance} \\
&= \bigvee\nolimits_{x \in R} f(x) \otimes f(x) \setminus f(z) && f \text{ is open and Lemma 5.13} \\
&\leq \bigvee\nolimits_{x' \in R'} x' \otimes x' \setminus f(z) && f[R] \subseteq R' \\
&= \gamma'(f(z)) && \text{Lemma 5.16(2).}
\end{aligned}$$

Postcomposition with ω again yields $\bar{\gamma}' f \geq (f \boxtimes_{\mathrm{C}} f)\bar{\gamma}$. Hence, we obtain $\bar{\gamma}' f = (f \boxtimes_{\mathrm{C}} f)\bar{\gamma}$, so f is a pure coalgebra morphism. Moreover, it is clear that the counit condition from Definition 5.20(1) is equivalent to the unit conditions from (Open), since (1) $e \dashv \epsilon$ and $f^*(e) \dashv \epsilon' \cdot f$, and (2) adjoints are unique, so either equations holds iff the other one does.

(2) If $\rho\colon R \to R'$ is a corelational morphism of pure coalgebras, then

$$
\begin{aligned}
\rho(x \setminus z) &= \rho(x \multimap \gamma(z)) && \text{Lemma 5.16(1)}\\
&\leq \rho(x) \multimap (\rho \otimes_{\mathrm{C}} \rho)(\gamma(z)) && \text{Proposition 5.6(3)}\\
&\leq \rho(x) \multimap \gamma'(\rho(z)) && \rho \text{ corelational morphism}\\
&= \rho(x) \setminus \rho(z) && \text{Lemma 5.16(1).}
\end{aligned}
$$

Conversely, if $\rho\colon R \to R'$ is a corelational morphism of residuation algebras, then using Lemma 5.16(2) twice, and that $\rho[R] \subseteq R'$, we obtain

$$(\rho \otimes_{\mathrm{C}} \rho)\gamma(z) = \bigvee\nolimits_{x\in R} \rho(x) \otimes \rho(x \setminus z) \leq \bigvee\nolimits_{x\in R} \rho(x) \otimes \rho(x) \setminus \rho(z) \leq \gamma'(\rho(z)).$$

For the respective counits we identify the neutral element $e \in R$ with the **JSL**-morphism $e\colon 2 \to R$ to compute

$$
\begin{aligned}
& & \epsilon \leq \epsilon' \cdot \rho & \\
&\iff & \forall x\colon \epsilon(x) \leq \epsilon'(\rho(x)) & \\
&\iff & \forall x, y\colon y \leq \epsilon(x) \Rightarrow y \leq \epsilon'(\rho(x)) & \\
&\iff & \forall x, y\colon e(y) \leq x \Rightarrow e'(y) \leq \rho(x) & \qquad e \dashv \epsilon, e' \dashv \epsilon'\\
&\iff & \forall x\colon e \leq x \Rightarrow e' \leq \rho(x) & \qquad y \in \{0,1\} \text{ and } e(0) = e'(0) = 0\\
&\iff & e' \leq \rho(e), &
\end{aligned}
$$

where in the last step we set $x = e$ for the downward implication, and the upward direction is simply monotonicity of ρ. ☐

Theorem 5.22. *The following categories are isomorphic*

$$\mathbf{Coalg}(V_\wedge) \cong \mathbf{ResACDL}, \quad \mathbf{Comon} \cong \mathbf{DerACDL}, \quad \mathbf{RelComon} \cong \mathbf{RelDerACDL}.$$

Proof. The three isomorphisms are given on objects by swapping between residuals and comultiplication (note that the residual unit is a left adjoint of the counit of the comultiplication), and act as identity on morphisms. Lemma 5.17 and 5.18 and Proposition 5.21 show that they are well-defined. ☐

From Theorem 5.22 we obtain a dual characterization of ordered monoids; it restricts to a duality between ordinary monoids and derivation CABAs. Recall that a *relational morphism* of ordered monoids M and N is a total order-relation $\rho\colon M \to \mathcal{D}^+N$ (where $\mathcal{D}$ is the downset monad) making the following diagrams commute laxly:

$$
\begin{tikzcd}
M \times M \arrow[rr, "\cdot_M"] \arrow[d, "\rho\times\rho"'] \arrow[drr, phantom, "\leq"] & & M \arrow[d, "\rho"] \\
\mathcal{D}N \times \mathcal{D}N \arrow[r, "\hat{\delta}"] & \mathcal{D}(N \times N) \arrow[r, "\mathcal{D}(\cdot_N)"] & \mathcal{D}N
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "1_M"] \arrow[d, "1_N"'] \arrow[dr, phantom, "\leq"] & M \arrow[d, "\rho"] \\
N \arrow[r, "\eta"] & \mathcal{D}N
\end{tikzcd}
\tag{5.3}
$$

Theorem 5.23.

(1) *The category of ordered monoids is dually equivalent to the category of derivation ACDLs (or ACDL-comonoids):*

$$\mathbf{OrdMon} \;\simeq^{\mathrm{op}}\; \mathbf{Comon} \;\cong\; \mathbf{DerACDL}.$$

(2) *The category of ordered monoids and relational morphisms is dually equivalent to the category of derivation ACDLs (or comonoids) and corelational morphisms:*

$$\mathbf{RelOrdMon} \;\simeq^{\mathrm{op}}\; \mathbf{RelComon} \;\cong\; \mathbf{RelDerACDL}.$$

Proof. Both statements follow by extending the equivalences from Theorem 5.22 with the extended duality applied to the setting Equation 5.1 established in Section 5.1: For item (1) we get that ordered monoids $M \times M \to M$ are dual to comonoids $C \to C + C$ in **ACDL**, and for item (2), this means that relational monoid morphisms $M \to \mathcal{D}^+N$ as in 5.3 dualize precisely to corelational morphisms $V_\vee N \to V_\vee M$ of comonoids. ☐

Remark 5.24. Both Theorem 5.22 and Theorem 5.23 restrict to finite carriers. All finite lattices are complete and $\mathbf{Res}_\mathrm{f} = \mathbf{ResACDL}_\mathrm{f}$, so, writing $\mathbf{Der}_\mathrm{f}$ for $\mathbf{DerACDL}_\mathrm{f}$, we get the equivalences

$$\mathbf{OrdMon}_\mathrm{f} \simeq^{\mathrm{op}} \mathbf{Comon}_\mathrm{f} \cong \mathbf{Der}_\mathrm{f} \quad \text{and} \quad \mathbf{RelOrdMon}_\mathrm{f} \simeq^{\mathrm{op}} \mathbf{RelComon}_\mathrm{f} \cong \mathbf{RelDer}_\mathrm{f}.$$

5.4. **Locally Finite Residuation Algebras.** We now extend the correspondence between residuation algebras and coalgebras from Section 5.3 from complete to non-complete carriers. The main challenge arises from the reliance on adjoints for the constructions in Section 5.3, whose existence is of course not ensured for arbitrary distributive lattices as carriers. We tackle this problem by considering *locally finite* structures, allowing us to extend the comultiplication Construction 5.15 from finite subalgebras to the whole lattice.

We start with the motivating example from automata theory: the residuals of regular languages from Example 5.9.(4).

Example 5.25. It is well known that the Boolean algebra $\mathrm{Reg}\,\Sigma$ of regular languages dualizes under Stone duality to the Stone space $\overline{\Sigma^*}$ of *profinite words*[1] (see e.g. Pippenger [Pip97]). The space $\overline{\Sigma^*}$ can be constructed as the limit in the category of Stone spaces of the diagram of all finite quotient monoids of Σ^*, regarded as discrete spaces. It is equipped with a continuous monoid structure $\mu\colon \overline{\Sigma^*} \times \overline{\Sigma^*} \to \overline{\Sigma^*}$ extending the concatenation of words. We calculate below that its dual comultiplication on regular languages under Stone duality is given as follows:

$$\begin{aligned} \gamma\colon \mathrm{Reg}\,\Sigma \quad &\to \quad \mathrm{Reg}\,\Sigma + \mathrm{Reg}\,\Sigma \\ L \quad &\mapsto \quad \bigvee\nolimits_{[v]\in \mathrm{Syn}_L} [v] \otimes [v] \setminus L, \end{aligned} \tag{5.4}$$

where Syn_L is the *syntactic monoid* of L; its elements are the equivalence classes of the *syntactic congruence relation* relation on Σ^* defined by

$$[v] = [w] \qquad \text{iff} \qquad \forall K, K' \subseteq \Sigma^*\colon v \in K \setminus L \;/\; K' \iff w \in K \setminus L \;/\; K'.$$

The Stone monoid $\overline{\Sigma^*}$ is *profinite*, that is, it is the (cofiltered) limit of the diagram of its finite continuous monoid quotients. Therefore, by duality, $\mathrm{Reg}\,\Sigma$ is the filtered colimit of its finite sub-coalgebras $\mathcal{P}M \cong \widehat{M} \hookrightarrow \widehat{\overline{\Sigma^*}} \cong \mathrm{Reg}\,\Sigma$ dual to finite monoid quotients $\overline{\Sigma^*} \twoheadrightarrow M$.

[1] This space is commonly denoted $\widehat{\Sigma^*}$ in the literature; we use the notation $\overline{\Sigma^*}$ to avoid a clash with notation $\widehat{(-)}$ for the dual equivalence.

This means that, given a regular language L, we can compute the value of $\gamma(L)$ using $\mathrm{Syn}(L)$ as in the following diagram:

$$\begin{tikzcd}
\mathrm{Reg}\,\Sigma \arrow[r, "\cong"] \arrow[rrrr, bend left, "\gamma"] & \mathrm{Cl}(\overline{\Sigma^*}) \arrow[r, "\mu^{-1}"] & \mathrm{Cl}(\overline{\Sigma^*} \times \overline{\Sigma^*}) \arrow[r, "\cong"] & \mathrm{Cl}(\overline{\Sigma^*}) + \mathrm{Cl}(\overline{\Sigma^*}) \arrow[r, "\cong"] & \mathrm{Reg}\,\Sigma + \mathrm{Reg}\,\Sigma \\
& \mathcal{P}\mathrm{Syn}_L \arrow[u, hook, "h^{-1}"] \arrow[r, "\mu^{-1}"] & \mathcal{P}(\mathrm{Syn}_L \times \mathrm{Syn}_L) \arrow[u, hook, "(h\times h)^{-1}"] \arrow[r, "\cong"] & \mathcal{P}\mathrm{Syn}_L + \mathcal{P}\mathrm{Syn}_L \arrow[u, hook, "h^{-1}+h^{-1}"] &
\end{tikzcd}$$

We denote by $[w]$ the syntactic congruence class of w with respect to L to compute

$$\begin{aligned}
\mu^{-1}(L) &= \mu^{-1}(h^{-1}(h[L])) && L \text{ recognized by } M \\
&= (h \times h)^{-1}(\mu^{-1}(h[L])) && h^{-1} \text{ is a coalgebra homomorphism} \\
&= (h \times h)^{-1}(\{(m,n) \mid mn \in h[L]\}) && \\
&= (h \times h)^{-1}(\bigcup_{mn \in h[L]} \{(m,n)\}) && \\
&\mapsto (h^{-1} + h^{-1})(\bigvee\nolimits_{mn \in h[L]} \{m\} \otimes \{n\}) && \mathcal{P}(M \times M) \cong \mathcal{P}M + \mathcal{P}M \\
&= \bigvee\nolimits_{mn \in h[L]} h^{-1}[\{m\}] \otimes h^{-1}[\{n\}] && \\
&= \bigvee\nolimits_{h(v)h(w) \in h[L]} h^{-1}h[v] \otimes h^{-1}h[w] && h \text{ surjective} \\
&= \bigvee\nolimits_{vw \in L} [v] \otimes [w] && \text{syntactic equivalence classes} \\
&= \bigvee\nolimits_{v \in \Sigma^*} \bigvee\nolimits_{w \in v^{-1}L} [v] \otimes [w] && \text{definition of } v^{-1}L \\
&= \bigvee\nolimits_{v \in \Sigma^*} [v] \otimes (\bigvee\nolimits_{w \in v^{-1}L} [w]) && (\star) \\
&= \bigvee\nolimits_{v \in \Sigma^*} [v] \otimes v^{-1}L && (\diamond) \\
&= \bigvee\nolimits_{[v] \in \mathrm{Syn}_L} [v] \otimes [v] \setminus L. && (\#)
\end{aligned}$$

Step $(\star)$ uses that L is regular and so the join $\bigvee_{w \in v^{-1}L}[w]$ is finite; at step $(\diamond)$ we insert the equality $v^{-1}L = \bigcup_{w \in v^{-1}L}[w]$, which holds by the definition of syntactic congruence; and at step $(\#)$ we use the definition of residuals in $\mathrm{Reg}\,\Sigma$ as derivatives, and that L is regular.

The key to extending the duality of Remark 5.24 is the notion of *residuation ideal* of a residuation algebra. It was introduced by Gehrke [Geh16b] to characterize quotients of Priestley topological algebras. In particular, she has shown [Geh16a, Thm. 15] that the Stone monoid quotient Syn_L of $\overline{\Sigma^*}$ dualizes to the residuation ideal generated by $L \in \mathrm{Reg}\,\Sigma$

Definition 5.26. A *residuation ideal* of a residuation algebra R is a sublattice $I \hookrightarrow R$ closed under derivatives w.r.t. arbitrary elements of R:

$$\forall z \in R, x \in I \colon x \setminus z \in I \text{ and } z \,/\, x \in I.$$

In particular, every residuation ideal is a residuation subalgebra of R. We denote the residuation ideal generated by a subset $X \subseteq R$ by $\backslash X /$.

Example 5.25 suggests a path to extending our constructions of Section 5.3 from complete to more general residuation algebras: extend them *locally*, that is, by considering suitable finite substructures. Note that in the formula (5.4) for the comultiplication on regular languages it is crucial that the residuation ideal $\backslash\{L\}/$ generated by a single regular language L is *finite*, so that the join (#) is defined.

Definition 5.27.

(1) A residuation algebra R is *locally finite* if every finite subset of R is contained in a finite residuation ideal of R.

(2) A $U_\wedge$-coalgebra C is *locally finite* if every finite subset of C is contained in a finite subcoalgebra of C. The category of locally finite comonoids is denoted $\mathbf{Comon}_{\mathrm{lf}}$.

Note that not every residuation algebra is locally finite, consider for example an infinite Boolean algebra in Example 5.9.(2).

Proposition 5.28.

(1) *Every locally finite residuation algebra R induces a locally finite $U_\wedge$-coalgebra $\gamma_\backslash \colon U_\wedge R \to U_\wedge(R \otimes R)$ where*

$$\gamma_\backslash(z) = (\iota_A \otimes \iota_A)(\gamma_A(z)) = \bigvee\nolimits_{x \in A} \iota_A(x) \otimes \iota_A(x \setminus z) = \bigvee\nolimits_{p \in \mathcal{J}A} \iota_A(p) \otimes \iota_A(p \setminus z),$$

for every finite residuation ideal $\iota_A \colon A \hookrightarrow R$ containing z (here γ_A is the comultiplication on A as in Construction 5.15).

(2) *Every locally finite $U_\wedge$-coalgebra (C, γ) induces a locally finite residuation algebra where*

$$x \setminus_\gamma z \;=\; \iota_A(x \multimap \gamma(z)), \qquad z \mathbin{/_\gamma} x \;=\; \iota_A(\gamma(z) \multimapinv x),$$

for every finite subcoalgebra $\iota_A \colon A \hookrightarrow C$ containing x, z (here $\setminus_A$ is the residual on A as given by Construction 5.15). The residuals have a canonical presentation as

$$x \setminus_\gamma z \;=\; \iota_z(\iota_z^*(x) \setminus z) \qquad \textit{and} \qquad z \mathbin{/_\gamma} x \;=\; \iota_z(z \mathbin{/} \iota_z^*(x))$$

where $\iota_z \colon \langle z \rangle \to C$ is the smallest (finite) subcoalgebra containing z.

(3) *These constructions are mutually inverse:*

$$\gamma_{\setminus_\gamma} \;=\; \gamma \qquad \textit{and} \qquad \setminus_{\gamma_\setminus} \;=\; \setminus.$$

Proof.

(1a) We first show that the comultiplication $\gamma_\backslash = (\iota_A \otimes \iota_A)(\gamma(z))$ is well-defined, that is, it does not depend on the residuation ideal A containing z. We first prove an auxiliary statement:

Lemma 5.29. *If $\iota_I = \iota_K \cdot \iota \colon I \hookrightarrow K \hookrightarrow R$ are finite residuation ideals containing z, then*

$$(\iota_I \otimes \iota_I)(\gamma_I(z)) = (\iota_K \otimes \iota_K)(\gamma_K(\iota(z))). \tag{5.5}$$

Proof. Since $I \subseteq K$, it is clear that

$$\begin{aligned}(\iota_I \otimes \iota_I)(\gamma_I(z)) &= (\iota_I \otimes \iota_I)(\bigvee\nolimits_{x \in I} x \otimes x \setminus z) \\ &\leq (\iota_K \otimes \iota_K)(\bigvee\nolimits_{x \in K} x \otimes x \setminus z) \\ &= (\iota_K \otimes \iota_K)(\gamma_K(\iota(z))).\end{aligned}$$

For the reverse inequality, note that for every join-prime $q \in \mathcal{J}K$ we find $p \in \mathcal{J}I$ such that $q \leq \iota(p)$, since

$$q \leq \top = \iota(\top) = \iota(\bigvee_{p\in\mathcal{J}I} p) = \bigvee_{p\in\mathcal{J}I} \iota(p)$$

and q is join-prime. We use this to calculate

$$\begin{aligned}
(\iota_K \otimes \iota_K)(\gamma_K(\iota(z))) &= \bigvee_{q\in\mathcal{J}K} \iota_K(q) \otimes \iota_K(q \setminus \iota(z)) \\
&= \bigvee_{p\in\mathcal{J}I} \bigvee_{q\leq\iota(p)} \iota_K(q) \otimes \iota_K(q \setminus \iota(z)) && \text{idempotence of join} \\
&= \bigvee_{p\in\mathcal{J}I} \bigvee_{q\leq\iota(p)} \iota_K(q) \otimes \iota_I(\iota^*(q) \setminus z) && (*) \\
&= \bigvee_{p\in\mathcal{J}I} \bigvee_{q\leq\iota(p)} \iota_I\iota^*(q) \otimes \iota_I(\iota^*(q) \setminus z) && (**) \\
&\leq \bigvee_{p\in\mathcal{J}I} \iota_I(p) \otimes \iota_I(p \setminus z) && (***) \\
&= (\iota_I \otimes \iota_I)(\gamma_I(z)).
\end{aligned}$$

For step $(*)$, we use that for $p \in \mathcal{J}I, q \in \mathcal{J}K$ such that $q \leq \iota(p)$, the following holds:

$$\begin{aligned}
\iota_K(q \setminus \iota(z)) &= \iota_K(q) \setminus \iota_K(\iota(z)) \\
&= \iota_K(q) \setminus \iota_I(z) \\
&= \iota_I(\iota_I^*(\iota_K(q)) \setminus z) && \iota_I \text{ finite (open) residuation morphism} \\
&= \iota_I(\iota^*(\iota_K^*(\iota_K(q))) \setminus z) && \iota_I = \iota_K\iota \\
&= \iota_I(\iota^*(q) \setminus z) && \iota_K \text{ embedding.}
\end{aligned}$$

Simlarly, for step $(**)$ we use have

$$\iota_K = \iota_K \cdot \mathrm{id} \leq \iota_K \cdot \iota \cdot \iota^* = \iota_I \cdot \iota^*.$$

Lastly, for step $(***)$ note that for every $q \in \mathcal{J}K$ we have $\iota^*(q) \in \mathcal{J}(I)$: indeed, $\iota^*(q) \leq x \vee y$ in I implies $q \leq \iota(x) \vee \iota(y)$ in K, hence $q \leq \iota(x)$ or $q \leq \iota(y)$, so $\iota^*(q) \leq x$ or $\iota^*(q) \leq y$. In particular for all $q \in \mathcal{J}K$ we find some $p \in \mathcal{J}I$ with

$$\iota_I\iota^*(q) \otimes \iota_I(\iota^*(q) \setminus z) = \iota_I(p) \otimes (\iota_I(p) \setminus z),$$

which implies $(***)$. ☐

Now, for well-definedness of $\gamma_\setminus$, if I, I' are finite residuation ideals containing z they are both contained in a finite residuation ideal K, since that R is locally finite; we write $\iota\colon I \hookrightarrow K \hookleftarrow I' :\iota'$ for the inclusion maps. Now we have

$$\begin{aligned}
(\iota_I \otimes \iota_I)(\gamma_I(z)) &= (\iota_K \otimes \iota_K)(\iota \otimes \iota)(\gamma(z)) && (5.5) \\
&= (\iota_K \otimes \iota_K)(\gamma_K(\iota(z))) && \iota \text{ coalgebra morphism} \\
&= (\iota_K \otimes \iota_K)(\gamma_K(\iota'(z))) && \iota, \iota' \text{ subcoalgebras of } K \\
&= (\iota_{I'} \otimes \iota_{I'})(\gamma_{I'}(z)) && \text{backwards.}
\end{aligned}$$

This shows that the mapping

$$\gamma_\setminus\colon R \mapsto R \otimes R, \qquad z \mapsto (\iota_I \otimes \iota_I)(\gamma(z))$$

does not depend on the choice of the residuation ideal I.

(1b) We show that the mapping $\gamma_\backslash$ indeed yields a $U_\wedge$-coalgebra structure, that is, it preserves finite meets. Let $F \subseteq R$ be a finite subset. By local finiteness we find a residuation ideal I containing F. Now we simply use that both the comultiplication on I and $\iota_I \otimes \iota_I$ preserve finite meets:

$$\gamma_\backslash(\bigwedge_{x\in F} x) = (\iota_I \otimes \iota_I)(\gamma(\bigwedge_{x\in F} x)) = \bigwedge_{x\in F}(\iota_I \otimes \iota_I)(\gamma(x)) = \bigwedge_{x\in F} \gamma_\backslash(x).$$

(1c) The coalgebra is easily seen to be locally finite, since for every finite subset $X \subseteq R$ we find a finite residuation ideal I containing X, and the corresponding coalgebra structure on I is by definition a subcoalgebra of $(R, \gamma_\backslash)$.

(2a) We first show that for finite subcoalgebras A, A' of R containing both x, z we have

$$\iota_A(x \setminus z) = \iota_{A'}(x \setminus z).$$

First, let $\iota_B \cdot \iota\colon A \hookrightarrow B \hookrightarrow R$ be finite subcoalgebras. Then

$$\begin{aligned}
\iota_A(x \setminus z) &= \iota_A(x \multimap \gamma(z)) \\
&= \iota_B(\iota(x \multimap \gamma(z))) \\
&= \iota_B(\iota(x) \multimap (\iota \otimes \iota)(\gamma(z))) && \text{embeddings preserve } \multimap \\
&= \iota_B(\iota(x) \multimap \gamma(\iota(z))) && \iota \text{ coalgebra morphism} \\
&= \iota_B(\iota(x) \setminus \iota(z)).
\end{aligned}$$

From this it follows that

$$\iota_B(x \setminus \iota(z)) = \iota_B(x \multimap (\iota \otimes \iota)(\gamma(z))) = \iota_B(\iota(\iota^*(x) \multimap \gamma(z))) = \iota_A(\iota^*(x) \setminus z). \tag{5.6}$$

Now let $A \hookrightarrow C$ be a finite subcoalgebra containing x and z. Then A certainly contains the (finite) subcoalgebra $\langle z \rangle$ generated by z; we write $\iota\colon \langle z \rangle \hookrightarrow A$ for its inclusion into A. We now obtain the canonical expression, where the last step uses (5.6)

$$x \setminus_\gamma z = \iota_A(x \setminus z) = \iota_A(x \setminus \iota(z)) = \iota_{\langle z \rangle}(\iota^*(x) \setminus z)$$

For general finite subcoalgebras $A, A' \hookrightarrow R$ containing x, z we find an upper bound $\iota\colon A \hookrightarrow B \hookleftarrow A' : \iota'$ and compute

$$\iota_A(x \setminus z) = \iota_B(\iota(x) \setminus \iota(z)) = \iota_B(\iota'(x) \setminus \iota'(z)) = \iota_{A'}(x \setminus z).$$

(2b) The proof that the residuals preserve finite meets in the covariant component is analogous to the proof for the comultiplication.

(2c) The residuation algebra structure induced by γ is locally finite: Since the coalgebra (C, γ) is locally finite, every finite subset $F \subseteq C$ is contained in a finite subcoalgebra $A \hookrightarrow C$. The finite residuation algebra structure on A given by Lemma 5.16(2) makes A a residuation subalgebra of $(C, \setminus_\gamma)$ containing F.

(2d) It remains to verify the residuation property:

$$\begin{aligned}
y \leq x \backslash_\gamma z \iff & y \leq \iota_z(\iota_z^*(x) \backslash_z z) && \text{definition } \backslash_\gamma \\
\iff & y \leq \iota_z(\iota_z^* x \multimap \gamma z) && \text{definition } \backslash_z \\
\iff & \iota_z^* y \leq \iota_z^* x \multimap \gamma z && \iota_z^* \dashv \iota_z \\
\iff & \iota_z^* x \otimes \iota_z^* y \leq \gamma z && \iota_z^* x \otimes (-) \dashv \iota_z^* x \multimap (-) \\
\iff & \iota_z^* x \leq \gamma z \multimapinv \iota_z^* y && (-) \otimes \iota_z^* y \dashv (-) \multimapinv \iota_z^* y \\
\iff & x \leq \iota_z(\gamma z \multimapinv \iota_z^* y) && \iota_z^* \dashv \iota_z \\
\iff & x \leq \iota_z(z /_z \iota_z^* y) && \text{definition } /_z \\
\iff & x \leq z /_\gamma y && \text{definition } /_\gamma
\end{aligned}$$

(3) The translations are inverse since they are liftings of the translations between the operators on the finite substructures: We first show that $\gamma_{\backslash_\gamma} = \gamma$. For every $z \in R$, the subcoalgebra $\iota_z \colon \langle z \rangle \hookrightarrow R$ generated by z is a residuation ideal of $\backslash_\gamma$: For all $x \in R$, we have

$$x \backslash_\gamma z = \iota_z(\iota_z^*(x) \multimap \gamma(z)) \in \langle z \rangle.$$

We can therefore choose it as a residuation ideal containing z in the definition of $\gamma_{\backslash_\gamma}$ to get

$$\gamma_{\backslash_\gamma}(z) = (\iota_z \otimes \iota_z)(\gamma_{\backslash_\gamma}(z)) = (\iota_z \otimes \iota_z)(\gamma(z)) = \gamma(z).$$

An analogous argument proves $\backslash_{\gamma_\backslash} = \backslash$. □

Proposition 5.28 shows that every locally finite residuation algebra carries a unique $U_\wedge$-coalgebra structure and vice versa. We may thus translate at will between the residuals and comultiplication as in the complete case and omit the subscripts. We extend Lemmas 5.17 and 5.18 to locally finite structures:

Lemma 5.30. *Let R be a locally finite residuation algebra.*

(1) *Finite residuation ideals correspond to finite subcoalgebras.*

(2) *The residuals are associative iff the comultiplication is coassociative.*

(3) *It is prime-unital iff the comultiplication is prime-counital.*

(4) *The comultiplication is pure iff every finite residuation ideal is pure (see Definition 5.19).*

Proof.

(1) If $\iota_I \colon I \hookrightarrow R$ is a finite residuation ideal, then by definition its comultiplication makes I a subcoalgebra of (R, γ).

In the reserve direction, let $A \hookrightarrow R$ be a finite subcoalgebra. The residuation algebra structure on A due to Lemma 5.16(1) together with the definition $x \setminus z = \iota_A(x \setminus z)$ of the residuation algebra structure on R show that A is a residuation subalgebra of R. To show that A is a residuation ideal, let $x \in R$ and $z \in A$. There exists a finite subcoalgebra B containing x, z; we denote the inclusion map by $\iota \colon A \hookrightarrow B$. By (5.6) we then have

$$x \setminus z = \iota_B(x \setminus z) = \iota_A(\iota^*(x) \setminus z),$$

which states that $x \setminus z$ lies in A.

(2) First, let γ be the coassociative comultiplication, and let $x, y, z \in R$. By local finiteness these elements are contained in a finite coassociative subcoalgebra $\iota_A \colon A \hookrightarrow R$. Then by Lemma 5.30(1), ι_A, is an associative finite residuation ideal of R, whence

$$x \setminus (z \mathbin{/} y) = \iota_A(x \setminus (z \mathbin{/} y)) = \iota_A((x \setminus z) \mathbin{/} y) = (x \setminus z) \mathbin{/} y.$$

The other direction works analogously: if the residuals are associative and $z \in R$, then it is contained in a finite associative residuation ideal $\iota_I \colon I \hookrightarrow R$. So I is a finite associative subcoalgebra of R and we have

$$\begin{aligned}(\bar{\gamma} \boxtimes \mathrm{id})(\bar{\gamma}(\iota_I(z))) &= (\iota_I \boxtimes \iota_I \boxtimes \iota_I)((\bar{\gamma} \boxtimes \mathrm{id})(\bar{\gamma}(z))) \\ &= (\iota_I \boxtimes \iota_I \boxtimes \iota_I)((\mathrm{id} \boxtimes \bar{\gamma})(\bar{\gamma}(z))) \\ &= (\mathrm{id} \boxtimes \bar{\gamma})(\bar{\gamma}(\iota_I(z))),\end{aligned}$$

proving that $\bar{\gamma}$ is associative.

(3) Let $\epsilon \colon R \to 2$ be a counit for the comultiplication

$$\bar{\gamma} = \omega \cdot \gamma \colon U_\wedge R \to U_\wedge(R \otimes R) \cong U_\wedge R \boxtimes U_\wedge R$$

with left adjoint element $e \in \mathbf{JSL}(2, R) \cong R$. Note that $\epsilon \cdot \iota_A$ is a counit for every subcoalgebra $\iota_A \colon A \hookrightarrow R$: Let $z \in A$, then, since ϵ is a unit for $\bar{\gamma}$, we have

$$\iota_A(((\epsilon \cdot \iota_A) \boxtimes \mathrm{id}_A)(\bar{\gamma}(z))) = (\epsilon \boxtimes \mathrm{id})(\iota_A \boxtimes \iota_A)(\bar{\gamma}(z)) = (\epsilon \boxtimes \mathrm{id})(\bar{\gamma}(\iota_A(z))) = \iota_A(z).$$

But by the implication (1) $\Rightarrow$ (2) of Lemma 5.18 this means that the left adjoint $e_A := \iota_A^*(e) \in A$ of $\epsilon \cdot \iota_A$ is a unit for the finite residuation ideal A of R. So for all $z \in R$ we have

$$e \setminus z = \iota_z(\iota_z^*(e) \setminus z) = \iota_z(e_z \setminus z) = \iota_z(z) = z,$$

so e is a unit for the residuals. The case for the right residual $/$ is dual.

For the other direction let $e \in R$ be a unit for the residuals with right adjoint $\epsilon \colon R \to 2$. For every residuation ideal $\iota_I \colon I \hookrightarrow R$ the element $\iota_I^*(e) \in I$ is the unit of I: The embedding trivially is an open residuation morphism, and we have

$$\iota_I(\iota_I^*(e) \setminus z) = e \setminus \iota_I(z) = \iota_I(z) \qquad \text{for every } z \in I.$$

So the subcoalgebra structure I has the counit $\epsilon \cdot \iota_I$. Now given $z \in R$, we pick a residuation ideal I containing z using that R is locally finite. Then we have

$$\begin{aligned}(\epsilon \boxtimes \mathrm{id})(\bar{\gamma}(z)) &= (\epsilon \boxtimes \mathrm{id})(\bar{\gamma}(\iota_I(z))) \\ &= (\epsilon \boxtimes \mathrm{id})((\iota_I \boxtimes \iota_I)(\bar{\gamma}(z))) \\ &= \iota_I((\epsilon\iota_I \boxtimes \mathrm{id})(\bar{\gamma}(z))) \\ &= \iota_I(z) = z.\end{aligned}$$

This shows that ϵ is a counit for $\bar{\gamma}$.

(4) If the comultiplication is pure, associative and has a counit, then this holds for every finite subcoalgebra. Every finite residuation ideal I of R is a finite pure subcoalgebra, which therefore is a derivation algebra.

Conversely, if every finite residuation ideal is a derivation algebra, then we only have to show that the comultiplication preserves finite joins, since it already is coassociative and has a counit. The join of finitely many elements is taken in some finite subcoalgebra A. But A is a finite residuation ideal and therefore by assumption a derivation algebra, and the comultiplication preserves finite joins. □

Remark 5.31. Lemma 5.30(4) characterizes locally finite residuation algebras with a pure comultiplication. By extended duality, its dual Priestley relation is functional. Gehrke [Geh16b, Prop. 3.15] has presented a necessary and sufficient condition for a general residuation algebra R to have a functional dual relation, namely *join-preservation at primes*:

$$\forall F \in \mathbf{DL}(R,2)\colon \forall a \in F\colon \forall b,c \in R\colon \exists a' \in F\colon a \setminus (b \vee c) \leq (a' \setminus b) \vee (a' \setminus c).$$

One can indeed show that every locally finite residuation algebra with a pure comultiplication (Lemma 5.30(4)) is join-preserving at primes: If F is a prime filter on C and $a \in F$, then we obtain by local finiteness for all $b, c \in C$ a finite *pure* residuation ideal $\iota\colon I \hookrightarrow C$ containing a, b, c. In this residuation ideal we have $a = \bigvee K$ for join-primes $K \subseteq \mathcal{J}I \subseteq C$. Since F is prime, some $a' \in K$ lies in F, and it satisfies

$$\begin{aligned}
a \setminus (b \vee c) &= \iota(a \setminus (b \vee c)) \\
&= \iota((\textstyle\bigvee K) \setminus (b \vee c)) \\
&\leq \iota(a' \setminus (b \vee c)) && (-) \setminus x \text{ antimonotone} \\
&= \iota(a' \setminus b \vee a' \setminus c) && I \text{ is pure} \\
&= \iota(a' \setminus b) \vee \iota(a' \setminus c) \\
&= (a' \setminus b) \vee (a' \setminus c) && a', b, c \in I.
\end{aligned}$$

This shows that the residuals are join-preserving at primes.

Definition 5.32. A residuation algebra R is a *derivation algebra* if it is locally finite, associative, prime-unital and every finite residuation ideal I is pure. This yields full subcategories

$$\mathbf{Der} \hookrightarrow \mathbf{Res} \qquad \text{and} \qquad \mathbf{RelRes} \hookrightarrow \mathbf{RelDer}.$$

Note that a derivation algebra with a finite carrier is precisely a finite derivation ACDL as defined in Definition 5.19(2). Recall from Definition 5.20 the definition morphism for $U_\wedge$-coalgebras and comonoids.

Proposition 5.33. *Let R, R' be locally finite residuation algebras with units.*

(1) *A lattice morphism $f \in \mathbf{DL}(R, R')$ is a residuation morphism iff it is a prime-counital morphism of $U_\wedge$-coalgebras.*

(2) *If R, R' are comonoids, a finite join-preserving function $\rho \in \mathbf{JSL}(R, R')$ is a corelational residuation morphism iff it is a corelational comonoid morphism.*

Proof.

(1) First, let $f\colon R \to R'$ be a residuation morphism. For $z \in R$ we choose a finite residuation ideal $I' \hookrightarrow R'$ containing $f(z)$. Since f is a residuation morphism we have by the (Back) condition that for every $y \in I'$ there exists some $x_{y,z}$ with $y \leq f(x_{y,z})$ and $y \setminus f(z) = f(x_{y,z} \setminus z)$. We now choose a finite ideal $I \hookrightarrow R$ containing z and all $x_{y,z}$ for $y \in I'$. We therefore have

$$\begin{aligned}
\gamma_\setminus(f(z)) &= \textstyle\bigvee_{y \in I'} y \otimes y \setminus f(z) \\
&\leq \textstyle\bigvee_{y \in I', y \leq f(x_{y,z})} f(x_{y,z}) \otimes f(x_{y,z} \setminus z) \\
&= (f \otimes f)(\textstyle\bigvee_{y \in I', y \leq f(x_{y,z})} x_{y,z} \otimes x_{y,z} \setminus z) \\
&\leq (f \otimes f)(\textstyle\bigvee_{x \in I} x \otimes x \setminus z) \\
&= (f \otimes f)(\gamma_\setminus(z)).
\end{aligned}$$

For the reverse inequality let $z \in R$ be contained in the finite residuation ideal I. We choose a finite residuation ideal $J \hookrightarrow R'$ containing $f(z)$ and all $f(x), x \in I$, and use (Forth):

$$\begin{aligned}(f \otimes f)(\gamma_{\setminus}(z)) &= \bigvee\nolimits_{x\in I} f(x) \otimes f(x \setminus z)\\ &\leq \bigvee\nolimits_{x\in I} f(x) \otimes f(x) \setminus f(z)\\ &\leq \bigvee\nolimits_{y\in J} y \otimes y \setminus f(z)\\ &= \gamma_{\setminus}(f(z)),\end{aligned}$$

This proves that f is a morphism of $U_\wedge$-coalgebras.

Conversely, let $f\colon R \to R'$ be a morphism of $U_\wedge$-coalgebras. For every $z \in R$ the morphism f restricts to the finite subcoalgebras generated by $z, f(z)$ as $f_z\colon \langle z\rangle \to \langle f(z)\rangle$. If we denote the respective inclusions by $\iota_z\colon \langle z\rangle \hookrightarrow R$ and $\iota_{f(z)}\colon \langle f(z)\rangle \hookrightarrow R'$, then this is equivalent to saying that $f \cdot \iota_z = \iota_{f(z)} \cdot f_z$. From the unit of ι_z we thus get

$$f \leq f \cdot \iota_z \cdot \iota_z^* = \iota_{f(z)} \cdot f_z \cdot \iota_z^*,$$

which under transposition is equivalent to

$$\iota_{f(z)}^* \cdot f \leq f_z \cdot \iota_z^*. \tag{5.7}$$

This entails the (Forth) condition:

$$\begin{aligned}f(x \setminus_\gamma z) &= f(\iota_z(\iota_z^*(x) \multimap \gamma(z))) && \text{def. } \setminus_\gamma\\ &= \iota_{f(z)}(f_z(\iota_z^*(x) \multimap \gamma(z))) && f_z \text{ restriction}\\ &\leq \iota_{f(z)}(f_z(\iota_z^*(x)) \multimap (f_z \otimes f_z)\gamma(z))) && \multimap \text{ Proposition 5.6(3)}\\ &= \iota_{(z)}(f_z(\iota_z^*(x)) \multimap \gamma(f_z(z))) && f_z \text{ coalgbra morphism}\\ &\leq \iota_{f(z)}(\iota_{f(z)}^*(f(x)) \multimap \gamma(f_z(z))) && (5.7) + \text{contravariance}\\ &\leq \iota_{f(z)}(\iota_{f(z)}^*(f(x)) \multimap \gamma(f(z))) && f_z(z) = f(z)\\ &= f(x) \setminus_\gamma f(z).\end{aligned}$$

To verify the (Back) condition, let $y \in R', z \in R$ and put

$$x_{y,z} = \iota_z(f_z^*(\iota_{f(z)}^*(y))).$$

Then

$$y \leq \iota_{f(z)}\iota_{f(z)}^*(y) \leq \iota_{f(z)}(f_z(f_z^*(\iota_{f(z)}^*(y)))) = f(\iota_z(f_z^*(\iota_{f(z)}^*(y)))) = f(x_{y,z})$$

and

$$\begin{aligned}y \setminus_\gamma f(z) &= \iota_{f(z)}(\iota_{f(z)}^*(y) \multimap \gamma(f_z(z))) && \text{def. } \setminus_\gamma\\ &= \iota_{f(z)}(\iota_{f(z)}^*(y) \multimap (f_z \otimes f_z)(\gamma(z))) && f_z \text{ coalgebra morphism}\\ &= \iota_{f(z)}(f_z(f_z^*(\iota_{f(z)}^*(y)) \multimap \gamma(z))) && \text{Proposition 5.6(3)}\\ &= f(\iota_z(f_z^*(\iota_{f(z)}^*) \multimap \gamma(z))) && \iota_{f(z)} \cdot f_z = f \cdot \iota_z\\ &= f(\iota_z(\iota_z^*(\iota_z(f_z^*(\iota_{f(z)}^*(y)))) \multimap \gamma(z))) && \iota_z^*\iota_z = \mathrm{id}\\ &= f_z(x_{y,z} \setminus_\gamma z) && \text{def. } x_{y,z}.\end{aligned}$$

For the prime-(co-)unitality conditions we split the pointwise equality $\forall x\colon \epsilon'(f(x)) = \epsilon(x)$ into $\epsilon'(f(x)) \leq \epsilon(x)$ and $\epsilon(x) \leq \epsilon'(f(x))$. These are equivalent to $e' \leq f(x) \Rightarrow e \leq x$ and $e \leq x \Rightarrow e' \leq f(x)$, respectively, combining to the desired condition $\forall x\colon e' \leq f(x) \Leftrightarrow e \leq x$.

(2) Now let R, R' be comonoids and let $f \in \mathbf{JSL}(R, R')$ be a corelational comonoid morphism. For $x, z \in R$ we choose a finite subcomonoid $\iota\colon I \hookrightarrow R$ that contains x, z and a finite subcomonoid $\iota'\colon I' \hookrightarrow R'$ (with comultiplication γ') containing $\rho[I]$. Then ρ restricts to a corelational morphism $\rho\colon I \to I'$ of finite comonoids. By Theorem 5.22 ρ is a corelational morphism of finite residuation algebras, so it satisfies $\rho(a \setminus_\gamma b) \leq \rho(a) \setminus_{\gamma'} \rho(b)$ for all $a, b \in I$. We therefore get

$$\rho(x \setminus_\gamma z) = \rho(\iota(x \setminus_\gamma z)) = \iota'(\rho(x \setminus_\gamma z)) \leq \iota'(\rho(x) \setminus_{\gamma'} \rho(z)) = \rho(x) \setminus_{\gamma'} (z).$$

which proves that f is a corelational residuation morphism. To verify $e' \leq \rho(e)$ we again choose finite subcoalgebras A, A' with $e \in A$ and $\rho[A] \cup \{e'\} \subseteq A'$. Since ρ is prime-counital it satisfies $\epsilon \leq \epsilon' \cdot \rho$ and therefore also $\epsilon \cdot \iota_A \leq \epsilon' \cdot \rho \cdot \iota_A = \epsilon' \cdot \iota_{A'} \cdot \rho$. As $\epsilon \cdot \iota_A$ and $\epsilon' \cdot \iota_{A'}$ are the counits for A and A', respectively, its restriction $\rho\colon A \to A'$ is thus also prime-counital and whence satisfies $\iota_A^*(e) \leq \rho(\iota_{A'}^*(e'))$ for the corresponding units of the residuals on A, A'. But $e \in A$ and $e' \in A'$, so this equation simplifies to the desired $e \leq \rho(e')$.

Conversely, if f is a corelational residuation morphism choose for $z \in R$ a finite residuation ideal $\iota\colon I \hookrightarrow R$ containing z and a finite residuation ideal $\iota'\colon I' \hookrightarrow R$ containing $\rho[I]$. Then we have

$$\begin{aligned}
(\rho \otimes \rho)(\gamma(z)) &= (\rho \otimes \rho)(\gamma(\iota(z))) \\
&= (\rho \otimes \rho)(\iota \otimes \iota)(\gamma(z)) \\
&= (\iota' \otimes \iota')(\rho \otimes \rho)(\gamma(z)) \\
&\leq (\iota' \otimes \iota')\gamma(\rho(z)) \\
&= \gamma(\rho(z)),
\end{aligned}$$

so ρ is a corelational residuation morphism. To show that $\epsilon \leq \epsilon' \cdot \rho$, take $z \in R$ with ideals I, I' chosen as before. Recall that $\iota^*(e)$ is a unit of the residuation ideal I and $\epsilon \cdot \iota$ is a counit for the corresponding subcoalgebra. Since ρ is prime-unital it satisfies

$$e' \leq \rho(e) \leq \rho(\iota(\iota^*(e))) = \iota'(\rho(\iota^*(e)))$$

which is equivalent to $\iota'^*(e') \leq \rho(\iota^*(e))$. Since $\rho\colon I \to I'$ is a corelational residuation morphism it is a corelational morphism of the coalgebra structures on I, I' and we thus get

$$\epsilon(z) = \epsilon(\iota(z)) \leq \epsilon'(\iota'(\rho(z))) = \epsilon'(\rho(\iota(z))) = \epsilon'(\rho(z)). \qquad \square$$

Remark 5.34. The proof of Proposition 5.33 gives an alternative formulation of the (Back) condition for locally finite residuation algebras as

$$y \setminus f(z) = f((\iota_z \cdot f_z^* \cdot \iota_{f(z)}^*)(y) \setminus z).$$

Here we choose the existentially quantified $x_{y,z}$ in (Back) via f_z^* *locally*:

$$
\begin{tikzcd}
x_{y,z} \arrow[rr, mapsto] \arrow[dr, phantom, "\in"] & & f(x_{y,z}) \arrow[r, phantom, "\geq"] \arrow[d, phantom, "\in"] & y \arrow[ddd, mapsto] \arrow[dl, phantom, "\ni"] \\
& R \arrow[r, "f"] \arrow[d, shift right=1.5ex, "\iota_z^*"'] \arrow[d, phantom, "\dashv"] & S \arrow[d, shift right=1.5ex, "\iota_{f(z)}^*"'] \arrow[d, phantom, "\dashv"] & \\
& \backslash z/ \arrow[u, shift right=1.5ex, "\iota_z"'] \arrow[r, shift right, "f_z"'] \arrow[r, phantom, "\perp"] \arrow[dl, phantom, "\ni"] & \backslash f(z)/ \arrow[u, shift right=1.5ex, "\iota_{f(z)}"'] \arrow[l, shift right, "f_z^*"'] \arrow[dr, phantom, "\in"] & \\
f_z^*(\iota_{f(z)}^*(y)) \arrow[uuu, mapsto] & & & \iota_{f(z)}^*(y) \arrow[lll, mapsto]
\end{tikzcd}
$$

Compare this with Equation (Open) satisfied by open residuation morphisms, where the existence of a global left adjoint f^* allows one to choose $x_{y,z} = f^*(y)$ *globally*, that is, independently of z.

We then have the following extension of Theorem 5.22 to locally finite structures:

Theorem 5.35.

(1) *The category of locally finite residuation algebras and residuation morphisms is isomorphic to the category of locally finite prime-unital $U_\wedge$-coalgebras and pure coalgebra morphisms.*

(2) *This isomorphism restricts to an isomorphism between the full subcategories of derivation algebras and locally finite comonoids.*

(3) *The category of derivation algebras and corelational residuation morphisms is isomorphic to the category of locally finite comonoids with relational morphisms.*

Proof. Immediate from Lemma 5.30 and Proposition 5.33. □

5.5. **Duality Theory for Locally Finite Residuation Algebras.** We now gathered the ingredients to present the first application of our abstract extended duality (Theorem 3.8): a categorical duality between *profinite ordered monoids* and *derivation algebras*. Recall that a profinite ordered monoid is a codirected limit of finite ordered monoids; like in the order-discrete setting, they are equivalent to *Priestley monoids*, viz. monoids in the cartesian category **Priest** (Proposition 5.40). This result is a non-trivial restriction of Gehrke's duality [Geh16a, Geh16b] between Priestley-topological algebras and residuation algebras.

Conceptually, this general duality is an extension of the finite duality $\mathbf{OrdMon}_\mathrm{f} \simeq^{\mathrm{op}} \mathbf{Comon}_\mathrm{f} \cong \mathbf{Der}_\mathrm{f}$ by forming suitable completions. We start by investigating the Ind- and Pro-completions of the categories involved in the finite duality Remark 5.24.

Remark 5.36. The *Ind-completion* (or *free completion under filtered colimits*) of a small category $\mathbf{C}$ is given by a category $\mathrm{Ind}(\mathbf{C})$ with filtered (equivalently directed) colimits and a full embedding $I\colon \mathbf{C} \hookrightarrow \mathrm{Ind}(\mathbf{C})$ such that every functor $F\colon \mathbf{C} \to \mathbf{D}$ into a category $\mathbf{D}$ with filtered colimits extends to a functor $\overline{F}\colon \mathrm{Ind}(\mathbf{C}) \to \mathbf{D}$, unique up to natural isomorphism, such that $F = \overline{F} \cdot I$. To show that a category $\mathbf{D}$ is the Ind-completion of a full subcategory $\mathbf{C}$, it suffices to prove the following (see e.g. [ACMU21, Thm. A.4]):

(1) $\mathbf{D}$ has filtered colimits,

(2) every object of $\mathbf{D}$ is a filtered colimit of objects of $\mathbf{C}$, and

(3) every object C of $\mathbf{C}$ is finitely presentable in $\mathbf{D}$, that is, the functor $\mathbf{D}(C,-)\colon \mathbf{D} \to \mathbf{Set}$ preserves filtered colimits.

The *Pro-completion* (or *free completion under cofiltered limits*) of $\mathbf{C}$ is defined dually.

Profinite ordered monoids form the Pro-completion of the category of finite ordered monoids. Dually, lattice comonoids (and therefore also derivation algebras by Theorem 5.35(2)) form Ind-completions of their respective subcategories of finite objects:

Proposition 5.37. *The category of locally finite comonoids forms the Ind-completion of the category of finite comonoids:*

$$\mathbf{Comon}_{\mathrm{lf}} \simeq \mathrm{Ind}(\mathbf{Comon}_{\mathrm{f}}).$$

Proof. (a) We first show that filtered colimits of lattice comonoids are formed in $\mathbf{Set}$. First, since $\mathbf{DL}$ is a category of algebras over a finitary signature, filtered colimits in $\mathbf{DL}$ are formed in $\mathbf{Set}$. Second, since $+\colon \mathbf{DL}\times\mathbf{DL} \to \mathbf{DL}$ is a finitary functor (colimits commute with colimits) filtered colimits in the category of $+$-coalgebras are also formed in $\mathbf{Set}$. As comonoids are a full subcategory of $+$-coalgebras it suffices to show that the filtered colimit in $+$-coalgebras of lattice comonoids is again a comonoid, which is a straightforward verification: Let $d_i\colon D_i \to D_i + D_i, i \in I$ be a cofiltered diagram of comonoids, and let $d\colon D \to D + D$ be their colimit in the category of $+$-coalgebras with colimit injections $\kappa_i\colon D_i \to D$. Since the colimit D is formed in $\mathbf{Set}$, there exists for every $x \in D$ some $i \in I$ and $x_i \in D_i$ with $\kappa_i(x_i) = x$. But then

$$\begin{aligned}
(d+\mathrm{id})(d(x)) &= (d+\mathrm{id})d(\kappa_i(x_i)) && \\
&= (d+\mathrm{id})(\kappa_i+\kappa_i)(d_i(x_i)) && \kappa_i \text{ comonoid morphism}\\
&= ((d\cdot\kappa_i)+\kappa_i)(d_i(x_i)) && \\
&= (((\kappa_i+\kappa_i)\cdot d_i)+\kappa_i)(d_i(x_i)) && \kappa_i \text{ comonoid morphism}\\
&= (\kappa_i+\kappa_i+\kappa_i)(d_i+\mathrm{id})(d_i(x_i)) && \\
&= (\kappa_i+\kappa_i+\kappa_i)(\mathrm{id}+d_i)(d_i(x_i)) && d_i \text{ comonoid}\\
&= \cdots && \text{backwards}\\
&= (\mathrm{id}+d)(d(x)),
\end{aligned}$$

so D is coassociative. Counitality works similarly for $\epsilon\colon D \to 2$ with $\epsilon(x) = \epsilon(\kappa_i(x_i)) = \epsilon_i(x_i)$. This proves that filtered colimits of comonoids are formed in $\mathbf{Set}$.

(b) To prove that the category $\mathbf{Comon}_{\mathrm{lf}}$ is the Ind-completion of its full subcategory $\mathbf{Comon}_{\mathrm{f}}$, we verify the conditions of Remark 5.36:

(1) $\mathbf{Comon}_{\mathrm{lf}}$ has filtered colimits: filtered colimits of comonoids are formed in $\mathbf{Set}$ by (a). Moreover, a filtered colimit $c_i\colon C_i \to C$ $(i \in I)$ of locally finite comonoids is locally finite: Given $x \in C$ one has $x = c_i(x_i)$ for some $i \in I$ and $x_i \in C_i$. Thus $x_i \in C_i'$ for some finite subcomonoid C_i' of C_i, and so x_i lies in the finite subcomonoid $c_i[C_i'] \subseteq C$.

(2) Every locally finite comonoid is the directed union of the diagram of all its finite subcomonoids. This follows again from (a), since this is clearly a directed union in $\mathbf{Set}$.

(3) Every finite comonoid is finitely presentable: A finite comonoid can be regarded as a coalgebra $C \to 2\times(C+C)$ for the functor $FX = 2\times(X+X)$ on $\mathbf{DL}$ by pairing its counit and comultiplication. Every finite F-coalgebra is finitely presentable in the category of all F-coalgebras [AP04, Lemma 3.2]. This implies the corresponding statement for finite comonoids since they form a full subcategory of the category of F-coalgebras. ∎

On the dual side, we require a property of Priestley monoids (Proposition 5.40 below), namely that every Priestley monoid is profinite, that is, it is a cofiltered limit of finite ordered monoids with the discrete topology. The corresponding result for (unordered) Stone monoids is well known [Joh82, Thm. VI.2.9]; its ordered version is analogous, and we give a full proof for the convenience of the reader.

We need some auxiliary results: First, we recall a well-known characterization for quotients of ordered algebras, see e.g. [MU19, Sec. B.2]. Recall that a preorder $\sqsubseteq$ on an ordered monoid A is *stable* if $\sqsubseteq$ refines the order on A (i.e. $\leq$ is included in $\sqsubseteq$) and the multiplication is monotone: $a \sqsubseteq b, a' \sqsubseteq b'$ implies $aa' \sqsubseteq bb'$.

Lemma 5.38. *Let X be an ordered monoid. Then ordered monoid quotients of X are in bijective correspondence with stable preorders on X.*

Proof. A quotient $e\colon A \twoheadrightarrow B$ induces a stable preorder by putting $a \sqsubseteq a'$ iff $e(a) \leq e(a')$ for $a, a' \in A$. Conversely, from a stable preorder $\sqsubseteq$ on $A \times A$ we obtain an equivalence relation on A by identifying $a \sim a'$ iff $a \sqsubseteq a'$ and $a' \sqsubseteq a$. Monotonicity of the multiplication then ensures that multiplication on equivalence classes is well-defined. The refinement property ensures that the canonical projection, which maps an element to its equivalence class, is order-preserving. It is a routine verification to check that these constructions are inverses. $\square$

Lemma 5.39. *Let X be a Priestley monoid. If $x \not\leq y$, then there exists a finite Priestley monoid quotient $f\colon X \twoheadrightarrow M$ such that $fx \not\leq fy$.*

We modify the proof of a corresponding statement for Stone algebras given by Johnstone [Joh82, Ch. VI, Sec. 2.7].

Proof. Let $x \not\leq y$ be elements of a Priestley monoid X, we show that there exists a quotient of $f\colon X \twoheadrightarrow M$ into some finite discretely-topologized ordered monoid M satisfying $f(x) \not\leq f(y)$. Since the underlying Priestley space of X is profinite, there exists a continuous surjection $e \in \mathbf{Priest}(X, A)$ such that A is a finite discretely topologized poset, satisfying $e(x) \not\leq e(y)$. We denote the preorder on X induced by e by $\sqsubseteq$ (i.e. $x \sqsubseteq y$ iff $e(x) \leq e(y)$). We define $\preceq$ by

$$x \preceq y \quad \text{iff} \quad \forall u, v \in X\colon uxv \sqsubseteq uyv,$$

and denote its corresponding equivalence relation by $\approx$.

(1) We prove that $\preceq$ is a stable preorder. First we show that $\preceq$ refines $\leq$: if $x \leq y$, then $\forall u, v \in X\colon uxv \leq uyv$ since the multiplication is monotone. Moreover, we have $uxv \sqsubseteq uyv$ for all $u, v \in X$, since $\leq$ is contained in $\sqsubseteq$. It follows that multiplication is $\preceq$-monotone: if $x \preceq y$ and $x' \preceq y'$, then for all $u, v \in X$ we have

$$u(xx')v = ux(x'v) \sqsubseteq uy(x'v) = (uy)x'v \sqsubseteq (uy)y'v = u(yy')v,$$

whence $xx' \preceq yy'$.

(2a) We prove that the equivalence relation $\approx \subseteq X \times X$ is open. The equivalence relation induced by $\sqsubseteq$ is denoted by $\equiv$, that is, $x \equiv y$ iff $x \sqsubseteq y \sqsubseteq x$ iff $e(x) = e(y)$. Note that $\equiv$ is open since it is the preimage of the (open) diagonal of A under $e \times e$. Now let $m \approx n$. By openness of $\equiv$ and continuity of the multiplication we obtain for all $u, v \in X$ open neighbourhoods $U_u, U_v, V_{u,v}$ and $W_{u,v}$ of u, v, m, n, respectively, such that

$$\forall u' \in U_u, v' \in U_v, m' \in V_{u,v}, n' \in W_{u,v}\colon u'm'v' \equiv u'n'v'. \tag{5.8}$$

By compactness of X we have $X \times X = \bigcup_{u,v} U_u \times U_v = \bigcup_{i=1}^{n} U_{u_i} \times U_{v_i}$ for some n and u_i, v_i. Now set $V = \bigcap_{i=1}^{n} V_{u_i,v_i}$ and $W = \bigcap_{i=1}^{n} W_{u_i,v_i}$. Then $V \times W$ is an open neighbourhood of

(m,n) satisfying $V \times W \subseteq \approx$: for $(m',n') \in V \times W$ we have for all $u',v' \in X$ some i with $(u',v') \in U_{u_i} \times U_{v_i}$. Using (5.8) we have $u'm'v' \equiv u'n'v'$, proving $m' \approx n'$.

(2b) We prove that every equivalence class $[x]_\approx$ is open. For every $y \in [x]_\approx$, there exists, since $\approx$ is open by (2a), a basic open $U \times V \subseteq \approx$ with $(x,y) \in U \times V$. For every $y' \in V$ we have $(x,y') \in U \times V \subseteq \approx$, so V is an open neighbourhood of y that is contained in $[x]_\approx$, proving that $\approx$ is open.

(3) It follows that the quotient f induced by $\preceq$ has the desired properties: (3a) It is a homomorphism of ordered monoids by Lemma 5.38, since $\preceq$ is a stable preorder. (3c) The codomain $X/{\approx}$ is finite and discrete, since equivalence relations with open equivalence classes are discrete, so $X/{\approx}$ is finite by compactness. (3d) Finally we show that f order-separates x and y: For the sake of contradiction suppose $f(x) \leq f(y)$. By definition we have $x \preceq y$, implying $x = 1x1 \sqsubseteq 1y1 = y$, a contradiction. □

Proposition 5.40. *Every Priestley monoid is profinite:*

$$\mathbf{PriestMon} \cong \mathbf{ProfOrdMon}.$$

Proof. For every $X \in \mathbf{PriestMon}$ we prove that $X \cong \lim D_X$, where D_X is the canonical codirected diagram for X over all finite discretely-topologized ordered monoid quotients of X; in particular, this shows that X is profinite.

First note that the limit of a diagram D in **PriestMon** is formed by taking the limit $L = \lim |D|$ of the underlying diagram $|-| \cdot D$ in **StoneMon** and equipping it with the product order. Then all projections are monotone, and this order makes L totally order-disconnected: if $(x_e)_e, (y_e)_e \in L$ with $(x_e)_e \not\leq (y_e)_e$ then there exists by definition of the order some quotient $e\colon X \twoheadrightarrow X_e$ with $x_e \not\leq y_e$. Hence $p_e^{-1}[\uparrow x_e]$ is a clopen upset of L containing $(x_e)_e$ but not $(y_e)_e$. This shows that L is a Priestley monoid, and it is easy to verify that it satisfies the universal property in **PriestMon**.

We have to prove that the canonical homomorphism $\phi\colon X \to \lim D_X$ is an isomorphism of Priestley monoids, or equivalently, that it is a continuous surjective order-embedding. Continuity and order-preservation are given since ϕ is a morphism in **Priest**.

That ϕ is an order-embedding follows from it being order-reflecting: $\phi(x) \leq \phi(y) \Rightarrow x \leq y$. By contraposition it suffices to show $x \not\leq y \Rightarrow \phi(x) \not\leq \phi(y)$, but the right side of the implication is equivalent to finding an ordered monoid quotient $f\colon X \twoheadrightarrow A$ into a finite Priestley monoid with $f(x) \not\leq f(y)$. This is precisely the content of Lemma 5.39.

To show that ϕ is surjective, we use a general property of codirected limits of compact Hausdorff spaces [RZ10, Lemma 1.1.5]: if $f_i\colon X \twoheadrightarrow X_i$ is a compatible family of surjections into a codirected diagram X_i, then the induced mapping $f\colon X \to \lim X_i$ is surjective. □

By duality we immediately obtain the following result:

Corollary 5.41. *Every comonoid is locally finite:* $\mathbf{Comon} \cong \mathbf{Comon}_{\mathrm{lf}}$.

Definition 5.42. Let X and Y be Priestley monoids. A *Priestley relational morphism* $X \to Y$ is a total Priestley relation $\rho\colon X \to \mathbb{V}_{\downarrow} Y$ such that

$$\rho(x)\rho(x') \subseteq \rho(xx') \qquad \text{and} \qquad 1_N \in \rho(1_M).$$

Theorem 5.43.

(1) *The category of derivation algebras is dually equivalent to the category of Priestley monoids*

$$\mathbf{Der} \cong \mathbf{Comon} \simeq^{\mathrm{op}} \mathbf{PriestMon}.$$

(2) *The duality from item (1) extends to Priestley relational morphisms:*

$$\mathbf{RelDer} \;\cong\; \mathbf{RelComon} \;\simeq^{\mathrm{op}}\; \mathbf{RelPriestMon}.$$

Proof.
(1) We assemble all the steps of the (dual) equivalences: The category of profinite ordered monoids is the Pro-completion of the category of finite ordered monoids [ACMU21, Prop. 2.10]. Since the category of finite ordered monoids is dual to the category of finite comonoids (Theorem 5.23 restricted to finite structures), the Pro-completion of the former is dual to the Ind-completion of the latter. By Proposition 5.37, the latter is equivalent to the category of locally finite comonoids – but this category is, by Theorem 5.35(2), equivalent to the category of derivation algebras:

$$\mathbf{PriestMon} \;\simeq\; \mathrm{Pro}(\mathbf{OrdMon}_{\mathrm{f}}) \;\simeq^{\mathrm{op}}\; \mathrm{Ind}(\mathbf{Comon}_{\mathrm{f}}) \;\simeq\; \mathbf{Comon}_{\mathrm{lf}} \;\simeq\; \mathbf{Der}.$$

(2) A Priestley relational morphism from M to N is precisely a total Priestley relation $\rho\colon M \to \mathbb{V}_{\downarrow}N$ such that the following diagrams commute laxly as indicated.

$$\begin{tikzcd}
M \times M \arrow[rr, "\cdot_M"] \arrow[d, "\rho\times\rho"'] \arrow[drr, phantom, "\lightning"] & & M \arrow[d, "\rho"] \\
\mathbb{V}_{\downarrow}N \times \mathbb{V}_{\downarrow}N \arrow[r, "\hat{\delta}"] & \mathbb{V}_{\downarrow}(N \times N) \arrow[r, "\mathbb{V}_{\downarrow}(\cdot_N)"] & \mathbb{V}_{\downarrow}N
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "1_M"] \arrow[d, "1_N"'] \arrow[dr, phantom, "\lightning"] & M \arrow[d, "\rho"] \\
N \arrow[r, "\eta"] & \mathbb{V}_{\downarrow}N
\end{tikzcd}$$

Recall that $\mathbb{V}_{\downarrow} \cong \widehat{F}_{\vee}\hat{U}_{\vee}$ for $U_{\vee}\colon \mathbf{DL} \to \mathbf{JSL}$, so under extended duality ρ dualizes precisely to a corelational morphism of comonoids:

$$\begin{tikzcd}
U_{\vee}\hat{M} \otimes U_{\vee}\hat{M} \arrow[dr, phantom, "\lightning"] & U_{\vee}\hat{M} \arrow[l, "U_{\vee}(\hat{\cdot}_M)"'] \\
U_{\vee}\hat{N} \otimes U_{\vee}\hat{N} \arrow[u, "\hat{\rho}^{-}\otimes\hat{\rho}^{-}"] & U_{\vee}\hat{N} \arrow[l, "U_{\vee}(\hat{\cdot}_M)"'] \arrow[u, "\hat{\rho}^{-}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[dr, phantom, "\lightning"] & U_{\vee}\hat{M} \arrow[l, "U_{\vee}(\hat{1}_M)"'] \\
 & U_{\vee}\hat{N} \arrow[ul, "U_{\vee}(\hat{1}_N)"] \arrow[u, "\hat{\rho}^{-}"']
\end{tikzcd}$$

Together with Theorem 5.35(3) this extends the duality established in item (1). □

Remark 5.44. Theorem 5.43 restricts to a duality between the category of profinite monoids and Stone relational morphisms and the category of Boolean derivation algebras and corelational residuation morphisms.

6. Duality for the Category of Small Categories

As a final application of the abstract extended duality framework, we derive a concrete description of $\mathbf{Cat}^{\mathrm{op}}$, the dual of the category of small categories and functors. We instantiate the parameters of Assumption 3.5 to the Diagram (5.1).

It is well known that small categories can be described in an object-free way as partial monoids [ML98, FJSZ23, SS67] with an additional *locality* condition. For our purposes it will be convenient to describe partial monoids more generally as monoids in the monoidal category **Rel**.

Notation 6.1. Given a ternary relation $r\colon X \times Y \to \mathcal{P}Z$, we obtain the relation

$$\bigcup \cdot \mathcal{P}(r) \cdot \hat{\delta}\colon \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}(X \times Y) \to \mathcal{P}\mathcal{P}Z \to \mathcal{P}Z, \qquad (A, B) \mapsto \bigcup_{x\in A, y\in B} r(x, y). \quad (6.1)$$

Abusing notation, we denote the map (6.1) also by $r\colon \mathcal{P}X \times \mathcal{P}Y \to \mathcal{P}Z$. We write $r_@(x,y)$ if $r(x,y) \neq \emptyset$, and similarly $x\ r_@\ y$ if r is used as an infix operator. Furthermore, we identify singleton sets $\{x\} \subseteq X$ with their unique inhabitants $x \in X$.

Definition 6.2.

(1) A *relational monoid* consists of a carrier set M, a subset $E \hookrightarrow M$ of *identities* and a *multiplication relation* $\circ\colon M \times M \to \mathcal{P}M$ such that the following diagrams commute in $\mathbf{Rel}$:

$$\begin{tikzcd}
M \bar{\times} M \bar{\times} M \arrow[r, "\mathrm{id}\bar{\times}\circ"] \arrow[d, "\circ\bar{\times}\mathrm{id}"'] & M \bar{\times} M \arrow[d, "\circ"] \\
M \bar{\times} M \arrow[r, "\circ"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \bar{\times} M \arrow[r, phantom, "\cong"] \arrow[d, "E\bar{\times}\mathrm{id}"] & M \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{id}"] & M \bar{\times} 1 \arrow[d, "\mathrm{id}\bar{\times}E"] \\
M \bar{\times} M \arrow[r, "\circ"] & M & M \bar{\times} M \arrow[l, "\circ"']
\end{tikzcd}$$

Using Notation 6.1, the diagrams above read

$$\forall x, y, z \in M\colon (x \circ y) \circ z = x \circ (y \circ z), \qquad E \circ x = x = x \circ E.$$

(2) A relational monoid is *partial* if $\circ$ is single-valued, and it is *local* [FJSZ23] if it satisfies

$$x \circ_@ y \text{ and } v \in y \circ z \implies x \circ_@ v.$$

(3) A *functorial* morphism $h\colon (M, \circ, E) \to (M', \circ, E')$ of partial monoids is a pure morphism of binary $J_{\mathcal{P}}$-operators (that is, $\mathcal{P}(h)(x \circ y) = h(x) \circ h(y)$) satisfying $h(E) \subseteq E'$. The category of relational monoids and functorial morphisms is denoted by $\mathbf{RelMon}$.

Every monoid is obviously also a relational monoid. An example of a partial monoid that is not local is given by $(\mathcal{P}(X), \circ, \emptyset)$, $|X| \geq 1$, with $A \circ B = A + B$ if A, B are disjoint and $A \circ B = \emptyset$ otherwise.

Remark 6.3.

(1) The full subcategory of $\mathbf{RelMon}$ consisting of local partial monoids is equivalent to the category of small categories [ML98, FJSZ23, SS67]:

$$\mathbf{RelMon}_{\mathrm{fun,loc}} \simeq \mathbf{Cat}.$$

From right to left, a small category $\mathbf{C}$ is mapped to the relational monoid $\operatorname{Mor}\mathbf{C}$ with identities $\{\mathrm{id}_C \mid C \in \mathbf{C}\}$. It is easy to see that $\operatorname{Mor}\mathbf{C}$ is a local partial monoid.

For left to right, one first has to show that in a partial monoid M every element $f \in M$ has unique left and right units $e^l_f, e^r_f \in E$. The equivalence then sends a monoid M to the category whose objects are given by the units E and with morphisms

$$\hom(e, e') = \{f \in M \mid e = e^l_f,\ e' = e^r_f\}.$$

(2) The multiplication of the relational monoid corresponding to a small category is single-valued, that is, it factorizes as

$$\operatorname{Mor}(\mathbf{C}) \times \operatorname{Mor}(\mathbf{C}) \to \operatorname{Mor}(\mathbf{C}) + \{\bot\} \hookrightarrow \mathcal{P}(\operatorname{Mor}(\mathbf{C})),$$

its set of units can be any element of $\mathcal{P}(\operatorname{Mor}(\mathbf{C}))$. This is the reason why we work with the full powerset in our assumptions (5.1) in lieu of the maybe submonad.

From Section 5.1 we already know relational monoids correspond to certain associative residuation CABAs, so it remains to single out the images of local partial monoids.

Definition 6.4.

(1) A residuation CABA is *functional* if for all atoms $a \in \mathcal{A}(R)$ the $\mathbf{CSL}_{\bigwedge}$-morphisms $a \setminus (-)\colon R \to R$ also preserve non-empty joins. It is *local* if it satisfies $x^? \setminus x^? = \top$, where

$$(-)^? \in \mathbf{CSL}_{\bigvee}(R, R), \qquad x \mapsto \neg(x \setminus \bot).$$

A residuation CABA is *categorical* if it has a (not necessarily atomic!) unit $e \in R$ and is associative, functional and local.

(2) A (non-unital) *morphism* from a residuation CABA R to R' is a complete residuation algebra morphism $h\colon R \to R'$, that is, h preserves all joins and meets and satisfies $x' \setminus f(z) = f(f^*(x') \setminus z)$. If R, R' are unital residuation CABAs, then a morphism is *lax unital* if it satisfies $e' \leq f(e)$.

We denote the category of residuation CABAs by **ResCABA** and its subcategory of categorical residuation CABAs and lax unital morphisms by **CatResCABA**.

Theorem 6.5. *The category of small categories is dually equivalent to the category of categorical residuation CABAs:*

$$\mathbf{Cat} \simeq^{\mathrm{op}} \mathbf{CatResCABA}.$$

Proof. Restricting the results from Section 5.1 to the order-discrete setting, we get a duality

$$\mathbf{ResCABA} \cong \mathbf{Coalg}(V_\wedge) = \mathrm{Op}^{1,2}_{V_\wedge}(\mathbf{CABA}) \simeq^{\mathrm{op}} \mathrm{Op}^{2,1}_{J_\mathcal{P}}(\mathbf{Set}). \tag{6.2}$$

From right to left, the dual of a $J_\mathcal{P}$-algebra $\circ\colon M \times M \to \mathcal{P}M$ is given by the residuation CABA with carrier $\widehat{M} \cong \mathcal{P}(M)$, whose left residual is given by

$$\setminus\colon \mathcal{P}(M)^\partial \boxtimes_{\mathrm{C}} \mathcal{P}(M) \to \mathcal{P}(M), \qquad A \boxtimes C \mapsto \{b \mid A \circ b \subseteq C\}.$$

It remains to show that this duality restricts to a duality between the category of local partial monoids and functorial homomorphisms, and the category of categorical residuation CABAs and lax unital morphisms.

(1) We start by showing that the duality restricts on objects: a binary $J_\mathcal{P}$-algebra $\circ\colon M \bar{\times} M \to \mathcal{P}M$ is a local, partial monoid iff its dual residuation CABA $\widehat{M}$ is unital, associative, functional and local.

(1a) Partiality of multiplication corresponds to functionality of $\widehat{M}$: By Lemma 5.17 restricted to non-empty joins, the residuation CABA $\widehat{M}$ is functional iff its $V_\wedge$-coalgebra structure preserves non-empty joins. An argument analogous to Corollary 4.7 then gives that a $V_\wedge$-coalgebra preserves non-empty joins iff its dual order-relation is a partial map. Combining these two equivalences we get that the residuation CABA $\widehat{M}$ is functional if its dual relation $\circ\colon M \times M \to \mathcal{P}(M)$ is a partial map.

(1b) By Lemma 5.18 the residuation CABA $\widehat{M}$ has a unit $e \in \widehat{M}$ iff its comultiplication has a counit $\epsilon \vdash e$ iff its dual algebra M has a unit $E = e$. By the same lemma, $\widehat{M}$ is associative iff its comultiplication is coassociative iff M is associative.

(1c) We show that the residuation CABA $\widehat{M}$ satisfies $\forall A \in \widehat{M}\colon A^? \setminus A^? = \top$ iff its dual algebra M is local: We prepare by making some observations about the locality of the residuation CABA $\widehat{M}$. Note that in $\widehat{M}$ we have $\bot = \emptyset$ and $\top = M$. For $A \in \widehat{M}$ we have

$$A^? = \neg(A \setminus \bot) = \{n \mid \neg(A \circ n \subseteq \emptyset)\} = \{n \mid A \circ_@ n\} = \{n \mid \exists m \in A\colon m \circ_@ n\}. \tag{6.3}$$

In particular, on atoms $m \in \widehat{M}$ membership in $m^?$ simplifies to the condition

$$n \in m^? \iff m \circ_@ n. \tag{6.4}$$

We rewrite the locality condition using the adjunction $B \circ (-) \dashv B \setminus (-)$:

$$A^? \setminus A^? = M \iff M \subseteq A^? \setminus A^? \iff A^? \circ M \subseteq A^?, \tag{6.5}$$

where $\circ\colon \widehat{M} \otimes_{\mathrm{C}} \widehat{M} \to \widehat{M}$ is the $V_\vee$-algebra structure on $\widehat{M}$ corresponding to the extension $\mathcal{P}M \times \mathcal{P}M \to \mathcal{P}M$ of the partial monoid multiplication.

We start with the direction that if $\widehat{M}$ satisfies (6.5), then M is local: For this, let $x, y \in M$ such that $x \circ_@ y$ and $v \in y \circ z$. We have to show that $x \circ_@ v$. We first apply (6.4) to $x \circ_@ y$ to obtain $y \in x^?$. Monotonicity of $\circ$ in both arguments then yields the first inclusion below and (6.5) the second one:

$$v \in y \circ z \subseteq x^? \circ M \subseteq x^?.$$

But by (6.4) this means that $x \circ_@ v$, as required.

For the other direction, we prove that if M is local, then $\widehat{M}$ is local. By (6.5) it suffices to prove $\forall A \in \widehat{M}\colon A^? \circ M \subseteq A^?$. Let $v \in A^? \circ M$. Then there exist $y \in A^?$, $z \in M$ such that $v \in y \circ z$. Since $y \in A^?$, there exists, by (6.3), some $x \in A$ with $x \circ_@ y$. Locality of M now implies that $x \circ_@ v$, so $v \in A^?$ as required using (6.3) again.

(2) Finally, we show that the duality in (6.2) restricts on morphisms as claimed. Let $f\colon M \to M'$ be a pure morphism of $J_{\mathcal{P}}$-algebras with dual morphism $h = f^{-1}\colon \widehat{M'} \to \widehat{M}$ of residuation CABAs. We have to show that f is functorial iff h is lax unital: By (1b) the unit E of a partial monoid M dualizes to the residual unit $E \in \widehat{M}$. Now f is a functorial morphism of partial monoids iff $f[E] \subseteq E'$ iff $E \subseteq f^{-1}[E'] = h(E')$ iff h is lax unital. □

7. Conclusion and Future Work

We have presented an abstract approach to extending Stone-type dualities based on adjunctions between monoidal categories and instantiated it to recover classical extended Stone and Priestley duality, along with a generalization of it to relational morphisms. Guided by these foundations, we have investigated residuation and derivation algebras, leading to a new duality for Priestley monoids, and we extended this duality to include relational morphisms. In addition, we have derived a new dual characterization of the category of small categories.

Relational morphisms are an important tool in algebraic language theory, notably for characterizing language operations algebraically. For instance, Straubing [Str81] first showed that relational morphisms are tightly connected to the concatenation product and the star operation on regular languages; see also the surveys by Pin [Pin88, Pin11]. In future work, we intend to apply our duality-theoretic insights on relational morphisms to illuminate, and possibly recover, these results from a categorical perspective, much in the spirit of the duality-theoretic view of Eilenberg's Variety Theorem by Gehrke et al. [GGP08] and the categorical works it has inspired (see e.g. [Sal17, UACM17, Boj15, Blu21]).

Another goal is to apply our duality framework beyond classical Stone and Priestley dualities. Specifically, we aim to develop an extended duality theory for the recently developed *nominal* Stone duality [BMU23], which would enable a generalization of our present results on residuation algebras to the nominal setting with applications to data languages.

A conceptually rather different dual characterization of the category of profinite monoids and continuous monoid morphisms in terms of *semi-Galois categories* has been provided by Uramoto [Ura16]. Extending this result to relational morphisms, similar to our Theorem 5.43, is another interesting point for future work.

Potential applications of our abstract approach to extended Stone duality are not limited to algebraic language theory. In Section 4.3 we have used the compositionality of extended (Stone) duality to recover results from modal correspondence theory by purely categorical methods. We hope that the monoidal approach might bring a new impulse into the historic endeavor of correspondence theory. In future work we will investigate the expressiveness of this idea, that is, study which relational properties can be captured by suitable inequations of operators. Applying this approach to more complex modal axioms that, for example, involve negation or implication or combine multiple modalities, can be expected to be non-trivial.

Our applications so far were based on Stone and Priestley duality, but the general framework of abstract extended duality applies far beyond this setting. For instance, Furber and Jacobs [FJ15] showed how to extend the duality between C^*-algebras and compact Hausdorff spaces ('Gelfand duality') to a 'probabilistic Gelfand duality', which emerges by employing a weaker notion of morphism between C^*-algebras and by replacing the category of compact Hausdorff spaces by the Kleisli category of the Radon monad. This result seems to fit perfectly into our approach of extending dualities. A thorough instantiation of the results from [FJ15] to our framework could not only place these results in a larger categorical context but also uncover new results in probabilistic duality theory.

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