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DECIDING THE EXISTENCE OF INTERPOLANTS AND DEFINITIONS IN FIRST-ORDER MODAL LOGIC

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Abstract. None of the first-order modal logics between K and $\mathsf{S5}$ under the constant domain semantics enjoys Craig interpolation or projective Beth definability, even in the language restricted to a single individual variable. It follows that deciding the existence of a Craig interpolant for a given implication or of an explicit definition for a given predicate cannot be directly reduced to deciding the validity of an implication, as in classical first-order and many other logics. Our concern here is the decidability and computational complexity of the interpolant and definition existence problems. We first consider two decidable fragments of first-order modal logic $\mathsf{S5}$: the one-variable fragment $\mathsf{Q^1S5}$ and its extension $\mathsf{S5}_{\mathcal{ALC}^u}$ that combines $\mathsf{S5}$ and the description logic $\mathcal{ALC}$ with the universal role. We prove that interpolant and definition existence in $\mathsf{Q^1S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$ is decidable in coN2ExpTime, being 2ExpTime-hard, while uniform interpolant existence is undecidable. These results transfer to the two-variable fragment FO^2 of classical first-order logic without equality. We also show that interpolant and definition existence in the one-variable fragment $\mathsf{Q^1K}$ of first-order modal logic K is non-elementary decidable, while uniform interpolant existence is again undecidable.

1. Introduction

First-order modal logics and their fragments are well-established formalisms in computational logic. For many decades, they have been used, e.g., as first-order temporal logics in program verification, policy monitoring, and databases [Krö87, BKM10, HP18, CT18], as epistemic, temporal, and standpoint description logics [DLN⁺98, LWZ08, AKK⁺17, ÁRS22], as spatio-temporal logics [KKWZ07], and as logics of knowledge and belief [BL09, Wan17, LPRW22, WWS22]. By now, significant progress has been made in understanding entailment in these 'two-dimensional' logics, in particular its computational complexity; see, e.g., [GKWZ03, HK15, LPRW23, AMO24] and references therein. However, very little is known about

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the decidability and complexity of fundamental algorithmic problems that can go beyond entailment. For example, the following reasoning tasks have been used in different areas of computer science:

***definition existence*:** Given a formula φ, a predicate P, and a signature σ, does there exist an *explicit definition* χ of P modulo φ in σ in the sense that $\varphi \models \forall \boldsymbol{x}\, (P(\boldsymbol{x}) \leftrightarrow \chi(\boldsymbol{x}))$ and $sig(\chi) \subseteq \sigma$, where $sig(\chi)$ comprises the non-logical symbols in χ. Such definitions can, for instance, support query evaluation in databases [TW11, BLtCT16].

***interpolant existence*:** Given formulas φ and ψ, does there exist a *Craig interpolant* χ for φ and ψ in the sense that $\models \varphi \to \chi$, $\models \chi \to \psi$, and $sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$. Craig interpolants are applied, for instance, in model checking and concept learning [McM18, JLPW22].

***conservative extensions/uniform interpolant verification*:** Given formulas φ and χ such that $\models \varphi \to \chi$ and $\sigma = sig(\chi) \subseteq sig(\varphi)$, is χ a *σ-uniform interpolant* of φ in the sense that χ is an interpolant for φ and ψ whenever $\models \varphi \to \psi$ and $sig(\varphi) \cap sig(\psi) \subseteq \sigma$? In this case, φ is known as a *conservative extension* of χ. These notions are used, for instance, for modular knowledge base design and modularisation [BKL$^+$16].

***uniform interpolant existence*:** Given a formula φ and a signature σ, does there exist a σ-uniform interpolant of φ? Uniform interpolants are, for instance, a mechanism for forgetting symbols from a knowledge base [EK19, GKL23, KLWW09, Koo20].

Investigating these problems for first-order modal logics (FOMLs) poses particular challenges. In contrast to many other logical formalisms, FOMLs typically do not enjoy the Craig interpolation property (CIP) as $\models \varphi \to \psi$ does not necessarily entail the existence of an interpolant χ for φ and ψ. Nor do they enjoy the projective Beth definability property (BDP) according to which implicit definability of a predicate in a given signature implies its explicit definability as required in definition existence. For logics with the CIP, the interpolant existence problem trivially reduces to checking validity of implications. Similarly, for logics with the BDP, checking explicit definability reduces to checking the validity of implications representing implicit definability. These trivial reductions of existence problems to validity problems do not work for logics without the CIP or BDP. Fine [Fin79] showed that no FOML with constant domain models (a standard assumption) between the first-order modal logic K of all Kripke frames and the first-order modal logic $\mathsf{S5}$ of all universal Kripke frames enjoys the CIP or BDP.

Example 1.1 (based on [Fin79])**.** Interpreting $\Box$ as the $\mathsf{S5}$-modality 'always', let T consist of the following axioms, where rep stands for the proposition 'replaceable':

$$\mathsf{rep} \to \Diamond\forall x\, \big(\mathsf{inPower}(x) \to \Box(\mathsf{rep} \to \neg\mathsf{inPower}(x))\big),$$

$$\neg\mathsf{rep} \to \Box\exists x\, \big(\mathsf{inPower}(x) \land \Box(\neg\mathsf{rep} \to \mathsf{inPower}(x))\big).$$

Then rep is true at a world w satisfying T iff there is a world w' where all those who were in power at w lose it. It follows that rep is implicitly defined via $\mathsf{inPower}$. However, we shall see in Example 3.4 that there is no explicit definition of rep via $\mathsf{inPower}$ in FOML. ⊣

Fine's example shows that the CIP and BDP fail already in typical decidable fragments of FOML lying between the one-variable fragment and full FOML. Because of their wide use, 'repairing' the CIP and BDP has become a major research challenge. For instance, it is shown in [Fit02, ABM03] that by adding second-order quantifiers or the machinery of hybrid logic constructors to FOML, one obtains logics with the CIP and BDP. The price, however, is that these extensions are undecidable even if applied to decidable fragments of FOML.

In this article, we take a fundamentally different, non-uniform approach. Instead of repairing the CIP and BDP by enriching the language, we stay within its original boundaries and explore the possibility of checking the existence of interpolants and definitions even though the reduction to validity via the CIP and BDP is blocked.

We first focus on two decidable fragments of first-order $\mathsf{S5}$: its one-variable fragment $\mathsf{Q}^1\mathsf{S5}$ illustrated in Example 1.1 and $\mathsf{S5}_{\mathcal{ALC}^u}$, the FOML obtained by combining $\mathsf{S5}$ and the description logic (DL) $\mathcal{ALC}^u$, which extends the basic DL $\mathcal{ALC}$—a notational variant of multimodal logic K—with the universal role. In $\mathsf{S5}_{\mathcal{ALC}^u}$, we admit the application of modal operators to concepts but not to roles, and so consider a typical monodic fragment of FOML, in which modal operators are only applied to formulas with at most one free variable [HWZ00, WZ01]. $\mathsf{Q}^1\mathsf{S5}$ is a fragment of $\mathsf{S5}_{\mathcal{ALC}^u}$, and satisfiability is NExpTime-complete for both languages [GKWZ03].

Our first main result is:

Theorem 1.2. *The interpolant and definition existence problems in* $\mathsf{Q}^1\mathsf{S5}$ *and* $\mathsf{S5}_{\mathcal{ALC}^u}$ *are decidable in* coN2ExpTime*, being* 2ExpTime*-hard.*

Thus, interpolant and definition existence is still decidable but harder than satisfiability by about one exponential. The proof is based on 'component-wise' bisimulations that replace standard FOML bisimulations in our characterisation of interpolant and definition existence. For the upper bound, we show that whenever there are component-wise bisimilar models witnessing non-existence of interpolants/definitions, then there are component-wise bisimilar models of at most double-exponential size. The proof is inspired by the recent upper bound proofs of interpolant existence in the two-variable first-order logic FO^2 [JW21] but requires a very different notion of type reflecting the two-dimensional flavour of first-order modal logic. The lower bound proof combines the interpolation counterexample of [MA98], the exponential grid generation from [HKK$^+$03, GJL15], and the representation of exponential-space bounded alternating Turing machines from [JW21].

Our result can be used to solve an open problem for FO^2 without equality. The known 2ExpTime-lower bound proof for interpolant existence in FO^2 uses equality in a critical way. Utilising a close link between $\mathsf{Q}^1\mathsf{S5}$ and equality-free FO^2, we obtain the lower bound of the following theorem as a corollary to Theorem 1.2, answering an open question of [JW21] (the upper bound is a straightforward consequence of the proof in [JW21]):

Theorem 1.3. *The interpolant and definition existence problems in equality-free* FO^2 *are decidable in* coN2ExpTime*, being* 2ExpTime*-hard.*

We then consider uniform interpolant existence and conservative extension and show that they behave rather differently to interpolation existence:

Theorem 1.4. *The uniform interpolant existence and conservative extension problems in* $\mathsf{Q}^1\mathsf{S5}$ *and* $\mathsf{S5}_{\mathcal{ALC}^u}$ *are both undecidable.*

The proof extends a reduction proving undecidability of conservative extensions for FO^2 (with and without equality) from [JLM$^+$17]. As a corollary of our proof, we obtain that uniform interpolant existence in FO^2 (with and without equality) is undecidable, settling an open problem from [JLM$^+$17].

Finally, we consider the one-variable fragment $\mathsf{Q}^1\mathsf{K}$ of first-order modal logic K determined by the class of all Kripke frames and prove the following:

Theorem 1.5. (*i*) *The interpolant and definition existence problems in* $\mathsf{Q}^1\mathsf{K}$ *are decidable in non-elementary time.*

(*ii*) *The uniform interpolant existence and conservative extensions problems in* $\mathsf{Q}^1\mathsf{K}$ *are both undecidable.*

The non-elementary upper bound is established using the fact that $\mathsf{Q}^1\mathsf{K}$ has finitely many non-equivalent formulas of bounded modal depth. The undecidability result is proved by adapting the undecidability proof for $\mathsf{Q}^1\mathsf{S5}$.

Related work on interpolant existence. The non-uniform approach to Craig interpolants and explicit definitions has only very recently been studied for a small number of modal and description logics, and also related decidable fragments of first-order logic such as the guarded and two-variable fragment [JW21], classical description and modal logics with nominals [AJM+23], various Horn logics [BLtCT16, FKW22], modal logics of linear frames [KWZ23b], and modal logics with the derivative operator [KWZ24]. For all these logics either the complexity of interpolant and explicit definition existence goes up by one exponential compared to validity or stays the same as validity even without the Craig interpolation property. An example of a decidable fragment of FO with undecidable interpolant and explicit definition existence is given in [WZ24]. Some decidability results on separating disjoint regular languages using FO-definable languages [Hen88, HRS10, PZ16] can be interpreted as results about interpolant existence in linear temporal logic *LTL*. An overview of the non-uniform approach to Craig interpolants and the relationship to separation of formal languages is given in [KWZ26].

The non-uniform investigation of *uniform* interpolants started much earlier in description logic with complexity results for uniform interpolant existence [LSW12, LW11] and size upper bounds for uniform interpolants if they exist [NR14]. The practical computation of uniform interpolants is an active research area for many years [KWW09, KS15, ZS16]; see [ZFA+18, Koo20] for recent system descriptions.

Structure. The article is organised as follows. Section 2 reminds the reader of the syntax and semantics of the one-variable fragments $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ of two basic first-order modal logics K and $\mathsf{S5}$, as well as defines bisimulations between their models. Section 3 introduces and illustrates the main notions we are concerned with here—Craig and uniform interpolants, explicit definitions, conservative extensions—and provides their model-theoretic characterisations. Section 4 establishes the upper and lower bounds for the interpolant and explicit definition existence problems in $\mathsf{Q}^1\mathsf{S5}$ and Section 5 proves that the conservative extension and uniform interpolant existence problems in $\mathsf{Q}^1\mathsf{S5}$ are undecidable. Section 6 extends the results of the previous two sections to the modal description logic $\mathsf{S5}_{\mathcal{ALC}^u}$. Section 7 establishes decidability of the interpolant and definition existence in $\mathsf{Q}^1\mathsf{K}$ and undecidability of conservative extension and uniform interpolant existence. Finally, Section 8 discusses further research and some open problems that arise from this work.

Some technical details are omitted from the main part of the article and can be found in the appendix. Appendix A discusses the connections between $\mathsf{Q}^1\mathsf{S5}$ and FO^2, and uses them to prove the lower bound of Theorem 1.3. Appendix B gives polynomial-time reductions of various interpolant existence problems modulo an ontology to the IEP.

2. Preliminaries

Logics. The formulas of the *one-variable fragment* FOM^1 *of first-order modal logic* are built from unary predicate symbols $\boldsymbol{p} \in \mathcal{P}$ in a countably-infinite set $\mathcal{P}$ and a single individual variable x using $\top$, $\neg$, $\wedge$, $\exists x$, and the possibility operator $\Diamond$. The other Booleans, $\forall x$, and the necessity operator $\Box$ are defined as standard abbreviations. A *signature* is any finite set $\sigma \subseteq \mathcal{P}$; the signature $sig(\varphi)$ of a formula φ comprises the predicate symbols in φ. If $sig(\varphi) \subseteq \sigma$, we call φ a *σ-formula*. By $sub(\varphi)$ we denote the closure under single negation of the set of subformulas of φ, and by $|\varphi|$ the cardinality of $sub(\varphi)$.

We interpret FOM^1-formulas in (*Kripke*) *models* with *constant domains* of the form $\mathfrak{M} = (W, R, D, I)$, where $W \neq \emptyset$ is a set of *worlds*, $R \subseteq W \times W$ an *accessibility relation* on W, $D \neq \emptyset$ an (FO-)*domain* of $\mathfrak{M}$, and $I(w)$ is an *interpretation* of the $\boldsymbol{p} \in \mathcal{P}$ over D, for each $w \in W$, that is, $\boldsymbol{p}^{I(w)} \subseteq D$. The *truth-relation* $\mathfrak{M}, w, d \models \varphi$, for any $w \in W$, $d \in D$ and FOM^1-formula φ, is defined inductively by taking

– $\mathfrak{M}, w, d \models \top$,
– $\mathfrak{M}, w, d \models \boldsymbol{p}(x)$ iff $d \in \boldsymbol{p}^{I(w)}$, for all $\boldsymbol{p} \in \mathcal{P}$,
– $\mathfrak{M}, w, d \models \exists x\, \varphi$ iff there is $d' \in D$ with $\mathfrak{M}, w, d' \models \varphi$,
– $\mathfrak{M}, w, d \models \Diamond\varphi$ iff there is $w' \in W$ with $(w, w') \in R$ and $\mathfrak{M}, w', d \models \varphi$,

and the standard clauses for $\neg$ and $\wedge$. If φ is a sentence (i.e., every occurrence of x in φ is in the scope of $\exists$), then $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}, w, d' \models \varphi$, for any $d, d' \in D$, and so we can omit d and write $\mathfrak{M}, w \models \varphi$. Similarly, we can write $\mathfrak{M}, d \models \psi$ if every $\boldsymbol{p}$ in ψ is in the scope of $\Diamond$.

The set of formulas φ with $\mathfrak{M}, w, d \models \varphi$, for all $\mathfrak{M}$, w, d, is denoted by $\mathsf{Q}^1\mathsf{K}$; it is the FOM^1-extension of the modal logic K. Those φ that are true everywhere in all models $\mathfrak{M}$ with $R = W \times W$ comprise $\mathsf{Q}^1\mathsf{S5}$, the FOM^1-extension of the modal logic $\mathsf{S5}$. Let L be one of these two logics. A *knowledge base* (KB), K, is any finite set of sentences. We say that K (*locally*) *entails* φ *in* L and write $K \models_L \varphi$ if $\mathfrak{M}, w \models K$ implies $\mathfrak{M}, w, d \models \varphi$, for any L-model $\mathfrak{M}$ and any w and d in it. Shortening $\emptyset \models_L \varphi$ to $\models_L \varphi$ (i.e., $\varphi \in L$), we note that $K \models_L \varphi$ iff $\models_L (\bigwedge_{\psi \in K} \psi \to \varphi)$, reducing KB-entailment in L to L-validity, which is known to be coNExpTime-complete [Mar99].

We refer the reader to [BG07, FM12] for detailed introductions to first-order modal logic in general and to [GKWZ03, Kur07] for decidable fragments of first-order modal logics.

Bisimulations. Given two models $\mathfrak{M} = (W, R, D, I)$ with w, d and $\mathfrak{M}' = (W', R', D', I')$ with w', d', we write $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, for a signature σ, if the same σ-formulas are true at w, d in $\mathfrak{M}$ and at w', d' in $\mathfrak{M}'$. We characterise $\equiv_\sigma$ using bisimulations. Namely, a relation

$$\boldsymbol{\beta} \subseteq (W \times D) \times (W' \times D')$$

is called a *σ-bisimulation* between models $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $((w, d), (w', d')) \in \boldsymbol{\beta}$ and $\boldsymbol{p} \in \sigma$:

(a) $\mathfrak{M}, w, d \models \boldsymbol{p}$ iff $\mathfrak{M}', w', d' \models \boldsymbol{p}$;
(w) if $(w, v) \in R$, then there is v' such that $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}$, and if $(w', v') \in R'$, then there is v such that $(w, v) \in R$ and $((v, d), (v', d')) \in \boldsymbol{\beta}$;
(d) for every $e \in D$, there is $e' \in D'$ such that $((w, e), (w', e')) \in \boldsymbol{\beta}$ and, for every $e' \in D'$, there is $e \in D$ such that $((w, e), (w', e')) \in \boldsymbol{\beta}$.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are *σ-bisimilar* and write $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ if there is a σ-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{M}$ and $\mathfrak{M}'$ with $((w, d), (w', d')) \in \boldsymbol{\beta}$. The next characterisation is proved in a standard way using ω-saturated models [CK98, GO07]:

Lemma 2.1. *For any signature σ and ω-saturated models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d',*

$$\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d' \quad \textit{iff} \quad \mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'.$$

The direction from right to left holds for arbitrary models.

Modal products and succinct notation. As observed by [Waj33], $\mathsf{S5}$ is a notational variant of the one-variable fragment FO^1 of first-order logic, FO: just drop x from $\exists x$ and $\boldsymbol{p}(x)$ in FO^1-formulas, treating $\exists$ as a possibility operator and $\boldsymbol{p}$ as a *propositional variable*. The same operation transforms FOM^1-formulas into more succinct *bimodal formulas* with $\Diamond$ interpreted over the (W, R) 'dimension' and $\exists$ over the $(D, D \times D)$ 'dimension'. This way we view the FOM^1-extensions of $\mathsf{S5}$ and K as two-dimensional *products* of modal logics: $\mathsf{S5} \times \mathsf{S5}$ and $\mathsf{K} \times \mathsf{S5}$. The former is known to be the 'equality and substitution-free' fragment of two-variable fragment FO^2 of FO [GKWZ03] (see Appendix A for details); the latter is embedded into FO by the *standard translation* * defined inductively by taking $\boldsymbol{p}^* = \boldsymbol{q}(z, x)$, $(\neg\varphi)^* = \neg\varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$, $(\exists\varphi)^* = \exists x\, \varphi^*$, $(\Diamond\varphi)^* = \exists y\, (R(z, y) \wedge \varphi^*\{y/z\})$, where y is a fresh variable not occurring in φ^* and $\{y/z\}$ means a substitution of y in place of z.

From now on, we write FOM^1-formulas as *bimodal* ones: for example, $\exists\Box\boldsymbol{p}$ stands for $\exists x\, \Box\boldsymbol{p}(x)$. By a formula we mean an FOM^1-formula unless indicated otherwise; a logic, L, is one of $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{Q}^1\mathsf{K}$, again unless stated otherwise.

3. Main Notions and Characterisations

We now introduce the main notions studied in this article and provide their model-theoretic characterisations. We start with interpolants and explicit definitions.

Craig interpolants. A formula χ is an *interpolant* of formulas φ and ψ *in a logic* L if $sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$, $\models_L \varphi \to \chi$ and $\models_L \chi \to \psi$. L enjoys the *Craig interpolation property* (*CIP*) if an interpolant for φ and ψ exists whenever $\models_L \varphi \to \psi$. One of our main concerns here is the *interpolant existence problem* (*IEP*) for L: decide if given φ and ψ have an interpolant in L. For logics with the CIP, the IEP reduces to validity, and so is not interesting. This is the case for many logics including propositional $\mathsf{S5}$ and K, but not for FOMLs with constant domain between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ [Fin79, MA98].

We note that besides the interpolants introduced above, one can also consider interpolants for the *global* consequence relation $\models^g_L$ that is defined by setting $\varphi \models^g_L \psi$ if, for all L-models $\mathfrak{M}$, whenever $\mathfrak{M}, w, d \models \varphi$ for all w, d in $\mathfrak{M}$, then $\mathfrak{M}, w, d \models \psi$ for all w, d in $\mathfrak{M}$. A formula χ is a *global deductive interpolant* of φ and ψ in L if $sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$, $\varphi \models^g_L \chi$ and $\chi \models^g_L \psi$. While the relationship between the respective CIPs is well understood [Fus26], the relationship between the respective IEPs remains to be investigated. In this paper, we are only concerned with 'local' interpolants, but note that, for $\mathsf{Q}^1\mathsf{S5}$, the global IEP can be polynomially reduced to the local IEP because $\varphi \models^g_{\mathsf{Q}^1\mathsf{S5}} \psi$ iff $\Box\forall\varphi \models_{\mathsf{Q}^1\mathsf{S5}} \psi$; see also the discussion of interpolants modulo ontologies in Section 6.

Explicit definitions. Given formulas φ, ψ and a signature σ, an *explicit σ-definition of ψ modulo φ in L* is a σ-formula χ such that $\models_L \varphi \to (\psi \leftrightarrow \chi)$. The *explicit σ-definition existence problem* (*EDEP*) for L is to decide, given φ, ψ and σ, whether there exists an explicit σ-definition of ψ modulo φ in L. The EDEP reduces trivially to entailment for logics enjoying the *projective Beth definability property* (*BDP*) according to which ψ is explicitly σ-definable modulo φ in L iff it is implicitly σ-definable modulo φ in the sense that $\{\varphi, \varphi'\} \models_L \psi \leftrightarrow \psi'$, where φ' and ψ' result from φ and ψ by uniformly replacing all non-σ-symbols with fresh ones [CK98]. Again, many logics including propositional $\mathsf{S5}$ and K enjoy the BDP while FOMLs with constant domains between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ do not.

Note that, in many applications, φ in our formulation of the EDEP corresponds to a KB K and ψ is a predicate $\boldsymbol{p}$. Then the problem whether there exists an explicit σ-definition of $\boldsymbol{p}$ modulo K is the problem of deciding whether there is χ with $sig(\chi) \subseteq \sigma$ and $K \models_L \forall x(\boldsymbol{p}(x) \leftrightarrow \chi(x))$. This problem trivially translates to the EDEP using our discussion of KBs above. In more detail, this view of the EDEP is discussed in Section 6 in the context of $\mathsf{S5}_{\mathcal{ALC}^u}$.

Lemma 2.1 together with the fact that FOM^1 is a fragment of FO are used to obtain, again in a standard way, the following criterion of interpolant existence. We call formulas φ and ψ *σ-bisimulation consistent* in L if there exist L-models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d' such that $\mathfrak{M}, w, d \models \varphi$, $\mathfrak{M}', w', d' \models \psi$ and $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.

Theorem 3.1. *For any FOM^1-formulas φ and ψ, the following conditions are equivalent*:

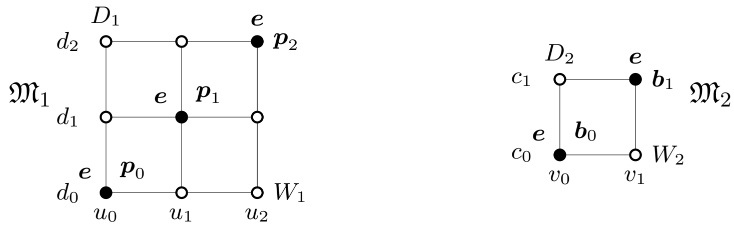
– *there does not exist an interpolant of φ and ψ in L*;
– *φ and $\neg\psi$ are $sig(\varphi) \cap sig(\psi)$-bisimulation consistent in L.*

Proof. Suppose φ and ψ do not have an interpolant in L and $\sigma = sig(\varphi) \cap sig(\psi)$. Consider the set Ξ of σ-formulas χ with $\models_L \varphi \to \chi$. By compactness, we have an ω-saturated model $\mathfrak{M}$ of L with w and d such that $\mathfrak{M}, w, d \models \chi$, for all $\chi \in \Xi$, and $\mathfrak{M}, w, d \models \neg\psi$. Take the set Ξ' of σ-formulas χ with $\mathfrak{M}, w, d \models \chi$ and an ω-saturated model $\mathfrak{M}'$ with $\mathfrak{M}', w', d' \models \Xi'$ and $\mathfrak{M}', w', d' \models \varphi$, for some w' and d'. Then $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, and so $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ by Lemma 2.1. The converse implication is straightforward. ⊣

Example 3.2. For any $n < \omega$, [MA98] constructed two FOM^1-formulas φ and ψ with $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi$ and $sig(\varphi) \cap sig(\psi) = \{\boldsymbol{e}\}$ that have no interpolant in the n-variable $\mathsf{Q}^n\mathsf{S5}$. For $n = 1$, the formulas φ and ψ look as follows:

$$\begin{aligned}\varphi \;&=\; \boldsymbol{p}_0 \land \Diamond\exists(\boldsymbol{p}_1 \land \Diamond\exists\boldsymbol{p}_2) \land \Box\forall\big[(\boldsymbol{e} \leftrightarrow \boldsymbol{p}_0 \lor \boldsymbol{p}_1 \lor \boldsymbol{p}_2) \land \textstyle\bigwedge_{i\neq j}(\boldsymbol{p}_i \to \neg\boldsymbol{p}_j) \land \\ &\qquad\qquad \bigwedge_{0\leq i\leq 2} \big(\boldsymbol{p}_i \to \Box(\boldsymbol{e} \to \boldsymbol{p}_i) \land \forall(\boldsymbol{e} \to \boldsymbol{p}_i)\big)\big],\\ \psi \;&=\; \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \lor \boldsymbol{b}_1) \to \Diamond\exists\big(\boldsymbol{b}_0 \land \Diamond(\neg\boldsymbol{e} \land \exists\boldsymbol{b}_0)\big) \lor \Diamond\exists\big(\boldsymbol{b}_1 \land \Diamond(\neg\boldsymbol{e} \land \exists\boldsymbol{b}_1)\big).\end{aligned}$$

To see that φ and $\neg\psi$ are $\{\boldsymbol{e}\}$-bisimulation consistent in $\mathsf{Q}^1\mathsf{S5}$, take the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ depicted below with $\mathfrak{M}_1, u_0, d_0 \models \varphi$ and $\mathfrak{M}_2, v_0, c_0 \models \neg\psi$. (In our pictures, the worlds are always shown along the horizontal axis and the domain elements along the vertical one, giving points of the form (w, d).) The relation $\boldsymbol{\beta}$ connecting each $\boldsymbol{e}$-point in $\mathfrak{M}_1$ with each $\boldsymbol{e}$-point in $\mathfrak{M}_2$, and similarly for $\neg\boldsymbol{e}$-points, is an $\{\boldsymbol{e}\}$-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that $\big((u_0, d_0), (v_0, c_0)\big) \in \boldsymbol{\beta}$. ⊣

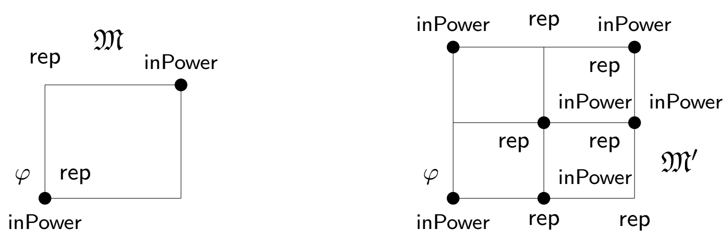


Similarly to Theorem 3.1 we obtain the following criterion of explicit definition existence:

Theorem 3.3. *For any φ, ψ, σ, the following are equivalent*:

– *there is no explicit σ-definition of ψ modulo φ in L*;
– *$\varphi \land \psi$ and $\varphi \land \neg\psi$ are σ-bisimulation consistent in L.*

Example 3.4. Suppose φ is the conjunction of the two KB axioms from Example 1.1, $\sigma = \{\mathsf{inPower}\}$, and $\psi = \mathsf{rep}$.[1] Then the second condition of Theorem 3.3 holds for the $\mathsf{Q^1S5}$-models shown below, in which (w, d) in $\mathfrak{M}$ is σ-bisimilar to (w', d') in $\mathfrak{M}'$ iff (w, d) and (w', d') agree on σ.



It follows that rep has no definition via inPower modulo φ in $\mathsf{Q^1S5}$. ⊣

The IEP and EDEP are closely related [GM05]. Here, we only require the following:

Theorem 3.5. *For any $L \in \{\mathsf{Q^1S5}, \mathsf{Q^1K}\}$, the EDEP for L and the IEP for L are polynomially reducible to each other.*

Proof. The EDEP is polynomially reducible to the IEP by a standard trick [GM05]: a formula ψ has an explicit σ-definition modulo φ in L iff the formulas $\varphi \land \psi$ and $\varphi^\sigma \to \psi^\sigma$ have an interpolant in L, where φ^σ, ψ^σ are obtained by replacing each variable $\boldsymbol{p} \notin \sigma$ with a fresh variable $\boldsymbol{p}^\sigma$. Indeed, any σ-formula χ with $\models_L \varphi \to (\psi \leftrightarrow \chi)$ is an interpolant of $\varphi \land \psi$ and $\varphi^\sigma \to \psi^\sigma$ in L. Conversely, any interpolant of $\varphi \land \psi$ and $\varphi^\sigma \to \psi^\sigma$ is an explicit σ-definition of ψ modulo φ in L.

For the other reduction, we observe first that the decision problem for L is polynomially reducible to the EDEP because, for $\psi = \boldsymbol{p} \notin sig(\varphi)$ and $\sigma = \emptyset$, we have $\models_L \neg\varphi$ iff there is an explicit σ-definition of ψ modulo φ in L. Then we use Theorems 3.1 and 3.3 to show that formulas φ, ψ have an interpolant in L iff $\models_L \varphi \to \psi$ and there is a $sig(\varphi) \cap sig(\psi)$-definition of ψ modulo $\psi \to \varphi$ in L. Indeed, it suffices to observe that, for any L-models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d', we have $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{M}', w', d' \models \neg\psi$ iff $\mathfrak{M}, w, d \models (\psi \to \varphi) \land \psi$ and $\mathfrak{M}', w', d' \models (\psi \to \varphi) \land \neg\psi$. ⊣

[1] As our $\mathsf{FOM^1}$ has no 0-ary predicates, the *proposition* rep is given as $\forall x\, \mathsf{rep}(x)$ assuming that we have $\models_L \forall x\, \mathsf{rep}(x) \lor \forall x\, \neg\mathsf{rep}(x)$.

We next define conservative extensions, an important notion in the context of ontology modules and modularisation [GHKS08, BKL+16].

Conservative extensions. Given formulas φ and ψ, we call φ an *L-conservative extension of* ψ if (a) $\models_L \varphi \to \psi$ and (b) $\models_L \varphi \to \chi$ implies $\models_L \psi \to \chi$, for any χ with $sig(\chi) \subseteq sig(\psi)$. In many applications, ψ is given by a KB K and φ is obtained by adding fresh axioms to K [GLW06, KWZ10]. (The translation of our results to the language of KBs is obvious.) The next example shows that this notion of conservative extension is syntax-dependent in the sense that it is not robust under the addition of fresh predicates.

Example 3.6. Consider the formulas

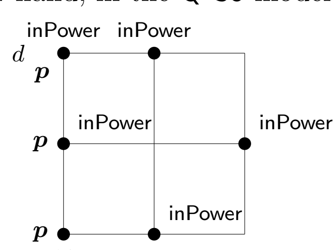
$$\varphi = \mathsf{rep} \wedge \Diamond\forall\big(\mathsf{inPower} \to \Box(\mathsf{rep} \to \neg\mathsf{inPower})\big),$$
$$\psi = \Box\forall(\Diamond\mathsf{inPower} \wedge \Diamond\neg\mathsf{inPower} \wedge \exists\mathsf{inPower} \wedge \exists\neg\mathsf{inPower}).$$

We claim that $\varphi \wedge \psi$ is a conservative extension of ψ in $\mathsf{Q^1S5}$. Indeed, condition (a) of the definition of conservative extension is trivial. To show (b), suppose $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi \to \chi$, for some χ such that $sig(\chi) \subseteq \{\mathsf{inPower}\} = \sigma$. We need to prove $\models_{\mathsf{Q^1S5}} \psi \to \chi$. Suppose ψ is true somewhere in a $\mathsf{Q^1S5}$-model $\mathfrak{N}$. By the definition of ψ, it is true everywhere in $\mathfrak{N}$. Consider the relation $\boldsymbol{\beta}$ that connects each $\mathsf{inPower}$-point in $\mathfrak{N}$ with each $\mathsf{inPower}$-point in $\mathfrak{M}$ from Example 3.4, and each $\neg\mathsf{inPower}$-point in $\mathfrak{N}$ with each $\neg\mathsf{inPower}$-point in $\mathfrak{M}$. It follows from the definition of ψ and the structure of $\mathfrak{M}$ that $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{N}$ and $\mathfrak{M}$. The reader can check that $\mathfrak{M}, w \models \varphi \wedge \psi$, and so $\mathfrak{M}, w, d \models \chi$ and $\mathfrak{M}, w, e \models \chi$. As $\boldsymbol{\beta}$ is a σ-bisimulation, we obtain that χ is true everywhere in $\mathfrak{N}$, establishing (b).

Now, let $\psi' = \psi \wedge (\mathsf{p} \vee \neg\mathsf{p})$, for a fresh proposition p. Then $\varphi \wedge \psi'$ is not a conservative extension of ψ' as witnessed by the formula χ below

$$\chi = \neg\big(\mathsf{p} \wedge \Box\exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \to \mathsf{inPower}))\big).$$

Indeed, we have $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi' \to \chi$. For suppose $\mathfrak{M}, w \models \varphi \wedge \psi'$, and so $w \models \mathsf{rep}$. Then, by φ, there is a world u with $u \models \forall(\mathsf{inPower} \to \Box(\mathsf{rep} \to \neg\mathsf{inPower}))$. By ψ', there is a domain element d with $u, d \models \mathsf{inPower}$, from which $w, d \models \neg\mathsf{inPower}$. Moreover, this is the case for all d with $u, d \models \mathsf{inPower}$. Now, if $w \models \neg\mathsf{p}$, we have $w \models \chi$. So let $w \models \mathsf{p}$. Then $u \not\models \exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \to \mathsf{inPower}))$ because if we had $u, d' \models \mathsf{inPower}$ for some d', then $w, d' \models \neg\mathsf{inPower}$, which is a contradiction. Thus, we obtain $w \models \chi$, which proves $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi' \to \chi$. On the other hand, in the $\mathsf{Q^1S5}$-model shown in the picture below,



ψ' is true at w while χ is false, and so $\not\models_{\mathsf{Q^1S5}} \psi' \to \chi$. ⊣

If in the definition of conservative extension we require property (b) to hold for all χ with $sig(\chi) \cap sig(\varphi) \subseteq sig(\psi)$, then φ is called a *strong L-conservative extension of* ψ. As observed by [JLM+17], the difference between conservative and strong conservative extensions is closely related to the failure of the CIP: if L enjoys the CIP, then L-conservative extensions coincide with strong L-conservative extensions. The problem of deciding whether a given φ is a (strong) conservative extension of a given ψ will be referred to as (S)CEP. The study of the complexity of the (S)CEP for DLs and modal logics started with [GLW06] and [GLWZ06]; see [BLR+19, JLM22] for more recent work.

Uniform interpolants. Given a signature σ, we call a formula χ a *σ-uniform interpolant of a formula φ in L* if $sig(\chi) = \sigma$ and φ is a strong L-conservative extension of χ. Observe that χ is then an interpolant of φ and ψ in L for any ψ with $\models_L \varphi \to \psi$ and $sig(\varphi) \cap sig(\psi) = \sigma$.

A logic L has the *uniform interpolation property* (*UIP*) if, for any φ and σ, there is a σ-uniform interpolant of φ in L. The UIP entails the CIP but not the other way round. For example, modal logic $\mathsf{S4}$ and description logic $\mathcal{ALC}^u$ enjoy the CIP but not the UIP [GZ95, LW11]. This leads to the *uniform interpolant existence problem* (*UIEP*): given φ and σ, decide whether φ has a σ-uniform interpolant in L. We refer the reader to the discussion of related work on interpolant existence in the introduction for a brief survey of the work done on the UIEP in description logic. This work is mostly motivated by the observation that a σ-uniform interpolant of a formula φ can be seen as the result of forgetting all non-σ-symbols from φ. Forgetting was first introduced in [LR94]. Note that the SCEP is equivalent to verifying whether a given formula is a uniform interpolant.

4. Deciding the IEP and EDEP in $\mathsf{Q^1S5}$

In this section, we prove Theorem 1.2 for $\mathsf{Q^1S5}$, stating that IEP and EDEP in $\mathsf{Q^1S5}$ are decidable in coN2ExpTime and 2ExpTime-hard. As a corollary (established in Appendix A), we obtain the lower bound of Theorem 1.3 stating that the IEP and EDEP in equality-free $\mathsf{FO^2}$ are 2ExpTime-hard.

4.1. **Upper bound.** Suppose we want to check whether φ and ψ have an interpolant in $\mathsf{Q^1S5}$. By Theorem 3.1, this is not the case iff there are $\mathsf{Q^1S5}$-models $\mathfrak{M}_1$ with w_1, d_1 and $\mathfrak{M}_2$ with w_2, d_2 such that $\mathfrak{M}_1, w_1, d_1 \models \varphi$, $\mathfrak{M}_2, w_2, d_2 \models \neg\psi$, and $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$. We are going to show that if such $\mathfrak{M}_i$ do exist, they can be chosen to be of double-exponential size in $|\varphi| + |\psi|$.

To begin with, as $R = W \times W$ in any $\mathsf{Q^1S5}$-model $\mathfrak{M} = (W, R, D, I)$, in this section we drop R and write simply $\mathfrak{M} = (W, D, I)$. Fix φ, ψ and $\sigma = sig(\varphi) \cap sig(\psi)$. Denote by $sub_\exists(\varphi, \psi)$ the closure under single negation of the set of formulas of the form $\exists\xi$ in $sub(\varphi, \psi) = sub(\varphi) \cup sub(\psi)$. The *world-type* of $w \in W$ in $\mathfrak{M} = (W, D, I)$ is defined as

$$\mathsf{wt}_\mathfrak{M}(w) = \{\rho \in sub_\exists(\varphi, \psi) \mid \mathfrak{M}, w \models \rho\}.$$

A set $\mathsf{wt} \subseteq sub_\exists(\varphi, \psi)$ is called a *world-type* in $\mathfrak{M}$ if it is the world-type of some $w \in W$.

Similarly, let $sub_\Diamond(\varphi, \psi)$ be the closure under single negation of the set of formulas of the form $\Diamond\xi$ in $sub(\varphi, \psi)$. The *domain-type* of $d \in D$ in $\mathfrak{M}$ is the set

$$\mathsf{dt}_\mathfrak{M}(d) = \{\rho \in sub_\Diamond(\varphi, \psi) \mid \mathfrak{M}, d \models \rho\}.$$

A set $\mathsf{dt} \subseteq sub_\Diamond(\varphi, \psi)$ is called a *domain-type* in $\mathfrak{M}$ if it is the domain-type of some $d \in D$.

The *full type* of $(w, d) \in W \times D$ in $\mathfrak{M}$ is the set

$$\mathsf{ft}_\mathfrak{M}(w, d) = \{\rho \in sub(\varphi, \psi) \mid \mathfrak{M}, w, d \models \rho\}.$$

A set $\mathsf{ft} \subseteq sub(\varphi, \psi)$ is called a *full type* in $\mathfrak{M}$ if it is the full type of some (w, d) in $\mathfrak{M}$.

The main technique of this section generalises the following construction that shows how, given any $\mathsf{Q^1S5}$-model $\mathfrak{M}$ satisfying a formula φ, we can construct from the world and domain types in $\mathfrak{M}$ a model $\mathfrak{M}'$ satisfying φ and having exponential size in $|\varphi|$. Intuitively, as a first approximation, we could start by taking the worlds W' (domain D') in $\mathfrak{M}'$ to comprise all the world- (domain-) types in $\mathfrak{M}$. But then we might have w, w' and d, d' with $\mathsf{wt}_{\mathfrak{M}}(w) = \mathsf{wt}_{\mathfrak{M}}(w')$, $\mathsf{dt}_{\mathfrak{M}}(d) = \mathsf{dt}_{\mathfrak{M}}(d')$ and different truth-values of some variables $\boldsymbol{p}$ at (w, d) and (w', d') in $\mathfrak{M}$. To deal with this issue, we introduce, as shown in the example below, sufficiently many copies of each world- and domain-type so that we can accommodate *all* possible truth-values in $\mathfrak{M}$ of the $\boldsymbol{p}$ in φ. (It is to be noted that there are many alternative ways of introducing copies of the wt and dt to define W' and D' and the truth-value of $\boldsymbol{p}$ at pairs of such copies. For instance, one could swap the role of W and D or give a symmetric construction introducing $2n$ copies of each wt and dt. We opted for the representation below as it generalises well to σ-bisimulation-consistency for $\mathsf{Q^1S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$, and it admits transparent inductive proofs.)

Example 4.1. Let $\mathfrak{M}, w, d \models \varphi$, for $\mathfrak{M} = (W, D, I)$, and let n be the number of full types in $\mathfrak{M}$ (over $sub(\varphi)$) and $[n] = \{1, \dots, n\}$. Define D' to be a set that contains n distinct copies of each dt in $\mathfrak{M}$ over $sub_{\Diamond}(\varphi)$, denoting the kth copy by dt^k. For any pair wt, dt in $\mathfrak{M}$, let $\Pi_{\mathsf{wt},\mathsf{dt}}$ denote the set of functions from $[n]$ *onto* the set of full types ft in $\mathfrak{M}$ with $\mathsf{wt} = \mathsf{ft} \cap sub_{\exists}(\varphi)$ and $\mathsf{dt} = \mathsf{ft} \cap sub_{\Diamond}(\varphi)$. Let Π denote the set of functions π mapping each pair wt, dt in $\mathfrak{M}$ to an element of $\Pi_{\mathsf{wt},\mathsf{dt}}$. For $\pi \in \Pi$ we set $\pi_{\mathsf{wt},\mathsf{dt}} = \pi(\mathsf{wt}, \mathsf{dt})$. Then let $\Pi^{\dagger} \subseteq \Pi$ be a smallest set for which the following condition holds: for any $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}}(u, e)$ and $k \in [n]$, there exists $\pi \in \Pi^{\dagger}$ with $\pi_{\mathsf{wt}_{\mathfrak{M}}(u),\mathsf{dt}_{\mathfrak{M}}(e)}(k) = \mathsf{ft}$. We claim that $|\Pi^{\dagger}| \leq n^2$. Indeed, for any $k \in [n]$ and any full type ft in $\mathfrak{M}$, we will include just a single function $f^{k,\mathsf{ft}} \in \Pi$ in $\Pi^{\dagger}$. Assume k and $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}}(u, e)$ are given. Then we can choose $f^{k,\mathsf{ft}}$ to be any function mapping pairs dt, wt into $\Pi_{\mathsf{wt},\mathsf{dt}}$ such that $f^{k,\mathsf{ft}}(\mathsf{wt}_{\mathfrak{M}}(u), \mathsf{dt}_{\mathfrak{M}}(e))(k) = \mathsf{ft}$. The resulting $\Pi^{\dagger}$ is as required.

We set $W' = \{\mathsf{wt}^{\pi}_{\mathfrak{M}}(u) \mid u \in W,\ \pi \in \Pi^{\dagger}\}$, treating each $\mathsf{wt}^{\pi}_{\mathfrak{M}}(u)$ as a fresh π-copy of $\mathsf{wt}^{\pi}_{\mathfrak{M}}(u)$. Then both $|W'|$ and $|D'|$ are exponential in $|\varphi|$. Define a model $\mathfrak{M}' = (W', D', I')$ by taking $\mathfrak{M}', \mathsf{wt}^{\pi}, \mathsf{dt}^k \models \boldsymbol{p}$ iff $\boldsymbol{p} \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$, for all $\pi \in \Pi^{\dagger}$ and wt, dt in $\mathfrak{M}$. We show by induction that $\mathfrak{M}', \mathsf{wt}^{\pi}, \mathsf{dt}^k \models \rho$ iff $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$, for any $\rho \in sub(\varphi)$. The basis of induction and the Boolean cases are straightforward.

Case $\rho = \exists \xi$. If $\mathsf{wt}^{\pi}, \mathsf{dt}^k \models \rho$, there is $\mathsf{dt}'^{k'}$ with $\mathsf{wt}^{\pi}, \mathsf{dt}'^{k'} \models \xi$. By IH, $\xi \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$, so $\rho \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$ and $\rho \in \mathsf{wt}$, whence $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. Conversely, let $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u, e)$. Then there is e' with $\mathfrak{M}, u, e' \models \xi$, and so $\xi \in \mathsf{ft}(u, e')$. Let $\mathsf{dt}' = \mathsf{dt}_{\mathfrak{M}}(e')$. As $\pi_{\mathsf{wt},\mathsf{dt}'}$ is surjective, there is k' with $\pi_{\mathsf{wt},\mathsf{dt}'}(k') = \mathsf{ft}(u, e')$, and so $\xi \in \pi_{\mathsf{wt},\mathsf{dt}'}(k')$. By IH, $\mathsf{wt}^{\pi}, \mathsf{dt}'^{k'} \models \xi$, and so $\mathsf{wt}^{\pi}, \mathsf{dt}^k \models \rho$.

Case $\rho = \Diamond \xi$. If $\mathsf{wt}^{\pi}, \mathsf{dt}^k \models \rho$, there exists $\mathsf{wt}'^{\pi'}$ with $\mathsf{wt}'^{\pi'}, \mathsf{dt}^k \models \xi$. By IH, $\xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$, so $\Diamond \xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$ and $\rho \in \mathsf{dt}$, whence $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. Conversely, if $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u, e)$, there is u' with $\mathfrak{M}, u', e \models \xi$. Let $\mathsf{wt}' = \mathsf{wt}_{\mathfrak{M}}(u')$. By the choice of $\Pi^{\dagger}$, it has π' with $\pi'_{\mathsf{wt}',\mathsf{dt}}(k) = \mathsf{ft}_{\mathfrak{M}}(u', e)$. Then $\xi \in \pi'_{\mathsf{wt}',\mathsf{dt}}(k)$, so $\mathsf{wt}'^{\pi'}, \mathsf{dt} \models \xi$ by IH and $\mathsf{wt}^{\pi}, \mathsf{dt} \models \Diamond \xi$. ⊣

Note that Example 4.1 is of interest beyond illustrating our method as it provides a new and short proof of the exponential finite model property of $\mathsf{Q^1S5}$ [GKV97] (equivalently, the exponential finite product model property of $\mathsf{S5} \times \mathsf{S5}$).

In order to be able to introduce more complex 'data structures' that allow us to extend the construction above from satisfiability to σ-bisimulation consistency, we start by giving a simpler, yet equivalent, definition of bisimulation between $\mathsf{Q^1S5}$-models.

Given a signature σ and $(w,d) \in W \times D$, the *literal σ-type* $\ell^\sigma_{\mathfrak{M}}(w,d)$ of (w,d) in $\mathfrak{M}$ is the set

$$\{\boldsymbol{p} \in \sigma \mid \mathfrak{M}, w, d \models \boldsymbol{p}\} \cup \{\neg \boldsymbol{p} \mid \boldsymbol{p} \in \sigma, \quad \mathfrak{M}, w, d \not\models \boldsymbol{p}\}.$$

A pair $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ of relations $\boldsymbol{\beta}_W \subseteq W_1 \times W_2$ and $\boldsymbol{\beta}_D \subseteq D_1 \times D_2$ is called a σ-$\mathsf{S5}$-*bisimulation* between $\mathfrak{M}_1 = (W_1, D_1, I_1)$ and $\mathfrak{M}_2 = (W_2, D_2, I_2)$ when the following conditions hold:

($\mathsf{s5}_W$) if $(w_1, w_2) \in \boldsymbol{\beta}_W$ then, for any $d_1 \in D_1$, there is $d_2 \in D_2$ such that $(d_1, d_2) \in \boldsymbol{\beta}_D$ and $\ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(w_2, d_2)$, and the other way round;

($\mathsf{s5}_D$) if $(d_1, d_2) \in \boldsymbol{\beta}_D$ then, for any $w_1 \in W_1$, there is $w_2 \in W_2$ such that $(w_1, w_2) \in \boldsymbol{\beta}_W$ and $\ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(w_2, d_2)$, and the other way round.

We say that $\mathfrak{M}_1, w_1, d_1$ and $\mathfrak{M}_2, w_2, d_2$ are σ-$\mathsf{S5}$-*bisimilar* and write $\mathfrak{M}_1, w_1, d_1 \sim^{\mathsf{S5}}_\sigma \mathfrak{M}_2, w_2, d_2$ if there exists a σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ with $(w_1, w_2) \in \boldsymbol{\beta}_W$, $(d_1, d_2) \in \boldsymbol{\beta}_D$ and $\ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(w_2, d_2)$. Note that in this case $\mathrm{dom}(\boldsymbol{\beta}_W) = W_1$, $\mathrm{ran}(\boldsymbol{\beta}_W) = W_2$, $\mathrm{dom}(\boldsymbol{\beta}_D) = D_1$, and $\mathrm{ran}(\boldsymbol{\beta}_D) = D_2$.

Theorem 4.2. $\mathfrak{M}_1, w_1, d_1 \sim^{\mathsf{S5}}_\sigma \mathfrak{M}_2, w_2, d_2$ *iff* $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$.

Proof. ($\Rightarrow$) Suppose $\mathfrak{M}_1, w_1, d_1 \sim^{\mathsf{S5}}_\sigma \mathfrak{M}_2, w_2, d_2$ is witnessed by $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$. Define $\boldsymbol{\beta}$ by setting $((v_1, e_1), (v_2, e_2)) \in \boldsymbol{\beta}$ iff $(v_1, v_2) \in \boldsymbol{\beta}_W$, $(e_1, e_2) \in \boldsymbol{\beta}_D$ and $\ell^\sigma_{\mathfrak{M}_1}(v_1, e_1) = \ell^\sigma_{\mathfrak{M}_2}(v_2, e_2)$. It follows that $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$. We show that $\boldsymbol{\beta}$ satisfies **(a)**, **(w)** and **(d)**. Let $((v_1, e_1), (v_2, e_2)) \in \boldsymbol{\beta}$. Then **(a)** follows from $\ell^\sigma_{\mathfrak{M}_1}(v_1, e_1) = \ell^\sigma_{\mathfrak{M}_2}(v_2, e_2)$. To show **(w)**, take any $u_1 \in W_1$. As $(e_1, e_2) \in \boldsymbol{\beta}_D$, there is u_2 with $(u_1, u_2) \in \boldsymbol{\beta}_W$ and $\ell^\sigma_{\mathfrak{M}_1}(u_1, e_1) = \ell^\sigma_{\mathfrak{M}_2}(u_2, e_2)$ by **($\mathsf{s5}_D$)**, from which $((u_1, e_1), (u_2, e_2)) \in \boldsymbol{\beta}$. The other implication in **(w)** is symmetric. Finally, consider any $c_1 \in D_1$. By **($\mathsf{s5}_W$)**, there exists c_2 such that $(c_1, c_2) \in \boldsymbol{\beta}_D$ and $\ell^\sigma_{\mathfrak{M}_1}(v_1, c_1) = \ell^\sigma_{\mathfrak{M}_2}(v_2, c_2)$, from which $((v_1, c_1), (v_2, c_2)) \in \boldsymbol{\beta}$. This and a symmetric argument establish **(d)**.

($\Leftarrow$) Let $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$ be witnessed by $\boldsymbol{\beta}$. Set

$$\boldsymbol{\beta}_W = \{(v_1, v_2) \mid \exists e_1, e_2\, ((v_1, e_1), (v_2, e_2)) \in \boldsymbol{\beta}\}, \ \boldsymbol{\beta}_D = \{(e_1, e_2) \mid \exists v_1, v_2\, ((v_1, e_1), (v_2, e_2)) \in \boldsymbol{\beta}\}.$$

Then $(w_1, w_2) \in \boldsymbol{\beta}_W$, $(d_1, d_2) \in \boldsymbol{\beta}_D$, $\ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(w_2, d_2)$. To show **($\mathsf{s5}_W$)**, suppose that $(v_1, v_2) \in \boldsymbol{\beta}_W$ and $c_1 \in D_1$. Then there are e_1, e_2 with $((v_1, e_1), (v_2, e_2)) \in \boldsymbol{\beta}$, and so, by **(d)**, there is c_2 with $((v_1, c_1), (v_2, c_2)) \in \boldsymbol{\beta}$, from which $(c_1, c_2) \in \boldsymbol{\beta}_D$ and $\ell^\sigma_{\mathfrak{M}_1}(v_1, c_1) = \ell^\sigma_{\mathfrak{M}_2}(v_2, c_2)$. Condition ($\mathsf{s5}_D$) is proved similarly using **(w)**. $\dashv$

For any $w_1 \in W_1$, $w_2 \in W_2$, we write $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ if there is a σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $(w_1, w_2) \in \boldsymbol{\beta}_W$. Similarly, for any $d_1 \in D_1$, $d_2 \in d_2$, we write $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$ if there is a σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $(d_1, d_2) \in \boldsymbol{\beta}_D$. We omit $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and write simply $(w_1, d_1) \sim_\sigma (w_2, d_2)$, $w_1 \sim_\sigma w_2$, $d_1 \sim_\sigma d_2$ if understood.

Example 4.3. Consider $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\sigma = \{\boldsymbol{e}\}$ from Example 3.2. Then $(W_1 \times W_1, D_1 \times D_1)$ is a σ-$\mathsf{S5}$-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_1$ witnessing $(u_i, d_i) \sim_\sigma (u_j, d_j)$ and $(u_k, d_l) \sim_\sigma (u_m, d_n)$, for $i, j, k, l, m, n \in \{0, 1, 2\}$, $k \neq l$, $m \neq n$. The pair $(W_1 \times W_2, D_1 \times D_2)$ is a σ-$\mathsf{S5}$-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ that witnesses $(u_i, d_i) \sim_\sigma (v_j, c_j)$ and $(u_k, d_l) \sim_\sigma (v_m, c_n)$, for all $i, k, l \in \{0, 1, 2\}$ with $k \neq l$, and $j, m, n \in \{0, 1\}$ with $m \neq n$ (cf. $\boldsymbol{\beta}$ in Example 3.2). $\dashv$

Suppose that $\mathfrak{M}_i = (W_i, D_i, I_i)$, for $i = 1, 2$, are σ-$\mathsf{S5}$-bisimilar models with pairwise disjoint W_i and D_i. By the definitions,

$$\sim_\sigma \text{ is an equivalence relation on } W_1 \cup W_2, \text{ and also on } D_1 \cup D_2. \tag{4.1}$$

Also, for all $w_1 \in W_1$, $w_2 \in W_2$, $d_1 \in D_1$, $d_2 \in D_2$,

$$\text{if } w_1 \sim_\sigma w_2 \text{ then there is } e_2 \in D_2 \text{ with } d_1 \sim_\sigma e_2 \text{ and } \ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(w_2, e_2), \text{ and the other way round;} \quad (4.2)$$

$$\text{if } d_1 \sim_\sigma d_2 \text{ then there is } v_2 \in W_2 \text{ with } w_1 \sim_\sigma v_2 \text{ and } \ell^\sigma_{\mathfrak{M}_1}(w_1, d_1) = \ell^\sigma_{\mathfrak{M}_2}(v_2, d_2), \text{ and the other way round.} \quad (4.3)$$

Now we are in a position to define the necessary 'data structures'. For any $w \in W_1 \cup W_2$ and $i \in \{1, 2\}$, we set

$$T_i(w) = \{\mathsf{wt}_{\mathfrak{M}_i}(v) \mid v \in W_i,\ v \sim_\sigma w\} \quad (4.4)$$

and call $\mathsf{wm}(w) = (T_1(w), T_2(w))$ the *world mosaic* of w in $\mathfrak{M}_1, \mathfrak{M}_2$. The pair of the form $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$, for $w \in W_i$, is called the *i-world point* of w in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *world mosaic*, wm, and an *i-world point*, wp_i, in $\mathfrak{M}_1$, $\mathfrak{M}_2$ are defined as the world mosaic and i-world point of some $w \in W_1 \cup W_2$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$ (in the latter case, $w \in W_i$).

Similarly, for any $d \in D_1 \cup D_2$ and $i \in \{1, 2\}$, we set

$$S_i(d) = \{\mathsf{dt}_{\mathfrak{M}_i}(e) \mid e \in D_i,\ e \sim_\sigma d\} \quad (4.5)$$

and call $\mathsf{dm}(d) = (S_1(d), S_2(d))$ the *domain mosaic* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. If $d \in D_i$, the pair $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$ is called the *i-domain point* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *domain mosaic*, dm, and an *i-domain point*, dp_i, in $\mathfrak{M}_1$, $\mathfrak{M}_2$ are defined as the domain mosaic and i-domain point of some $d \in D_1 \cup D_2$. As follows from (4.4), (4.5) and (4.1),

(wm) $u \sim_\sigma v$ implies $\mathsf{wm}(u) = \mathsf{wm}(v)$,

(dm) $d \sim_\sigma e$ implies $\mathsf{dm}(d) = \mathsf{dm}(e)$.

Observe that the number of distinct wp_i and dp_i is at most double-exponential in $|\varphi| + |\psi|$.

Example 4.4. (a) Take $\mathfrak{M}_1$, $\mathfrak{M}_2$ from Example 3.2, $\sigma = \{\boldsymbol{e}\}$ and $\tau = \{\boldsymbol{e}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{b}_0, \boldsymbol{b}_1\}$. Then $\mathsf{wt}_{\mathfrak{M}_1}(u_i)$ and $\mathsf{dt}_{\mathfrak{M}_2}(c_i)$ contain, respectively, the sets

$$\{\exists(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\exists\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\exists\boldsymbol{p}_j \mid j \neq i\}, \quad \{\Diamond(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\Diamond\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\Diamond\boldsymbol{p}_j \mid j \neq i\}.$$

The σ-$\mathsf{S5}$-bisimulations from Example 4.3 give $\mathsf{wm}(u_0) = \mathsf{wm}(u_1) = \mathsf{wm}(u_2) = \mathsf{wm}(v_0) = \mathsf{wm}(v_1)$, so $\mathfrak{M}_1, \mathfrak{M}_2$ have $\mathsf{wm} = (\{\mathsf{wt}_{\mathfrak{M}_1}(u_i) \mid i = 0, 1, 2\}, \{\mathsf{wt}_{\mathfrak{M}_2}(v_i) \mid i = 0, 1\})$ as the only world mosaic. $\mathfrak{M}_1$ has three distinct 1-world points $(\mathsf{wt}_{\mathfrak{M}_1}(u_i), \mathsf{wm})$, for $i = 0, 1, 2$; $\mathfrak{M}_2$ has two 2-world points. Similarly, $\mathfrak{M}_1$, $\mathfrak{M}_2$ define one domain mosaic, $\mathfrak{M}_1$ has three distinct 1-domain points and $\mathfrak{M}_2$ has two 2-domain points.

(b) It can happen that non-bisimilar domain elements give the same domain-point. Consider the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ below and suppose that $sub_\Diamond(\varphi, \psi)$ has no formulas with $\exists$

in the scope of $\Diamond$, $\sigma = \{\boldsymbol{a}\}$ and $sig(\varphi, \psi) = \{\boldsymbol{a}, \boldsymbol{p}\}$. Then $\mathsf{dt}_{\mathfrak{M}_1}(d) = \mathsf{dt}_{\mathfrak{M}_1}(d')$ but $d \not\sim_\sigma d'$ because $\Diamond(\boldsymbol{a} \wedge \exists\neg\boldsymbol{a})$ is true at d and false at d'; likewise, we have $\mathsf{dt}_{\mathfrak{M}_2}(e) = \mathsf{dt}_{\mathfrak{M}_2}(e')$ but $e \not\sim_\sigma e'$. Since $d \sim_\sigma e$ and $d' \sim_\sigma e'$, we then have $\mathsf{dm}(d) = (\{\mathsf{dt}_{\mathfrak{M}_1}(d)\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e)\}) = \mathsf{dm}(e)$, $\mathsf{dm}(d') = (\{\mathsf{dt}_{\mathfrak{M}_1}(d')\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e')\}) = \mathsf{dm}(e')$, $\mathsf{dp}_1(d) = \mathsf{dp}_1(d')$, and $\mathsf{dp}_2(e) = \mathsf{dp}_2(e')$. ⊣

Suppose $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1, d_1 \models \varphi$ and $\mathfrak{M}_2, w_2, d_2 \models \neg\psi$. We construct $\mathfrak{M}'_i = (W'_i, D'_i, I'_i)$, $i = 1, 2$, witnessing σ-bisimulation consistency of φ and $\neg\psi$ and having at most double-exponential size in $|\varphi| + |\psi|$. Intuitively, W'_i and D'_i consist of copies of the

i-world and, respectively, i-domain points in $\mathfrak{M}_1$, $\mathfrak{M}_2$ rather than copies of the world- and domain-types as in Example 4.1. Then we obtain the required σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ by including in $\boldsymbol{\beta}_W$ and $\boldsymbol{\beta}_D$ exactly those 1/2-world and, respectively, 1/2-domain points that share the same world and domain mosaic.

Let n be the number of full types occurring either in $\mathfrak{M}_1$ or $\mathfrak{M}_2$ (over $sub(\varphi, \psi)$) and $[n] = \{1, \dots, n\}$. For $i = 1, 2$, set

$$D'_i = \{\mathsf{dp}_i^k \mid \mathsf{dp}_i \text{ an } i\text{-domain point in } \mathfrak{M}_1,\, \mathfrak{M}_2,\, k \in [n]\},$$

treating dp_i^k as the kth copy of dp_i and assuming all of the copies to be distinct. Next, we define W'_i, $i = 1, 2$, using the sets $\Pi_{\mathsf{wp}_i, \mathsf{dp}_i}$ of all *surjective* functions of the form

$$\pi_{\mathsf{wp}_i, \mathsf{dp}_i} \colon [n] \to \{\mathsf{ft}_{\mathfrak{M}_i}(w, d) \mid (w, d) \in W_i \times D_i, \quad \mathsf{wp}_i = \mathsf{wp}_i(w),\, \mathsf{dp}_i = \mathsf{dp}_i(d)\}.$$

Observe that, for any $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$, $k \in [n]$, and $\pi_{\mathsf{wp}_i, \mathsf{dp}_i} \in \Pi_{\mathsf{wp}_i, \mathsf{dp}_i}$ we have $\mathsf{wt} = \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) \cap sub_{\exists}(\varphi, \psi)$ and $\mathsf{dt} = \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) \cap sub_{\Diamond}(\varphi, \psi)$. On the other hand, it might happen that $\pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$, but $\mathsf{wm} \neq \mathsf{wm}(w)$ or $\mathsf{dm} \neq \mathsf{dm}(d)$.

Denote by Π_i the set of all functions π mapping every pair $\mathsf{wp}_i, \mathsf{dp}_i$ to an element of $\Pi_{\mathsf{wp}_i, \mathsf{dp}_i}$ and set $\pi_{\mathsf{wp}_i, \mathsf{dp}_i} = \pi(\mathsf{wp}_i, \mathsf{dp}_i)$. Let $\Pi_i^\dagger \subseteq \Pi_i$ be a smallest set such that, for any full type ft in $\mathfrak{M}_i$ and any $k \in [n]$, if $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$, for some $(w, d) \in W_i \times D_i$, then there exists $\pi \in \Pi_i^\dagger$ with $\pi_{\mathsf{wp}_i(w), \mathsf{dp}_i(d)}(k) = \mathsf{ft}$. We claim that $|\Pi_i^\dagger| \leq n^2$, $i = 1, 2$. Indeed, for any $k \in [n]$ and any full type ft in $\mathfrak{M}_i$, we will include just a single function $f^{k, \mathsf{ft}} \in \Pi_i$ in $\Pi_i^\dagger$. Assume k and $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(u, e)$ are given. Then we can choose $f^{k, \mathsf{ft}}$ to be any function mapping pairs $\mathsf{wp}_i, \mathsf{dp}_i$ into $\Pi_{\mathsf{wp}_i, \mathsf{dp}_i}$ such that, for any w, d with $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$, we have $f^{k, \mathsf{ft}}(\mathsf{wp}_i(w), \mathsf{dp}_i(d))(k) = \mathsf{ft}$, where $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$ and $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$. Observe that there might be w', d' with $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(w', d')$ but $\mathsf{wp}_i(w') \neq \mathsf{wp}_i(w)$ or $\mathsf{dp}_i(d') \neq \mathsf{dp}_i(d)$. Nevertheless, such pairs (w, d), (w', d') do not give conflicting requirements on the choice of $f^{k, \mathsf{ft}}$. The constructed $\Pi_i^\dagger$ is as required.

Then we set

$$W'_i = \{\mathsf{wp}_i^\pi \mid \mathsf{wp}_i \text{ an } i\text{-world point in } \mathfrak{M}_1,\, \mathfrak{M}_2,\, \pi \in \Pi_i^\dagger\},$$

treating wp_i^π as a fresh π-copy of wp_i. Clearly, both $|D'_i|$ and $|W'_i|$ are double-exponential in $|\varphi| + |\psi|$. Finally, we set

$$\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \boldsymbol{p} \quad \text{iff} \quad \boldsymbol{p} \in \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) \tag{4.6}$$

and define $\boldsymbol{\beta}_W \subseteq W'_1 \times W'_2$ and $\boldsymbol{\beta}_D \subseteq D'_1 \times D'_2$ by taking $(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2}) \in \boldsymbol{\beta}_W$ iff $\mathsf{wm}_1 = \mathsf{wm}_2$, where $\mathsf{wp}_i = (\mathsf{wt}_i, \mathsf{wm}_i)$, for $i = 1, 2$; and similarly $(\mathsf{dp}_1^{k_1}, \mathsf{dp}_2^{k_2}) \in \boldsymbol{\beta}_D$ iff $\mathsf{dm}_1 = \mathsf{dm}_2$, where $\mathsf{dp}_i = (\mathsf{dt}_i, \mathsf{dm}_i)$, for $i = 1, 2$.

Lemma 4.5. (i) $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$ *iff* $\rho \in \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k)$, *for every* $\rho \in sub(\varphi, \psi)$.

(ii) *The pair* $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$ *is a* σ-$\mathsf{S5}$-*bisimulation between* $\mathfrak{M}'_1$ *and* $\mathfrak{M}'_2$.

Proof. (i) The proof is by induction on the construction of ρ, with the basis given by (4.6). For the induction step, suppose first that $\rho = \exists \xi$. If $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$, then there is $\mathsf{dp}_i'^{k'}$ such that $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'} \models \xi$. By IH, $\xi \in \pi_{\mathsf{wp}_i, \mathsf{dp}'_i}(k')$ and $\exists \xi \in \pi_{\mathsf{wp}_i, \mathsf{dp}'_i}(k')$. Then $\exists \xi \in \mathsf{wt}$ for $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, and so $\rho \in \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k)$. Conversely, suppose $\rho \in \pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k)$, where $\pi_{\mathsf{wp}_i, \mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$ with $\mathsf{wp}_i = \mathsf{wp}_i(w)$ and $\mathsf{dp}_i = \mathsf{dp}_i(d)$. Then there is d' with $\mathfrak{M}_i, w, d' \models \xi$. Let $\mathsf{dp}'_i = \mathsf{dp}_i(d')$. As $\pi_{\mathsf{wp}_i, \mathsf{dp}_i}$ is surjective, there is k' with $\xi \in \pi_{\mathsf{wp}_i, \mathsf{dp}'_i}(k')$. By IH, $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'} \models \xi$. It follows that $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$.

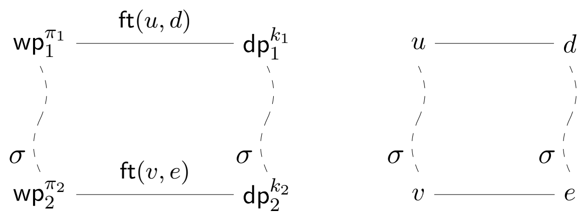
Next, let $\rho = \Diamond\xi$. Suppose $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$. Then there is $\mathsf{wp}_i'^{\pi'}$ with $\mathfrak{M}'_i, \mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i^k \models \xi$. By IH, $\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$ and $\Diamond\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$. Then $\Diamond\xi \in \mathsf{dt}$ for $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$. It follows that $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$. Conversely, let $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w,d)$ with $\mathsf{wp}_i = \mathsf{wp}_i(w)$ and $\mathsf{dp}_i = \mathsf{dp}_i(d)$. Then there is w' with $\mathfrak{M}_i, w', d \models \xi$. By the choice of $\Pi_i^\dagger$, there exists $\pi' \in \Pi_i^\dagger$ such that $\pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k) = \mathsf{ft}_{\mathfrak{M}_i}(w', d)$, where $\mathsf{wp}'_i = \mathsf{wp}_i(w')$. Then $\xi \in \pi'_{\mathsf{wp}'_i,\mathsf{dp}_i}(k)$, and so $\mathfrak{M}'_i, \mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i \models \xi$ by IH, whence $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i \models \Diamond\xi$.

The induction step for the Booleans is straightforward.

(*ii*) To check $(\mathbf{s5}_W)$, suppose $(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2}) \in \boldsymbol{\beta}_W$ with $\mathsf{wp}_i = (\mathsf{wt}_i, \mathsf{wm}_i)$, for $i = 1, 2$, so $\mathsf{wm}_1 = \mathsf{wm}_2$. Take any $\mathsf{dp}_1^{k_1} \in D'_1$ with $\mathsf{dp}_1 = (\mathsf{dt}_1, \mathsf{dm}_1)$. We need to find $\mathsf{dp}_2^{k_2} \in D'_2$ such that $\mathsf{dp}_2 = (\mathsf{dt}_2, \mathsf{dm}_2)$, $\mathsf{dm}_1 = \mathsf{dm}_2$ and $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for all $\boldsymbol{p} \in \sigma$.

Suppose $\pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1) = \mathsf{ft}_{\mathfrak{M}_1}(u,d)$, for some $(u,d) \in W_1 \times D_1$. Then $\mathsf{wp}_1 = \mathsf{wp}_1(u) = (\mathsf{wt}_{\mathfrak{M}_1}(u), \mathsf{wm}(u))$, $\mathsf{wm}_1 = \mathsf{wm}(u) = (T_1(u), T_2(u))$, and $\mathsf{dp}_1 = \mathsf{dp}_1(d) = (\mathsf{dt}_{\mathfrak{M}_1}(d), \mathsf{dm}(d))$ and $\mathsf{dm}_1 = \mathsf{dm}(d) = (S_1(d), S_2(d))$. As $\mathsf{wm}_1 = \mathsf{wm}_2$, we have $\mathsf{wt}_2 \in T_2(u)$ by (4.1). Thus, by (4.4) and **(wm)**, there is $v \in W_2$ with $u \sim_\sigma v$ and $\mathsf{wp}_2 = (\mathsf{wt}_{\mathfrak{M}_2}(v), \mathsf{wm}(v))$.

Now, by (4.2), there exists $e \in D_2$ with $d \sim_\sigma e$ and $\ell^\sigma_{\mathfrak{M}_1}(u,d) = \ell^\sigma_{\mathfrak{M}_2}(v,e)$. By **(dm)**, $\mathsf{dm}(d) = \mathsf{dm}(e)$. Since all functions $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ are surjective, there exists $k_2 \in [n]$ with $\pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2) = \mathsf{ft}_{\mathfrak{M}_2}(v,e)$, implying that $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for all $\boldsymbol{p} \in \sigma$.



The other implication in $(\mathbf{s5}_W)$ is similar.

To check $(\mathbf{s5}_D)$, let $(\mathsf{dp}_1^{k_1}, \mathsf{dp}_2^{k_2}) \in \boldsymbol{\beta}_D$ with $\mathsf{dp}_i = (\mathsf{dt}_i, \mathsf{dm}_i)$, $i = 1, 2$, so $\mathsf{dm}_1 = \mathsf{dm}_2$. Take any $\mathsf{wp}_1^{\pi^1} \in W'_1$ with $\mathsf{wp}_1 = (\mathsf{wt}_1, \mathsf{wm}_1)$. We need to find $\mathsf{wp}_2^{\pi^2} \in W'_2$ with $\mathsf{wp}_2 = (\mathsf{wt}_2, \mathsf{wm}_2)$ such that $\mathsf{wm}_1 = \mathsf{wm}_2$ and $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for every $\boldsymbol{p} \in \sigma$.

Let $\pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1) = \mathsf{ft}_{\mathfrak{M}_1}(u,d)$, for some $(u,d) \in W_1 \times D_1$. Then $\mathsf{wp}_1 = \mathsf{wp}_1(u) = (\mathsf{wt}_{\mathfrak{M}_1}(u), \mathsf{wm}(u))$, $\mathsf{wm}_1 = \mathsf{wm}(u) = (T_1(u), T_2(u))$, and $\mathsf{dp}_1 = \mathsf{dp}_1(d) = (\mathsf{dt}_{\mathfrak{M}_1}(d), \mathsf{dm}(d))$ and $\mathsf{dm}_1 = \mathsf{dm}(d) = (S_1(d), S_2(d))$. As $\mathsf{dm}_1 = \mathsf{dm}_2$, we have $\mathsf{dt}_2 \in S_2(u)$ by (4.1). Thus, by (4.5) and **(dm)**, there exists $e \in D_2$ such that $e \sim_\sigma d$ and $\mathsf{dp}_2 = (\mathsf{dt}_{\mathfrak{M}_2}(e), \mathsf{dm}(e))$. By (4.3), there is $v \in W_2$ with $u \sim_\sigma v$ and $\ell^\sigma_{\mathfrak{M}_1}(u,d) = \ell^\sigma_{\mathfrak{M}_2}(v,e)$. By **(wm)**, we have $\mathsf{wm}(u) = \mathsf{wm}(v)$. By the choice of $\Pi_2^\dagger$, there is $\pi^2 \in \Pi_2^\dagger$ such that $\pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2) = \mathsf{ft}_{\mathfrak{M}_2}(v,e)$, which implies that $\boldsymbol{p} \in \pi^1_{\mathsf{wp}_1,\mathsf{dp}_1}(k_1)$ iff $\boldsymbol{p} \in \pi^2_{\mathsf{wp}_2,\mathsf{dp}_2}(k_2)$, for all $\boldsymbol{p} \in \sigma$. The other implication in $(\mathbf{s5}_D)$ is similar. ⊣

The construction and lemmas above yield:

Theorem 4.6. *Any formulas φ and $\neg\psi$ are $sig(\varphi) \cap sig(\psi)$-bisimulation consistent in* $\mathsf{Q^1S5}$ *iff there are witnessing* $\mathsf{Q^1S5}$*-models of size double-exponential in $|\varphi| + |\psi|$.*

Proof. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be $\mathsf{Q^1S5}$-models with $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1, d_1 \models \varphi$, $\mathfrak{M}_2, w_2, d_2 \models \neg\psi$, where $\sigma = sig(\varphi) \cap sig(\psi)$. Consider the models $\mathfrak{M}'_1$, $\mathfrak{M}'_2$ with σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_D)$. For $i = 1, 2$, let $\mathsf{wp}_i = \mathsf{wp}_i(w_i) = (\mathsf{wt}_{\mathfrak{M}_i}(w_i), \mathsf{wm}(w_i))$ and let $\mathsf{dp}_i = \mathsf{dp}_i(d_i) = (\mathsf{dt}_{\mathfrak{M}_i}(d_i), \mathsf{dm}(w_i))$. By the choice of $\Pi_i^\dagger$, we have π^i with $\pi^i_{\mathsf{wp}_i,\mathsf{dp}_i}(1) = \mathsf{ft}_{\mathfrak{M}_i}(w_i, d_i)$.

Then $\mathfrak{M}'_1, \mathsf{wp}_1^{\pi^1}, \mathsf{dp}_1^1 \models \varphi$ and $\mathfrak{M}'_2, \mathsf{wp}_2^{\pi^2}, \mathsf{dp}_2^1 \models \neg\psi$ by Lemma 4.5 (i). Since $w_1 \sim_\sigma w_2$ and $d_1 \sim_\sigma d_2$, **(wm)** and **(dm)** imply $\mathsf{wm}(w_1) = \mathsf{wm}(w_2)$ and $\mathsf{dm}(d_1) = \mathsf{dm}(d_2)$. By Lemma 4.5 (ii), $(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2}) \in \boldsymbol{\beta}_W$ and $(\mathsf{dp}_1^{\pi^1}, \mathsf{dp}_2^{\pi^2}) \in \boldsymbol{\beta}_D$, and so $\mathfrak{M}'_1, \mathsf{wp}_1^{\pi^1}, \mathsf{dp}_1^1 \sim_\sigma \mathfrak{M}'_2, \mathsf{wp}_2^{\pi^2}, \mathsf{dp}_2^1$ by Theorem 4.2. ⊣

Theorems 4.6, 3.1 and 3.5 give the upper bound result in Theorem 1.2 for $\mathsf{Q^1S5}$, stating that both IEP and EDEP for $\mathsf{Q^1S5}$ are decidable in coN2ExpTime.

4.2. **Lower bound.** We next prove the lower bound in Theorem 1.2 for $\mathsf{Q^1S5}$, stating that the IEP and EDEP for $\mathsf{Q^1S5}$ are both 2ExpTime-hard.

We reduce the word problem for languages recognised by exponentially space bounded alternating Turing machines (ATMs). It is well-known that there are 2^n-space bounded ATMs for which the recognised language is 2ExpTime-hard [CKS81].

A 2^n*-space bounded ATM* is a tuple $M = (Q, q_0, \Gamma, \Delta)$, whose set Q of states is partitioned to $\forall$-states and $\exists$-states, with the initial state q_0 being a $\forall$-state; Γ is the tape alphabet containing the blank symbol $\flat$; and $\Delta \colon Q \times \Gamma \to \mathcal{P}\big(Q \times \Gamma \times \{L, R\}\big)$ is the transition function such that $|\Delta(q, a)|$ is always either 0 or 2, and $\forall$-states and $\exists$-states alternate on every computation path. $\forall$- and $\exists$-*configurations* are represented by 2^n-long sequences of symbols from $\Gamma \cup (Q \times \Gamma)$, with a single symbol in the sequence being from $Q \times \Gamma$.

Similarly to [JW21], we use the following (slightly non-standard) acceptance condition. An *accepting computation-tree* is an infinite tree of configurations such that $\forall$-configurations always have 2 children, and $\exists$-configurations always have 1 child (marked by 0 or 1). We say that M *accepts an input word* $\overline{a} = (a_0, a_1, \dots, a_{n-1})$ if there is an accepting computation-tree with the configuration $c_{init} = \big((q_0, a_0), a_1, \dots, a_{n-1}, \flat, \dots, \flat\big)$ at its root. Note that, starting from the standard ATM acceptance condition defined via accepting states, this can be achieved by assuming that the 2^n-space bounded ATM terminates on every input and then modifying it to enter an infinite loop from the accepting state.

Given a 2^n-space bounded ATM M and an input word $\overline{a}$ of length n, we will construct in polytime formulas φ and ψ such that

(1) $\models_{\mathsf{Q^1S5}} \varphi \to \psi$, and
(2) M accepts $\overline{a}$ iff $\varphi, \neg\psi$ are σ-bisimulation consistent in $\mathsf{Q^1S5}$, where $\sigma = \mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$.

By Theorems 3.5 and 3.1, it follows that both IEP and EDEP are 2ExpTime-hard for $\mathsf{Q^1S5}$.

One aspect of our construction is similar to that of [AJM+21, JW21]: we also represent accepting computation-trees as binary trees whose nodes are coloured by predicates in σ. However, unlike the formalisms in the cited work, $\mathsf{Q^1S5}$ cannot express the uniqueness of properties, and so the remaining ideas are novel. One part of φ 'grows' 2^n-many copies of σ-coloured binary trees, using a technique from 2D propositional modal logic [HKK+03, GJL15]. Another part of φ colours the tree-nodes with non-σ-symbols to ensure that, in the mth tree, for each $m < 2^n$, the content of the mth tape-cell is properly changing during the computation. Then we use ideas from Example 3.2 to make sure that the generated 2^n-many trees are all σ-bisimilar, and so represent the same accepting computation-tree.

We begin with defining the conjuncts (4.7)–(4.39) of φ. We will use three counters, B, U and V, each counting modulo 2^n and implemented using $2n$-many unary predicate symbols: $\boldsymbol{h}_0^A, \dots, \boldsymbol{h}_{n-1}^A$, $\boldsymbol{v}_0^A, \dots, \boldsymbol{v}_{n-1}^A$ for $A \in \{B, U, V\}$. We write equ^A for $\bigwedge_{i<n}(\boldsymbol{h}_i^A \leftrightarrow \boldsymbol{v}_i^A)$, and write $[A\!=\!m]$ for $m < 2^n$, if equ^A holds and the $\boldsymbol{h}^A$- and $\boldsymbol{v}^A$-sequences represent m in binary. We use $[A\!<\!m]$ if $[A\!=\!k]$ for some $k < m$, and we use $[A\!\neq\!m]$ if $[A\!=\!k]$ for some $k \neq m$.

We write succ^A for expressing that '$\boldsymbol{h}^A$-value $=$ $\boldsymbol{v}^A$-value+1 (mod 2^n)':

$$\bigvee_{i<n}\Big(\boldsymbol{h}_i^A \wedge \neg\boldsymbol{v}_i^A \wedge \bigwedge_{j<i}(\neg\boldsymbol{h}_j^A \wedge \boldsymbol{v}_j^A) \wedge \bigwedge_{i<j<n}(\boldsymbol{h}_j^A \leftrightarrow \boldsymbol{v}_j^A)\Big) \vee \bigwedge_{i<n}(\neg\boldsymbol{h}_i^A \wedge \boldsymbol{v}_i^A).$$

We express, for $A \in \{B, U, V\}$, that the $\boldsymbol{h}^A$-predicates are 'modally-stable' and the $\boldsymbol{v}^A$-predicates are 'FO-stable':

$$\Box\forall \bigwedge_{i<n}\big((\boldsymbol{h}_i^A \to \Box\boldsymbol{h}_i^A) \wedge (\neg\boldsymbol{h}_i^A \to \Box\neg\boldsymbol{h}_i^A)\big), \tag{4.7}$$

$$\Box\forall \bigwedge_{i<n}\big((\boldsymbol{v}_i^A \to \forall\boldsymbol{v}_i^A) \wedge (\neg\boldsymbol{v}_i^A \to \forall\neg\boldsymbol{v}_i^A)\big). \tag{4.8}$$

We use the B-counter to generate 2^n-many 'special' equ^B-points 'coloured' by a fresh predicate $\boldsymbol{r}$ for the root-node of the trees representing the computation. The succ^B-points used in generating the $\boldsymbol{r}$-points will be marked by a fresh predicate $\boldsymbol{n}^B$ (for 'next B'):

$$[B=0] \wedge \boldsymbol{r}, \tag{4.9}$$

$$\Box\forall\big(\boldsymbol{r} \wedge [B \neq 2^n - 1] \to \exists\boldsymbol{n}^B\big), \tag{4.10}$$

$$\Box\forall(\boldsymbol{n}^B \to \Diamond\boldsymbol{r}), \tag{4.11}$$

$$\Box\forall(\boldsymbol{r} \to \mathsf{equ}^B), \tag{4.12}$$

$$\Box\forall(\boldsymbol{n}^B \to \mathsf{succ}^B). \tag{4.13}$$

Then, at each $\boldsymbol{r}$-point, we 'grow' an infinite binary rooted tree that we will use to represent the accepting computation-tree of M on $\overline{a}$ as follows. The binary tree is divided into 2^n-long 'linear' levels (where each node has one child only): each linear 2^n-long subpath within such a level represents a configuration. In addition, the infinite binary tree is branching to two at the last node of the linear subpath representing each $\forall$-configuration (see more details in the proof of Lemma 4.8 below).

We grow this infinite binary tree with the help of the U-counter. Nodes of this infinite 'U-tree' are marked by a fresh predicate $\boldsymbol{t}$, and the succ^U-points used in generating the $\boldsymbol{t}$-points will be marked by a fresh predicate $\boldsymbol{n}^U$. First, we generate a computation-tree 'skeleton' of alternating $\forall$- and $\exists$-levels, and with appropriate branching. We use fresh predicates $\boldsymbol{q}_\forall$ and $\boldsymbol{q}_\exists^i$, $i = 0, 1$, to mark the levels, and an additional predicate $\boldsymbol{z}$ to enforce two different children at $\forall$-levels. Given any formula χ, we write $\mathsf{next}(\chi)$ for $\forall\big(\boldsymbol{n}^U \to \Box(\boldsymbol{t} \to \chi)\big)$. We add the following conjuncts, for $i = 0, 1$:

$$\Box\forall\big(\boldsymbol{r} \to [U=0] \wedge \boldsymbol{q}_\forall \wedge \boldsymbol{t}\big), \tag{4.14}$$

$$\Box\forall(\boldsymbol{t} \to \exists\boldsymbol{n}^U), \tag{4.15}$$

$$\Box\forall(\boldsymbol{n}^U \to \Diamond\boldsymbol{t}), \tag{4.16}$$

$$\Box\forall(\boldsymbol{t} \to \mathsf{equ}^U), \tag{4.17}$$

$$\Box\forall(\boldsymbol{n}^U \to \mathsf{succ}^U), \tag{4.18}$$

$$\Box\forall\big(\boldsymbol{t} \wedge [U \neq 2^n - 1] \wedge \boldsymbol{q}_\forall \to \mathsf{next}(\boldsymbol{q}_\forall)\big), \tag{4.19}$$

$$\Box\forall\big(\boldsymbol{t} \wedge [U \neq 2^n - 1] \wedge \boldsymbol{q}_\exists^i \to \mathsf{next}(\boldsymbol{q}_\exists^i)\big), \tag{4.20}$$

$$\Box\forall\big(\boldsymbol{t} \wedge [U = 2^n - 1] \wedge \boldsymbol{q}_\forall \to \exists(\boldsymbol{n}^U \wedge \boldsymbol{z}) \wedge \exists(\boldsymbol{n}^U \wedge \neg\boldsymbol{z})\big), \tag{4.21}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[U=2^n-1]\wedge\boldsymbol{q}_\forall\to\mathsf{next}(\boldsymbol{q}_\exists^0\vee\boldsymbol{q}_\exists^1)\big),\tag{4.22}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[U=2^n-1]\wedge\boldsymbol{q}_\exists^i\to\mathsf{next}(\boldsymbol{q}_\forall)\big).\tag{4.23}$$

Next, for each $\gamma\in\Gamma\cup(Q\times\Gamma)$, we introduce a fresh predicate $\boldsymbol{s}_\gamma$. We initialise the computation on input $\overline{a}=(a_0,a_1,\dots,a_{n-1})$, where $a_i\neq\flat$ for $i<n$:

$$\Box\forall\Big(\boldsymbol{r}\to\boldsymbol{s}_{(q_0,a_0)}\wedge\mathsf{next}(\boldsymbol{s}_{a_1}\wedge\dots\mathsf{next}(\boldsymbol{s}_{a_{n-1}}\wedge\mathsf{next}(\boldsymbol{s}_\flat))\dots)\Big),\tag{4.24}$$

$$\Box\forall\big(\boldsymbol{t}\wedge\boldsymbol{s}_\flat\wedge[U\neq 2^n-1]\to\mathsf{next}(\boldsymbol{s}_\flat)\big).\tag{4.25}$$

Next, we ensure that the subsequent configurations are properly represented. Using the V-counter, we ensure that, for each $m<2^n$, the U-tree that is grown at the mth $\boldsymbol{r}$-point properly describes the 'evolution' of the mth tape-cell's content during the accepting computation. We begin with ensuring that the V-counter increases along the U-counter, and with initialising it as 2^n-1-m of the value m of the B-counter:

$$\Box\forall(\boldsymbol{t}\to\mathsf{equ}^V),\tag{4.26}$$

$$\Box\forall(\boldsymbol{n}^U\to\mathsf{succ}^V),\tag{4.27}$$

$$\Box\forall\Big[\boldsymbol{r}\to\bigwedge_{i<n}\big((\boldsymbol{h}_i^B\leftrightarrow\neg\boldsymbol{h}_i^V)\wedge(\boldsymbol{v}_i^B\leftrightarrow\neg\boldsymbol{v}_i^V)\big)\Big].\tag{4.28}$$

Below we enforce the proper evolution of the 'middle' section of the 2^n-long tape (when $0<m<2^n-1$), the two missing cells at the beginning and the end of the tape can be handled similarly.

In order to do this, we represent the transition function Δ of M by two partial functions

$$f_i\colon\big(\Gamma\cup(Q\times\Gamma)\big)^3\to\big(\Gamma\cup(Q\times\Gamma)\big),\quad\text{for } i=0,1,$$

giving the next content of the middle-cell for each triple of cells. We ensure that the domain of the f_i is proper by taking, for all (q,a) with $|\Delta(q,a)|=0$, the conjunct

$$\Box\forall\neg\boldsymbol{s}_{(q,a)}.\tag{4.29}$$

For each $\overline{\gamma}=(\gamma_0,\gamma_1,\gamma_2)\in\big(\Gamma\cup(Q\times\Gamma)\big)^3$ in the domain of any of the f_i, we write $\mathsf{cells}_{\overline{\gamma}}$ for

$$\boldsymbol{s}_{\gamma_0}\wedge\exists\Big[\boldsymbol{n}^U\wedge\Diamond\Big(\boldsymbol{t}\wedge\big(\boldsymbol{s}_{\gamma_1}\wedge\exists(\boldsymbol{n}^U\wedge\Diamond\boldsymbol{s}_{\gamma_2})\big)\Big)\Big].$$

In addition to the $\boldsymbol{s}_\gamma$ variables, for some $\gamma\in\Gamma\cup(Q\times\Gamma)$, we will use additional variables $\boldsymbol{s}_\gamma^0$, $\boldsymbol{s}_\gamma^1$, and $\boldsymbol{s}_\gamma^+$, and have the conjuncts, for $i=0,1$ and $\overline{\gamma}$ in the domain of any of the f_i:

$$\Box\forall\big(\boldsymbol{t}\wedge[V=2^n-1]\wedge[U<2^n-2]\wedge\mathsf{cells}_{\overline{\gamma}}\wedge\boldsymbol{q}_\forall\to\mathsf{next}(\boldsymbol{s}^0_{f_0(\overline{\gamma})}\wedge\boldsymbol{s}^1_{f_1(\overline{\gamma})})\big),\tag{4.30}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[V=2^n-1]\wedge[U<2^n-2]\wedge\mathsf{cells}_{\overline{\gamma}}\wedge\boldsymbol{q}_\exists^i\to\mathsf{next}(\boldsymbol{s}^i_{f_i(\overline{\gamma})})\big),\tag{4.31}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[U\neq 2^n-1]\wedge\boldsymbol{s}_\gamma^i\to\mathsf{next}(\boldsymbol{s}_\gamma^i)\big),\tag{4.32}$$

$$\Box\forall\Big(\boldsymbol{t}\wedge[U=2^n-1]\wedge\boldsymbol{q}_\forall\wedge\boldsymbol{s}_\gamma^0\to\forall\big(\boldsymbol{n}^U\wedge\boldsymbol{z}\to\Box(\boldsymbol{t}\to\boldsymbol{s}_\gamma^+)\big)\Big),\tag{4.33}$$

$$\Box\forall\Big(\boldsymbol{t}\wedge[U=2^n-1]\wedge\boldsymbol{q}_\forall\wedge\boldsymbol{s}_\gamma^1\to\forall\big(\boldsymbol{n}^U\wedge\neg\boldsymbol{z}\to\Box(\boldsymbol{t}\to\boldsymbol{s}_\gamma^+)\big)\Big),\tag{4.34}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[U=2^n-1]\wedge\boldsymbol{q}_\exists^i\wedge\boldsymbol{s}_\gamma^i\to\mathsf{next}(\boldsymbol{s}_\gamma^+)\big),\tag{4.35}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[V\neq 2^n-1]\wedge\boldsymbol{s}_\gamma^+\to\mathsf{next}(\boldsymbol{s}_\gamma^+)\big),\tag{4.36}$$

$$\Box\forall\big(\boldsymbol{t}\wedge[V=2^n-1]\wedge\boldsymbol{s}_\gamma^+\to\mathsf{next}(\boldsymbol{s}_\gamma)\big).\tag{4.37}$$

Finally, we introduce a fresh predicate $\boldsymbol{e}$ that will 'interact' with the formula ψ. We add conjuncts to φ ensuring that each of the generated U-trees stays within the 'B-domain' of its root $\boldsymbol{r}$-point (meaning every node of these trees is an $\boldsymbol{e}$-point having the same B-value):

$$\Box\forall(\boldsymbol{t} \vee \boldsymbol{n}^U \to \boldsymbol{e}), \tag{4.38}$$

$$\Box\forall(\boldsymbol{e} \to \mathsf{equ}^B). \tag{4.39}$$

By this, we have completed the definition of φ.

Next, using the second formula of Example 3.2, we define the formula ψ such that $sig(\varphi) \cap sig(\psi)$ is the set

$$\sigma = \{\boldsymbol{e}, \boldsymbol{r}, \boldsymbol{n}^U, \boldsymbol{z}, \boldsymbol{t}, \boldsymbol{q}_\forall, \boldsymbol{q}_\exists^0, \boldsymbol{q}_\exists^1\} \cup \{\boldsymbol{s}_\gamma \mid \gamma \in \Gamma \cup (Q \times \Gamma)\}.$$

We let

$$\psi \;=\; \chi \wedge \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1) \to \Diamond\exists\big(\boldsymbol{b}_0 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists\boldsymbol{b}_0)\big) \vee \Diamond\exists\big(\boldsymbol{b}_1 \wedge \Diamond(\neg\boldsymbol{e} \wedge \exists\boldsymbol{b}_1)\big),$$

where $\chi = \bigwedge_{\boldsymbol{p}\in\sigma\setminus\{\boldsymbol{e}\}}(\boldsymbol{p} \to \boldsymbol{p})$ and $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ are fresh predicates.

Lemma 4.7. *If $n > 1$ then $\models_{\mathsf{Q^1S5}} \varphi \to \psi$.*

Proof. Suppose $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1)$ for some model $\mathfrak{M} = (W, D, I)$. Then, by (4.9)–(4.13), we have at least $2^n > 3$ different $\boldsymbol{r}$-points $(w_0, d_0), \dots, (w_{2^n-1}, d_{2^n-1})$ in $W \times D$, with the respective B-values $0, \dots, 2^n - 1$. As $\boldsymbol{r}$-points are also $\boldsymbol{e}$-points by (4.14) and (4.38), the pigeonhole principle implies that there are $i \neq j < 2^{n-1}$, $k \in \{0, 1\}$ such that $\mathfrak{M}, w_i, d_i \models \boldsymbol{b}_k$ and $\mathfrak{M}, w_j, d_j \models \boldsymbol{b}_k$. Then $\mathfrak{M}, w_j, d_i \models \neg\boldsymbol{e}$ by (4.7), (4.8) and (4.39), and so $\mathfrak{M}, w_0, d_0 \models \psi$. ⊣

Lemma 4.8. *If M accepts $\overline{a}$ then φ, $\neg\psi$ are σ-bisimulation consistent in* $\mathsf{Q^1S5}$.

Proof. Let $\mathfrak{T} = (T, S_0, S_1, \boldsymbol{q}_\forall, \boldsymbol{q}_\exists^0, \boldsymbol{q}_\exists^1, \boldsymbol{s}_\gamma)_{\gamma\in\Gamma\cup(Q\times\Gamma)}$ be the infinite binary tree-shaped FO-structure with root $r \in T$ and binary predicates S_0, S_1, that represents the accepting computation-tree of M on $\overline{a}$ as discussed after formula (4.13) above, that is, configurations are represented by subpaths of 2^n-many nodes linked by S_0. Every node of the 2^n-long subpath representing a $\forall$-configuration is coloured by $\boldsymbol{q}_\forall$. The last node representing a $\forall$-configuration has one S_i-child, for each of $i = 0, 1$, where the representations of the two subsequent $\exists$-configurations start. For $i = 0, 1$, if it is the i-child of an $\exists$-configuration c that is present in the computation-tree, then every node of the 2^n-long subpath representing c is coloured by $\boldsymbol{q}_\exists^i$ (see Fig. 1 for an example). The last node representing an $\exists$-configuration has one S_0-child, where the representation of the next configuration starts. Nodes representing a configuration are also coloured with $\boldsymbol{s}_\gamma$ for the corresponding symbol γ from $\Gamma \cup (Q \times \Gamma)$.

We begin by defining a model $\mathfrak{M} = (W, D, I)$ making φ true. Take $2 \cdot 2^n$ many disjoint copies W_m and D_m, $m < 2^n$, of T and let $W = \bigcup_{m<2^n} W_m$ and $D = \bigcup_{m<2^n} D_m$. For each $m < 2^n$ and $t \in T$, let w_m^t and d_m^t denote the copy of t in W_m and D_m, respectively. We define I first for the symbols in σ. For all $m < 2^n$, $t \in T$ and $\boldsymbol{p} \in \{\boldsymbol{q}_\forall, \boldsymbol{q}_\exists^0, \boldsymbol{q}_\exists^1\} \cup \{\boldsymbol{s}_\gamma \mid \gamma \in \Gamma \cup (Q \times \Gamma)\}$, we let

$$\boldsymbol{e}^{I(w_m^t)} = \{d_m^{t'} \mid t' \in T\}, \tag{4.40}$$

$$\boldsymbol{r}^{I(w_m^t)} = \begin{cases} \{d_m^t\}, & \text{if } t = r, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{4.41}$$

$$(\boldsymbol{n}^U)^{I(w_m^t)} = \{d_m^{t'} \mid S_0(t, t') \text{ or } S_1(t, t')\}, \tag{4.42}$$

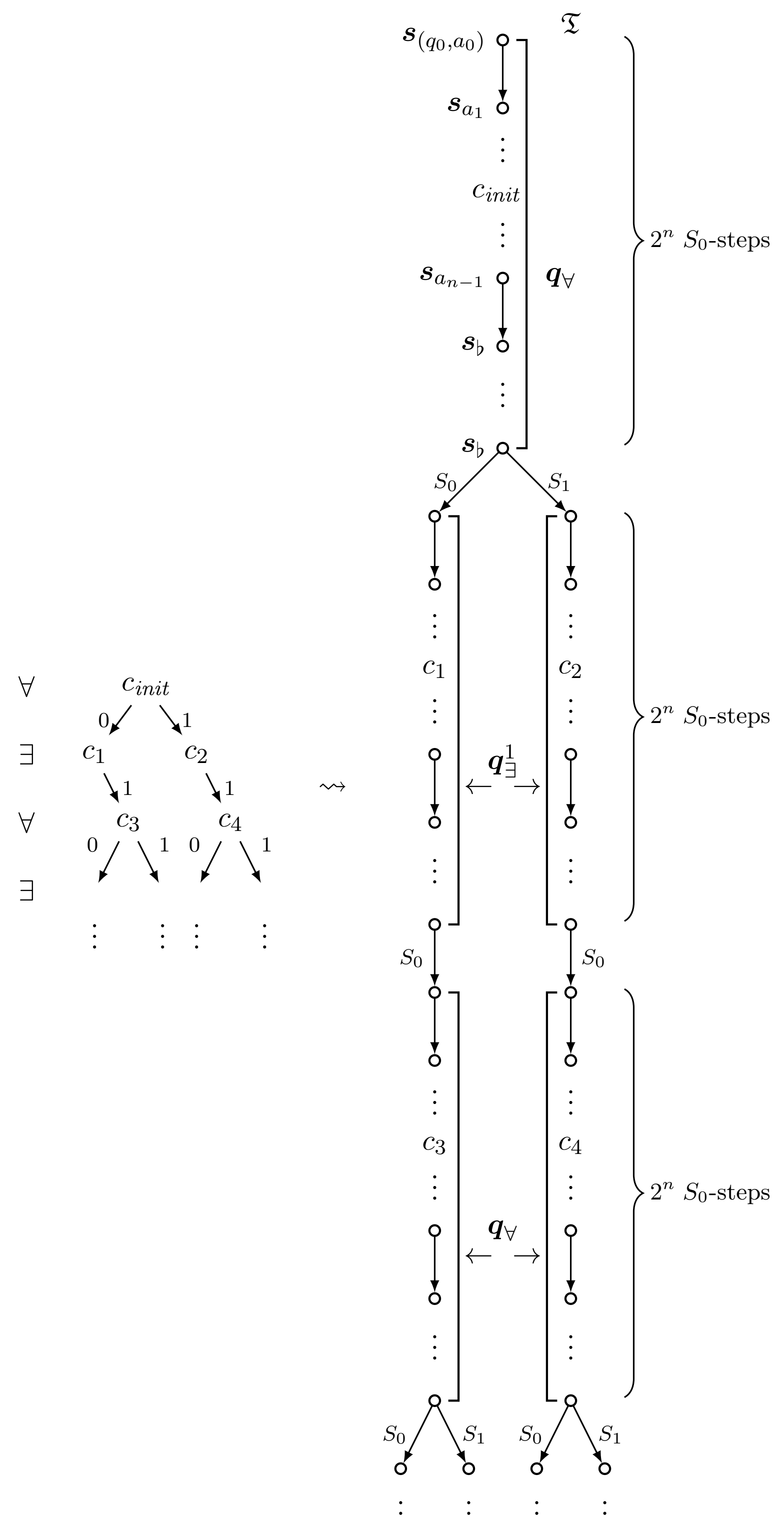


Figure 1: Representing accepting computation-trees.

$$\boldsymbol{z}^{I(w_m^t)} = \{d_m^{t'} \mid S_0(t,t')\}, \tag{4.43}$$

$$\boldsymbol{t}^{I(w_m^t)} = \{d_m^t\}, \tag{4.44}$$

$$\boldsymbol{p}^{I(w_m^t)} = \begin{cases} \{d_m^t\}, & \text{if } \boldsymbol{p}(t) \text{ holds in } \mathfrak{T}, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4.45}$$

Next, we define I for the symbols not in σ. The $\boldsymbol{h}_i^B$- and $\boldsymbol{v}_i^B$-predicates, for $i < n$, set up a binary counter counting from 0 to $2^n - 1$ on pairs (w_0^r, d_0^r), …, $(w_{2^n-1}^r, d_{2^n-1}^r)$ in such a way that they are

- stable within each $W_m \times D_m$, $m < 2^n$: if $\mathfrak{M}, w_m^r, d_m^r \models \boldsymbol{h}_i^B$ then $\mathfrak{M}, w, d \models \boldsymbol{h}_i^B$ for all $w \in W_m$, $d \in D_m$; if $\mathfrak{M}, w_m^r, d_m^r \models \boldsymbol{v}_i^B$ then $\mathfrak{M}, w, d \models \boldsymbol{v}_i^B$ for all $w \in W_m$, $d \in D_m$;
- the $\boldsymbol{h}_i^B$-predicates are modally-stable: if $\mathfrak{M}, w, d \models \boldsymbol{h}_i^B$ for some $w \in W$ and $d \in D$ then $\mathfrak{M}, w', d \models \boldsymbol{h}_i^B$ for all $w' \in W$;
- the $\boldsymbol{v}_i^B$-predicates are FO-stable: if $\mathfrak{M}, w, d \models \boldsymbol{v}_i^B$ for some $w \in W$ and $d \in D$ then $\mathfrak{M}, w, d' \models \boldsymbol{v}_i^B$ for all $d' \in D$.

We let, for all $m < 2^n$ and $t \in T$,

$$(\boldsymbol{n}^B)^{I(w_m^t)} = \begin{cases} \{d_{m+1}^t\}, & \text{if } m < 2^n - 1 \text{ and } t = r, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For each $m < 2^n$, the $\boldsymbol{h}_i^U$- and $\boldsymbol{v}_i^U$-predicates, for $i < n$, set up a binary counter counting from 0 modulo 2^n infinitely along the levels of the tree $\mathfrak{T}$, on pairs of the form (w_m^t, d_m^t), for $t \in T$. The $\boldsymbol{h}_i^U$-predicates are modally-stable, while the $\boldsymbol{v}_i^U$-predicates are FO-stable, in the above sense.

Then, for each $m < 2^n$, the modally-stable $\boldsymbol{h}_i^V$- and the FO-stable $\boldsymbol{v}_i^V$-predicates set up a binary counter counting from $2^n - 1 - m$ modulo 2^n infinitely along the levels of the tree $\mathfrak{T}$, on pairs of the form (w_m^t, d_m^t), for $t \in T$. Also, we extend the FO-structure $\mathfrak{T}$ to $\mathfrak{T}_m^+$ by adding unary predicates $\boldsymbol{s}_\gamma^0$, $\boldsymbol{s}_\gamma^1$, $\boldsymbol{s}_\gamma^+$, for $\gamma \in \Gamma \cup (Q \times \Gamma)$, see Fig. 2. For all $m < 2^n$, $t \in T$, $\boldsymbol{p} \in \{\boldsymbol{s}_\gamma^0, \boldsymbol{s}_\gamma^1, \boldsymbol{s}_\gamma^+ \mid \gamma \in \Gamma \cup (Q \times \Gamma)\}$, we let

$$\boldsymbol{p}^{I(w_m^t)} = \begin{cases} \{d_m^t\}, & \text{if } \boldsymbol{p}(t) \text{ holds in } \mathfrak{T}_m^+, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is readily checked that $\mathfrak{M}, w_0^r, d_0^r \models \varphi$.

Next, we define a model $\hat{\mathfrak{M}} = (\hat{W}, \hat{D}, \hat{I})$ making $\neg\psi$ true. We take four disjoint copies $\hat{W}_0, \hat{W}_1$ and $\hat{D}_0, \hat{D}_1$ of T and let $\hat{W} = \hat{W}_0 \cup \hat{W}_1$ and $\hat{D} = \hat{D}_0 \cup \hat{D}_1$. For each $k < 2$ and $t \in T$, let $\hat{w}_m^t$ and $\hat{d}_m^t$ denote the copy of t in $\hat{W}_k$ and $\hat{D}_k$, respectively. Now, for symbols in σ we define $\hat{I}$ similarly to I in (4.40)–(4.45) above. For symbols not in σ the only ones with non-empty $\hat{I}$-extensions are the $\boldsymbol{b}_i$, for $i < 2$: For all $i, k < 2$, $t \in \mathfrak{T}$, we let

$$\boldsymbol{b}_i^{\hat{I}(\hat{w}_k^t)} = \begin{cases} \{\hat{d}_i^{t'} \mid t' \in T\}, & \text{if } k = i, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is readily checked that $\hat{\mathfrak{M}}, \hat{w}_0^r, \hat{d}_0^r \models \neg\psi$.

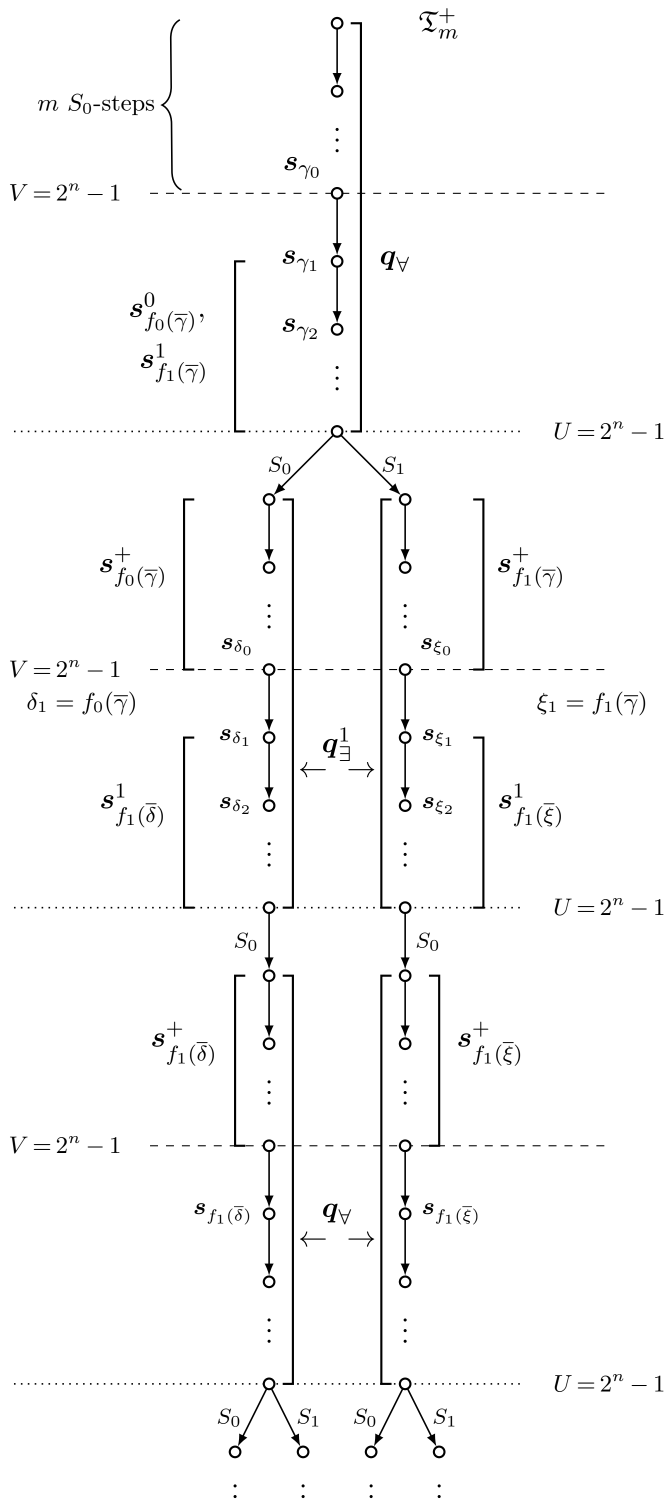
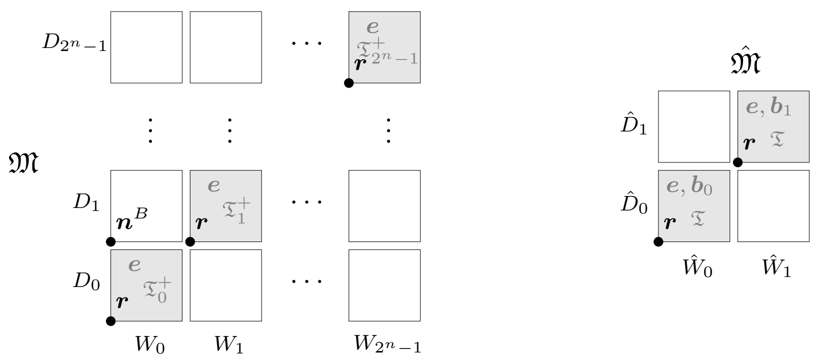


Figure 2: Passing information from one configuration to the next.



Finally, we define a relation $\boldsymbol{\beta} \subseteq (W \times D) \times (\hat{W} \times \hat{D})$ by taking, for any $w, d, \hat{w}, \hat{d}$, $\big((w,d),(\hat{w},\hat{d})\big) \in \boldsymbol{\beta}$ iff there exist $t, t' \in T$, $m, m' < 2^n$, $k, k' < 2$ such that $w = w^t_m$, $d = d^{t'}_{m'}$, $\hat{w} = \hat{w}^t_k$, $\hat{d} = \hat{d}^{t'}_{k'}$, and $m = m'$ iff $k = k'$. It is not hard to show that $\boldsymbol{\beta}$ is a σ-bisimulation between $\mathfrak{M}$ and $\hat{\mathfrak{M}}$ with $\big((w^r_0, d^r_0),(\hat{w}^r_0,\hat{d}^r_0)\big) \in \boldsymbol{\beta}$. ⊣

Lemma 4.9. *If $n > 1$ and φ, $\neg\psi$ are σ-bisimulation consistent, then M accepts $\bar{a}$.*

Proof. Let $\mathfrak{M}, w_0, d_0$ and $\mathfrak{M}', w'_0, d'_0$ be models with $\mathfrak{M}, w_0, d_0 \models \varphi$, $\mathfrak{M}', w'_0, d'_0 \models \neg\psi$, and $\mathfrak{M}, w_0, d_0 \sim_\sigma \mathfrak{M}', w'_0, d'_0$. Let $(w_0, d_0), \dots, (w_{2^n-1}, d_{2^n-1})$ be subsequent $\boldsymbol{r}$-points in $\mathfrak{M}$ generated by (4.9)–(4.13), with the respective B-values $0, \dots, 2^n - 1$. By (4.14) and (4.38), we have $\mathfrak{M}, w_i, d_i \models \boldsymbol{e}$, for $i < 2^n$. We claim that for all $i, j < 2^n$ there are w_{ij}, d_{ij} such that

$$\mathfrak{M}, w_{ij}, d_{ij} \models [B = i], \tag{4.46}$$

$$\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}, w_j, d_j. \tag{4.47}$$

Indeed, to begin with, there are w'_i, d'_i such that $\mathfrak{M}, w_i, d_i \sim_\sigma \mathfrak{M}', w'_i, d'_i$, and so $\mathfrak{M}', w'_i, d'_i \models \boldsymbol{e}$. Let $k < 2^n$ and $k \neq i, j$. As the B-values of (w_i, d_i) and (w_k, d_k) are different, $\mathfrak{M}, w_k, d_i \models \neg\boldsymbol{e}$ follows by (4.39). Thus, there exist w'_k, d'_k such that $\mathfrak{M}', w'_k, d'_i \models \neg\boldsymbol{e}$, $\mathfrak{M}', w'_k, d'_k \models \boldsymbol{e}$, and $\mathfrak{M}, w_k, d_k \sim_\sigma \mathfrak{M}', w'_k, d'_k$. Similarly, there exist w'_j, d'_j such that $\mathfrak{M}', w'_j, d'_k \models \neg\boldsymbol{e}$, $\mathfrak{M}', w'_j, d'_j \models \boldsymbol{e}$, and $\mathfrak{M}, w_j, d_j \sim_\sigma \mathfrak{M}', w'_j, d'_j$, see Fig. 3.

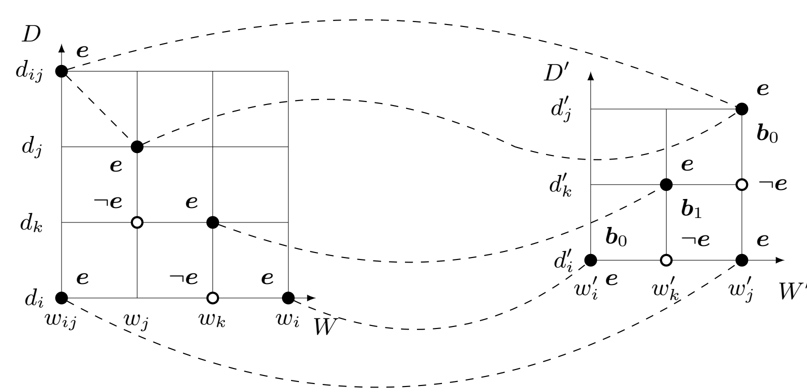


Figure 3: Enforcing 2^n σ-bisimilar trees.

As $\mathfrak{M}', w'_0, d'_0 \models \neg\psi$, we have $\mathfrak{M}', w'_i, d'_i \models \boldsymbol{b}_s$ for $s = 0$ or $s = 1$. Suppose $s = 0$ (the other case is similar). It follows from $\neg\psi$ that $\mathfrak{M}', w'_k, d'_k \models \boldsymbol{b}_1$ and $\mathfrak{M}', w'_j, d'_j \models \boldsymbol{b}_0$.

Then $\mathfrak{M}', w'_j, d'_i \models \boldsymbol{e}$ also follows from $\neg\psi$. As $\mathfrak{M}, w_i, d_i \sim_\sigma \mathfrak{M}', w'_i, d'_i$, there are w_{ij}, d_{ij} with $\mathfrak{M}, w_{ij}, d_i \models \boldsymbol{e}$, $\mathfrak{M}, w_{ij}, d_{ij} \models \boldsymbol{e}$ and $\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}', w'_j, d'_j$. Thus, $\mathfrak{M}, w_{ij}, d_{ij} \sim_\sigma \mathfrak{M}, w_j, d_j$, and so (4.47) holds. We also have $\mathfrak{M}, w_{ij}, d_{ij} \models \mathsf{equ}^B$ by (4.39), and so (4.46) follows from (4.8).

In particular, (4.46) and (4.47) imply that, for each $m < 2^n$, there exist w^+_m, d^+_m such that $\mathfrak{M}, w^+_m, d^+_m \models [B = m]$ and $\mathfrak{M}, w^+_m, d^+_m \sim_\sigma \mathfrak{M}, w_0, d_0$. Take the U-tree $\mathfrak{T}_0$ grown from (w_0, d_0) by (4.14)–(4.23). Then it has a σ-bisimilar copy $\mathfrak{T}'_m$ grown from each (w^+_m, d^+_m). Choose a computation-tree 'skeleton' from $\mathfrak{T}_0$ determined by its $\boldsymbol{q}^i_\exists$ labels. As $\boldsymbol{e}, \boldsymbol{r}, \boldsymbol{z}, \boldsymbol{n}^U, \boldsymbol{t}, \boldsymbol{q}_\forall, \boldsymbol{q}^i_\exists \in \sigma$, the formulas (4.24)–(4.37) imply that the $\boldsymbol{s}_\gamma$ labels in $\mathfrak{T}'_m$ properly describe the 'evolution' of the mth tape-cell's content via the chosen 'skeleton'. As all $\boldsymbol{s}_\gamma$ are in σ, using its $\boldsymbol{s}_\gamma$ labels and (4.29), we can extract an accepting computation-tree from $\mathfrak{T}_0$ (with all tape-cell contents evolving properly). ⊣

This completes the lower bound proof of Theorem 1.2 for $\mathsf{Q}^1\mathsf{S5}$. The lower bound result of Theorem 1.3 for equality-free FO^2 is proved in Appendix A.

Note that the 2ExpTime lower bound results of Theorem 1.2 hold even if we want to decide, for any FOM^1-formulas φ and ψ, whether an interpolant or an explicit definition exists not only in $\mathsf{Q}^1\mathsf{S5}$ but in any finite-variable fragment of first-order $\mathsf{S5}$. More precisely, we claim that, for any $n, \ell < \omega$ with $2^n > \ell + 1$, given a 2^n-space bounded ATM M and an input word $\overline{a}$ of length n, there exist polytime FOM^1-formulas φ and ψ_ℓ such that

(1) $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi_\ell$, and
(2) M accepts $\overline{a}$ iff φ, $\neg\psi_\ell$ are σ-bisimulation consistent in the ℓ-variable fragment of quantified $\mathsf{S5}$, where $\sigma = sig(\varphi) \cap sig(\psi_\ell)$.

Indeed, φ is the same as before and ψ_ℓ is similar to $\psi = \psi_1$ above: we just divide $\boldsymbol{e}$ not into two but $\ell + 1$ parts, using fresh variables $\boldsymbol{b}_0, \dots, \boldsymbol{b}_\ell$. The proof that these modifications work is similar to the proof above.

5. The (S)CEP and UIEP in $\mathsf{Q}^1\mathsf{S5}$ are Undecidable

We now turn to the (strong) conservative extension and uniform interpolant existence problems, which, in contrast to interpolant existence, turn out to be undecidable. We show the following refinement of Theorem 1.4.

Theorem 5.1. (*i*) *The (S)CEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*
(*ii*) *The UIEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*

The undecidability proof for the CEP is by adapting an undecidability proof for the CEP of FO^2 in [JLM⁺17]. The main new idea here is the generation of arbitrary large binary trees within $\mathsf{Q}^1\mathsf{S5}$-models that can then be forced to be grids in case one does not have a (strong) conservative extension. The proof is by reduction of the following undecidable tiling problem. By a *tiling system* we mean a tuple $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow)$, where T is a finite set of *tiles* with $\boldsymbol{o}, \boldsymbol{z}^\uparrow, \boldsymbol{z}^\rightarrow \in T$, and $H, V \subseteq T \times T$ are horizontal and vertical *matching relations*. We say that $\mathfrak{T}$ *has a solution* if there exists a triple (n, m, τ), where $0 < n, m < \omega$ and $\tau \colon \{0, \dots, n-1\} \times \{0, \dots, m-1\} \to T$, such that the following hold, for all $i < n$ and $j < m$:

(t1) if $i < n - 1$ then $(\tau(i,j), \tau(i+1,j)) \in H$;
(t2) if $j < m - 1$ then $(\tau(i,j), \tau(i,j+1)) \in V$;
(t3) $\tau(i,j) = \boldsymbol{o}$ iff $i = j = 0$;

(t4) $\tau(i,j) = \boldsymbol{z}^{\rightarrow}$ iff $i = n-1$, and $\tau(i,j) = \boldsymbol{z}^{\uparrow}$ iff $j = m-1$.

The reader can easily show by reduction of the halting problem for Turing machines that it is undecidable whether a given tiling system has a solution; cf. [vEB97].

For any tiling system $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^{\uparrow}, \boldsymbol{z}^{\rightarrow})$, we show how to construct in polytime formulas φ and ψ such that $\mathfrak{T}$ has a solution iff $\varphi \wedge \psi$ is not a (strong) conservative extension of φ. For any model $\mathfrak{M} = (W, D, I)$, we mark the points on the finite grid to be tiled by a predicate $\boldsymbol{g}$, that is, we let $\boldsymbol{g}^{\mathfrak{M}} = \{(w,d) \in W \times D \mid d \in \boldsymbol{g}^{I(w)}\}$. Then we define the intended 'horizontal' and 'vertical' neighbour relations $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$ on the grid by setting

$$R_h^{\mathfrak{M}} = \big\{\big((w,d),(w',d')\big) \in \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}} \mid (w',d) \models \boldsymbol{x},\ (w,d) \models \neg \boldsymbol{z}^{\rightarrow}\big\}, \tag{5.1}$$

$$R_v^{\mathfrak{M}} = \big\{\big((w,d),(w',d')\big) \in \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}} \mid (w,d') \models \boldsymbol{y},\ (w,d) \models \neg \boldsymbol{z}^{\uparrow}\big\}. \tag{5.2}$$

We set, for any formula χ: $\Box_h \chi = \neg \Diamond_h \neg \chi$, $\Box_v \chi = \neg \Diamond_v \neg \chi$,

$$\Diamond_h \chi = \boldsymbol{g} \wedge \neg \boldsymbol{z}^{\rightarrow} \wedge \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \chi)\big), \qquad \Diamond_v \chi = \boldsymbol{g} \wedge \neg \boldsymbol{z}^{\uparrow} \wedge \exists\big(\boldsymbol{y} \wedge \Diamond(\boldsymbol{g} \wedge \chi)\big).$$

Now φ uses the following conjuncts to generate the grid:

$$\boldsymbol{o} \wedge \boldsymbol{g} \wedge \Box\forall\big(\boldsymbol{g} \wedge \neg(\boldsymbol{z}^{\uparrow} \wedge \boldsymbol{z}^{\rightarrow}) \rightarrow \Diamond \boldsymbol{x}\big) \wedge \Box\forall(\boldsymbol{x} \rightarrow \exists \boldsymbol{g}),$$

$$\Box\forall(\boldsymbol{g} \wedge \neg \boldsymbol{z}^{\uparrow} \rightarrow \exists \boldsymbol{y}), \tag{5.3}$$

$$\Box\forall(\boldsymbol{y} \rightarrow \Diamond \boldsymbol{g}). \tag{5.4}$$

Next, we regard each tile $\boldsymbol{t} \in T$ as a fresh predicate, and we add the following conjuncts to φ, expressing the constraints for the tiles:

$$\Box\forall(\boldsymbol{g} \wedge \neg \Diamond_h \top \rightarrow \boldsymbol{z}^{\rightarrow}), \tag{5.5}$$

$$\Box\forall(\boldsymbol{g} \leftrightarrow \bigvee_{\boldsymbol{t} \in T} \boldsymbol{t}), \tag{5.6}$$

$$\Box\forall \bigwedge_{\boldsymbol{t} \neq \boldsymbol{t}'} (\boldsymbol{t} \rightarrow \neg \boldsymbol{t}'), \tag{5.7}$$

$$\Box\forall\big(\boldsymbol{t} \rightarrow \Box_h \bigvee_{(\boldsymbol{t},\boldsymbol{t}') \in H} \boldsymbol{t}'\big), \tag{5.8}$$

$$\Box\forall\big(\boldsymbol{t} \rightarrow \Box_v \bigvee_{(\boldsymbol{t},\boldsymbol{t}') \in V} \boldsymbol{t}'\big), \tag{5.9}$$

$$\Box\forall(\boldsymbol{g} \rightarrow \neg \Diamond_h \boldsymbol{o} \wedge \neg \Diamond_v \boldsymbol{o}), \tag{5.10}$$

$$\Box\forall\big((\boldsymbol{z}^{\rightarrow} \rightarrow \Box_v \boldsymbol{z}^{\rightarrow}) \wedge (\Diamond_v \boldsymbol{z}^{\rightarrow} \rightarrow \boldsymbol{z}^{\rightarrow})\big), \tag{5.11}$$

$$\Box\forall\big((\boldsymbol{z}^{\uparrow} \rightarrow \Box_h \boldsymbol{z}^{\uparrow}) \wedge (\Diamond_h \boldsymbol{z}^{\uparrow} \rightarrow \boldsymbol{z}^{\uparrow})\big). \tag{5.12}$$

Let $\sigma = sig(\varphi) = \{\boldsymbol{g}, \boldsymbol{x}, \boldsymbol{y}\} \cup \{\boldsymbol{t} \mid \boldsymbol{t} \in T\}$.

Note that we have not yet forced $R_h^{\mathfrak{M}}, R_v^{\mathfrak{M}}$ to form a grid-like structure on $\boldsymbol{g}^{\mathfrak{M}}$-points. We say that a $\boldsymbol{g}^{\mathfrak{M}}$-point (w,d) is *confluent* if, for every $R_h^{\mathfrak{M}}$-successor (w_h, d_h) and every $R_v^{\mathfrak{M}}$-successor (w_v, d_v) of (w,d), there is (w', d') that is both an $R_v^{\mathfrak{M}}$-successor of (w_h, d_h) and an $R_h^{\mathfrak{M}}$-successor of (w_v, d_v). Forcing the grid to be finite and confluence of all grid-points are achieved using the formula ψ, which contains two additional predicates, $\boldsymbol{q}$ and $\boldsymbol{s}$, behaving like second-order variables over grid-points. We set

$$\psi = \boldsymbol{q} \wedge \Box\forall\big(\boldsymbol{q} \rightarrow \Diamond_h \boldsymbol{q} \vee \Diamond_v \boldsymbol{q} \vee (\Diamond_v \Box_h \boldsymbol{s} \wedge \Diamond_h \Box_v \neg \boldsymbol{s})\big).$$

It is readily seen that if $\mathfrak{M}, w_0, d_0 \models \varphi$, for some model $\mathfrak{M}$, then the following are equivalent:

(c1) $\mathfrak{M}', w_0, d_0 \models \psi$, for some model $\mathfrak{M}' = (W, D, I')$ with I' the same as I on all predicates save possibly $\boldsymbol{q}$ and $\boldsymbol{s}$ (we call such an $\mathfrak{M}'$ a *variant* of $\mathfrak{M}$);

(c2) $\mathfrak{M}$ contains an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-path starting at (w_0, d_0) or a non-confluent $\boldsymbol{g}^{\mathfrak{M}}$-point accessible from (w_0, d_0) via an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-path.

Lemma 5.2. *If $\mathfrak{T}$ has a solution, then $\varphi \wedge \psi$ is not a conservative extension of φ.*

Proof. Let (n, m, τ) be a solution to $\mathfrak{T}$. We enumerate the points of the $n \times m$-grid starting from the first horizontal row $(0,0), \dots, (n-1, 0)$, then continuing with the second row $(0,1), \dots, (n-1, 1)$, etc. We define a model $\mathfrak{N} = (W, D, J)$ with $W = D = \{0, \dots, nm-1\}$ that represents this enumeration as follows (remember that $\boldsymbol{o}, \boldsymbol{z}^{\uparrow}, \boldsymbol{z}^{\rightarrow} \in T$, $\boldsymbol{z}^{\uparrow}$ marks the tiles of last row, and $\boldsymbol{z}^{\rightarrow}$ marks the tiles of the last column). For all $k < nm$ and $\boldsymbol{t} \in T$,

$$\boldsymbol{g}^{J(k)} = \{k\}, \tag{5.13}$$

$$\boldsymbol{x}^{J(k)} = \begin{cases} \{k-1\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{5.14}$$

$$\boldsymbol{y}^{J(k)} = \begin{cases} \{k+n\}, & \text{if } k < nm - n, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{5.15}$$

$$\boldsymbol{t}^{J(k)} = \begin{cases} \{k\}, & \text{if } k = jn + i \text{ and } \tau(i,j) = \boldsymbol{t}, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{5.16}$$

For $n = m = 3$, the model $\mathfrak{N}$ (without tiles other than $\boldsymbol{o}, \boldsymbol{z}^{\rightarrow}, \boldsymbol{z}^{\uparrow}$) and the relations $R_h^{\mathfrak{N}}$, $R_v^{\mathfrak{N}}$ are illustrated in Fig. 4. It is easy to check that $\mathfrak{N}, 0, 0 \models \varphi$ and $\mathfrak{N}', 0, 0 \models \neg\psi$, for any variant $\mathfrak{N}'$ of $\mathfrak{N}$.

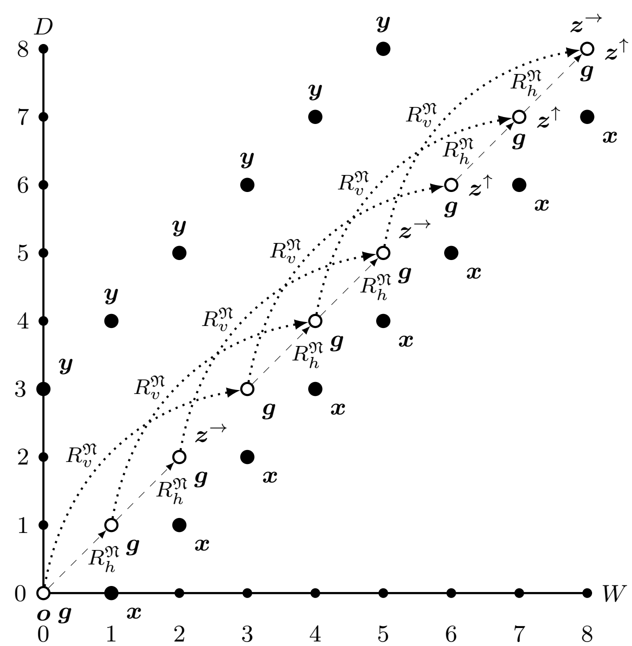


Figure 4: The model $\mathfrak{N}$.

We construct a formula χ such that $sig(\chi) = \sigma$, $\models_{\mathsf{Q^1S5}} \psi \wedge \varphi \to \neg\chi$ but $\not\models_{\mathsf{Q^1S5}} \varphi \to \neg\chi$, which means that $\varphi \wedge \psi$ is not a (strong) conservative extension of φ. Intuitively, the formula

χ characterises the model $\mathfrak{N}$ at $(0,0)$. First, for every $(i,j) \in W \times D$, we construct a formula $\varphi_{i,j}$ such that, for all $(i',j') \in W \times D$,

$$\mathfrak{N}, i', j' \models \varphi_{i,j} \quad \text{iff} \quad (i',j') = (i,j). \tag{5.17}$$

For instance, we can set inductively

$$\begin{aligned} \varphi_{0,0} &= \boldsymbol{o}, \\ \varphi_{i+1,i+1} &= \boldsymbol{g} \wedge \exists(\boldsymbol{x} \wedge \Diamond\varphi_{i,i}), \\ \varphi_{i,j} &= \exists\varphi_{i,i} \wedge \Diamond\varphi_{j,j}, \text{ for } i \neq j. \end{aligned}$$

Now let

$$\chi_{i,j} = \varphi_{i,j} \wedge \bigwedge_{\boldsymbol{p}\in\sigma,\ \mathfrak{N},i,j\models\boldsymbol{p}} \boldsymbol{p} \ \wedge \bigwedge_{\boldsymbol{p}\in\sigma,\ \mathfrak{N},i,j\models\neg\boldsymbol{p}} \neg\boldsymbol{p}, \tag{5.18}$$

and let χ be the conjunction of

$$\chi_{0,0}, \tag{5.19}$$

$$\Box\forall(\chi_{i,i} \to \Box_h\chi_{i+1,i+1}), \text{ for } i < nm-1, \tag{5.20}$$

$$\Box\forall(\chi_{i,i} \to \Box_v\chi_{i+n,i+n}), \text{ for } i < nm-n, \tag{5.21}$$

$$\Box\forall(\chi_{i,i} \to \Diamond\chi_{j,i}), \text{ for } i,j < nm, \tag{5.22}$$

$$\Box\forall(\chi_{i,i} \to \exists\chi_{i,j}), \text{ for } i,j < nm, \tag{5.23}$$

$$\Box\forall(\Diamond\chi_{l,i} \wedge \exists\chi_{i,j} \to \chi_{i,i}), \text{ for } i,j,l < nm. \tag{5.24}$$

Using (5.17), it is easy to see that $\mathfrak{N}, 0, 0 \models \chi$, and so $\varphi \wedge \chi$ is satisfiable, i.e., $\not\models_{\mathsf{Q^1S5}} \varphi \to \neg\chi$. Now suppose that $\mathfrak{M}$ is any model such that $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \chi$ for some w_0, d_0. Using the equivalence **(c1)** $\Leftrightarrow$ **(c2)**, we show that $\mathfrak{M}, w_0, d_0 \models \neg\psi$, which implies $\models_{\mathsf{Q^1S5}} \psi \wedge \varphi \to \neg\chi$.

To begin with, by (5.1), (5.2), the definition of $\mathfrak{N}$, and (5.18)–(5.21), there cannot exist an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain. Now suppose there is an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain from (w_0, d_0) to some node (w,d) with an $R_h^{\mathfrak{M}}$-successor (w_1,d_1) and $R_v^{\mathfrak{M}}$-successor (w_2,d_2). Then $\mathfrak{M}, w, d \models \chi_{i,i}$ for some i, $\mathfrak{M}, w_1, d_1 \models \chi_{i+1,i+1}$ and $\mathfrak{M}, w_2, d_2 \models \chi_{i+n,i+n}$, by (5.19)–(5.21). By (5.22) and (5.23), there exist d_1' with $\mathfrak{M}, w_1, d_1' \models \chi_{i+1,i+n+1}$, and w_2' with $\mathfrak{M}, w_2', d_2 \models \chi_{i+n+1,i+n}$. By (5.24), $\mathfrak{M}, w_2', d_1' \models \chi_{i+n+1,i+n+1}$. Moreover, as by (5.18), $\boldsymbol{z}^{\uparrow}$ is not a conjunct of $\chi_{i+1,i+1}$, and $\boldsymbol{z}^{\rightarrow}$ is not a conjunct of $\chi_{i+n,i+n}$, we have that $\boldsymbol{y}$ is a conjunct of $\chi_{i+1,i+n+1}$, and $\boldsymbol{x}$ is a conjunct of $\chi_{i+n+1,i+n}$. Thus, by (5.1) and (5.2), $((w_1,d_1),(w_2',d_1')) \in R_v^{\mathfrak{M}}$ and $((w_2,d_2)(w_2',d_1')) \in R_h^{\mathfrak{M}}$, and so (w,d) is confluent. ⊣

We say that a formula α is a *model conservative extension* of a formula β if $\models_L \alpha \to \beta$ and, for any model $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \beta$, there exists a model $\mathfrak{M}'$ with $\mathfrak{M}', w, d \models \alpha$, which coincides with $\mathfrak{M}$ except for the interpretation of the predicates in $sig(\alpha) \setminus sig(\beta)$. Clearly, if α is a model conservative extension of β, then α is also a strong conservative extension of β. Thus, if $\varphi \wedge \psi$ in our proof is not a conservative extension of φ, then $\varphi \wedge \psi$ is not a model conservative extension of φ.

Lemma 5.3. *If $\varphi \wedge \psi$ is not a conservative extension of φ, then $\mathfrak{T}$ has a solution.*

Proof. Consider a model $\mathfrak{M} = (W, D, I)$ such that $\mathfrak{M}, w, d \models \varphi$ but $\mathfrak{M}', w, d \models \neg\psi$ in any variant $\mathfrak{M}'$ of $\mathfrak{M}$. Using the equivalence **(c1)** $\Leftrightarrow$ **(c2)**, one can easily find within $\mathfrak{M}$ a finite grid-shaped (with respect to $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$) submodel, which gives a solution to $\mathfrak{T}$.

For instance, we can start by taking an $R_h^{\mathfrak{M}}$-path of $\boldsymbol{g}^{\mathfrak{M}}$-points

$$(w,d) = (w_0^0, d_0^0) R_h^{\mathfrak{M}} (w_1^0, d_1^0) R_h^{\mathfrak{M}} \dots R_h^{\mathfrak{M}} (w_{n-1}^0, d_{n-1}^0),$$

for some $n > 0$, that ends with the first point (w^0_{n-1}, d^0_{n-1}) such that $\mathfrak{M}, w^0_{n-1}, d^0_{n-1} \models \boldsymbol{z}^{\rightarrow}$. Such a path must exist by $\neg$**(c2)** and (5.5). The chosen points (w^0_i, d^0_i) form the first row of the required grid.

Next, observe that $\Box\forall(\boldsymbol{g} \land \neg\Diamond_v\top \to \boldsymbol{z}^{\uparrow})$ follows from (5.3) and (5.4). So, similarly to the above, by $\neg$**(c2)** we can take an $R^{\mathfrak{M}}_v$-path

$$(w^0_0, d^0_0) R^{\mathfrak{M}}_v (w^1_0, d^1_0) R^{\mathfrak{M}}_v \dots R^{\mathfrak{M}}_h (w^{m-1}_0, d^{m-1}_0),$$

for some $m > 0$, that ends with the first point (w^{m-1}_0, d^{m-1}_0) such that $\mathfrak{M}, w^{m-1}_0, d^{m-1}_0 \models \boldsymbol{z}^{\uparrow}$. It forms the first column of the grid. By $\neg$**(c2)** again, the point (w^0_0, d^0_0) is confluent, and so we find (w^1_1, d^1_1) with

$$(w^0_1, d^0_1) R^{\mathfrak{M}}_v (w^1_1, d^1_1) \quad \text{and} \quad (w^1_0, d^1_0) R^{\mathfrak{M}}_h (w^1_1, d^1_1).$$

Similarly, we find the remaining $\boldsymbol{g}^{\mathfrak{M}}$-points (w^j_i, d^j_i) for the whole $n \times m$-grid. By (5.6) and (5.7), each (w^j_i, d^j_i) makes exactly one tile $\boldsymbol{t} \in T$ true. By (5.8) and (5.9), the matching conditions of **(t1)** and **(t2)** are satisfied. By (5.10), we have **(t3)**. Finally, **(t4)** is satisfied by (5.11) and (5.12). ⊣

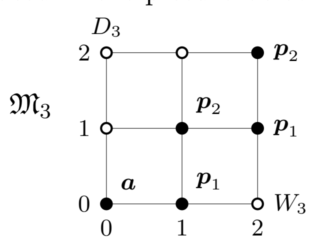
This completes the proof of Theorem 5.1 (*i*).

The undecidability proof for the UIEP merges the counterexample to the UIP from Example 5.4 below with the formulas constructed to prove the undecidability of CEP above.

Example 5.4. Suppose $\sigma = \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}$ and

$$\varphi_0 = \Box\forall\big(\boldsymbol{a} \to \Diamond(\boldsymbol{p}_1 \land \boldsymbol{b})\big) \land \Box\forall\big(\boldsymbol{p}_1 \land \boldsymbol{b} \to \exists(\boldsymbol{p}_2 \land \boldsymbol{b})\big) \land \Box\forall\big(\boldsymbol{p}_2 \land \boldsymbol{b} \to \Diamond(\boldsymbol{p}_1 \land \boldsymbol{b})\big).$$

To show that $\boldsymbol{a} \land \varphi_0$ has no σ-uniform interpolant in $\mathsf{Q^1S5}$, for every positive $r < \omega$, we define a formula χ_r inductively by taking $\chi_0 = \top$ and $\chi_{r+1} = \boldsymbol{p}_1 \land \exists(\boldsymbol{p}_2 \land \Diamond\chi_r)$. Then $\models_{\mathsf{Q^1S5}} \boldsymbol{a} \land \varphi_0 \to \Diamond\chi_r$ for all $r > 0$. Thus, if ϱ were a σ-uniform interpolant of $\boldsymbol{a} \land \varphi_0$, then $\models_{\mathsf{Q^1S5}} \varrho \to \Diamond\chi_r$ would follow for all $r > 0$. Consider a model $\mathfrak{M}_r = (W_r, D_r, I_r)$ with $W_r = D_r = \{0, \dots, r-1\}$, in which $\boldsymbol{a}$ is true at $(0,0)$, $\boldsymbol{p}_1$ at $(k, k-1)$, and $\boldsymbol{p}_2$ at (k,k), for $0 < k < r$, as illustrated in the picture below:



$$\mathfrak{M}_3, 0, 0 \models \chi_2$$
$$\mathfrak{M}_3, 0, 0 \not\models \chi_3$$

Then $\mathfrak{M}_r, 0, 0 \not\models \Diamond\chi_r$, for any $r > 0$, and so $\mathfrak{M}_r, 0, 0 \not\models \varrho$. On the other hand, $\mathfrak{M}_r, 0, 0 \models \Diamond\chi_{r'}$ for all $r' < r$. Now consider the ultraproduct $\prod_U \mathfrak{M}_r$ with a non-principal ultrafilter U on $\omega \setminus \{0\}$ (we refer the reader to [CK98] for the definition and relevant properties of ultrafilters and ultraproducts). As each $\Diamond\chi_{r'}$ is true at $(0,0)$ in almost all $\mathfrak{M}_r$, it follows from the properties of ultraproducts that, for a suitable $\overline{0}$ and all $r > 0$, we have $\prod_U \mathfrak{M}_r, \overline{0}, \overline{0} \models \boldsymbol{a} \land \neg\varrho \land \Diamond\chi_r$. One can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{M}_r$ so that $\mathfrak{M}, \overline{0}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{M}$. Then $\mathfrak{M} \models \boldsymbol{a} \land \varphi_0 \land \neg\varrho$, contrary to $\models_{\mathsf{Q^1S5}} \boldsymbol{a} \land \varphi_0 \to \varrho$, for any uniform interpolant ϱ of $\boldsymbol{a} \land \varphi_0$. ⊣

We now come to the proof of Theorem 5.1 (*ii*). Take $\mathfrak{T}$, φ, and ψ from the proof of Theorem 5.1 (*i*). Using fresh predicates $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}_1, \boldsymbol{p}_2$, and φ_0 from Example 5.4, set

$$\varphi' = \varphi \land (\boldsymbol{p}_1 \to \boldsymbol{p}_1) \land (\boldsymbol{p}_2 \to \boldsymbol{p}_2), \qquad \psi' = (\psi \lor \boldsymbol{a}) \land \varphi_0.$$

Let $\sigma = sig(\varphi')$. We show that it is undecidable whether there exists a σ-uniform interpolant of $\varphi' \wedge \psi'$ in $\mathsf{Q^1S5}$.

Lemma 5.5. *If there is no σ-uniform interpolant of $\varphi' \wedge \psi'$ in $\mathsf{Q^1S5}$, then $\mathfrak{T}$ has a solution.*

Proof. As the assumption implies that $\varphi' \wedge \psi'$ is not a model conservative extension of φ', the proof is a straightforward variant of the proof of Lemma 5.3. Consider a model $\mathfrak{M} = (W, D, I)$ with $\mathfrak{M}, w, d \models \varphi'$ but $\mathfrak{M}', w, d \models \neg\psi'$ in any variant $\mathfrak{M}'$ of $\mathfrak{M}$ obtained by interpreting the predicates $\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{a}, \boldsymbol{b}$. In particular, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are both interpreted as $\emptyset$ in all worlds, then $\mathfrak{M}', w, d \models \varphi_0$, and so $\mathfrak{M}', w, d \models \neg\psi$ follows. But this case is considered in the proof of Lemma 5.3. ⊣

Lemma 5.6. *If $\mathfrak{T}$ has a solution, then there is no σ-uniform interpolant of $\varphi' \wedge \psi'$ in $\mathsf{Q^1S5}$.*

Proof. Assume that (n, m, τ) is a solution to $\mathfrak{T}$. Take the formulas χ and χ_s constructed in the proof of Lemma 5.2 and in Example 5.4, respectively. As it was shown, we have $\models_{\mathsf{Q^1S5}} \varphi \wedge \psi \to \neg\chi$ and $\models_{\mathsf{Q^1S5}} \boldsymbol{a} \wedge \varphi_0 \to \chi_s$ for all $s > 0$. Thus, $\models_{\mathsf{Q^1S5}} \varphi' \wedge \psi' \to (\chi \to \chi_s)$ for all $s > 0$. Therefore, if ϱ were a uniform interpolant of $\varphi' \wedge \psi'$, then we would have

$$\models_{\mathsf{Q^1S5}} \varrho \to (\chi \to \Diamond\chi_s) \text{ for all } s > 0. \tag{5.25}$$

On the other hand, we combine $\mathfrak{N} = (W, D, J)$ from the proof of Lemma 5.2 with the models $\mathfrak{M}_s = (W_s, D_s, I_s)$ constructed in Example 5.4. For every $s \geq nm$, we define a model $\mathfrak{N}_s = (W_s, D_s, J_s)$ as follows. For every $k < s$, we let

$$\boldsymbol{p}^{J_s(k)} = \begin{cases} \boldsymbol{p}^{J(k)}, & \text{if } k < nm \text{ and } \boldsymbol{p} \in sig(\varphi), \\ \boldsymbol{p}^{I_s(k)}, & \text{if } \boldsymbol{p} \in \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

As shown in the proof of Lemma 5.2, $\mathfrak{N}, 0, 0 \models \varphi \wedge \chi$, so we have $\mathfrak{N}_s, 0, 0 \models \varphi' \wedge \chi$. As it was shown in Example 5.4 that $\mathfrak{M}_s, 0, 0 \not\models \chi_s$ for all $s > 0$, we have $\mathfrak{N}_s, 0, 0 \not\models \chi_s$ for all $s \geq nm$. So it follows from (5.25) that $\mathfrak{N}_s, 0, 0 \models \neg\varrho$. Note that also $\mathfrak{N}_s, 0, 0 \models \chi_{s'}$ for all $s > s' \geq nm$. Now consider the ultraproduct $\prod_U \mathfrak{N}_s$ with a non-principal ultrafilter U on $\omega \setminus \{0, \dots, nm-1\}$. As each $\chi_{s'}$ is true in almost all $\mathfrak{N}_s, 0, 0$, it follows from the properties of ultraproducts [CK98] that $\prod_U \mathfrak{N}_s, \overline{0}, \overline{0} \models \boldsymbol{a} \wedge \neg\varrho \wedge \chi_{s'}$ for all $s' > 0$, for a suitable $\overline{0}$. But then one can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{N}_s$ so that $\mathfrak{N}, \overline{0}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{N}$. Then $\mathfrak{N}, \overline{0}, \overline{0} \models \boldsymbol{a} \wedge \varphi_0 \wedge \varphi' \wedge \neg\varrho$ and so $\mathfrak{N}, \overline{0}, \overline{0} \models \varphi' \wedge \psi' \wedge \neg\varrho$. As $\models_{\mathsf{Q^1S5}} \varphi' \wedge \psi' \to \varrho$ should hold for a uniform interpolant ϱ of $\varphi' \wedge \psi'$, we have derived a contradiction. ⊣

This completes the proof of Theorem 5.1 (*ii*).

Remark 5.7. We can translate Example 5.4 into FO^2 by taking $\sigma = \{A, R, P\}$ and

$$\varphi_0 = \forall x(A(x) \to \exists y(R(x,y) \wedge B(y) \wedge P(y)) \wedge \forall x(B(x) \wedge P(x) \to \exists y(R(x,y) \wedge B(y) \wedge P(y))).$$

Then $A(x) \wedge \varphi_0$ has no σ-uniform interpolant in FO^2, which is shown similarly to the proof in Example 5.4. It follows that FO^2 does not have the UIP. This example can be merged with the proof of the undecidability of the CEP in FO^2 from [JLM+17]—in the same way as we combined Example 5.4 with the undecidability proof for the UIEP in $\mathsf{Q^1S5}$—to show that the UIEP is undecidable in FO^2 (with and without =). The latter problem has so far remained open.

6. The Modal Description Logic $\mathsf{S5}_{\mathcal{ALC}^u}$

Next, we extend the results of Sections 4 and 5 to the modal description logic $\mathsf{S5}_{\mathcal{ALC}^u}$, where $\mathcal{ALC}^u$ is the basic description logic $\mathcal{ALC}$ with the universal role [BHLS17]. In other words, $\mathcal{ALC}^u$ is a notational variant of multimodal logic K with the universal modality and can be regarded as a fragment of FO^2. (An example showing that it does not enjoy the CIP will be given at the end of the section.)

The *concepts* of $\mathsf{S5}_{\mathcal{ALC}^u}$ are constructed from *concept names* $A \in \mathcal{C}$, *role names* $R \in \mathcal{R}$, for some countably-infinite and disjoint sets $\mathcal{C}$ and $\mathcal{R}$, and a distinguished *universal role* U by means of the following grammar:

$$C \;:=\; A \;\mid\; \top \;\mid\; C \sqcap C' \;\mid\; \neg C \;\mid\; \exists R.C \;\mid\; \exists U.C \;\mid\; \Diamond C.$$

A *signature* σ is any finite set of concept and role names. As usual, the universal role is regarded as a logical symbol and not part of any signature. The signature $sig(C)$ of a concept C comprises the concept and role names in C, again excluding the universal role. If $sig(C) \subseteq \sigma$, then C is called a *σ-concept*. We interpret $\mathsf{S5}_{\mathcal{ALC}^u}$ in models $\mathfrak{M} = (W, \Delta, I)$, where $I(w)$ is an interpretation of the concept and role names at each world $w \in W$ over domain $\Delta \neq \emptyset$: $A^{I(w)} \subseteq \Delta$, $R^{I(w)} \subseteq \Delta \times \Delta$, and $U^{I(w)} = \Delta \times \Delta$. The *truth-relation* $\mathfrak{M}, w, d \models C$ is defined by taking

- $\mathfrak{M}, w, d \models \top$,
- $\mathfrak{M}, w, d \models A$ iff $d \in A^{I(w)}$,
- $\mathfrak{M}, w, d \models \exists S.C$ iff there is $(d, d') \in S^{I(w)}$ such that $\mathfrak{M}, w, d' \models C$,
- $\mathfrak{M}, w, d \models \Diamond C$ iff there is $w' \in W$ with $\mathfrak{M}, w', d \models C$,

and the standard clauses for Boolean $\sqcap$ and $\neg$. An expression of the form $C \sqsubseteq D$ is called a *concept inclusion*. We sometimes use more conventional $C^{I(w)} = \{d \in \Delta \mid \mathfrak{M}, w, d \models C\}$, writing $\mathfrak{M}, w \models C \sqsubseteq D$ if $C^{I(w)} \subseteq D^{I(w)}$, and $\models C \sqsubseteq D$ if $\mathfrak{M}, w \models C \sqsubseteq D$ for all $\mathfrak{M}$ and w. The problem of deciding whether $\models C \sqsubseteq D$, for given C and D, is coNExpTime-complete [GKWZ03].

An *interpolant for $C \sqsubseteq D$ in* $\mathsf{S5}_{\mathcal{ALC}^u}$ is a concept E such that $sig(E) \subseteq sig(C) \cap sig(D)$, $\models C \sqsubseteq E$, and $\models E \sqsubseteq D$. The IEP for $\mathsf{S5}_{\mathcal{ALC}^u}$ is to decide whether a given concept inclusion $C \sqsubseteq D$ has an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$.

Remark 6.1. Typical applications of description logics use reasoning modulo *ontologies*, which are finite sets, $\mathcal{O}$, of concept inclusions. We then set $\mathcal{O} \models C \sqsubseteq D$ iff whenever $\mathfrak{M}, w \models \alpha$ for all $\alpha \in \mathcal{O}$, then $\mathfrak{M}, w \models C \sqsubseteq D$. Reasoning modulo ontologies is reducible to the ontology-free case by the following equivalence:

$$\mathcal{O} \models C \sqsubseteq D \quad \text{iff} \quad \models \top \sqsubseteq \bigsqcup_{C' \sqsubseteq D' \in \mathcal{O}} \exists U.(C' \sqcap \neg D') \sqcup \forall U.(\neg C \sqcup D).$$

where $\sqcup$ is dual to $\sqcap$. The following problems can easily be reduced to the IEP in polynomial time (see Appendix B):

***IEP modulo ontologies*:** Given an ontology $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$, does there exist a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$?

***ontology interpolant existence* (*OIEP*):** Given an ontology $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$, is there an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$?

***EDEP modulo ontologies*:** Given an ontology $\mathcal{O}$, a signature σ, and a concept name A, does there exist a concept C such that $sig(C) \subseteq \sigma$ and $\mathcal{O} \models A \equiv C$?

Explicit definitions have been proposed for query rewriting in ontology-based data access [FKN13, TW21], developing and maintaining ontology alignments [GPT16], and ontology engineering [tCCMV06]. The IEP is fundamental for robust modularisations and decompositions of ontologies [KLWW09, BKL+16].

Our main result in this section is the following:

Theorem 6.2. *The IEP, EDEP (modulo ontologies), and the OIEP are all decidable in* coN2ExpTime*, being* 2ExpTime*-hard.*

We begin by formulating a model-theoretic characterisation of interpolant existence in $\mathsf{S5}_{\mathcal{ALC}^u}$ in terms of the following generalisation of σ-$\mathsf{S5}$-bisimulations for $\mathsf{Q}^1\mathsf{S5}$ from Section 4.1. Similar bisimulations have been used [WS17] to characterise the fragment $\mathsf{S5}_{\mathcal{ALC}}$ of $\mathsf{S5}_{\mathcal{ALC}^u}$ without the universal role as a bisimulation-invariant fragment of full first-order modal logic $\mathsf{S5}$ over finite and unrestricted models.

A *σ-bisimulation* between models $\mathfrak{M}_i = (W_i, \Delta_i, I_i)$, $i = 1, 2$, is any triple $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ with $\boldsymbol{\beta}_W \subseteq W_1 \times W_2$, $\boldsymbol{\beta}_\Delta \subseteq \Delta_1 \times \Delta_2$, and $\boldsymbol{\beta} \subseteq (W_1 \times \Delta_1) \times (W_2 \times \Delta_2)$ if

- **(w)** for any $(w_1, w_2) \in \boldsymbol{\beta}_W$ and $d_1 \in \Delta_1$, there is $d_2 \in \Delta_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $d_2 \in \Delta_2$,
- **(d)** for any $(d_1, d_2) \in \boldsymbol{\beta}_\Delta$ and $w_1 \in W_1$, there is $w_2 \in W_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $w_2 \in W_2$,
- **(c)** $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ implies both $(w_1, w_2) \in \boldsymbol{\beta}_W$ and $(d_1, d_2) \in \boldsymbol{\beta}_\Delta$,

and the following hold for all $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$:

- **(a)** $\mathfrak{M}_1, w_1, d_1 \models A$ iff $\mathfrak{M}_2, w_2, d_2 \models A$, for all $A \in \sigma$;
- **(r)** if $(d_1, e_1) \in S^{I(w_1)}$ and $sig(S) \in \sigma$, then there is $e_2 \in \Delta_2$ with $(d_2, e_2) \in S^{I(w_2)}$ and $((w_1, e_1), (w_2, e_2)) \in \boldsymbol{\beta}$, and the other way round.

We write $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$ to say that there is a σ-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ for which $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$; $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ says that there is a σ-bisimulation $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ with $(w_1, w_2) \in \boldsymbol{\beta}_W$; and $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$ that there is $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ with $(d_1, d_2) \in \boldsymbol{\beta}_\Delta$. The usage of $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$ is as in Section 2 but for any σ-concepts, $\mathfrak{M}_1, w_1 \equiv_\sigma \mathfrak{M}_2, w_2$ means that the same σ-concepts of the form $\exists U.C$ are true at w_1 in $\mathfrak{M}_1$ and at w_2 in $\mathfrak{M}_2$, and $\mathfrak{M}_1, d_1 \equiv_\sigma \mathfrak{M}_2, d_2$ means that the same σ-concepts of the form $\Diamond C$ are true at d_1 in $\mathfrak{M}_1$ and at d_2 in $\mathfrak{M}_2$. The following is an $\mathsf{S5}_{\mathcal{ALC}^u}$-analogue of Lemma 2.1:

Lemma 6.3. *For any ω-saturated $\mathsf{S5}_{\mathcal{ALC}^u}$-models $\mathfrak{M}_1$ with w_1, d_1 and $\mathfrak{M}_2$ with w_2, d_2,*

- $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$ *iff* $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$,
- $\mathfrak{M}_1, w_1 \equiv_\sigma \mathfrak{M}_2, w_2$ *iff* $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$,
- $\mathfrak{M}_1, d_1 \equiv_\sigma \mathfrak{M}_2, d_2$ *iff* $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$.

The implication from right to left holds for arbitrary models.

Proof. The proof of the direction from right to left is by a straightforward induction on the construction of concepts C, using **(a)** for concept names, **(r)** for $\exists R.C$, **(c)** and **(w)** for $\exists U.C$, and **(c)** and **(d)** for $\Diamond C$. For the converse direction, we define $\boldsymbol{\beta}_W$, $\boldsymbol{\beta}_\Delta$, and $\boldsymbol{\beta}$ via $\equiv_\sigma$ in the obvious way. Then we observe that $\mathfrak{M}_1, w_1, d_1 \equiv_\sigma \mathfrak{M}_2, w_2, d_2$ implies $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ and $\mathfrak{M}_1, d_1 \equiv_\sigma \mathfrak{M}_2, d_2$. Hence we obtain **(c)**. We obtain **(w)** and **(d)** using saturatedness, **(a)** is trivial, and **(r)** follows again from saturatedness. ⊣

The criterion below—in which σ-bisimulation consistency is defined as in Section 3 with concepts C, D in place of formulas φ, ψ—is an $\mathsf{S5}_{\mathcal{ALC}^u}$-analogue of Theorem 3.1:

Theorem 6.4. *The following conditions are equivalent for any concepts C and D:*
– *there does not exist an interpolant for $C \sqsubseteq D$ in $\mathsf{S5}_{\mathcal{ALC}^u}$;*
– *C and $\neg D$ are $sig(C) \cap sig(D)$-bisimulation consistent.*

We now extend the construction of Section 4.1 from $\mathsf{Q^1S5}$ to $\mathsf{S5}_{\mathcal{ALC}^u}$. In contrast to $\mathsf{Q^1S5}$, we now have to deal with more involved σ-bisimulations between the respective first-order models $I(w_1)$ and $I(w_2)$ (satisfying conditions (**a**) and (**r**)). To this end we introduce *full mosaics* (sets of full types realised in σ-bisimilar pairs (w, d)) and *full points* (full mosaics with a distinguished full type). The range of the surjections π used to construct W' and D' then consists of full points rather than full types. This provides us with the data structure to define σ-bisimilar first-order models $I(w)$ when required. This construction establishes an upper bound on the size of models witnessing bisimulation consistency:

Theorem 6.5. *For any concepts C and D, there does not exist an interpolant for $C \sqsubseteq D$ in $\mathsf{S5}_{\mathcal{ALC}^u}$ iff there are models witnessing that C and $\neg D$ are $sig(C) \cap sig(D)$-bisimulation consistent of size double-exponential in $|C| + |D|$.*

Proof. Given concepts C and D, we define the sets $sub(C, D)$, $sub_\Diamond(C, D)$, and $sub_\exists(C, D)$ as in Section 4.1 regarding $\exists U$ as the $\mathsf{S5}_{\mathcal{ALC}^u}$-counterpart of $\exists$ in $\mathsf{Q^1S5}$. The *world-type* $\mathsf{wt}_{\mathfrak{M}}(w)$ of $w \in W$ in $\mathfrak{M} = (W, \Delta, I)$, the *domain-type* $\mathsf{dt}_{\mathfrak{M}}(d)$ of $d \in \Delta$, and the *full type* $\mathsf{ft}_{\mathfrak{M}}(w, d)$ of (w, d) in $\mathfrak{M}$ are also defined as in Section 4.1. Observe that we have $\mathsf{wt}_{\mathfrak{M}}(w) = \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{wt}}$ and $\mathsf{dt}_{\mathfrak{M}}(d) = \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{dt}}$, where

$$\mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{wt}} = sub_\exists(C, D) \cap \mathsf{ft}_{\mathfrak{M}}(w, d), \quad \mathsf{ft}_{\mathfrak{M}}(w, d)^{\mathsf{dt}} = sub_\Diamond(C, D) \cap \mathsf{ft}_{\mathfrak{M}}(w, d).$$

Now, suppose that $\sigma = sig(C) \cap sig(D)$, $\mathfrak{M}_i = (W_i, \Delta_i, I_i)$, for $i = 1, 2$, have pairwise disjoint W_i and Δ_i, $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$ with $\mathfrak{M}_1, w_1, d_1 \models C$, and $\mathfrak{M}_2, w_2, d_2 \models \neg D$. For $w \in W_1 \cup W_2$, we define the *world mosaic* $\mathsf{wm}(w) = (T_1(w), T_2(w))$ by (4.4) and the *i-world point* $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$ of w in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Using (4.5) with Δ_i in place of D_i, we define the *domain mosaic* $\mathsf{dm}(d) = (S_1(d), S_2(d))$ and *i-domain point* $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$ of $d \in \Delta_1 \cup \Delta_2$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Then, for $(w, d) \in (W_1 \times \Delta_1) \cup (W_2 \times \Delta_2)$, we set

$$F_i(w, d) = \{\mathsf{ft}_{\mathfrak{M}_i}(v, e) \mid (v, e) \in W_i \times \Delta_i, (v, e) \sim_\sigma (w, d)\}$$

calling $\mathsf{fm}(w, d) = (F_1(w, d), F_2(w, d))$ the *full mosaic* and $\mathsf{fp}_i(w, d) = (\mathsf{ft}_{\mathfrak{M}_i}(w, d), \mathsf{fm}(w, d))$ the *i-full point* of (w, d) in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Given $\mathsf{fm} = (F_1, F_2)$, we set

$$\mathsf{fm}^{\mathsf{wt}} = (\{\mathsf{ft}^{\mathsf{wt}} \mid \mathsf{ft} \in F_1\}, \{\mathsf{ft}^{\mathsf{wt}} \mid \mathsf{ft} \in F_2\}), \quad \mathsf{fm}^{\mathsf{dt}} = (\{\mathsf{ft}^{\mathsf{dt}} \mid \mathsf{ft} \in F_1\}, \{\mathsf{ft}^{\mathsf{dt}} \mid \mathsf{ft} \in F_2\}).$$

Lemma 6.6. *Suppose* $\mathsf{fm} = \mathsf{fm}(w, d)$, $\mathsf{wm} = \mathsf{wm}(w)$, *and* $\mathsf{dm} = \mathsf{dm}(d)$. *Then* $\mathsf{fm}^{\mathsf{wt}} = \mathsf{wm}$ *and* $\mathsf{fm}^{\mathsf{dt}} = \mathsf{dm}$.

Proof. The inclusions $\mathsf{wm} \subseteq \mathsf{fm}^{\mathsf{wt}}$ and $\mathsf{dm} \subseteq \mathsf{fm}^{\mathsf{dt}}$ follow from (**w**) and (**d**). Indeed, suppose $\mathsf{wm} = (T_1(w), T_2(w))$, $\mathsf{wt} \in T_i(w)$, and $\mathsf{fm} = (F_1(w, d), F_2(w, d))$. By definition, there exists $v \in W_i$ with $v \sim_\sigma w$ such that $\mathsf{wt} = \mathsf{wt}_{\mathfrak{M}_i}(v)$. By (**w**), there exists $e \in \Delta_i$ with $(w, d) \sim_\sigma (v, e)$. But then $\mathsf{ft}_{\mathfrak{M}_i}(v, e) \in F_i(w, d)$ and $\mathsf{ft}_{\mathfrak{M}_i}(v, e)^{\mathsf{wt}} = \mathsf{wt}$, as required. The second claim is proved in the same way using (**d**).

The converse inclusions $\mathsf{fm}^{\mathsf{wt}} \subseteq \mathsf{wm}$ and $\mathsf{fm}^{\mathsf{dt}} \subseteq \mathsf{dm}$ follow from (**c**). To see this, let $\mathsf{fm} = (F_1(w, d), F_2(w, d))$, $\mathsf{wm} = (T_1(w), T_2(w))$, and $\mathsf{dm} = (S_1(d), S_2(d))$. Let $\mathsf{ft} \in F_i(w, d)$.

Then there are $(v, e) \in W_i \times \Delta_i$ with $(v, e) \sim_\sigma (w, d)$. Therefore, by **(c)**, $v \sim_\sigma w$ and $e \sim_\sigma d$, and so $\mathsf{wt} \in T_i(w)$ and $\mathsf{dt} \in S_i(d)$, as required. ⊣

As in Section 4.1, we construct models $\mathfrak{M}'_1 = (W'_1, \Delta'_1, I'_1)$ and $\mathfrak{M}'_2 = (W'_2, \Delta'_2, I'_2)$ from copies of i-world points wp_i and i-domain points dp_i in $\mathfrak{M}_1$, $\mathfrak{M}_2$. Let $n = m_1 \times m_2$, where m_1 and m_2 are the number of full types and, respectively, full mosaics over $sub(C, D)$ in $\mathfrak{M}_1$, $\mathfrak{M}_2$. For $i = 1, 2$, we set

$$\Delta'_i = \{\mathsf{dp}_i^k \mid \mathsf{dp}_i \text{ an } i\text{-domain point in } \mathfrak{M}_1, \mathfrak{M}_2,\ k \in [n]\}.$$

For an i-world point wp_i and an i-domain point dp_i, let

$$L^i_{\mathsf{wp}_i,\mathsf{dp}_i} = \{\mathsf{fp}_i(w, d) \mid \mathsf{wp}_i = \mathsf{wp}_i(w),\ \mathsf{dp}_i = \mathsf{dp}_i(d),\ (w, d) \in W_i \times \Delta_i\}.$$

We define W'_i using the set $\Pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ of surjective functions of the form

$$\pi_{\mathsf{wp}_i,\mathsf{dp}_i} \colon [n] \to L^i_{\mathsf{wp}_i,\mathsf{dp}_i}.$$

Let Π_i denote the set of all functions π mapping any pair $\mathsf{wp}_i, \mathsf{dp}_i$ to an element of $\Pi_{\mathsf{wp}_i,\mathsf{dp}_i}$. We set $\pi_{\mathsf{wp}_i,\mathsf{dp}_i} = \pi(\mathsf{wp}_i, \mathsf{dp}_i)$ and then let $\Pi_i^\dagger \subseteq \Pi_i$ be a smallest set such that, for any $\mathsf{fp}_i = \mathsf{fp}_i(w, d) = (\mathsf{ft}_{\mathfrak{M}_i}(w, d), \mathsf{fm}(w, d))$, $\mathsf{wp}_i = \mathsf{wp}_i(w)$, and $\mathsf{dp}_i = \mathsf{dp}_i(d)$ with $(w, d) \in \mathfrak{M}_i$ and any $k \in [n]$, there is $\pi \in \Pi_i^\dagger$ with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{fp}_i$. It can be seen in the same way as in the proof for $\mathsf{Q}^1\mathsf{S5}$ in Section 4.1 that $|\Pi_i^\dagger| \le n^2$, for $i = 1, 2$.

Now let

$$W'_i = \{\mathsf{wp}_i^\pi \mid \ \mathsf{wp}_i \text{ an } i\text{-world point in } \mathfrak{M}_1, \mathfrak{M}_2 \text{ and } \pi \in \Pi_i^\dagger\}.$$

Note that $|\Delta'_i|$ and $|W'_i|$ are double-exponential in $|C| + |D|$. It remains to define the extensions of concept and role names in $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$. When defining them and investigating their properties, we use the following notation:

$$C^{\mathfrak{M}'_i} = \{(w, d) \mid d \in C^{I'_i(w)}\}, \quad R^{\mathfrak{M}'_i} = \{((w, d), (w, d')) \mid (d, d') \models R^{I'_i(w)}\}$$

and similarly for $C^{\mathfrak{M}_i}$ and $R^{\mathfrak{M}_i}$. We set

$$(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in A^{\mathfrak{M}'_i} \ \text{ iff } \ A \in \mathsf{ft} \ \text{ for } \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm}).$$

The definition of $R^{\mathfrak{M}'_i}$ is more involved. Call a pair $\mathsf{ft}_1, \mathsf{ft}_2$ *R-coherent* if $\exists R.C \in \mathsf{ft}_1$ whenever $C \in \mathsf{ft}_2$, for all $\exists R.E \in sub(C, D)$, and call $\mathsf{ft}_1, \mathsf{ft}_2$ *R-witnessing* if they are R-coherent and $\mathsf{ft}_1^{\mathsf{wt}} = \mathsf{ft}_2^{\mathsf{wt}}$. For full mosaics $\mathsf{fm} = (F_1, F_2)$ and $\mathsf{fm}' = (F'_1, F'_2)$ and a role $R \in \sigma$, we set $\mathsf{fm} \preceq_R \mathsf{fm}'$ if there exist functions $f_i \colon F_i \to F'_i$, $i = 1, 2$, such that, for all $\mathsf{ft} \in F_i$, the pair ft, $f_i(\mathsf{ft})$ is R-witnessing. Now suppose that $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$ and $\pi_{\mathsf{wp}_i,\mathsf{dp}'_i}(k') = (\mathsf{ft}', \mathsf{fm}')$. For $R \in \sigma$, we set

$$((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k), (\mathsf{wp}_i^\pi, \mathsf{dp}_i^{k'})) \in R^{\mathfrak{M}'_i} \quad \text{iff} \quad \mathsf{fm} \preceq_R \mathsf{fm}' \text{ and } \mathsf{ft}, \mathsf{ft}' \text{ is } R\text{-witnessing.} \tag{6.1}$$

For $R \notin \sigma$, we omit the condition $\mathsf{fm} \preceq_R \mathsf{fm}'$ from (6.1).

Lemma 6.7. *Suppose $E \in sub(C, D)$. Then we have $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}'_i}$ iff $E \in \mathsf{ft}$, for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$.*

Proof. The proof is by induction on the construction of E. The basis of induction follows from the definition, and the inductive step for the Booleans is trivial.

Let $E = \exists U.E'$. Suppose first $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}'_i}$ and $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft}, \mathsf{fm})$. By definition, there exists ${\mathsf{dp}'_i}^{k'}$ with $(\mathsf{wp}_i^\pi, {\mathsf{dp}'_i}^{k'}) \in E'^{\mathfrak{M}'_i}$. By IH, $E' \in \mathsf{ft}'$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}'_i}(k') = (\mathsf{ft}', \mathsf{fm}')$. Then $\exists U.E' \in \mathsf{ft}'$, and so $\exists U.E' \in \mathsf{ft}'^{\mathsf{wt}}$. As $\mathsf{ft}^{\mathsf{wt}} = \mathsf{ft}'^{\mathsf{wt}}$, we obtain $E \in \mathsf{ft}$, as required.

Conversely, let $E = \exists U.E' \in \mathsf{ft}$, for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there is d' with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w,d')$. Let $\mathsf{dp}_i' = \mathsf{dp}_i(d')$ and $\mathsf{fm}' = \mathsf{fm}(w,d')$. As $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}$ is surjective, there is k' with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}',\mathsf{fm}')$. Then, by IH, we obtain $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$, and so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$.

Let $E = \Diamond E'$. Suppose first that $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$. Let $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$ and $\mathsf{dp}_i = (\mathsf{dt},\mathsf{dm})$. By definition, there is $\mathsf{wp}_i'^{\pi'}$ with $(\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i^k) \in E'^{\mathfrak{M}_i'}$. By IH, $E' \in \mathsf{ft}'$ for $\pi'_{\mathsf{wp}_i',\mathsf{dp}_i}(k) = (\mathsf{ft}',\mathsf{fm}')$. By definition, $\Diamond E' \in \mathsf{ft}'$, and so $\Diamond E' \in \mathsf{ft}'^{\mathsf{dt}} = \mathsf{dt}$. But then, again by definition, $E \in \mathsf{ft}$, as required.

Conversely, let $E = \Diamond E' \in \mathsf{ft}$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there is w' with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w',d)$. Let $\mathsf{wp}_i' = \mathsf{wp}_i(w')$ and $\mathsf{fm}' = \mathsf{fm}(w',d)$. By definition of $\Pi_i^\dagger$, there is $\pi' \in \Pi_i^\dagger$ with $\pi'_{\mathsf{wp}_i',\mathsf{dp}_i}(k) = (\mathsf{ft}',\mathsf{fm}')$. By IH, $(\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i^k) \in E'^{\mathfrak{M}_i'}$, and so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$.

Let $E = \exists R.E'$. Suppose that $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ and $R \in \sigma$. Let $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$. By definition, there exists $\mathsf{dp}_i'^{k'}$ with $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k)(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'})) \in R^{\mathfrak{M}_i'}$ and $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$. By IH, $E' \in \mathsf{ft}'$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}',\mathsf{fm}')$. By the definition of $R^{\mathfrak{M}_i'}$, we obtain that ft,ft' are R-coherent. But then $E \in \mathsf{ft}$, as required. The case $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$ and $R \notin \sigma$ is similar.

Conversely, let $E = \exists R.E' \in \mathsf{ft}$ for $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$. Take (w,d) with $\mathsf{ft} = \mathsf{ft}(w,d)$ and $\mathsf{fm} = \mathsf{fm}(w,d)$. By definition, there exists e with $E' \in \mathsf{ft}'$ for $\mathsf{ft}' = \mathsf{ft}(w,e)$ and $((w,d),(w,e)) \in R^{\mathfrak{M}_i}$. Let $\mathsf{dp}_i' = \mathsf{dp}_i(e)$ and $\mathsf{fm}' = \mathsf{fm}(w,e)$. Assume first that $R \in \sigma$. We define functions f_j, $j = 1,2$ witnessing $\mathsf{fm} \preceq_R \mathsf{fm}'$.

Suppose $\mathsf{ft} \in F_j$, where $\mathsf{fm} = (F_1,F_2)$. Then we find $(w',d') \sim_\sigma (w,d)$ with $\mathsf{ft} = \mathsf{ft}(w',d')$. By **(r)**, $(w,d),(w,e) \in R^{\mathfrak{M}_i}$ and $(w',d') \sim_\sigma (w,d)$ give us e' with $(w',d'),(w',e') \in R^{\mathfrak{M}_i}$ and $(w,e) \sim_\sigma (w',e')$. Let $\mathsf{ft}' = \mathsf{ft}(w',e')$. We define the required f_j by taking $f_j(\mathsf{ft}) = (\mathsf{ft}')$.

Since $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}$ is surjective, there exists k' such that $\pi_{\mathsf{wp}_i,\mathsf{dp}_i'}(k') = (\mathsf{ft}',\mathsf{fm}')$. By IH, $(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'}) \in E'^{\mathfrak{M}_i'}$. By the definition of $R^{\mathfrak{M}_i'}$, $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k),(\mathsf{wp}_i^\pi, \mathsf{dp}_i'^{k'})) \in R^{\mathfrak{M}_i'}$, so $(\mathsf{wp}_i^\pi, \mathsf{dp}_i^k) \in E^{\mathfrak{M}_i'}$. The case $R \notin \sigma$ is similar. ⊣

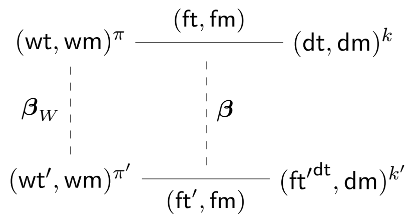
We define $\boldsymbol{\beta}_W \subseteq W_1' \times W_2'$, $\boldsymbol{\beta}_\Delta \subseteq \Delta_1' \times \Delta_2'$, and $\boldsymbol{\beta} \subseteq (W_1' \times \Delta_1') \times (W_2' \times \Delta_2')$ by taking

– $((\mathsf{wt},\mathsf{wm})^\pi, (\mathsf{wt}',\mathsf{wm}')^{\pi'}) \in \boldsymbol{\beta}_W$ iff $\mathsf{wm} = \mathsf{wm}'$;
– $((\mathsf{dt},\mathsf{dm})^k, (\mathsf{dt}',\mathsf{dm}')^{k'}) \in \boldsymbol{\beta}_\Delta$ iff $\mathsf{dm} = \mathsf{dm}'$;
– $((\mathsf{wp}_i^\pi, \mathsf{dp}_i^k),(\mathsf{wp}_i'^{\pi'}, \mathsf{dp}_i'^{k'})) \in \boldsymbol{\beta}$ iff $\mathsf{wp}_i = (\mathsf{wt},\mathsf{wm})$, $\mathsf{wp}_i' = (\mathsf{wt}',\mathsf{wm}')$, $\mathsf{dp}_i = (\mathsf{dt},\mathsf{dm})$, and $\mathsf{dp}_i' = (\mathsf{dt}',\mathsf{dm}')$ with $\mathsf{wm} = \mathsf{wm}'$, $\mathsf{dm} = \mathsf{dm}'$, $\mathsf{fm} = \mathsf{fm}'$, and $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = (\mathsf{ft},\mathsf{fm})$ and $\pi'_{\mathsf{wp}_i',\mathsf{dp}_i'}(k') = (\mathsf{ft}',\mathsf{fm}')$.

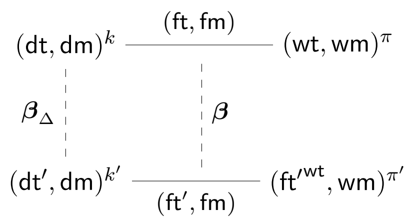
Lemma 6.8. *The triple $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ is a σ-bisimulation between $\mathfrak{M}_1'$ and $\mathfrak{M}_2'$.*

Proof. We show that $(\boldsymbol{\beta}_W, \boldsymbol{\beta}_\Delta, \boldsymbol{\beta})$ satisfies conditions **(w)**, **(d)**, **(c)**, **(a)**, and **(r)**.

(w) Suppose $((\mathsf{wt},\mathsf{wm})^\pi, (\mathsf{wt}',\mathsf{wm}')^{\pi'}) \in \boldsymbol{\beta}_W$ and $(\mathsf{dt},\mathsf{dm})^k \in \Delta_i'$. We need to find $(\mathsf{dt}',\mathsf{dm}')^{k'}$ with $(((\mathsf{wt},\mathsf{wm})^\pi, (\mathsf{dt},\mathsf{dm})^k), ((\mathsf{wt}',\mathsf{wm}')^{\pi'}, (\mathsf{dt}',\mathsf{dm}')^{k'})) \in \boldsymbol{\beta}$. We have $\mathsf{wm} = \mathsf{wm}'$ and set $(\mathsf{ft},\mathsf{fm}) = \pi_{\mathsf{wt},\mathsf{wm},\mathsf{dt},\mathsf{dm}}(k)$. Assume $\mathsf{fm} = (F_1,F_2)$. By Lemma 6.6, there exists $\mathsf{ft}' \in F_i$ with $\mathsf{ft}'^{\mathsf{wt}} = \mathsf{wt}'$. As the component $\pi'_{\mathsf{wt}',\mathsf{wm},\mathsf{ft}'^{\mathsf{dt}},\mathsf{dm}}$ of π' is surjective, there exists $k' \in [n]$ such that $\pi'_{\mathsf{wt}',\mathsf{wm},\mathsf{ft}'^{\mathsf{dt}},\mathsf{dm}}(k') = (\mathsf{ft}',\mathsf{fm})$. Then $(\mathsf{ft}'^{\mathsf{dt}},\mathsf{dm})^{k'}$ is as required; see the picture below.

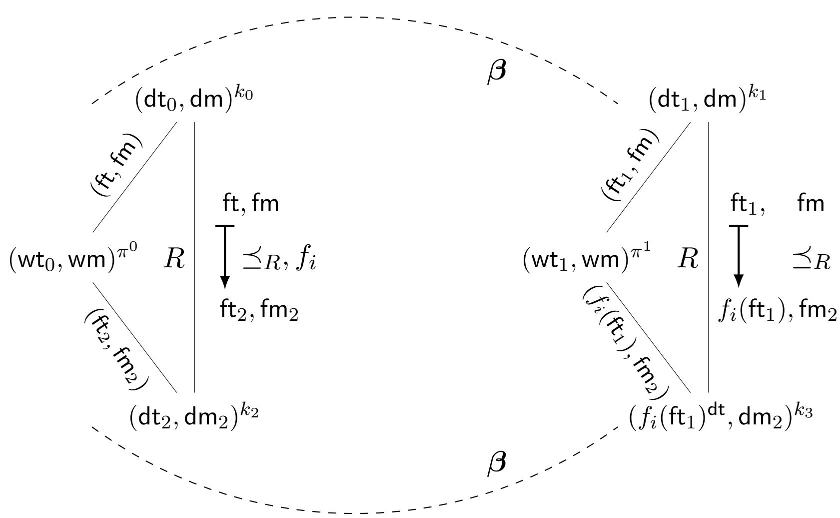


(d) Suppose that $((\mathsf{dt},\mathsf{dm})^k, (\mathsf{dt}',\mathsf{dm}')^{k'}) \in \boldsymbol{\beta}_\Delta$ and $(\mathsf{wt},\mathsf{wm})^\pi \in W_i'$. We need to construct $(\mathsf{wt}',\mathsf{wm}')^{\pi'}$ with $(((\mathsf{wt},\mathsf{wm})^\pi,(\mathsf{dt},\mathsf{dm})^k),((\mathsf{wt}',\mathsf{wm}')^{\pi'},(\mathsf{dt}',\mathsf{dm}')^{k'})) \in \boldsymbol{\beta}$. We have $\mathsf{dm} = \mathsf{dm}'$ and set $(\mathsf{ft},\mathsf{fm}) = \pi_{\mathsf{wt},\mathsf{wm},\mathsf{dt},\mathsf{dm}}(k)$. Assume $\mathsf{fm} = (F_1, F_2)$. By Lemma 6.6, there exists $\mathsf{ft}' \in F_i$ with $\mathsf{ft}'^{\mathsf{dt}} = \mathsf{dt}'$. By the construction of $\Pi_i^\dagger$, there exists $\pi' \in \Pi_i^\dagger$ such that, for the component $\pi'_{\mathsf{wt}'^{\mathsf{wt}},\mathsf{wm},\mathsf{ft}',\mathsf{dm}}$ of π', we have $\pi'_{\mathsf{wt}'^{\mathsf{wt}},\mathsf{wm},\mathsf{ft}',\mathsf{dm}}(k') = (\mathsf{ft}',\mathsf{fm})$. Then $(\mathsf{ft}'^{\mathsf{wt}},\mathsf{wm})^{\pi'}$ is as required; see the picture below.



Condition **(c)** follows from the definition of $\boldsymbol{\beta}_W$, $\boldsymbol{\beta}_\Delta$, $\boldsymbol{\beta}$; and condition **(a)** follows from the definition of $\boldsymbol{\beta}$.

(r) Suppose $R \in \sigma$, $\boldsymbol{\beta}$ contains $(((\mathsf{wt}_0,\mathsf{wm})^{\pi^0},(\mathsf{dt}_0,\mathsf{dm})^{k_0}),((\mathsf{wt}_1,\mathsf{wm})^{\pi^1},(\mathsf{dt}_1,\mathsf{dm})^{k_1}))$ and $R^{\mathfrak{M}_i}$ contains $(((\mathsf{wt}_0,\mathsf{wm})^{\pi^0},(\mathsf{dt}_0,\mathsf{dm})^{k_0}),((\mathsf{wt}_0,\mathsf{wm})^{\pi^0},(\mathsf{dt}_2,\mathsf{dm}_2)^{k_2}))$. Assume also that $(\mathsf{ft},\mathsf{fm}) = \pi^0_{\mathsf{wt}_0,\mathsf{wm},\mathsf{dt}_0,\mathsf{dm}}(k_0)$. Then there exists ft_1 such that $(\mathsf{ft}_1,\mathsf{fm}) = \pi^1_{\mathsf{wt}_1,\mathsf{wm},\mathsf{dt}_1,\mathsf{dm}}(k_1)$. Moreover, for $(\mathsf{ft}_2,\mathsf{fm}_2) = \pi^0_{\mathsf{wt}_0,\mathsf{wm},\mathsf{dt}_2,\mathsf{dm}_2}(k_2)$, we have $\mathsf{fm} \preceq_R \mathsf{fm}_2$ and we may assume that $f_i(\mathsf{ft}) = \mathsf{ft}_2$, for the function f_i witnessing $\mathsf{fm} \preceq_R \mathsf{fm}_2$. Then $(f_i(\mathsf{ft}_1)^{\mathsf{dt}},\mathsf{dm}_2)^{k_3}$ with $k_3 \in [n]$ such that $\pi^1_{\mathsf{wt}_1,\mathsf{wm},f_i(\mathsf{ft}_1)^{\mathsf{dt}},\mathsf{dm}_2}(k_3) = (f_i(\mathsf{ft}_1),\mathsf{fm}_2)$ is as required; see the picture below.



This completes the proof of the lemma. ⊣

Theorem 6.5 follows. ⊣

This result gives the upper bound of Theorem 6.2. The lower one follows from the same lower bound for $\mathsf{Q}^1\mathsf{S5}$ by treating FOM^1-formulas φ, ψ as role-free $\mathsf{S5}_{\mathcal{ALC}^u}$-concepts and, using Theorems 3.1 and 6.4, one can readily show that φ and ψ have an interpolant in $\mathsf{Q}^1\mathsf{S5}$ iff they have an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$.

The (strong) conservative extension problem, (S)CEP, and the uniform interpolant existence problem, UIEP, in $\mathsf{S5}_{\mathcal{ALC}^u}$ are defined in the obvious way. Using the same argument as for interpolation, the undecidability of the (S)CEP and UIEP in $\mathsf{S5}_{\mathcal{ALC}^u}$ follows directly from the undecidability of both problems for $\mathsf{Q}^1\mathsf{S5}$. Note that, for the component logics—propositional $\mathsf{S5}$ and description logic $\mathcal{ALC}^u$—the CEP is coNExpTime and 2ExpTime-complete, respectively [GLWZ06, JLM+17].

7. First-Order Modal Logic $\mathsf{Q}^1\mathsf{K}$

We consider the one-variable first-order modal logic $\mathsf{Q}^1\mathsf{K}$ and show Theorem 1.5. By the *modal depth* $md(\varphi)$ of a FOM^1-formula φ we mean the maximal number of nestings of $\Diamond$ in φ; if φ has no modal operators, then $md(\varphi) = 0$. Formulas of modal depth k can be characterised using a finitary version of bisimulations, called k-bisimulations, that are defined below.

For a signature σ and two models $\mathfrak{M} = (W, R, D, I)$ and $\mathfrak{M}' = (W', R', D', I')$, a sequence $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ of relations $\boldsymbol{\beta}_i \subseteq (W \times D) \times (W' \times D')$ is a *σ-k-bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $\boldsymbol{p} \in \sigma$ and $((w, d), (w', d')) \in \boldsymbol{\beta}_i$: **(a)** and **(d)** from Section 2 as well as

(w′) if $i > 0$ and $(w, v) \in R$, then there is v' with $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}_{i-1}$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are *σ-k-bisimilar* and write $\mathfrak{M}, w, d \sim_\sigma^k \mathfrak{M}', w', d'$ if there is a σ-k-bisimulation $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ between $\mathfrak{M}$ and $\mathfrak{M}'$ with $((w, d), (w', d')) \in \boldsymbol{\beta}_k$. We write $\mathfrak{M}, w, d \equiv_\sigma^k \mathfrak{M}', w', d'$ when $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}', w', d' \models \varphi$, for every σ-formula φ with $md(\varphi) \leq k$.

We now define formulas $\tau_{\mathfrak{M},\sigma}^k$ generalising the characteristic formulas of [GO07], that describe every model $\mathfrak{M}$ up to σ-k-bisimulations. For $\mathfrak{M} = (W, R, D, I)$ and signature σ, let

$$t_{\mathfrak{M},\sigma}^0(w, d) = \bigwedge \{\boldsymbol{p} \in \sigma \mid \mathfrak{M}, w, d \models \boldsymbol{p}\} \wedge \bigwedge \{\neg \boldsymbol{p} \mid \boldsymbol{p} \in \sigma,\ \mathfrak{M}, w, d \not\models \boldsymbol{p}\},$$
$$\tau_{\mathfrak{M},\sigma}^0(w, d) = t_{\mathfrak{M},\sigma}^0(w, d) \wedge \bigwedge_{e \in D} \exists t_{\mathfrak{M},\sigma}^0(w, e) \wedge \forall \bigvee_{e \in D} t_{\mathfrak{M},\sigma}^0(w, e),$$

and let $\tau_{\mathfrak{M},\sigma}^{k+1}(w, d)$ be the conjunction of the formulas below:

$$t_{\mathfrak{M},\sigma}^0(w, d) \wedge \bigwedge_{(w,v) \in R} \Diamond \tau_{\mathfrak{M},\sigma}^k(v, d) \wedge \Box \bigvee_{(w,v) \in R} \tau_{\mathfrak{M},\sigma}^k(v, d),$$
$$\bigwedge_{e \in D} \exists \big(t_{\mathfrak{M}}^0(w, e) \wedge \bigwedge_{(w,v) \in R} \Diamond \tau_{\mathfrak{M}}^k(v, e) \wedge \Box \bigvee_{(w,v) \in R} \tau_{\mathfrak{M}}^k(v, e)\big),$$
$$\forall \bigvee_{e \in D} \big(t_{\mathfrak{M}}^0(w, e) \wedge \bigwedge_{(w,v) \in R} \Diamond \tau_{\mathfrak{M}}^k(v, e) \wedge \Box \bigvee_{(w,v) \in R} \tau_{\mathfrak{M}}^k(v, e)\big).$$

The following lemma says that $\tau_{\mathfrak{M},\sigma}^k(w, d)$ is the strongest formula of modal depth k that is true at w, d in $\mathfrak{M}$:

Lemma 7.1. *For any models $\mathfrak{M}$ with w, d and $\mathfrak{N}$ with v, e, and any $k < \omega$, the following conditions are equivalent*:

(*i*) $\mathfrak{N}, v, e \equiv^k_\sigma \mathfrak{M}, w, d$;

(*ii*) $\mathfrak{N}, v, e \models \tau^k_{\mathfrak{M},\sigma}(w, d)$;

(*iii*) $\mathfrak{N}, v, e \sim^k_\sigma \mathfrak{M}, w, d$.

Proof. The proof is by induction over k. For $k = 0$ the equivalences hold by definition. Assume the equivalences have been shown for k. For $(i) \Rightarrow (ii)$ assume that $\mathfrak{N}, v, e \equiv^{k+1}_\sigma \mathfrak{M}, w, d$. Obviously $\mathfrak{M}, w, d \models \tau^{k+1}_{\mathfrak{M},\sigma}(w, d)$ and $\tau^{k+1}_{\mathfrak{M},\sigma}(w, d)$ has modal depth $k + 1$. Hence $\mathfrak{N}, v, e \models \tau^{k+1}_{\mathfrak{M},\sigma}(w, d)$, as required. $(ii) \Rightarrow (iii)$ Assume $\mathfrak{N}, v, e \models \tau^{k+1}_{\mathfrak{M},\sigma}(w, d)$. Then define $\boldsymbol{\beta}_i$ for $i \leq k + 1$ by taking

$$\boldsymbol{\beta}_i = \big\{\big((v', e'), (w', d')\big) \mid \mathfrak{N}, v', e' \models \tau^i_{\mathfrak{M},\sigma}(w', d')\big\}.$$

It is readily seen that $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_{k+1}$ is a σ-$(k + 1)$-bisimulation. Hence $\mathfrak{N}, v, e \sim^{k+1}_\sigma \mathfrak{M}, w, d$. $(iii) \Rightarrow (i)$ holds by definition of σ-k-bisimulations. ⊣

For any $k \geq 0$ and formula φ with $md(\varphi) \leq k$, we now set

$$\exists^{\sim\sigma,k}\varphi \;=\; \bigvee_{\mathfrak{M},w,d \models \varphi} \tau^k_{\mathfrak{M},\sigma}(w, d).$$

Thus, for any $\mathfrak{N}, v, e$, we have $\mathfrak{N}, v, e \models \exists^{\sim\sigma,k}\varphi$ iff there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{N}, v, e \sim^k_\sigma \mathfrak{M}, w, d$, i.e., $\exists^{\sim\sigma,k}$ is an existential depth restricted bisimulation quantifier [DL06, Fre06]. Clearly, $\models_{\mathsf{Q}^1\mathsf{K}} \varphi \to \exists^{\sim\sigma,k}\varphi$.

Theorem 7.2. *The following conditions are equivalent, for any formulas φ and ψ and $k, k' \geq 0$ with $md(\varphi) = k$, $md(\psi) = k'$, and $n = \max\{k, k'\}$*:

(a) *there is χ such that $sig(\chi) \subseteq \sigma$, $\models_{\mathsf{Q}^1\mathsf{K}} \varphi \to \chi$, and $\models_{\mathsf{Q}^1\mathsf{K}} \chi \to \psi$;*

(b) $\models_{\mathsf{Q}^1\mathsf{K}} \exists^{\sim\sigma,n}\varphi \to \psi$.

Proof. (a) $\Rightarrow$ (b) If $\not\models_{\mathsf{Q}^1\mathsf{K}} \exists^{\sim\sigma,n}\varphi \to \psi$, then there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \exists^{\sim\sigma,n}\varphi$ and $\mathfrak{M}, w, d \models \neg\psi$. By the definition of $\exists^{\sim\sigma,n}\varphi$, we then have $\mathfrak{M}, w, d \models \tau^n_{\mathfrak{M}',\sigma}(w', d')$ and $\mathfrak{M}', w', d' \models \varphi$, for some model $\mathfrak{M}'$, w', d'. By Lemma 7.1, $\mathfrak{M}', w', d' \sim^n_\sigma \mathfrak{M}, w, d$. Using a standard unfolding argument, we may assume that (W, R) in $\mathfrak{M}$ and (W', R') in $\mathfrak{M}'$ are tree-shaped with respective roots w, w'. As φ and ψ have modal depth $\leq n$, we may also assume that the depth of (W, R) and (W', R') is $\leq n$. But then $\mathfrak{M}', w', d' \sim_\sigma \mathfrak{M}, w, d$, contrary to (a). The implication (b) $\Rightarrow$ (a) is trivial. ⊣

We do not know whether $\exists^{\sim\sigma,k}\varphi$ is equivalent to a formula whose size can be bounded by an elementary function in $|\sigma|$, $|\varphi|$, k. For pure $\mathcal{ALC}$, it is indeed equivalent to an exponential-size concept [tCCMV06].

Condition (b) in Theorem 7.2 gives an obvious non-elementary algorithm for checking whether given formulas have an interpolant in $\mathsf{Q}^1\mathsf{K}$. Thus, by Theorem 3.5, we obtain:

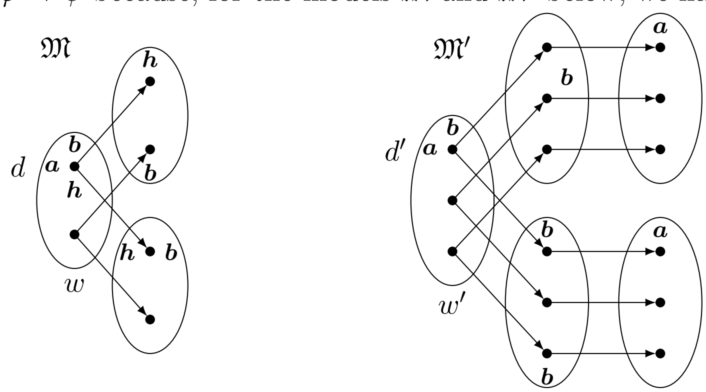
Theorem 7.3. *The IEP and EDEP for $\mathsf{Q}^1\mathsf{K}$ are decidable in non-elementary time.*

The proof above seems to give a hint that the UIEP for $\mathsf{Q}^1\mathsf{K}$ might also be decidable as (an analogue of) $\exists^{\sim\sigma,k}\varphi$ of modal depth $md(\varphi)$ is a uniform interpolant of any propositional modal formula φ in K [Vis96]. The next example illustrates why this is not the case for 'two-dimensional' $\mathsf{Q}^1\mathsf{K}$.

Example 7.4. Suppose $\sigma = \{\boldsymbol{a}, \boldsymbol{b}\} = \mathrm{sig}(\psi)$,

$$\varphi = \forall(\boldsymbol{a} \leftrightarrow \boldsymbol{b} \leftrightarrow \boldsymbol{h}) \wedge \forall(\boldsymbol{h} \leftrightarrow \Box\boldsymbol{h} \leftrightarrow \Diamond\boldsymbol{h}) \wedge \Diamond\forall(\boldsymbol{b} \leftrightarrow \boldsymbol{h}),$$
$$\psi = \forall(\boldsymbol{a} \leftrightarrow \Box\Box\boldsymbol{a} \leftrightarrow \Diamond\Diamond\boldsymbol{a}) \wedge \Box\Diamond\top \to \Diamond\forall(\boldsymbol{b} \leftrightarrow \Diamond\boldsymbol{a}).$$

Intuitively, φ at a world w says (using the 'help' predicate $\boldsymbol{h}$) that there is an R-successor v such that, for every e, we have $v, e \models \boldsymbol{b}$ iff $w, e \models \boldsymbol{b}$. The premise of ψ at w says two things: first, at distance two, R-successors u of w, for every e, we have $u, e \models \boldsymbol{a}$ iff $w, e \models \boldsymbol{a}$; and second, every R-successor of w, in particular v, also has an R-successor u. These conditions imply that, for every e, we have $v, e \models \boldsymbol{b}$ iff $u, e \models \boldsymbol{a}$, and so $\models_{\mathsf{Q^1K}} \varphi \to \psi$. On the other hand, $\not\models_{\mathsf{Q^1K}} \exists^{\sim\sigma,1}\varphi \to \psi$ because, for the models $\mathfrak{M}$ and $\mathfrak{M}'$ below, we have $\mathfrak{M}, w, d \models \varphi$



and $\mathfrak{M}', w', d' \not\models \psi$ but $\mathfrak{M}, w, d \sim^1_\sigma \mathfrak{M}', w', d'$ (a σ-1-bisimulation connects all points in the roots w and w' that agree on σ, and all points in the depth 1 worlds that agree on σ). ⊣

In fact, by adapting the undecidability proof for $\mathsf{Q^1S5}$, we prove the following.

Theorem 7.5. *(i) The (S)CEP for* $\mathsf{Q^1K}$ *is undecidable.*
(ii) The UIEP for $\mathsf{Q^1K}$ *is undecidable.*

For the proof of (i), for any tiling system $\mathfrak{T}$, we show how to construct in polytime formulas φ and ψ such that $\mathfrak{T}$ has a solution iff $\varphi \wedge \psi$ is not a (strong) conservative extension of φ.

Let $\mathfrak{T} = (T, H, V, \boldsymbol{o}, \boldsymbol{z}^{\uparrow}, \boldsymbol{z}^{\rightarrow})$ be a tiling system. To prove undecidability of strong conservative extensions we work with models $\mathfrak{M} = (W, R, D, I)$ of modal depth 1 having a root $r \in W$ and R-successors $W' = W \setminus \{r\}$ of r. We encode the finite grid to be tiled on $W' \times D$ in essentially the same way as previously on the whole $W \times D$. In particular, $\boldsymbol{g}^{\mathfrak{M}} \subseteq W' \times D$ and $R_h^{\mathfrak{M}}, R_v^{\mathfrak{M}} \subseteq \boldsymbol{g}^{\mathfrak{M}} \times \boldsymbol{g}^{\mathfrak{M}}$ are defined as in (5.1) and (5.2) before. We cannot, however, define the modalities $\Diamond_h\chi$ and $\Diamond_v\chi$ using $\mathsf{FOM^1}$-formulas, as we cannot directly refer from (w, d) to (w', d) in our model $\mathfrak{M}$. Instead, we have to 'speak about' $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$ from the viewpoint of points of the form (r, d), for the root r of (W, R). For instance, $\Box\forall(\chi_1 \to \Diamond_h\chi_2)$ is expressed using

$$\forall\big[\Diamond(\boldsymbol{g} \wedge \chi_1 \wedge \neg\boldsymbol{z}^{\rightarrow}) \to \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \chi_2)\big)\big]$$

and $\Box\forall(\chi_1 \to \Diamond_v\chi_2)$ using

$$\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{g} \wedge \chi_1 \wedge \neg\boldsymbol{z}^{\uparrow})\big) \to \Diamond(\boldsymbol{g} \wedge \chi_2)\big].$$

We now define the new formula φ in detail. The following conjuncts generate the grid:

$$\begin{aligned}
&\Diamond(\boldsymbol{o} \wedge \boldsymbol{g}),\\
&\forall\big[\Diamond\big(\boldsymbol{g} \wedge \neg(\boldsymbol{z}^{\uparrow} \wedge \boldsymbol{z}^{\rightarrow})\big) \rightarrow \Diamond \boldsymbol{x}\big],\\
&\Box\forall(\boldsymbol{x} \rightarrow \exists \boldsymbol{g}),\\
&\Box\forall(\boldsymbol{g} \wedge \neg \boldsymbol{z}^{\uparrow} \rightarrow \exists \boldsymbol{y}),\\
&\forall(\Diamond \boldsymbol{y} \rightarrow \Diamond \boldsymbol{g}).
\end{aligned}$$

The constraints on the tiles are expressed by the following conjuncts:

$$\begin{aligned}
&\forall\big(\Diamond \boldsymbol{g} \wedge \neg\Diamond \boldsymbol{x} \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big),\\
&\Box\forall(\boldsymbol{g} \leftrightarrow \bigvee_{\boldsymbol{t}\in T} \boldsymbol{t}) \wedge \Box\forall \bigwedge_{\boldsymbol{t}\neq \boldsymbol{t}'} (\boldsymbol{t} \rightarrow \neg \boldsymbol{t}'),\\
&\forall\big[\Diamond(\boldsymbol{t} \wedge \neg \boldsymbol{z}^{\rightarrow}) \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \bigvee_{(\boldsymbol{t},\boldsymbol{t}')\in H} \boldsymbol{t}')\big)\big],\\
&\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{t} \wedge \neg \boldsymbol{z}^{\uparrow})\big) \rightarrow \Box(\boldsymbol{g} \rightarrow \bigvee_{(\boldsymbol{t},\boldsymbol{t}')\in V} \boldsymbol{t}')\big],\\
&\forall\big(\Diamond(\boldsymbol{y} \wedge \exists \boldsymbol{z}^{\rightarrow}) \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big),\\
&\forall\big[\Diamond \boldsymbol{z}^{\rightarrow} \rightarrow \Box\big(\boldsymbol{y} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\rightarrow})\big)\big],\\
&\forall\big[\Diamond \boldsymbol{z}^{\uparrow} \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\uparrow})\big)\big],\\
&\forall\big(\Diamond(\boldsymbol{x} \wedge \exists \boldsymbol{z}^{\uparrow}) \rightarrow \Box(\boldsymbol{g} \rightarrow \boldsymbol{z}^{\uparrow})\big).
\end{aligned}$$

Finally, we take a fresh predicate $\boldsymbol{p}_0$ and add the conjunct $\boldsymbol{p}_0 \rightarrow \boldsymbol{p}_0$ to φ.

We now aim to construct a formula ψ for which, as previously, we have **(c1)** $\Leftrightarrow$ **(c2)**. This is slightly more involved, as we cannot directly express case distinctions using disjunction and nested 'modalities'. We require auxiliary predicates to achieve this: we use $\boldsymbol{a}_h$ to encode that $\boldsymbol{q}$ is true in an R_h-successor of a $\boldsymbol{q}$-node, $\boldsymbol{a}_v$ to encode that $\boldsymbol{q}$ is true in an R_v-successor of a $\boldsymbol{q}$-node, and $\boldsymbol{b}', \boldsymbol{b}''$ are used to encode that a $\boldsymbol{q}$-node is not confluent. In detail, ψ starts with the conjunct

$$\Diamond(\boldsymbol{g} \wedge \boldsymbol{q}).$$

Next we add a conjunct making a case distinction between $\boldsymbol{a}_h$, $\boldsymbol{a}_v$, and $\boldsymbol{b}'$:

$$\Box\forall\big(\boldsymbol{q} \rightarrow (\neg \boldsymbol{z}^{\rightarrow} \wedge \boldsymbol{a}_h) \vee (\neg \boldsymbol{z}^{\uparrow} \wedge \exists(\boldsymbol{y} \wedge \boldsymbol{a}_v)) \vee (\neg \boldsymbol{z}^{\rightarrow} \wedge \neg \boldsymbol{z}^{\uparrow} \wedge \exists(\boldsymbol{y} \wedge \boldsymbol{b}'))\big).$$

The next two conjuncts state the consequences of $\boldsymbol{a}_h$ and $\boldsymbol{a}_v$, respectively:

$$\begin{aligned}
&\forall\big[\Diamond(\boldsymbol{q} \wedge \boldsymbol{a}_h) \rightarrow \Diamond\big(\boldsymbol{x} \wedge \exists(\boldsymbol{g} \wedge \boldsymbol{q})\big)\big],\\
&\forall\big[\Diamond\big(\boldsymbol{y} \wedge \exists(\boldsymbol{q} \wedge \boldsymbol{a}_v)\big) \rightarrow \Diamond(\boldsymbol{g} \wedge \boldsymbol{q})\big].
\end{aligned}$$

The next conjunct forces $\boldsymbol{s}$ to be true in the horizontal successors of a vertical successor:

$$\forall\big[\Diamond(\boldsymbol{y} \wedge \exists \boldsymbol{b}') \rightarrow \Box\big(\boldsymbol{x} \rightarrow \forall(\boldsymbol{g} \rightarrow \boldsymbol{s})\big)\big].$$

Finally, the following formulas force $\neg \boldsymbol{s}$ in the horizontal successors of a vertical successor:

$$\begin{aligned}
&\forall\big[\Diamond(\boldsymbol{q} \wedge \boldsymbol{b}') \rightarrow \Diamond\big(\boldsymbol{x} \wedge \forall(\boldsymbol{y} \rightarrow \boldsymbol{b}'')\big)\big],\\
&\forall\big(\Diamond(\boldsymbol{y} \wedge \boldsymbol{b}'') \rightarrow \Box(\boldsymbol{g} \rightarrow \neg \boldsymbol{s})\big).
\end{aligned}$$

It is not difficult to show that the equivalence **(c1)** $\Leftrightarrow$ **(c2)** holds for φ and ψ.

Lemma 7.6. *If $\mathfrak{T}$ has a solution, then $\varphi \wedge \psi$ is not a conservative extension of φ.*

Proof. The proof is obtained by modifying the proof of Lemma 5.2. To begin with, we modify the model $\mathfrak{N} = (W, D, J)$ defined in that proof by adding a root world to W from where every other world is accessible in one R-step. More precisely, we define a new model $\mathfrak{N} = (W, R, D, J)$ as follows. We let $D = W' = \{0, \dots, nm-1\}$, $W = W' \cup \{r\}$, $R = \{(r, w) \mid w \in W'\}$, and J is defined by (5.13)–(5.16), plus having $\boldsymbol{p}^{J(r)} = \emptyset$ for $\boldsymbol{p} \in \{\boldsymbol{g}, \boldsymbol{x}, \boldsymbol{y}\} \cup T$, and $\boldsymbol{p}_0^{J(w)} = \emptyset$ for all $w \in W$.

It is straightforward to check that $\mathfrak{N}, r, 0 \models \varphi$ and $\mathfrak{N}, r, 0 \models \neg\psi$. First, we show that $\varphi \wedge \psi$ is not a *strong* conservative extension of φ. We construct a formula χ with $sig(\chi) \cap sig(\psi) \subseteq sig(\varphi)$ such that $\varphi \wedge \chi$ is satisfiable but $\models_{\mathsf{Q^1S5}} \varphi \wedge \chi \to \neg\psi$. It then follows that $\varphi \wedge \psi$ is not a strong conservative extension of φ. The formula χ provides a description of the model $\mathfrak{N}$ at $(r, 0)$. We take, for every $(i, j) \in W \times D$, a fresh predicate $\boldsymbol{p}_{i,j}$ and extend $\mathfrak{N}$ to $\mathfrak{N}'$ by setting, for all $(i', j') \in W \times D$,

$$\mathfrak{N}', i', j' \models \boldsymbol{p}_{i,j} \quad \text{iff} \quad (i', j') = (i, j). \tag{7.1}$$

Now let $\sigma' = sig(\varphi) \cup \{\boldsymbol{p}_{i,j} \mid (i, j) \in W \times D\}$, and set

$$\chi_{i,j} = \bigwedge_{\boldsymbol{p} \in \sigma',\ \mathfrak{N}, i, j \models \boldsymbol{p}} \boldsymbol{p} \ \wedge \bigwedge_{\boldsymbol{p} \in \sigma',\ \mathfrak{N}, i, j \models \neg\boldsymbol{p}} \neg\boldsymbol{p}. \tag{7.2}$$

Let χ be the conjunction of of the following formulas:

$$\chi_{r,0} \wedge \Box(\boldsymbol{o} \wedge \boldsymbol{g} \to \chi_{0,0}), \tag{7.3}$$

$$\forall\big[\Diamond\chi_{i,i} \to \Box\big(\boldsymbol{x} \to (\chi_{i+1,i} \wedge \forall(\boldsymbol{g} \to \chi_{i+1,i+1}))\big)\big], \text{ for } i < nm-1, \tag{7.4}$$

$$\forall\big[\Diamond(\boldsymbol{y} \wedge \exists\chi_{i,i}) \to \Box(\boldsymbol{y} \to \chi_{i,i+n}) \wedge \Box(\boldsymbol{g} \to \chi_{i+n,i+n})\big], \text{ for } i < nm-n, \tag{7.5}$$

$$\Box\forall(\chi_{i,i} \to \exists\chi_{i,j}), \text{ for } i, j < nm, \tag{7.6}$$

$$\forall(\Diamond\chi_{i,i} \to \Diamond\chi_{j,i}), \text{ for } i, j < nm, \tag{7.7}$$

$$\forall(\Diamond\chi_{i,j} \to \chi_{r,j}), \text{ for } i, j < nm, \tag{7.8}$$

$$\forall\big(\chi_{r,j} \to \Box(\exists\chi_{l,k} \to \chi_{l,j})\big), \text{ for } j, k, l < nm. \tag{7.9}$$

It is easy to see that $\mathfrak{N}', r, 0 \models \chi$, and so $\varphi \wedge \chi$ is satisfiable. Now suppose that $\mathfrak{M}$ is any model such that $\mathfrak{M}, w_0, d_0 \models \varphi \wedge \chi$ for some w_0, d_0. We show that $\mathfrak{M}, w_0, d_0 \models \neg\psi$. Observe that if $\big((w, d), (w', d')\big) \in R_h^{\mathfrak{M}}$ and $\mathfrak{M}, w, d \models \chi_{i,i}$, then $\mathfrak{M}, w', d' \models \chi_{i+1,i+1}$, by (7.4), and if $\big((w, d), (w', d')\big) \in R_v^{\mathfrak{M}}$ and $\mathfrak{M}, w, d \models \chi_{i,i}$, then $\mathfrak{M}, w', d' \models \chi_{i+n,i+n}$, by (7.5). Hence, there cannot be an infinite $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain.

Now suppose there is an $R_h^{\mathfrak{M}} \cup R_v^{\mathfrak{M}}$-chain from (w_0, d_0) to some node (w, d) which has an $R_h^{\mathfrak{M}}$-successor (w_1, d_1) and $R_v^{\mathfrak{M}}$-successor (w_2, d_2). Then $\mathfrak{M}, w, d \models \chi_{i,i}$ for some i, $\mathfrak{M}, w_1, d_1 \models \chi_{i+1,i+1}$ and $\mathfrak{M}, w_2, d_2 \models \chi_{i+n,i+n}$, by (7.3)–(7.5).

There exist d_1' with $\mathfrak{M}, w_1, d_1' \models \chi_{i+1,i+n+1}$, and w_2' with $\mathfrak{M}, w_2', d_2 \models \chi_{i+n+1,i+n}$, by (7.6) and (7.7). Then $\mathfrak{M}, w_0, d' \models \chi_{r,i+n+1}$ by (7.8). By (7.9) for $l = j = i+n+1$ and $k = i+n$, we obtain $\mathfrak{M}, w_2', d_1' \models \chi_{i+n+1,i+n+1}$. Moreover, as $\boldsymbol{z}^{\uparrow}$ is not a conjunct of $\chi_{i+1,i+1}$, and $\boldsymbol{z}^{\rightarrow}$ is not a conjunct of $\chi_{i+n,i+n}$, we have that $\boldsymbol{y}$ is a conjunct of $\chi_{i+1,i+n+1}$, and $\boldsymbol{x}$ is a conjunct of $\chi_{i+n+1,i+n}$. Thus, $\big((w_1, d_1), (w_2', d_1')\big) \in R_v^{\mathfrak{M}}$ and $\big((w_2, d_2)(w_2', d_1')\big) \in R_h^{\mathfrak{M}}$, and so (w, d) is confluent.

We next aim to prove that $\varphi \wedge \psi$ is not a (necessarily strong) conservative extension of φ. In this case, we are not allowed to use the fresh predicates $\boldsymbol{p}_{i,j}$ in the formula χ to achieve (7.1). We instead use the predicate $\boldsymbol{p}_0 \in sig(\varphi)$ to uniquely characterise the points of $\mathfrak{N}$. Take a bijection f from $\{0,\dots,n-1\} \times \{0,\dots,m-1\}$ to $\{0,\dots,nm-1\}$ and set, for all $(i,j) \in W \times D$:

$$\varphi_{i,j} = \Diamond^{f(i,j)+1}\boldsymbol{p}_0, \quad \varphi_{r,j} = \neg\exists \boldsymbol{g} \wedge \Diamond\varphi_{0,j}, \quad \text{for } i,j < nm.$$

Modify the model $\mathfrak{N} = (W,R,D,J)$ constructed above to $\mathfrak{N}^+ = (W^+,R^+,D,J^+)$ by adding an nm-long R-chain to each leaf $i < nm$ of (W,R), and define $\boldsymbol{p}_0^{J^+(w)}$ for each $w \in W^+ \setminus W$ such that we still have, for all 'old' points $(i',j') \in W \times D$, the analogue of (5.17):

$$\mathfrak{N}^+, i', j' \models \varphi_{i,j} \quad \text{iff} \quad (i',j') = (i,j).$$

Now let $\sigma^- = sig(\varphi) \setminus \{\boldsymbol{p}_0\}$ and set, for all $(i,j) \in W \times D$,

$$\chi'_{i,j} = \varphi_{i,j} \wedge \bigwedge_{\boldsymbol{p}\in\sigma^-,\ \mathfrak{N},i,j\models\boldsymbol{p}} \boldsymbol{p} \ \wedge \bigwedge_{\boldsymbol{p}\in\sigma^-,\ \mathfrak{N},i,j\models\neg\boldsymbol{p}} \neg\boldsymbol{p}.$$

Finaly, define χ' with $sig(\chi') \subseteq sig(\varphi)$ by replacing $\chi_{i,j}$ in (7.3)–(7.9) by $\chi'_{i,j}$. ⊣

Lemma 7.7. *If $\varphi \wedge \psi$ is not a model conservative extension of φ, then $\mathfrak{T}$ has a solution.*

Proof. Consider a model $\mathfrak{M} = (W,R,D,I)$ such that $\mathfrak{M},w,d \models \varphi$ but $\mathfrak{M}',w,d \models \neg\psi$ in any extension $\mathfrak{M}'$ of $\mathfrak{M}$ obtained by interpreting the predicates $\boldsymbol{q},\boldsymbol{s}$. Similarly, to the proof of Lemma 5.3, by using the equivalence **(c1)** $\Leftrightarrow$ **(c2)**, one can easily find within $\mathfrak{M}$ a finite grid-shaped (with respect to $R_h^{\mathfrak{M}}$ and $R_v^{\mathfrak{M}}$) submodel, which gives a solution to $\mathfrak{T}$. ⊣

This completes the proof of Theorem 7.5 (*i*). We next prove Theorem 7.5 (*ii*) based on a modification of Example 5.4 for the case of $\mathsf{Q^1K}$.

Example 7.8. Let φ_0

$$\varphi_0 = \forall\big(\Diamond\boldsymbol{a} \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big) \wedge \Box\forall\big(\boldsymbol{p}_1 \wedge \boldsymbol{b} \to \exists(\boldsymbol{p}_2 \wedge \boldsymbol{b})\big) \wedge \forall\big(\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{b}) \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big)$$

and let $\sigma = \{\boldsymbol{a},\boldsymbol{p}_1,\boldsymbol{p}_2\}$. We show that there is no σ-uniform interpolant of $\Diamond\boldsymbol{a} \wedge \varphi_0$ in $\mathsf{Q^1K}$.

For every $s > 0$, we define a formula χ_s as follows. Take fresh predicates $\boldsymbol{a}_s^h$ and $\boldsymbol{a}_s^v$ and construct χ_s stating that s-many steps of the $\boldsymbol{p}_1,\boldsymbol{p}_2$-ladder have been constructed by forcing $\boldsymbol{a}_1^h,\dots,\boldsymbol{a}_s^h$ to be true horizontally and $\boldsymbol{a}_1^v,\dots,\boldsymbol{a}_s^v$ to be true vertically in the corresponding steps. In detail, we let

$$\chi'_1 = \forall(\Diamond\boldsymbol{a} \to \Box\boldsymbol{a}_1^h) \wedge \Box\forall(\boldsymbol{a}_1^h \wedge \boldsymbol{p}_1 \to \forall\boldsymbol{a}_1^v),$$
$$\chi_1 = \chi'_1 \to \exists\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{a}_1^v),$$

and for $s > 0$,

$$\chi'_{s+1} = \chi'_s \wedge \forall\big(\Diamond(\boldsymbol{a}_s^v \wedge \boldsymbol{p}_2) \to \Box\boldsymbol{a}_{s+1}^h\big) \wedge \Box\forall(\boldsymbol{a}_{s+1}^h \wedge \boldsymbol{p}_1 \to \forall\boldsymbol{a}_{s+1}^v),$$
$$\chi_{s+1} = \chi'_{s+1} \to \exists\Diamond(\boldsymbol{p}_2 \wedge \boldsymbol{a}_{s+1}^v).$$

Then $\models_{\mathsf{Q^1K}} \Diamond\boldsymbol{a} \wedge \varphi_0 \to \chi_s$ for all $s > 0$. Thus, if ϱ were a σ-uniform interpolant of $\Diamond\boldsymbol{a} \wedge \varphi_0$, then $\models_{\mathsf{Q^1K}} \varrho \to \chi_s$ would follow, for all $s > 0$.

On the other hand, for $s > 0$, we modify the model $\mathfrak{M}_s = (W_s,D_s,I_s)$ defined in Example 5.4 by adding a root world to W_s from where every other world is accessible in one

R_s-step. More precisely, we define a new model $\mathfrak{M}_s = (W_s, R_s, D_s, I_s)$ as follows. We let $W'_s = D_s = \{0, \dots, s-1\}$, $W_s = W'_s \cup \{r\}$, $R_s = \{(r, w) \mid w \in W'_s\}$, and I_s is defined by

$$\boldsymbol{a}^{I_s(k)} = \begin{cases} \{0\}, & \text{if } k = 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{7.10}$$

$$\boldsymbol{p}_1^{I_s(k)} = \begin{cases} \{k-1\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{7.11}$$

$$\boldsymbol{p}_2^{I_s(k)} = \begin{cases} \{k\}, & \text{if } k > 0, \\ \emptyset, & \text{otherwise;} \end{cases} \tag{7.12}$$

plus having $\boldsymbol{p}^{I_s(r)} = \emptyset$ for $\boldsymbol{p} \in \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{a}_1^h, \dots, \boldsymbol{a}_s^h, \boldsymbol{a}_1^v, \dots, \boldsymbol{a}_s^v\}$ and, for all $0 < i \leq s$ and all $k < s$,

$$(\boldsymbol{a}_i^h)^{I_s(k)} = \{i-1\}, \qquad (\boldsymbol{a}_i^v)^{I_s(k)} = \begin{cases} \{0, \dots, s-1\}, & \text{if } 0 < k = i < s, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for every $s > 0$, $\mathfrak{M}_s, r, 0 \not\models \chi_s$ and so $\mathfrak{M}_s, r, 0 \models \neg\varrho$. Also, $\mathfrak{M}_s, r, 0 \models \chi_{s'}$ for all $s' < s$. Now consider the ultraproduct $\prod_U \mathfrak{M}_s$ with U a non-principal ultrafilter on $\omega \setminus \{0\}$. As each $\chi_{s'}$ is true in almost all $\mathfrak{M}_s, r, 0$, it follows from the properties of ultraproducts [CK98] that $\prod_U \mathfrak{M}_s, \overline{r}, \overline{0} \models \Diamond\boldsymbol{a} \wedge \neg\varrho \wedge \chi_{s'}$ for all $s' > 0$, for some suitable $\overline{r}, \overline{0}$. But then one can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{M}_s$ such that $\mathfrak{M}, \overline{r}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{M}$. Then $\mathfrak{M} \models \Diamond\boldsymbol{a} \wedge \varphi_0 \wedge \neg\varrho$ and as $\models_{\mathsf{Q}^1\mathsf{K}} \Diamond\boldsymbol{a} \wedge \varphi_0 \to \varrho$ should hold for a uniform interpolant ϱ of $\Diamond\boldsymbol{a} \wedge \varphi_0$, we have derived a contradiction. ⊣

The proof of Theorem 7.5 (*ii*) is now by combining the construction of Theorem 7.5 and Example 7.8 in exactly the same way as the construction of Theorem 5.1 and Example 5.4 were combined in the proof of Theorem 5.1 (*ii*).

8. Outlook

Craig interpolation and Beth definability have been studied extensively for most logical systems, let alone those with applications in computing such as KR, verification, and databases. In fact, one of the first questions typically asked about a logic L of interest is whether L has interpolants for *all* valid implications $\varphi \to \psi$. Some L enjoy this property, while others miss it. This paper and preceding [JW21, AJM$^+$23] open a new, *non-uniform* perspective on interpolation/definability for the latter type of L by regarding formulas φ and ψ as input and deciding whether they have an interpolant in L. We refer the reader to [KWZ26] for an overview of current research and open questions from this perspective.

In the context of first-order modal logics, challenging open questions that arise from this work are:

- What is the tight complexity of the IEP for $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$?
- Is the non-elementary upper bound for the IEP in $\mathsf{Q}^1\mathsf{K}$ optimal? (Note that just like $\mathsf{Q}^1\mathsf{S5}$-validity, $\mathsf{Q}^1\mathsf{K}$-validity is known to be coNExpTime-complete [Mar99].)
- Is the IEP decidable for $\mathsf{K}_{\mathcal{ALC}^u}$?
- More generally, what happens if we replace $\mathsf{S5}$ and K by other standard modal logics, e.g., $\mathsf{S4}$, multimodal $\mathsf{S5}$, or the linear temporal logic *LTL*, and/or use in place of $\mathcal{ALC}^u$ more expressive DLs containing, for instance, nominals or role inclusions, and/or consider other

monodic fragments of first-order modal logics such as the monodic guarded or two-variable fragment?
- It would also be of interest to investigate variants of the IEP in more detail. For instance, for the logics considered here, does a Lyndon interpolant exist iff an arbitrary interpolant exists? If not, is the existence of Lyndon interpolants also decidable and of the same complexity? Is global deductive interpolant existence decidable for $\mathsf{Q}^1\mathsf{K}$?

A different line of research is computing interpolants. For logics with the CIP, this is typically done using resolution, tableau, or sequent calculi as , e.g., in [Fit96, KV17, Kuz18, Wer21]. It remains to be seen if interpolants in the logics without the CIP considered in this article can be extracted from tableau or resolution proofs in these calculi or those designed specifically for monodic fragments [LSWZ01, LSWZ02, DFK06]. A more recent approach is based on type-elimination known from complexity proofs for modal and guarded logics [BtCV16, tC22]. The question whether these proofs can be turned into an algorithm computing interpolants in, say, $\mathsf{Q}^1\mathsf{S5}$ is non-trivial and open. More generally, one can try to develop calculi for the consequence relation '$\varphi \models \psi$ iff there are no $sig(\varphi) \cap sig(\psi)$-bisimilar models satisfying φ and $\neg\psi$' and use them to compute interpolants; see [BvB99] for a model-theoretic account of such consequence relations for infinitary logics without the CIP.

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Appendix A. Connections with FO^2

We begin by discussing the connections between $\mathsf{Q}^1\mathsf{S5}$ and FO^2. Then we prove the lower bound result of Theorem 1.3, stating that interpolant and definition existence in FO^2 without equality are 2ExpTime-hard.

The *atoms* of FO^2 are of the form $x = y$, $\boldsymbol{p}(x, y)$, $\boldsymbol{p}(y, x)$, $\boldsymbol{p}(x, x)$, and $\boldsymbol{p}(y, y)$, with $\boldsymbol{p}$ ranging over binary predicate symbol in $\mathcal{P}$.[2] A *signature* is any finite set $\sigma \subseteq \mathcal{P}$. FO^2-*formulas* are built up from atoms using $\neg, \wedge, \exists x, \exists y$. We consider two proper fragments of FO^2: in the *equality-free* fragment, we do not have atoms of the form $x = y$, and in the *equality- and substitution-free* fragment, the only available atoms are of the form $\boldsymbol{p}(y, x)$. We interpret FO^2-formulas in usual FO-*models* of the form $\mathfrak{A} = (A^{\mathfrak{A}}, \boldsymbol{p}^{\mathfrak{A}})_{\boldsymbol{p} \in \mathcal{P}}$, where $A^{\mathfrak{A}}$ is a nonempty set and $\boldsymbol{p}^{\mathfrak{A}} \subseteq A^{\mathfrak{A}} \times A^{\mathfrak{A}}$, for each $\boldsymbol{p} \in \mathcal{P}$.

Now, fix some signature σ. We connect two FO-models $\mathfrak{A}, \mathfrak{B}$ with three different kinds of σ-bisimulations, depending on the chosen fragment $\mathcal{L}$ of FO^2, as follows. Given $\mathcal{L}$, let $Lit_{\mathcal{L}(\sigma)}$ denote the set of available *literals* for $\boldsymbol{p} \in \sigma$ (atoms and negated atoms) in $\mathcal{L}$. Given $\mathfrak{A}$ and $a, a' \in A^{\mathfrak{A}}$, we define

$$\ell_{\mathfrak{A}}^{\mathcal{L}(\sigma)}(a, a') = \big\{\ell \in Lit_{\mathcal{L}(\sigma)} \mid \mathfrak{A} \models \ell[a/y, a'/x]\big\}.$$

A relation $\boldsymbol{\beta} \subseteq (A^{\mathfrak{A}} \times A^{\mathfrak{A}}) \times (B^{\mathfrak{B}} \times B^{\mathfrak{B}})$ is a σ-*bisimulation between* $\mathfrak{A}$ *and* $\mathfrak{B}$ *in* $\mathcal{L}$ if the following hold, for all $\big((a, a'), (b, b')\big) \in \boldsymbol{\beta}$:

[2] It is shown in [GKV97] that, in FO^2, one can replace relations of arbitrary arity by binary relations as far the complexity of satisfiability is concerned. This has been extended to bisimulation consistency in [JW21]. We therefore consider FO^2 with binary relations only.

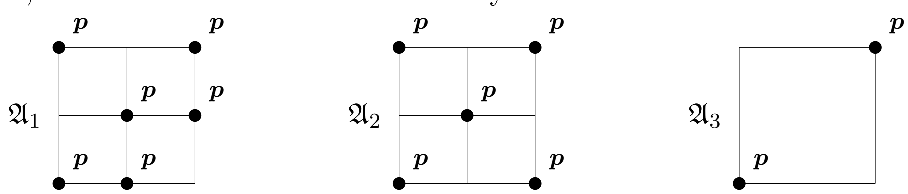
(1) $\ell_{\mathfrak{A}}^{\mathcal{L}(\sigma)}(a,a') = \ell_{\mathfrak{B}}^{\mathcal{L}(\sigma)}(b,b')$;
(2) for every $a'' \in A^{\mathfrak{A}}$ there is $b'' \in B^{\mathfrak{B}}$ such that $\big((a,a''),(b,b'')\big) \in \boldsymbol{\beta}$, and the other way round;
(3) for every $a'' \in A^{\mathfrak{A}}$ there is $b'' \in B^{\mathfrak{B}}$ such that $\big((a'',a'),(b'',b')\big) \in \boldsymbol{\beta}$, and the other way round.

If $\big((a,a'),(b,b')\big) \in \boldsymbol{\beta}$ for some $\boldsymbol{\beta}$, then we say that $\mathfrak{A}, a, a'$ and $\mathfrak{B}, b, b'$ are *σ-bisimilar in $\mathcal{L}$*.

Given FO-models $\mathfrak{A}, \mathfrak{B}$ and $a, a' \in A^{\mathfrak{A}}$, $b, b' \in B^{\mathfrak{B}}$, we write $\mathfrak{A}, a, a' \equiv_{\mathcal{L}(\sigma)} \mathfrak{B}, b, b'$ whenever $\mathfrak{A} \models \varphi[a/y, a'/x]$ iff $\mathfrak{B} \models \varphi[b/y, b'/x]$ hold for all $\mathcal{L}(\sigma)$-formulas φ. Then we have the following well-known equivalence, for any pair $\mathfrak{A}, \mathfrak{B}$ of saturated models:

$$\mathfrak{A}, a, a' \equiv_{\mathcal{L}(\sigma)} \mathfrak{B}, b, b' \quad \text{iff} \quad \mathfrak{A}, a, a' \text{ and } \mathfrak{B}, b, b' \text{ are } \sigma\text{-bisimilar in } \mathcal{L}.$$

Clearly, if $\mathfrak{A}, a, a'$ and $\mathfrak{B}, b, b'$ are σ-bisimilar in FO^2, then they are σ-bisimilar in the equality-free fragment, and if they are σ-bisimilar in the equality-free fragment, then they are σ-bisimilar in the equality- and substitution-free fragment. However, as the models below show, the converse directions do not always hold.



Here, $\mathfrak{A}_1$ and $\mathfrak{A}_3$ are $\{\boldsymbol{p}\}$-bisimilar in the equality- and substitution-free fragment of FO^2, but not in the equality-free fragment: $\forall x \forall y\big(\boldsymbol{p}(x,y) \leftrightarrow \boldsymbol{p}(y,x)\big) \land \forall x \boldsymbol{p}(x,x)$ is true in $\mathfrak{A}_3$, while it is not true in $\mathfrak{A}_1$. $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are $\{\boldsymbol{p}\}$-bisimilar in the equality-free fragment, but not in FO^2: $\forall x \forall y\big(\boldsymbol{p}(x,y) \to x = y\big)$ is true in $\mathfrak{A}_3$, while it is not true in $\mathfrak{A}_2$.

Next, we discuss connections between $\mathsf{Q}^1\mathsf{S5}$ and the three fragments of FO^2 introduced above. With a slight abuse of notation, we consider the predicate symbols in $\mathcal{P}$ (and thus in any signature σ) as unary symbols when dealing with FOM^1 and binary ones when dealing with (fragments of) FO^2. We translate each FOM^1-formula φ to an FO^2-formula $\varphi^\dagger$ by taking

$$\boldsymbol{p}(x)^\dagger = \boldsymbol{p}(y,x), \quad (\neg\varphi)^\dagger = \neg\varphi^\dagger, \quad (\varphi \land \psi)^\dagger = \varphi^\dagger \land \psi^\dagger, \quad (\Diamond\varphi)^\dagger = \exists y \varphi^\dagger, \quad (\exists\varphi)^\dagger = \exists x \varphi^\dagger.$$

Observe that the image of this translation is in the equality- and substitution-free fragment of FO^2. It is easy to show the following:

Lemma A.1. *For all FOM^1-formulas φ, ψ and FOM^1-signature σ,*

(i) $\models_{\mathsf{Q^1S5}} \varphi$ *iff* $\models_{\mathsf{FO}^2} \varphi^\dagger$;
(ii) *φ and ψ are σ-bisimulation consistent in $\mathsf{Q}^1\mathsf{S5}$ iff $\varphi^\dagger$ and $\psi^\dagger$ are σ-bisimulation consistent in the equality- and substitution-free fragment of FO^2.*

Proof. Follows straightforwardly from the observations (a) and (b) below:

(a) FO-models for the equality- and substitution-free fragment of FO^2 and *square* $\mathsf{Q}^1\mathsf{S5}$-models (W, D, I) with $|W| = |D|$ are in 1–1 correspondence in the following sense:

– For every FO-model $\mathfrak{A}$, take the square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}_{\mathfrak{A}} = (A^{\mathfrak{A}}, A^{\mathfrak{A}}, I)$ where, for all $a, b \in A^{\mathfrak{A}}$ and $\boldsymbol{p} \in \mathcal{P}$, $b \in \boldsymbol{p}^{I(a)}$ iff $(a,b) \in \boldsymbol{p}^{\mathfrak{A}}$. Then we have

$$\mathfrak{M}_{\mathfrak{A}}, a, b \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi^\dagger[a/y, b/x].$$

– For every square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M} = (W, D, I)$ and every bijection $f \colon D \to W$, take the FO-model $\mathfrak{A}_{\mathfrak{M},f} = (D, \boldsymbol{p}^{\mathfrak{A}_{\mathfrak{M},f}})$ where, for all $a, b \in D$ and $\boldsymbol{p} \in \mathcal{P}$, $(a,b) \in \boldsymbol{p}^{\mathfrak{A}_{\mathfrak{M},f}}$ iff $b \in \boldsymbol{p}^{I(f(a))}$.

Then we have

$$\mathfrak{M}, f(a), b \models \varphi \quad \text{iff} \quad \mathfrak{A}_{\mathfrak{M},f} \models \varphi^{\dagger}[a/y, b/x]. \tag{A.1}$$

(*b*) For all $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M}$, w, d, there exists a square $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}'$, w', d' such that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are bisimilar in $\mathsf{Q}^1\mathsf{S5}$ (see, e.g., [GKWZ03, Prop.3.12]). ⊣

Proof of Theorem 1.3, lower bound. For the equality- and substitution-free fragment of FO^2, 2ExpTime-hardness is now a straightforward consequence of Lemma A.1. Indeed, we can simply use the †-translations of the formulas used in Section 4.2. In order to prove 2ExpTime-hardness for the equality-free fragment, we need an additional step. Namely, we need to show that there are suitable bijections between the FO- and modal domains of each of the two (square) $\mathsf{Q}^1\mathsf{S5}$-models constructed in the proof of Lemma 4.8 such that the resulting FO-models are not only σ-bisimilar in the equality- and substitution-free fragment of FO^2, but are also σ-bisimilar in the equality-free fragment. In fact, we claim that they are σ-bisimilar in full FO^2, and so the lower bound result of Theorem 1.3 generalises that of [JW21]. To this end, take the σ-bisimulation $\boldsymbol{\beta}$ in $\mathsf{Q}^1\mathsf{S5}$ between the $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M} = (W, D, I)$ and $\hat{\mathfrak{M}} = (\hat{W}, \hat{D}, \hat{I})$ defined in the proof of Lemma 4.8. Now define a bijection $f\colon D \to W$ by taking $f(d^t_m) = w^t_m$ for all $m < 2^n$, $t \in T$, and a bijection $\hat{f}\colon \hat{D} \to \hat{W}$ by taking $\hat{f}(\hat{d}^t_k) = \hat{w}^t_k$ for all $k < 2$, $t \in T$. Using that (*i*) the respective restrictions of the FO-models $\mathfrak{A}_{\mathfrak{M},f}$ to D_m and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}}$ to $\hat{D}_k$ are σ-isomorphic for any $m < 2^n$, $k < 2$, and (*ii*) for all $\boldsymbol{p} \in \sigma$, we have $\mathfrak{A}_{\mathfrak{M},f} \not\models \boldsymbol{p}[y/d^t_m, x/d^{t'}_{m'}]$ if $m \neq m'$, and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}} \not\models \boldsymbol{p}[y/\hat{d}^t_k, x/\hat{d}^{t'}_{k'}]$ if $k \neq k'$, it is straightforward to see that the relation

$$\boldsymbol{\beta}^{f,\hat{f}} = \big\{ \big((a,b),(\hat{a},\hat{b})\big) \in (D \times D) \times (\hat{D} \times \hat{D}) \mid \big((f(a),b),\hat{f}(\hat{a}),\hat{b})\big) \in \boldsymbol{\beta} \big\}$$

is a σ-bisimulation between $\mathfrak{A}_{\mathfrak{M},f}$ and $\mathfrak{A}_{\hat{\mathfrak{M}},\hat{f}}$ in FO^2. ⊣

Appendix B. Proofs for Section 6

Here, we give polynomial-time reductions of the interpolant existence problems modulo ontologies (introduced in Remark 6.1) to the IEP. Call a concept E with $sig(E) \subseteq \sigma$ a *σ-interpolant for a concept inclusion* $C \sqsubseteq D$ if $\models C \sqsubseteq E$ and $\models E \sqsubseteq D$. The problem to decide whether a σ-interpolant exists for $C \sqsubseteq D$ can be reduced in polytime to interpolant existence, as the following two conditions are easily seen to be equivalent, for any σ and concept inclusion $C \sqsubseteq D$:

– there exists a σ-interpolant for $C \sqsubseteq D$;
– there exists an interpolant for $C' \sqsubseteq D'$, where C', D' are obtained from C, D by replacing in C all symbols not in σ by fresh symbols not in C and D and adding, for all $A \in \sigma$ and $R \in \sigma$, the conjuncts $A \sqcup \neg A$ and $\exists R.\top \sqcup \neg\exists R.\top$ to C and D.

For an ontology $\mathcal{O}$, take the concept

$$\mathcal{O}^c = \bigwedge_{C \sqsubseteq D \in \mathcal{O}} \forall U.(\neg C \sqcup D).$$

Recall that the IEP modulo ontologies is to decide, given an ontology $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$, whether there exists a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$. For the reduction, assume $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$ are given. Then the following conditions are equivalent:

– there exists a σ-interpolant for $\mathcal{O}^c \sqcap C \sqsubseteq \neg\mathcal{O}^c \sqcup D$;

– there exists a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$.

Next, recall that ontology interpolant existence is to decide, given an ontology $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$, whether there is an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$. For the reduction, assume $\mathcal{O}$, a signature σ, and a concept inclusion $C \sqsubseteq D$ are given. Then the following conditions are equivalent:

– there exists a σ-interpolant for $\mathcal{O}^c \sqsubseteq \neg C \sqcup D$;
– there exists an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$.

Finally, we recall that the DEP modulo ontologies is to decide, given an ontology $\mathcal{O}$, a signature σ, and a concept name A, whether there exists a σ-concept C such that $\mathcal{O} \models A \equiv C$. We reduce this problem to the IEP modulo ontologies. Suppose an ontology $\mathcal{O}$, a signature σ, and a concept name A are given. We may assume that $A \notin \sigma$ since otherwise the problem is trivial. Now let $\mathcal{O}'$ be the result of replacing in $\mathcal{O}$ all symbols X that are not in σ by fresh X'. Then the following conditions are equivalent:

– there exists a σ-concept C such that $\mathcal{O} \cup \mathcal{O}' \models A \sqsubseteq C$ and $\mathcal{O} \cup \mathcal{O}' \models C \sqsubseteq A'$;
– there exists a σ-concept C such that $\mathcal{O} \models A \equiv C$.

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