



ON THE DESCRIPTIVE COMPLEXITY OF GROUPS WITHOUT ABELIAN NORMAL SUBGROUPS

JOSHUA A. GROCHOW ^{a,b} AND MICHAEL LEVET ^c

^a Department of Computer Science, University of Colorado Boulder, 1111 Engineering Dr, ECOT
717, 430 UCB, Boulder, CO 80309, USA
e-mail address: jgrochow@colorado.edu

^b Department of Mathematics, University of Colorado Boulder, Campus Box 395, Boulder, CO 80309
USA

^c Department of Computer Science, College of Charleston, 66 George Street, Charleston, SC 29424,
USA
e-mail address: levetm@cofc.edu

ABSTRACT. In this paper, we explore the descriptive complexity theory of finite groups by examining the power of the second Ehrenfeucht–Fraïssé bijective pebble game in Hella’s (*Ann. Pure Appl. Log.*, 1989) hierarchy. This is a Spoiler–Duplicator game in which Spoiler can place up to two pebbles each round. While it trivially solves graph isomorphism, it may be nontrivial for finite groups, and other ternary relational structures. We first provide a novel generalization of Weisfeiler–Leman (WL) coloring, which we call *2-ary WL*. We then show that 2-ary WL is equivalent to the second Ehrenfeucht–Fraïssé bijective pebble game in Hella’s hierarchy.

Our main result is that, in the pebble game characterization, only $O(1)$ pebbles and $O(1)$ rounds are sufficient to identify all groups without Abelian normal subgroups (a class of groups for which isomorphism testing is known to be in P; Babai, Codenotti, & Qiao, ICALP 2012). We actually show that 7 pebbles and 7 rounds suffice. In particular, we show that within the first few rounds, Spoiler can force Duplicator to select an isomorphism between two such groups at each subsequent round. By Hella’s results (*ibid.*), this is equivalent to saying that these groups are identified by formulas in first-order logic with generalized 2-ary quantifiers, using only $O(1)$ variables and $O(1)$ quantifier depth.

1. INTRODUCTION

¹ Descriptive complexity theory studies the relationship between the complexity of describing a given problem in some logic, and the complexity of solving the problem by an algorithm. When the problems involved are isomorphism problems, Immerman and Lander [IL90] showed that complexity of a logical sentence describing the isomorphism type of a graph was essentially the same as the Weisfeiler–Leman coloring dimension of that graph, and the complexity of an Ehrenfeucht–Fraïssé pebble game (see also [CFI92]).

¹A preliminary version of this work previously appeared in GandALF 2023 [GL23]. Here, we provide full proofs of (i) the equivalence between 2-ary WL and Hella’s 2-ary pebble game, and (ii) key lemmas for establishing our main result Theorem 1.1. Due to space constraints, these details were omitted in the extended abstract that appeared in GandALF.

It is a well-known open question whether there is a logic that exactly captures the complexity class P on unordered (unlabeled) structures; on ordered structures such a logic was given by Immerman [Imm86] and Vardi [Var82]. The difference between these two settings is essentially the GRAPH CANONIZATION problem, whose solution allows one to turn an unordered graph into an ordered graph in an isomorphism-preserving way.

One natural approach in trying to capture P on unordered structures is thus to attempt to extend first-order logic FO by generalized quantifiers (cf. Mostowski [Mos57] and Lindstrom [Lin66]) in the hopes that the augmented logics can characterize finite graphs up to isomorphism, thus reducing the unordered case to the previously solved ordered case. A now-classical approach, initiated by Immerman [Imm86], was to augment fixed-point logic with counting quantifiers, which can be analyzed in terms of an equivalence induced by (variable confined) fragments of first-order logic with counting. However, Cai, Fürer, & Immerman [CFI92] showed that $FO + LFP$ plus counting does not capture P on finite graphs. More generally, Flum & Grohe have characterized when FO plus counting captures P on unordered structures [FG00].

The approach of Cai, Fürer, & Immerman (*ibid.*, see also [IL90]) was to prove a three-way equivalence: between (1) counting logics, (2) the higher-dimensional Weisfeiler–Leman coloring procedure, and (3) Ehrenfeucht–Fraïssé pebble games. Ehrenfeucht–Fraïssé pebble games [Ehr61, Fra54] have long been an important tool in proving the inexpressibility of certain properties in various logics; in this case, they used such games to show that the logics could not express the difference between certain pairs of non-isomorphic graphs. Consequently, Cai, Fürer, & Immerman ruled out the Weisfeiler–Leman algorithm as a polynomial-time isomorphism test for graphs, which resolved a long-standing open question in isomorphism testing. Nonetheless, the Weisfeiler–Leman coloring procedure is a key subroutine in many algorithms for GRAPH ISOMORPHISM, including Babai’s quasi-polynomial-time algorithm [Bab16]. It is thus interesting to study its properties and its distinguishing power.

While the result of Cai, Fürer, & Immerman ruled out Weisfeiler–Leman as a polynomial-time isomorphism test for graphs, for *groups* it remains an interesting open question. The general WL procedure for groups was introduced by Brachter & Schweitzer [BS20] and has been studied in several papers since then [BS22, GL26, CL24, Col23, Lev23, Vag23, Bra23, CRP24]. Outside the scope of WL, it is known that GROUP ISOMORPHISM is AC^0 -reducible to GRAPH ISOMORPHISM, and there is no AC^0 reduction in the opposite direction [CTW13]. For this and other reasons, GROUP ISOMORPHISM is believed to be the easier of the two problems, so it is possible that WL—and more generally, tools from descriptive complexity—could yield stronger results for groups than for graphs.

On graphs, which are binary relational structures, if Spoiler is allowed to pebble two elements per turn, then Spoiler can win on any pair of non-isomorphic graphs. However, groups are ternary relational structures (the relation is $\{(a, b, c) : ab = c\}$), so such a game may yield nontrivial insights into the descriptive complexity of finite groups. Hella [Hel89, Hel96] introduced such games in a more general context, and showed that allowing Spoiler to pebble q elements per round corresponded to the generalized q -ary quantifiers of Mostowski [Mos57] and Lindstrom [Lin66]. When $q = 1$, Hella shows that this pebble game is equivalent in power to the FO plus counting logics mentioned above. Our focus in this paper is to study the power of the $q = 2$ -ary game for identifying finite groups.

Main Results. In this paper, we initiate the study of Hella’s 2-ary Ehrenfeucht–Fraïssé-style pebble game, in the setting of groups. Our main result is that this pebble game efficiently

characterizes isomorphism in a class of groups for which isomorphism testing is known to be in P , but only by quite a nontrivial algorithm, which we discuss after stating our result.

Theorem 1.1. *Let G be a group with no Abelian normal subgroups (a.k.a. Fitting-free or semisimple), and let H be arbitrary. If $G \not\cong H$, then Spoiler has a winning strategy for the Ehrenfeucht–Fraïssé game at the second level of Hella’s hierarchy, using 7 pebbles and 7 rounds.*

Every group G can be written as an extension of its solvable radical $\text{Rad}(G)$ by the quotient $G/\text{Rad}(G)$, which does not have Abelian normal subgroups. As such, the latter class of groups is quite natural, both group-theoretically and computationally. Computationally, it has been used in algorithms for general finite groups both in theory (e.g., [BB99, BBS09]) and in practice (e.g., [CH03]). Isomorphism testing in this family of groups can be solved efficiently in practice [CH03], and is known to be in P through a series of two papers [BCGQ11, BCQ12].

In proving Thm. 1.1, we show that with the use of only a few (at most 7) pebbles, Spoiler can effectively force Duplicator to select an isomorphism of G and H . We contrast this with the setting of ordinary (1-ary) Weisfeiler–Leman (which is equivalent to the 1-ary pebble game): subsequent to the preprint of our paper [GL23], Brachter improved the current-best upper bound on the classical 1-ary WL-dimension from the trivial $\log n$ down to $O(\log \log n)$ [Bra23]. Furthermore, we do not have any lower bounds on the WL-dimension for semisimple groups.

Further Related Work. Despite the fact that Weisfeiler–Leman is insufficient to place GRAPH ISOMORPHISM (GI) into PTIME, it remains an active area of research. For instance, Weisfeiler–Leman is a key subroutine in Babai’s quasipolynomial-time GI algorithm [Bab16]. Furthermore, Weisfeiler–Leman has led to advances in simultaneously developing both efficient isomorphism tests and the descriptive complexity theory for finite graphs; see for instance, [Gro17, GV06, KM20, KPS19, GK19, GN23, GK21, KSS22, AF90, AF97, Ros09]. Weisfeiler–Leman also has close connections to the Sherali–Adams hierarchy in linear programming [GO15].

The complexity of the GROUP ISOMORPHISM (GPI) problem is a well-known open question. In the Cayley (multiplication) table model, GPI belongs to $\text{NP} \cap \text{coAM}$. The generator-enumerator algorithm, attributed to Tarjan in 1978 [Mil78], has time complexity $n^{\log_p(n)+O(1)}$, where n is the order of the group and p is the smallest prime dividing n . This bound has escaped largely unscathed: Rosenbaum [Ros13] (see [LGR16, Sec. 2.2]) improved this to $n^{(1/4)\log_p(n)+O(1)}$. And even the impressive body of work on practical algorithms for this problem, led by Eick, Holt, Leedham-Green and O’Brien (e.g., [BEO02, ELGO02, BE99, CH03]) still results in an $n^{\Theta(\log n)}$ -time algorithm in the general case (see [Wil19, Page 2]). In the past several years, there have been significant advances on algorithms with worst-case guarantees on the serial runtime for special cases of this problem including Abelian groups [Kav07, Vik96, Sav80], direct product decompositions [Wil12, KN09], groups with no Abelian normal subgroups [BCGQ11, BCQ12], coprime and tame group extensions [LG09, QST11, BQ12, GQ15], low-genus p -groups and their quotients [LW12, BMW17], Hamiltonian groups [DS19], and groups of almost all orders [DW22].

Key motivation for GPI is due to its close relation to GI. In the Cayley (verbose) model, GPI reduces to GI [ZKT85], while GI reduces to the succinct GPI problem [HL74, Mek81] (recently simplified [HQ21]). In light of Babai’s breakthrough result that GI

is quasipolynomial-time solvable [Bab16], GPI in the Cayley model is a key barrier to improving the complexity of GI. Both verbose GPI and GI are considered to be candidate NP-intermediate problems, that is, problems that belong to NP, but are neither in P nor NP-complete [Lad75]. There is considerable evidence suggesting that GI is not NP-complete [Sch88, BH92, IPZ01, Bab16, KST92, AK06]. As verbose GPI reduces to GI, this evidence also suggests that GPI is not NP-complete. It is also known that GI is strictly harder than GPI under AC^0 reductions [CTW13].

While the descriptive complexity of graphs has been extensively studied, the work on the descriptive complexity of groups is scant compared to the algorithmic literature on GROUP ISOMORPHISM (GPI). There has been work relating first order logics and groups [NT17], as well as work examining the descriptive complexity of finite Abelian groups [Gom10]. Recently, Brachter & Schweitzer [BS20] introduced three variants of Weisfeiler–Leman for groups, including corresponding logics and pebble games. These pebble games correspond to the first level of Hella’s hierarchy [Hel89, Hel96]. In particular, Brachter & Schweitzer showed that 3-dimensional Weisfeiler–Leman can distinguish p -groups arising from the CFI graphs [CFI92] via Mekler’s construction [Mek81], suggesting that $FO + LFP + C$ may indeed capture PTIME on groups. Determining whether even $o(\log n)$ -dimensional Weisfeiler–Leman can resolve GPI is an open question.

The use of Weisfeiler–Leman for groups is quite new. To the best of our knowledge, using Weisfeiler–Leman for GROUP ISOMORPHISM testing was first attempted by Brooksbank, Grochow, Li, Qiao, & Wilson [BGL⁺19]. Brachter & Schweitzer [BS20] subsequently introduced three variants of Weisfeiler–Leman for groups that more closely resemble that of graphs. In particular, Brachter & Schweitzer [BS20] characterized their algorithms in terms of logics and Ehrenfeucht–Fraïssé pebble games. The relationship between the works of Brachter & Schweitzer and Brooksbank, Grochow, Li, Qiao, & Wilson [BGL⁺19] is an interesting question.

In subsequent work, Brachter & Schweitzer [BS22] further developed the descriptive complexity of finite groups. They showed in particular that low-dimensional Weisfeiler–Leman can detect key group-theoretic invariants such as composition series, radicals, and quotient structure. Furthermore, they also showed that Weisfeiler–Leman can identify direct products in polynomial-time, provided it can also identify the indecomposable direct factors in polynomial-time. Grochow & Levet [GL26] extended this result to show that Weisfeiler–Leman can compute direct products in parallel, provided it can identify each of the indecomposable direct factors in parallel. Additionally, Grochow & Levet showed that constant-dimensional Weisfeiler–Leman can in a constant number of rounds identify coprime extensions $H \ltimes N$, where the normal Hall subgroup N is Abelian and the complement H is $O(1)$ -generated. This placed isomorphism testing into L; the previous bound for isomorphism testing in this family was P [QST11]. Grochow & Levet also ruled out $FO + LFP$ as a candidate logic for capturing PTIME on finite groups, by showing that the count-free Weisfeiler–Leman algorithm cannot even identify Abelian groups in polynomial-time.

2. PRELIMINARIES

2.1. Groups. Unless stated otherwise, all groups are assumed to be finite. For a set X , let $\text{Sym}(X)$ be the group of permutations on X . The *socle* of a group G , denoted $\text{Soc}(G)$, is the subgroup generated by the minimal normal subgroups of G . If G has no Abelian normal

subgroups, then $\text{Soc}(G)$ decomposes uniquely as the direct product of non-Abelian simple factors. The Classification of Finite Simple Groups provides that, if S is a non-Abelian simple group, then there exist $x, y \in S$ such that $S = \langle x, y \rangle$, and this is the only way in which our results depend on the Classification. The *normal closure* of a subset $S \subseteq G$, denoted $\text{ncl}(S)$, is the smallest normal subgroup of G that contains S .

We say that a normal subgroup $N \trianglelefteq G$ *splits* in G if there exists a subgroup $H \leq G$ such that $H \cap N = \{1\}$ and $G = HN$. The conjugation action of H on N allows us to express multiplication of G in terms of pairs $(h, n) \in H \times N$. We note that the conjugation action of H on N induces a group homomorphism $\theta : H \rightarrow \text{Aut}(N)$ mapping $h \mapsto \theta_h$, where $\theta_h : N \rightarrow N$ sends $\theta_h(n) = hnh^{-1}$. So given (H, N, θ) , we may define the group $H \rtimes_{\theta} N$ on the set $\{(h, n) : h \in H, n \in N\}$ with the product $(h_1, n_1)(h_2, n_2) = (h_1 h_2, \theta_{h_2^{-1}}(n_1) n_2)$. We refer to the decomposition $G = H \rtimes_{\theta} N$ as a *semidirect product* decomposition. When the action θ is understood, we simply write $G = H \rtimes N$. If θ is trivial, then $H \trianglelefteq G$ and $G = H \times N$. In this case, we say that H is a *direct complement* of N in G . The *wreath product* $N \wr \text{Sym}(k)$ is the semidirect product $\text{Sym}(k) \rtimes N^k$, where $\text{Sym}(k)$ acts by permuting the copies of N .

For $g, h \in G$, the *commutator* $[g, h] := ghg^{-1}h^{-1}$. If $X, Y \leq G$, then $[X, Y] = \langle [x, y] : x \in X, y \in Y \rangle$.

2.2. Pebbling Game. We recall the bijective pebble game of Hella [Hel89, Hel96], in the context of WL on graphs as that is likely more familiar to more readers. This game is often used to show that two graphs X and Y cannot be distinguished by k -WL. The game is an Ehrenfeucht–Fraïssé game, with two players: Spoiler and Duplicator. Each player begins with $k + 1$ pebbles, p_1, \dots, p_{k+1} for Spoiler and p'_1, \dots, p'_{k+1} for Duplicator, which are placed beside the graph. Each round proceeds as follows.

- (1) Spoiler chooses $i \in [k + 1]$, and picks up pebbles p_i, p'_i .
- (2) Duplicator chooses a bijection $f : V(X) \rightarrow V(Y)$.
- (3) Spoiler places p_i on some vertex $v \in V(X)$. Then p'_i is placed on $f(v)$.
- (4) We check the winning condition, which will be formalized later.

Let v_1, \dots, v_m be the vertices of X pebbled at the end of step 1, and let v'_1, \dots, v'_m be the corresponding pebbled vertices of Y . Spoiler wins precisely if the map $v \mapsto v'$ does not extend to an isomorphism of the induced subgraphs $X[\{v_1, \dots, v_m\}]$ and $Y[\{v'_1, \dots, v'_m\}]$. Duplicator wins otherwise. There is one additional rule: if X and Y do not have the same number of vertices, then we say that Spoiler wins before round 1 begins, or “after 0 rounds.” We note that X and Y are not distinguished by the first r rounds of k -WL if and only if Duplicator wins the first r rounds of the $(k + 1)$ -pebble game [Hel89, Hel96, CFI92].

Hella [Hel89, Hel96] exhibited a hierarchy of (q -ary) pebble games where, for $q \geq 1$, Spoiler could pebble a sequence of $1 \leq j \leq q$ elements $(v_1, \dots, v_j) \mapsto (f(v_1), \dots, f(v_j))$ in a single round. The case of $q = 1$ corresponds to the case of Weisfeiler–Leman. As remarked by Hella [Hel96, p. 6, just before §4], the q -ary game immediately identifies all relational structures of arity $\leq q$. For example, the $q = 2$ game on graphs solves GI: for if two graphs X and Y are non-isomorphic, then any bijection $f : V(X) \rightarrow V(Y)$ that Duplicator selects must map an adjacent pair of vertices u, v in X to a non-adjacent pair $f(u), f(v)$ in Y or vice-versa. Spoiler immediately wins by pebbling $(u, v) \mapsto (f(u), f(v))$. However, as groups are ternary relational structures (the relation being $\{(a, b, c) : a, b, c \in G, ab = c\}$), the $q = 2$ case can, at least in principle, be non-trivial on groups.

Brachter & Schweitzer [BS20] adapted Hella’s [Hel89, Hel96] pebble games in the $q = 1$ case to the setting of groups, obtaining three different versions. Their Version III involves reducing to graphs and playing the pebble game on graphs, so we don’t consider it further here. Versions I and II are both played on the groups G and H directly.

Both versions are played identically as for graphs, with the only difference being the winning condition. We recall the following standard definitions in order to describe these winning conditions.

Definition 2.1. Let G, H be two groups. Given k -tuples $\bar{g} = (g_1, \dots, g_k) \in G^k$ and $\bar{h} = (h_1, \dots, h_k) \in H^k$, we say (\bar{g}, \bar{h}) ...

- (1) ...*gives a well-defined map* if $g_i = g_j \Leftrightarrow h_i = h_j$ for all $i \neq j$;
- (2) ...*are partially isomorphic* or *give a partial isomorphism* if they give a well-defined map, and for all i, j, k we have $g_i g_j = g_k \Leftrightarrow h_i h_j = h_k$;
- (3) ...*are marked isomorphic* or *give a marked isomorphism* if it gives a well-defined map, and the map that sends $g_i \mapsto h_i$ for all i extends to an isomorphism $\langle g_1, \dots, g_k \rangle \rightarrow \langle h_1, \dots, h_k \rangle$.

Let v_1, \dots, v_m be the group elements of G pebbled at the end of step 1, and let v'_1, \dots, v'_m be the corresponding pebbled vertices of H . In Version I, Spoiler wins precisely if (\bar{v}, \bar{v}') does not give a partial isomorphism, and in Version II Spoiler wins precisely if (\bar{v}, \bar{v}') does not give a marked isomorphism.

Notation: If f is the bijection chosen by Duplicator, and Spoiler pebbles $g \in G$, resp. the pair $g_1, g_2 \in G$, we write “Spoiler pebbles $g \mapsto f(g)$ ”, resp. “Spoiler pebbles $(g_1, g_2) \mapsto (f(g_1), f(g_2))$ ”.

Both Versions I and II may be generalized to allow Spoiler to pebble up to q group elements at a single round, for some $q \geq 1$. Mimicking the proof above for $q = 2$ for graphs, we have that $q = 3$ is sufficient to solve GPI in a single round. The distinguishing power, however, of the $q = 2$ game for groups remains unclear, and is the main subject of this paper. As we are interested in the round complexity, we introduce the following notation.

Definition 2.2 (Notation for pebbles, rounds, arity, and WL version). Let $k \geq 2, r \geq 1, q \in [3]$, and $J \in \{I, II\}$. Denote $(k, r)\text{-WL}_J^q$ to be the k -pebble, r -round, q -ary Version J pebble game.

We refer to q as the *arity* of the pebble game, as it corresponds to the arity of generalized quantifiers in a logic whose distinguishing power is equivalent to that of the game:

Remark 2.3 (Equivalence with logics with generalized 2-ary quantifiers). Hella [Hel89] describes the game (essentially the same as our description, but with no restriction on number of pebbles, and a transfinite number of rounds) for general q at the bottom of p. 244, for arbitrary relational structures. We restrict to the case of $q = 2$, a finite number of pebbles and rounds, and the (relational) language of groups. Hella proves that this game is equivalent to first-order logic with arbitrary q -ary quantifiers in [Hel89, Thm. 2.5].

We will frequently use the following observation without mention.

Observation 2.4. In the 2-ary Version II pebble game, Duplicator must select a bijection that preserves the identity, or Spoiler wins in that round. If Spoiler has two pebbles in hand, Duplicator must select a bijection that preserves inverses, or Spoiler wins in that round.

Proof. Suppose not. If Duplicator selects a bijection that does not preserve the identity element, then Spoiler places a pebble on $1 \mapsto f(1) \neq 1$, and wins at the end of that round as $1 \cdot 1 = 1$, but $f(1) \cdot f(1) \neq f(1)$.

Next, suppose Spoiler has two pebbles in hand, and Duplicator selects a bijection $f : G \rightarrow H$ such that $f(g^{-1}) \neq f(g)^{-1}$. Spoiler pebbles $(g, g^{-1}) \mapsto (f(g), f(g^{-1}))$. Now $gg^{-1} = 1$, while $f(g)f(g^{-1}) \neq 1$. So Spoiler wins at the end of this round. \square

3. HIGHER-ARITY WEISFEILER-LEMAN-STYLE COLORING CORRESPONDING TO HIGHER ARITY PEBBLE GAMES

Brachter & Schweitzer [BS20] introduced Versions I and II of 1-ary WL, which are equivalent up to a small additive constant in the WL-dimension [BS20] and $O(\log n)$ rounds [GL26]. For the purpose of comparison, we introduce Versions I and II of 2-ary WL. We will see later that only one additional round suffices in the 2-ary case (see Thm. 3.6). The differences in Versions I and II of WL (both the 1-ary and 2-ary variants) arise from whether the group is viewed as a structure with a ternary relation (Version I) or as a structure with a binary function (Version II).

To give our definition of 2-ary color refinement, we introduce some notation and terminology. Given a k -tuple $\bar{x} = (x_1, \dots, x_k) \in G^k$, a pair of distinct indices $i, j \in [k]$, and a pair of group elements y, z , we define $\bar{x}_{(i,j) \leftarrow (y,z)}$ to be the k -tuple \bar{x}' that agrees with \bar{x} on all indices besides i, j , and with $x'_i = y, x'_j = z$. If $i = j$, we require $y = z$, and we denote this $\bar{x}_{i \leftarrow y}$. Two graphs Γ_1, Γ_2 , with edge-colorings $c_i : E(\Gamma_i) \rightarrow C$ to some color set C (for $i = 1, 2$) are color isomorphic if there is a graph isomorphism $\varphi : V(\Gamma_1) \rightarrow V(\Gamma_2)$ that also preserves colors, in the sense that $c_1((u, v)) = c_2((\varphi(u), \varphi(v)))$ for all edges $(u, v) \in E(\Gamma_1)$. We note that the color names for the group elements must be consistent with the color names for the complete graphs in the refinement steps. Beyond this consistency, the names of the colors themselves are not relevant.

Definition 3.1 (2-ary k -dimensional Weisfeiler-Leman coloring). Let G, H be two groups of the same order, let $k \geq 1$.

- (Initial coloring, Version I) For all k -tuples $\bar{x}, \bar{y} \in G^k \cup H^k$, $\chi_0^{2,I}(\bar{x}) = \chi_0^{2,I}(\bar{y})$ iff \bar{x}, \bar{y} are partially isomorphic.
- (Initial coloring, Version II) For all k -tuples $\bar{x}, \bar{y} \in G^k \cup H^k$, $\chi_0^{2,II}(\bar{x}) = \chi_0^{2,II}(\bar{y})$ iff \bar{x}, \bar{y} have the same marked isomorphism type.
- (Color refinement) Given a coloring $\chi : G^k \cup H^k \rightarrow C$, the color refinement operator R defines a new coloring $R(\chi)$ as follows. For each k -tuple $\bar{x} \in G^k$ (resp., H^k), we define an edge-colored graph $\Gamma_{\bar{x}, \chi, i, j}$. If $i = j$, it is the graph on vertex set $V(\Gamma_{\bar{x}, \chi, i, i}) = G$ (resp., H) with all self-loops and no other edges, where the color of each self-loop (g, g) is $\chi(\bar{x}_{i \leftarrow g})$. If $i \neq j$, it is the complete directed graph with self-loops on vertex set G (resp., H), where the color of each edge (y, z) is $\chi(\bar{x}_{(i,j) \leftarrow (y,z)})$. For an edge-colored graph Γ , we use $[\Gamma]$ to denote its edge-colored isomorphism class. We then define

$$R(\chi)(\bar{x}) = (\chi(\bar{x}); [\Gamma_{\bar{x}, \chi, 1, 1}], [\Gamma_{\bar{x}, \chi, 1, 2}], \dots, [\Gamma_{\bar{x}, \chi, k, k-1}], [\Gamma_{\bar{x}, \chi, k, k}]).$$

That is, the new color consists of the old color, as well as the tuple of $\binom{k+1}{2}$ edge-colored isomorphism types of the graphs $\Gamma_{\bar{x}, \chi, i, j}$.

The refinement operator may be iterated: $R^t(\chi) := R(R^{t-1}(\chi))$, and we define the *stable refinement* of χ as $R^t(\chi)$ where the partition induced by $R^t(\chi)$ on $G^k \cup H^k$ is the same as that induced by $R^{t+1}(\chi)$. We denote the stable refinement by $R^\infty(\chi)$.

Finally, for $J \in \{I, II\}$ and all $r \geq 0$, we define $\chi_{r+1}^{2,J} = R(\chi_r^{2,J})$, and $\chi_\infty^{2,J} := R^\infty(\chi_0^{2,J})$.

Remark 3.2. Since it was one of our stumbling blocks in coming up with this generalized coloring, we clarify here how this indeed generalizes the usual 1-ary WL coloring procedure. In the 1-ary “oblivious” k -WL procedure (see [Gro21, §5], equivalent to ordinary WL), the color of a k -tuple \bar{x} is refined using its old color, together with a k -tuple of multisets

$$(\{\{\chi(\bar{x}_{1 \leftarrow y}) : y \in G\}\}, \{\{\chi(\bar{x}_{2 \leftarrow y}) : y \in G\}\}, \dots, \{\{\chi(\bar{x}_{k \leftarrow y}) : y \in G\}\}).$$

For each i , note that two multisets $\{\{\chi(\bar{x}_{i \leftarrow y}) : y \in G\}\}$ and $\{\{\chi(\bar{x}'_{i \leftarrow y}) : y \in G\}\}$ are equal iff the graphs $\Gamma_{\bar{x}, \chi, i, i}$ and $\Gamma_{\bar{x}', \chi, i, i}$ are color-isomorphic. That is, edge-colored graphs with only self-loops and no other edges are essentially the same, up to isomorphism, as multisets. Our procedure generalizes this by also considering graphs with other edges, which (as we’ll see in the proof of equivalence below) are used to encode the choice of 2 simultaneous pebbles by Spoiler in each move of the game.

Theorem 3.3. *Let G, H be two groups of order n , with $\bar{x} = (x_1, \dots, x_k) \in G^k$, $\bar{y} = (y_1, \dots, y_k) \in H^k$. Starting from the initial pebbling $x_i \mapsto y_i$ for all $i = 1, \dots, k$, Spoiler has a winning strategy in the k -pebble, r -round, 2-ary Version J pebble game (for $J \in \{I, II\}$) iff $\chi_r^{2,J}(\bar{x}) \neq \chi_r^{2,J}(\bar{y})$.*

Proof. By induction on r . The base case, $r = 0$, is built into the definition: the initial colors of \bar{x}, \bar{y} agree iff $x_i = x_j \Leftrightarrow y_i = y_j$ and in Version I, $x_i x_j = x_\ell \Leftrightarrow y_i y_j = y_\ell$, or in Version II the pebbled map is a marked isomorphism. But these are precisely the conditions for Spoiler to lose without making a move, so both directions are established for $r = 0$.

Now suppose $r > 0$, and the equivalence is established for $r - 1$. For ease of notation, throughout the proof we denote colorings $\chi_r^{2,J}$ by χ^J instead, and we denote the graphs $\Gamma_{\bar{x}, \chi_r^J, i, j}$ by $\Gamma_{\bar{x}, r, i, j}$ instead ($q = 2$ and the dependence on J still being implied).

(\Leftarrow) Suppose $\chi_r^J(\bar{x}) \neq \chi_r^J(\bar{y})$. If $\chi_{r-1}^J(\bar{x}) \neq \chi_{r-1}^J(\bar{y})$, then Spoiler has a winning strategy in $r - 1$ rounds by the inductive hypothesis. Otherwise, we have $\chi_{r-1}^J(\bar{x}) = \chi_{r-1}^J(\bar{y})$. But since their colors differ at the r -th round, there must exist $i, j \in [k]$ such that $\Gamma_{\bar{x}, r-1, i, j}$ is not color-isomorphic to $\Gamma_{\bar{y}, r-1, i, j}$. Spoiler picks up the pebbles i and j . Now, no matter what bijection φ Duplicator chooses, φ is not a color isomorphism on these graphs, so there exists an edge $(x'_i, x'_j) \in E(\Gamma_{\bar{x}, r-1, i, j})$ whose color differs from that of the edge $(\varphi(x'_i), \varphi(x'_j)) \in E(\Gamma_{\bar{y}, r-1, i, j})$. (Note: there is no concern that the latter is not an edge of $\Gamma_{\bar{y}, r-1, i, j}$, for φ is automatically an uncolored graph isomorphism: when $i \neq j$ the graphs are complete, and when $i = j$ the graphs consist of n isolated vertices with self-loops, so any bijection works.) Spoiler places the i -th pebble on $x'_i \mapsto \varphi(x'_i) =: y'_i$ and the j -th pebble on $x'_j \mapsto \varphi(x'_j) =: y'_j$.

By the definition of the edge coloring, the fact that the edges (x'_i, x'_j) and (y'_i, y'_j) receive different colors in the $\Gamma_{\bullet, r-1, i, j}$ graphs means that $\chi_{r-1}^J(\bar{x}_{(i,j) \leftarrow (x'_i, x'_j)}) \neq \chi_{r-1}^J(\bar{y}_{(i,j) \leftarrow (y'_i, y'_j)})$. Now, by the inductive hypothesis applied to $\bar{x}_{(i,j) \leftarrow (x'_i, x'_j)}$ and $\bar{y}_{(i,j) \leftarrow (y'_i, y'_j)}$, Spoiler can win in at most $r - 1$ additional rounds.

(\Rightarrow) Suppose $\chi_r^J(\bar{x}) = \chi_r^J(\bar{y})$. We show that Duplicator has a strategy that does not lose through round r . On the first round, Spoiler picks up pebbles i and j (not necessarily distinct). Since $\chi_r^J(\bar{x}) = \chi_r^J(\bar{y})$, we have that $\Gamma_{\bar{x}, r-1, i, j}$ is color-isomorphic to $\Gamma_{\bar{y}, r-1, i, j}$, say by the isomorphism $\varphi: G = V(\Gamma_{\bar{x}, r-1, i, j}) \rightarrow V(\Gamma_{\bar{y}, r-1, i, j}) = H$. Duplicator uses the isomorphism

φ as their chosen bijection. Suppose Spoiler places the pebbles on $x'_i \mapsto \varphi(x'_i) =: y'_i$ and $x'_j \mapsto \varphi(x'_j) =: y'_j$. First, we claim that Duplicator has not yet lost. For the fact that φ is a color isomorphism means that the colors of the edge $(x'_i, x'_j) \in E(\Gamma_{\bar{x}, r-1, i, j})$ and $(y'_i, y'_j) \in E(\Gamma_{\bar{y}, r-1, i, j})$ are the same. But these colors are precisely $\chi_{r-1}^J(\bar{x}_{(i,j) \leftarrow (x'_i, x'_j)})$ and $\chi_{r-1}^J(\bar{y}_{(i,j) \leftarrow (y'_i, y'_j)})$. Since χ_{r-1}^J refines χ_0^J , we also have $\chi_0^J(\bar{x}_{(i,j) \leftarrow (x'_i, x'_j)}) = \chi_0^J(\bar{y}_{(i,j) \leftarrow (y'_i, y'_j)})$; since the 0-th coloring precisely matches the condition for Duplicator not to lose, Duplicator has not yet lost.

It remains to show that Duplicator can continue not to lose for an additional $r-1$ rounds. But this now follows from the inductive hypothesis applied to $\bar{x}_{(i,j) \leftarrow (x'_i, x'_j)}$ and $\bar{y}_{(i,j) \leftarrow (y'_i, y'_j)}$, for we have already established that they receive the same color under χ_{r-1}^J . \square

Corollary 3.4. *For two groups G, H of the same order and any $k \geq 1$, the following are equivalent:*

- (1) *The 2-ary k -pebble game does not distinguish two groups G, H*
- (2) *The multisets of stable colors on G^k and H^k are the same, that is, $\{\{\chi_\infty^{2,J}(\bar{x}) : \bar{x} \in G^k\}\} = \{\{\chi_\infty^{2,J}(\bar{y}) : \bar{y} \in H^k\}\}$*
- (3) $\chi_\infty^{2,J}(\underbrace{(1_G, \dots, 1_G)}_k) = \chi_\infty^{2,J}(\underbrace{(1_H, \dots, 1_H)}_k)$.

The analogous result holds in the $q = 1$ case, going back to [BS20].

Proof. (2 \Rightarrow 3) Note that the identity tuples $(1, \dots, 1)$ are the unique tuples of their color, since they are the only tuples (x_1, \dots, x_k) such that $x_i^2 = x_i$ for all i (Version I), and the only tuples for which $\langle x_1, \dots, x_k \rangle = 1$ (Version II). Thus, if the multisets of stable colors are equal, the colors of $(1_G, \dots, 1_G)$ and $(1_H, \dots, 1_H)$ must be equal.

(3 \Rightarrow 2) Suppose $\chi_\infty^{2,J}((1_G, \dots, 1_G)) = \chi_\infty^{2,J}((1_H, \dots, 1_H))$. Claim: for all $0 \leq \ell \leq k$,

$$\begin{aligned} & \{\{\chi_\infty^{2,J}(x_1, \dots, x_\ell, 1_G, \dots, 1_G) : x_1, \dots, x_\ell \in G\}\} \\ &= \{\{\chi_\infty^{2,J}(x_1, \dots, x_\ell, 1_H, \dots, 1_H) : x_1, \dots, x_\ell \in H\}\}. \end{aligned} \quad (*)$$

For notational simplicity, for $\bar{x} \in G^\ell$, define \bar{x}^+ as the k -tuple $(x_1, \dots, x_\ell, 1_G, \dots, 1_G)$, and similarly *mutatis mutandis* for $\bar{y} \in H^\ell$.

For $\ell = 0$, $(*)$ is precisely our assumption (3).

Suppose $(*)$ is true for some $0 \leq \ell < k$. We show that it remains true for $\min\{\ell + 2, k\}$. For notational simplicity, we assume $\ell + 2 \leq k$; the proof in the case $k = \ell + 1$ is similar.

By the inductive hypothesis, there is a bijection $\varphi_\ell: G^\ell \rightarrow H^\ell$ such that for all $\bar{x} \in G^\ell$,

$$\chi_\infty^{2,J}(\bar{x}^+) = \chi_\infty^{2,J}(\varphi_\ell(\bar{x})^+)$$

For each $\bar{x} \in G^\ell$, by the definition of the coloring, there is a color-isomorphism $\psi = \psi_{\bar{x}}: \Gamma_{\bar{x}^+, \infty, \ell+1, \ell+2} \rightarrow \Gamma_{\varphi_\ell(\bar{x})^+, \infty, \ell+1, \ell+2}$ (technically the subscripts here should be “ $\infty - 1$,” but because it is the stable coloring, the colored graphs are the same as what we have written). By the definition of the colors on the edges, this means that for all $x_{\ell+1}, x_{\ell+2} \in G$, we have that $\chi_\infty^J(x_1, \dots, x_\ell, x_{\ell+1}, x_{\ell+2}, 1_G, \dots, 1_G) = \chi_\infty^J(\varphi_\ell(\bar{x}), \psi(x_{\ell+1}), \psi(x_{\ell+2}), 1_H, \dots, 1_H)$ (where here we have slightly abused our parentheses, but in a way that should be clear). Thus, we may extend φ_ℓ to a color-preserving bijection $\varphi_{\ell+2}$ by defining

$$\varphi_{\ell+2}(\bar{x}, x_{\ell+1}, x_{\ell+2}) = (\varphi_\ell(\bar{x}), \psi_{\bar{x}}(x_{\ell+1}), \psi_{\bar{x}}(x_{\ell+2})) \quad \forall \bar{x} \in G^\ell, \forall x_{\ell+1}, x_{\ell+2} \in G.$$

This establishes $(*)$ for $\ell + 2$ (assuming $\ell + 2 \leq k$). Consequently, this establishes the claim for all $\ell \leq k$, and thus that (3) implies (2).

(3 \Leftrightarrow 1) The 2-ary k -pebble game starting from empty initial configurations is equivalent, move-for-move, with the 2-ary k -pebble game starting from the configuration $(1_G, 1_G, \dots, 1_G) \mapsto (1_H, 1_H, \dots, 1_H)$. By Thm. 3.3, the 2-ary k -pebble game thus does not distinguish G from H iff $\chi_\infty^{2,J}((1_G, \dots, 1_G)) = \chi_\infty^{2,J}((1_H, \dots, 1_H))$. \square

Remark 3.5. For arbitrary relational structures with relations of arity $q + 1$, the q -ary pebble game may still be nontrivial, as pointed out in Hella [Hel96, p. 6, just before §4]. Our coloring procedure generalizes in the following way to this more general setting, and the proof of the equivalence between the coloring procedure and Hella’s pebble game is the same as the above, *mutatis mutandis*. The main change is that for a q -ary pebble game, instead of just considering a graph on edges of size 1 (when $i = j$) or 2 (when $i \neq j$), we consider a q' -uniform directed hypergraph, where each hyperedge consists of a list of q' vertices, for all $1 \leq q' \leq q$. This gives a coloring equivalent of the logical and game characterizations provided by Hella; this trifecta is partly why we feel it is justified to call this a “higher-arity Weisfeiler–Leman” coloring procedure.

We note that there has been some work on equivalences with specific binary and higher-arity quantifiers—see for instance, the invertible map game of Dawar & Holm [DH12] which generalizes rank logic, in which Spoiler can place multiple pebbles, but the bijections Duplicator selects must satisfy additional structure. Subsequently, Dawar & Vagnozzi [DV20] provided a generalization of Weisfeiler–Leman that further subsumes the invertible map game. We note that Dawar & Vagnozzi’s “ $WL_{k,r}$ ”, although it looks superficially like our r -ary k -WL, is in fact quite different: in particular, their refinement step “flattens” a multiset of multisets into its multiset union, which loses information compared to our 2-ary (resp., r -ary) game; indeed, they show that their $WL_{*,r}$ is equivalent to ordinary (1-ary) WL for any fixed r , whereas already 2-ary WL can solve GI. In general, the relationship between Hella’s 2-ary game and the works of Dawar & Holm and Dawar & Vagnozzi remains open.

3.1. Equivalence between 2-ary (k, r) -WL Versions I and II. In this section we show that, up to additive constants in the number of pebbles and rounds, 2-ary WL Versions I and II are equivalent in their distinguishing power. For two different WL versions W, W' , we write $W \preceq W'$ to mean that if W distinguishes two groups G and H , then so does W' .

Theorem 3.6. *Let $k \geq 2, r \geq 1$. We have that:*

$$(k, r)\text{-}WL_I^2 \preceq (k, r)\text{-}WL_{II}^2 \preceq (k + 2, r + 1)\text{-}WL_I^2.$$

Proof. For the first statement, note that if $\bar{x} \mapsto \bar{y}$ is a marked isomorphism, then it is a partial isomorphism as well. After that, each color refinement step in the two versions are the same, so at whatever round Version I distinguishes G from H , Version II will distinguish G from H at that round (or possibly sooner).

For the second statement, suppose that $(k, r)\text{-}WL_{II}^2$ distinguishes G from H . Let $r_0 \leq r$ be the minimum number of rounds after which Spoiler has a winning strategy in this game. We show how Spoiler can win in the $(k + 2, r_0 + 1)\text{-}WL_I^2$ game. For the first r_0 rounds, Spoiler makes exactly the same moves it would make in response to Duplicator’s bijections in the Version II game. At the end of the r_0 round, Spoiler has pebbled a map $\bar{x} \mapsto \bar{y}$ of k -tuples that is not a marked isomorphism, but may still be a partial isomorphism. On

the next round, Spoiler picks up the two pebbles $k + 1, k + 2$. Let φ be the bijection Duplicator chooses. Since the pebbled map is not a marked isomorphism, there is a word w such that $w(\varphi(x_1), \dots, \varphi(x_k)) \neq \varphi(w(\bar{x}))$. Let w be such a word of minimal length ℓ , and write $w(\bar{x}) = x_{i_1}^{\pm 1} w'(\bar{x})$, where w' has length $\ell - 1$. Spoiler places the two pebbles on $w'(\bar{x}) \mapsto \varphi(w'(\bar{x}))$ and $w(\bar{x}) \mapsto \varphi(w(\bar{x}))$. Now, since pebbles $i_1, k + 1, k + 2$ are on $x_{i_1}, w'(\bar{x}), w(\bar{x})$, resp., and $x_{i_1}^{\pm 1} w'(\bar{x}) = w(\bar{x})$ but $\varphi(x_{i_1})^{\pm 1} \varphi(w'(\bar{x})) = w(\varphi(\bar{x})) \neq \varphi(w(\bar{x}))$, Spoiler wins the Version I game. \square

4. DESCRIPTIVE COMPLEXITY OF SEMISIMPLE GROUPS

In this section, we show that the $(O(1), O(1))\text{-WL}_{II}^2$ pebble game can identify groups with no Abelian normal subgroups, also known as semisimple groups. We begin with some preliminaries.

4.1. Preliminaries. Semisimple groups are motivated by the following characteristic filtration:

$$1 \leq \text{Rad}(G) \leq \text{Soc}^*(G) \leq \text{PKer}(G) \leq G,$$

which arises in the computational complexity community where it is known as the Babai–Beals filtration [BB99], as well as in the development of practical algorithms for computer algebra systems (cf., [CH03]). We now explain the terms of this chain. Here, $\text{Rad}(G)$ is the *solvable radical*, which is the unique maximal solvable normal subgroup of G . The socle of a group, denoted $\text{Soc}(G)$, is the subgroup generated by all the minimal normal subgroups of G . $\text{Soc}^*(G)$ is the preimage of the socle $\text{Soc}(G/\text{Rad}(G))$ under the natural projection map $\pi : G \rightarrow G/\text{Rad}(G)$. To define PKer , we start by examining the action on $\text{Soc}(G/\text{Rad}(G)) \cong \text{Soc}^*(G)/\text{Rad}(G)$ that is induced by the action of G on $\text{Soc}^*(G)$ by conjugation. As $\text{Soc}^*(G)/\text{Rad}(G) \cong \text{Soc}(G/\text{Rad}(G))$ is the direct product of finite, non-Abelian simple groups T_1, \dots, T_k , this action permutes the k simple factors, yielding a homomorphism $\varphi : G \rightarrow S_k$. The kernel of this action is denoted $\text{PKer}(G)$.

When $\text{Rad}(G)$ is trivial, G has no Abelian normal subgroups (and vice versa). We refer to such groups as *semisimple* (following [BCGQ11, BCQ12]) or *trivial-Fitting* (following [CH03]). As a semisimple group G has no Abelian normal subgroups, we have that $\text{Soc}(G)$ is the direct product of non-Abelian simple groups. As the conjugation action of G on $\text{Soc}(G)$ permutes the direct factors of $\text{Soc}(G)$, there exists a faithful permutation representation $\alpha : G \rightarrow G^* \leq \text{Aut}(\text{Soc}(G))$. G is determined by $\text{Soc}(G)$ and the action α . Let H be a semisimple group with the associated action $\beta : H \rightarrow \text{Aut}(\text{Soc}(H))$. We have that $G \cong H$ precisely if $\text{Soc}(G) \cong \text{Soc}(H)$ via an isomorphism that makes α equivalent to β in the sense introduced next.

We now introduce the notion of permutational isomorphism, which is our notion of equivalence for α and β . Let A and B be finite sets, and let $\pi : A \rightarrow B$ be a bijection. For $\sigma \in \text{Sym}(A)$, let $\sigma^\pi \in \text{Sym}(B)$ be defined by $\sigma^\pi := \pi^{-1} \sigma \pi$. For a set $\Sigma \subseteq \text{Sym}(A)$, denote $\Sigma^\pi := \{\sigma^\pi : \sigma \in \Sigma\}$. Let $K \leq \text{Sym}(A)$ and $L \leq \text{Sym}(B)$ be permutation groups. A bijection $\pi : A \rightarrow B$ is a *permutational isomorphism* $K \rightarrow L$ if $K^\pi = L$.

The following lemma, applied with $R = \text{Soc}(G)$ and $S = \text{Soc}(H)$, gives a precise characterization of semisimple groups in terms of the associated actions.

Lemma 4.1 [BCGQ11, Lemma 3.1], cf. [CH03, §3]. *Let G and H be groups, with $R \triangleleft G$ and $S \triangleleft H$ groups with trivial centralizers. Let $\alpha : G \rightarrow G^* \leq \text{Aut}(R)$ and $\beta : H \rightarrow H^* \leq \text{Aut}(S)$ be faithful permutation representations of G and H via the conjugation action on R and S , respectively. Let $f : R \rightarrow S$ be an isomorphism. Then f extends to an isomorphism $\hat{f} : G \rightarrow H$ if and only if f is a permutational isomorphism between G^* and H^* ; and if so, $\hat{f} = \alpha f^* \beta^{-1}$, where $f^* : G^* \rightarrow H^*$ is the isomorphism induced by f .*

Remark 4.2. We note that Lem. 4.1 depends only on the assumption that the centralizers $C_G(R)$ and $C_H(S)$ are trivial. In particular, Lem. 4.1 does not depend on the Classification of Finite Simple Groups (CFSG); the only dependence our results have on CFSG is through its consequence that all finite simple groups are 2-generated.

We also need the following standard group-theoretic lemmas. The first provides a key condition for identifying whether a non-Abelian simple group belongs in the socle. Namely, if $S_1 \cong S_2$ are non-Abelian simple groups where S_1 is in the socle and S_2 is not in the socle, then the normal closures of S_1 and S_2 are non-isomorphic. In particular, the normal closure of S_1 is a direct product of non-Abelian simple groups, while the normal closure of S_2 is not a direct product of non-Abelian simple groups. We will apply this condition later when S_1 is a simple direct factor of $\text{Soc}(G)$; in which case, the normal closure of S_1 is of the form S_1^k . We include the proofs of these two lemmas for completeness.

Lemma 4.3. *Let G be a finite semisimple group. A subgroup $S \leq G$ is contained in $\text{Soc}(G)$ if and only if the normal closure of S is a direct product of non-Abelian simple groups.*

Proof. Let N be the normal closure of S . Since the socle is normal in G and N is the smallest normal subgroup containing S , we have that S is contained in $\text{Soc}(G)$ if and only if N is.

Suppose first that S is contained in the socle. Since $\text{Soc}(G)$ is normal and contains S , by the definition of N we have that $N \leq \text{Soc}(G)$. As N is a normal subgroup of G , contained in $\text{Soc}(G)$, it is a direct product of minimal normal subgroups of G , each of which is a direct product of non-Abelian simple groups.

Conversely, suppose N is a direct product of non-Abelian simple groups. We proceed by induction on the size of N . If N is minimal normal in G , then N is contained in the socle by definition. If N is not minimal normal, then it contains a proper subgroup $M \subsetneq N$ such that M is normal in G , hence also $M \trianglelefteq N$. However, as N is a direct product of non-Abelian simple groups T_1, \dots, T_k , the only subgroups of N that are normal in N are direct products of subsets of $\{T_1, \dots, T_k\}$, and all such normal subgroups have direct complements. Thus we may write $N = L \times M$ where both L, M are nontrivial, hence strictly smaller than N , and both L and M are direct product of non-Abelian simple groups.

We now argue that L must also be normal in G . Since conjugating N by $g \in G$ is an automorphism of N , we have that $N = gLg^{-1} \times gMg^{-1}$. Since M is normal in G , the second factor here is just M , so we have $N = gLg^{-1} \times M$. But since the direct complement of M in N is unique (since N is a direct product of *non-Abelian* simple groups), we must have $gLg^{-1} = L$. Thus L is normal in G .

By induction, both L and M are contained in $\text{Soc}(G)$, and thus so is N . We conclude since $S \leq N$. \square

Lemma 4.4. *Let $S_1, \dots, S_k \leq G$ be non-Abelian simple subgroups such that for all distinct $i, j \in [k]$ we have $[S_i, S_j] = 1$. Then $\langle S_1, \dots, S_k \rangle = S_1 S_2 \cdots S_k = S_1 \times \cdots \times S_k$.*

Proof. By induction on k . The base case $k = 1$ is vacuously true. Suppose $k \geq 2$ and that the result holds for $k - 1$. Then $T := S_1 S_2 \cdots S_{k-1} = S_1 \times \cdots \times S_{k-1}$. Now, since S_k commutes with each S_i , and they generate T , we have that $[S_k, T] = 1$. Hence T is contained in the normalizer (or even the centralizer) of S_k , so $TS_k = S_k T = \langle T, S_k \rangle$, and S_k and T are normal subgroups of TS_k . As $TS_k = \langle T, S_k \rangle$ and T, S_k are both normal subgroups of TS_k with $[T, S_k] = 1$, we have that TS_k is a central product of T and S_k . As $Z(T) = Z(S_k) = 1$, it is their direct product. \square

4.2. Main Results. We show that the second Ehrenfeucht–Fraïssé game in Hella’s hierarchy can identify both $\text{Soc}(G)$ and the conjugation action when G is semisimple. We first show that this pebble game can identify whether a group is semisimple. Namely, if G is semisimple and H is not semisimple, then Spoiler can distinguish G from H .

Proposition 4.5. *Let G be a semisimple group of order n , and let H be an arbitrary group of order n . If H is not semisimple, then Spoiler can win in the $(3, 2)$ - WL_{II}^2 game.*

Proof. Recall that a group is semisimple if and only if it contains no Abelian normal subgroups. As H is not semisimple, $\text{Soc}(H) = A \times T$, where A is a nontrivial subgroup that is the direct product of elementary Abelian groups and T is a direct product of non-Abelian simple groups. We show that Spoiler can win using at most 3 pebbles and 2 rounds. Let $f : G \rightarrow H$ be the bijection that Duplicator selects. Let $a \in A$. So $\text{ncl}_H(a) \leq A$. Let $b := f^{-1}(a) \in G$, and let $B := \text{ncl}_G(b)$. As G is semisimple, we have that B is not Abelian. Spoiler first pebbles $b \mapsto f(b) = a$.

So there exist $g_1, g_2 \in G$ such that $g_1 b g_1^{-1}$ and $g_2 b g_2^{-1}$ do not commute (for B is generated by $\{g b g^{-1} : g \in G\}$, and if they all commuted then B would be Abelian). Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. If $f'(g_i^{-1}) \neq f'(g_i)^{-1}$ for either $i = 1, 2$, Spoiler uses the two pebbles in hand to win that round, by Observation 2.4, for a total of 3 pebbles and 2 rounds.

Otherwise, Spoiler pebbles $(g_1, g_2) \mapsto (f'(g_1), f'(g_2))$. As $\text{ncl}(f(b)) \leq A$ is Abelian, $f'(g_1)f(b)f'(g_1)^{-1}$ and $f'(g_2)f(b)f'(g_2)^{-1}$ commute. Spoiler now wins, again with a total of 3 pebbles and 2 rounds. \square

We now apply Lemma 4.3 to show that Duplicator must map the direct factors of $\text{Soc}(G)$ to isomorphic direct factors of $\text{Soc}(H)$.

Lemma 4.6. *Let G, H be finite groups of order n . Let $\text{Fac}(\text{Soc}(G))$ denote the set of simple direct factors of $\text{Soc}(G)$. Let $S \in \text{Fac}(\text{Soc}(G))$ be a non-Abelian simple group, with $S = \langle x, y \rangle$. If Duplicator selects a bijection $f : G \rightarrow H$ such that:*

- (a) $S \not\cong \langle f(x), f(y) \rangle$, then Spoiler can win in the $(2, 1)$ - WL_{II}^2 game; or
- (b) $f(S) \neq \langle f(x), f(y) \rangle$, then Spoiler can win in the $(3, 2)$ - WL_{II}^2 pebble game.

Note that Lemma 4.6(b) stipulates that, in order not to lose, Duplicator must map the elements in S precisely to the elements in $\langle f(x), f(y) \rangle$. Furthermore, S and $\langle f(x), f(y) \rangle$ must be isomorphic. However, in order to avoid losing, this lemma does not yet establish that it is necessary for Duplicator to choose a bijection f such that the restriction $f|_S : S \rightarrow f(S)$ is an isomorphism.

Proof.

- (a) If $S \not\cong \langle f(x), f(y) \rangle$, then Spoiler pebbles $(x, y) \mapsto (f(x), f(y))$ and immediately wins.
- (b) Suppose that $S \cong \langle f(x), f(y) \rangle$. By part (a), we may assume that the map $(x, y) \mapsto (f(x), f(y))$ extends to an isomorphism; otherwise Spoiler immediately wins. So suppose that $f(S) \neq \langle f(x), f(y) \rangle$. Note that as $S \cong \langle f(x), f(y) \rangle$, we have that $|S| = |\langle f(x), f(y) \rangle|$. So $f(S) \subseteq \langle f(x), f(y) \rangle$ if and only if $f(S) = \langle f(x), f(y) \rangle$.

Now by the assumption that $f(S) \neq \langle f(x), f(y) \rangle$, there exists an element $b \in S$ such that $f(b) \notin \langle f(x), f(y) \rangle$. Let $a \in S$ such that $a \neq 1$ and $f(a) \in \langle f(x), f(y) \rangle$. Spoiler pebbles $(a, b) \mapsto (f(a), f(b))$. Let $f' : G \rightarrow H$ be the bijection that Duplicator chooses on the next round. Let $g, h \in G$ such that $f'(g) = f(x)$ and $f'(h) = f(y)$. Spoiler now pebbles $(g, h) \mapsto (f(x), f(y))$. As $S \cong \langle f(x), f(y) \rangle$ by assumption, if $T = \langle g, h \rangle$ is not isomorphic to S , then Spoiler wins immediately. Furthermore, as $f(a) \in \langle f(x), f(y) \rangle$, we may assume that $a \in \langle g, h \rangle$; otherwise, Spoiler immediately wins.

We now claim that $T \leq \text{Soc}(G)$. As $\text{Soc}(G)$ is normal in G , we have that $T \cap \text{Soc}(G)$ is normal in T . As T is simple and intersects with $\text{Soc}(G)$ non-trivially (namely, $a \in T \cap \text{Soc}(G)$), it follows that $T \leq \text{Soc}(G)$. Now as $S \trianglelefteq \text{Soc}(G)$ and $T \leq \text{Soc}(G)$, we have by similar argument (using the fact that $a \in S$) that $T = S$. So $S = \langle g, h \rangle$. Now $a, b \in S = \langle g, h \rangle$; however, $f(b) \notin \langle f(x), f(y) \rangle$. So Spoiler wins. \square

Proposition 4.7. *Let G, H be semisimple groups of order n . Let $f : G \rightarrow H$ be the bijection Duplicator selects. If there exists $S \in \text{Fac}(\text{Soc}(G))$ such that $f(S) \notin \text{Fac}(\text{Soc}(H))$ or $f(S) \not\cong S$, then Spoiler can win in the $(4, 3)$ - WL^2_{II} pebble game.*

Proof. As G is semisimple, we have that $S = \langle x, y \rangle$ is non-Abelian. Let $f : G \rightarrow H$ be the bijection Duplicator selects. We may assume that $f(S) \cong S$ (though $f|_S$ need not be an isomorphism); otherwise, Spoiler can win with 2 pebbles and 1 round by pebbling the generators (x, y) for S . Furthermore, we may assume that $f(S) = \langle f(x), f(y) \rangle$ setwise; otherwise, Spoiler can win with 3 pebbles and 2 rounds by Lem. 4.6.

Suppose that $f(S)$ is not a direct factor of $\text{Soc}(H)$. Spoiler pebbles $(x, y) \mapsto (f(x), f(y))$. We now have the following cases.

- **Case 1:** Suppose that $f(S)$ is not contained in $\text{Soc}(H)$. Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. As $(x, y) \mapsto (f(x), f(y))$ have been pebbled, if $f'|_S$ is not an isomorphism, then Spoiler wins with 1 additional pebble and 1 additional round, for a total of 3 pebbles and 2 rounds: for, since $\{x, y\}$ generates S , if $f'|_S$ is not an isomorphism then there is some $z \in S$ such that $(x, y, z) \mapsto (f(x), f(y), f'(z))$ does not extend to an isomorphism, and Spoiler can pebble such a z .

Next, as $f(S) = \langle f(x), f(y) \rangle$, we must have that at least one of $f(x), f(y)$ is not in $\text{Soc}(H)$, and therefore $f'(S) \supseteq \{f(x), f(y)\}$ is also not contained in $\text{Soc}(H)$. As $S \trianglelefteq \text{Soc}(G)$, the normal closure $\text{ncl}(S)$ is minimal normal in G [Isa08, Exercise 2.A.7]. But since $f'(S)$ is not even contained in $\text{Soc}(H)$, we have by Lemma 4.3 that $\text{ncl}(f'(S))$ is not a direct product of non-Abelian simple groups, so $\text{ncl}(S) \not\cong \text{ncl}(f'(S))$. We note that $\text{ncl}(S) = \langle \{gSg^{-1} : g \in G\} \rangle$.

As $\text{ncl}(f'(S))$ is not isomorphic to a direct power of S , there is some conjugate $T = gSg^{-1} \neq S$ such that $f'(T)$ does not commute with $f'(S)$, by Lemma 4.4. Yet since $S \leq \text{Soc}(G)$, T and S do commute. Spoiler pebbles $g \mapsto f'(g)$. Spoiler has now pebbled x, y, g which generate a subgroup containing $\langle S, T \rangle = S \times T \leq G$ and $S \times T \cong S \times S$. However, as the image is not isomorphic to $S \times S$, the map $(x, y, g) \mapsto (f(x), f(y), f'(g))$ does not extend to an isomorphism of $\langle x, y, g \rangle$ with $\langle f(x), f(y), f'(g) \rangle$, as any such isomorphism

would have to map $S \times T \leq \langle x, y, g \rangle$ to a subgroup of H isomorphic to $S \times S$. So Spoiler now wins, with a total of 3 pebbles and 2 rounds.

- **Case 2:** Suppose now that $f(S) \leq \text{Soc}(H)$, but that $f(S)$ is not normal in $\text{Soc}(H)$. As $f(S)$ is not normal in $\text{Soc}(H)$, there exists $T = \langle a, b \rangle \in \text{Fac}(\text{Soc}(H))$ such that T does not normalize $f(S)$. Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. Again, we may assume $f'|_S$ is an isomorphism. By Lem. 4.6, we may assume without loss of generality that for each $T \in \text{Fac}(\text{Soc}(G))$, $f'(T) \cong T$ (though $f'|_T$ need not be an isomorphism) and, by the previous case, that $f'(T) \leq \text{Soc}(H)$. Otherwise, Spoiler wins with 2 additional pebbles and 2 additional rounds. Note that while Case 1 prescribes 3 additional pebbles, we may at the next round reuse pebbles on x, y , and only one such pebble is needed at the next round. This brings our total down to 2 additional pebbles. Let $g = (f')^{-1}(a)$ and $h = (f')^{-1}(b)$. If either g or h is not in $\text{Soc}(G)$, then by Lem. 4.6 (b), Spoiler can use the two pebbles in hand this round, plus one additional pebble and one additional round to win, for a total of 4 pebbles and 3 rounds. Otherwise, as S is normal in $\text{Soc}(G)$, we note that $gSg^{-1} = S$ and $hSh^{-1} = S$. Spoiler pebbles $(g, h) \mapsto (a, b)$. The map $(x, y, g, h) \mapsto (f(x), f(y), a, b)$ does not extend to an isomorphism of $\langle x, y, g, h \rangle$ and $\langle f(x), f(y), a, b \rangle$, since g, h normalize $S = \langle x, y \rangle$ but a, b cannot both normalize $f(S) = \langle f(x), f(y) \rangle$ (since T does not). Spoiler now wins. In this case, we have used at most 4 pebbles and 3 rounds.

The result follows. \square

Lemma 4.8. *Let G, H be semisimple groups of order n , let S be a non-Abelian simple group in $\text{Fac}(\text{Soc}(G))$. Let $f, f' : G \rightarrow H$ be two bijections selected by Duplicator at two different rounds. If $f(S) \cap f'(S) \neq 1$, then $f(S) = f'(S)$, or Spoiler can win in the $(4, 3)$ - WL_{II}^2 pebble game.*

Proof. By Prop. 4.7, both $f(S)$ and $f'(S)$ must be simple normal subgroups of $\text{Soc}(H)$ (or Spoiler wins with 4 pebbles and 3 rounds). Since they intersect nontrivially, but distinct simple normal subgroups of $\text{Soc}(H)$ intersect trivially, the two must be equal. \square

We next introduce the notion of weight.

Definition 4.9. Let $\text{Soc}(G) = S_1 \times \cdots \times S_k$ where each S_i is a simple normal subgroup of $\text{Soc}(G)$. For any $s \in \text{Soc}(G)$, write $s = s_1 s_2 \cdots s_k$ where each $s_i \in S_i$, and define the *weight* of s , denoted $\text{wt}(s)$, as the number of i 's such that $s_i \neq 1$. For $s \notin \text{Soc}(G)$ we define $\text{wt}(s) = \infty$.

Note that the definition of weight is well-defined since the S_i are the unique subsets of $\text{Soc}(G)$ that are simple normal subgroup of $\text{Soc}(G)$, so the decomposition $s = s_1 s_2 \cdots s_k$ is unique up to the order of the factors.²

Lemma 4.10 (Weight Lemma). *Let G, H be semisimple groups of order n . If Duplicator selects a bijection that does not preserve the weight of every element of G , then Spoiler can win in the $(4, 4)$ - WL_{II}^2 game. In particular, Spoiler can win the $(4, 4)$ - WL_{II}^2 game if Duplicator selects a bijection that does not map $\text{Soc}(G)$ bijectively to $\text{Soc}(H)$.*

²This is essentially a particular instance of the “rank lemma” from [GL26], which intuitively states that WL detects in $O(\log n)$ rounds the set of elements for a given subgroup provided that it also identifies the generators. As we are now in the setting of 2-ary WL we give the full proof, which also has tighter bounds on the number of rounds.

Proof. The identity is the unique element of weight 0, which must be sent to the identity because it is the unique element e satisfying $e^2 = e$. Otherwise Spoiler can win by pebbling the identity with 1 pebble in 1 round. The case of weight 1 is precisely Proposition 4.7, which needs only 4 pebbles and 3 rounds.

Suppose for the sake of contradiction that f does not preserve weight. Among the elements $s \in G$ such that $\text{wt}(f(s)) \neq \text{wt}(s)$, choose one of minimal weight w . We must have $w \geq 2$, since we have already handled the cases of weight 0 and 1. Furthermore, we may assume w is finite; if $|\text{Soc}(G)| \neq |\text{Soc}(H)|$, then we may choose G to be the one with larger socle, whereby some finite-weight element must get sent to an infinite-weight element, and if the socles have the same size, then since f is a bijection, if it doesn't preserve the weight of some infinite-weight element, it must also map some finite-weight element to an infinite-weight element.

Since f is a bijection from the elements of weight $< w$ in $\text{Soc}(G)$ to the elements of weight $< w$ in $\text{Soc}(H)$ (by the choice of s), and $\text{wt}(f(s)) \neq w$, we must have $\text{wt}(f(s)) > w$ since f is a bijection.

Without loss of generality, write $\text{Soc}(G) = S_1 \times \cdots \times S_k$ where the S_i are the simple normal subgroups of $\text{Soc}(G)$ and $s \in S_1 \times S_2 \times \cdots \times S_w$, where we can write $s = s_1 s_2 \cdots s_w$ with all $s_i \neq 1$. Since s is a smallest-weight element whose weight is not preserved, $f(s_2 s_3 \cdots s_w)$ must have weight precisely $w - 1$. Now $f(s_1)$ has weight 1, so $f(s_1) f(s_2 s_3 \cdots s_w)$ has weight either $w - 2$, $w - 1$ or w . But since $\text{wt}(f(s)) > w$, we have $f(s_1 s_2 \cdots s_w) = f(s) \neq f(s_1) f(s_2 s_3 \cdots s_w)$, since the two elements have different weights. Now Spoiler pebbles the pair $(s, s_2 s_3 \cdots s_w) \mapsto (f(s), f(s_2 \cdots s_w))$.

At the next round, Duplicator selects another bijection f' . We still have $\text{wt}(f'(s)) = \text{wt}(f(s)) > w$, $\text{wt}(f'(s_2 \cdots s_w)) = \text{wt}(f(s_2 \cdots s_w)) = w - 1$, and (by Proposition 4.7) $\text{wt}(f'(s_1)) = 1$. Note that although Proposition 4.7 uses 4 pebbles and 3 rounds, after the first round of that protocol, Spoiler may reuse the pebbles that were previously on s and $s_2 s_3 \cdots s_w$. Since the first round of that protocol was the second round in this proof, the total here would be 4 pebbles and 4 rounds. Thus, again, we have $\text{wt}(f'(s_1) f'(s_2 s_3 \cdots s_w)) \in \{w, w - 1, w - 2\}$, so we must have $f'(s_1 s_2 \cdots s_w) = f'(s) \neq f'(s_1) f'(s_2 \cdots s_w)$. Spoiler now pebbles s_1 and wins with 4 pebbles and 2 rounds. In either case, Spoiler used at most 4 pebbles and at most 4 rounds. \square

Lemma 4.11. *Let G and H be semisimple groups with isomorphic socles. Let $S_1, S_2 \in \text{Fac}(\text{Soc}(G))$ be distinct. Let $f : G \rightarrow H$ be the bijection that Duplicator selects. If there exist $x_i \in S_i$ such that $f(x_1 x_2) \neq f(x_1) f(x_2)$, then Spoiler can win in the $(4, 5)$ - WL_{II}^2 pebble game.*

Proof. By Lem. 4.10, we may assume that $\text{wt}(s) = \text{wt}(f(s))$ for all $s \in \text{Soc}(G)$; otherwise, Spoiler wins with at most 4 pebbles and 4 rounds. As $f(x_1 x_2)$ has weight 2, $f(x_1 x_2)$ belongs to the direct product of two simple factors in $\text{Fac}(\text{Soc}(H))$, so it can be written $f(x_1 x_2) = y_1 y_2$ with each y_i in distinct simple factors in $\text{Fac}(\text{Soc}(H))$. Without loss of generality suppose that $y_1 \neq f(x_1)$. Spoiler pebbles $(x_1, x_1 x_2) \mapsto (f(x_1), f(x_1 x_2))$. Now $\text{wt}(x_1^{-1} \cdot x_1 x_2) = 1$, while $\text{wt}(f(x_1)^{-1} \cdot f(x_1 x_2)) \geq 2$. (Note that we cannot quite yet directly apply Lem. 4.10, because we have not yet identified a single element x such that $\text{wt}(x) \neq \text{wt}(f(x))$.)

On the next round, Duplicator selects another bijection f' . Spoiler now pebbles $x_2 \mapsto f'(x_2)$. Suppose that f' preserves weight. Otherwise, by Lem. 4.10, Spoiler wins with 4 pebbles, reusing existing pebbles on the board, and 4 additional rounds for a total of 5 rounds.

Because $\text{wt}(x_1^{-1} \cdot x_1 x_2) = 1$ but $\text{wt}(f(x_1)^{-1} f(x_1 x_2)) \geq 2$, and f' preserves weight, we have $f'(x_2) \neq f'(x_1)^{-1} f'(x_1 x_2)$. Thus, the pebbled map $(x_1, x_2, x_1 x_2) \mapsto (f'(x_1), f(x_2), f(x_1 x_2))$ does not extend to an isomorphism, and so Spoiler wins with 3 pebbles and 2 rounds.

In total, Spoiler used at most 4 pebbles and at most 5 rounds. \square

Recall that if G is semisimple, then $G \cong G^* \leq \text{Aut}(\text{Soc}(G))$ (see Section 4.1). Now each minimal normal subgroup $N \trianglelefteq G$ is of the form $N = S^k$, where S is a non-Abelian simple group. So $\text{Aut}(N) = \text{Aut}(S) \wr \text{Sym}(k)$. In particular,

$$G \leq \prod_{\substack{N \trianglelefteq G \\ N \text{ is minimal normal}}} \text{Aut}(N).$$

So if $g \in G$, then the conjugation action of g on $\text{Soc}(G)$ acts by (i) automorphism on each simple direct factor of $\text{Soc}(G)$, and (ii) by permuting the direct factors of $\text{Soc}(G)$. Provided generators of the direct factors of the socle are pebbled, Spoiler can detect inconsistencies of the automorphism action. However, doing so directly would be too expensive as there could be $\Theta(\log |G|)$ generators, so we employ a more subtle approach with a similar outcome. By Lem. 4.10, Duplicator must select bijections $f : G \rightarrow H$ that preserve weight. We use Lem. 4.10 in tandem with the fact that the direct factors of the socle commute to effectively pebble the set of all the generators at once. Namely, suppose that $\text{Fac}(\text{Soc}(G)) = \{S_1, \dots, S_k\}$, where $S_i = \langle x_i, y_i \rangle$. Let $x := x_1 \cdots x_k$ and $y := y_1 \cdots y_k$. We will show that it suffices for Spoiler to pebble (x, y) rather than individually pebbling generators for each S_i (this will still allow the factors to be permuted, but that is all).

Lemma 4.12. *Let G and H be semisimple groups with isomorphic socles, with $\text{Fac}(\text{Soc}(G)) = \{S_1, \dots, S_m\}$, with $S_i = \langle x_i, y_i \rangle$. Let $f : G \rightarrow H$ be the bijection that Duplicator selects, and suppose that (i) for all i , $f(S_i) \cong S_i$ (though $f|_{S_i}$ need not be an isomorphism) and $f(S_i) \in \text{Fac}(\text{Soc}(H))$, (ii) for every $s \in G$, $\text{wt}(s) = \text{wt}(f(s))$, and (iii) for all i , $f(S_i) = \langle f(x_i), f(y_i) \rangle$.*

Now suppose that Spoiler pebbles $(x_1 \cdots x_m, y_1 \cdots y_m) \mapsto (f(x_1 \cdots x_m), f(y_1 \cdots y_m))$. As f preserves weight, we may write $f(x_1 \cdots x_m) = h_1 \cdots h_m$ and $f(y_1 \cdots y_m) = z_1 \cdots z_m$ with $h_i, z_i \in f(S_i)$ for all i .

Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at any subsequent round in which the pebbles used above have not moved. If any of the following hold, then Spoiler can win in the WL_{II}^2 pebble game with 4 additional pebbles and 5 additional rounds (beyond the number of current pebbles and rounds, when this lemma is applied):

- (a) *f' does not satisfy conditions (i)–(iii),*
- (b) *there exists an $i \in [m]$ such that $f'(x_i) \notin \{h_1, \dots, h_m\}$ or $f'(y_i) \notin \{z_1, \dots, z_m\}$*
- (c) *there exists an $i \in [m]$ such that $f'|_{S_i}$ is not an isomorphism*
- (d) *there exists $g \in G$ and $i \in [m]$ such that $gS_i g^{-1} = S_i$ and for some $x \in S_i$, $f'(g x g^{-1}) \neq f'(g) f'(x) f'(g)^{-1}$*

Furthermore, cases (a) and (b) instead only require 4 additional pebbles and 4 additional rounds (rather than 5 additional rounds).

Proof. We prove each part in turn:

- (a) If f' does not satisfy (i)–(iii) then Spoiler can win with 4 additional pebbles and 4 additional rounds by, respectively, Prop. 4.7, Lem. 4.10, and Lem. 4.6.
- (b) Suppose there exists an $i \in [m]$ such that for all $j \in [m]$, $f'(x_i) \neq h_j$. Spoiler pebbles $(x_i, x_1 \cdots x_m x_i^{-1}) \mapsto (f'(x_i), f'(x_1 \cdots x_m x_i^{-1}))$. Now $\text{wt}(x_1 \cdots x_m \cdot x_i^{-1}) = m - 1$, while

$\text{wt}(f(x_1 \cdots x_m) \cdot f'(x_i)^{-1}) = m$, since $f'(x_i) \notin \{h_1, \dots, h_m\}$. Since f' preserves weight (by part (a)), $f'(x_1 \cdots x_m) f'(x_i)^{-1} \neq f'(x_1 \cdots x_m x_i^{-1})$. As all three of these elements have been pebbled, Spoiler wins. Similarly if instead some y_i has $f'(y_i) \notin \{z_1, \dots, z_m\}$. In total, Spoiler used at most 4 additional pebbles and 4 additional rounds (by part (a)).

- (c) First, by (b), we may assume that $f'(x_i) = h_j$ and $f'(y_i) = z_j$ for some j . Otherwise, Spoiler wins with 4 additional pebbles and 4 additional rounds. Now suppose that for some $i \in [k]$, $f'|_{S_i}$ is not an isomorphism. Then there is some word $w(x, y)$ such that $f'(w(x_i, y_i)) \neq w(f'(x_i), f'(y_i))$. We may further assume $w(x_i, y_i) \neq 1$, since if f' preserves all such words, then f' must preserve *all* words and thereby be an isomorphism.

Spoiler pebbles $g = w(x_i, y_i)$. Let $f'' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. Since $f'(S_i)$ and $f''(S_i)$ intersect at $f''(g) \neq 1$, by Lem. 4.8, we have $f'(S_i) = f''(S_i)$; otherwise, Spoiler wins with 2 additional pebbles and 3 additional rounds. Note that while Lem. 4.8 prescribes 4 additional pebbles, we may reuse the two pebbles on $x_1 \cdots x_m$ and $y_1 \cdots y_m$.

Recall from the first paragraph in the proof for (c), we have that $f'(x_i) = h_j$ and $f'(y_i) = z_j$ for some j . Since $f''(x_i) \in f''(S_i) = f'(S_i)$, and by (b) we must have $f''(x_i) \in \{h_1, \dots, h_m\}$, only one of which is in $f'(S_i)$, we must have $f''(x_i) = h_j$ as well (in particular, $f''(x_i) = f'(x_i)$). Similarly, $f''(y_i) = f'(y_i) = z_j$. Otherwise, Spoiler wins with 3 additional pebbles (reusing the pebble on g) and 4 additional rounds, for a total of 5 rounds. Spoiler now pebbles $(x_i, y_i) \mapsto (h_j, z_j)$. The pebbled map $(x_i, y_i, g) \mapsto (f'(x_i), f'(y_i), f'(g))$ does not extend to an isomorphism by the choice of g , and so Spoiler wins with at most 3 additional pebbles and 5 additional rounds.

- (d) By (c), we may assume that $f'|_{S_i}$ is an isomorphism; otherwise, Spoiler wins with 3 additional pebbles and 5 additional rounds. In this case, Spoiler pebbles $(g, gxg^{-1}) \mapsto (f'(g), f'(gxg^{-1}))$. Let $f'' : G \rightarrow H$ be the bijection that Duplicator selects on the next round. As we pebbled $gxg^{-1} \neq 1$, we have that $f''(S_i)$ and $f'(S_i)$ overlap at $f'(gxg^{-1})$, so we have $f''(S_i) = f'(S_i)$ by Lem. 4.8 (otherwise, by similar argument as in part (c), Spoiler wins with 2 additional pebbles and 3 additional rounds).

As in the previous part, by (b), we have that $f''(x_i) = f'(x_i) = h_j$ and $f''(y_i) = f'(y_i) = z_j$. Otherwise, Spoiler may win. The strategy in (b) requires Spoiler to use 3 additional pebbles, and 4 additional rounds. As the first round of (b) is the second round here in part (d), this yields a total of 5 rounds. Spoiler now pebbles $(x_i, y_i) \mapsto (h_j, z_j)$. Now, as $x \in \langle x_i, y_i \rangle$ there is a word w such that $x = w(x_i, y_i)$. Suppose \bar{f} is a map extending our pebbled map $(x_i, y_i, g, gxg^{-1}) \mapsto (h_j, z_j, f'(g), f'(gxg^{-1}))$. If $\bar{f}(g^{-1}) \neq \bar{f}(g)^{-1}$, then \bar{f} is not an isomorphism. If $\bar{f}(x) \neq w(\bar{f}(x_i), \bar{f}(y_i))$, then \bar{f} is also not an isomorphism. But if $\bar{f}(g^{-1}) = \bar{f}(g)^{-1}$ and $\bar{f}(x) = w(\bar{f}(x_i), \bar{f}(y_i)) = w(h_j, z_j) = f'(x)$, then we have $\bar{f}(gxg^{-1}) \neq \bar{f}(g)\bar{f}(x)\bar{f}(g)^{-1}$ by assumption, so again \bar{f} is not an isomorphism. Thus, there is no isomorphism \bar{f} extending our pebbled map, and Spoiler wins with at most 4 additional pebbles and 5 additional rounds. \square

Lem. 4.12 provides enough to establish that Spoiler can force Duplicator to select at each round a bijection that restricts to an isomorphism on the socles.

Proposition 4.13. *Let G and H be semisimple groups with isomorphic socles, with $\text{Fac}(\text{Soc}(G)) = \{S_1, \dots, S_m\}$ and $S_i = \langle x_i, y_i \rangle$. Let $f : G \rightarrow H$ be the bijection that Duplicator selects at some round, and suppose that (i) for all i , $f(S_i) \cong S_i$ (though $f|_{S_i}$ need not be*

an isomorphism) and $f(S_i) \in \text{Fac}(\text{Soc}(H))$, (ii) for every $s \in G$, $\text{wt}(s) = \text{wt}(f(s))$, and (iii) for all i , $f(S_i) = \langle f(x_i), f(y_i) \rangle$. Now suppose that Spoiler pebbles $(x_1 \cdots x_m, y_1 \cdots y_m) \mapsto (f(x_1 \cdots x_m), f(y_1 \cdots y_m))$.

Let $f': G \rightarrow H$ be the bijection that Duplicator selects at any subsequent round in which the pebbles used above have not moved. Then $f'|_{\text{Soc}(G)}: \text{Soc}(G) \rightarrow \text{Soc}(H)$ must be an isomorphism, or Spoiler can win in 5 more rounds using at most 5 more pebbles in the WL_{II}^2 pebble game.

Proof. By Lemma 4.10 f must map $\text{Soc}(G)$ bijectively to $\text{Soc}(H)$ and must preserve weights (otherwise, Spoiler wins with 4 pebbles and 4 rounds). Suppose that $f|_{\text{Soc}(G)}: \text{Soc}(G) \rightarrow \text{Soc}(H)$ is not an isomorphism. The only way this is possible is if $f|_{\text{Soc}(G)}$ is not a homomorphism. So there exist $s, s' \in \text{Soc}(G)$ such that $f(s)f(s') \neq f(ss')$. We claim that this implies that there are $t_i \in S_i$ for all $i = 1, \dots, m$ such that $f(t_1 t_2 \cdots t_m) \neq f(t_1)f(t_2) \cdots f(t_m)$.

For suppose otherwise. Write $s = s_1 s_2 \cdots s_m$ and $s' = s'_1 s'_2 \cdots s'_m$ with $s_i, s'_i \in S_i$ for all i . Then the failure to be a homomorphism is rewritten as:

$$f(s_1 s_2 \cdots s_m) f(s'_1 s'_2 \cdots s'_m) \neq f(s_1 \cdots s_m s'_1 \cdots s'_m) = f(s_1 s'_1 s_2 s'_2 \cdots s_m s'_m).$$

Then by our supposition, the LHS of the preceding displayed equation is:

$$f(s_1)f(s_2) \cdots f(s_m)f(s'_1)f(s'_2) \cdots f(s'_m),$$

while the RHS is $f(s_1 s'_1)f(s_2 s'_2) \cdots f(s_m s'_m)$.

But, by Lem. 4.12 (c), f is an isomorphism on each simple normal subgroup of $\text{Soc}(G)$ (otherwise, Spoiler wins with 4 additional pebbles and 5 additional rounds), so the latter is

$$f(s_1)f(s'_1)f(s_2)f(s'_2) \cdots f(s_m)f(s'_m).$$

Furthermore, since the simple normal subgroups of $\text{Soc}(G)$ commute with one another, their images must commute in $\text{Soc}(H)$ (otherwise Spoiler can pebble two elements that commute in $\text{Soc}(G)$ but whose images don't commute in $\text{Soc}(H)$ and win), the LHS is equal to the same, a contradiction. Thus we have some $t_i \in S_i$ for $i = 1, \dots, m$ such that $f(t_1 t_2 \cdots t_m) \neq f(t_1)f(t_2) \cdots f(t_m)$.

Suppose $t = t_1 \cdots t_m$ has weight w , and without loss of generality, re-index the S_i and t_i so that $t \in S_1 \times \cdots \times S_w$ with $t = t_1 t_2 \cdots t_w$. We still have $f(t_1 \cdots t_w) \neq f(t_1) \cdots f(t_w)$. If $f(t)$ is not in $f(S_1) \times f(S_{i_2}) \times \cdots \times f(S_{i_w})$ for any $(w-1)$ tuple of distinct indices i_2, i_3, \dots, i_w , then Spoiler pebbles t and t_1 . On the next round, Duplicator picks another f' with $f'(t_1) = f(t_1) \in S_1$ and $f'(t) = f(t) \notin f(S_1) \times f(S_{i_2}) \times \cdots \times f(S_{i_w})$ for any $(w-1)$ tuple of distinct indices. So $f'(t)$ is not the product of something in $f(S_1)$ with any element of weight $w-1$. But by Lemma 4.10, $f(t_2 t_3 \cdots t_w)$ must have weight $w-1$. Otherwise, Spoiler wins with 3 additional pebbles beyond those on (t, t_1) and 4 additional rounds, bringing our total to 5 additional pebbles and 5 additional rounds. While Lem. 4.10 prescribes 4 additional pebbles, we may reuse the pebble on t_1 .

Furthermore, we have $f(S_1) = f'(S_1)$ by Lem. 4.8, or Spoiler can win using 3 additional pebbles and 3 additional rounds, by reusing the pebble on t . So we have $f'(t) \neq f'(t_1)f'(t_2 t_3 \cdots t_w)$. Spoiler may now pebble $t_2 t_3 \cdots t_w$ and win (3 pebbles, 2 rounds). Thus, $f(t)$ must be the product of something in $f(S_1)$ with some element of weight $w-1$. So there is some $w-1$ -tuple of distinct indices i_2, \dots, i_w such that $f(t) \in f(S_1) \times f(S_{i_2}) \times \cdots \times f(S_{i_w})$.

Now repeat the same argument for any $j \in \{1, \dots, w\}$ taking the place of 1. We thus find that for any such j , $f(t)$ must be in the direct product of $f(S_j)$ and $w-1$ -many other $f(S_i)$'s. Thus $f(t)$ must be in $f(S_1) \times f(S_2) \times \cdots \times f(S_w)$, since it has weight

exactly w . So we have $f(t) = f(t'_1)f(t'_2)\cdots f(t'_w)$ for some $t'_i \in S_i$. But we also have $f(t) \neq f(t_1)f(t_2)\cdots f(t_w)$. Thus, there must be some i such that $t_i \neq t'_i$. Without loss of generality (by re-indexing if needed), for simplicity of notation we may suppose that $i = 1$. Spoiler pebbles $(t, t_1) \mapsto (f(t), f(t_1))$. On the next round, Duplicator selects a new bijection f' with $f'(t) = f(t)$ and $f'(t_1) = f(t_1)$. Since S_1 is the unique simple normal subgroup of $\text{Soc}(G)$ containing t_1 , and similarly $f(S_1)$ is the unique simple normal subgroup of $\text{Soc}(H)$ containing $f(t_1)$, by Lem. 4.8 we have $f'(S_1) = f(S_1)$. Otherwise, Spoiler wins with 3 additional pebbles and 3 additional rounds, for a total of 5 pebbles and 4 rounds. While Lem. 4.8 prescribes 4 additional pebbles, we may reuse the pebble on t .

Finally, we claim that $f'(t_1t_2\cdots t_w) \neq f'(t_1)f'(t_2t_3\cdots t_w)$. Suppose otherwise. Then by Lemma 4.10, $\text{wt}(f'(t)) = \text{wt}(f(t)) = w$, and $\text{wt}(f'(t_2t_3\cdots t_w)) = w - 1$ (or Spoiler wins with 3 additional pebbles and 4 additional rounds; for the former, Spoiler can re-use the pebble on t_1 , and for the latter Spoiler can re-use the pebble on t). Now, if $f'(t_2t_3\cdots t_w)$ has a non-identity component in the $f(S_1)$ factor of $\text{Soc}(H)$, then there must be some other $i \in \{2, \dots, w\}$ such that $f'(t_2t_3\cdots t_w)$ has trivial projection onto $f(S_i)$. Note that $f'(t_1) = f(t_1) \in f(S_1)$ thus also has trivial projection onto $f(S_i)$. But then this would contradict the fact that $\text{wt}(f'(t)) = w$. So the projection of $f'(t_2t_3\cdots t_w)$ onto $f(S_1)$ (as a quotient of $\text{Soc}(H)$) must be trivial. Thus, the component of $f'(t) = f(t)$ in $f(S_1)$ is $f(t'_1)$, but the component of $f'(t_1)f'(t_2\cdots t_w)$ in $f(S_1)$ is precisely $f'(t_1) = f(t_1) \neq f(t'_1)$. Thus $f'(t) \neq f'(t_1)f'(t_2\cdots t_w)$ as claimed. As t, t_1 are already pebbled, Spoiler now pebbles $t_2t_3\cdots t_w$, and wins (3 pebbles, 2 rounds). The worst strategy Spoiler needed to use was 5 pebbles and 5 rounds (coming from our application of Lem. 4.10 in the fourth paragraph). \square

Remark 4.14. Brachter & Schweitzer [BS22, Lemma 5.22] previously showed that (1-ary) Weisfeiler–Leman can decide whether two groups have isomorphic socles. However, their results did not solve the search problem; that is, they did not show Duplicator must select bijections that restrict to an isomorphism on the socle even in the case for semisimple groups. This contrasts with Prop. 4.13, where we show that 2-ary WL effectively solves the search problem. This is an important ingredient in our proof that the $(7, 7)\text{-WL}_{II}^2$ pebble game solves isomorphism for semisimple groups.

We obtain as a corollary of Lem. 4.12 and Lem. 4.13 that if G and H are semisimple, then Duplicator must select bijections that restrict to isomorphisms of $\text{PKer}(G)$ and $\text{PKer}(H)$.

Corollary 4.15. *Let G and H be semisimple groups of order n . Let $\text{Fac}(\text{Soc}(G)) := \{S_1, \dots, S_m\}$, and suppose that $S_i = \langle x_i, y_i \rangle$. Let $x := x_1 \cdots x_m$ and $y := y_1 \cdots y_m$. Let $f : G \rightarrow H$ be the bijection that Duplicator selects. Spoiler begins by pebbling $(x, y) \mapsto (f(x), f(y))$. Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at any subsequent round in which the preceding pebbles have not moved. If $f'|_{\text{PKer}(G)} : \text{PKer}(G) \rightarrow \text{PKer}(H)$ is not an isomorphism, then Spoiler can win with 5 additional pebbles and 5 additional rounds in the WL_{II}^2 pebble game.*

Proof. We have the following cases.

- **Case 1:** Suppose that $f'(\text{PKer}(G)) \neq \text{PKer}(H)$. Then without loss of generality, there exists $g \in \text{PKer}(G)$ such that $f'(g) \notin \text{PKer}(H)$. By Proposition 4.7 we may assume that $f'(S_i) \in \text{Fac}(\text{Soc}(H))$ for all $S_i \in \text{Fac}(\text{Soc}(G))$, or Spoiler can win with 2 additional pebbles beyond those on (x, y) , which we reuse, and 3 additional rounds.

Let $i \in [m]$ such that $f'(g) \cdot f'(S_i) \cdot f'(g)^{-1} \neq f'(S_i)$. Spoiler pebbles $(g, x_i) \mapsto (f'(g), f'(x_i))$. Let $f'' : G \rightarrow H$ be the bijection that Duplicator selects at the next round.

As $x_i \mapsto f'(x_i)$ is pebbled, we have that $f''(S_i) = f'(S_i)$; otherwise, by Lem. 4.8, Spoiler wins with 2 additional pebbles from the start of this corollary (re-using the pebbles on two of x, y, g) and 3 additional rounds. By Lem. 4.12 (b), we may assume that $f''(y_i) = f'(y_i)$. Otherwise, Spoiler wins with 2 additional pebbles (reusing the pebbles on g and x_i , bringing us to a total of 4 pebbles) and 4 additional rounds (for a total of 5 rounds). Spoiler now pebbles $y_i \mapsto f''(y_i) = f'(y_i)$. As $g \in \text{PKer}(G)$, we have that $g \cdot S_i \cdot g^{-1} = S_i$, while $f'(g) \cdot f'(S_i) \cdot f'(g)^{-1} \neq f'(S_i)$. Since the pebbles on x_i, y_i fix their images in $f'(S_i)$, and we have also pebbled g , there is no marked isomorphism that extends the pebbling, and thus Spoiler wins, with a total of at most 4 additional pebbles and 5 additional rounds.

- **Case 2:** Suppose now that $f'(\text{PKer}(G)) = \text{PKer}(H)$, but that $f'|_{\text{PKer}(G)}$ is not an isomorphism. As G and H are semisimple, so are $\text{PKer}(G)$ and $\text{PKer}(H)$. Thus, if $f'|_{\text{PKer}(G)}$ fails to be an isomorphism, then by Lem. 4.1, there exists $g \in \text{PKer}(G)$ and $s \in \text{Soc}(G)$ such that $f'(gsg^{-1}) \neq f'(g)f'(s)f'(g)^{-1}$. By Prop. 4.13, f' must be an isomorphism on the socle (or Spoiler wins with 5 additional pebbles and 5 additional rounds), so if we write $s = s_1 s_2 \dots s_m$ with each $s_i \in S_i$, then we must have some $i \in [m]$ such that $f'(gs_i g^{-1}) \neq f'(g)f'(s_i)f'(g)^{-1}$. Since $g \in \text{PKer}(G)$, we have $gS_i g^{-1} = S_i$ as well. This case is handled precisely by Lem. 4.12 (d), in which only 4 additional pebbles and 5 additional rounds. Thus, for Case 2, we have used at most 5 additional pebbles and 5 additional rounds. \square

We now show that if G and H are not permutationally equivalent in their actions on $\text{Fac}(\text{Soc}(G))$ (resp., $\text{Fac}(\text{Soc}(H))$), then Spoiler can win.

Lemma 4.16. (Same assumptions as Lem. 4.12.) *Let G and H be semisimple groups with isomorphic socles, with $\text{Fac}(\text{Soc}(G)) = \{S_1, \dots, S_m\}$ and $S_i = \langle x_i, y_i \rangle$. Let $f : G \rightarrow H$ be the bijection that Duplicator selects at some round, and suppose that (i) for all i , $f(S_i) \cong S_i$ (though $f|_{S_i}$ need not be an isomorphism) and $f(S_i) \in \text{Fac}(\text{Soc}(H))$, (ii) for every $s \in G$, $\text{wt}(s) = \text{wt}(f(s))$, and (iii) for all i , $f(S_i) = \langle f(x_i), f(y_i) \rangle$. Now suppose that Spoiler pebbles $(x_1 \dots x_m, y_1 \dots y_m) \mapsto (f(x_1 \dots x_m), f(y_1 \dots y_m))$.*

Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. Suppose that there exist $g \in G$ and $i \in [m]$ such that $f'(gS_i g^{-1}) = f'(S_j)$, but $f'(g)f'(S_i)f'(g)^{-1} = f'(S_k)$ for some $k \neq j$. Then Spoiler can win with 4 additional pebbles and 4 additional rounds in the WL_{II}^2 pebble game.

Furthermore, if g is already pebbled, then Spoiler can win with 3 additional pebbles and 5 additional rounds.

Proof. We consider the following cases.

- **Case 1:** Suppose first that $i = j$. In this case, Spoiler pebbles $(g, x_i) \mapsto (f'(g), f'(x_i))$. Let $f'' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. As $x_i \mapsto f'(x_i)$ is pebbled, we have that $f''(S_i) = f'(S_i)$ by Lem. 4.8. Otherwise, Spoiler wins with 3 additional pebbles, reusing the pebble on g , and 3 additional rounds (bringing our total to 4 rounds). Spoiler now pebbles $y_i \mapsto f''(y_i)$. The map $(g, x_i, y_i) \mapsto (f'(g), f'(x_i), f''(y_i))$ does not extend to a marked isomorphism, as conjugation by g sends S_i to itself, but conjugation by $f'(g)$ does not send $f'(S_i)$ to itself. Thus, in this case, Spoiler wins with at most 4 additional pebbles and 4 additional rounds (which is achieved via the application of Lem. 4.8).
- **Case 2:** Suppose $i \neq j$ but $i = k$. This is symmetric to the preceding case, by swapping the roles of G and H .

- **Case 3:** Suppose now that i, j, k are all distinct. By Lem. 4.11, we have that $f'(x_i x_j) = f'(x_i) f'(x_j)$ (or Spoiler wins with 4 additional pebbles and 5 additional rounds). Spoiler begins by pebbling $(g, x_i x_j) \mapsto (f'(g), f'(x_i x_j))$. Let $f'' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. We have the following cases.
 - **Case 3(a):** Suppose that $f''(S_i) = f'(S_i)$. As $x_i x_j \mapsto f'(x_i) f'(x_j)$ is pebbled, we have necessarily that $f''(x_j) \in f'(S_j)$, and hence that $f''(S_j) = f'(S_j)$ by Lem. 4.8. Otherwise, Spoiler wins with 3 additional pebbles, reusing the pebble on g , and 3 additional rounds (for a total of 4 rounds). In this case, Spoiler pebbles $(x_i, y_i) \mapsto (f''(x_i), f''(y_i))$. As $g S_i g^{-1} = S_j$, but $f'(g) f''(S_i) f'(g)^{-1} \neq f''(S_j)$, the map $(g, x_i, y_i, x_i x_j) \mapsto (f'(g), f''(x_i), f''(y_i), f'(x_i x_j))$ does not extend to an isomorphism. So Spoiler wins with at most 4 additional pebbles and 4 additional rounds.
 - **Case 3(b):** Suppose that $f''(S_i) \neq f'(S_i)$. As $x_i x_j \mapsto f'(x_i) f'(x_j)$ is pebbled, we have necessarily that $f''(S_i) = f'(S_j)$ and $f''(S_j) = f'(S_i)$; otherwise, by Prop. 4.7, Spoiler wins with 3 additional pebbles, reusing the pebble on g , and 4 additional rounds. We now have the following sub-cases.
 - * **Case 3(b).i:** Suppose first that $g S_j g^{-1} = S_i$. By assumption, $g S_i g^{-1} = S_j$, but $f'(g) f'(S_i) f'(g)^{-1} = f'(S_k)$ (where $k \notin \{i, j\}$). So the conjugation map $\sigma_g : a \mapsto g a g^{-1}$ swaps S_i and S_j , while the conjugation map $\sigma_{f'(g)}$ does not swap $f'(S_i)$ and $f'(S_j)$. It follows that the map $(g, x_j, y_j, x_i x_j) \mapsto (f'(g), f''(x_j), f''(y_j), f'(x_i x_j))$ does not extend to an isomorphism. Spoiler now pebbles $(x_j, y_j) \mapsto (f''(x_j), f''(y_j))$. Thus, Spoiler wins with at most 4 additional pebbles and 2 additional rounds.
 - * **Case 3(b).ii:** Suppose now that $g S_j g^{-1} \neq S_i$. By assumption, as $g S_i g^{-1} = S_j$, we have that $g^{-1} S_j g = S_i$. Now recall that $g \mapsto f'(g)$ is pebbled. As the conjugation map $x \mapsto f'(g)^{-1} \cdot x \cdot f'(g)$ is a bijection and $f'(g) f'(S_i) f'(g)^{-1} = f'(S_k)$ (where again, $k \notin \{i, j\}$), we have that $f'(g)^{-1} f'(S_j) f'(g) \neq f'(S_i)$.

Suppose first that $f'(g)^{-1} f'(S_j) f'(g) = f'(S_j)$. Spoiler now pebbles $(x_i, y_i) \mapsto (f''(x_i), f''(y_i))$. As $g S_i g^{-1} \neq S_i$ and $f''(S_i) = f'(S_j)$, the map $(g, x_i, y_i, x_i x_j) \mapsto (f'(g), f''(x_i), f''(y_i), f'(x_i x_j))$ does not extend to an isomorphism. So Spoiler wins with at most 4 pebbles and 2 rounds.

Suppose now that $f'(g)^{-1} f'(S_j) f'(g) \neq f'(S_j)$. By the first paragraph of Case 3(b).ii, we have that $f'(g)^{-1} f'(S_j) f'(g) \neq f'(S_i)$. As $g^{-1} S_j g = S_i$, we have that $f''(g^{-1} x_j g) \in f''(S_i)$. So in particular, $f''(g^{-1} x_j g) \neq f'(g)^{-1} f''(x_j) f'(g)$. Spoiler now pebbles $(x_j, g^{-1} x_j g) \mapsto (f''(x_j), f''(g^{-1} x_j g))$. So Spoiler wins with at most 4 pebbles and 2 rounds.

In total, Spoiler used at most 4 additional pebbles and 5 additional rounds. \square

Theorem 4.17. *Let G be a semisimple group and H an arbitrary group of order n , not isomorphic to G . Then Spoiler has a winning strategy in the $(7, 7)$ - WL_I^2 pebble game.*

Proof. If H is not semisimple, then by Prop. 4.5, Spoiler wins with 3 pebbles and 2 rounds. So we now suppose H is semisimple.

Let $\text{Fac}(\text{Soc}(G)) = \{S_1, \dots, S_k\}$, and let x_i, y_i be generators of S_i for each $i \in [k]$. Let f_0 be the bijection chosen by Duplicator. Spoiler pebbles $(x_1 x_2 \cdots x_k, y_1 y_2, \dots, y_k) \mapsto (f_0(x_1 \cdots x_k), f_0(y_1 \cdots y_k))$. On subsequent rounds, we thus have satisfied the hypotheses of Lem. 4.12 and Prop. 4.13. Spoiler will never move these pebbles, and thus all subsequent

bijections chosen by Duplicator must restrict to isomorphisms on the socle (or Spoiler wins with at most 5 additional pebbles and 5 additional rounds. Let f be the bijection chosen by Duplicator at the next round. In particular, we may assume that f restricts to an isomorphism on the socle, or Spoiler wins, having used at most 7 pebbles and 6 rounds by Prop. 4.13.

By Lem. 4.1, we have that $G \cong H$ if and only if there is an isomorphism $\mu: \text{Soc}(G) \rightarrow \text{Soc}(H)$ that induces a permutational isomorphism $\mu^*: G^* \rightarrow H^*$. Thus, since $G \not\cong H$, there must be some $g \in G$ and $s \in \text{Soc}(G)$ such that $f(gsg^{-1}) \neq f(g)f(s)f(g)^{-1}$. Write $s = s_1 \cdots s_k$ with each $s_i \in S_i$ (not necessarily nontrivial). We claim that there exists some i such that $f(gs_i g^{-1}) \neq f(g)f(s_i)f(g)^{-1}$. For suppose not, then we have

$$\begin{aligned} f(gsg^{-1}) &= f(gs_1 g^{-1} gs_2 g^{-1} \cdots gs_k g^{-1}) \\ &= f(gs_1 g^{-1})f(gs_2 g^{-1}) \cdots f(gs_k g^{-1}) \quad (\text{by Prop. 4.13}) \\ &= f(g)f(s_1)f(g)^{-1}f(g)f(s_2)f(g)^{-1} \cdots f(g)f(s_k)f(g)^{-1} \\ &= f(g)f(s_1 \cdots s_k)f(g)^{-1} = f(g)f(s)f(g)^{-1}, \end{aligned}$$

a contradiction. For simplicity of notation, without loss of generality we may assume $i = 1$, so we now have $f(gs_1 g^{-1}) \neq f(g)f(s_1)f(g)^{-1}$.

We break the argument into cases:

- (1) If $gs_1 g^{-1} \in S_1$, then we have $gS_1 g^{-1} = S_1$ (any two distinct simple normal factors of the socle intersect trivially), we have by Lem. 4.12 (d) that Spoiler can win with at most 4 additional pebbles (for a total of 6 pebbles) and 5 additional rounds (for a total of 6 rounds).
- (2) If $gs_1 g^{-1} \in S_j$ for $j \neq 1$ and $f(g)f(s_1)f(g)^{-1} \notin f(S_j)$, we have by Lem. 4.16 that Spoiler can win with at most 4 additional pebbles (for a total of 6 pebbles) and 5 additional rounds (for a total of 6 rounds).
- (3) Suppose now that $gs_1 g^{-1} \in S_j$ for some $j \neq 1$ and $f(g)f(s_1)f(g)^{-1} \in f(S_j)$. Spoiler begins by pebbling $(g, gs_1 g^{-1}) \mapsto (f(g), f(gs_1 g^{-1}))$. Let $f' : G \rightarrow H$ be the bijection that Duplicator selects at the next round. As $gs_1 g^{-1} \in S_j$ is pebbled, we have that $f'(S_j) = f(S_j)$ by Lem. 4.8 (or Spoiler wins with 2 additional pebbles, by re-using 2 of the 4 already placed pebbles, except for the one on $gs_1 g^{-1}$, and 3 additional rounds). Now by assumption, $gS_1 g^{-1} = S_j$ and $f(g)f(S_1)f(g)^{-1} = f(S_j)$. So as $g \mapsto f(g)$ is pebbled, we claim that we may assume $f'(S_1) = f(S_1)$. For suppose not; then we have $g^{-1}S_j g = S_1$ but $f'(g)^{-1}f'(S_j)f'(g) = f(g)^{-1}f(S_j)f(g) = f(S_1) \neq f'(S_1)$. But then Spoiler can win with 3 additional pebbles (for a total of 7 pebbles) and 5 additional rounds (for a total of 7 rounds) by Lem. 4.16. Thus we have $f'(S_1) = f(S_1)$.

In particular, we have that $f'(x_1) = f(x_1)$ and $f'(y_1) = f(y_1)$, by the same argument as in the proof of Lem. 4.12 (c) (or Spoiler wins with 3 additional pebbles and 5 additional rounds, bringing us to a total of 7 pebbles and 7 rounds). As $S_1 = \langle x_1, y_1 \rangle$, we have by Lem. 4.12 (c) that $f'(s_1) = f(s_1)$ (or Spoiler wins with 3 additional pebbles and 5 additional rounds, bringing us to a total of 7 pebbles and 7 rounds). Spoiler now pebbles $(x_1, y_1) \mapsto (f'(x_1), f'(y_1))$. As the pebbled map $(g, x_1, y_1, gs_1 g^{-1}) \mapsto (f(g), f'(x_1), f'(y_1), f'(gs_1 g^{-1}))$ does not extend to an isomorphism, Spoiler wins using 6 pebbles and 3 rounds. Thus, Spoiler used at most 7 pebbles and 7 rounds.

In the worst case (Case 3), we use at most 7 pebbles and 7 rounds. □

5. CONCLUSION

We exhibited a novel Weisfeiler–Leman algorithm that provides an algorithmic characterization of the second Ehrenfeucht–Fraïssé game in Hella’s [Hel89, Hel96] hierarchy. We also showed that this Ehrenfeucht–Fraïssé game can identify groups without Abelian normal subgroups using $O(1)$ pebbles and $O(1)$ rounds. In particular, within the first few rounds, Spoiler can force Duplicator to select an isomorphism at each subsequent round. This effectively solves the search problem in the pebble game characterization.

Our work leaves several directions for further research.

Question 5.1. Can the constant-dimensional 2-ary Weisfeiler–Leman algorithm be implemented in time $n^{o(\log n)}$?

Question 5.2. What is the (1-ary) Weisfeiler–Leman dimension of groups without Abelian normal subgroups?

Question 5.3. Show that the second Ehrenfeucht–Fraïssé game in Hella’s hierarchy can identify coprime extensions of the form $H \times N$ with both H, N Abelian (the analogue of [QST11]). More generally, an analogue of Babai–Qiao [BQ12] would be to show that when $|H|, |N|$ are coprime and N is Abelian, that Spoiler can distinguish $H \times N$ from any non-isomorphic group using a constant number of pebbles that is no more than that which is required to identify H (or the maximum of that of H and a constant independent of N, H).

Question 5.4. Let $p > 2$ be prime, and let G be a p -group with bounded genus. Show that in the second Ehrenfeucht–Fraïssé game in Hella’s hierarchy, Spoiler has a winning strategy using a constant number of pebbles. This is a descriptive complexity analogue of [BMW17, IQ19]. It would even be of interest to start with the case where G has bounded genus over a field extension K/\mathbb{F}_p of bounded degree.

In the setting of groups, Hella’s hierarchy collapses to some $q \leq 3$, since 3-ary WL can identify all ternary relational structures, including groups. It remains open to determine whether this hierarchy collapses further to either $q = 1$ or $q = 2$. Even if it does not collapse, it would also be of interest to determine whether the 1-ary and 2-ary games are equivalent. Algorithmically, this is equivalent to determining whether 1-ary and 2-ary WL have the same distinguishing power.

Question 5.5. Does there exist an infinite family of non-isomorphic pairs of groups $\{(G_n, H_n)\}$ for which Spoiler requires $\omega(1)$ pebbles to distinguish G_n from H_n ? We ask this question for the Ehrenfeucht–Fraïssé games at both the first and second levels of Hella’s hierarchy.

Recall that the game at the first level of Hella’s hierarchy is equivalent to Weisfeiler–Leman [CFI92, Hel89, Hel96], and so a lower bound against either of these games provides a lower bound against Weisfeiler–Leman, which, as of this writing, remains open in the setting of groups. More generally, it would also be of interest to investigate Hella’s hierarchy on higher arity structures. For a q -ary relational structure, the q -ary pebble game suffices to decide isomorphism. Are there interesting, natural classes of higher arity structures for which Hella’s hierarchy collapses further to some level $q' < q$?

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REFERENCES

- [AF90] Miklos Ajtai and Ronald Fagin. Reachability is harder for directed than for undirected finite graphs. *Journal of Symbolic Logic*, 55(1):113–150, 1990. doi:10.2307/2274958.
- [AF97] Sanjeev Arora and Ronald Fagin. On winning strategies in Ehrenfeucht–Fraïssé games. *Theoretical Computer Science*, 174(1):97–121, 1997. doi:10.1016/S0304-3975(96)00015-1.
- [AK06] V. Arvind and Piyush P. Kurur. Graph isomorphism is in SPP. *Information and Computation*, 204(5):835–852, 2006. doi:10.1016/j.ic.2006.02.002.
- [Bab16] László Babai. Graph isomorphism in quasipolynomial time [extended abstract]. In *STOC’16—Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, pages 684–697. ACM, New York, 2016. Preprint of full version at arXiv:1512.03547v2 [cs.DS]. doi:10.1145/2897518.2897542.
- [BB99] László Babai and Robert Beals. A polynomial-time theory of black box groups I. In *Groups St Andrews 1997 in Bath, I*, 1999.
- [BBS09] László Babai, Robert Beals, and Ákos Seress. Polynomial-time theory of matrix groups. In *Proceedings of the Forty-First Annual ACM Symposium on Theory of Computing*, STOC ’09, page 55–64. Association for Computing Machinery, 2009. doi:10.1145/1536414.1536425.
- [BCGQ11] László Babai, Paolo Codenotti, Joshua A. Grochow, and Youming Qiao. Code equivalence and group isomorphism. In *Proceedings of the Twenty-Second Annual ACM–SIAM Symposium on Discrete Algorithms (SODA11)*, pages 1395–1408, Philadelphia, PA, 2011. SIAM. doi:10.1137/1.9781611973082.107.
- [BCQ12] László Babai, Paolo Codenotti, and Youming Qiao. Polynomial-time isomorphism test for groups with no abelian normal subgroups - (extended abstract). In *International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 51–62, 2012. doi:10.1007/978-3-642-31594-7_5.
- [BE99] Hans Ulrich Besche and Bettina Eick. Construction of finite groups. *J. Symb. Comput.*, 27(4):387–404, 1999. doi:10.1006/jsco.1998.0258.
- [BEO02] Hans Ulrich Besche, Bettina Eick, and E.A. O’Brien. A millennium project: Constructing small groups. *Intern. J. Alg. and Comput*, 12:623–644, 2002. doi:10.1142/S0218196702001115.
- [BGL⁺19] Peter A. Brooksbank, Joshua A. Grochow, Yinan Li, Youming Qiao, and James B. Wilson. Incorporating Weisfeiler–Leman into algorithms for group isomorphism. arXiv:1905.02518 [cs.CC], 2019.
- [BH92] Harry Buhrman and Steven Homer. Superpolynomial circuits, almost sparse oracles and the exponential hierarchy. In R. K. Shyamasundar, editor, *Foundations of Software Technology and Theoretical Computer Science, 12th Conference, New Delhi, India, December 18-20, 1992, Proceedings*, volume 652 of *Lecture Notes in Computer Science*, pages 116–127. Springer, 1992. doi:10.1007/3-540-56287-7_99.
- [BMW17] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. A fast isomorphism test for groups whose Lie algebra has genus 2. *Journal of Algebra*, 473:545–590, 2017. doi:10.1016/j.jalgebra.2016.12.007.
- [BQ12] László Babai and Youming Qiao. Polynomial-time isomorphism test for groups with Abelian Sylow towers. In *29th STACS*, pages 453 – 464. Springer LNCS 6651, 2012. doi:10.4230/LIPIcs.STACS.2012.453.

- [Bra23] Jendrik Brachter. *Combinatorial approaches to the group isomorphism problem*. PhD thesis, TU Darmstadt, Sept. 2023. URL: https://tuprints.ulb.tu-darmstadt.de/26387/1/thesis_brachter.pdf.
- [BS20] Jendrik Brachter and Pascal Schweitzer. On the Weisfeiler–Leman dimension of finite groups. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8–11, 2020*, pages 287–300. ACM, 2020. doi:10.1145/3373718.3394786.
- [BS22] Jendrik Brachter and Pascal Schweitzer. A Systematic Study of Isomorphism Invariants of Finite Groups via the Weisfeiler–Leman Dimension. In Shiri Chechik, Gonzalo Navarro, Eva Rotenberg, and Grzegorz Herman, editors, *30th Annual European Symposium on Algorithms (ESA 2022)*, volume 244 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–27:14, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.ESA.2022.27.
- [CFI92] Jin-Yi Cai, Martin Fürer, and Neil Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica*, 12(4):389–410, 1992. Originally appeared in SFCS '89. doi:10.1007/BF01305232.
- [CH03] John J. Cannon and Derek F. Holt. Automorphism group computation and isomorphism testing in finite groups. *J. Symb. Comput.*, 35:241–267, March 2003. doi:10.1016/S0747-7171(02)00133-5.
- [CL24] Nathaniel A. Collins and Michael Levet. Count-free Weisfeiler–Leman and group isomorphism. *International Journal of Algebra and Computation*, 2024. doi:10.1142/S0218196724500103.
- [Col23] Nathaniel A. Collins. Weisfeiler–Leman and group isomorphism, 2023. Undergraduate Thesis; University of Colorado Boulder. URL: https://scholar.colorado.edu/concern/undergraduate_honors_theses/6m311q80b.
- [CRP24] Gang Chen, Qing Ren, and Ilia Ponomarenko. On multidimensional Schur rings of finite groups. *Journal of Group Theory*, 27(1):61–88, 2024. doi:10.1515/jgth-2023-0032.
- [CTW13] Arkadev Chattopadhyay, Jacobo Torán, and Fabian Wagner. Graph isomorphism is not AC^0 -reducible to group isomorphism. *ACM Trans. Comput. Theory*, 5(4):Art. 13, 13, 2013. Preliminary version appeared in FSTTCS '10; ECCC Tech. Report TR10-117. doi:10.1145/2540088.
- [DH12] Anuj Dawar and Bjarki Holm. Pebble games with algebraic rules. In Artur Czumaj, Kurt Mehlhorn, Andrew Pitts, and Roger Wattenhofer, editors, *Automata, Languages, and Programming*, pages 251–262, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg. doi:10.1007/978-3-642-31585-5_25.
- [DS19] Bireswar Das and Shivdutt Sharma. Nearly linear time isomorphism algorithms for some non-abelian group classes. In René van Bevern and Gregory Kucherov, editors, *Computer Science – Theory and Applications*, pages 80–92, Cham, 2019. Springer International Publishing. doi:10.1007/s00224-020-10010-z.
- [DV20] Anuj Dawar and Danny Vagnozzi. Generalizations of k -dimensional Weisfeiler–Leman stabilization. *Moscow Journal of Combinatorics and Number Theory*, 9:229–252, 10 2020. doi:10.2140/moscow.2020.9.229.
- [DW22] Heiko Dietrich and James B. Wilson. Polynomial-time isomorphism testing for groups of most finite orders, 2022. doi:10.1109/F0CS52979.2021.00053.
- [Ehr61] A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fund. Math.*, 49:129–141, 1960/61. doi:10.4064/fm-49-2-129-141.
- [ELGO02] Bettina Eick, C. R. Leedham-Green, and E. A. O’Brien. Constructing automorphism groups of p -groups. *Comm. Algebra*, 30(5):2271–2295, 2002. doi:10.1081/AGB-120003468.
- [FG00] Jörg Flum and Martin Grohe. On fixed-point logic with counting. *The Journal of Symbolic Logic*, 65(2):777–787, 2000. doi:10.2307/2586569.
- [Fra54] Roland Fraïssé. Sur quelques classifications des systèmes de relations. *Publ. Sci. Univ. Alger. Sér. A*, 1:35–182 (1955), 1954.
- [GK19] Martin Grohe and Sandra Kiefer. A Linear Upper Bound on the Weisfeiler–Leman Dimension of Graphs of Bounded Genus, 2019. doi:10.4230/LIPIcs.ICALP.2019.117.
- [GK21] Martin Grohe and Sandra Kiefer. Logarithmic weisfeiler-leman identifies all planar graphs. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12–16, 2021, Glasgow, Scotland*

- (*Virtual Conference*), volume 198 of *LIPICs*, pages 134:1–134:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPICs.ICALP.2021.134.
- [GL23] Joshua A. Grochow and Michael Levet. On the descriptive complexity of groups without abelian normal subgroups (extended abstract). *Electronic Proceedings in Theoretical Computer Science*, 390:185–202, September 2023. doi:10.4204/eptcs.390.12.
- [GL26] Joshua A. Grochow and Michael Levet. On the parallel complexity of group isomorphism via Weisfeiler–Leman. *Journal of Computer and System Sciences*, 156:103703, 2026. Preliminary version appeared in the proceedings of FCT 2023, DOI:10.1007/978-3-031-43587-4_17. doi:10.1016/j.jcss.2025.103703.
- [GN23] Martin Grohe and Daniel Neuen. Canonisation and definability for graphs of bounded rank width. *ACM Trans. Comput. Log.*, 24(1):6:1–6:31, 2023. doi:10.1145/3568025.
- [GO15] Martin Grohe and Martin Otto. Pebble games and linear equations. *J. Symb. Log.*, 80(3):797–844, 2015. doi:10.1017/jsl.2015.28.
- [Gom10] Walid Gomaa. Descriptive complexity of finite abelian groups. *IJAC*, 20:1087–1116, 12 2010. doi:10.1142/S0218196710006047.
- [GQ15] Joshua A. Grochow and Youming Qiao. Polynomial-time isomorphism test of groups that are tame extensions - (extended abstract). In *Algorithms and Computation - 26th International Symposium, ISAAC 2015, Nagoya, Japan, December 9-11, 2015, Proceedings*, pages 578–589, 2015. doi:10.1007/978-3-662-48971-0_49.
- [Gro17] Martin Grohe. *Descriptive complexity, canonisation, and definable graph structure theory*, volume 47 of *Lecture Notes in Logic*. Association for Symbolic Logic, Ithaca, NY; Cambridge University Press, Cambridge, 2017. doi:10.1017/9781139028868.
- [Gro21] Martin Grohe. The logic of graph neural networks. In *LICS '21: Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2021. Preprint arXiv:2104.14624 [cs.LG]. doi:10.1109/LICS52264.2021.9470677.
- [GV06] Martin Grohe and Oleg Verbitsky. Testing graph isomorphism in parallel by playing a game. In Michele Bugliesi, Bart Preneel, Vladimiro Sassone, and Ingo Wegener, editors, *Automata, Languages and Programming, 33rd International Colloquium, ICALP 2006, Venice, Italy, July 10-14, 2006, Proceedings, Part I*, volume 4051 of *Lecture Notes in Computer Science*, pages 3–14. Springer, 2006. doi:10.1007/11786986_2.
- [Hel89] Lauri Hella. Definability hierarchies of generalized quantifiers. *Annals of Pure and Applied Logic*, 43(3):235 – 271, 1989. doi:10.1016/0168-0072(89)90070-5.
- [Hel96] Lauri Hella. Logical hierarchies in PTIME. *Information and Computation*, 129(1):1–19, 1996. doi:10.1006/inco.1996.0070.
- [HL74] Hermann Heineken and Hans Liebeck. The occurrence of finite groups in the automorphism group of nilpotent groups of class 2. *Arch. Math. (Basel)*, 25:8–16, 1974. doi:10.1007/BF01238631.
- [HQ21] Xiaoyu He and Youming Qiao. On the Baer–Lovász–Tutte construction of groups from graphs: Isomorphism types and homomorphism notions. *Eur. J. Combin.*, 98:103404, 2021. doi:10.1016/j.ejc.2021.103404.
- [IL90] Neil Immerman and Eric Lander. Describing graphs: A first-order approach to graph canonization. In Alan L. Selman, editor, *Complexity Theory Retrospective: In Honor of Juris Hartmanis on the Occasion of His Sixtieth Birthday, July 5, 1988*, pages 59–81. Springer New York, New York, NY, 1990. doi:10.1007/978-1-4612-4478-3_5.
- [Imm86] Neil Immerman. Relational queries computable in polynomial time. *Inf. Control.*, 68(1-3):86–104, 1986. doi:10.1016/S0019-9958(86)80029-8.
- [IPZ01] Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001. doi:10.1006/jcss.2001.1774.
- [IQ19] Gábor Ivanyos and Youming Qiao. Algorithms based on *-algebras, and their applications to isomorphism of polynomials with one secret, group isomorphism, and polynomial identity testing. *SIAM J. Comput.*, 48(3):926–963, 2019. doi:10.1137/18M1165682.
- [Isa08] I. Isaacs. *Finite Group Theory*. American Mathematical Society, 2008.
- [Kav07] T. Kavitha. Linear time algorithms for abelian group isomorphism and related problems. *Journal of Computer and System Sciences*, 73(6):986 – 996, 2007. doi:10.1016/j.jcss.2007.03.013.

- [KM20] Sandra Kiefer and Brendan D. McKay. The iteration number of colour refinement. In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 73:1–73:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.73.
- [KN09] Neeraj Kayal and Timur Nezhmetdinov. Factoring groups efficiently. In Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris Nikolettseas, and Wolfgang Thomas, editors, *Automata, Languages and Programming*, pages 585–596, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg. doi:10.1007/978-3-642-02927-1_49.
- [KPS19] Sandra Kiefer, Ilia Ponomarenko, and Pascal Schweitzer. The Weisfeiler–Leman dimension of planar graphs is at most 3. *J. ACM*, 66(6), November 2019. doi:10.1145/3333003.
- [KSS22] Sandra Kiefer, Pascal Schweitzer, and Erkal Selman. Graphs identified by logics with counting. *ACM Trans. Comput. Log.*, 23(1):1:1–1:31, 2022. doi:10.1145/3417515.
- [KST92] Johannes Köbler, Uwe Schöning, and Jacobo Torán. Graph isomorphism is low for PP. *Comput. Complex.*, 2:301–330, 1992. doi:10.1007/BF01200427.
- [Lad75] Richard E. Ladner. On the structure of polynomial time reducibility. *J. ACM*, 22(1):155–171, January 1975. doi:10.1145/321864.321877.
- [Lev23] Michael Levet. *On the Combinatorial and Logical Complexities of Algebraic Structures*. PhD thesis, University of Colorado Boulder, 2023. URL: <https://www.proquest.com/openview/4c04a0a6191b835ac0bc7580faa4b6ca/1?pq-origsite=gscholar&cbl=18750&diss=y>.
- [LG09] François Le Gall. Efficient isomorphism testing for a class of group extensions. In *Proc. 26th STACS*, pages 625–636, 2009. doi:10.4230/LIPIcs.STACS.2009.1830.
- [LGR16] François Le Gall and David J. Rosenbaum. On the group and color isomorphism problems. arXiv:1609.08253 [cs.CC], 2016.
- [Lin66] P. Lindström. First order predicate logic with generalized quantifiers. *Theoria*, 32(3):186–195, 1966. doi:10.1111/j.1755-2567.1966.tb00600.x.
- [LW12] Mark L. Lewis and James B. Wilson. Isomorphism in expanding families of indistinguishable groups. *Groups - Complexity - Cryptology*, 4(1):73–110, 2012. doi:10.1515/gcc-2012-0008.
- [Mek81] Alan H. Mekler. Stability of nilpotent groups of class 2 and prime exponent. *The Journal of Symbolic Logic*, 46(4):781–788, 1981. doi:10.2307/2273227.
- [Mil78] Gary L. Miller. On the $n^{\log n}$ isomorphism technique (a preliminary report). In *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing*, STOC '78, pages 51–58, New York, NY, USA, 1978. Association for Computing Machinery. doi:10.1145/800133.804331.
- [Mos57] Andrzej Mostowski. On a generalization of quantifiers. *Fundamenta Mathematicae*, 44(1):12–36, 1957. URL: <http://eudml.org/doc/213418>.
- [NT17] André Nies and Katrin Tent. Describing finite groups by short first-order sentences. *Israel J. Math.*, 221(1):85–115, 2017. doi:10.1007/s11856-017-1563-2.
- [QST11] Youming Qiao, Jayalal M. N. Sarma, and Bangsheng Tang. On isomorphism testing of groups with normal Hall subgroups. In *Proc. 28th STACS*, pages 567–578, 2011. doi:10.4230/LIPIcs.STACS.2011.567.
- [Ros09] Benjamin Rossman. Ehrenfeucht–Fraïssé games on random structures. In Hiroakira Ono, Makoto Kanazawa, and Ruy J. G. B. de Queiroz, editors, *Logic, Language, Information and Computation, 16th International Workshop, WoLLIC 2009, Tokyo, Japan, June 21-24, 2009. Proceedings*, volume 5514 of *Lecture Notes in Computer Science*, pages 350–364. Springer, 2009. doi:10.1007/978-3-642-02261-6_28.
- [Ros13] David J. Rosenbaum. Bidirectional collision detection and faster deterministic isomorphism testing. arXiv:1304.3935 [cs.DS], 2013.
- [Sav80] C. Savage. An $O(n^2)$ algorithm for abelian group isomorphism. Technical report, North Carolina State University, 01 1980.
- [Sch88] Uwe Schöning. Graph isomorphism is in the low hierarchy. *Journal of Computer and System Sciences*, 37(3):312 – 323, 1988. doi:10.1016/0022-0000(88)90010-4.
- [Vag23] Danny Vagnozzi. *Variations on the theme of higher dimensional Weisfeiler–Leman algorithms*. PhD thesis, University of Cambridge, 2023. doi:10.17863/CAM.95245.
- [Var82] Moshe Y. Vardi. The complexity of relational query languages (extended abstract). In Harry R. Lewis, Barbara B. Simons, Walter A. Burkhard, and Lawrence H. Landweber, editors, *Proceedings*

- of the 14th Annual ACM Symposium on Theory of Computing, May 5-7, 1982, San Francisco, California, USA, pages 137–146. ACM, 1982. doi:10.1145/800070.802186.
- [Vik96] Narayan Vikas. An $O(n)$ algorithm for abelian p -group isomorphism and an $O(n \log n)$ algorithm for abelian group isomorphism. *Journal of Computer and System Sciences*, 53(1):1–9, 1996. doi:10.1006/jcss.1996.0045.
- [Wil12] James B. Wilson. Existence, algorithms, and asymptotics of direct product decompositions, I. *Groups - Complexity - Cryptology*, 4(1), Jan 2012. doi:10.1515/gcc-2012-0007.
- [Wil19] James B. Wilson. The threshold for subgroup profiles to agree is logarithmic. *Theory of Computing*, 15(19):1–25, 2019. doi:10.4086/toc.2019.v015a019.
- [ZKT85] V. N. Zemlyachenko, N. M. Korneenko, and R. I. Tyshkevich. Graph isomorphism problem. *J. Soviet Math.*, 29(4):1426–1481, May 1985. doi:10.1007/BF02104746.