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WEAK SIMPLICIAL BISIMILARITY AND MINIMISATION FOR POLYHEDRAL MODEL CHECKING

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Abstract. The work described in this paper builds on the polyhedral semantics of the *Spatial Logic for Closure Spaces* (SLCS) and the geometric spatial model checker PolyLogicA. Polyhedral models are central in domains that exploit mesh processing, such as 3D computer graphics. A discrete representation of polyhedral models is given by cell poset models, which are amenable to geometric spatial model checking using $\mathtt{SLCS}_\eta$, a weaker version of SLCS. In this work we show that the mapping from polyhedral models to cell poset models preserves and reflects $\mathtt{SLCS}_\eta$. We also propose weak simplicial bisimilarity on polyhedral models and weak $\pm$-bisimilarity on cell poset models, where by "weak" we mean that the relevant equivalence is coarser than the corresponding one for SLCS, leading to a greater reduction of the size of models and thus to more efficient model checking.

We show that the proposed bisimilarities enjoy the Hennessy-Milner property, i.e. two points are weakly simplicial bisimilar iff they are logically equivalent for $\mathtt{SLCS}_\eta$. Similarly, two cells are weakly $\pm$-bisimilar iff they are logically equivalent in the poset-model interpretation of $\mathtt{SLCS}_\eta$. Furthermore we present a model minimisation procedure and prove that it correctly computes the minimal model with respect to weak $\pm$-bisimilarity, i.e. with respect to logical equivalence of $\mathtt{SLCS}_\eta$. The procedure works via an encoding into LTSs and then exploits branching bisimilarity on those LTSs, exploiting the minimisation capabilities as included in the mCRL2 toolset. Various examples show the effectiveness of the approach.

Key words and phrases: Bisimulation relations; Spatial bisimilarity; Spatial logics; Logical equivalence; Spatial model checking; Polyhedral models; Model minimisation.

* This paper is an extended version of [BCG+24a].

The authors are listed in alphabetical order, as they equally contributed to the work presented in this paper.

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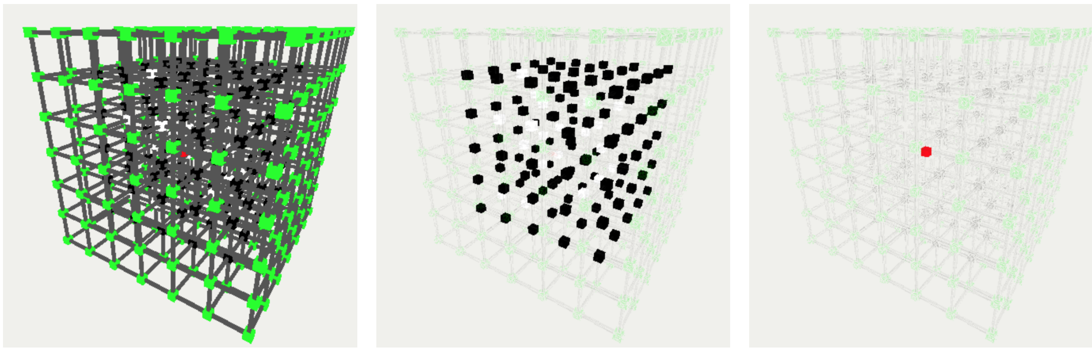


(a) (b) (c)

Figure 1. 3D maze (1a), black and white rooms (1b) and red rooms (1c) in the 3D maze (source [BCG+22]).

1. Introduction and Related Work

Spatial and spatio-temporal model checking have recently been successfully employed in a variety of application areas, including Collective Adaptive Systems [CLM+16, CGG+18, AAV24, ADT24], signal analysis [NBC+18], image analysis [CLLM16, HJK+15, BBC+20], and polyhedral modelling [BCG+22, CGL+23a, BCG+24a, BCG+24b]. Interest in these methods for spatial analysis is increasing in Computer Science and in other domains, including initially unanticipated ones, such as medical imaging [BCLM19b, BBC+21].

Spatial model checking is a global technique: it comprises the automatic verification of properties, expressed in a suitable spatial logic, such as the Spatial Logic for Closure Spaces (`SLCS`) [CLLM14, CLLM16], for each point of a spatial model. The logic `SLCS` has been defined originally for closure models, i.e. models based on Čech closure spaces [Čech66], a generalisation of topological spaces, and model checking algorithms have been developed for finite closure models also in combination with discrete time, leading to spatio-temporal model checking [CGG+18]. The spatial model checker `VoxLogicA`, proposed in [BCLM19b], is very efficient in checking properties of large images – represented as symmetric finite closure models – expressed in `SLCS` [BCLM19b, BCLM19a, BBC+21]. For example, the automatic segmentation via a suitable `SLCS` formula characterising the white matter of the brain in a 3D MRI image consisting of circa 12M voxels (i.e. $256 \times 256 \times 181$), requires approximately 10 seconds, using `VoxLogicA` on a desktop computer [BCLM19a].[1]

In [CLMV22, CLMV25] several bisimulations for finite closure spaces have been studied, with the aim to improve the efficiency of model checking via model minimisation. These notions cover a spectrum from CM-bisimilarity, an equivalence based on *proximity* — similar to and inspired by topo-bisimilarity for topological models [BB07] — to CMC-bisimilarity, CM-bisimilarity specialisation for quasi-discrete closure models, and CoPa-bisimilarity, an equivalence based on *conditional reachability*. Each of these bisimilarities has been equipped with its logical characterisation.

[1]Intel Core i9-9900K processor (with 8 cores and 16 threads) and 32GB of RAM. Note that `VoxLogicA` checks such logical specifications for *every* point in the model exploiting parallel execution, memoization, and state-of-the-art imaging libraries [BCLM19b].

The spatial model checking techniques mentioned above, targeting grid-based structures, have been extended to *polyhedral models* [BCG+22, LQ23]. Polyhedra are subsets in $\mathbb{R}^n$ generated by simplicial complexes, i.e. finite collections of simplexes satisfying certain conditions. A simplex is the convex hull of a set of affinely independent points. Given a set PL of proposition letters, a polyhedral model is obtained from a polyhedron by assigning a polyhedral subset to each proposition letter $p \in \texttt{PL}$, namely those points that "satisfy" proposition p. Polyhedral models in $\mathbb{R}^3$ can be used for (approximately) representing objects in continuous 3D space. This is typical of many 3D visual computing techniques, where an object is split into suitable geometric parts of different size. Such ways of splitting of an object are known as mesh techniques and include triangular surface meshes or tetrahedral volume meshes (see [LPZ12]). Interestingly, polyhedral models can conveniently be represented by discrete structures, the so-called *cell poset models*: each point of the polyhedron is mapped to a (unique) "cell", i.e. an element of the associated cell poset model. Cell poset models, being a particular case of Kripke models, are amenable to discrete model checking.

In [BCG+22], a variant of SLCS for polyhedral models, called $\texttt{SLCS}_\gamma$ in the sequel, as well as a geometric model checking algorithm have been proposed. The latter has been implemented in the PolyLogicA model checker, together with PolyVisualizer, a tool for visualising and inspecting polyhedral models (see [BCG+22] for details). Example 1.1 below gives an idea of the framework of spatial model checking using PolyLogicA.

Example 1.1. Figure 1a shows a "3D maze" example originating from [BCG+22]. The maze consists of "rooms" that are connected by "corridors". The rooms come in four colours: white, black, green, and red for only one room. The cells of white, black, green, red rooms satisfy (only) predicate letter **white**, **black**, **green**, **red**, respectively. Predicate letter **corridor** is satisfied by (all and only the cells of) corridors. The green rooms are all situated at the outer boundary of the maze and represent the surroundings of the maze that can be reached via an exit. The white, black, and red rooms and related corridors are situated inside the maze and form the maze itself. Figure 1b shows all the white and black rooms. Figure 1c shows the red room. The corridors between rooms are dark grey. Valid paths through the maze should only pass by white/red rooms and related corridors to reach a green room without passing by black rooms or corridors that connect to black rooms. All the images shown in Figure 1 are generated by PolyLogicA and can be visualised (and inspected by) PolyVisualizer: the result of a model checking session is presented by showing an image where the cells that satisfy the formula of interest are shown opaque, while the rest of the image is shown transparent in the background. For instance, in Figure 1b the result of model checking the simple $\texttt{SLCS}_\gamma$ formula **black** $\vee$ **white** by PolyLogicA is shown, and similarly for Figure 1c and formula **red**. ♣

$\texttt{SLCS}_\gamma$ can express spatial properties of points lying in polyhedral models, and, in particular, *conditional reachability* properties. Besides negation and conjunction, $\texttt{SLCS}_\gamma$ provides the γ reachability operator. Informally, a point x in a polyhedral model satisfies the conditional reachability formula $\gamma(\Phi_1, \Phi_2)$ if there is a topological path starting from x, ending in a point y satisfying Φ_2, and such that all the intermediate points of the path between x and y satisfy Φ_1. Note that neither x nor y is required to satisfy Φ_1. Many interesting properties, such as proximity (in the topological sense, i.e. "being in the topological closure of") or "being surrounded by" can be expressed using reachability (see [BCG+22]).

Moreover, in [BCG+22] *simplicial bisimilarity* (denoted by $\sim_\triangle$ in the sequel) has been proposed for polyhedral models, and it has been shown that it enjoys the Hennessy-Milner

Property (HMP) with respect to $\mathtt{SLCS}_\gamma$. In [CGL+23a] $\pm$-*bisimilarity* (denoted by $\sim_\pm$ in the sequel) has been proposed for cell poset models, that also enjoys the HMP for $\mathtt{SLCS}_\gamma$.

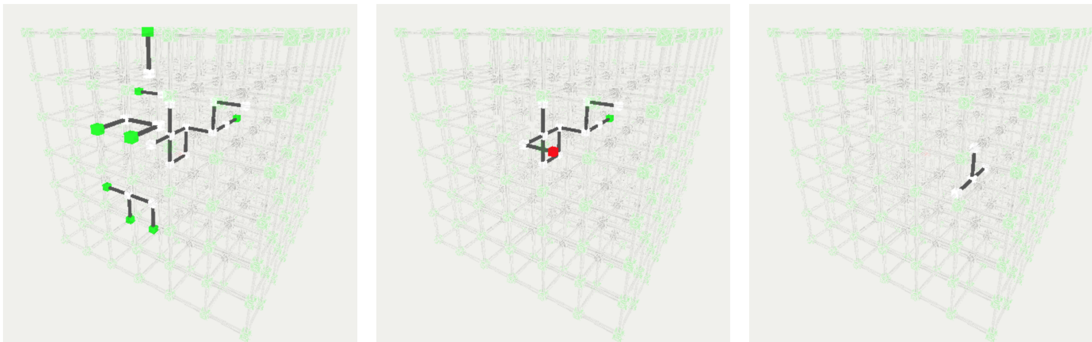
In this paper we introduce a weaker version of conditional reachability, denoted by η. A point x in a polyhedral model satisfies the conditional reachability formula $\eta(\Phi_1, \Phi_2)$ if there is a topological path starting from x, ending in a point y satisfying Φ_2, and x and all the intermediate points of the path between x and y satisfy Φ_1. Thus now x is required to satisfy Φ_1. The operator η can be expressed using γ and we will show that the logic where γ has been replaced by η — $\mathtt{SLCS}_\eta$, in the sequel — is strictly weaker than $\mathtt{SLCS}_\gamma$ in the sense that it distinguishes fewer points than $\mathtt{SLCS}_\gamma$. Furthermore, as mentioned above, $\mathtt{SLCS}_\gamma$ can express proximity — that boils down to the standard *possibility* modality $\Diamond$ in the poset model interpretation — whereas $\mathtt{SLCS}_\eta$ cannot. We show that the mapping from a polyhedral model to its cell poset model preserves and reflects $\mathtt{SLCS}_\eta$: a point satisfies a formula of $\mathtt{SLCS}_\eta$ if and only if the cell which it is mapped to satisfies the formula[2]. This result paves the way to the definition and implementation of model checking techniques for $\mathtt{SLCS}_\eta$ on polyhedral models, by working on their discrete representations.

Model reduction for cell poset models, as a means for *improving model checking efficiency* is our main concern in the present work. In particular, we are interested in techniques based on *spatial* bisimilarity. For that purpose we introduce *weak simplicial bisimilarity* on polyhedral models ($\approx_\triangle$) showing that it enjoys the HMP with respect to $\mathtt{SLCS}_\eta$ — $\approx_\triangle$ coincides with the logical equivalence $\equiv_\eta$ as induced by $\mathtt{SLCS}_\eta$ — and a notion of bisimulation equivalence for cell poset models, namely *weak* $\pm$-*bisimilarity* ($\approx_\pm$, to be read as 'weak plus-minus' bisimilarity) such that two points in the polyhedral model are weakly simplicial bisimilar if and only if their cells are weakly $\pm$-bisimilar. We show that also on cell poset models the HMP holds: $\approx_\pm$ coincides with $\equiv_\eta$.

The reason why we are interested in $\mathtt{SLCS}_\eta$ is that it characterises bisimilarities — in the polyhedral model and the associated poset model — that are coarser than simplicial bisimilarity and $\pm$-bisimilarity, respectively (thence the adjective "weak" in the names of the two bisimilarities). This allows for greater model reduction, as we will see, for instance, in Example 4.14 and Figure 7. At the same time, interesting reachability properties can be expressed in $\mathtt{SLCS}_\eta$, as shown, for instance, by the following example.

Example 1.2. Let us consider again the polyhedral model of Figure 1a. Suppose we are interested in all those white rooms from which an exit (i.e. green room) can be reached without passing by black rooms or corridors connected to black rooms. Moreover, we want to know which route — in the sense of rooms and corridors — one can follow from each such white room for reaching an exit. We start by defining some auxiliary formulas: a cell satisfies formula $\eta(\mathbf{corridor}, \mathbf{white}) \wedge \neg\eta(\mathbf{corridor}, \mathbf{green} \vee \mathbf{black} \vee \mathbf{red})$ if it belongs to a corridor and from such a cell only (cells of) white rooms — i.e. neither green, nor black, nor red — can be reached via the corridor. For the sake of readability, we name such a formula `CorWW`. Formula `CorWG`, defined as $\eta(\mathbf{corridor}, \mathbf{white}) \wedge \eta(\mathbf{corridor}, \mathbf{green})$, is satisfied by those cells of corridors between white and green rooms. Next, we define formula `WtG` that characterises the cells of white rooms, corridors between white rooms, and corridors between white and green rooms, by which one can reach a green room, i.e. without passing by black rooms or corridors connected to black rooms: $\mathtt{WtG} = \eta((\mathbf{white} \vee \mathtt{CorWW} \vee \mathtt{CorWG}), \mathbf{green})$. Keeping in mind that in the answer to our model checking query we want to see the green exits

[2]A similar feature was shown to hold for $\mathtt{SLCS}_\gamma$ in [BCG+22].



(A) (B) (C)

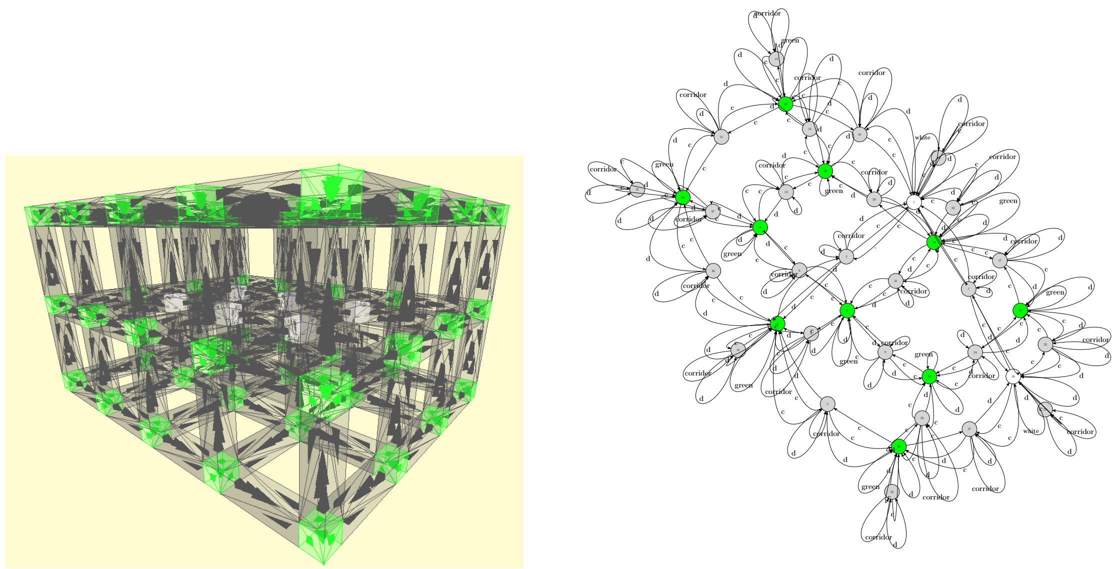
FIGURE 2. Spatial model checking results of the properties Q1 (2a), Q2 (2b) and Q3 (2c) for the 3D maze of Figure 1. (source: [BCG+22]).

as well, we define the complete query Q1 by $\mathtt{WtG} \vee \eta(\mathbf{green}, \mathtt{WtG})$. The result of PolyLogicA applied on Q1 and the "maze" is shown in Figure 2a.

Suppose now we are interested in showing the white rooms, and connecting corridors, from which both a green room and the red room can be reached, without having to pass by black rooms (and related corridors), i.e. we want to show if and how one can reach an exit from the red room. The relevant query Q2 is given by the formula $\eta((\mathtt{Q1} \vee \mathtt{CorWR}), \mathbf{red}) \vee \eta((\mathbf{red} \vee \mathtt{CorWR}), \mathtt{Q1})$ where CorWR stands for $\eta(\mathbf{corridor}, \mathbf{white}) \wedge \eta(\mathbf{corridor}, \mathbf{red})$. The result of the model checking session is shown in Figure 2b.

Finally, Figure 2c shows the white rooms, and related corridors, from which it is *not* possible to reach a green room without having to pass by a black room and is the result of model checking the formula Q3 defined as $(\mathbf{white} \vee \mathtt{CorWW}) \wedge \neg\mathtt{WtG}$. ♣

Building upon the theoretical results for $\mathtt{SLCS}_\eta$, weak simplicial bisimilarity and weak $\pm$-bisimilarity, we introduce a minimisation procedure based on weak $\pm$-bisimilarity, namely *weak $\pm$-minimisation*. The procedure uses an encoding of cell poset models into labelled transition systems (LTSs) following an approach that is similar to that presented in [CGL+23b] for finite closure models. More precisely, in the case of cell poset models, there is a one-to-one correspondence between the states of the LTS and the cells of the poset model. It is shown that two cells are weakly $\pm$-bisimilar in the poset model if and only if they — as states of the encoded LTS — are branching bisimulation equivalent. This provides an effective way for computing the equivalence classes for the set of cells, from which the minimal model is built, on which $\mathtt{SLCS}_\eta$ model checking can be safely performed. In fact, efficient LTS minimisation tools are available for branching bisimulation, such as the one provided by the mCRL2 toolset [GJKW17]. As we will see in Section 7, this can lead to a drastic reduction of the size of the spatial model, thus increasing the practical efficiency of spatial model checking. Figure 3a shows an example of a maze, composed of 6,145 cells of three colours: white, green, and grey — for corridors. This model is reduced to an LTS consisting of only 38 states, which is a reduction of two orders of magnitude. The different white, green and grey states of the minimised LTS represent the various equivalence classes of cells in the original polyhedral model. Even if this is a synthetic example, chosen on purpose for its symmetry properties, it illustrates the potential of the approach. Figure 3b only gives a first visual impression of spatial minimisation for polyhedra. We postpone the discussion of the details to Section 7.



(A) Maze (B) Minimal LTS

FIGURE 3. A maze (3a) and its respective minimal LTS (3b).

In conclusion, in the present paper, we focus on *model reduction* — as a way of improving model checking efficiency — and spatial *reachability* — rather than *proximity*. In particular, we are interested in a framework for model reduction with the following features:

(1) It should be *sound* and *complete*, i.e. be based on a notion of bisimilarity that enjoys the Hennessy-Milner Property (HMP) so that completeness and soundness of the optimised model checking procedure — via model reduction — are guaranteed.
(2) It should be *optimal* with respect to the logic of interest, in the sense of yielding the minimal model with respect to the equivalence induced by the logic of interest, but also a *useful* one. In this respect we have been inspired by the use of branching bisimilarity in the context of LTSs: branching bisimilarity — that is weaker than strong bisimilarity — enjoys the HMP with respect to CTL* without X (next) — that is weaker than full CTL* — and both the equivalence and its logical characterisation are widely used in concurrency theory and its applications. In essence, weak simplicial bisimilarity in the context of spatial logic is a re-interpretation in space of branching bisimilarity in the context of temporal logic. Similarly, $\mathtt{SLCS}_\eta$ can be seen as the spatial counterpart of CTL$^* \setminus X$.
(3) It should exploit *existing tools* for minimisation via bisimulation, since at present powerful and efficient model minimisation techniques and tools are available for branching bisimilarity minimisation.

As we mentioned above, the fact that logical equivalence $\equiv_\eta$ is *coarser* than $\equiv_\gamma$ implies that poset model minimisation modulo $\equiv_\eta$ results in models that can be smaller than those obtained modulo $\equiv_\gamma$, and this is one reason why we focus on $\mathtt{SLCS}_\eta$ in the present paper. As is to be expected, we do not have a general measure of the "gain", in terms of percentage of reduction in the number of cells of the input models, when using $\equiv_\eta$ instead of $\equiv_\gamma$, because this depends on the specific model.

Furthermore, we show that $\mathtt{SLCS}_\eta$ is of interest for reasoning about reachability, which is an essential feature in topological structures, as illustrated by the examples presented in this paper. There are also additional notions that can easily be expressed using the η modality such as "double reachability" and "being surrounded". The former are properties like "there is a path (from the point of interest) reaching — while passing only through points satisfying Φ_1 — a point satisfying Φ_2 that can (also) be reached from a point satisfying Φ_3 via a path passing through points satisfying Φ_2". By exploiting the non-directionality of topological paths, this can be expressed by the following $\mathtt{SLCS}_\eta$ formula:

$$\eta(\Phi_1, \eta(\Phi_2, \Phi_3)).$$

A formula like the above can be used for modelling an emergency egress situation — e.g. in a building modelled as a polyhedral model — in which, for instance, Φ_1 characterises points in a building (such as the one schematised by the polyhedral model shown in Figure 1) that are accessible to somebody to be rescued in that building (including the place where the person is located), but are not accessible to a rescue team; Φ_3 characterises the place where the rescue team is located while Φ_2 characterises points that are accessible to the rescue team (here we assume that Φ_3 implies Φ_2 — if not, just replace Φ_3 with $\Phi_3 \wedge \Phi_2$). The team and those to be rescued can thus meet in a point satisfying the nested η-formula $\eta(\Phi_2, \Phi_3)$.

The notion of "being surrounded" can be expressed using the η modality as described below. We say that starting from a point x that satisfies Φ_1 one cannot "escape" from Φ_1 without "passing through" Φ_2 — i.e. is "surrounded" by Φ_2 — if any path starting from x and reaching a point that does not satisfy Φ_1 must first pass through Φ_2. More precisely, x must satisfy Φ_1 and there is no path from x leading to a point satisfying neither Φ_1 nor Φ_2 without first passing through a point satisfying Φ_2. In $\mathtt{SLCS}_\eta$ this is captured by the following formula:

$$\Phi_1 \wedge \neg\eta(\neg\Phi_2, \neg(\Phi_1 \vee \Phi_2)).$$

Note that if x itself satisfies Φ_2, then starting from x one cannot escape from Φ_1 without passing through Φ_2.[3]

Below, we summarise the main contributions of this paper:

- presentation of $\mathtt{SLCS}_\eta$, a spatial logic for polyhedral models which is weaker than $\mathtt{SLCS}_\gamma$;
- introduction of *weak simplicial bisimilarity* on polyhedral models ($\approx_\triangle$) and showing that it enjoys the HMP with respect to $\mathtt{SLCS}_\eta$;
- introduction of *weak* $\pm$*-bisimilarity* on cell poset models ($\approx_\pm$) with the corresponding HMP result;
- introduction of a novel cell poset model minimisation procedure based on weak $\pm$-bisimilarity — and exploiting an encoding to LTSs and branching bisimilarity — including the formal proof of its correctness;
- proof-of-concept of the practical potential and effectiveness of this approach through a prototype toolchain and spatial model checking examples. It is shown that the cell poset models can be drastically reduced by several orders of magnitude.

[3]As we will see in Section 3, the spatial properties discussed above can be expressed also in $\mathtt{SLCS}_\gamma$ (see Lemma 3.5).

The first three items above have been presented originally in [BCG+24a] where only some of the proofs of the relevant results where shown: in the present paper, all proofs are presented in detail. The last two items above are original contributions.

The paper is structured as follows. We provide a summary of necessary background information in Section 2. Section 3 introduces $\texttt{SLCS}_\eta$ and addresses its relationship with $\texttt{SLCS}_\gamma$. It is also shown that $\texttt{SLCS}_\eta$ is preserved and reflected by the mapping $\mathbb{F}$ from polyhedral models to finite cell poset models. Weak simplicial bisimilarity and weak $\pm$-bisimilarity are defined in Section 4 where it is also shown that they enjoy the HMP with respect to the interpretation of $\texttt{SLCS}_\eta$ on polyhedral models and on finite poset models, respectively. The minimisation procedure, based on weak $\pm$-bisimilarity and exploiting its relationship with branching bisimulation equivalence, is defined in Section 5 where its correctness is also addressed. The procedure is currently implemented by means of an experimental toolchain using mCRL2 and is introduced in Section 6. Examples of use of the toolchain are presented in Section 7. Conclusions and a discussion on future work are reported in Section 8.

Finally, in Appendix A detailed proofs are provided and, in Appendix B, an additional minimisation example is shown.

2. Background and Notation

In this section we introduce notation and recall necessary background information, the relevant details of the language $\texttt{SLCS}_\gamma$, its polyhedral and poset models, the truth-preserving map $\mathbb{F}$ between these models, simplicial bisimilarity and $\pm$-bisimilarity.

For sets X and Y, a function $f : X \to Y$, and subsets $A \subseteq X$ and $B \subseteq Y$ we define $f(A)$ and $f^{-1}(B)$ as $\{f(a) \,|\, a \in A\}$ and $\{a \,|\, f(a) \in B\}$, respectively. The *restriction* of f on A is denoted by $f|A$. The powerset of X is denoted by $\mathbf{2}^X$. For a binary relation $R \subseteq X \times X$ we let $R^- = \{(y,x) \,|\, (x,y) \in R\}$ denote its converse and let $R^\pm$ denote $R \cup R^-$. For partial orders $\preceq$ we will use the standard notation $\succeq$ for $\preceq^-$ and $x \prec y$ whenever $x \preceq y$ and $x \neq y$ (and similarly for $x \succ y$). If R is an equivalence relation on A, we let A/R denote the *quotient* of A via R. In the remainder of the paper we assume that a set PL of *proposition letters* is fixed. The sets of natural numbers and of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We use the standard interval notation: for $x, y \in \mathbb{R}$ we let $[x, y]$ be the set $\{r \in \mathbb{R} \,|\, x \leq r \leq y\}$, $[x, y) = \{r \in \mathbb{R} \,|\, x \leq r < y\}$, and so on. Intervals of $\mathbb{R}$ are equipped with the Euclidean topology inherited from $\mathbb{R}$. We use a similar notation for intervals over $\mathbb{N}$: for $n, m \in \mathbb{N}$, $[m; n]$ denotes the set $\{i \in \mathbb{N} \,|\, m \leq i \leq n\}$, $[m; n) = \{i \in \mathbb{N} \,|\, m \leq i < n\}$, and so on. Finally, for topological space (X, τ) and $A \subseteq X$ we let $\mathcal{C}_T(A)$ denote the topological closure of A.

Below we recall some basic notions, assuming that the reader is familiar with topological spaces, Kripke models, and posets.

2.1. Polyhedral Models and Cell Poset Models.

A *simplex* σ of dimension d is the convex hull of a set $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ of $d+1$ affinely independent points in $\mathbb{R}^m$, with $d \leq m$, i.e. $\sigma = \{\lambda_0\mathbf{v_0} + \ldots + \lambda_d\mathbf{v_d} \,|\, \lambda_0, \ldots, \lambda_d \in [0,1] \text{ and } \sum_{i=0}^d \lambda_i = 1\}$. For instance, a segment AB together with its end-points A and B is a simplex in $\mathbb{R}^m$, for $m \geq 1$. Any subset of the set $\{\mathbf{v_0}, \ldots, \mathbf{v_d}\}$ of points characterising a simplex σ induces a simplex σ' in turn, and we write $\sigma' \sqsubseteq \sigma$, noting that $\sqsubseteq$ is a partial order, e.g. $A \sqsubseteq A \sqsubseteq AB$, $B \sqsubseteq B \sqsubseteq AB$ and $AB \sqsubseteq AB$. The *barycentre* b_σ of σ is defined as follows: $b_\sigma = \sum_{i=0}^d \frac{1}{d+1}\mathbf{v_i}$.

The *relative interior* $\widetilde{\sigma}$ of a simplex σ is the same as σ "without its borders", i.e. the set $\{\lambda_0\mathbf{v_0} + \ldots + \lambda_d\mathbf{v_d} \mid \lambda_0, \ldots, \lambda_d \in (0,1] \text{ and } \sum_{i=0}^{d} \lambda_i = 1\}$. For instance, the open segment $\widetilde{AB}$, without the end-points A and B is the relative interior of segment AB. The relative interior of a simplex is often called a *cell* and is equal to the topological interior taken inside the affine hull of the simplex.[4] A partial order is defined on cells: we say that $\widetilde{\sigma_1} \preccurlyeq \widetilde{\sigma_2}$ if and only if $\widetilde{\sigma_1} \subseteq \mathcal{C}_T(\widetilde{\sigma_2})$ where, we recall, $\mathcal{C}_T$ denotes the topological closure operator. It is easy to see that $\preccurlyeq$ is indeed a partial order. Note furthermore that $\sqsubseteq$ and $\preccurlyeq$ are compatible, in the sense that $\widetilde{\sigma_1} \preccurlyeq \widetilde{\sigma_2}$ if and only if $\sigma_1 \sqsubseteq \sigma_2$. In the above example, we have $\widetilde{A} \preccurlyeq \widetilde{A} \preccurlyeq \widetilde{AB}, \widetilde{B} \preccurlyeq \widetilde{B} \preccurlyeq \widetilde{AB}$, and $\widetilde{AB} \preccurlyeq \widetilde{AB}$.

A *simplicial complex* K is a finite collection of simplexes of $\mathbb{R}^m$ such that: (i) if $\sigma \in K$ and $\sigma' \sqsubseteq \sigma$ then also $\sigma' \in K$; (ii) if $\sigma, \sigma' \in K$ and $\sigma \cap \sigma' \neq \emptyset$, then $\sigma \cap \sigma' \sqsubseteq \sigma$ and $\sigma \cap \sigma' \sqsubseteq \sigma'$. The *cell poset* of simplicial complex K is $(\widetilde{K}, \preccurlyeq)$ where $\widetilde{K}$ is the set $\{\, \widetilde{\sigma} \mid \sigma \in K\}$, and $\preccurlyeq$ is the union of the partial orders on the cells of the simplexes of K.

The polyhedron $|K|$ of K is the set-theoretic union of the simplexes in K. Note that $|K|$ inherits the topology of $\mathbb{R}^m$ and that $\widetilde{K}$ forms a partition of polyhedron $|K|$. Note furthermore that different simplicial complexes can give rise to the same polyhedron.

A *polyhedral model* is a pair $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$ where $P = |K|$ for some simplicial complex K and $\mathcal{V}_\mathcal{P} : \mathtt{PL} \to \mathbf{2}^P$ maps every proposition letter $p \in \mathtt{PL}$ to the set of points of P satisfying p. It is required that, for all $p \in \mathtt{PL}$, $\mathcal{V}_\mathcal{P}(p)$ is always a union of cells in $\widetilde{K}$. A poset model is a triple $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ where $(W, \preccurlyeq)$ is a poset that is equipped with a valuation function $\mathcal{V}_\mathcal{F} : \mathtt{PL} \to \mathbf{2}^W$. Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$ with $P = |K|$, for some simplicial complex K, we say that $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ is the *cell poset model* of $\mathcal{P}$ relative to K if and only if $W = \widetilde{K}$, $(\widetilde{K}, \preccurlyeq)$ is the cell poset of K, and, for all $\widetilde{\sigma} \in \widetilde{K}$, we have: $\widetilde{\sigma} \in \mathcal{V}_\mathcal{F}(p)$ if and only if $\widetilde{\sigma} \subseteq \mathcal{V}_\mathcal{P}(p)$. We will omit to specify "relative to K" if this is clear from the context. For all $x \in P$, we let $\mathbb{F}(x)$ denote the unique cell $\widetilde{\sigma} \in \widetilde{K}$ such that $x \in \widetilde{\sigma}$. Note that $\mathbb{F}(x)$ is well defined, since $\widetilde{K}$ is a partition of $|K|$, and that $\mathbb{F} : P \to \widetilde{K}$ is a continuous function [BMMP18, Corollary 3.4]. With slight overloading, we let $\mathbb{F}(\mathcal{P})$ denote the cell poset model of $\mathcal{P}$. In the following, when we say that $\mathcal{F}$ is a cell poset model, we mean that there exist a simplicial complex K and a polyhedral model $\mathcal{P} = (|K|, \mathcal{V}_\mathcal{P})$ such that $\mathcal{F} = \mathbb{F}(\mathcal{P})$. Finally, note that poset models are a subclass of Kripke models.

Figure 4 shows a polyhedral model. There are three proposition letters, **red**, **green**, and **grey**, shown by different colours (4a). The model is "unpacked" into its cells in Figure 4b. The latter are collected in the cell poset model, whose Hasse diagram is shown in Figure 4c.

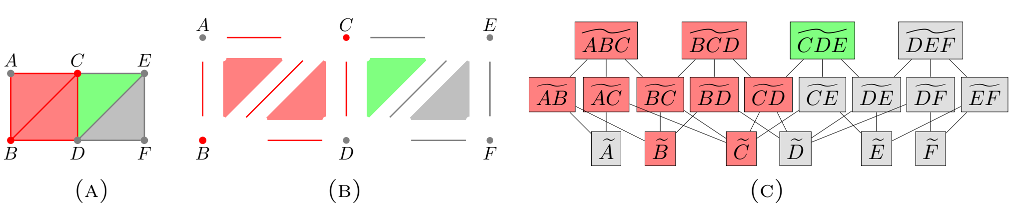
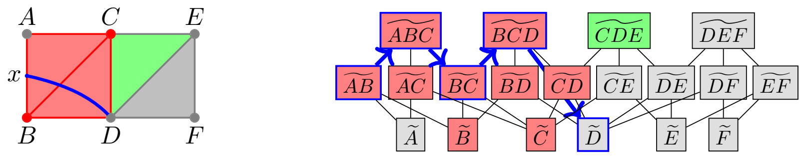


FIGURE 4. A polyhedral model $\mathcal{P}_4$ (4a) with its cells (4b) and the Hasse diagram of the related cell poset (4c).

[4]But note that the relative interior of a simplex composed of just a single point is the point itself and not the empty set.



(A) (B)

FIGURE 5. (5a) A topological path π from a point x to vertex D in the polyhedral model $\mathcal{P}_4$ of Figure 4a. (5b) The corresponding $\pm$-path $(\widetilde{A}, \widetilde{ABC}, \widetilde{BC}, \widetilde{BCD}, \widetilde{D})$, in blue, in the Hasse diagram of the cell poset model $\mathbb{F}(\mathcal{P})$. Note that the $\pm$-path does not pass through $\widetilde{CD}$ but it goes directly from $\widetilde{BCD}$ to $\widetilde{D}$. This reflects the fact that, for small $\epsilon > 0$ we have $\pi(1-\epsilon) \in \widetilde{BCD}$ while $\pi(1) = D$ and $\pi([0,1]) \cap \widetilde{CD} = \emptyset$.

2.2. **Paths.** In a topological space (X, τ), a *topological path* from $x \in X$ is a total, continuous function $\pi : [0,1] \to X$ such that $\pi(0) = x$. We call $\pi(0)$ and $\pi(1)$ the *starting point* and *ending point* of π, respectively, while $\pi(r)$ is an *intermediate point* of π, for all $r \in (0,1)$. Figure 5a shows a path from a point x in the open segment $\widetilde{AB}$ to point D in the polyhedral model of Figure 4a.

Topological paths relevant for our work are represented in cell posets by so-called $\pm$-paths, a subclass of undirected paths [BCG+22]. For technical reasons[5] in this paper we extend the definition given in [BCG+22] to general Kripke frames.

Given a Kripke frame (W, R), an *undirected path* of length $\ell \in \mathbb{N}$ from w is a total function $\pi : [0;\ell] \to W$ such that $\pi(0) = w$ and, for all $i \in [0;\ell)$, $R^{\pm}(\pi(i), \pi(i+1))$. The *starting point* and *ending point* are $\pi(0)$ and $\pi(\ell)$, respectively, while $\pi(i)$ is an intermediate point, for all $i \in (0;\ell)$. For an undirected path π of length ℓ we often use the sequence notation $(w_i)_{i=0}^{\ell}$ where $w_i = \pi(i)$ for $i \in [0;\ell]$.

Given paths $\pi' = (w'_i)_{i=0}^{\ell'}$ and $\pi'' = (w''_i)_{i=0}^{\ell''}$, with $w'_{\ell'} = w''_0$, the *sequentialisation* $\pi' \cdot \pi'' : [0;\ell' + \ell''] \to W$ of π' with π'' is the path from w'_0 defined as follows:

$$(\pi' \cdot \pi'')(i) = \begin{cases} \pi'(i), \text{ if } i \in [0;\ell'], \\ \pi''(i - \ell'), \text{ if } i \in [\ell';\ell' + \ell'']. \end{cases}$$

For a path $\pi = (w_i)_{i=0}^{\ell}$ and $k \in [0;\ell]$ we define the k-shift of π, denoted by $\pi{\uparrow}k$, as follows: $\pi{\uparrow}k = (w_{j+k})_{j=0}^{\ell-k}$ and, for $0 < m \leq \ell$, we let $\pi{\leftarrow}m$ denote the path obtained from π by inserting a copy of $\pi(m)$ immediately before $\pi(m)$ itself. In other words, we have: $\pi{\leftarrow}m = (\pi|[0;m]) \cdot ((\pi(m), \pi(m)) \cdot (\pi{\uparrow}m))$. Finally, any path $\pi|[0;k]$, for some $k \in [0;\ell]$, is a *(non-empty) prefix* of π.

An undirected path $\pi : [0;\ell] \to W$ is a $\pm$*-path* if and only if $\ell \geq 2$, $R(\pi(0), \pi(1))$ and $R^{-}(\pi(\ell-1), \pi(\ell))$.

Example 2.1. The $\pm$-path $(\widetilde{AB}, \widetilde{ABC}, \widetilde{BC}, \widetilde{BCD}, \widetilde{D})$, drawn in blue in Figure 5b, passes through the same cells, and in the same order, as the topological path from x in the polyhedral model $\mathcal{P}_4$ of Figure 4 shown in Figure 5a (source [CGL+23a]). ♣

[5] We are interested in model checking structures resulting from the minimisation, via bisimilarity, of cell poset models, and such structures are often just (reflexive) Kripke models rather than poset models.

Note that a topological path could, in principle, pass through some cells infinitely often. Such paths are not relevant for our theory since they play no role in the semantics of the logic and have no impact on weak simplicial bisimilarity, neither on the proofs of related results and, consequently, we are not interested in representing them. We will come back to this issue in Section 4.

In the context of this paper it is often convenient to use a generalisation of $\pm$-paths, so-called "down paths", $\downarrow$-paths for short: a $\downarrow$-path from w, of length $\ell \geq 1$, is an undirected path π from w of length ℓ such that $R^{-}(\pi(\ell-1), \pi(\ell))$. Finally, it is also convenient to use a subclass of $\pm$-paths, namely $\Updownarrow$-paths (to be read "up-down paths"): an $\Updownarrow$-*path* from w, of length 2ℓ, for $\ell \geq 1$, is a $\pm$-path π of length 2ℓ such that $R(\pi(2i), \pi(2i+1))$ and $R^{-}(\pi(2i+1), \pi(2i+2))$, for all $i \in [0;\ell)$.

Clearly, every $\Updownarrow$-path is also a $\pm$-path and every $\pm$-path is also a $\downarrow$-path. The following lemmas ensure that in *reflexive* Kripke frames $\Updownarrow$-, $\pm$-, and $\downarrow$-paths can be safely used interchangeably since for every $\pm$-path there is an $\Updownarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 2.2 below, proven in Appendix A.1). Furthermore, for every $\downarrow$-path there is a $\Updownarrow$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 2.3 below, proven in Appendix A.2). Finally, for every $\downarrow$-path there is a $\pm$-path with the same starting and ending points and with the same set of intermediate points, occurring in the same order (Lemma 2.4 below, proven in Appendix A.3).

Lemma 2.2. *Given a reflexive Kripke frame (W, R) and a $\pm$-path $\pi : [0;\ell] \to W$, there is a $\Updownarrow$-path $\pi' : [0;\ell'] \to W$, for some ℓ', and a total, surjective, monotonic non-decreasing function $f : [0;\ell'] \to [0;\ell]$ such that $\pi'(j) = \pi(f(j))$ for all $j \in [0;\ell']$.* □

Lemma 2.3. *Given a reflexive Kripke frame (W, R) and a $\downarrow$-path $\pi : [0;\ell] \to W$, there is a $\Updownarrow$-path $\pi' : [0;\ell''] \to W$, for some ℓ', and a total, surjective, monotonic non-decreasing function $f : [0;\ell'] \to [0;\ell]$ such that $\pi'(j) = \pi(f(j))$ for all $j \in [0;\ell']$.* □

Lemma 2.4. *Given a reflexive Kripke frame (W, R) and a $\downarrow$-path $\pi : [0;\ell] \to W$, there is a $\pm$-path $\pi' : [0;\ell''] \to W$, for some ℓ', and a total, surjective, monotonic, non-decreasing function $f : [0;\ell'] \to [0;\ell]$ with $\pi'(j) = \pi(f(j))$ for all $j \in [0;\ell']$.* □

2.3. The Logic $\mathtt{SLCS}_\gamma$ and Related Bisimilarities.

In [BCG+22], $\mathtt{SLCS}_\gamma$, a version of $\mathtt{SLCS}$ for polyhedral models, has been presented that consists of predicate letters, negation, conjunction, and the single modal operator γ, expressing conditional reachability. The satisfaction relation for $\gamma(\Phi_1, \Phi_2)$, for a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, and $x \in P$, as defined in [BCG+22], is recalled below:

$$\begin{array}{lcl} \mathcal{P}, x \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a topological path } \pi : [0,1] \to |K| \text{ exists such that } \pi(0) = x, \\ & & \mathcal{P}, \pi(1) \models \Phi_2, \text{and } \mathcal{P}, \pi(r) \models \Phi_1 \text{ for all } r \in (0,1). \end{array}$$

We also recall the interpretation of $\mathtt{SLCS}_\gamma$ on poset models. The satisfaction relation for $\gamma(\Phi_1, \Phi_2)$, for a poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ and $w \in W$, is as follows:

$$\begin{array}{lcl} \mathcal{F}, w \models \gamma(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a } \pm\text{-path } \pi : [0;\ell] \to W \text{ exists such that } \pi(0) = w, \\ & & \mathcal{F}, \pi(\ell) \models \Phi_2, \text{and } \mathcal{F}, \pi(i) \models \Phi_1 \text{ for all } i \in (0;\ell). \end{array}$$

In [BCG+22] it has also been shown that, for all $x \in P$ and $\mathtt{SLCS}_\gamma$ formulas Φ, we have: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$. In addition, *simplicial bisimilarity*, a novel notion of bisimilarity for polyhedral models, has been defined. It is based on the notion of *simplicial path*: given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$, with $P = |K|$ for some simplicial complex K, a topological path π in P is *simplicial* if and only if there is a finite sequence $r_0 = 0 < \ldots < r_k = 1$ of values in [0,1] and cells $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_k \in \widetilde{K}$ such that, for all $i \in [1; k]$ we have that $\pi((r_{i-1}, r_i)) \subseteq \widetilde{\sigma}_i$.[6]

Definition 2.5. Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$, with $P = |K|$ for some simplicial complex K, a symmetric binary relation $Z \subseteq P \times P$ is a *simplicial bisimulation* if, for all $x_1, x_2 \in P$, whenever $Z(x_1, x_2)$ holds, we have that:

(1) $\mathcal{V}_{\mathcal{P}}^{-1}(x_1) = \mathcal{V}_{\mathcal{P}}^{-1}(x_2)$ and
(2) for each simplicial path π_1 from x_1 there is a simplicial path π_2 from x_2, such that $Z(\pi_1(r), \pi_2(r))$ for all $r \in [0, 1]$.

Two points $x_1, x_2 \in P$ are *simplicial bisimilar*, written $x_1 \sim_{\triangle}^{\mathcal{P}} x_2$, if there exists a simplicial bisimulation Z such that $Z(x_1, x_2)$. •

It has been shown that simplicial bisimilarity enjoys the classical Hennessy-Milner property: two points $x_1, x_2 \in P$ are simplicial bisimilar if and only if they satisfy the same $\texttt{SLCS}_\gamma$ formulas, i.e. they are equivalent with respect to the logic $\texttt{SLCS}_\gamma$, written $x_1 \equiv_\gamma^{\mathcal{P}} x_2$.

The result has been extended to $\pm$-*bisimilarity* on finite poset models, a notion of bisimilarity based on $\pm$-paths: given finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$, $w_1, w_2 \in W$ are $\pm$-bisimilar, written $x_1 \sim_{\pm}^{\mathcal{F}} x_2$, if and only if they satisfy the same $\texttt{SLCS}_\gamma$ formulas, i.e. $x_1 \equiv_\gamma^{\mathcal{F}} x_2$ (see [CGL+23a] for details). In summary, we have:

$$x_1 \sim_{\triangle}^{\mathcal{P}} x_2 \text{ iff } x_1 \equiv_\gamma^{\mathcal{P}} x_2 \text{ iff } \mathbb{F}(x_1) \equiv_\gamma^{\mathbb{F}(\mathcal{P})} \mathbb{F}(x_2) \text{ iff } \mathbb{F}(x_1) \sim_{\pm}^{\mathbb{F}(\mathcal{P})} \mathbb{F}(x_2).$$

In Section 4 we show a similar result for a *weaker* logic introduced in the next section, and originally presented in [BCG+24a]. Finally, in [BCG+22] it has been shown that the classical modality $\diamond$ can be expressed using γ. We recall that for polyhedral model $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$ and for poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$, the semantics of $\diamond\Phi$ is defined as follows:

$$\mathcal{P}, x \models \diamond\, \Phi \quad \Leftrightarrow \quad x \in \mathcal{C}_T(\{\, x' \in P \,|\, \mathcal{P}, x' \models \Phi \,\})$$

$$\mathcal{F}, w \models \diamond\Phi \quad \Leftrightarrow \quad w' \in W \text{ exists such that } w \preccurlyeq w' \text{ and } \mathcal{F}, w' \models \Phi.$$

It turns out that $\diamond\Phi$ is equivalent to $\gamma(\Phi, \texttt{true})$, for all $\texttt{SLCS}_\gamma$ formulas Φ.

We close this section with a small example.

Example 2.6. With reference to Figure 4a, we have that no red point, call it y, in the open segment CD is simplicial bisimilar to the red point C. In fact, although both y and C satisfy $\gamma(\mathbf{green}, \texttt{true})$, we have that C satisfies also $\gamma(\mathbf{grey}, \texttt{true})$, which is not the case for y. Similarly, with reference to Figure 4c, cell $\widetilde{C}$ satisfies $\gamma(\mathbf{grey}, \texttt{true})$, which is not satisfied by $\widetilde{CD}$. ♣

[6]Essentially, simplicial paths have been introduced for avoiding to have to deal with "bad" paths, e.g. paths that can oscillate infinitely often between a set of cells.

2.4. Labelled Transition Systems and Related Bisimilarities.

Definition 2.7. A *labelled transition system*, LTS for short, is a tuple $(S, L, \longrightarrow)$ where S is a non-empty set of *states*, L is a non-empty set of *transition labels* and $\longrightarrow \subseteq S \times L \times S$ is the transition relation. •

For $\tau \in L$ denoting the "silent" action we let $t \xrightarrow{\tau^*} t'$ whenever $t = t'$ or there are $t_0, \ldots, t_n$, for $n > 0$ such that $t_0 = t$, $t_n = t'$ and $t_i \xrightarrow{\tau} t_{i+1}$ for $i \in [0; n)$.

Definition 2.8 (Strong Bisimulation and Strong Equivalence)**.** Given an LTS $\mathbb{S} = (S, L, \longrightarrow)$ a binary relation $B \subseteq S \times S$ is a *strong bisimulation* if, for all $s_1, s_2 \in S$, if $B(s_1, s_2)$ then the following holds:

(1) if $s_1 \xrightarrow{\lambda} s_1'$ for some λ and s_1, then s_2' exists such that $s_2 \xrightarrow{\lambda} s_2'$ and $B(s_1', s_2')$, and
(2) if $s_2 \xrightarrow{\lambda} s_2'$ for some λ and s_2, then s_1' exists such that $s_1 \xrightarrow{\lambda} s_1'$ and $B(s_1', s_2')$.

We say that s_1 and s_2 are *strongly equivalent* in $\mathbb{S}$, written $s_1 \sim^{\mathbb{S}} s_2$ if a strong bisimulation B exists such that $B(s_1, s_2)$. •

It has been shown that $\sim^{\mathbb{S}}$ is the union of all strong bisimulations in $\mathbb{S}$, it is the largest strong bisimulation and it is an equivalence relation [Mil89].

Definition 2.9 (Branching Bisimulation and Equivalence)**.** Given an LTS $\mathbb{S} = (S, L, \longrightarrow)$ such that $\tau \in L$ a binary relation $B \subseteq S \times S$ is a *branching bisimulation* iff, for all $s, t, s' \in S$, and $\lambda \in L$, whenever $B(s, t)$ and $s \xrightarrow{\lambda} s'$, it holds that: (i) $B(s', t)$ and $\lambda = \tau$, or (ii) $B(s, \bar{t}), B(s', t')$ and $t \xrightarrow{\tau^*} \bar{t}$, $\bar{t} \xrightarrow{\lambda} t'$, for some $\bar{t}, t' \in S$.

Two states $s, t \in S$ are called *branching bisimilar* in $\mathbb{S}$, written $s \leftrightarroweq_b^{\mathbb{S}} t$ if $B(s, t)$ for some branching bisimulation B for S. •

It has been shown that $\leftrightarroweq_b^{\mathbb{S}}$ is the union of all branching bisimulations in $\mathbb{S}$, it is the largest branching bisimulation and it is an equivalence relation [GW96].

We will omit the superscript $\mathbb{S}$ in $\sim^{\mathbb{S}}$ and $\leftrightarroweq_b^{\mathbb{S}}$ when this will not cause confusion.

3. Weak SLCS on Polyhedral Models

In this section we introduce $\texttt{SLCS}_\eta$, a logic for polyhedral models that is weaker than $\texttt{SLCS}_\gamma$, yet is still capable of expressing interesting conditional reachability properties. We present also an interpretation of the logic on finite poset models.

Definition 3.1 (Weak SLCS on polyhedral models - $\texttt{SLCS}_\eta$)**.** The abstract language of $\texttt{SLCS}_\eta$ is the following:

$$\Phi ::= p \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2 \mid \eta(\Phi_1, \Phi_2).$$

The satisfaction relation of $\texttt{SLCS}_\eta$ with respect to a given polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, $\texttt{SLCS}_\eta$ formula Φ, and point $x \in P$ is defined recursively on the structure of Φ as follows:

$$\begin{array}{lcl}
\mathcal{P}, x \models p & \Leftrightarrow & x \in \mathcal{V}_\mathcal{P}(p); \\
\mathcal{P}, x \models \neg\Phi & \Leftrightarrow & \mathcal{P}, x \models \Phi \text{ does not hold}; \\
\mathcal{P}, x \models \Phi_1 \wedge \Phi_2 & \Leftrightarrow & \mathcal{P}, x \models \Phi_1 \text{ and } \mathcal{P}, x \models \Phi_2; \\
\mathcal{P}, x \models \eta(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a topological path } \pi : [0,1] \to P \text{ exists such that} \\
 & & \pi(0) = x,\ \mathcal{P}, \pi(1) \models \Phi_2, \text{ and } \mathcal{P}, \pi(r) \models \Phi_1 \text{ for all } r \in [0,1).
\end{array}$$

•

Remark 3.2. It is worth pointing out that the definition of the satisfaction relation of $\mathtt{SLCS}_\eta$ does *not* depend on the specific simplicial complex K that generates the polyhedron $P = |K|$. In other words: given polyhedral models $\mathcal{P}' = (P, \mathcal{V}_{\mathcal{P}'})$ with $P = |K'|$ and $\mathcal{P}'' = (P, \mathcal{V}_{\mathcal{P}''})$ with $P = |K''| = |K'|$ and $\mathcal{V}_{\mathcal{P}'} = \mathcal{V}_{\mathcal{P}''}$, for all $\mathtt{SLCS}_\eta$ formulas Φ and $x \in P$ the following holds: $\mathcal{P}', x \models \Phi$ iff $\mathcal{P}'', x \models \Phi$. ⁂

As usual, disjunction ($\vee$) is derived as the dual of $\wedge$. Note that the only difference between $\eta(\Phi_1, \Phi_2)$ and $\gamma(\Phi_1, \Phi_2)$ is that the former requires that *also the first element* of a path witnessing the formula satisfies Φ_1, hence the use of the left closed interval $[0, 1)$ here. Although this might seem at first sight only a very minor difference, it has considerable consequences: η cannot express $\Diamond$, which, instead, can be expressed in terms of γ (see Remark 3.8 and Remark 3.18 below).

Definition 3.3 ($\mathtt{SLCS}_\eta$ Logical Equivalence)**.** Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$, with $P = |K|$ for some simplicial complex K, and $x_1, x_2 \in P$, we say that x_1 and x_2 are *logically equivalent* with respect to $\mathtt{SLCS}_\eta$, written $x_1 \equiv_\eta^{\mathcal{P}} x_2$, if and only if, for all $\mathtt{SLCS}_\eta$ formulas Φ, it holds that $\mathcal{P}, x_1 \models \Phi$ if and only if $\mathcal{P}, x_2 \models \Phi$. •

In the following, we will refrain from indicating the model $\mathcal{P}$ explicitly as a superscript of $\equiv_\eta^{\mathcal{P}}$ when it is clear from the context. Below, we show that $\mathtt{SLCS}_\eta$ can be encoded into $\mathtt{SLCS}_\gamma$ so that the latter is at least as expressive as the former.

Definition 3.4. We define the encoding $\mathcal{E}$ of $\mathtt{SLCS}_\eta$ into $\mathtt{SLCS}_\gamma$ as follows:

$$\begin{array}{lllllll} \mathcal{E}(p) & = & p & \qquad & \mathcal{E}(\Phi_1 \wedge \Phi_2) & = & \mathcal{E}(\Phi_1) \wedge \mathcal{E}(\Phi_2) \\ \mathcal{E}(\neg\Phi) & = & \neg\mathcal{E}(\Phi) & \qquad & \mathcal{E}(\eta(\Phi_1, \Phi_2)) & = & \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2)) \end{array}$$
•

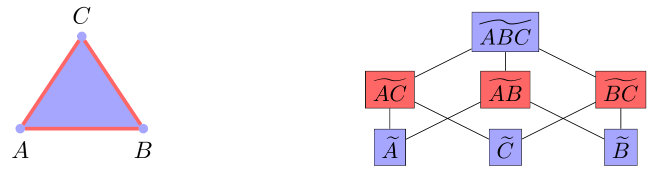
The following lemma is easily proven by structural induction on Φ (see Appendix A.4).

Lemma 3.5. *Let $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$, with $P = |K|$ for some simplicial complex K, be a polyhedral model, $x \in P$, and Φ a $\mathtt{SLCS}_\eta$ formula. Then $\mathcal{P}, x \models \Phi$ if and only if $\mathcal{P}, x \models \mathcal{E}(\Phi)$.* □

A direct consequence of Lemma 3.5 is that $\mathtt{SLCS}_\eta$ is weaker than $\mathtt{SLCS}_\gamma$.

Proposition 3.6. *Let $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$, with $P = |K|$ for some simplicial complex K, be a polyhedral model. For all $x_1, x_2 \in P$ the following holds: if $x_1 \equiv_\gamma x_2$ then $x_1 \equiv_\eta x_2$.* □

Remark 3.7. The converse of Proposition 3.6 does *not* hold, as shown by the polyhedral model $\mathcal{P}_6 = (P_6, \mathcal{V}_{\mathcal{P}_6})$ in Figure 6a, where P_6 is the simplex K_6 generated by points A, B, and C, i.e. the triangle ABC, and $\mathcal{V}_{\mathcal{P}_6}$ is specified by the colours in the figure. It is easy to see that, for all $x \in \widetilde{ABC}$, we have $A \not\equiv_\gamma x$ and $A \equiv_\eta x$. Let, in fact, $x \in \widetilde{ABC}$. Clearly, $A \not\equiv_\gamma x$ since $\mathcal{P}_6, A \models \gamma(\mathbf{red}, \mathtt{true})$ whereas $\mathcal{P}_6, x \not\models \gamma(\mathbf{red}, \mathtt{true})$. It can easily be shown, by induction on the structure of formulas, that $A \equiv_\eta x$ for all $x \in \widetilde{ABC}$ (see Appendix A.5). As an additional, a bit more complex, example, let us consider the polyhedral model $\mathcal{P}_4$ of Figure 4. It is easy to see that every $x \in \widetilde{CE}$ satisfies $\gamma(\mathbf{green}, \mathtt{true})$, while for no $y \in \widetilde{DEF}$ we have $\mathcal{P}_4, y \models \gamma(\mathbf{green}, \mathtt{true})$. So, for all such x and y, we have $x \not\equiv_\gamma y$. On the other hand, as we will see in Example 4.14 of Section 4 (on page 21), cells $\widetilde{CE}$ and $\widetilde{DEF}$ will fall in the same equivalence class of $\equiv_\eta$ on $\mathbb{F}(\mathcal{P}_4)$ and so, by Theorem 3.20 below — guaranteeing that $\mathtt{SLCS}_\eta$ is preserved and reflected by mapping $\mathbb{F}$ — and Theorem 5.11 of Section 5 — stating correctness of $\pm$-minimisation — we get that $x \equiv_\eta y$. The above reasoning can be generalised to any pair of points $x \in \widetilde{D} \cup \widetilde{E} \cup \widetilde{CE} \cup \widetilde{DE}$ and $y \in \widetilde{F} \cup \widetilde{DF} \cup \widetilde{EF} \cup \widetilde{DEF}$: we have $x \equiv_\eta y$ but $x \not\equiv_\gamma y$. ⁂



(A) (B)

FIGURE 6. A polyhedral model (6a) $\mathcal{P}_6$, and the Hasse diagram of its cell poset model (6b).

Remark 3.8. The example of Figure 6a is useful also for showing that the classical topological interpretation of the modal logic operator $\Diamond$ cannot be expressed in $\texttt{SLCS}_\eta$. Clearly, in the model of the figure, we have $\mathcal{P}_6, A \models \Diamond\textbf{red}$ while $\mathcal{P}_6, x \models \Diamond\textbf{red}$ for no $x \in \widetilde{ABC}$. On the other hand, $A \equiv_\eta x$ holds for all $x \in \widetilde{ABC}$, as we have just seen in Remark 3.7. So, if $\Diamond$ were expressible in $\texttt{SLCS}_\eta$, then A and x should have agreed on $\Diamond\textbf{red}$ for each $x \in \widetilde{ABC}$. ⋇

Below, we re-interpret $\texttt{SLCS}_\eta$ on finite Kripke models instead of polyhedral models. The only difference from Definition 3.1 is, of course, the fact that η-formulas are defined using $\pm$-paths instead of topological ones.

Definition 3.9 ($\texttt{SLCS}_\eta$ on finite Kripke models)**.** The satisfaction relation of $\texttt{SLCS}_\eta$ with respect to a given finite Kripke model $\mathcal{K} = (W, R, \mathcal{V}_\mathcal{K})$, an $\texttt{SLCS}_\eta$ formula Φ, and an element $w \in W$, is defined recursively on the structure of Φ:

$$
\begin{array}{lll}
\mathcal{K}, w \models p & \Leftrightarrow & w \in \mathcal{V}_\mathcal{K}(p);\\
\mathcal{K}, w \models \neg\Phi & \Leftrightarrow & \mathcal{K}, w \not\models \Phi;\\
\mathcal{K}, w \models \Phi_1 \wedge \Phi_2 & \Leftrightarrow & \mathcal{K}, w \models \Phi_1 \text{ and } \mathcal{K}, w \models \Phi_2;\\
\mathcal{K}, w \models \eta(\Phi_1, \Phi_2) & \Leftrightarrow & \text{a } \pm\text{-path } \pi : [0;\ell] \to W \text{ exists such that}\\
 & & \pi(0) = w,\ \mathcal{K}, \pi(\ell) \models \Phi_2, \text{ and } \mathcal{K}, \pi(i) \models \Phi_1 \text{ for all } i \in [0;\ell).
\end{array}
$$
•

Remark 3.10. We recall here that $\pm$-paths are defined on general Kripke frames, of which finite posets are a subclass. The reason why in Definition 3.9 we use finite Kripke models, instead of restricting it to finite poset models, stems from the fact that the result of minimisation of a finite poset model, modulo weak $\pm$-bisimilarity, is, in general, not guaranteed to be again a poset model, whereas it is guaranteed to be a (reflexive) finite Kripke model. As we will see in Section 5, the fact that the minimal model is not necessarily a poset model does not affect correctness of the minimisation procedure, and so it does not constitute a problem for the optimised model checking method presented in this paper. In the rest of this section, as well as in Section 4, we will anyway be interested in poset models, so that we will restrict the relevant results to the latter. ⋇

The following result, proven in Appendix A.6, states that to evaluate an $\texttt{SLCS}_\eta$ formula $\eta(\Phi_1, \Phi_2)$ in a poset model, it does not matter whether one considers $\pm$-paths or $\downarrow$-paths.

Proposition 3.11. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, $w \in W$, and $\texttt{SLCS}_\eta$ formulas Φ_1 and Φ_2, the following statements are equivalent:*

(1) *There exists a $\pm$-path $\pi : [0;\ell] \to W$ for some ℓ with $\pi(0) = w$, $\mathcal{F}, \pi(\ell) \models \Phi_2$, and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0;\ell)$.*

(2) *There exists a* $\downarrow$*-path* $\pi : [0;\ell'] \to W$ *for some* ℓ' *with* $\pi(0) = w$, $\mathcal{F}, \pi(\ell') \models \Phi_2$, *and* $\mathcal{F}, \pi(i) \models \Phi_1$ *for all* $i \in [0;\ell')$. □

Definition 3.12 (Logical Equivalence)**.** Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ and elements $w_1, w_2 \in W$ we say that w_1 and w_2 are *logically equivalent* with respect to $\mathtt{SLCS}_\eta$, written $w_1 \equiv_\eta^{\mathcal{F}} w_2$, if and only if, for all $\mathtt{SLCS}_\eta$ formulas Φ, it holds that $\mathcal{F}, w_1 \models \Phi$ if and only if $\mathcal{F}, w_2 \models \Phi$. •

Again, in the following, we will refrain from indicating the model $\mathcal{F}$ explicitly in $\equiv_\eta^{\mathcal{F}}$ when it is clear from the context. It is useful to define a "characteristic" $\mathtt{SLCS}_\eta$ formula $\chi(w)$ that is satisfied by all and only those elements w' with $w' \equiv_\eta w$, as shown in Appendix A.7.

Definition 3.13. Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$, $w_1, w_2 \in W$, define $\mathtt{SLCS}_\eta$ formula δ_{w_1,w_2} as follows: if $w_1 \equiv_\eta w_2$, then set $\delta_{w_1,w_2} = \mathtt{true}$, otherwise pick some $\mathtt{SLCS}_\eta$ formula ψ such that $\mathcal{F}, w_1 \models \psi$ and $\mathcal{F}, w_2 \models \neg\psi$, and set $\delta_{w_1,w_2} = \psi$. For $w \in W$ define $\chi(w) = \bigwedge_{w' \in W} \delta_{w,w'}$. •

Proposition 3.14. *Given a finite poset model* $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$*, for* $w_1, w_2 \in W$*, it holds that* $\mathcal{F}, w_2 \models \chi(w_1)$ *if and only if* $w_1 \equiv_\eta w_2$*.* □

The following lemma is the poset model counterpart of Lemma 3.5 (see Appendix A.8):

Lemma 3.15. *Let* $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ *be a finite poset model,* $w \in W$*, and* Φ *an* $\mathtt{SLCS}_\eta$ *formula. Then* $\mathcal{F}, w \models \Phi$ *if and only if* $\mathcal{F}, w \models \mathcal{E}(\Phi)$*.* □

Thus we get, as for the interpretation on polyhedral models, that $\mathtt{SLCS}_\eta$ on finite poset models is weaker than $\mathtt{SLCS}_\gamma$:

Proposition 3.16. *Let* $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ *be a finite poset model. For all* $w_1, w_2 \in W$ *the following holds: if* $w_1 \equiv_\gamma w_2$ *then* $w_1 \equiv_\eta w_2$*.* □

Remark 3.17. As expected, the converse of Proposition 3.16 does not hold, as shown by the poset model $\mathbb{F}(\mathcal{P}_6)$ of Figure 6b. Clearly, $\widetilde{A} \not\equiv_\gamma \widetilde{ABC}$. In fact $\mathbb{F}(\mathcal{P}_6), \widetilde{A} \models \gamma(\mathbf{red}, \mathtt{true})$ whereas $\mathbb{F}(\mathcal{P}_6), \widetilde{ABC} \not\models \gamma(\mathbf{red}, \mathtt{true})$. On the other hand, it can be easily shown, by induction on the structure of formulas, that $\widetilde{A} \equiv_\eta \widetilde{ABC}$ (see Appendix A.9). With reference to the polyhedral model $\mathcal{P}_4$ of Figure 4, its poset model $\mathbb{F}(\mathcal{P}_4) = (W_4, \preccurlyeq, \mathcal{V}_{\mathbb{F}(\mathcal{P}_4)})$, and Example 4.14 of Section 4, we have that $\widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{CE}, \widetilde{DE}, \widetilde{DF}, \widetilde{EF}$, and $\widetilde{DEF}$ are all equivalent according to weak $\pm$-bisimilarity. We invite the reader to check that, letting $\phi_0, \phi_1, \phi_2, \psi_1, \psi_2, \psi_3$, and ψ_4 be defined as

$$
\begin{aligned}
\phi_0 &= \gamma(\mathbf{green}, \mathtt{true}) & \psi_1 &= \neg\phi_0 \\
\phi_1 &= \gamma(\neg\phi_0, \mathtt{true}) & \psi_2 &= \phi_0 \wedge \neg\phi_1 \\
\phi_2 &= \gamma(\phi_0 \wedge \neg\phi_1, \mathtt{true}) & \psi_3 &= \phi_1 \wedge \phi_1 \wedge \neg\phi_2 \\
& & \psi_4 &= \phi_0 \wedge \phi_1 \wedge \phi_2
\end{aligned}
$$

we have

$\mathcal{P}_4, \widetilde{DF} \models \neg\phi_0$, and the same holds for $\widetilde{DEF}, \widetilde{EF}$ and $\widetilde{F}$,
$\mathcal{P}_4, \widetilde{CE} \models \neg\phi_0 \wedge \neg\phi_1$,
$\mathcal{P}_4, \widetilde{DE} \models \neg\phi_0 \wedge \phi_1 \wedge \neg\phi_2$, and
$\mathcal{P}_4, \widetilde{E} \models \neg\phi_0 \wedge \phi_1 \wedge \phi_2$.

As a consequence, each of ψ_1, ψ_2, ψ_3, and ψ_4 cannot be true in conjunction with any of the others and so, the classes $\{\widetilde{DF}, \widetilde{DEF}, \widetilde{EF}, \widetilde{F}\}$, $\{\widetilde{CE}\}$, $\{\widetilde{DE}\}$, and $\{\widetilde{E}\}$ must definitely be distinct in the quotient of W_4 modulo $\equiv_\gamma$. ⋇

Remark 3.18. As for the case of the continuous interpretation of $\mathtt{SLCS}_\eta$, the example of Figure 6b is useful also for showing that the classical modal logic operator $\Diamond$ cannot be expressed in $\mathtt{SLCS}_\eta$. Clearly, in the model of the figure, we have $\mathbb{F}(\mathcal{P}_6), \widetilde{A} \models \Diamond\mathbf{red}$ while $\mathbb{F}(\mathcal{P}_6), \widetilde{ABC} \not\models \Diamond\mathbf{red}$. On the other hand $\widetilde{A} \equiv_\eta \widetilde{ABC}$ holds, as we have just seen in Remark 3.17. So, if $\Diamond$ were expressible in $\mathtt{SLCS}_\eta$, then $\widetilde{A}$ and $\widetilde{ABC}$ should have agreed on $\Diamond\mathbf{red}$. ⋇

The following result, proven in Appendix A.10, is useful to set up a bridge between the continuous and the discrete interpretations of $\mathtt{SLCS}_\eta$.

Lemma 3.19. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, for all $x \in P$ and formulas Φ of $\mathtt{SLCS}_\eta$ the following holds: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$.* □

As a direct consequence of Lemma 3.15 and Lemma 3.19 we get, by Theorem 3.20 below, proven in A.11, the bridge between the continuous and the discrete interpretations of $\mathtt{SLCS}_\eta$:

Theorem 3.20. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, for all $x \in P$ and formulas Φ of $\mathtt{SLCS}_\eta$ it holds that: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$.* □

This theorem allows one to go back and forth between the polyhedral model and the corresponding poset model without losing anything expressible in $\mathtt{SLCS}_\eta$.

4. Weak Simplicial Bisimilarity

In this section, we introduce weak versions of simplicial bisimilarity and $\pm$-bisimilarity and we show that they coincide with logical equivalence induced by $\mathtt{SLCS}_\eta$ in polyhedral and poset models, respectively. We are looking for a notion of bisimilarity that enjoys the HMP with respect to $\mathtt{SLCS}_\eta$, i.e. that coincides with $\equiv_\eta$. We already know that simplicial bisimilarity $\sim_\triangle$ enjoys the HMP with respect $\mathtt{SLCS}_\gamma$, i.e. $\sim_\triangle = \equiv_\gamma$ and, moreover, that $\equiv_\eta$ is weaker than $\equiv_\gamma$. Here, by "weaker" we mean coarser, i.e. one that includes simplicial bisimilarity, in the sense of set inclusion, $\equiv_\gamma \subset \equiv_\eta$.

A natural step in the search for such a notion of bisimilarity is to reconsider the definition of simplicial bisimilarity, recalled in Section 2.3 (see Definition 2.5), and seek to weaken its conditions. Of course, the first condition cannot be relaxed in any meaningful way: equivalent points must at least satisfy the same predicate letters. Let us thus focus on the second condition, namely the one concerning topological paths. The condition requires that as "one moves on" π_2 using cursor r, the corresponding point on π_1, i.e. $\pi_1(r)$, must be related by Z to the current point in π_2, namely $\pi_2(r)$. The points in π_2 and π_1, while one moves the cursor r, must go "hand in hand" in Z.

One way of relaxing the above condition is to require only that (2.a) the ending points of π_1 and π_2 are related — i.e. $Z(\pi_1(1), \pi_2(1))$ — and (2.b) for each other point y_2 of π_2, there is a point y_1 of π_1, different from $\pi_1(1)$, such that y_1 and y_2 are related — i.e. for each $r_2 \in [0, 1)$ there is $r_1 \in [0, 1)$ such that $Z(\pi_1(r_1), \pi_2(r_2))$.

Interestingly, it turns out that the bisimilarity induced by a definition of bisimulation relation where condition (2) is relaxed as above, coincides exactly with $\equiv_\eta$, the logical equivalence induced by $\mathtt{SLCS}_\eta$! In practice, we do not even need the notion of simplicial path, in the sense that the actual definition, given below, is based on general topological paths and characterises an equivalence relation — which we call *weak simplicial bisimilarity*, $\approx_\triangle$ — that coincides with $\equiv_\eta$, as guaranteed by Theorem 4.9. The proof of this theorem, as well as those of all results related to $\approx_\triangle$, does not require the use of simplicial paths.

Definition 4.1 (Weak Simplicial Bisimulation)**.** Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, a symmetric relation $Z \subseteq |K| \times |K|$ is a *weak simplicial bisimulation* if, for all $x_1, x_2 \in |K|$, whenever $Z(x_1, x_2)$, it holds that:

(1) $\mathcal{V}_\mathcal{P}^{-1}(\{x_1\}) = \mathcal{V}_\mathcal{P}^{-1}(\{x_2\})$;

(2) for each topological path π_1 from x_1, there is a topological path π_2 from x_2 such that $Z(\pi_1(1), \pi_2(1))$ and for all $r_2 \in [0,1)$ there is $r_1 \in [0,1)$ such that $Z(\pi_1(r_1), \pi_2(r_2))$.

Two points $x_1, x_2 \in P$ are weakly simplicial bisimilar, written $x_1 \approx_\triangle^\mathcal{P} x_2$, if there is a weak simplicial bisimulation Z such that $B(x_1, x_2)$. •

Example 4.2. With reference to Figure 6a, the binary relation Z composed of all those pairs of points that have the same colour, i.e.

$$Z = \left(\widetilde{AB} \cup \widetilde{BC} \cup \widetilde{AC}\right)^2 \cup \left(ABC^2 \setminus \left(\widetilde{AB} \cup \widetilde{BC} \cup \widetilde{AC}\right)^2\right)$$

is a weak simplicial bisimulation. Take, for example, any pair $(x, y) \in \widetilde{AB} \times \widetilde{BC}$: both x and y satisfy only one predicate letter, namely **red**. In addition, let π_x be any topological path starting from x and such that $\pi_x(1)$ is red. Then it is easy to see, just by visual inspection, that one can find a path π_y from y such that $\pi_y(1)$ is red and, for each intermediate point of π_y there is in π_x an intermediate point of the same colour. The reasoning for the case in which $\pi_x(1)$ is blue is similar. Thus $x \approx_\triangle y$. The reasoning can be extended to all pairs in Z: actually $\approx_\triangle$ coincides with Z for the polyhedral model of Figure 6a.

As an additional example, let us consider the polyhedral model $\mathcal{P}_4$ of Figure 4a and points A and D therein. It is easy to see that there is no weak simplicial bisimulation Z such that $Z(A, D)$. Suppose such a Z exists. Take π_1 from D such that, $\pi_1(r) = D$ for all $r \in [0, \bar{r}]$, and $\emptyset \subset \pi_1((\bar{r}, 1]) \subset \widetilde{CDE}$, for some $\bar{r} \in (0,1)$. Clearly, any π_2 from A should be such that $\pi_2(1) \in \widetilde{CDE}$, otherwise $Z(\pi_1(1), \pi_2(1))$ would not hold. But any topological path starting from A and ending in $\widetilde{CDE}$ would necessarily pass by red points, and for any such red point $\pi_2(r_2)$ for some $r_2 \in (0,1)$ there would be no $r_1 \in (0,1)$ such that $Z(\pi_1(r_1), \pi_2(r_2))$, since no point of π_1 is red. As one would expect, we have also that $\mathcal{P}_4, D \models \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green})$ whereas $\mathcal{P}_4, A \not\models \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green})$. ♣

Definition 4.3 below rephrases Definition 4.1 for finite posets and discrete paths and it settles the finite poset counterpart of weak simplicial bisimilarity, namely weak ±-bisimilarity, a weaker version of ±-bisimilarity introduced in [CGL+23a]. The second condition in the definition deals with ↓-paths. In particular, for a weak ±-bisimulation Z on a poset model, it is required that, for all nodes w_1, w_2 of the poset, whenever $Z(w_1, w_2)$, for each ↓-path $\pi_1 = (w_1, u_1, d_1)$, there is a ±-path[7] π_2 from w_2 of some length $\ell_2 \geq 2$ such that (ii.a) the ending elements of π_1 and π_2 are related — i.e. $Z(d_1, \pi_2(\ell_2))$ — and (ii.b) for each other element v_2 of π_2 there is an element v_1 of π_1, different from $\pi_1(2)$, such that v_1 and v_2 are related — i.e. for all $j \in [0; \ell_2)$, there is $i \in [0; 2)$ such that $Z(\pi_1(i), \pi_2(j))$. In other words,

[7] Recall that ±-paths are a subclass of ↓-paths.

since $\pi_1(0) = w_1$ and $\pi_1(1) = u_1$, it is required that $Z(w_1, \pi_2(j))$ or $Z(u_1, \pi_2(j))$ holds for all $j \in [0; \ell_2)$. Note that it is sufficient to consider $\downarrow$-paths of length 2 starting from w_1. As shown by Theorem 4.12, the resulting relation $\approx_\triangle$ coincides with $\equiv_\eta$.

Definition 4.3 (Weak $\pm$-bisimulation)**.** Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, a symmetric binary relation $Z \subseteq W \times W$ is a weak $\pm$-bisimulation if, for all $w_1, w_2 \in W$, whenever $Z(w_1, w_2)$, it holds that:

(1) $\mathcal{V}_\mathcal{F}^{-1}(\{w_1\}) = \mathcal{V}_\mathcal{F}^{-1}(\{w_2\})$;
(2) for each $u_1, d_1 \in W$ such that $w_1 \preccurlyeq^\pm u_1 \succcurlyeq d_1$ there is a $\pm$-path $\pi_2 : [0; \ell_2] \to W$ from w_2 such that $Z(d_1, \pi_2(\ell_2))$ and, for all $j \in [0; \ell_2)$, it holds that $Z(w_1, \pi_2(j))$ or $Z(u_1, \pi_2(j))$.

We say that w_1 is weakly $\pm$-bisimilar to w_2, written $w_1 \approx_\pm^\mathcal{F} w_2$ if there is a weak $\pm$-bisimulation Z such that $Z(w_1, w_2)$. •

For example, all red cells in the Hasse diagram of Figure 6b are weakly $\pm$-bisimilar and all blue cells are weakly $\pm$-bisimilar.

The following lemma shows that, in a polyhedral model $\mathcal{P}$, weak simplicial bisimilarity $\approx_\triangle^\mathcal{P}$, as given by Definition 4.1, is stronger than $\equiv_\eta$ – logical equivalence with respect to $\texttt{SLCS}_\eta$:

Lemma 4.4. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, for all $x_1, x_2 \in P$, the following holds: if $x_1 \approx_\triangle^\mathcal{P} x_2$ then $x_1 \equiv_\eta x_2$.* □

Proof. By induction on the structure of the formulas. We consider only the case $\eta(\Phi_1, \Phi_2)$. Suppose $x_1 \approx_\triangle x_2$ and $\mathcal{P}, x_1 \models \eta(\Phi_1, \Phi_2)$. Then there is a topological path π_1 from x_1 such that $\mathcal{P}, \pi_1(1) \models \Phi_2$ and $\mathcal{P}, \pi_1(r_1) \models \Phi_1$ for all $r_1 \in [0, 1)$. Since $x_1 \approx_\triangle x_2$, then there is a topological path π_2 from x_2 such that $\pi_1(1) \approx_\triangle \pi_2(1)$ and for each $r_2 \in [0, 1)$ there is $r_1' \in [0, 1)$ such that $\pi_1(r_1') \approx_\triangle \pi_2(r_2)$. By the Induction Hypothesis, we get $\mathcal{P}, \pi_2(1) \models \Phi_2$ and, for each $r_2 \in [0, 1)$ $\mathcal{P}, \pi_2(r_2) \models \Phi_1$. Thus $\mathcal{P}, x_2 \models \eta(\Phi_1, \Phi_2)$. □

Furthermore, logical equivalence induced by $\texttt{SLCS}_\eta$ is stronger than weak simplicial-bisimilarity, as implied by Lemma 4.8 below, which uses the following auxiliary lemmas, proven in Appendix A.12, Appendix A.13, and Appendix A.14 respectively.

Lemma 4.5. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ and weak $\pm$-bisimulation $Z \subseteq W \times W$, for all w_1, w_2 such that $Z(w_1, w_2)$, the following holds: for each $\downarrow$-path $\pi_1 : [0; k_1] \to W$ from w_1 there is a $\downarrow$-path $\pi_2 : [0; k_2] \to W$ from w_2 such that $Z(\pi_1(k_1), \pi_2(k_2))$ and, for each $j \in [0; k_2)$, exists $i \in [0; k_1)$ such that $Z(\pi_1(i), \pi_2(j))$.* □

Lemma 4.6. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, and associated cell poset model $\mathbb{F}(\mathcal{P}) = (W, \preccurlyeq, \mathcal{V}_{\mathbb{F}(\mathcal{P})})$, for any $\downarrow$-path $\pi : [0; \ell] \to W$, there is a topological path $\pi' : [0, 1] \to |K|$ such that: (i) $\mathbb{F}(\pi'(0)) = \pi(0)$, (ii) $\mathbb{F}(\pi'(1)) = \pi(\ell)$, and (iii) for all $r \in (0, 1)$ exists $i < \ell$ such that $\mathbb{F}(\pi'(r)) = \pi(i)$.* □

Lemma 4.7. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, and associated cell poset model $\mathbb{F}(\mathcal{P}) = (W, \preccurlyeq, \mathcal{V}_{\mathbb{F}(\mathcal{P})})$, for any topological path $\pi : [0, 1] \to |K|$ the following holds: $\mathbb{F}(\pi([0, 1]))$ is a connected subposet of W and there are $k > 0$ and a $\downarrow$-path $\hat{\pi} : [0; k] \to W$ from $\mathbb{F}(\pi(0))$ to $\mathbb{F}(\pi(1))$ such that, for all $i \in [0; k)$, $r \in [0, 1)$ exists with $\hat{\pi}(i) = \mathbb{F}(\pi(r))$.* □

Lemma 4.8. *In a given polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, it holds that $\equiv_\eta$ is a weak simplicial bisimulation.*

Proof. Let $x_1, x_2 \in |K|$ be such that $x_1 \equiv_\eta x_2$. The first condition of Definition 4.1 is clearly satisfied since $x_1 \equiv_\eta x_2$. Suppose π_1 is a topological path from x_1. By Lemma 4.7, $\mathbb{F}(\pi_1([0,1]))$ is a connected subposet of $\widetilde{K}$ and a $\downarrow$-path $\hat{\pi}_1 : [0;k_1] \to \widetilde{K}$ from $\mathbb{F}(\pi_1(0))$ to $\mathbb{F}(\pi_1(1))$ exists such that, for all $i \in [0;k_1)$, $r_1 \in [0,1)$ exists with $\hat{\pi}_1(i) = \mathbb{F}(\pi_1(r_1))$. We also know that $\mathbb{F}(x_1) \equiv_\eta \mathbb{F}(x_2)$, as a consequence of Theorem 3.20, since $x_1 \equiv_\eta x_2$. In addition, due to Lemma 4.11 below, we also know that $\mathbb{F}(x_1) \approx_\pm \mathbb{F}(x_2)$. By Lemma 4.5, we get that there is a $\downarrow$-path $\hat{\pi}_2 : [0;k_2] \to \widetilde{K}$ such that $\hat{\pi}_1(k_1) \equiv_\eta \hat{\pi}_2(k_2)$ and, for each $j \in [0;k_2)$, $i \in [0;k_1)$ exists such that $\hat{\pi}_1(i) \equiv_\eta \hat{\pi}_2(j)$. By Lemma 4.6, it follows that there is topological path π_2 from x_2 satisfying the three conditions of the lemma and, again by Theorem 3.20, we have that $\pi_2(1) \equiv_\eta \pi_1(1)$. In addition, for any $r_2 \in [0,1)$, since $\mathbb{F}(\pi_2(r_2)) = \hat{\pi}_2(j)$ for $j \in [0;k_2)$ (condition (ii) of Lemma 4.6) there is $i \in [0;k_1)$ such that $\hat{\pi}_1(i) \equiv_\eta \hat{\pi}_2(j)$. Finally, by construction, there is $r_1 \in [0,1)$ such that $\mathbb{F}(\pi_1(r_1)) = \hat{\pi}_1(i)$. By Theorem 3.20, we arrive at $\pi_1(r_1) \equiv_\eta \pi_2(r_2)$. □

On the basis of Lemma 4.4 and Lemma 4.8, we have that the largest weak simplicial bisimulation exists, it is a weak simplicial bisimilarity, it is an equivalence relation, and it coincides with logical equivalence in the polyhedral model induced by $\texttt{SLCS}_\eta$, thus establishing the HMP for $\approx_\triangle^{\mathcal{P}}$ with respect to $\texttt{SLCS}_\eta$:

Theorem 4.9. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, and $x_1, x_2 \in P$, the following holds: $x_1 \equiv_\eta^{\mathcal{P}} x_2$ if and only if $x_1 \approx_\triangle^{\mathcal{P}} w_2$.* □

Similar results hold for poset models. The following lemma shows that, in every finite poset model $\mathcal{F}$, weak $\pm$-bisimilarity (Definition 4.3) is stronger than logical equivalence with respect to $\texttt{SLCS}_\eta$, i.e. $\approx_\pm^{\mathcal{F}} \subseteq \equiv_\eta^{\mathcal{F}}$:

Lemma 4.10. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, for all $w_1, w_2 \in W$, if $w_1 \approx_\pm^{\mathcal{F}} w_2$ then $w_1 \equiv_\eta^{\mathcal{F}} w_2$.*

Proof. By induction on formulas. We consider only the case $\eta(\Phi_1, \Phi_2)$. Suppose $w_1 \approx_\pm w_2$ and $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$. Then, there is (a $\pm$-path and so) a $\downarrow$-path π_1 from w_1 of some length k_1 such that $\mathcal{F}, \pi_1(k_1) \models \Phi_2$ and for all $i \in [0;k_1)$ it holds that $\mathcal{F}, \pi_1(i) \models \Phi_1$. By Lemma 4.5, we know that a $\downarrow$-path π_2 from w_2 exists of some length k_2 such that $\pi_1(k_1) \approx_\pm \pi_2(k_2)$ and for all $j \in [0;k_2)$ exists $i \in [0;k_1)$ such that $\pi_1(i) \approx_\pm \pi_2(j)$. By the Induction Hypothesis, we then get that $\mathcal{F}, \pi_2(k_2) \models \Phi_2$ and for all $j \in [0;k_2)$ we have $\mathcal{F}, \pi_2(j) \models \Phi_1$. This implies that $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$. □

Furthermore, logical equivalence induced by $\texttt{SLCS}_\eta$ is stronger than weak $\pm$-bisimilarity, i.e. $\equiv_\eta^{\mathcal{F}} \subseteq \approx_\pm^{\mathcal{F}}$, as implied by the following:

Lemma 4.11. *In a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, $\equiv_\eta^{\mathcal{F}}$ is a weak $\pm$-bisimulation.*

Proof. If $w_1 \equiv_\eta w_2$, then the first requirement of Definition 4.3 is trivially satisfied. We prove that $\equiv_\eta$ satisfies the second requirement of Definition 4.3. Suppose $w_1 \equiv_\eta w_2$ and let u_1, d_1 be as in the above-mentioned requirement. This implies that $\mathcal{F}, w_1 \models \eta(\chi(w_1) \vee \chi(u_1), \chi(d_1))$, where, we recall, $\chi(w)$ is the 'characteristic formula' for w as in Definition 3.13. Since $w_1 \equiv_\eta w_2$, we also have that $\mathcal{F}, w_2 \models \eta(\chi(w_1) \vee \chi(u_1), \chi(d_1))$ holds. This in turn means that a $\downarrow$-path π_2 of some length k_2 from w_2 exists such that $\mathcal{F}, \pi_2(k_2) \models \chi(d_1)$ and for all $j \in [0;k_2)$ we have $\mathcal{F}, \pi_2(j) \models \chi(w_1) \vee \chi(u_1)$, i.e. $\mathcal{F}, \pi_2(j) \models \chi(w_1)$ or $\mathcal{F}, \pi_2(j) \models \chi(u_1)$. Consequently, by Proposition 3.14, we have: $\pi_2(k_2) \equiv_\eta d_1$ and, for all $j \in [0;k_2)$, $\pi_2(j) \equiv_\eta w_1$ or $\pi_2(j) \equiv_\eta u_1$, so that the second condition of the definition is fulfilled. □

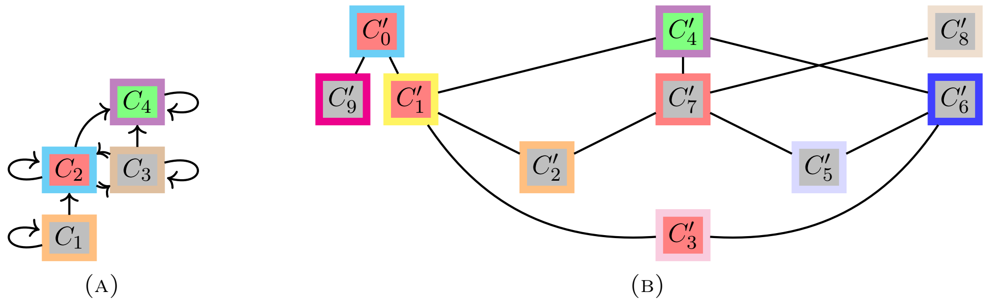


FIGURE 7. The minimal model $\mathbb{F}(\mathcal{P}_4)_{\min}$, modulo weak $\pm$-bisimilarity (7a), and modulo $\pm$-bisimilarity (7b), of the cell poset model $\mathbb{F}(\mathcal{P}_4)$ of Figure 4c. Note that the minimal model modulo $\pm$-bisimilarity is a poset model and so it is represented by its Hasse diagram.

On the basis of Lemma 4.10 and Lemma 4.11, we have that the largest weak $\pm$-bisimulation exists, it is a weak $\pm$-bisimilarity, it is an equivalence relation, and it coincides with logical equivalence in the finite poset induced by $\mathtt{SLCS}_\eta$:

Theorem 4.12. *For every finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F}), w_1, w_2 \in W$, the following holds: $w_1 \equiv_\eta^\mathcal{F} w_2$ if and only if $w_1 \approx_\pm^\mathcal{F} w_2$.* □

By this we have established the HMP for $\approx_\pm$ with respect to $\mathtt{SLCS}_\eta$.

Recalling that, by Theorem 3.20, given polyhedral model $\mathcal{P} = (|K|, \mathcal{V}_\mathcal{P})$ for all $x \in |K|$ and $\mathtt{SLCS}_\eta$ formula Φ, we have that $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$, we get the following final result:

Corollary 4.13. *Given a polyhedral model $\mathcal{P} = (P, \mathcal{V}_\mathcal{P})$, with $P = |K|$ for some simplicial complex K, for all $x_1, x_2 \in P$ the following holds:*

$$x_1 \approx_\triangle^\mathcal{P} x_2 \text{ iff } x_1 \equiv_\eta^\mathcal{P} x_2 \text{ iff } \mathbb{F}(x_1) \equiv_\eta^{\mathbb{F}(\mathcal{P})} \mathbb{F}(x_2) \text{ iff } \mathbb{F}(x_1) \approx_\pm^{\mathbb{F}(\mathcal{P})} \mathbb{F}(x_2). \quad \square$$

This says that $\mathtt{SLCS}_\eta$-equivalence in a polyhedral model is the same as weak simplicial bisimilarity, which maps by $\mathbb{F}$ to the weak $\pm$-bisimilarity in the corresponding poset model, where the latter coincides with the $\mathtt{SLCS}_\eta$-equivalence.

In the example below, and in the sequel, whenever we show a graphical representation of a minimal model in a figure, we use the following convention: each node of the Kripke model is coloured according to the predicate letter satisfied by the cells belonging to the equivalence class represented by the node — obviously, since all such cells are weakly $\pm$-bisimilar, they all satisfy the same predicate letters[8] — whereas the colour of the *border* of the node identifies the equivalence class itself, and is, therefore, unique within the model. Note that the colour of the borders of the nodes have only an illustrative purpose. In particular, they are not related to the colours expressing the evaluation of proposition letters.

Example 4.14. Figure 7a shows the minimal model $\mathbb{F}(\mathcal{P}_4)_{\min}$, modulo $\approx_\pm$, of the poset model $\mathbb{F}(\mathcal{P}_4)$ shown in Figure 4c. $\mathbb{F}(\mathcal{P}_4)_{\min}$ is built using the procedure that will be described in detail in Section 5. Note that $\mathbb{F}(\mathcal{P}_4)_{\min}$ is not a poset model, but it is a reflexive Kripke model. As we can see in the figure, we have four equivalence classes. More

[8] In the examples, for the sake of readability, each cell satisfies a single predicate letter, namely its "colour".

specifically, the classes are: $C_1 = \{\widetilde{A}\}$, represented by the grey node with orange border, $C_2 = \{\widetilde{B}, \widetilde{C}, \widetilde{AB}, \widetilde{AC}, \widetilde{BC}, \widetilde{BD}, \widetilde{CD}, \widetilde{ABC}, \widetilde{BCD}\}$, represented by the red node with cyan border, $C_3 = \{\widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{CE}, \widetilde{DE}, \widetilde{DF}, \widetilde{EF}, \widetilde{DEF}\}$, represented by the grey node with brown border, and, finally, $C_4 = \{\widetilde{CDE}\}$, represented by the green node with violet border.

As we will see in Section 5, the fact that $\widetilde{D} \preccurlyeq \widetilde{CD}$ holds, with $\widetilde{D} \in C_3$ and $\widetilde{CD} \in C_2$, implies that (C_3, C_2) belongs to the accessibility relation $R_{\min}$ of the Kripke model $\mathbb{F}(\mathcal{P}_4)_{\min}$. Similarly, we have that the fact that $\widetilde{C} \preccurlyeq \widetilde{CE}$ holds, with $\widetilde{C} \in C_2$ and $\widetilde{CE} \in C_3$, implies that $(C_2, C_3) \in R_{\min}$. With the same rationale, since $\widetilde{D} \preccurlyeq \widetilde{D}$ holds, we have that $(C_3, C_3) \in R_{\min}$. Finally, since $\widetilde{A} \preccurlyeq \widetilde{AB}$ and $\widetilde{CD} \preccurlyeq \widetilde{CDE}$, we get that $\{(C_1, C_2), (C_2, C_4)\} \subseteq R_{\min}$ whereas we can see from Figure 7a that $(C_1, C_4) \notin R_{\min}$. The presence of cycles as the above, as well as the fact that transitivity of the accessibility relation is not guaranteed, imply that the minimal model of a poset model, modulo $\approx_{\triangle}$, is not necessarily a poset model. Anyway, it is guaranteed, by construction, to be a reflexive Kripke model.

Note that cell $\widetilde{A}$ of the poset model of Figure 4c is in a different equivalence class, namely C_1, than any other grey cell of the poset model: the latter cells belong to C_3. In fact, it is easy to see that there is no weak $\pm$-bisimulation Z such that $Z(\widetilde{A}, w)$ for any $w \in C_3$. This is because condition (2) of Definition 4.3 cannot be satisfied, as shown in the sequel. Suppose for instance $Z(\widetilde{A}, \widetilde{D})$ for some weak bisimulation relation Z. Then, with reference to Definition 4.3, take $w_1 = \widetilde{D}$ and $u_1 = d_1 = \widetilde{CDE}$: clearly $w_1 \preccurlyeq^{\pm} u_1 \succcurlyeq d_1$. Any π_2 from $\widetilde{A}$ should end in $\widetilde{CDE}$, otherwise $B(d_1, \pi_2(\ell_2))$ would not hold, since $\mathcal{V}^{-1}_{\mathbb{F}(\mathcal{P}_4)}(d_1) = \mathbf{green}$ and $\mathcal{V}_{\mathbb{F}(\mathcal{P}_4)}(\mathbf{green}) = \{\widetilde{CDE}\}$. But any path from $\widetilde{A}$ and ending in $\widetilde{CDE}$ would necessarily pass by a cell, say $\pi_2(j)$, for some $j \in (0; \ell_2)$ such that $\pi_2(j) \in \mathcal{V}_{\mathbb{F}(\mathcal{P}_4)}(\mathbf{red})$. For such a j we would have that neither $Z(w_1, \pi_2(j))$ would hold, since $w_1 = \widetilde{D} \notin \mathcal{V}_{\mathbb{F}(\mathcal{P}_4)}(\mathbf{red})$, nor $Z(u_1, \pi_2(j))$, for the same reason. So, there exists no weak $\pm$-bisimulation containing $(\widetilde{A}, \widetilde{D})$. And, in fact, we also have that $\mathbb{F}(\mathcal{P}_4), \widetilde{D} \models \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green})$ whereas $\mathbb{F}(\mathcal{P}_4), \widetilde{A} \not\models \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green})$.

As another example, suppose $Z(\widetilde{A}, \widetilde{DEF})$ for some weak bisimulation relation Z and let $w_1 = u_1 = \widetilde{DEF}$ and $d_1 = \widetilde{D}$. Any π_2 from from $\widetilde{A}$ should necessarily end in a grey cell. But such a cell cannot be $\widetilde{A}$, since we already know that no $\pm$-bisimulation can contain $(\widetilde{A}, \widetilde{D})$. And, on the other hand, if $\pi_2(\ell_2) \in C_3$, then we would have a similar problem as above, with the unavoidable red elements of π_2. From the logical perspective, we see that $\mathbb{F}(\mathcal{P}_4), \widetilde{DEF} \models \eta(\mathbf{grey}, \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green}))$ whereas $\mathbb{F}(\mathcal{P}_4), \widetilde{A} \not\models \eta(\mathbf{grey}, \eta(\mathbf{green} \vee \mathbf{grey}, \mathbf{green}))$. The reasoning for all the other cases is similar. Finally, the reader can easily check that both $\mathbb{F}(\mathcal{P}_4), \widetilde{D} \models \eta(\mathbf{grey} \vee \mathbf{red}, \mathbf{red})$ and $\mathbb{F}(\mathcal{P}_4), \widetilde{E} \models \eta(\mathbf{grey} \vee \mathbf{red}, \mathbf{red})$. Actually, any grey point satisfies the above formula.

Weak $\pm$-bisimilarity ensures that, for each $\pm$-path in the poset model, there is a corresponding $\pm$-path in the minimal model and vice-versa. For instance the $\pm$-path $(C_2, C_2, C_2, C_2, C_3)$ in the minimal model corresponds to $\pm$-path $(\widetilde{AB}, \widetilde{ABC}, \widetilde{BC}, \widetilde{BCD}, \widetilde{D})$ in the poset model — witnessing, in both cases, $\mathbb{F}(\mathcal{P}_4), \widetilde{AB} \models \eta(\mathbf{red}, \mathbf{grey})$. The correspondence, of course, is not unique: for instance, the above $\pm$-path in the minimal model corresponds also to the $\pm$-path $(\widetilde{AB}, \widetilde{ABC}, \widetilde{C}, \widetilde{CD}, \widetilde{D})$.

Finally, in Figure 7b the minimal model of $\mathbb{F}(\mathcal{P}_4)$ with respect to $\equiv_{\gamma}$ is shown. Note that the minimal model is a poset model and, in fact, in the figure its Hasse diagram is shown. We have 10 equivalence classes, namely $C'_0 = \{\widetilde{B}, \widetilde{AB}, \widetilde{AC}, \widetilde{BC}, \widetilde{BD}, \widetilde{ABC}, \widetilde{BCD}\}$, $C'_1 = \{\widetilde{CD}\}$, $C'_2 = \{\widetilde{D}\}$, $C'_3 = \{\widetilde{C}\}$, $C'_4 = \{\widetilde{CDE}\}$, $C'_5 = \{\widetilde{E}\}$, $C'_6 = \{\widetilde{CE}\}$, $C'_7 = \{\widetilde{DE}\}$, $C'_8 = \{\widetilde{F}, \widetilde{DF}, \widetilde{EF}, \widetilde{DEF}\}$, and $C'_9 = \{\widetilde{A}\}$. ♣

5. Building the Minimal Model Modulo Logical Equivalence

In this section we present a minimisation procedure for finite poset models modulo weak $\pm$-bisimilarity or, equivalently, modulo $\equiv_\eta$. Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, the procedure consists of three steps:

Step 1: The poset model $\mathcal{F}$ is encoded as an LTS denoted $\mathbb{S}_C(\mathcal{F})$. The set of states of $\mathbb{S}_C(\mathcal{F})$ is W itself. The encoding is such that it is ensured that logically equivalent elements of $\mathcal{F}$ are mapped into branching bisimilar states of $\mathbb{S}_C(\mathcal{F})$. Thus, for $w_1, w_2 \in W$ that are logically equivalent with respect to $\texttt{SLCS}_\eta$ in the poset model $\mathcal{F}$, i.e. $w_1 \equiv^\mathcal{F}_\eta w_2$, we have that they are branching bisimilar as states in the LTS $\mathbb{S}_C(\mathcal{F})$, i.e. $w_1 \leftrightarroweq^{\mathbb{S}_C(\mathcal{F})}_b w_2$.

Step 2: The LTS $\mathbb{S}_C(\mathcal{F})$ is reduced modulo branching bisimilarity using available software tools, such as mCRL2 [GJKW17]. This step yields the set of equivalence classes of W for $\leftrightarroweq^{\mathbb{S}_C(\mathcal{F})}_b$. Because of the correspondence of logical equivalence and branching bisimilarity, we obtain $W/{\equiv^\mathcal{F}_\eta}$.

Step 3: The minimal model $\mathcal{F}_{\min} = (W_{\min}, R_{\min}, \mathcal{V}_{\mathcal{F}_{\min}})$ is built. It turns out that this model is not necessarily a poset model (see the example in Figure 7a). However, it is a reflexive Kripke model where $W_{\min} = W/{\equiv^\mathcal{F}_\eta}$, $R_{\min}$ is a relation induced by the ordering $\preccurlyeq$ of $\mathcal{F}$, and, most importantly, $\texttt{SLCS}_\eta$ is preserved and reflected, i.e. for each $w \in W$ and $\texttt{SLCS}_\eta$ formula Φ the following holds: $\mathcal{F}, w \models \Phi$ if and only if $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$.

In the remainder of this section we focus on Step 1 and Step 3.

5.1. The Encoding of $\mathcal{F}$ as $\mathbb{S}_C(\mathcal{F})$.

We obtain the LTS $\mathbb{S}_C(\mathcal{F}) = (S, L, \rightarrow)$ from the poset $\mathcal{F}$ as specified in Definition 5.1 below. $\mathbb{S}_C(\mathcal{F})$ is an LTS representing each node $w \in W$ of $\mathcal{F}$ as a distinct state. So, we put $S = W$. For example, the set of states of the LTS $\mathbb{S}_C(\mathcal{F}_8)$ of Figure 8d is $\{\widetilde{D}, \widetilde{E}, \widetilde{F}, \widetilde{DE}, \widetilde{EF}\}$, i.e. the same as that of the nodes of $\mathcal{F}_8 = \mathbb{F}(\mathcal{P}_8)$.

The set L of transition labels includes all predicate letters in $\texttt{PL}$, plus the "silent move" τ, typical of LTSs in concurrency theory, and the two special labels $\mathbf{c}$ and $\mathbf{d}$, the meaning of which will be discussed later. In our example of Figure 8, we have $L = \{\mathbf{blue}, \mathbf{red}, \tau, \mathbf{c}, \mathbf{d}\}$. We use transitions in $\mathbb{S}_C(\mathcal{F})$ for several purposes, as follows. For each state w, the fact that w (represents a node of $\mathcal{F}$ that) satisfies a predicate letter p is represented by a self-loop: each predicate letter $p \in \texttt{PL}$ such that $w \in \mathcal{V}_\mathcal{F}(p)$ is represented in $\mathbb{S}_C(\mathcal{F})$ by a transition from w to itself, labelled by p (Rule (PLC)). The transitions labelled by τ relate those states in $\mathbb{S}_C(\mathcal{F})$ representing nodes in $\mathcal{F}$ that are related by $\preccurlyeq$ or by $\succcurlyeq$ and satisfy the same set of predicate letters (Rule (TAU)). Intuitively, this represents in the LTS the fact that "nothing changes" when moving from one such node w to another one, w' (including w itself).

On the contrary, the fact that two states w and w' represent "adjacent" nodes of $\mathcal{F}$ — i.e. nodes related by $\preccurlyeq^\pm$ — which do *not* satisfy the same set of predicate letters, is modelled by transitions $w \xrightarrow{\mathbf{c}} w'$ and $w' \xrightarrow{\mathbf{c}} w$, where $\mathbf{c}$ stands for "change", with the obvious meaning (see Rule (CNG)).

Finally, Rule (DWN) makes sure that whenever $w \succcurlyeq w'$ in $\mathcal{F}$, a transition labelled $\mathbf{d}$ goes from (the state representing) w to (that representing) w'. The label $\mathbf{d}$ stands for "down". "Marking" the pair (w, w') with the transition $w \xrightarrow{\mathbf{d}} w'$ is relevant for identifying (the end of) $\downarrow$-paths. Recall that such paths are the most fundamental ones for the semantics and the properties of $\texttt{SLCS}_\eta$. We invite the reader to check that all the transitions in the LTS of Figure 8d are generated according to the above mentioned rules.

Definition 5.1. For a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ and symbols $\tau, \mathbf{c}, \mathbf{d} \notin \texttt{PL}$, the LTS $\mathbb{S}_C(\mathcal{F})$ is defined by $\mathbb{S}_C(\mathcal{F}) = (S, L, \rightarrow)$ where

- the set of states S is the set W;
- the set of labels L consists of $\texttt{PL} \cup \{\tau, \mathbf{c}, \mathbf{d}\}$;
- the transition relation $\rightarrow$ is the smallest relation on $S \times L \times S$ induced by the following transition rules.

$$\text{(PLC)}\ \frac{w \in \mathcal{V}_\mathcal{F}(p)}{w \xrightarrow{p} w} \qquad \text{(TAU)}\ \frac{w \preccurlyeq^{\pm} w' \quad \mathcal{V}_\mathcal{F}^{-1}(\{w\}) = \mathcal{V}_\mathcal{F}^{-1}(\{w'\})}{w \xrightarrow{\tau} w'}$$

$$\text{(CNG)}\ \frac{w \preccurlyeq^{\pm} w' \quad \mathcal{V}_\mathcal{F}^{-1}(\{w\}) \neq \mathcal{V}_\mathcal{F}^{-1}(\{w'\})}{w \xrightarrow{\mathbf{c}} w'} \qquad \text{(DWN)}\ \frac{w \succcurlyeq w'}{w \xrightarrow{\mathbf{d}} w'} \quad \bullet$$

In order to show that the above definition establishes that $w_1 \equiv_\eta^\mathcal{F} w_2$ if and only if $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, it is convenient to consider an intermediate structure, that is an LTS too. We denote this second LTS by $\mathbb{S}_A(\mathcal{F})$. This structure helps in the proofs to separate concerns related to the various equivalences that are involved. Suppose that nodes w_1 and w_2 of $\mathcal{F}$ are encoded by the states s_1 and s_2 in $\mathbb{S}_A(\mathcal{F})$, respectively. We will have that w_1 and w_2 are logically equivalent in $\mathcal{F}$ with respect to $\texttt{SLCS}_\eta$ if and only if states s_1 and s_2 are strongly bisimilar (in the classical sense [Mil89]) in $\mathbb{S}_A(\mathcal{F})$, written $s_1 \sim^{\mathbb{S}_A(\mathcal{F})} s_2$. Furthermore, it will hold that s_1 and s_2 are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$ if and only if w_1 and w_2 are branching bisimilar in $\mathbb{S}_C(\mathcal{F})$, thus providing the correctness of the construction.

LTS $\mathbb{S}_A(\mathcal{F})$ is more abstract than $\mathbb{S}_C(\mathcal{F})$ in the sense that all the nodes of $\mathcal{F}$ that satisfy the same proposition letters and that are connected via $\preccurlyeq^{\pm}$ are mapped to the same state of $\mathbb{S}_A(\mathcal{F})$. Thus, intuitively, a state of $\mathbb{S}_A(\mathcal{F})$ corresponds to a class of states of $\mathbb{S}_C(\mathcal{F})$. This is a class of states representing nodes w and w' in $\mathcal{F}$ for which "nothing changes" when moving from w to w', as discussed above. More precisely, define $\Theta = \{\, \mathcal{V}_\mathcal{F}^{-1}(\{w\}) \mid w \in W \,\}$ and consider, for $\alpha \in \Theta$, the α-connected components of $\mathcal{F}$. Then, each state s of $\mathbb{S}_A(\mathcal{F})$ is an α-connected component of $\mathcal{F}$, for some α as above. So, we group together all the nodes in W that can reach one another only via a path in $\mathcal{F}$ composed of elements all satisfying exactly the same proposition letters. The above intuition is formalised by the following definition.

Definition 5.2. Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, we define relation $\rightleftharpoons \subseteq W \times W$ as the set of pairs (w_1, w_2) such that an undirected path π of some length ℓ exists with $\pi(0) = w_1, \pi(\ell) = w_2$, and $\mathcal{V}_\mathcal{F}^{-1}(\{\pi(i)\}) = \mathcal{V}_\mathcal{F}^{-1}(\{\pi(j)\})$, for all $i, j \in [0; \ell]$. $\bullet$

The relevant definitions lead straightforwardly to the following observation.

Proposition 5.3. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ be a finite poset model. Then $\rightleftharpoons$ is an equivalence relation on W.* $\square$

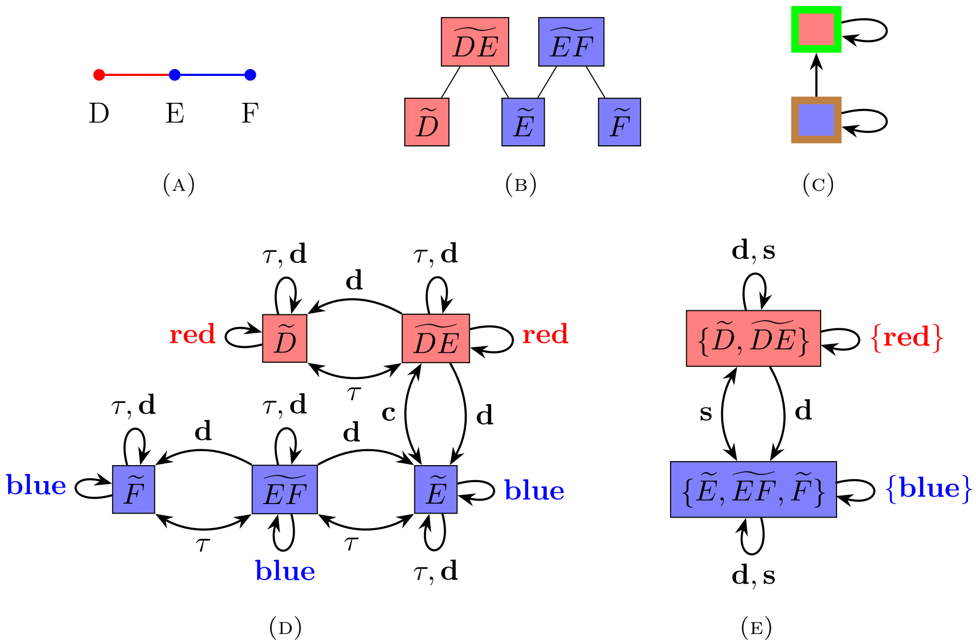


FIGURE 8. (8a) A polyhedral model $\mathcal{P}_8$; (8b) Hasse diagram of the poset model $\mathcal{F}_8 = \mathbb{F}(\mathcal{P}_8)$; (8c) minimal Kripke model $\mathcal{F}_{8\min}$; (8d) the LTS $\mathbb{S}_C(\mathcal{F}_8)$ obtained from $\mathcal{F}_8$ by the encoding of Definition 5.1; (8e) The LTS $\mathbb{S}_A(\mathcal{F}_8)$ obtained from $\mathcal{F}_8$ by the encoding of Definition 5.4. Note that whenever $w \stackrel{\ell}{\longrightarrow} w'$ and $w' \stackrel{\ell}{\longrightarrow} w$ a "double transition" $w \stackrel{\ell}{\longleftrightarrow} w'$ is drawn in the figure between w and w'.

The encoding to the more "abstract" LTS is defined in Definition 5.4 below. The states of $\mathbb{S}_C(\mathcal{F})$ are the equivalence classes of W modulo the equivalence relation $\rightleftharpoons$, i.e. $S = W/\rightleftharpoons$. With reference to Figure 8, we obtain two states, namely $\{\widetilde{D}, \widetilde{DE}\}$ and $\{\widetilde{E}, \widetilde{F}, \widetilde{EF}\}$, as shown in Figure 8e. The set L of transition labels includes the powerset of the set of predicate letters in $\mathbf{2}^{\mathtt{PL}}$, plus the two special labels $\mathbf{s}, \mathbf{d}$. In our example of Figure 8, we have $L_8 = \{\emptyset, \{\mathbf{blue}\}, \{\mathbf{red}\}, \{\mathbf{blue}, \mathbf{red}\}, \mathbf{s}, \mathbf{d}\}$.

Similarly to Rule (PLC) for the definition of $\mathbb{S}_C(\mathcal{F})$, Rule (PL) induces a self-loop in each state of $\mathbb{S}_A(\mathcal{F})$ (representing equivalence class) $[w]_{\rightleftharpoons}$. This transition is labelled with the *set* of predicate letters $\mathcal{V}_{\mathcal{F}}^{-1}(w)$ satisfied by the elements of the class. Note that, by definition of $\rightleftharpoons$, all the elements of such an equivalence class satisfy the same set of predicate letters. Transitions labelled by $\mathbf{d}$ (Rule (Down)) have the same interpretation as in the definition of $\mathbb{S}_C(\mathcal{F})$ while those labelled by $\mathbf{s}$ (Rule (Step)) model a single step in $\preccurlyeq^{\pm}$, regardless of there being "a change" or not.

Definition 5.4. Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$, and $\mathbf{s}, \mathbf{d} \notin \mathtt{PL}$, we define the LTS $\mathbb{S}_A(\mathcal{F}) = (S, L, \rightarrow)$ where

- the set S of states is the quotient $W/\rightleftharpoons$ of W modulo $\rightleftharpoons$;
- the set L of labels is $\mathbf{2}^{\mathtt{PL}} \cup \{\mathbf{s}, \mathbf{d}\}$;

- the transition relation is the smallest relation on $W \times L \times W$ induced by the following transition rules:

$$\text{(PL)}\ [w]_{\rightleftharpoons} \overset{\mathcal{V}_{\mathcal{F}}^{-1}(\{w\})}{\longrightarrow} [w]_{\rightleftharpoons}$$

$$\text{(Step)}\ \frac{w \preccurlyeq^{\pm} w'}{[w]_{\rightleftharpoons} \overset{\mathbf{s}}{\longrightarrow} [w']_{\rightleftharpoons}} \qquad \text{(Down)}\ \frac{w \succcurlyeq w'}{[w]_{\rightleftharpoons} \overset{\mathbf{d}}{\longrightarrow} [w']_{\rightleftharpoons}}$$

•

The following theorem ensures that any two elements w_1 and w_2 of a finite poset model $\mathcal{F}$ are logically equivalent in $\mathcal{F}$ with respect to $\mathtt{SLCS}_\eta$ if and only if their equivalence classes $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$. The theorem uses the following lemma, proven in Appendix A.15:

Lemma 5.5. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ and $w_1, w_2 \in W$ the following holds: if $w_1 \rightleftharpoons w_2$, then $w_1 \equiv_\eta w_2$.* □

Theorem 5.6. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ be a finite poset model. For all $w_1, w_2 \in W$ it holds that $w_1 \equiv_\eta^{\mathcal{F}} w_2$ if and only if $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$.*

Proof. We first prove that if $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ then $w_1 \equiv_\eta^{\mathcal{F}} w_2$. We proceed by induction on $\mathtt{SLCS}_\eta$ formulas and consider only the case $\eta(\Phi_1, \Phi_2)$, since the other cases are straightforward. Suppose $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ and $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$. Since $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$, there is (a $\pm$-path, and so, by Proposition 3.11) a $\downarrow$-path π_1 from w_1 of some length $\ell_1 \geqslant 1$ such that $\mathcal{F}, \pi_1(\ell_1) \models \Phi_2$ and $\mathcal{F}, \pi_1(i) \models \Phi_1$ for all $i \in [0; \ell_1)$. At this point, we use induction on ℓ_1, together with structural induction on the formulas, for showing that also $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$ holds.

Base case: $\ell_1 = 1$.
In this case we have $\mathcal{F}, w_1 \models \Phi_1$ and $\mathcal{F}, \pi_1(1) \models \Phi_2$, with $w_1 \succcurlyeq \pi_1(1)$. Moreover, by the Induction Hypothesis on formulas, we also have $\mathcal{F}, w_2 \models \Phi_1$. In addition, by Rule (Down), we get $[w_1]_{\rightleftharpoons} \overset{\mathbf{d}}{\longrightarrow} [\pi_1(1)]_{\rightleftharpoons}$. Since $[w_1]_{\rightleftharpoons} \sim [w_2]_{\rightleftharpoons}$ by hypothesis, we also get $[w_2]_{\rightleftharpoons} \overset{\mathbf{d}}{\longrightarrow} [w_2']_{\rightleftharpoons}$, for some $[w_2']_{\rightleftharpoons}$ with $[w_2']_{\rightleftharpoons} \sim [\pi_1(1)]_{\rightleftharpoons}$. Note that, by definition of $\rightleftharpoons$ and since $[w_2]_{\rightleftharpoons} \overset{\mathbf{d}}{\longrightarrow} [w_2']_{\rightleftharpoons}$, there is a path π_2' from w_2 of some length ℓ_2' such that $\pi_2'(j) \rightleftharpoons w_2$ for all $j \in [0; \ell_2']$ and $\pi_2'(\ell_2') \succcurlyeq w_2''$, with $w_2'' \in [w_2']_{\rightleftharpoons}$. Recalling that $\mathcal{F}, w_2 \models \Phi_1$, by Lemma 5.5, we also get that $\mathcal{F}, \pi_2'(j) \models \Phi_1$ for all $j \in [0; \ell_2']$. Recalling also that $\mathcal{F}, \pi_1(1) \models \Phi_2$, again by the Induction Hypothesis on formulas, from $[w_2']_{\rightleftharpoons} \sim [\pi_1(1)]_{\rightleftharpoons}$, we get $\mathcal{F}, w_2' \models \Phi_2$ and, by Lemma 5.5, we also get $\mathcal{F}, w_2'' \models \Phi_2$. Consider now path $\pi_2 : [0; \ell_2' + 1] \to W$ defined as follows:

$$\pi_2(j) = \begin{cases} \pi_2'(j) & \text{if } j \in [0; \ell_2'], \\ w_2'' & \text{if } j = \ell_2' + 1. \end{cases}$$

Clearly, π_2 is a $\downarrow$-path from w_2 since π_2' is an undirected path and $\pi_2(\ell_2') \succcurlyeq \pi_2(\ell_2' + 1)$. Furthermore, we have shown above that $\mathcal{F}, \pi_2(\ell_2' + 1) \models \Phi_2$ and $\mathcal{F}, \pi_2(j) \models \Phi_1$ for all $j \in [0; \ell_2' + 1)$.

Thus, we have that $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$, witnessed by π_2.

Induction step: We assume the assertion holds for $\ell_1 = n$, for $n \geqslant 1$ and we show it holds

for $\ell_1 = n+1$.
Since $w_1 \preccurlyeq^{\pm} \pi_1(1)$, by Rule (Step), we have that $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [\pi_1(1)]_{\rightleftharpoons}$, and since, by hypothesis, $[w_1]_{\rightleftharpoons} \sim [w_2]_{\rightleftharpoons}$, we also know that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$ for some w_2' such that $[w_2']_{\rightleftharpoons} \sim [\pi_1(1)]_{\rightleftharpoons}$. Furthermore, $\mathcal{F}, \pi_1(1) \models \eta(\Phi_1, \Phi_2)$ since $\ell_1 \geqslant 2$ and that this is witnessed by $\pi_1 \uparrow 1$, which is a $\downarrow$-path of length n. Thus, by the Induction Hypothesis on ℓ_1, we get that $\mathcal{F}, w_2' \models \eta(\Phi_1, \Phi_2)$ since $[w_2']_{\rightleftharpoons} \sim [\pi_1(1)]_{\rightleftharpoons}$ (see above). From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$, by Rule (Step), we know that $w \in [w_2]_{\rightleftharpoons}$ and $w' \in [w_2']_{\rightleftharpoons}$ exist such that $w \preccurlyeq^{\pm} w'$. Since $w \in [w_2]_{\rightleftharpoons}$ an undirected path π_2' exists from w_2 to w, of some length ℓ_2', such that $\pi_2'(j) \rightleftharpoons w_2$ for all $j \in [0; \ell_2']$. By the Induction Hypothesis on formulas, we know that $\mathcal{F}, w_2 \models \Phi_1$, and so, by Lemma 5.5, we get also $\mathcal{F}, \pi_2'(j) \models \Phi_1$ for all $j \in [0; \ell_2']$. Moreover, since $\mathcal{F}, w_2' \models \eta(\Phi_1, \Phi_2)$ (see above) and $w' \rightleftharpoons w_2'$, again by Lemma 5.5, we get $\mathcal{F}, w' \models \eta(\Phi_1, \Phi_2)$. This means that there is a $\pm$-path π_2'' from w' of some length ℓ_2'' witnessing $\mathcal{F}, w' \models \eta(\Phi_1, \Phi_2)$. Define π_2 as follows: $\pi_2' \cdot (w, w') \cdot \pi_2''$. It is easy to see that π_2 is a $\downarrow$-path witnessing $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$.

Now we prove that if $w_1 \equiv_\eta^{\mathcal{F}} w_2$ then $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$. We do this by showing that the following binary relation B on W is a strong bisimulation:

$$B = \{(s_1, s_2) \in S \times S \,|\, \text{there are } w_1 \in s_1,\, w_2 \in s_2 \text{ such that } w_1 \equiv_\eta w_2\}.$$

Let, without loss of generality, $s_1 = [w_1]_{\rightleftharpoons}$ and $s_2 = [w_2]_{\rightleftharpoons}$, for some $w_1, w_2 \in W$ with $w_1 \equiv_\eta w_2$ and suppose $B([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$, with $w_1 \equiv_\eta w_2$. We distinguish three cases:

Case A: $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ with $\alpha \in \mathbf{2}^{\mathsf{PL}}$.
If $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ for some $\alpha \in \mathbf{2}^{\mathsf{PL}}$ and $w_1' \in W$, then, by Rule (PL), we know that $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$. Furthermore, since $w_1 \equiv_\eta w_2$, we also know that $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \alpha$. In addition, again by Rule (PL), we get that $[w_2]_{\rightleftharpoons} \xrightarrow{\alpha} [w_2]_{\rightleftharpoons}$ and, by hypothesis $B([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.

Case B: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Down) there are $w \in [w_1]_{\rightleftharpoons}$ and $w' \in [w_1']_{\rightleftharpoons}$ such that $w \succcurlyeq w'$. Note that (w, w') is a $\downarrow$-path witnessing $\mathcal{F}, w \models \eta(\chi(w), \chi(w'))$, where χ is as in Definition 3.13 on page 16. Since $w \rightleftharpoons w_1$, we have that $\mathcal{F}, w_1 \models \eta(\chi(w), \chi(w'))$ holds, by Lemma 5.5. Moreover, since, by hypothesis, $w_1 \equiv_\eta w_2$, we also have $\mathcal{F}, w_2 \models \eta(\chi(w), \chi(w'))$. Then a $\pm$-path $\pi : [0; \ell] \to W$ exists from w_2 such that $\mathcal{F}, \pi(\ell) \models \chi(w')$ and $\mathcal{F}, \pi(j) \models \chi(w)$ for all $j \in [0; \ell)$. This in turn, by Proposition 3.14, means that $\pi(\ell) \equiv_\eta w'$ and $\pi(j) \equiv_\eta w$ for all $j \in [0; \ell)$. By Lemma 5.5, since $w' \rightleftharpoons w_1'$, we get $w' \equiv_\eta w_1'$, and by transitivity, since $\pi(\ell) \equiv_\eta w'$ (see above), we also have $\pi(\ell) \equiv_\eta w_1'$. Similarly, we get $\pi(j) \equiv_\eta w \equiv_\eta w_1$, which implies $\mathcal{V}_{\mathcal{F}}^{-1}(\{\pi(j)\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$, for all $j \in [0; \ell)$. Recall that $w_1 \equiv_\eta w_2$, which implies $\mathcal{V}_{\mathcal{F}}^{-1}(w_2) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$ and so we get also $\mathcal{V}_{\mathcal{F}}^{-1}(\{\pi(j)\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$, for all $j \in [0; \ell)$. In addition, for all $j \in [0; \ell)$ we have that $\pi|[0; j]$ connects $\pi(0) = w_2$ to $\pi(j)$. This means that, for all $j \in [0; \ell)$, $\pi(j) \in [w_2]_{\rightleftharpoons} = [\pi(\ell-1)]_{\rightleftharpoons}$ and since $\pi(\ell-1) \succcurlyeq \pi(\ell)$, by Rule (Down) we deduce $[\pi(\ell-1)]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [\pi(\ell)]_{\rightleftharpoons}$, that is $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [\pi(\ell)]_{\rightleftharpoons}$. Recall that $\pi(\ell) \equiv_\eta w_1'$, so that, by definition of relation B, we finally get $B([w_1']_{\rightleftharpoons}, [\pi(\ell)]_{\rightleftharpoons})$.

Case C: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$
Suppose, finally, that $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$. We distinguish two cases:

Case C1: $w_1' \in [w_1]_{\rightleftharpoons}$. In this case, by Lemma 5.5, we have also $w_1' \equiv_\eta w_1$. Furthermore, $w_1 \equiv_\eta w_2$ by hypothesis, thus we get $w_1' \equiv_\eta w_2$. But then, since $w_2 \preccurlyeq^{\pm} w_2$, by Rule (Step), we know that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2]_{\rightleftharpoons}$ and since $w_1' \equiv_\eta w_2$, by definition of relation B, we finally get $B([w_1']_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.
Case C2: $w_1' \notin [w_1]_{\rightleftharpoons}$. We know there are $w \in [w_1]_{\rightleftharpoons}$ and $w' \in [w_1']_{\rightleftharpoons}$ such that $w \preccurlyeq^{\pm} w'$. Since $w \rightleftharpoons w_1$, then $\mathcal{V}_{\mathcal{F}}^{-1}(\{w\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$ and since $w' \rightleftharpoons w_1'$, then $\mathcal{V}_{\mathcal{F}}^{-1}(\{w'\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$. Furthermore, since $w \preccurlyeq^{\pm} w'$, there is path (w, w') connecting w with w'. So there is a path connecting w_1 to w_1' and if $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$ would hold, it could not be that $w_1' \notin [w_1]_{\rightleftharpoons}$. Consequently, it must be $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$, which in turn implies $w_1 \not\equiv_\eta w_1'$. We note that the following holds:

$$\mathcal{F}, w_1 \models \eta(\chi(w_1), \eta(\chi(w_1) \vee \chi(w_1'), \chi(w_1')))$$

and, since $w_1 \equiv_\eta w_2$ we also have

$$\mathcal{F}, w_2 \models \eta(\chi(w_1), \eta(\chi(w_1) \vee \chi(w_1'), \chi(w_1'))).$$

Let π be a $\pm$-path from w_2 witnessing the above formula and let k be the first index such that $\mathcal{F}, \pi(k) \models \chi(w_1')$. We have that, for all $j \in [0; k)$, $\mathcal{F}, \pi(j) \models \chi(w_1)$ and $\pi|[0; j]$ connects $\pi(0) = w_2$ to $\pi(j)$. Furthermore, for all such j, we have $\pi(j) \equiv_\eta w_1$, by Proposition 3.14, which entails $\mathcal{V}_{\mathcal{F}}^{-1}(\{\pi(j)\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$. Thus $\pi(j) \in [w_2]_{\rightleftharpoons}$ for all $j \in [0; k)$ and since $\pi(k-1) \preccurlyeq^{\pm} \pi(k)$ we have, by Rule (Step) $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [\pi(k)]_{\rightleftharpoons}$. Finally, recalling that, again by Proposition 3.14, $w_1' \equiv_\eta \pi(k)$, we get $B([w_1']_{\rightleftharpoons}, [\pi(k)]_{\rightleftharpoons})$. □

The following theorem ensures that $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ are strongly bisimilar in $\mathbb{S}_A(\mathcal{F})$ if and only if w_1 and w_2 are branching bisimilar in $\mathbb{S}_C(\mathcal{F})$. The theorem uses the following lemma, proven in Appendix A.16:

Lemma 5.7. *Consider a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$. Then for all $w_1, w_2 \in W$ the following holds: if $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, then $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$.* □

Theorem 5.8. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ be a finite poset model. For all $w_1, w_2 \in W$ it holds that $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ if and only if $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$.*

Proof. We first prove that if $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ then $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$. We show that the following relation is a branching bisimulation:

$$B_C = \{(w_1, w_2) \in W \times W \mid [w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}\}.$$

Let us assume $B_C(w_1, w_2)$. We have to consider a few cases:

Case A: $w_1 \xrightarrow{p} w_1$.
If $w_1 \xrightarrow{p} w_1$, then, by Rule (PLC), we have $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$. By definition of B_C and by hypothesis we know that $[w_1]_{\rightleftharpoons} \sim [w_2]_{\rightleftharpoons}$ and so, by Lemma 5.7, we get $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$. It follows then that $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$ and, again by Rule (PLC), we finally get $w_2 \xrightarrow{p} w_2$, which is the required mimicking step since $B(w_1, w_2)$.

Case B: $w_1 \xrightarrow{\tau} w_1'$.
If $w_1 \xrightarrow{\tau} w_1'$ for some $w_1' \in W$, then, by Rule (TAU), we know that $w_1 \preccurlyeq^{\pm} w_1'$, with $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$, which, by definition of $\rightleftharpoons$, means $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and since $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ by definition of B_C, given that $B_C(w_1, w_2)$, we get $[w_1']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$.

This, in turn, again by definition of B_C, means $B_C(w_1', w_2)$.

Case C: $w_1 \xrightarrow{\mathbf{c}} w_1'$.
If $w_1 \xrightarrow{\mathbf{c}} w_1'$ for some $w_1' \in W$, then, by Rule (CNG), we know that $w_1 \preccurlyeq^{\pm} w_1'$, with $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$, and, by Rule (Step), we have $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$. Since, by definition of B_C and by hypothesis, $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, we also have $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$ for some $[w_2']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$. From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2']_{\rightleftharpoons}$, by Rule (Step), we know there are $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$ such that $w_3 \preccurlyeq^{\pm} w_3'$. By Lemma 5.7, since $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$ by hypothesis and $[w_1']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2']_{\rightleftharpoons}$ (see above), we have $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$ and $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2'\})$ and since $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\})$ (see above), we get $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1'\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2'\})$. Consequently, since $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$, we also finally get that $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_3\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_3'\})$. Thus, by rule (CNG), we know that $w_3 \xrightarrow{\mathbf{c}} w_3'$. Now, since $w_3 \in [w_2]_{\rightleftharpoons}$, by definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know there are $s_0, \ldots s_n \in W$ with $s_0 = w_2$, $s_n = w_3$ such that $s_i \xrightarrow{\tau} s_{i+1}$ and $s_{i+1} \xrightarrow{\tau} s_i$, for all $i \in [0; n)$. We note that $B_C(w_1, s_i)$ for all $i \in [0; n]$. In fact for each $i \in [0; n]$ we have that $[s_i]_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ by definition of $\rightleftharpoons$ and we also know that $[w_2]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, since $B_C(w_1, w_2)$ by hypothesis. Thus we get $[s_i]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, i.e. $B_C(w_1, s_i)$. Furthermore, we also note that $B_C(w_1', w_3')$. In fact $[w_3']_{\rightleftharpoons} = [w_2']_{\rightleftharpoons}$, since $w_3' \in [w_2']_{\rightleftharpoons}$. Furthermore, $[w_2']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$ (see above). So, we get $[w_3']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$, i.e. $B_C(w_1', w_3')$. In conclusion, we have that if $w_1 \xrightarrow{\mathbf{c}} w_1'$ for some $w_1' \in W$, then $w_2 = s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} s_n = w_3 \xrightarrow{\mathbf{c}} w_3'$ with $B_C(w_1', w_3')$ and $B_C(w_1, s_i)$ for all $i \in [0; n]$.

Case D: $w_1 \xrightarrow{\mathbf{d}} w_1'$.
If $w_1 \xrightarrow{\mathbf{d}} w_1'$ for some $w_1' \in W$, then, by Rule (DWN), we know that $w_1 \succcurlyeq w_1'$, and, by Rule (Down), we have $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$. Since, by definition of B_C and by hypothesis, $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, we also have $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$ for some $[w_2']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$. From $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_2']_{\rightleftharpoons}$, by Rule (Down), we know there are $w_3 \in [w_2]_{\rightleftharpoons}$ and $w_3' \in [w_2']_{\rightleftharpoons}$ such that $w_3 \succcurlyeq w_3'$ and, by Rule (DWN) we know that $w_3 \xrightarrow{\mathbf{d}} w_3'$. Now, since $w_3 \in [w_2]_{\rightleftharpoons}$, by definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know there are $s_0, \ldots s_n \in W$ with $s_0 = w_2$, $s_n = w_3$ such that $s_i \xrightarrow{\tau} s_{i+1}$ and $s_{i+1} \xrightarrow{\tau} s_i$, for all $i \in [0; n)$. We note that $B_C(w_1, s_i)$ for all $i \in [0; n]$. In fact for each $i \in [0; n]$ we have that $[s_i]_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ by definition of $\rightleftharpoons$ and we also know that $[w_2]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, since $B_C(w_1, w_2)$ by hypothesis. Thus we get $[s_i]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1]_{\rightleftharpoons}$, i.e. $B_C(w_1, s_i)$. Furthermore, we also note that $B_C(w_1', w_3')$. In fact $[w_3']_{\rightleftharpoons} = [w_2']_{\rightleftharpoons}$, since $w_3' \in [w_2']_{\rightleftharpoons}$. In addition, $[w_2']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$ (see above). So, we get $[w_3']_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_1']_{\rightleftharpoons}$, i.e. $B_C(w_1', w_3')$. In conclusion, we have that if $w_1 \xrightarrow{\mathbf{d}} w_1'$ for some $w_1' \in W$, then $w_2 = s_0 \xrightarrow{\tau} s_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} s_n = w_3 \xrightarrow{\mathbf{d}} w_3'$ with $B_C(w_1', w_3')$ and $B_C(w_1, s_i)$ for all $i \in [0; n]$.

We now prove that if $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, then $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$. We show that the following relation is a strong bisimulation:

$$B_A = \{(s_1, s_2) \in S \times S \,|\, \text{there are } w_1 \in s_1,\, w_2 \in s_2 \text{ such that } w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2\}.$$

Let, without loss of generality, $s_1 = [w_1]_{\rightleftharpoons}$ and $s_2 = [w_2]_{\rightleftharpoons}$ for some $w_1, w_2 \in W$ with $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, and suppose $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$. We distinguish three cases:

Case A: $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ with $\alpha \in \mathbf{2}^{\mathsf{PL}}$:
By Rule (PL), if $[w_1]_{\rightleftharpoons} \xrightarrow{\alpha} [w_1']_{\rightleftharpoons}$ for $\alpha \in \mathbf{2}^{\mathsf{PL}}$ and $w_1' \in W$, then $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and $\alpha = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})$. On the one hand, if $p \in \alpha$ then $w_1 \xrightarrow{p} w_1$ by rule (PLC). Since $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$ it follows that $w_2 \xrightarrow{\tau} \ldots \xrightarrow{\tau} \bar{w}_2 \xrightarrow{p} w_2'$ for $\bar{w}_2, w_2' \in W$ such that $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{\bar{w}_2\})$, $\bar{w}_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$, and $w_2' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_1$. By rule (TAU), $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$. Thus, $\alpha \subseteq \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$. On the other hand, if $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$ then $w_2 \xrightarrow{p} w_2$ by rule (PLC). Since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ we have that $w_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} \bar{w}_1 \xrightarrow{p} w_1'$ for $\bar{w}_1, w_1' \in W$ such that $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{\bar{w}_1\})$, $\bar{w}_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, $w_1' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$. By rule (TAU) we obtain that $p \in \mathcal{V}_{\mathcal{F}}^{-1}(\{\bar{w}_1\})$. Thus, $p \in \alpha$. Hence, $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\}) \subseteq \alpha$. So, $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\}) = \alpha$. Therefore, $[w_2]_{\rightleftharpoons} \xrightarrow{\alpha} [w_2]_{\rightleftharpoons}$ by rule (PL). By assumption, $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$ for target states $[w_1]_{\rightleftharpoons}$ and $[w_2]_{\rightleftharpoons}$ as required.

Case B: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Down), we know that there are $w_3 \in [w_1]_{\rightleftharpoons}$ and $w_3' \in [w_1']_{\rightleftharpoons}$ such that $w_3 \succcurlyeq w_3'$. This implies, by Rule (DWN), that $w_3 \xrightarrow{\mathbf{d}} w_3'$. By definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know that there are $m \geqslant 0$ and $t_0, \ldots, t_m \in W$ with $t_0 = w_1$, $t_m = w_3$ such that $t_i \xrightarrow{\tau} t_{i+1}$ and $t_{i+1} \xrightarrow{\tau} t_i$, for all $i \in [0; m)$. This implies that $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, and consequently $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ by hypothesis. Furthermore, since $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, there are $n \geqslant 0$ and $v_0, \ldots, v_n, v_{n+1} \in W$ with $w_2 = v_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} v_n \xrightarrow{\mathbf{d}} v_{n+1}$, such that $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ and $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_i$ for all $i \in [0; n]$. Moreover, by Rule (DWN), we have $v_n \succcurlyeq v_{n+1}$ which imples, by Rule (Down), that $[v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [v_{n+1}]_{\rightleftharpoons}$. Note that, by construction of $\mathbb{S}_C(\mathcal{F})$ we also have $\mathcal{V}_{\mathcal{F}}^{-1}(w_2) = \mathcal{V}_{\mathcal{F}}^{-1}(v_0) = \ldots = \mathcal{V}_{\mathcal{F}}^{-1}(v_n)$ and so $[v_i] = [w_2]_{\rightleftharpoons}$ for all $i \in [0; n]$. Thus, $[w_2]_{\rightleftharpoons} = [v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{d}} [v_{n+1}]_{\rightleftharpoons}$. Furthermore, $B_A([w_3']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$ holds, since $w_3' \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ (see above) and, recalling that $[w_3']_{\rightleftharpoons} = [w_1']_{\rightleftharpoons}$, we also know that $B_A([w_1']_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$.

Case C: $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$
If $[w_1]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_1']_{\rightleftharpoons}$ for some $w_1' \in W$, then, by Rule (Step), we know that there are $w_3 \in [w_1]_{\rightleftharpoons}$ and $w_3' \in [w_1']_{\rightleftharpoons}$ such that $w_3 \preccurlyeq^{\pm} w_3'$. We distinguish two cases:
Case C1: $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_3\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_3'\})$.
If $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_3\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_3'\})$, then, by Rule (TAU), we know $w_3 \xrightarrow{\tau} w_3'$. But then, by definition of $\rightleftharpoons$, we get $[w_3]_{\rightleftharpoons} = [w_3']_{\rightleftharpoons}$ and since $[w_3]_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$ and $[w_3']_{\rightleftharpoons} = [w_1']_{\rightleftharpoons}$ (see above), we get $[w_1']_{\rightleftharpoons} = [w_1]_{\rightleftharpoons}$. On the other hand, since, trivially, $w_2 \preccurlyeq^{\pm} w_2$, by Rule (Step), we also get that $[w_2]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [w_2]_{\rightleftharpoons}$. Moreover, since by hypothesis, we also have $B_A([w_1]_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$, we finally get that also $B_A([w_1']_{\rightleftharpoons}, [w_2]_{\rightleftharpoons})$.

Case C2: $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_3\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w'_3\})$.
If $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_3\}) \neq \mathcal{V}_{\mathcal{F}}^{-1}(\{w'_3\})$, then, by Rule (CNG), we know $w_3 \xrightarrow{\mathbf{c}} w'_3$. By definition of $\rightleftharpoons$ and by construction of $\mathbb{S}_C(\mathcal{F})$ we know that there are $m \geqslant 0$ and $t_0, \ldots, t_m \in W$ with $t_0 = w_1$, $t_m = w_3$ such that $t_i \xrightarrow{\tau} t_{i+1}$ and $t_{i+1} \xrightarrow{\tau} t_i$, for all $i \in [0; m)$. This implies that $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, and consequently $w_2 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_3$, since $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$ by hypothesis. Furthermore, since $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$, there are $n \geqslant 0$ and $v_0, \ldots, v_n, v_{n+1} \in W$ with $w_2 = v_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} v_n \xrightarrow{\mathbf{c}} v_{n+1}$, such that $w'_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ and $w_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_i$ for all $i \in [0; n]$. Moreover, by Rule (CNG), we have $v_n \preccurlyeq^{\pm} v_{n+1}$ which imples, by Rule (Step), that $[v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [v_{n+1}]_{\rightleftharpoons}$. Note that, by construction of $\mathbb{S}_C(\mathcal{F})$ we also have $\mathcal{V}_{\mathcal{F}}^{-1}(w_2) = \mathcal{V}_{\mathcal{F}}^{-1}(v_0) = \ldots = \mathcal{V}_{\mathcal{F}}^{-1}(v_n)$ and so $[v_i] = [w_2]_{\rightleftharpoons}$ for all $i \in [0; n]$. Thus, $[w_2]_{\rightleftharpoons} = [v_n]_{\rightleftharpoons} \xrightarrow{\mathbf{s}} [v_{n+1}]_{\rightleftharpoons}$. Furthermore, $B_A([w'_3]_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$ holds, since $w'_3 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} v_{n+1}$ (see above) and, recalling that $[w'_3]_{\rightleftharpoons} = [w'_1]_{\rightleftharpoons}$, we also know that $B_A([w'_1]_{\rightleftharpoons}, [v_{n+1}]_{\rightleftharpoons})$. ☐

From Theorems 5.6 and 5.8 we finally obtain our claim:

Corollary 5.9. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ be a finite poset model. For all $w_1, w_2 \in W$ the following holds: $w_1 \equiv_\eta^{\mathcal{F}} w_2$ if and only if $w_1 \leftrightarroweq_b^{\mathbb{S}_C(\mathcal{F})} w_2$.* ☐

Now that we have characterised logical equivalence $\equiv_\eta$ for $\mathtt{SLCS}_\eta$ for the elements of a finite poset model $\mathcal{F}$ in terms of branching bisimilarity $\leftrightarroweq_b$ for the LTS $\mathbb{S}_C(\mathcal{F})$, we can compute the minimal LTS modulo branching bisimilarity with standard techniques available, such as branching bisimilarity minimisation provided by the `mCRL2` toolset.

5.2. **Building the Minimal Model.** Via the correspondence of $\mathtt{SLCS}_\eta$ logical equivalence for a poset model and branching bisimilarity of its encoding, one can obtain the equivalence classes of $\equiv_\eta$ by identifying the branching bisimilar states in the LTS. With the equivalence classes modulo $\equiv_\eta$ for the poset model available, we can consider the ensued quotient model. We obtain a Kripke model that is minimal with respect to $\equiv_\eta$, but which is not necessarily a poset model.

Definition 5.10 ($\mathcal{F}_{\min}$)**.** For a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ let the Kripke model $\mathcal{F}_{\min} = (W_{\min}, R_{\min}, \mathcal{V}_{\mathcal{F}_{\min}})$ have

- set of nodes $W_{\min} = W/{\equiv_\eta}$, the equivalence classes of W with respect to $\equiv_\eta$,
- accessibility relation $R_{\min} \subseteq W_{\min} \times W_{\min}$ satisfying
$$R_{\min}([w_1], [w_2]) \text{ if and only if } w'_1 \preccurlyeq w'_2 \text{ for some } w'_1 \equiv_\eta w_1 \text{ and } w'_2 \equiv_\eta w_2$$
for $w_1, w_2 \in W$, and
- valuation $\mathcal{V}_{\mathcal{F}_{\min}} : \mathtt{PL} \to \mathbf{2}^{W_{\min}}$ such that
$$\mathcal{V}_{\mathcal{F}_{\min}}(p) = \{\, [w] \in W_{\min} \mid w' \in \mathcal{V}_{\mathcal{F}}(p) \text{ for some } w' \equiv_\eta w \,\}$$
for $p \in \mathtt{PL}$. •

Clearly, $\mathcal{F}_{\min}$ is a finite reflexive Kripke model. Reflexivity of the accessibility relation $R_{\min}$ is immediate from reflexivity of the ordering $\preccurlyeq$. Furthermore, it is minimal with respect to $\mathtt{SLCS}_\eta$ by definition of $\equiv_\eta$ and $W/\equiv_\eta$. An example of the minimal Kripke model of the polyhedral model in Figure 8a is shown in Figure 8c. The following theorem ensures that the model defined above is sound and complete with respect to the logic, so that the minimisation procedure is correct.

Theorem 5.11. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ let $\mathcal{F}_{\min}$ be defined as in Definition 5.10. Then, for each $w \in W$ and $\mathtt{SLCS}_\eta$ formula Φ the following holds: $\mathcal{F}, w \models \Phi$ if and only if $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$.*

Proof. We first prove that $\mathcal{F}, w \models \Phi$ implies $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$. We proceed by induction on the structure of Φ and we show the proof only for $\Phi = \eta(\Phi_1, \Phi_2)$ the other cases being straightforward. Suppose $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$. This means there is a $\pm$-path π of some length $\ell \geqslant 2$ such that $\pi(0) = w$, $\mathcal{F}, \pi(\ell) \models \Phi_2$, and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0;\ell)$. Now define $\pi_{\min} : [0;\ell] \to W_{\min}$ with $\pi_{\min}(i) = [\pi(i)]$ for all $i \in [0;\ell]$. We show that $\pi_{\min}$ is a $\pm$-path with respect to $R_{\min}$. We have that $R_{\min}(\pi_{\min}(0), \pi_{\min}(1))$ by definition of $R_{\min}$ because $\pi(0) \in [\pi(0)] = \pi_{\min}(0)$, $\pi(1) \in [\pi(1)] = \pi_{\min}(1)$ and $\pi(0) \preccurlyeq \pi(1)$ by assumption. Similarly, we have that $R^-_{\min}(\pi_{\min}(\ell-1), \pi_{\min}(\ell))$ and also that $R^\pm_{\min}(\pi_{\min}(i), \pi_{\min}(i+1))$ for all $i \in (0;\ell-1)$. Furthermore, since $\mathcal{F}, \pi(\ell) \models \Phi_2$, by the Induction Hypothesis, we have that $\mathcal{F}_{\min}, \pi_{\min}(\ell) \models \Phi_2$. Similarly, we have that $\mathcal{F}_{\min}, \pi_{\min}(i) \models \Phi_1$ for all $i \in [0;\ell)$ since $\mathcal{F}, \pi(i) \models \Phi_1$. So $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \eta(\Phi_1, \Phi_2)$.

Now we prove that $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \Phi$ implies $\mathcal{F}, w \models \Phi$. Also in this case we proceed by induction on the structure of Φ and we show the proof only for $\Phi = \eta(\Phi_1, \Phi_2)$. Suppose $\mathcal{F}_{\min}, [w]_{\equiv_\eta} \models \eta(\Phi_1, \Phi_2)$. Hence there is a $\pm$-path $\pi_{\min}$ such that $\pi_{\min}(0) = [w]_{\equiv_\eta}$, $\mathcal{F}_{\min}, \pi(\ell_{\min}) \models \Phi_2$, and $\mathcal{F}_{\min}, \pi_{\min}(i) \models \Phi_1$ for all $i \in [0;\ell_{\min})$. Since $R_{\min}$ is reflexive, using Lemma 2.2, we know that there is also an $\Updownarrow$-path $\hat{\pi}_{\min}$ from $[w]_{\equiv_\eta}$ of some length $2k$, for $k \geqslant 1$, with the same starting-/ending points and the same intermediate points as $\pi_{\min}$ and that obviously witnesses $\eta(\Phi_1, \Phi_2)$ for $[w]_{\equiv_\eta}$. By induction on k, in the sequel, we show that there is a $\pm$-path π from w witnessing $\eta(\Phi_1, \Phi_2)$.

Base case: $k = 1$.
In this case, we have that $\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$, $\mathcal{F}_{\min}, \hat{\pi}_{\min}(0) \models \Phi_1$ $\mathcal{F}_{\min}, \hat{\pi}_{\min}(1) \models \Phi_1$, and $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \Phi_2$. Furthermore, since $\hat{\pi}_{\min}$ is an $\Updownarrow$-path with respect to $R_{\min}$, we know that

$$\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}, R_{\min}(\hat{\pi}_{\min}(0), \hat{\pi}_{\min}(1)), R^-_{\min}(\hat{\pi}_{\min}(1), \hat{\pi}_{\min}(2))$$

and, by definition of $R_{\min}$, there are $w_0 \in \hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$, $w_1', w_1'' \in \hat{\pi}_{\min}(1)$, and $w_2 \in \hat{\pi}_{\min}(2)$ such that $w_0 \preccurlyeq w_1'$ and $w_1'' \succcurlyeq w_2$. Moreover, by the Induction Hypothesis with respect to the structure of formulas, we have that $\mathcal{F}, w_0 \models \Phi_1$, $\mathcal{F}, w_1' \models \Phi_1$, $\mathcal{F}, w_1'' \models \Phi_1$, and $\mathcal{F}, w_2 \models \Phi_2$. Note that $\mathcal{F}, w_1'' \models \eta(\Phi_1, \Phi_2)$, witnessed by the following $\pm$-path: (w_1'', w_1'', w_2). But then we have that also $\mathcal{F}, w_1' \models \eta(\Phi_1, \Phi_2)$ holds since $w_1' \equiv_\eta w_1''$, recalling that $w_1', w_1'' \in \hat{\pi}_{\min}(1) \in W/\equiv_\eta$. There is then a $\pm$-path $\pi' : [0;\ell'] \to W$ from w_1' of some length ℓ' such that $\mathcal{F}, \pi'(\ell') \models \Phi_2$ and $\mathcal{F}, \pi'(i) \models \Phi_1$ for all $i \in [0;\ell')$. Furthermore, $w_0 \preccurlyeq w_1'$ by hypothesis and so $\pi = (w_0, w_1') \cdot \pi' : [0;\ell'+1] \to W$ is a $\pm$-path from w_0 witnessing $\mathcal{F}, w_0 \models \eta(\Phi_1, \Phi_2)$. Finally, recalling that $w, w_0 \in \hat{\pi}_{\min}(0) \in W/\equiv_\eta$, we know that $w \equiv_\eta w_0$ and so we have proven the assertion $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$.

Induction step: $k = n+1$ assuming the assertion holds for $k = n$, for $n > 0$.
Since $k > 1$, we know that $\mathcal{F}_{\min}, \hat{\pi}_{\min}(1) \models \Phi_1$ and $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \Phi_1 \wedge \neg\Phi_2$. Furthermore,

$$\hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}, R_{\min}(\hat{\pi}_{\min}(0), \hat{\pi}_{\min}(1)), R^-_{\min}(\hat{\pi}_{\min}(1), \hat{\pi}_{\min}(2))$$

because $\hat{\pi}_{\min}$ is an $\Updownarrow$-path. By definition of $R_{\min}$, there are $w_0 \in \hat{\pi}_{\min}(0) = [w]_{\equiv_\eta}$, $w_1', w_1'' \in \hat{\pi}_{\min}(1)$ and $w_2 \in \hat{\pi}_{\min}(2)$ such that $w_0 \preccurlyeq w_1'$ and $w_1'' \succcurlyeq w_2$. By the Induction Hypothesis with respect to the structure of the formula, we get that $\mathcal{F}, w_0 \models \Phi_1$, $\mathcal{F}, w_1' \models \Phi_1$, $\mathcal{F}, w_1'' \models \Phi_1$, and $\mathcal{F}, w_2 \models \Phi_1 \wedge \neg\Phi_2$. We consider now the $\updownarrow$-path $\hat{\pi}_{\min} \uparrow 2$ from $\hat{\pi}_{\min}(2)$ of length $2n$, noting that it witnesses $\eta(\Phi_1, \Phi_2)$, since so does $\hat{\pi}_{\min}$ and $k > 1$. In other words, we have that $\mathcal{F}_{\min}, \hat{\pi}_{\min}(2) \models \eta(\Phi_1, \Phi_2)$ with $w_2 \in \hat{\pi}_{\min}(2)$. By the Induction Hypothesis with respect to k, we then have that $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$. So there is a $\updownarrow$-path $\pi_2 : [0; \ell_2] \to W$ from w_2 of some length ℓ_2 such that $\mathcal{F}, \pi_2(\ell_2) \models \Phi_2$ and $\mathcal{F}, \pi_2(i) \models \Phi_1$ for $i \in [0; \ell_2)$. Note that $\mathcal{F}, \pi_2(0) \models \Phi_1$ as well, since $\pi_2(0) = w_2$ and $\mathcal{F}, w_2 \models \Phi_1 \wedge \neg\Phi_2$ (see above). Let us consider now the path $\pi'' = (w_1'', w_1'', w_2) \cdot \pi_2$. Such a path is an $\updownarrow$-path since so is π_2, and $w_1'' \succcurlyeq w_2$ by hypothesis. Note that $\updownarrow$-path π'' witnesses $\mathcal{F}, w_1'' \models \eta(\Phi_1, \Phi_2)$. But then we have that also $\mathcal{F}, w_1' \models \eta(\Phi_1, \Phi_2)$ holds since $w_1' \equiv_\eta w_1''$, recalling that $w_1', w_1'' \in \hat{\pi}_{\min}(1) \in W/\equiv_\eta$. Thus, we have that the following holds: $\mathcal{F}, w_1' \models \Phi_1 \wedge \eta(\Phi_1, \Phi_2)$. There is then a $\pm$-path $\pi' : [0; \ell'] \to W$ from w_1' of some length ℓ' such that $\mathcal{F}, \pi'(\ell') \models \Phi_2$ and $\mathcal{F}, \pi'(i) \models \Phi_1$ for all $i \in [0; \ell')$. Furthermore, $w_0 \preccurlyeq w_1'$ by hypothesis and so $\pi = (w_0, w_1') \cdot \pi' : [0; \ell' + 1] \to W$ is a $\pm$-path from w_0 witnessing $\mathcal{F}, w_0 \models \eta(\Phi_1, \Phi_2)$. Finally, recalling that $w, w_0 \in \hat{\pi}_{\min}(0) \in W/\equiv_\eta$, we know that $w \equiv_\eta w_0$ and so we have proven the assertion $\mathcal{F}, w \models \eta(\Phi_1, \Phi_2)$. □

Finally, the following theorem turns out to be useful for simplifying the procedure for the effective construction of $\mathcal{F}_{\min}$:

Theorem 5.12. *For any poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ and $\mathcal{F}_{\min}$ as of Definition 5.10 and for all $\alpha_1, \alpha_2 \in W_{\min}$, it holds that $R_{min}(\alpha_1, \alpha_2)$ if and only if $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of the minimal LTS obtained from $\mathbb{S}_C(\mathcal{F})$ via branching bisimilarity.*

Proof. In the sequel, we let $\mathbb{S}_C(\mathcal{F})/\underline{\leftrightarrow}_b$ denote the minimal LTS obtained from $\mathbb{S}_C(\mathcal{F})$ via branching bisimilarity. First of all, by Corollary 5.9, $W_{\min}$ coincides with the quotient of the set of states W of $\mathbb{S}_C(\mathcal{F})$ modulo branching bisimilarity. Now, suppose that $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of $\mathbb{S}_C(\mathcal{F})/\underline{\leftrightarrow}_b$. By standard construction of the minimal LTS modulo an equivalence on its state set, we know that $w_1 \in \alpha_1$ and $w_2 \in \alpha_2$ exist such that $w_2 \xrightarrow{\mathbf{d}} w_1$ is a transition of $\mathbb{S}_C(\mathcal{F})$. But then, by Rule (DWN), we get that $w_1 \preccurlyeq w_2$ and so, by definition of $\mathcal{F}_{\min}$, we finally get $R_{\min}(\alpha_1, \alpha_2)$. If, on the other hand, $R_{\min}(\alpha_1, \alpha_2)$ holds, then we know that there exist $w_1 \in \alpha_1$ and $w_2 \in \alpha_2$ such that $w_1 \preccurlyeq w_2$, by definition of $\mathcal{F}_{\min}$. But then, by Rule (DWN), we get that $w_2 \xrightarrow{\mathbf{d}} w_1$ is a transition of $\mathbb{S}_C(\mathcal{F})$. Again, by standard construction of the minimal LTS modulo an equivalence on its state set, we know that $\alpha_2 \xrightarrow{\mathbf{d}} \alpha_1$ is a transition of $\mathbb{S}_C(\mathcal{F})/\underline{\leftrightarrow}_b$. □

Remark 5.13. The fact that the minimal model might not be a poset model does not constitute a problem, at any (i.e. theoretical, implementation, user) level. More specifically, at the theoretical level, Theorem 5.11 guarantees that $\texttt{SLCS}_\eta$ interpreted on a finite poset model $\mathcal{F}$ is preserved and reflected by the minimisation result $\mathcal{F}_{\min}$, despite the finite reflexive Kripke model $\mathcal{F}_{\min}$ is not necessarily a poset model. The above, via Theorem 3.20, guarantees that $\texttt{SLCS}_\eta$ is preserved and reflected by the full chain of translations, from the polyhedral model $\mathcal{P}$ to the minimal model $\mathbb{F}(\mathcal{P})_{\min}$ via finite poset $\mathbb{F}(\mathcal{P})$.

In summary, we have:

$$\mathcal{P}, x \models \Phi \quad \text{iff} \quad \mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi \quad \text{iff} \quad \mathbb{F}(\mathcal{P})_{\min}, [\mathbb{F}(x)]_{\equiv_\eta} \models \Phi. \tag{5.1}$$

Taking the first and the last statements of (5.1) above we get the following: a point x of a polyhedral model $\mathcal{P}$, laying in a cell $\widetilde{\sigma}$ of $\mathcal{P}$, satisfies a $\texttt{SLCS}_\eta$ formula Φ in the polyhedral interpretation of Φ on $\mathcal{P}$ if and only if the node of the Kripke model $\mathbb{F}(\mathcal{P})_{\min}$ that (uniquely) represents the equivalence class $[\mathbb{F}(x)]_{\equiv_\eta}$ of $\mathbb{F}(x) = \widetilde{\sigma}$ modulo $\equiv_\eta$ (or, equivalently modulo weak $\pm$-bisimilarity) satisfies Φ in the relational interpretation of Φ on $\mathbb{F}(\mathcal{P})_{\min}$. At the implementation level, an experimental prototype of a variant of `PolyLogicA` has been developed that is capable to deal with general Kripke models and η semantics, as briefly discussed in Section 6 below. At the user level, we observe that the user deals only with the description of the polyhedral model $\mathcal{P}$ and the input formula Φ as input and the (figure showing the) cells satisfying Φ as output of model checking. All the details of the minimisation procedure are hidden to the user. ⋇

6. An Experimental Minimisation Toolchain

In this section we provide a brief overview of an experimental toolchain to study the minimisation procedure for polyhedral models and to illustrate the practical potential of the theory presented in the previous section. The further development and a thorough analysis of the toolchain will be the subject of future work. Figure 9 illustrates the elements of the toolchain that, starting from a polyhedral model in `json` format, produces the set of equivalence classes and the minimal Kripke model. The former may serve as input for the `PolyVisualizer` tool[9] [BCG$^+$22], a polyhedra visualizer, to inspect the results, whereas the latter can be used for spatial model checking. For that purpose, a variant of `PolyLogicA` is required, since minimal models may turn out not to be posets. In particular, they might not be transitive (see the discussion in Example 4.14 and in Section 5). In addition, the variant has to accomodate for the different semantics of the reachability operators γ and η. An experimental prototype of the tool has been developed and it is publicly available.[10] The complexity of the model checking algorithm is linear in the size of the model and the number of sub-formulas to be checked. A fully fledged implementation and efficiency study is left for future work.

The toolchain is also able to map the results obtained on the minimal Kripke model back to the original polyhedral model, because of the direct correspondence between the states of the Kripke model and the equivalence classes.

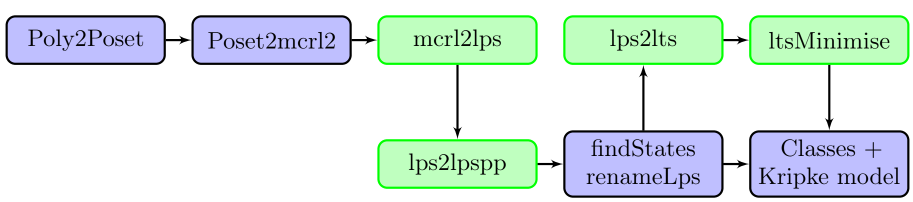


Figure 9. Toolchain for polyhedral model minimisation. Parts in green are command line operations of the `mCRL2` toolset. Parts in blue are developed in Python in the context of the current paper.

The toolchain uses several command line operations provided by the `mCRL2` toolset [BGK$^+$19] (shown in green in Figure 9) and a number of operations developed in the context of this paper (shown in blue in Figure 9). The prototype aims to demonstrate the feasibility of our approach from a qualitative perspective, providing support for examples that illustrate the practical usefulness of the theory. The operation `Poly2Poset` transforms the polyhedral model into a poset model. The operation `Poset2mcrl2` encodes the poset model into a `mCRL2`

[9] http://ggrilletti2.scienceontheweb.net/polyVisualizer/polyVisualizer_static_maze.html

specification of an LTS following the procedure defined in Definition 5.1. The operations `mcrl2lps` and `lps2lts` transform the encoding into a linearised LTS-representation which is then minimised (`ltsMinimise`) via branching bisimulation. The operation `lps2lpspp` provides a textual version of the linear process which is used to obtain the correspondence between internal state labels of the minimised LTS and the cells of the original polyhedral model present in the equivalence classes. The latter, in turn, are essential for the generation of the result files of model checking the minimised model and form the input to the `PolyVisualizer` (together with the original polyhedral model and a colour definition file). Figure 10 and Figure 11 in the next section show an example.[10] Maintaining the relation between internal state labels of the minimised LTS and the original states of the poset and polyhedral model is the most tricky part of the toolchain as such internal state labels are assigned dynamically in the `lps2lts` procedure. This aspect is dealt with by the `findStates` and `renameLps` procedures.

7. Minimisation at Work

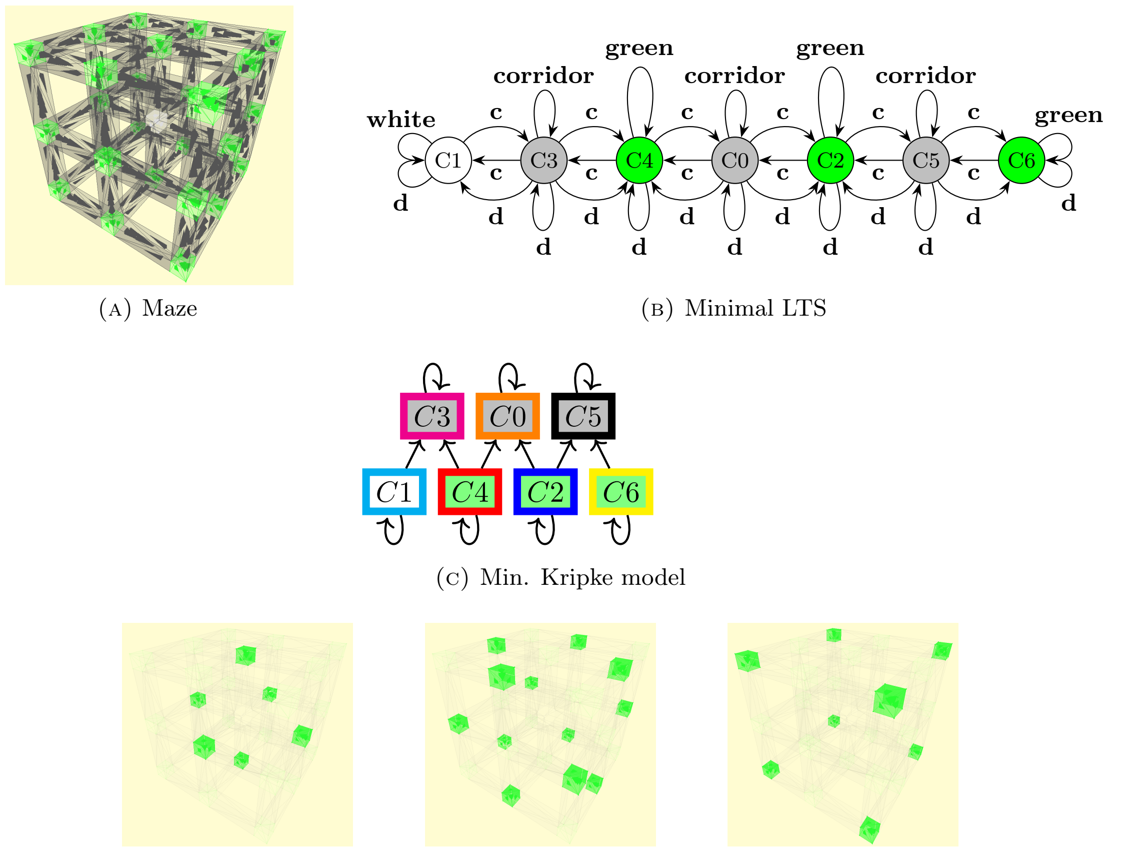
In this section, we show, as a proof of concept, an example of use of the experimental toolchain presented in Section 6. Figure 10a shows a simple symmetric 3D maze composed of one white room in the middle, 26 green rooms, and connecting grey corridors. Like in the previous examples, the cells of the white and green rooms satisfy only predicate letter **white** and **green**, respectively. Those of corridors satisfy only **corridor**. In total, the structure consists of 2,619 cells. We have chosen a symmetric structure on purpose. This makes it easy to interpret the various equivalence classes as nodes of the minimal Kripke model of this structure, shown in Figure 10c. Note the considerable reduction that is obtained: from 2,619 cells to just 7 in the minimal model (observe furthermore that, for this example, the minimal model is also a poset model).

Figure 10b shows the minimal LTS with respect to branching bisimilarity as produced by `mCRL2`.[11] The minimal Kripke model with respect to $\equiv_\eta$ obtained (see Theorem 5.12) from the LTS of Figure 10b is shown in Figure 10c. The Kripke model has seven nodes — of course, in direct correspondence with the seven states of the minimal LTS. Node `C1` represents the class of the cells of the white room and is coloured in white in the figure, three nodes (`C3`, `C0`, and `C5`) correspond to cells of corridors and are coloured in grey, and the other three (`C4`, `C2`, and `C6`) correspond to cells of green rooms, and are coloured in green. Green node `C4` (visualised on the original polyhedron in Figure 10d) represents the class of (the cells of) green rooms that are directly connected to the white room by a corridor. Green node `C2` (visualised in Figure 10e) represents the class of (the cells of) green rooms situated on the edges of the maze. Green node `C6` (visualised in Figure 10f) represents the class of green rooms situated at the corners of the maze.

It is not difficult to find $\mathtt{SLCS}_\eta$ formulas that distinguish the various green classes. For example, the cells in `C4` satisfy $\phi_1 = \eta(\mathbf{green} \vee \eta(\mathbf{corridor}, \mathbf{white}), \mathbf{white})$, whereas no cell in `C2` or `C6` satisfies ϕ_1. To distinguish class `C2` from `C6`, one can observe that cells in `C2` satisfy $\phi_2 = \eta(\mathbf{green} \vee \eta(\mathbf{corridor}, \phi_1), \phi_1)$ whereas those in `C6` do not satisfy ϕ_2. Figure 11

[10] The software and examples are available at `https://github.com/VoxLogicA-Project/Polyhedra-minimisation`.

[11] The numbering of the states is as generated by `mCRL2`.



(A) Maze (B) Minimal LTS

(C) Min. Kripke model

(D) C4 (E) C2 (F) C6

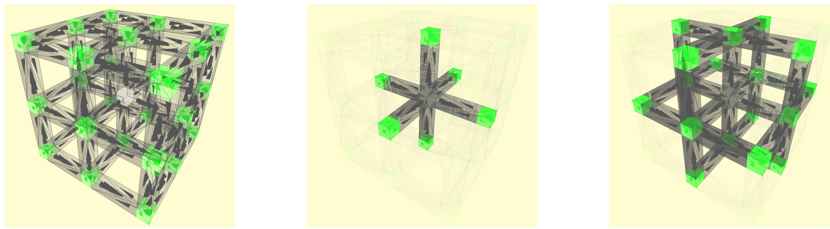
FIGURE 10. A maze with 27 rooms: 26 green and one white in the middle.

shows the result of `PolyLogicA` model checking for the formulas ϕ_1 (see Figure 11b) and ϕ_2 (see Figure 11c).[12]

Table 1 provides a detailed overview regarding the time performance of the various components of the toolchain (see Figure 9) on four models of the maze of different sizes.[13] In each model all green rooms form the outer frame of the maze and white rooms are positioned inside the maze. The table has one separate column for each maze. The first horizontal block shows the number of cells and vertices for the models, as well as the number of the equivalence classes. The names of the components of the toolchain are listed in the first column of the second horizontal block of the table. In the list two additional activities appear, namely, loading of the model (`loadData`) and the production of the equivalence classes and of the minimal Kripke model (`createJsonFiles` and `createModelFile`, respectively). The remaining columns show the computing time of each component, in seconds. The third block

[12]All tests were performed on a workstation equipped with an Intel(R) Core(TM) i9-9900K CPU @ 3.60 GHz (8 cores, 16 threads).

[13]Maze 3x3x3 is shown in Figure 10a, Maze 3x5x3 in Figure 12a (in Appendix B), and Maze 3x5x4 in Figure 3a.



(A) (B) ϕ_1 (C) ϕ_2

FIGURE 11. (11a) The 3D maze. Results of `PolyLogicA` model checking of the formulas ϕ_1 (11b) and ϕ_2 (11c) on the minimised model as they are shown to the user by `PolyVisualizer` — results are mapped back automatically by the procedure onto the full 3D maze.

TABLE 1. Performance for 3D maze example. All times are in seconds.

	Maze 3x3x3	Maze 3x5x3	Maze 3x5x4	Maze 5x5x5
Nr. of classes	7	21	38	21
Nr. of cells	2,619	3,568	6,145	13,375
Nr. of vertices	216	288	480	1,000
`poly2poset`	0.35	0.34	0.43	1.10
`loadData`	0.00	0.00	0.01	0.02
`poset2mcrl2`	0.16	0.30	0.42	0.95
`mcrl2lps`	1.71	3.51	5.42	23.72
`lps2lpspp`	0.24	0.41	0.57	1.95
`findStates`	0.17	0.31	0.41	4.18
`renamelps`	0.54	0.95	1.34	4.47
`lps2lts`	21.41	78.26	135.22	794.33
`ltsMinimise`	0.06	0.23	0.24	0.35
`createJsonFiles`	6.35	51.37	160.53	587.99
`createModelFile`	0.01	0.01	0.01	0.03
Model checking original model	8.76	24.90	64.50	671.30
Model checking minimised model	0.02	0.03	0.03	0.03

shows the model checking times for formulas ϕ_1 and ϕ_2, in the original as well as the minimal models.

Note the substantial reduction in size (several orders of magnitude) of the minimised model, where the number of states corresponds to the number of equivalence classes, compared to the full model (number of cells). This leads to a similar reduction in model checking time (see last two lines of Table 1). Clearly, the time for encoding (`poset2mcrl2`) and minimising (see `ltsMinimise`) the model is very small, whereas there seems to be a bottleneck of computing time needed for the `mCRL2` procedure `lps2lts`. However, the latter step may be avoided by implementing the encoding directly into the binary `mCRL2` LTS format. This requires usage of the `mCRL2 C++` *application programming interface*, and is left to future work.

In summary, the considerable reduction of the models and their relative model checking times are very encouraging, also considering that the minimised model, once obtained, can be used for multiple model checking sessions.

8. Conclusions

Polyhedral models are widely used in domains that exploit mesh processing such as 3D computer graphics. These models are typically huge, consisting of very many cells. Spatial model checking of such models is an interesting, novel approach to verify properties of such models and to visualise the results in a graphically appealing way. In previous work the polyhedral model checker PolyLogicA was developed for this purpose [BCG+22].

In [BCG+22] simplicial bisimilarity was proposed for polyhedral models — i.e. models of continuous space — while ±-bisimilarity, the corresponding equivalence for cell-poset models — discrete representations of polyhedral models — was first introduced in [CGL+23a]. In order to support large model reductions, in this paper the novel notions of weak simplicial bisimilarity and weak ±-bisimilarity have been presented, and the correspondence between the two has been studied. We have also presented $\mathtt{SLCS}_\eta$, a weaker version of the Spatial Logic for Closure Spaces on polyhedral models, and we have shown that simplicial bisimilarity enjoys the Hennessy-Milner property (Theorem 4.9). Furthermore, we have shown that the property holds for ±-bisimilarity on poset models and the interpretation of $\mathtt{SLCS}_\eta$ on such models (Theorem 4.12). $\mathtt{SLCS}_\eta$ can be used in the geometric spatial model checker PolyLogicA for checking spatial reachability properties of polyhedral models. Model checking results can be visualised by projecting them onto the original polyhedral structure, showing in a specific colour all the cells satisfying the property of interest.

In order to reduce model checking time and computing resources, we have proposed an effective procedure that computes the minimal model, modulo logical equivalence with respect to the logic $\mathtt{SLCS}_\eta$, of a polyhedral model. Such minimised models are also amenable to model checking with a variant of PolyLogicA dealing with general Kripke models and with the η modality.

The procedure has been formalised and proven correct. A prototype implementation of the procedure has been developed in the form of a toolchain, that also involves operations provided by the mCRL2 toolset, to study the practical feasibility of the approach and to identify possible bottlenecks. We have also shown how the model checking results of the minimal model can be projected back onto the original polyhedral model. This provides a direct 3D visual inspection of the results through the polyhedra visualizer PolyVisualizer.

In future work we aim at a more sophisticated implementation of the procedure, possibly using in a more direct way the minimisation operations provided by mCRL2 and integrating the various steps in the procedure. Such an implementation, would also enable us to experiment applying our methodology and supporting tools to real-world case studies. On the theoretical side, an interesting issue that is beyond the scope of the present paper, and that we would like to address in future work, is the relationship between $\mathtt{SLCS}_\eta$, $\mathtt{SLCS}_\gamma$, and $\Diamond$. Finally, we would be interested in extending $\mathtt{SLCS}_\eta/\mathtt{SLCS}_\gamma$ with additional operators, for example those concerning notions of distance, and in applying our spatial model checking framework to a larger number of case studies.

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APPENDIX A. DETAILED PROOFS

A.1. **Proof of Lemma 2.2.**

Lemma 2.2. *Given a reflexive Kripke frame (W, R) and a $\pm$-path $\pi : [0;\ell] \to W$, there is a $\Updownarrow$-path $\pi' : [0;\ell'] \to W$, for some ℓ', and a total, surjective, monotonic non-decreasing function $f : [0;\ell'] \to [0;\ell]$ such that $\pi'(j) = \pi(f(j))$ for all $j \in [0;\ell']$.*

Proof. We proceed by induction on the length ℓ of $\pm$-path π.
Base case: $\ell = 2$.
In this case, by definition of $\pm$-path, we have $R(\pi(0), \pi(1))$ and $R^-(\pi(1), \pi(2))$, which, by definition of $\Updownarrow$-path, implies that π itself is an $\Updownarrow$-path and $f : [0;\ell] \to [0;\ell]$ is just the identity function.

Induction step. We assume the assertion holds for all $\pm$-paths of length ℓ and we prove it for $\ell + 1$. Let $\pi : [0;\ell + 1] \to W$ be a $\pm$-path. Then $R^-(\pi(\ell), \pi(\ell + 1))$, since π is a $\pm$-path. We consider the following cases:
Case A: $R^-(\pi(\ell - 1), \pi(\ell))$ and $R^-(\pi(\ell), \pi(\ell + 1))$.
In this case, consider the prefix $\pi_1 = \pi|[0;\ell]$ of π, noting that π_1 is a $\pm$-path of length ℓ. By the Induction Hypothesis there is an $\Updownarrow$-path π'_1 of some length ℓ'_1 and a total, surjective, monotonic non-decreasing function $g : [0;\ell'_1] \to [0;\ell]$ such that $\pi'_1(j) = \pi_1(g(j)) = \pi(g(j))$ for all $j \in [0;\ell'_1]$. Note that $\pi'_1(\ell'_1) = \pi(\ell)$ so that the sequentialisation of π'_1 with the two-element path $(\pi(\ell), \pi(\ell + 1))$ is well-defined. Consider path $\pi' = (\pi'_1 \cdot (\pi(\ell), \pi(\ell + 1))) \leftarrow \ell'_1$, of length $\ell'_1 + 2$ consisting of π'_1 followed by $\pi(\ell)$ followed in turn by $\pi(\ell + 1)$. In other words, $\pi' = (\pi'_1(0) \ldots \pi'_1(\ell'_1), \pi(\ell), \pi(\ell + 1))$, with $\pi'_1(\ell'_1) = \pi(\ell)$ — recall that R is reflexive. It is easy to see that π' is an $\Updownarrow$-path and that function $f : [0;\ell'_1 + 2] \to [0;\ell + 1]$, with $f(j) = g(j)$ for $j \in [0;\ell'_1]$, $f(\ell'_1 + 1) = \ell$ and $f(\ell'_1 + 2) = \ell + 1$, is total, surjective, and monotonic non-decreasing.
Case B: $R(\pi(\ell - 1), \pi(\ell))$ and $R^-(\pi(\ell), \pi(\ell + 1))$.
In this case the prefix $\pi|[0;\ell]$ of π is *not* a $\pm$-path. We then consider the path consisting of prefix $\pi|[0;\ell-1]$ where we add a copy of $\pi(\ell-1)$, i.e. the path $\pi_1 = (\pi|[0;\ell-1]) \leftarrow (\ell-1)$ — we can do that because R is reflexive. Note that π_1 is a $\pm$-path and has length ℓ. By the Induction Hypothesis there is an $\Updownarrow$-path π'_1 of some length ℓ'_1 and a total, surjective, monotonic non-decreasing function $g : [0;\ell'_1] \to [0;\ell]$ such that $\pi'_1(j) = \pi_1(g(j)) = \pi(g(j))$ for all $j \in [0;\ell'_1]$. Consider path $\pi' = \pi'_1 \cdot (\pi(\ell - 1), \pi(\ell), \pi(\ell + 1))$, of length $\ell'_1 + 2$, that is well defined since $\pi'_1(\ell'_1) = \pi(\ell - 1)$ by definition of π_1. In other words, $\pi' = (\pi'_1(0), \ldots, \pi'_1(\ell'_1), \pi(\ell), \pi(\ell + 1))$, with $\pi'_1(\ell'_1) = \pi(\ell-1)$. Path π' is an $\Updownarrow$-path. In fact $\pi'|[0;\ell'_1] = \pi'_1$ is an $\Updownarrow$-path. Furthermore, $\pi'(\ell'_1) = \pi(\ell - 1)$, $R(\pi(\ell - 1), \pi(\ell))$, $R^-(\pi(\ell), \pi(\ell + 1))$ and $\pi(\ell + 1) = \pi'(\ell'_1 + 2)$. Finally, function $f : [0;\ell'_1 + 2] \to [0;\ell + 1]$, with $f(j) = g(j)$ for $j \in [0;\ell'_1]$, $f(\ell'_1 + 1) = \ell$ and $f(\ell'_1 + 2) = \ell + 1$, is total, surjective, and monotonic non-decreasing. ◻

A.2. **Proof of Lemma 2.3.**

Lemma 2.3. *Given a reflexive Kripke frame (W, R) and a $\downarrow$-path $\pi : [0;\ell] \to W$, there is an $\Updownarrow$-path $\pi' : [0;\ell'] \to W$, for some ℓ', and a total, surjective, monotonic non-decreasing function $f : [0;\ell'] \to [0;\ell]$ such that $\pi'(j) = \pi(f(j))$ for all $j \in [0;\ell']$.*

Proof. The proof is carried out by induction on the length ℓ of π.
Base case. $\ell = 1$. Suppose $\ell = 1$, i.e. $\pi : [0;1] \to W$ with $R^-(\pi(0), \pi(1))$. Then let $\pi' : [0;2] \to W$ be such that $\pi'(0) = \pi'(1) = \pi(0)$ and $\pi'(2) = \pi(1)$ — we can do that since R is reflexive — and $f : [0;2] \to [0;1]$ be such that $f(0) = f(1) = 0$ and $f(2) = 1$. Clearly π' is an $\updownarrow$-path and $\pi'(j) = \pi(f(j))$ for all $j \in [0;2]$.
Induction step. We assume the assertion holds for all $\downarrow$-paths of length ℓ and we prove it for $\ell + 1$. Let $\pi : [0; \ell + 1] \to W$ a $\downarrow$-path and suppose the assertion holds for all $\downarrow$-paths of length ℓ. In particular, it holds for $\pi \uparrow 1$, i.e., there is an $\updownarrow$-path π'' of some length ℓ'' with $\pi''(0) = \pi(1)$, and total, monotonic non-decreasing surjection $g : [0; \ell''] \to W$ such that $\pi''(j) = \pi(g(j))$ for all $j \in [0; \ell'']$. Suppose $R(\pi(0), \pi(1))$ does not hold. Then, since R is reflexive, we let $\pi' = (\pi(0), \pi(0), \pi(1)) \cdot \pi''$ and $f : [0; \ell'' + 2] \to [0; \ell + 1]$ with $f(0) = f(1) = 0$ and $f(j) = g(j - 2)$ for all $j \in [2; \ell'' + 2]$. If instead $R(\pi(0), \pi(1))$, then we let $\pi' = (\pi(0), \pi(1), \pi(1)) \cdot \pi''$ and $f : [0; \ell'' + 2] \to [0; \ell + 1]$ with $f(0) = 0, f(1) = 1$ and $f(j) = g(j - 2)$ for all $j \in [2; \ell'' + 2]$. □

A.3. **Proof of Lemma 2.4.**

Lemma 2.4. *Given a reflexive Kripke frame* (W, R) *and a* $\downarrow$*-path* $\pi : [0; \ell] \to W$*, there is a* $\pm$*-path* $\pi' : [0; \ell''] \to W$*, for some* ℓ'*, and a total, surjective, monotonic, non-decreasing function* $f : [0; \ell'] \to [0; \ell]$ *with* $\pi'(j) = \pi(f(j))$ *for all* $j \in [0; \ell']$*.*

Proof. The assertion follows directly from Lemma 2.3 since every $\updownarrow$-path is also a $\pm$-path. □

A.4. **Proof of Lemma 3.5.**

Lemma 3.5. *Let* $\mathcal{P} = (|K|, \mathcal{V}_{\mathcal{P}})$ *be a polyhedral model,* $x \in |K|$ *and* Φ *a* $\mathtt{SLCS}_\eta$ *formula. Then* $\mathcal{P}, x \models \Phi$ *iff* $\mathcal{P}, x \models \mathcal{E}(\Phi)$*.*

Proof. By induction on the structure of Φ. We consider only the case $\eta(\Phi_1, \Phi_2)$. Suppose $\mathcal{P}, x \models \eta(\Phi_1, \Phi_2)$. By definition there is a topological path π such that $\mathcal{P}, \pi(1) \models \Phi_2$ and $\mathcal{P}, \pi(r) \models \Phi_1$ for all $r \in [0, 1)$. By the Induction Hypothesis this is the same to say that $\mathcal{P}, \pi(1) \models \mathcal{E}(\Phi_2)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}(\Phi_1)$ for all $r \in [0, 1)$, i.e. $\mathcal{P}, x \models \mathcal{E}(\Phi_1)$, $\mathcal{P}, \pi(1) \models \mathcal{E}(\Phi_2)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}(\Phi_1)$ for all $r \in (0, 1)$. In other words, we have $\mathcal{P}, x \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$ that, by Definition 3.4 on page 14 means $\mathcal{P}, x \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$.

Suppose now $\mathcal{P}, x \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$, i.e. $\mathcal{P}, x \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, by Definition 3.4 on page 14. Since $\mathcal{P}, x \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, there is a path π such that $\mathcal{P}, \pi(1) \models \mathcal{E}(\Phi_2)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}(\Phi_1)$ for all $r \in (0, 1)$. Using the Induction Hypothesis we know the following holds: $\mathcal{P}, x \models \Phi_1$, $\mathcal{P}, \pi(1) \models \Phi_2$, and $\mathcal{P}, \pi(r) \models \Phi_1$ for all $r \in (0, 1)$, i.e. $\mathcal{P}, \pi(1) \models \Phi_2$ and $\mathcal{P}, \pi(r) \models \Phi_1$ for all $r \in [0, 1)$. So, we get $\mathcal{P}, x \models \eta(\Phi_1, \Phi_2)$. □

A.5. **Proof concerning the example of Remark 3.7.**

The assertion can be proven by induction on the structure of formulas. The case for proposition letters, negation and conjunction are straightforward and omitted.

Suppose $\mathcal{P}_6, A \models \eta(\Phi_1, \Phi_2)$. Then there is a topological path $\pi_A : [0,1] \to P_6$ from A such that $\mathcal{P}_6, \pi_A(1) \models \Phi_2$ and $\mathcal{P}_6, \pi_A(r) \models \Phi_1$ for all $r \in [0,1)$. Since $\mathcal{P}_6, A \models \Phi_1$, by the Induction Hypothesis, we have that $\mathcal{P}_6, x \models \Phi_1$ for all $x \in \widetilde{ABC}$. For each $x \in \widetilde{ABC}$, define $\pi_x : [0,1] \to P_6$ as follows, for arbitrary $v \in (0,1)$:

$$\pi_x(r) = \begin{cases} \frac{r}{v}A + \frac{v-r}{v}x, \text{ if } r \in [0,v), \\ \\ \pi_A(\frac{r-v}{1-v}), \text{ if } r \in [v,1]. \end{cases}$$

Function π_x is continuous. Furthermore, for all $y \in [0,v)$, we have that $\mathcal{P}_6, \pi_x(y) \models \Phi_1$, since $\pi_x(y) \in \widetilde{ABC}$. Also, for all $y \in [v,1)$ we have that $\mathcal{P}_6, \pi_x(y) \models \Phi_1$, since $\pi_x(y) = \pi_A(\frac{y-v}{1-v})$, $0 \leq \frac{y-v}{1-v} < 1$ and for $y \in [0,1)$ we have that $\mathcal{P}_6, \pi_A(y), \models \Phi_1$. Thus $\mathcal{P}_6, \pi_x(r) \models \Phi_1$ for all $r \in [0,1)$. Finally, $\pi_x(1) = \pi_A(1)$ and $\mathcal{P}_6, \pi_A(1) \models \Phi_2$ by hypothesis. Thus, π_x is a topological path that witnesses $\mathcal{P}_6, x \models \eta(\Phi_1, \Phi_2)$.

The proof of the converse is similar, using instead function $\pi_A : [0,1] \to P_6$ defined as follows, for arbitrary $v \in (0,1)$:

$$\pi_A(r) = \begin{cases} \frac{r}{v}p + \frac{v-r}{v}A, \text{ if } r \in [0,v), \\ \\ \pi_p(\frac{r-v}{1-v}), \text{ if } r \in [v,1]. \end{cases}$$

A.6. **Proof of Proposition 3.11.**

Proposition 3.11. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, $w \in W$, and* $\texttt{SLCS}_\eta$ *formulas Φ_1 and Φ_2, the following statements are equivalent:*

(1) *There exists a $\pm$-path $\pi : [0;\ell] \to W$ for some ℓ with $\pi(0) = w$, $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0;\ell)$.*
(2) *There exists a $\downarrow$-path $\pi : [0;\ell'] \to W$ for some ℓ' with $\pi(0) = w$, $\mathcal{F}, \pi(\ell') \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0;\ell)$.*

Proof. The equivalence of statements (1) and (2) follows directly from Lemma 2.4 and the fact that $\pm$-paths are also $\downarrow$-paths. □

A.7. **Proof of Proposition 3.14.**

Proposition 3.14. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$, for $w_1, w_2 \in W$, it holds that*

$$\mathcal{F}, w_2 \models \chi(w_1) \textit{ if and only if } w_1 \equiv_\eta w_2.$$

Proof. Suppose $w_1 \not\equiv_\eta w_2$, then we have $\mathcal{F}, w_2 \not\models \delta_{w_1,w_2}$, and so $\mathcal{F}, w_2 \not\models \bigwedge_{w \in W} \delta_{w_1,w}$. If, instead, $w_1 \equiv_\eta w_2$, then we have: $\delta_{w_1,w_1} \equiv \delta_{w_1,w_2} \equiv \texttt{true}$ by definition, since $w_1 \equiv_\eta w_1$ and $w_1 \equiv_\eta w_2$. Moreover, for any other w, we have that, in any case, $\mathcal{F}, w_1 \models \delta_{w_1,w}$ holds and since $w_1 \equiv_\eta w_2$, also $\mathcal{F}, w_2 \models \delta_{w_1,w}$ holds. So, in conclusion, $\mathcal{F}, w_2 \models \bigwedge_{w \in W} \delta_{w_1,w}$. □

A.8. **Proof of Lemma 3.15.**

Lemma 3.15. *Let $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ be a finite poset model, $w \in W$ and Φ a $\texttt{SLCS}_\eta$ formula. Then $\mathcal{F}, w \models \Phi$ iff $\mathcal{F}, w \models \mathcal{E}(\Phi)$.*

Proof. Similar to that of Lemma 3.5, but with reference to the finite poset intepretation of the logic. □

A.9. **Proof concerning the example of Remark 3.17.**

We prove the assertion by induction on the structure of formulas. The case for atomic proposition letters, negation and conjunction are straightforward and omitted. Suppose $\mathcal{F}, \widetilde{A} \models \eta(\Phi_1, \Phi_2)$. Then, there is a $\pm$-path π of some length $\ell \geq 2$ such that $\pi(0) = \widetilde{A}$, $\pi(\ell) \models \Phi_2$ and $\pi(i) \models \Phi_1$ for all $i \in [0; \ell)$. Since $\mathcal{F}, \widetilde{A} \models \Phi_1$, by the Induction Hypothesis, we have that $\mathcal{F}, \widetilde{ABC} \models \Phi_1$. Consider then path $\pi' = (\widetilde{ABC}, \widetilde{ABC}, \widetilde{A}) \cdot \pi$. Path π' is a $\pm$-path and it witnesses $\mathcal{F}, \widetilde{ABC} \models \eta(\Phi_1, \Phi_2)$.

Suppose now $\mathcal{F}, \widetilde{ABC} \models \eta(\Phi_1, \Phi_2)$ and let π be a $\pm$-path witnessing it. Then, path $(\widetilde{A}, \widetilde{ABC}, \widetilde{ABC}) \cdot \pi$ is a $\pm$-path witnessing $\mathcal{F}, \widetilde{A} \models \eta(\Phi_1, \Phi_2)$.

A.10. **Proof of Lemma 3.19.**

The proof of the lemma uses a similar result, for the γ operator, that we have already proven in [BCG+22] namely:

Theorem 4.4 of [BCG+22]. *Let $\mathcal{P} = (P, \mathcal{V}_{\mathcal{P}})$ be a polyhedral model and $x \in P$. Then, for every formula Φ of $\texttt{SLCS}_\gamma$ we have that: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$.*

Lemma 3.19. *Given a polyhedral model $\mathcal{P} = (|K|, \mathcal{V}_{\mathcal{P}})$, for all $x \in |K|$ and formulas Φ of $\texttt{SLCS}_\eta$ the following holds: $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$.*

Proof. The proof is by induction on the structure of Φ. We consider only the case $\eta(\Phi_1, \Phi_2)$. Suppose $\mathcal{P}, x \models \eta(\Phi_1, \Phi_2)$. By Lemma 3.5 we get $\mathcal{P}, x \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$ and then, by Definition 3.4, we have $\mathcal{P}, x \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, that is $\mathcal{P}, x \models \mathcal{E}(\Phi_1)$ and $\mathcal{P}, x \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$. Again by Lemma 3.5 on page 14, we get also $\mathcal{P}, x \models \Phi_1$ and so, by the Induction Hypothesis, we have $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi_1)$. Furthermore, by Theorem 4.4 of [BCG+22] we also get $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$. Thus we get $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, that is $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$.
Suppose now $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\eta(\Phi_1, \Phi_2))$. This means $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi_1) \wedge \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$, that is $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi_1)$ and $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$. By the Induction Hypothesis we get that $\mathcal{P}, x \models \Phi_1$. Furthermore, by Theorem 4.4 of [BCG+22] we also get $\mathcal{P}, x \models \gamma(\mathcal{E}(\Phi_1), \mathcal{E}(\Phi_2))$. This means that there is topological path π such that $\mathcal{P}, \pi(1) \models \mathcal{E}(\Phi_2)$ and $\mathcal{P}, \pi(r) \models \mathcal{E}(\Phi_1)$ for all $r \in (0, 1)$. Using Lemma 3.5 we also get $\mathcal{P}, \pi(1) \models \Phi_2$ and $\mathcal{P}, \pi(r) \models \Phi_1$ for all $r \in (0, 1)$ and since also $\mathcal{P}, x \models \Phi_1$ (see above), we get $\mathcal{P}, \pi(1) \models \Phi_2$ and $\mathcal{P}, \pi(r) \models \Phi_1$ for all $r \in [0, 1)$, that is $\mathcal{P}, x \models \eta(\Phi_1, \Phi_2)$. □

A.11. **Proof of Theorem 3.20.**

Theorem 3.20. *Given a polyhedral model $\mathcal{P} = (|K|, \mathcal{V}_{\mathcal{P}})$, for all $x \in |K|$ and formulas Φ of* $\mathtt{SLCS}_\eta$ *it holds that:* $\mathcal{P}, x \models \Phi$ *if and only if* $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$.

Proof. Using Lemma 3.19, we know that $\mathcal{P}, x \models \Phi$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$. Moreover, by Lemma 3.15, we know that $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \mathcal{E}(\Phi)$ if and only if $\mathbb{F}(\mathcal{P}), \mathbb{F}(x) \models \Phi$, which brings us to the result. □

A.12. **Proof of Lemma 4.5.**

Lemma 4.5. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ and weak $\pm$-bisimulation $Z \subseteq W \times W$, for all w_1, w_2 such that $Z(w_1, w_2)$, the following holds: for each $\downarrow$-path $\pi_1 : [0; k_1] \to W$ from w_1 there is a $\downarrow$-path $\pi_2 : [0; k_2] \to W$ from w_2 such that $Z(\pi_1(k_1), \pi_2(k_2))$ and for each $j \in [0; k_2)$ there is $i \in [0; k_1)$ such that $Z(\pi_1(i), \pi_2(j))$.*

Proof. Let $\pi_1 : [0; k_1] \to W$ be a $\downarrow$-path from w_1. By Lemma 2.3 on page 11 we know that there is an $\updownarrow$-path $\hat{\pi}_1 : [0; 2h] \to W$ and total, monotonic non-decreasing surjection $f : [0; 2h] \to [0; k_1]$ such that $\hat{\pi}_1(j) = \pi_1(f(j))$ for all $j \in [0; 2h]$. Furthermore, by Lemma A.1 below, we know that there is a $\downarrow$-path $\pi_2 : [0; k_2] \to W$ from w_2 such that $Z(\hat{\pi}_1(2h), \pi_2(k_2))$ and for each $j \in [0; k_2)$ there is $i \in [0; 2h)$ such that $Z(\hat{\pi}_1(i), \pi_2(j))$. In addition, $\hat{\pi}_1(0) = \pi_1(0) = w_1$, $Z(\pi_1(k_1), \pi_2(k_2))$ since $Z(\hat{\pi}_1(2h), \pi_2(k_2))$ and $\hat{\pi}_1(2h) = \pi_1(k_1)$. Finally, for each $j \in [0; k_2)$ there is $i \in [0; k_1)$ such that $Z(\pi_1(i), \pi_2(j))$, since there is $n \in [0; 2h)$ such that $Z(\hat{\pi}_1(n), \pi_2(j))$ and $f(n) = i$ for some $i \in [0; k_1)$. □

Lemma A.1. *Given a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$ and a weak $\pm$-bisimulation $Z \subseteq W \times W$, for all w_1, w_2 such that $Z(w_1, w_2)$, the following holds: for each $\updownarrow$-path $\pi_1 : [0; 2h] \to W$ from w_1 there is a $\downarrow$-path $\pi_2 : [0; k] \to W$ from w_2 such that $Z(\pi_1(2h), \pi_2(k))$ and for each $j \in [0; k)$ there is $i \in [0; 2h)$ such that $Z(\pi_1(i), \pi_2(j))$.*

Proof. We prove the assertion by induction on h.
Base case. $h = 1$.
If $h = 1$, the assertion follows directly from Definition 4.3 on page 19 where $w_1 = \pi_1(0), u_1 = \pi_1(1)$ and $d_1 = \pi_1(2)$.
Induction step. We assume the assertion holds for $\updownarrow$-paths of length $2h$ or less and we prove it for $\updownarrow$-paths of length $2(h+1)$.
Suppose π_1 is a $\updownarrow$-path of length $2h+2$ and consider $\updownarrow$-path $\pi_1' = \pi_1|[0; 2h]$. By the Induction Hypothesis, we know that there is a $\downarrow$-path $\pi_2' : [0; k'] \to W$ from w_2 such that $Z(\pi_1'(2h), \pi_2'(k'))$ and for each $j \in [0; k')$ there is $i \in [0; 2h)$ such that $Z(\pi_1'(i), \pi_2'(j))$. Clearly, this means that $Z(\pi_1(2h), \pi_2'(k'))$ and for each $j \in [0; k')$ there is $i \in [0; 2h)$ such that $Z(\pi_1(i), \pi_2'(j))$. Furthermore, since $Z(\pi_1(2h), \pi_2'(k'))$ and Z is a weak $\pm$-bisimulation, we also know that there is a $\downarrow$-path $\pi_2'' : [0; k''] \to W$ from $\pi_2'(k')$ such that $Z(\pi_1(2h+2), \pi_2''(k''))$ and for each $j \in [0; k'')$ there is $i \in [2h; 2h+2)$ such that $Z(\pi_1(i), \pi_2'(j))$. Let $\pi_2 : [0; k' + k''] \to W$ be defined as $\pi_2 = \pi_2' \cdot \pi_2''$. Clearly π_2 is a $\downarrow$-path, since so is π_2''. Furthermore $Z(\pi_1(2h+2), \pi_2(k'+k''))$ since $Z(\pi_1(2h+2), \pi_2''(k''))$ and $\pi_2''(k'') = \pi_2(k'+k'')$. Finally, it is straightforward to check for all $j \in [0; k'+k'')$ there is $i \in [0; 2h+2)$ such that $Z(\pi_1(i), \pi_2(j))$. □

A.13. **Proof of Lemma 4.6.**

Lemma 4.6. *Given a polyhedral model* $\mathcal{P} = (|K|, \mathcal{V}_\mathcal{P})$*, and associated cell poset model* $\mathbb{F}(\mathcal{P}) = (W, \preccurlyeq, \mathcal{V}_{\mathbb{F}(\mathcal{P})})$*, for any* $\downarrow$*-path* $\pi : [0;\ell] \to W$*, there is a topological path* $\pi' : [0,1] \to |K|$ *such that: (i)* $\mathbb{F}(\pi'(0)) = \pi(0)$*, (ii)* $\mathbb{F}(\pi'(1)) = \pi(\ell)$*, and (iii) for all* $r \in (0,1)$ *there is* $i < \ell$ *such that* $\mathbb{F}(\pi'(r)) = \pi(i)$*.*

Proof. Since π is a $\downarrow$-path, we have that either $\mathcal{C}_T(\mathbb{F}^{-1}(\pi(k-1))) \sqsubseteq \mathcal{C}_T(\mathbb{F}^{-1}(\pi(k)))$ or $\mathcal{C}_T(\mathbb{F}^{-1}(\pi(k))) \sqsubseteq \mathcal{C}_T(\mathbb{F}^{-1}(\pi(k-1)))$, for each $k \in (0;\ell]$[14]. It follows that there is a continuous map $\pi'_k : [\frac{k-1}{\ell}, \frac{k}{\ell}] \to |K|$ such that, in the first case, $\mathbb{F}(\pi'_k(\frac{k-1}{\ell})) = \pi(k-1)$ and $\pi'_k((\frac{k-1}{\ell}, \frac{k}{\ell}]) \subseteq \mathcal{C}_T(\mathbb{F}^{-1}(\pi(k)))$, while in the second case, $\pi'_k([\frac{k-1}{\ell}, \frac{k}{\ell})) \subseteq \mathcal{C}_T(\mathbb{F}^{-1}(\pi(k-1)))$ and $\mathbb{F}(\pi'_k(\frac{k}{\ell})) = \pi(k)$. In fact π'_k can be realised as a linear bijection to the line segment connecting the barycenters in the corresponding cell, either in $\mathbb{F}^{-1}(\pi(k))$ or in $\mathbb{F}^{-1}(\pi(k-1))$, respectively.

For each $k \in (0;\ell)$, both $\pi'_k(\frac{k}{\ell})$ and $\pi'_{k+1}(\frac{k}{\ell})$ coincide with the barycenter of $\mathbb{F}^{-1}(\pi(k))$, so that defining $\pi'(r) = \pi'_k(r)$ for $r \in [\frac{k-1}{\ell}, \frac{k}{\ell}]$ correctly defines a topological path (actually a piece-wise linear path), satisfying (i) and (ii). Finally since π is a $\downarrow$-path, $\pi(\ell) \preccurlyeq \pi(\ell-1)$, so that $\pi'([\frac{\ell-1}{\ell}, 1)) \subseteq \mathbb{F}^{-1}(\pi(\ell-1))$. This implies (iii) above. □

A.14. **Proof of Lemma 4.7.**

Lemma 4.7. *Given a polyhedral model* $\mathcal{P} = (|K|, \mathcal{V}_\mathcal{P})$*, and associated cell poset model* $\mathbb{F}(\mathcal{P}) = (W, \preccurlyeq, \mathcal{V}_{\mathbb{F}(\mathcal{P})})$*, for any topological path* $\pi : [0,1] \to |K|$ *the following holds:* $\mathbb{F}(\pi([0,1]))$ *is a connected subposet of* W *and there is* $k > 0$ *and a* $\downarrow$*-path* $\hat{\pi} : [0;k] \to W$ *from* $\mathbb{F}(\pi(0))$ *to* $\mathbb{F}(\pi(1))$ *such that for all* $i \in [0;k)$ *there is* $r \in [0,1)$ *with* $\hat{\pi}(i) = \mathbb{F}(\pi(r))$*.*

Proof. Continuity of $\mathbb{F} \circ \pi$ ensures that $\mathbb{F}(\pi([0,1]))$ is a connected subposet of W. Thus there is an undirected path $\hat{\pi} : [0;k] \to W$ from $\mathbb{F}(\pi(0))$ to $\mathbb{F}(\pi(1))$ of some length $k > 0$. In particular, $\hat{\pi}(k-1) \succcurlyeq \hat{\pi}(k)$, as shown in the sequel, by contradiction. Suppose that $\hat{\pi}(k-1) \prec \hat{\pi}(k)$. This would mean that there is $\epsilon < 1$, with $\pi(\epsilon) \in \mathbb{F}(\pi(\epsilon)) = \hat{\pi}(k-1)$, such that $\pi(r') \in \hat{\pi}(k) = \mathbb{F}(\pi(1))$ for no $r' \in (\epsilon, 1)$ — otherwise $\hat{\pi}(k-1) = \hat{\pi}(k)$ would hold. But the fact that no such an r' exists contradicts the fact that π is continuous, since continuity requires that for each neighbourhood $N_1(\pi(1))$ of $\pi(1)$ there is a neighbourhood $N_2(1) \subseteq [0,1]$ of 1 such that $\pi(t) \in N_1(\pi(1))$ whenever $t \in N_2(1)$. We thus conclude that $\hat{\pi}(k-1) \succcurlyeq \hat{\pi}(k)$, and so $\hat{\pi}_1$ is a $\downarrow$-path. By definition and connectedness of $\mathbb{F}(\pi([0,1]))$ we finally get that for all $i \in [0;k)$ there is $r \in [0,1)$ with $\hat{\pi}(i) = \mathbb{F}(\pi(r))$. □

A.15. **Proof of Lemma 5.5.**

Lemma 5.5. *Given a finite poset model* $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_\mathcal{F})$ *and* $w_1, w_2 \in W$ *the following holds: if* $w_1 \rightleftharpoons w_2$*, then* $w_1 \equiv_\eta w_2$*.*

Proof. By induction on the structure of $\texttt{SLCS}_\eta$ formulas. We show only the case for $\eta(\Phi_1, \Phi_2)$ since the others are straightforward. Suppose $\mathcal{F}, w_1 \models \eta(\Phi_1, \Phi_2)$. Then there is a $\pm$-path π from w_1 of some length ℓ such that $\mathcal{F}, \pi(\ell) \models \Phi_2$ and $\mathcal{F}, \pi(i) \models \Phi_1$ for all $i \in [0;\ell)$. In particular, we have that $\mathcal{F}, w_1 \models \Phi_1$. So, by the Induction Hypothesis, since $w_1 \rightleftharpoons w_2$, we get that also $\mathcal{F}, w_2 \models \Phi_1$. In addition, by definition of $\rightleftharpoons$, and given that $w_2 \rightleftharpoons w_1$,

[14]We recall here that $\sigma_1 \sqsubseteq \sigma_2$ iff $\widetilde{\sigma_1} \preccurlyeq \widetilde{\sigma_2}$ and that $\sigma = \mathcal{C}_T(\widetilde{\sigma})$.

there is an undirected path π' of some length ℓ' such that $\pi'(0) = w_2, \pi(\ell') = w_1$ and $\mathcal{V}_{\mathcal{F}}^{-1}(\{\pi'(i)\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{\pi'(j)\})$, for all $i, j \in [0;\ell']$. Note that, by definition of $\rightleftharpoons$, we have that $\pi'(k) \rightleftharpoons w_1$ for all $k \in [0;\ell']$. Thus, again by the Induction Hypothesis, we also get $\mathcal{F}, \pi'(k) \models \Phi_1$ for all $k \in [0;\ell']$. Clearly, the sequentialisation $\pi' \cdot \pi$ of π' with π is a $\downarrow$-path since π is a $\pm$-path. Furthermore, by Lemma 2.4, there is a $\pm$-path π'' with the same starting and ending points as $\pi' \cdot \pi$, and with the same set of intermediate points, occurring in the same order. Thus π'' witnesses $\mathcal{F}, w_2 \models \eta(\Phi_1, \Phi_2)$. □

A.16. **Proof of Lemma 5.7.**

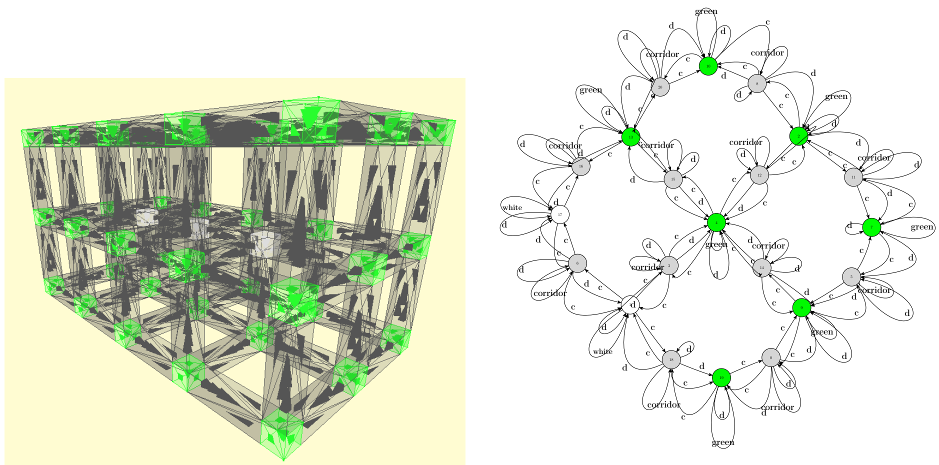
Lemma 5.7. *Consider a finite poset model $\mathcal{F} = (W, \preccurlyeq, \mathcal{V}_{\mathcal{F}})$. Then for all $w_1, w_2 \in W$ the following holds: if $[w_1]_{\rightleftharpoons} \sim^{\mathbb{S}_A(\mathcal{F})} [w_2]_{\rightleftharpoons}$, then $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$.*

Proof. By Rule (PL), we have $[w_1]_{\rightleftharpoons} \stackrel{\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})}{\longrightarrow} [w_1]_{\rightleftharpoons}$ and, by hypothesis, we also have $[w_2]_{\rightleftharpoons} \stackrel{\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\})}{\longrightarrow} [w_2']_{\rightleftharpoons}$, for some $[w_2']_{\rightleftharpoons} \sim [w_1]_{\rightleftharpoons}$. But then, using again Rule (PL), we get $[w_2']_{\rightleftharpoons} = [w_2]_{\rightleftharpoons}$ and $\mathcal{V}_{\mathcal{F}}^{-1}(\{w_1\}) = \mathcal{V}_{\mathcal{F}}^{-1}(\{w_2\})$. □

APPENDIX B. 3D MAZE EXAMPLE OF SECTION 7

Below, the spatial logic specification in ImgQL is shown, that was used for model checking the various maze-variants in Table 1 in Section 7 with PolyLogicA. ImgQL is the input language of PolyLogicA in which spatial logic properties of $\texttt{SLCS}_\eta$ can be expressed. In the specification below, first the polyhedral model is loaded in json format. After that, the atomic propositions **green**, **white** and **corridor** are defined. This is followed by a number of properties for the maze that should be self-explanatory. They include the formulas for ϕ_1 and ϕ_2 that were introduced in Section 7. Finally, the lines starting by save are defining which results to save in a file. Such files contain the name of a property and for each property a list of true/false items, one for each cell in the polyhedral model and in the order in which these cells are defined in that polyhedral model.

```
load model = "polyInput_Poset.json"

let green       = ap("G")
let white       = ap("W")
let corridor    = ap("corridor")


let greenOrWhite  = (green | white)

let oneStepToWhite   = eta((green | eta(corridor,white)),white)
let twoStepsToWhite  = eta((green | eta(corridor,oneStepToWhite)), oneStepToWhite) & (!oneStepToWhite)
let threeStepsToWhite = eta((green | eta(corridor,twoStepsToWhite)), twoStepsToWhite) &
                                          (!twoStepsToWhite) & (!oneStepToWhite)

let phi1 = eta((green | eta(corridor,white)),white)
let phi2 = eta((green | eta(corridor,oneStepToWhite)), oneStepToWhite)

save "green" green
save "white" white
save "corr" corridor
save "phi1" phi1
save "phi2" phi2
```



(A) Maze 3x5x3 (B) Minimised LTS

FIGURE 12. Maze of dimension 3x5x3 (Fig. 12a) and its respective minimal LTSs (Figs. 12b).

Figure 12 shows the 3x5x3 maze and its minimised LTS. Note that in the LTS not all transition labels are shown in order to avoid cluttering of the image. However, states corresponding to corridors, green rooms and white rooms, are shown in grey, green and white, respectively.