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SIMPLE CLASSES OF AUTOMATIC STRUCTURES

ACHIM BLUMENSATH

Masaryk University Brno
e-mail address: blumens@fi.muni.cz

Abstract. We study two subclasses of the class of automatic structures: automatic structures of polynomial growth and Presburger structures. We present algebraic characterisations of the groups and the equivalence structures in these two classes.

1. Introduction

Automatic structures, introduced in [Hod76, Hod82, KN95], form a quite large class of infinite structures with good algorithmic properties. Unfortunately these structures are less well-behaved from an algebraic perspective. In particular, the class lacks good closure properties. A persistent and non-trivial problem has been to obtain algebraic characterisations of when a structure of a certain kind admits an automatic presentation, or to prove that such a characterisation is not possible. There is quite a long list of papers devoted to this topic [Blu99, Del04, KRS05, KNRS07, NT08, Tsa11, FT13, HKLL13, AGKP14, Rub21]. As a further indication of the difficulty of the problem, let us also mention that it is Σ^1_1-complete to decide whether a given computable structure is automatic [BHK$^+$19].

Because of the difficulty of the full problem, several authors have introduced subclasses of automatic structures that are simpler to deal with. The most well-known such class is that of *unary automatic* structures [Blu99, KR01] which, unfortunately, turned out to be too simple to be of much interest. A second subclass is that of automatic structures of *polynomial growth,* which was introduced in [Bár07], see also [Hus16, GK20].

In the present article we consider two subclasses of automatic structures and we give characterisations of two kinds of algebras in these classes: groups and equivalence structures. The first class is that of structures of polynomial growth. Automatic equivalence structures of polynomial growth have already been characterised in [GK20], but the proof in that article contains an error caused by a confusion about what kind of coefficients the polynomials under consideration have. Below we provide a new proof of this result, together with a characterisation of automatic groups of polynomial growth.

The second class of automatic structures seems to be new: structures interpretable in Presburger arithmetic. This class properly contains the class of structures of polynomial growth. Also for this class we present characterisations of which groups and equivalence structures it contains.

The overview of the article is as follows. We start in Section 2 with recalling some preliminaries. Sections 3 and 4 introduce the two classes we will be studying: automatic

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structures of polynomial growth and Presburger structures. We present several algebraic characterisations of such structures in Sections 5 and 6. In the former, we study linear orders and groups, in the latter, equivalence structures.

2. Preliminaries

Let us fix notation and terminology. For $k < \omega$, we write $[k] := \{0, \dots, k-1\}$. We denote the *disjoint union* of two sets A and B by $A + B := \{0\} \times A \cup \{1\} \times B$. The *range* of a function $f : A \to B$ is $\operatorname{rng} f := f[A]$. We use three different orderings on Σ^*: $\leq_{\text{lex}}$ is the *lexicographic ordering,* $\leq_{\text{llex}}$ the *length-lexicographic* one, and $\leq_{\text{pf}}$ is the *prefix ordering.* For further details and all omitted proofs, we refer the reader to [Blu].

An automatic structure is a relational structure where the universe and each relation is given by a regular language over some alphabet. To formally define what it means for a relation to be regular, we encode tuples of words by a single word over the product alphabet.

Definition 2.1. Let Σ be an alphabet and $\Box \notin \Sigma$ a new letter.

(a) The *convolution* of words $w_0, \dots, w_{n-1} \in \Sigma^*$ is the word

$$w_0 \otimes \dots \otimes w_{n-1} := \begin{bmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{n-1,0} \end{bmatrix} \begin{bmatrix} a_{0,1} \\ a_{1,1} \\ \vdots \\ a_{n-1,1} \end{bmatrix} \cdots \begin{bmatrix} a_{0,l-1} \\ a_{1,l-1} \\ \vdots \\ a_{n-1,l-1} \end{bmatrix}$$

over the alphabet $(\Sigma + \{\Box\})^n$ where

$$l := \max_{i<n} |w_i|$$

and $a_{i,j}$ is the j-th letter of w_i or $a_{i,j} := \Box$ if w_i has less than j letters.

(b) A relation $R \subseteq \Sigma^* \times \dots \times \Sigma^*$ is *regular* if the language

$$L_R := \{\, w_0 \otimes \dots \otimes w_{n-1} \mid \langle w_0, \dots, w_{n-1} \rangle \in R \,\}$$

is regular.

(c) A relational Γ-structure $\mathfrak{A} = \langle A, \bar{R} \rangle$ is *automatic* if $\mathfrak{A} \cong \langle L_D, (L_R)_{R \in \Gamma} \rangle$ where L_D is a regular language over some alphabet Σ and all relations R are regular. In this case we call the structure $\langle L_D, (L_R)_{R \in \Gamma} \rangle$ an *automatic presentation* of $\mathfrak{A}$. Structures $\mathfrak{A}$ with functions are called automatic, if the corresponding relational structure is automatic that is obtained from $\mathfrak{A}$ by replacing each function by its graph. Usually we identify the elements a of an automatic structure with the words representing them. The length of this word is denoted by $\|a\|$. ⌟

The main reason automatic structures are so well-behaved algorithmically is the fact that their first-order theory is decidable. In fact, decidability holds for the following extension of first-order logic.

Definition 2.2. We denote first-order logic by FO, while FOC(U) is the extension of FO by the following quantifiers.

$\exists^\infty x \varphi(x)$	'There are infinitely many elements x satisfying φ'.
$\exists^{k,m} x \varphi(x)$	'The number of elements x satisfying φ is finite and congruent k modulo m'.
$\mathsf{U} X \varphi(X)$	'There exists an infinite relation X satisfying φ'.

where x is a first-order variable, X a second-order one (not necessarily monadic), and in the last case we require that X occurs only *negatively* in φ. ⌟

Decidability now follows from the following theorem (which is a combination of results from [KN95, Blu99, KRS04, KL10]).

Theorem 2.3. *Given an automatic structure $\mathfrak{A}$ (represented by a tuple of automata) and an* FOC(U)*-formula $\varphi(\bar{x})$ (without free second-order variables), one can effectively compute an automaton recognising the relation $\varphi^{\mathfrak{A}}$ defined by φ.*

As a consequence of this result, one can show that automatic structures are closed under a variety of logical operations. Let us introduce one of them.

Definition 2.4. Let L be a logic like FO or FOC(U) and let $k < \omega$.

(a) A *(k-dimensional) L-interpretation* is defined by a list of formulae

$$\tau = \langle \delta(\bar{x}), (\varphi_R(\bar{x}_0, \dots, \bar{x}_{n_R-1}))_{R \in \Sigma} \rangle$$

where Σ is a relational signature, δ and φ_R are L-formula over some signature Γ, n_R is the arity of the relation R, and $\bar{x}, \bar{x}_i$ are k-tuples of variables.

Given a Γ-structure $\mathfrak{A}$, such an interpretation defines a Σ-structure

$$\tau(\mathfrak{A}) := \langle \delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R \in \Sigma} \rangle$$

with universe

$$\delta^{\mathfrak{A}} := \{ \bar{a} \in A^k \mid \mathfrak{A} \models \delta(\bar{a}) \}$$

and relations

$$\varphi_R^{\mathfrak{A}} := \{ \langle \bar{a}_0, \dots, \bar{a}_{n_R-1} \rangle \in A^{kn_R} \mid \mathfrak{A} \models \varphi_R(\bar{a}_0, \dots, \bar{a}_{n_R-1}) \} .$$

(b) We say that a structure $\mathfrak{B}$ is *L-interpretable* in $\mathfrak{A}$ if $\mathfrak{B} = \tau(\mathfrak{A})$, for some L-interpretation τ. ⌟

Proposition 2.1 [Blu99]**.** *Let $\mathfrak{A}$ be an automatic structure and τ an* FOC(U)*-interpretation. Then $\tau(\mathfrak{A})$ is automatic.*

We can characterise automatic structures via interpretations in various structures. The most natural ones of these are the following ones. $\mathscr{P}_{\text{fin}}\langle \omega, \leq \rangle$ denotes the structure whose elements are all finite subsets of ω and that has two relations: inclusion $\subseteq$ and the relation $\leq$ defined by

$$A \leq B \quad :\text{iff} \quad A = \{a\} \text{ and } B = \{b\} \text{ for some } a \leq b .$$

The p-ary tree $\langle [p]^*, \leq_{\text{pf}}, (\text{suc}_k)_{k<p}, =_{\text{len}} \rangle$ has the prefix-order

$$u \leq_{\text{pf}} v \quad :\text{iff} \quad v = ux , \quad \text{for some } x \in [p]^*,$$

p successor functions $\text{suc}_k(v) := vk$, and the *equal-length* predicate

$$u =_{\text{len}} v \quad :\text{iff} \quad |u| = |v| .$$

Finally, $\langle \mathbb{N}, +, |_p \rangle$ is the expansion of Presburger Arithmetic by the divisibility predicate

$$k \mid_p m \quad :\text{iff} \quad k \text{ is a power of } p \text{ dividing } m .$$

Theorem 2.2 [Blu99, CL07]**.** *Let $\mathfrak{A}$ be a structure. The following statements are equivalent.*

(1) *$\mathfrak{A}$ is automatic.*
(2) *$\mathfrak{A}$ is* FO*-interpretable in $\mathscr{P}_{\text{fin}}\langle \omega, \leq \rangle$*

(3) $\mathfrak{A}$ *is* FO*-interpretable in* $\langle [p]^*, \leq_{\mathrm{pf}}, (\mathrm{suc}_k)_{k<p}, =_{\mathrm{len}} \rangle$*, for some* $p \geq 2$.
(4) $\mathfrak{A}$ *is* FO*-interpretable in* $\langle \mathbb{N}, +, |_p \rangle$*, for some* $p \geq 2$.

3. Automatic structures of polynomial growth

Characterising which structures have an automatic presentation is a very hard problem. To simplify the task, we introduce several subclasses of automatic structures where it is easier to prove characterisations. We start with the following one, which was first introduced in [Bár07].

Definition 3.1. A language $L \subseteq \Sigma^*$ has *polynomial growth* if there exists a polynomial $p(x)$ such that

$$\big|\{ w \in L \mid |w| \leq n \}\big| \leq p(n)\,, \quad \text{for all } n < \omega\,.$$

Similarly, we say that an automatic structure $\mathfrak{A}$ has *polynomial growth* if there exists a polynomial $p(x)$ such that

$$\big|\{ a \in A \mid \|a\| \leq n \}\big| \leq p(n)\,, \quad \text{for all } n < \omega\,.$$

In this case we also say that $\mathfrak{A}$ is *poly-growth automatic.* ⌟

Example. The infinite grid $\langle \mathbb{Z} \times \mathbb{Z}, E_0, E_1 \rangle$ is poly-growth automatic. (We can represent a point $\langle i, k \rangle \in \mathbb{Z} \times \mathbb{Z}$ by the word $a^i b^k$.) ⌟

The characterisation of automatic structures via interpretations from Theorem 2.2 can be transferred to poly-growth automatic structures as follows.

Theorem 3.2. *Let* $\mathfrak{A}$ *be a structure. The following statements are equivalent.*

(1) $\mathfrak{A}$ *is an automatic structure of polynomial growth.*
(2) $\mathfrak{A}$ *has an automatic presentation whose universe is a finite union of languages of the form*

$$u_0 v_0^* u_1 v_1^* \cdots u_{k-1} v_{k-1}^* u_k\,, \quad \textit{with } u_0, \dots, u_k, v_0, \dots, v_{k-1} \in \Sigma^*.$$

(3) $\mathfrak{A}$ *has an automatic presentation whose universe is a finite union of languages of the form*

$$a_0^{m_0} (b_0^{k_0})^* a_1^{m_1} (b_1^{k_1})^* \cdots a_{n-1}^{m_{n-1}} (b_{n-1}^{k_{n-1}})^* a_n^{m_n}$$

for distinct letters $a_0, \dots, a_n, b_0, \dots, b_{n-1}$ *(with distinct members of the union using disjoint alphabets).*
(4) $\mathfrak{A}$ *is (k-dimensionally)* FO*-interpretable in* $\langle \mathbb{N}, \leq, m \mid \cdot \rangle$*, for some* m, k.
(5) $\mathfrak{A}$ *is (k-dimensionally)* FO*-interpretable in* $\langle \omega, \leq \rangle$*, for some* k.

Proof. The equivalence (1) ⇔ (4) is Theorem 3.3.6 of [Bár07].

(1) ⇔ (2) follows since, according to [SYZS92], every regular language of polynomial growth can be written as a finite union of languages of the form

$$u_0 v_0^* u_1 v_1^* \cdots u_{k-1} v_{k-1}^* u_k\,, \quad \text{with } u_0, \dots, u_k, v_0, \dots, v_{k-1} \in \Sigma^*,$$

(3) ⇒ (2) is trivial. For (2) ⇒ (3), let E be the relation of all pairs $\langle w, w' \rangle$ where

$$w = u_0 v_0^{i_0} u_1 v_1^{i_1} \cdots u_{n-1} v_{n-1}^{i_{n-1}} u_n$$

and $$w' = a_0^{m_0} b_0^{k_0 i_0} a_1^{m_1} b_1^{k_1 i_1} \cdots a_{n-1}^{m_{n-1}} b_{n-1}^{k_{n-1} i_{n-1}} a_n^{m_n},$$

for $i_0, \dots, i_{n-1} < \omega$ and $m_j := |u_j|$ and $k_j := |v_j|$. Note that E is regular since we can write its convolution L_E as

$$[u_0 \otimes a_0^{m_0}][v_0 \otimes b_0^{k_0}]^*[u_1 \otimes a_1^{m_1}][v_1 \otimes b_1^{k_1}]^* \cdots \\ [u_{n-1} \otimes a_{n-1}^{m_{n-1}}][v_{n-1} \otimes b_{n-1}^{k_{n-1}}]^*[u_n \otimes a_n^{m_n}]\,.$$

We claim that the image of the presentation of $\mathfrak{A}$ under E is again an automatic presentation of $\mathfrak{A}$. Let $R \subseteq A^n$ be a relation of $\mathfrak{A}$ and let R' be its image under E. We have to show that R' is regular. Let Σ be the alphabet used by the given presentation of $\mathfrak{A}$ and let Γ be the alphabet such that $E \subseteq \Sigma^* \times \Gamma^*$. By Theorem 2.2, there exists an FO-formula $\varphi(\bar{x})$ defining R in the tree $\langle \Sigma^*, \leq_{\mathrm{pf}}, (\mathrm{suc}_a)_{a \in \Sigma}, =_{\mathrm{len}} \rangle$. Modifying φ slightly, we obtain an FO-formula $\varphi'(\bar{x})$ defining R in

$$\mathfrak{T} := \big\langle (\Sigma + \Gamma)^*, \leq_{\mathrm{pf}}, (\mathrm{suc}_a)_{a \in \Sigma + \Gamma}, =_{\mathrm{len}} \big\rangle\,.$$

We can define the image R' inside $\mathfrak{T}$ by the formula

$$\psi(\bar{x}) := \exists \bar{y} \Big[\varphi'(\bar{y}) \wedge \bigwedge_{i<n} E y_i x_i\Big]\,.$$

This implies that R' is regular.

(5) ⇒ (4) is trivial.

(4) ⇒ (5) We can define a 2-dimensional FO-interpretation of $\langle \mathbb{N}, \leq, m \mid \cdot \rangle$ in $\langle \omega, \leq \rangle$ by encoding the number $mk + i$ by the pair $\langle k, i \rangle$. This leads to the formulae

$$\begin{aligned} \delta(xx') &:= x' < m\,, \\ \varphi_{\leq}(xx', yy') &:= x < y \vee [x = y \wedge x' \leq y']\,, \\ \varphi_{m|\cdot}(xx') &:= x' = 0\,. \end{aligned}$$

□

Lemma 3.3. *The class of poly-growth automatic structures is closed under finite disjoint unions and finite direct products.*

Proof. If we have FO-interpretations of $\mathfrak{A}$ and $\mathfrak{B}$ in $\langle \omega, \leq \rangle$, we can use them to construct interpretations of $\mathfrak{A} + \mathfrak{B}$ and $\mathfrak{A} \times \mathfrak{B}$ in $\langle \omega, \leq \rangle$. □

4. Presburger Structures

Our second class is slightly larger than that of the poly-growth automatic structures. The definition is as follows.

Definition 4.1. (a) A *Presburger structure* is a structure $\mathfrak{A}$ for which there exists a (many-dimensional) FO-interpretation of $\mathfrak{A}$ in $\langle \mathbb{N}, + \rangle$.

(b) We say that a subset $S \subseteq \mathbb{N}^n$ is *Presburger-definable* if it is FO-definable in $\langle \mathbb{N}, + \rangle$. ⌟

Proposition 4.2. *Every poly-growth automatic structure is a Presburger structure and every Presburger structure is automatic.*

Proof. The first claim follows from Theorem 3.2 and the fact that there exists an FO-interpretation of $\langle \mathbb{N}, \leq, m \mid \cdot \rangle$ in $\langle \mathbb{N}, + \rangle$. The second claim follows by Theorem 2.2. □

To better understand Presburger structures, we need some results about which kinds of relations are definable in $\langle \mathbb{N}, + \rangle$.

Definition 4.3. (a) A function $\varphi : \mathbb{N}^m \to \mathbb{N}^n$ is *affine* if it is of the form

$$\varphi(\bar{x}) := u + \sum_{i<m} v_i x_i\,, \quad \text{for some } u, v_0, \dots, v_{m-1} \in \mathbb{N}^n.$$

(b) A set $S \subseteq \mathbb{N}^n$ is *semilinear* if it is of the form

$$S = \operatorname{rng} \varphi_0 \cup \dots \cup \operatorname{rng} \varphi_{k-1}\,,$$

for suitable affine functions $\varphi_0, \dots, \varphi_{k-1}$.

(c) We call S *simple* if it is of the form $S = \operatorname{rng} \varphi$ for some affine function $\varphi : \mathbb{N}^m \to \mathbb{N}^n$ such that the images $\varphi(e_0), \dots, \varphi(e_{m-1})$ of the unit vectors $e_0, \dots, e_{m-1}$ are linearly independent in $\mathbb{Q}^n$. ⌟

Definition 4.4. (a) For $k, m, p \in \mathbb{N}$ with $p > 1$, we write

$$k \mid_p m \quad :\text{iff} \quad k = p^n \mid m\,, \quad \text{for some } n \in \mathbb{N}\,.$$

(b) Two natural numbers $k, l \in \mathbb{N}$ are *multiplicatively independent* if the only integer solution to the equation $k^n = l^m$ is $n = 0 = m$. ⌟

It turns out that the Presburger-definable sets are exactly the semilinear ones.

Theorem 4.5. *Let $S \subseteq \mathbb{N}^n$. The following statements are equivalent.*

(1) *S is semilinear.*
(2) *S is* FO*-definable in $\langle \mathbb{N}, + \rangle$.*
(3) *S is* FOC(U)*-definable in $\langle \mathbb{N}, + \rangle$.*
(4) *There are multiplicatively independent numbers $k, l \geq 2$ such that S is* FO*-definable in both $\langle \mathbb{N}, +, |_k \rangle$ and $\langle \mathbb{N}, +, |_l \rangle$.*
(5) *There is some $m < \omega$ such that S is quantifier-free definable in the structure $\langle \mathbb{N}, +, \leq, m \mid \cdot, 0, 1 \rangle$.*

Proof. (1) ⇔ (2) is a classical result from [GS66].

(5) ⇒ (2) is trivial.

(2) ⇒ (5) holds since the structure $\langle \mathbb{N}, +, \leq, (m|\cdot)_{m<\omega}, 0, 1 \rangle$ admits quantifier elimination (see, e.g., [Mar02]).

(2) ⇒ (3) is trivial.

(3) ⇒ (4) Fix an FOC(U)-definable set $S \subseteq \mathbb{N}^n$. The structures $\mathfrak{N}_k := \langle \mathbb{N}, +, |_k \rangle$ and $\mathfrak{N}_l := \langle \mathbb{N}, +, |_l \rangle$ have automatic presentations based on, respectively, the k-ary encoding and the l-ary encoding. By Theorem 2.3, these presentations can be expanded to ones of, respectively, $\langle \mathfrak{N}_k, S \rangle$ and $\langle \mathfrak{N}_l, S \rangle$. Finally, it follows by Theorem 2.2 that S is FO-definable in both $\mathfrak{N}_k$ and $\mathfrak{N}_l$.

(4) ⇒ (2) is a classical result by Cobham and Semenov (see [DR21] for an introduction). □

The following result will be useful below to simplify semilinear sets.

Proposition 4.6 (Ito [Ito69], Eilenberg, Schützenberger [ES69])**.** *Every semilinear set $S \subseteq \mathbb{N}^n$ can be written as a disjoint union of finitely many simple semilinear sets.*

Our next aim is to derive a bound on the out-degree of a semilinear relation.

Definition 4.7. (a) A *vector partition function* is a function $f : \mathbb{N}^n \to \mathbb{N}$ that, for some matrix $A \in \mathbb{N}^{n \times m}$, maps a tuple $\bar{x} \in \mathbb{N}^n$ to the number of tuples $\bar{y} \in \mathbb{N}^m$ such that $\bar{x} = A\bar{y}$. We denote the vector partition function associated with A by $\psi_A : \mathbb{N}^n \to \mathbb{N}$. (Note that not every matrix has an associated vector partition function since the equation $\bar{x} = A\bar{y}$ might have infinitely many solutions.)

(b) A *generalised vector partition function* is a function of the form

$$g(\bar{x}) := \sum_{i<s} \psi_{A_i}(\bar{x} + \bar{c}_i)\,, \quad \text{for } A_i \in \mathbb{N}^{n\times m_i} \text{ and } \bar{c}_i \in \mathbb{N}^n.$$

(c) A function $g : \mathbb{N}^n \to \mathbb{N}$ is a *piecewise polynomial* if there exists a partition $\mathcal{S}$ of $\mathbb{N}^n$ into semilinear sets such that, for every $S \in \mathcal{S}$, the restriction $g \restriction S$ is a polynomial in $\mathbb{Q}[x_0, \dots, x_{n-1}]$. ⌟

Proposition 4.8 (Sturmfels [Stu95])**.** *Every vector partition function is piecewise polynomial.*

Corollary 4.9. *Every generalised vector partition function is piecewise polynomial.*

Definition 4.10. For $\bar{x}, \bar{a} \in \mathbb{N}^n$, we write

$$\bar{x}^{\bar{a}} := x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}\,.$$ ⌟

Proposition 4.11 (Woods [Woo15])**.** *Let $R \subseteq \mathbb{N}^k \times \mathbb{N}^l$ be a semilinear relation of finite out-degree. The function $d : \mathbb{N}^k \to \mathbb{N}$ mapping each tuple $\bar{u} \in \mathbb{N}^k$ to its R-out-degree is a generalised vector partition function.*

Proof. We give a simplified proof of the original, stronger statement from [Woo15]. With each relation $S \subseteq \mathbb{N}^{k+l}$ we associate the formal power-series

$$f_S(\bar{x}, \bar{y}) := \sum_{\langle \bar{c}, \bar{d}\rangle \in S} \bar{x}^{\bar{c}} \bar{y}^{\bar{d}}.$$

We can use Proposition 4.6 to write $R = S_0 \cup \dots \cup S_{n-1}$ as a finite disjoint union of simple semilinear sets $S_i = \operatorname{rng} \varphi_i$. Suppose that

$$\varphi_i(0) = \bar{u}_i \bar{u}'_i \quad \text{and} \quad \varphi_i(e_j) = v_{i,j} \bar{v}'_{i,j}\,, \quad \text{for } j < s_i\,.$$

A direct calculation shows that

$$f_{S_i}(\bar{x}, \bar{y}) = \frac{\bar{x}^{\bar{u}_i} \bar{y}^{\bar{u}'_i}}{(1 - \bar{x}^{\bar{v}_{i,0}} \bar{y}^{\bar{v}'_{i,0}}) \cdots (1 - \bar{x}^{\bar{v}_{i,s_i-1}} \bar{y}^{\bar{v}'_{i,s_i-1}})}\,.$$

Hence, we obtain

$$f_R(\bar{x}, \bar{y}) = \sum_{i<n} \frac{\bar{x}^{\bar{u}_i} \bar{y}^{\bar{u}'_i}}{(1 - \bar{x}^{\bar{v}_{i,0}} \bar{y}^{\bar{v}'_{i,0}}) \cdots (1 - \bar{x}^{\bar{v}_{i,s_i-1}} \bar{y}^{\bar{v}'_{i,s_i-1}})}\,,$$

which implies that

$$\begin{aligned}
\sum_{\bar{c} \in \mathbb{N}^k} d(\bar{c}) \bar{x}^{\bar{c}} &= \sum_{\bar{c} \in \mathbb{N}^k} \big|\{\, \bar{d} \mid \langle \bar{c}, \bar{d}\rangle \in R \,\}\big| \cdot x^{\bar{c}} \\
&= \sum_{\langle \bar{c}, \bar{d}\rangle \in R} \bar{x}^{\bar{c}} \\
&= \sum_{\langle \bar{c}, \bar{d}\rangle \in R} \bar{x}^{\bar{c}} \bar{1}^{\bar{d}} \\
&= f_R(\bar{x}, 1 \dots 1) \\
&= \sum_{i<n} \frac{\bar{x}^{\bar{u}_i}}{(1 - \bar{x}^{\bar{v}_{i,0}}) \cdots (1 - \bar{x}^{\bar{v}_{i,s_i-1}})}\,.
\end{aligned}$$

Since generalised vector partition functions are closed under addition, it is therefore sufficient to prove that, given a power-series of the form,

$$\sum_{\bar b} g(\bar b)\bar x^{\bar b} = \frac{\bar x^{\bar c}}{(1-\bar x^{\bar a_0})\cdots(1-\bar x^{\bar a_{m-1}})}\,,$$

the coefficient function g is a generalised vector partition function. In this case, we obtain

$$\begin{aligned}\sum_{\bar b} g(\bar b+\bar c)\bar x^{\bar b} &= \frac{1}{(1-\bar x^{\bar a_0})\cdots(1-\bar x^{\bar a_{m-1}})}\\ &= \Big[\sum_{\mu_0<\omega} \bar x^{\mu_0\bar a_0}\Big]\cdots\Big[\sum_{\mu_{m-1}<\omega} \bar x^{\mu_{m-1}\bar a_{m-1}}\Big]\\ &= \sum_{\mu_0,\dots,\mu_{m-1}<\omega} \bar x^{\mu_0\bar a_0+\dots+\mu_{m-1}\bar a_{m-1}}\,,\end{aligned}$$

which implies that $g(\bar z+\bar c)$ is equal to the number of tuples $\bar\mu \in \mathbb{N}^m$ satisfying

$$\bar z = \mu_0\bar a_0 + \dots + \mu_{m-1}\bar a_{m-1}\,.$$

Hence, $g(\bar z) = \psi_A(\bar z+\bar c)$, for some A and c. □

Example. Let $R \subseteq \mathbb{N}^2 \times \mathbb{N}$ be the set of all triples $\langle a,b,c\rangle$ such that c is an even number with $a \le c \le b$. This relation is definable in $\langle \mathbb{N},+\rangle$ and, hence, semilinear. Its out-degree is

$$d(a,b) := \begin{cases} 0 & \text{if } a > b\,,\\ \frac12(b-a)+1 & \text{for } a \le b \text{ and } a,b \text{ even},\\ \frac12(b-a) & \text{for } a\le b \text{ and } a,b \text{ odd},\\ \frac12(b-a+1) & \text{otherwise.}\end{cases}$$

In particular, note that $d \in \mathbb{Q}[a,b]$, but $d \notin \mathbb{N}[a,b]$. Finally, note that d is the a vector partition function associated with the equation

$$\begin{bmatrix} a\\ b\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1\\ 1 & 2 & 1\end{bmatrix} \cdot \begin{bmatrix} x\\ y\\ z\end{bmatrix}\,.$$

⌟

5. Growth Arguments

To prove that certain structures are not poly-growth automatic we can use the following growth argument, which follows from Theorem 2.3 together with a pumping argument originally due to Khoussainov and Nerode [KN95].

Definition 5.1. Let $\mathfrak{A}$ be a structure and $\varphi(\bar x, y)$ a formula. For a set $U \subseteq A$ and a number $n < \omega$, we define the set $N_\varphi(U,n)$ of *reachable elements at distance* n by

$$N_\varphi(U,0) := U\,,$$

and $\quad N_\varphi(U,n+1) := N_\varphi(U,n) \cup \{\, b \in A \mid \mathfrak{A} \models \varphi(\bar a, b) \text{ for some } \bar a \subseteq U \,\}$. ⌟

Lemma 5.2. *Let $\mathfrak{A}$ be an automatic structure. For every* FOC(U)*-formula $\varphi(\bar x;\bar z)$ of finite out-degree, there exists a constant k such that*

$$\mathfrak{A} \models \varphi(\bar a;\bar c) \quad \textit{implies} \quad \|\bar a\| \le \|\bar c\| + k\,, \quad \textit{for all } \bar a, \bar c\,.$$

Lemma 5.3. *Let $\mathfrak{A}$ be a poly-growth automatic structure, $U \subseteq A$ finite, and φ a formula of finite out-degree. Then there exist constants $d, k > 0$ such that*

$$|N_\varphi(U, n)| \leq n^d + k\,, \quad \text{for all } n < \omega\,.$$

Proof. Let $l := \max\{\, \|c\| \mid c \in U \,\}$. By Lemma 5.2, we can find a constant c such that

$$\|a\| \leq l + cn\,, \quad \text{for all } a \in N_\varphi(U, n)\,.$$

By assumption, there exists a polynomial $p(x)$ such that the universe of $\mathfrak{A}$ contains at most $p(n)$ words of length at most n. Consequently,

$$|N_\varphi(U, n)| \leq p(l + cn)\,.$$ □

Our first case study concerns linear orders. We start with ordinals.

Theorem 5.4. *An ordinal $\langle \alpha, \leq \rangle$ is poly-growth automatic if, and only if, $\alpha < \omega^\omega$.*

Proof. (⇒) It was shown in [Del04] that all automatic ordinals are smaller than ω^ω.

(⇐) It follows by Lemma 3.3 and Theorem 3.2 that the class of poly-growth automatic ordinals is closed under ordinal addition and multiplication. Furthermore, $\langle \omega, \leq \rangle$ is poly-growth automatic. □

Definition 5.5. Let $\mathfrak{A}$ be a coloured linear order.

(a) $\mathfrak{A}$ is *scattered* if the order of the rationals cannot be embedded into A.

(b) $\mathfrak{A}$ is *regular* if it can be (1-dimensionally) MSO-interpreted in the infinite binary tree $\langle \{0,1\}^*, \mathrm{suc}_0, \mathrm{suc}_1 \rangle$. ⌟

Proposition 5.6. *Let $\mathfrak{A}$ be a coloured linear order. If $\mathfrak{A}$ is regular and scattered, it is poly-growth automatic.*

Proof. It is a well-known result (see, e.g., Section VI.4 of [Blu]) that every scattered regular linear order $\mathfrak{A}$ can be constructed from finite linear orders using finite ordered sums and right multiplication by ω or ω^{op}. By Lemma 3.3 and Theorem 3.2, all of these operations preserve poly-growth automaticity. □

Example. The converse is not true. Let $\langle \omega, \leq, P \rangle$ be the order with

$$P := \{\, n(n+1)/2 \mid n < \omega \,\}\,.$$

This order has an automatic presentation $\langle a^*b^*, \leq_{\mathrm{lex}}, a^* \rangle$ with polynomial growth, but it is not regular. (It cannot be expressed using the operations from the proof of Proposition 5.6). ⌟

Instead of the converse, we can use Lemma 5.3 to prove the following weaker statement.

Definition 5.7. Let $\mathcal{Z}$ be the set consisting of all finite linear orders together with ω, ω^{op} (ω with the opposite ordering), and $\mathbb{Z}$. By induction on an ordinal α, we define classes VD_α of linear orders as follows.

$$\begin{aligned} \mathrm{VD}_0 &:= \{0, 1\}\,, \\ \mathrm{VD}_{\alpha+1} &:= \big\{\, \textstyle\sum_{i \in I} \mathfrak{A}_i \;\big|\; I \in \mathcal{Z},\ \mathfrak{A}_i \in \mathrm{VD}_\alpha \,\big\}\,, \\ \mathrm{VD}_\delta &:= \bigcup_{\alpha < \delta} \mathrm{VD}_\alpha\,, \quad \text{for limit ordinals } \delta\,. \end{aligned}$$

The VD-*rank* $\mathrm{VD}(\mathfrak{A})$ of a linear order $\mathfrak{A}$ is the least ordinal α with $\mathfrak{A} \in \mathrm{VD}_\alpha$. If no such ordinal exists, we set $\mathrm{VD}(\mathfrak{A}) := \infty$. ⌟

Proposition 5.8. *For every poly-growth automatic linear order $\mathfrak{A}$, we have $\mathrm{VD}(\mathfrak{A}) < \omega$.*

Proof. For the proof, we introduce a second rank for linear orders. The *condensation* $\mathrm{cn}(\mathfrak{A})$ of a linear oder $\mathfrak{A}$ is the quotient $\mathfrak{A}/{\sim}$ by the equivalence relation

$$x \sim y \quad :\text{iff} \quad \text{there are only finitely many elements between } x \text{ and } y\,.$$

For each ordinal α, we define the α-th iteration of cn by

$$\mathrm{cn}^0(\mathfrak{A}) := \mathfrak{A}\,, \quad \mathrm{cn}^{\alpha+1}(\mathfrak{A}) = \mathrm{cn}(\mathrm{cn}^\alpha(\mathfrak{A}))\,,$$

and, for a limit ordinal δ, $\mathrm{cn}^\delta(\mathfrak{A})$ is the colimit of the sequence $(\mathrm{cn}^\alpha(\mathfrak{A}))_{\alpha<\delta}$. The *finite condensation rank* $\mathrm{FC}(\mathfrak{A})$ of $\mathfrak{A}$ is the least ordinal α such that $\mathrm{cn}^{\alpha+1}(\mathfrak{A}) = \mathrm{cn}^\alpha(\mathfrak{A})$. It has been shown in [KRS05] that $\mathrm{FC}(\mathfrak{A})$ is finite, for every automatic linear order $\mathfrak{A}$. Furthermore, it is known that $\mathrm{FC}(\mathfrak{A}) = \mathrm{VD}(\mathfrak{A})$, for every scattered countable linear order (see, e.g., Section 5.3 of [Ros82]).

Hence, it is sufficient to show that every poly-growth automatic linear order is scattered. For a contradiction, suppose that there exists a poly-growth automatic linear order $\mathfrak{A}$ that is not scattered. Set $n := \mathrm{FC}(\mathfrak{A})$. Then $\mathrm{cn}^n(\mathfrak{A}) \cong \langle \mathbb{Q}, \leq \rangle$. Since we can FOC-interpret $\mathrm{cn}^n(\mathfrak{A})$ in $\mathfrak{A}$, the order $\langle \mathbb{Q}, \leq \rangle$ is poly-growth automatic. Let $\varphi(x, y, z)$ be the formula stating that z is the $\leq_{\mathrm{llex}}$-least element with $x < z < y$. Then φ has finite out-degree and

$$|N_\varphi(\{a, b\}, n)| = 2 + 2^{n-1}, \quad \text{for } a < b$$

(in the structure $\langle \mathbb{Q}, \leq \rangle$). A contradiction to Lemma 5.3. □

Next, let us take a look at groups and semigroups. A characterisation of all finitely generated Presburger groups follows immediately from the corresponding characterisation of automatic groups. (Here, an *automatic group* is an automatic structure that happens to be a group. There is also a commonly used notion of an automatic group due to Thurston [ECH+92], which is more restrictive and which we will not be dealing with in this article.)

Proposition 5.9. *Let $\mathfrak{G}$ be a finitely generated group. The following statements are equivalent.*

(1) *$\mathfrak{G}$ is a Presburger structure.*
(2) *$\mathfrak{G}$ is automatic.*
(3) *$\mathfrak{G}$ is virtually abelian.*

Proof. (2) ⇔ (3) has been proved as Theorem 8 of [OT05]; (1) ⇒ (2) is trivial; and (3) ⇒ (1) follows from Remark 4 in [OT05] where the authors construct an interpretation of $\mathfrak{G}$ in $\langle \mathbb{Z}, + \rangle$ and, hence, also in $\langle \mathbb{N}, + \rangle$ (see also Section XII.9 of [Blu]). □

The class of poly-growth automatic groups turns out to be much smaller. Before giving the characterisation, let us take a quick look at poly-growth automatic semigroups.

Lemma 5.10. *Let $\mathfrak{S}$ be a semigroup such that there exists an embedding of $\langle \mathbb{N} \setminus \{0\}, + \rangle$ into $\mathfrak{S}$. Then $\mathfrak{S}$ is not poly-growth automatic.*

Proof. Suppose that $\mathfrak{S} = \langle S, + \rangle$ is an automatic semigroup into which $\langle \mathbb{N} \setminus \{0\}, + \rangle$ can be embedded, and let c be the image of 1 under this embedding. By Lemma 3.2 of [KNRS07], there exists a constant k such that

$$\|nc\| \leq \|c\| + k \log_2 n\,, \quad \text{for all } n\,.$$

It follows that

$$n \leq 2^{(m-\|c\|)/k} \quad \text{implies} \quad \|nc\| \leq m\,.$$

Hence, the set $\{ a \in S \mid \|a\| \leq m \}$ contains at least $2^{(m-\|c\|)/k}$ elements and $\mathfrak{G}$ is not of polynomial growth. □

It turns out that the only poly-growth automatic groups are the finite ones.

Theorem 5.11. *A group is poly-growth automatic if, and only if, it is finite.*

Proof. Let $\mathfrak{G} = \langle G, \cdot, {}^{-1}, e \rangle$ be a poly-growth automatic group. By Lemma 5.2 there exists a constant k such that

- for every $a \in G$, there is some $b \in G$ with $\|a\| < \|b\| \leq \|a\| + k$,
- $\|ab\| \leq \max\{\|a\|, \|b\|\} + k$,
- $\|a^{-1}\| \leq \|a\| + k$.

Setting $m := \|e\|$, it follows that, for each $n < \omega$, there exists some element $a_n \in A$ of length

$$m + 4kn \leq \|a_n\| < m + 4kn + k\,.$$

Set $D_0 := \{e\}$,

$$C_n := \{ a_0^{s_0} \cdots a_{n-1}^{s_{n-1}} \mid s_0, \dots, s_{n-1} \in \{0,1\} \}\,,$$
$$D_n := \{ a^{-1}b \mid a, b \in C_n \}\,.$$

We claim that

(i) $\|c\| < m + 4k(n-1) + 2k$, for all $n > 0$ and $c \in C_n$,
(ii) $\|c\| < m + 4kn$, for all $c \in D_n$,
(iii) $|C_n| = 2^n$.

It follows that G contains at least 2^n elements of length at most $4kn$. A contradiction to the fact that $\mathfrak{G}$ has polynomial growth. Hence, it remains to prove the above claims.

(i) We proceed by induction on n. For $n = 1$, we have $\|e\| = m$ and $\|a_0\| < m+k \leq m+2k$. For the inductive step, let $c \in C_n$. If $c \in C_{n-1}$, the claim follows by inductive hypothesis. Otherwise, we can write $c = da_{n-1}$ with $d \in D_{n-1}$. Then $\|d\|, \|a_{n-1}\| < m + 4k(n-1) + k$ implies, by choice of k, that

$$\|c\| < m + 4k(n-1) + 2k\,.$$

(ii) Let $a, b \in C_n$. By (i), we have $\|a\|, \|b\| < m + 4k(n-1) + 2k$. By choice of k, this implies that

$$\|a^{-1}b\| < m + 4k(n-1) + 2k + 2k = m + 4kn\,.$$

(iii) Suppose that

$$a_0^{s_0} \cdots a_n^{s_n} = a_0^{t_0} \cdots a_n^{t_n}, \quad \text{for } s_0, \dots, s_n, t_0, \dots, t_n \in \{0,1\}\,.$$

We prove that $s_i = t_i$ by induction on n. Set

$$b := a_0^{s_0} \cdots a_{n-1}^{s_{n-1}} \quad \text{and} \quad c := a_0^{t_0} \cdots a_{n-1}^{t_{n-1}}\,.$$

If $s_n = t_n$, we obtain $b = c$ and the claim follows by inductive hypothesis. Otherwise, we may assume without loss of generality that $s_n = 0$ and $t_n = 1$. Hence,

$$b = ca_n \quad \text{implies} \quad a_n = c^{-1}b \in D_n\,.$$

By (iii), it follows that $\|a_n\| < m + 4kn$. A contradiction to our choice of a_n. □

6. Equivalence Structures

The aim of this last section is to prove characterisations both of Presburger equivalence relations and of poly-growth automatic equivalence relations.

Definition 6.1. (a) An *equivalence structure* is a structure of the form $\langle A, \sim\rangle$ where $\sim$ is an equivalence relation on A.

(b) Given a function $g : \mathbb{N}^n \to (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$, we denote by $\mathfrak{E}(g)$ the equivalence structure with exactly $|g^{-1}(k)|$ classes of size k, for each $k \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$. ⌟

Our aim is to prove the following two characterisations.

Theorem 6.2. *An equivalence structure* $\mathfrak{A}$ *is a Presburger structure if, and only if,*

$$\mathfrak{A} \cong \mathfrak{E}(g) + \mathfrak{C}$$

where g *is a generalised vector partition function and* $\mathfrak{C}$ *is a countable equivalence structure with only infinite classes.*

Theorem 6.3. *An equivalence structure* $\mathfrak{A}$ *is poly-growth automatic if, and only if, it can be written as a finite disjoint union of*

- *structures of the form* $\mathfrak{E}(p)$*, for polynomials* $p \in \mathbb{N}[\bar{x}]$*, and*
- *countable equivalence structures where every class is infinite.*

Remark. Theorem 6.3 was already stated in [GK20], but the proof in that article contained an error: for one direction the authors require a polynomial with natural coefficients, but the other direction only produces polynomials with rational ones. Below we will present a new, correct proof. ⌟

We start with a simple lemma that helps us to define interpretations of structures of the form $\mathfrak{E}(g)$ in well-ordered structures, i.e., structures where one of the relations is a well-ordering.

Lemma 6.4. *Let* $\mathfrak{A}$ *be a well-ordered structure and* $g : \mathbb{N}^n \to \mathbb{N}$ *a function. There exists an* FO*-interpretation of* $\mathfrak{E}(g)$ *in* $\mathfrak{A}$ *if, and only if, there are* $k, m < \omega$*, an injective function* $\sigma : \mathbb{N}^n \to A^k$*, and an* FO*-definable relation* $R \subseteq A^k \times A^m$ *such that*

$$d_R(\bar{a}) = \begin{cases} g(\bar{c}) & \text{if } \bar{a} = \sigma(\bar{c})\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where d_R *is the function mapping a tuple* $\bar{a}$ *to its* R*-out-degree.*

Proof. ($\Leftarrow$) Given R, we set

$$\langle \bar{x}, \bar{y}\rangle \sim \langle \bar{x}', \bar{y}'\rangle \quad :\text{iff} \quad \bar{x} = \bar{x}'\,.$$

Then $\langle R, \sim\rangle \cong \mathfrak{E}(g)$.

($\Rightarrow$) Let $\tau = \langle \delta(\bar{x}), \varphi(\bar{x}, \bar{y})\rangle$ be a k-dimensional FO-interpretation of $\mathfrak{E}(g) = \langle E, \sim\rangle$ in $\mathfrak{A}$ and let $\nu : \delta^{\mathfrak{A}} \to E$ be the corresponding isomorphism. By definition of $\mathfrak{E}(g)$, there exists a bijection $\rho : \mathbb{N}^n \to E/{\sim}$ such that

$$|\rho(\bar{k})| = g(\bar{k})\,, \quad \text{for all } \bar{k} \in \mathbb{N}^n.$$

Set $\approx := \varphi^{\mathfrak{A}}$ and let $P \subseteq \delta^{\mathfrak{A}}$ be the set containing the minimal (w.r.t. the lexicographic ordering induced by the well-ordering of $\mathfrak{A}$) element of each $\approx$-class. Then $R := {\approx} \cap (P \times \mathbb{N}^m)$ is FO-definable and the R-out-degree of an element $\bar{a} \in P$ is

$$\big|[\bar{a}]_{\approx}\big| = \big|[\nu(\bar{a})]_{\sim}\big| = g\big(\rho^{-1}\big([\nu(\bar{a})]_{\sim}\big)\big) = g\big((\rho^{-1} \circ q \circ \nu)(\bar{a})\big)\,,$$

where $q : E \to E/{\sim}$ is the projection. Since the restriction of $\rho^{-1} \circ q \circ \nu$ to P is bijective, we obtain the desired function σ by setting

$$\sigma := (\rho^{-1} \circ q \circ \nu \upharpoonright P)^{-1} : \mathbb{N}^n \to A^k. \qquad \square$$

Our characterization of Presburger equivalence structures can now be proved as follows.

Proof of Theorem 6.2. ($\Leftarrow$) Note that the equivalence structures $\mathfrak{C}_1 := \langle \mathbb{N}, E_1 \rangle$ and $\mathfrak{C}_\infty := \langle \mathbb{N}^2, E_\infty \rangle$ with

$$E_1 := \mathbb{N} \times \mathbb{N} \quad \text{and} \quad E_\infty := \{ \langle \langle n, i \rangle, \langle n, j \rangle \rangle \mid n, i, j \in \mathbb{N} \}$$

are Presburger structures. ($\mathfrak{C}_1$ has a single infinite class and $\mathfrak{C}_\infty$ has countably infinitely many.) Since Presburger structures are closed under finite disjoint unions, it therefore remains to show that $\mathfrak{E}(g)$ is Presburger, for every generalised vector partition function

$$g(\bar{x}) := \sum_{k<s} \psi_{A_k}(\bar{x} + \bar{c}_k)\,, \quad \text{for } A_k \in \mathbb{N}^{n \times m_k} \text{ and } \bar{c}_k \in \mathbb{N}^n.$$

Given such a function g, set $m := \max_k m_k$. The relation

$$R := \{ \langle \bar{x}, \bar{y}, k \rangle \in \mathbb{N}^n \times \mathbb{N}^m \times \mathbb{N} \mid k < s,\ A_k \bar{y} = \bar{x} + \bar{c}_k,\ y_i = 0 \text{ for } i \geq m_k \}$$

is Presburger definable and the out-degree of $\bar{x} \in \mathbb{N}^n$ is equal to

$$\sum_{k<s} \psi_{A_k}(\bar{x} + \bar{c}_k) = g(\bar{x})\,.$$

Consequently, we can use Lemma 6.4 to find an FO-interpretation of $\mathfrak{E}(g)$ in $\langle \mathbb{N}, +, \leq \rangle$.

($\Rightarrow$) Suppose that there exists a k-dimensional FO-interpretation of $\mathfrak{A}$ in $\langle \mathbb{N}, + \rangle$. Note that the substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ consisting of all finite equivalence classes can be defined by the FOC-formula

$$\varphi(x) := \neg \exists^\infty y [y \sim x]\,.$$

By Theorem 4.5, it therefore follows that $\mathfrak{A}_0$ is also a Presburger structure. Hence, it is sufficient to prove that $\mathfrak{A}_0 \cong \mathfrak{E}(g)$, for some generalised vector partition function. Let $P \subseteq A \subseteq \mathbb{N}^n$ be the set containing the $\leq_{\text{lex}}$-minimal element of every $\sim$-class. Since P is definable, so is the relation $R := {\sim} \cap (P \times A)$. It therefore follows by Proposition 4.11 that the function $d : \mathbb{N}^n \to \mathbb{N}$ mapping a tuple $\bar{k}$ to its R-out-degree is of the form

$$p(\bar{x}) := \sum_{i<s} \psi_{A_i}(\bar{x} + \bar{c}_i)\,.$$

Since

$$d(\bar{k}) = \begin{cases} \big|[\bar{k}]_\sim\big| & \text{if } \bar{k} \in P\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we further have $\mathfrak{A} \cong \mathfrak{E}(d)$. $\square$

We can make the description in Theorem 6.2 more explicit by replacing vector partition functions by certain polynomials.

Definition 6.5. A polynomial $p \in \mathbb{Q}[x_0, \dots, x_{n-1}]$ is *positive* if the associated polynomial function $\mathbb{Q}^n \to \mathbb{Q}$ restricts to a function $\mathbb{N}^n \to \mathbb{N} \setminus \{0\}$. ⌟

Lemma 6.6. *Let g be a generalised vector partition function. Then $\mathfrak{E}(g)$ can be written as a finite union of structures of the form $\mathfrak{E}(p)$, for positive polynomials p with integer coefficients.*

Proof. By Proposition 4.8, $g \in \mathbb{Q}[\bar{x}]$ is piecewise polynomial. Hence, there exists a finite partition $\mathcal{S}$ of $\mathbb{N}^n$ into semilinear sets and a family of polynomials $(q_S)_{S\in\mathcal{S}}$ such that

$$g \restriction S = q_S\,, \quad \text{for all } S \in \mathcal{S}\,.$$

By Proposition 4.6, we may assume that every $S \in \mathcal{S}$ is simple. For each $S \in \mathcal{S}$, fix an injective affine function φ_S with $S = \operatorname{rng}\varphi_S$. Then $q_S \circ \varphi_S$ is a polynomial in $\mathbb{Q}[\bar{x}]$. Furthermore, we have

$$\mathfrak{E}(g) \cong \sum_{S\in\mathcal{S}} \mathfrak{E}(q_S \circ \varphi_S)$$

by injectivity of φ_S.

To conclude the proof, it is therefore sufficient to show that every structure of the form $\mathfrak{E}(q)$ with $q \in \mathbb{Q}[\bar{x}]$ can be written as a finite disjoint union of structures $\mathfrak{E}(h)$ with positive $h \in \mathbb{Z}[\bar{x}]$. We can write $q = \frac{1}{\mu}q_0$ with $q_0 \in \mathbb{Z}[\bar{x}]$ and $0 < \mu < \omega$. For each tuple $\bar{c} \in [\mu]^n$, we obtain a polynomial

$$p_{\bar{c}}(\bar{x}) := q(\mu\bar{x} + \bar{c}) \in \mathbb{Z}[\bar{x}]\,.$$

(Note that the constant term of $p_{\bar{c}}$ belongs to $\mathbb{Z}$ since q induces a function $\mathbb{N}^n \to \mathbb{N}$.) Furthermore, we have

$$\mathfrak{E}(q) \cong \sum_{\bar{c}\in[\mu]^n} \mathfrak{E}(p_{\bar{c}})\,.$$

□

Let us turn to poly-growth automatic equivalence structures. One direction of Theorem 6.3 consists of the following lemma.

Lemma 6.7. *For every positive $p \in \mathbb{Q}_{\geq 0}[x_0, \dots, x_{n-1}]$, the structure $\mathfrak{E}(p)$ is poly-growth automatic.*

Proof. Suppose that

$$p = \tfrac{1}{\mu} \sum_{j<m} \lambda_j \bar{x}^{\bar{a}_j}\,, \quad \text{for } \mu, \lambda_0, \dots, \lambda_{m-1} \in \mathbb{N}\,.$$

Let $k_i := \max_j a_{j,i}$ be the maximal exponent of x_i in p, and let R be the relation of all tuples

$$\langle \bar{x}, \bar{y}_0 \dots \bar{y}_{n-1}, z, w\rangle \in \mathbb{N}^n \times \mathbb{N}^{k_0} \times \dots \times \mathbb{N}^{k_{n-1}} \times \mathbb{N} \times \mathbb{N}$$

such that

$$\begin{aligned} z &< m\,, \\ w &< \lambda_z\,, \\ y_{i,j} &< x_i\,, \quad \text{for } i < n \text{ and } j < a_{j,i}\,, \\ y_{i,j} &= 0\,, \quad \text{for } i < n \text{ and } a_{j,i} \leq j < k_i\,. \end{aligned}$$

Then R is definable in $\langle \omega, \leq\rangle$ and the R-out-degree of $\bar{x} \in \mathbb{N}^n$ is equal to

$$\sum_{j<m} \lambda_j \bar{x}^{\bar{a}_j}\,.$$

Consequently, we can use Lemma 6.4 to construct an FO-interpretation of $\mathfrak{E}(p_0)$ in $\langle\omega,\leq\rangle$, where $p_0 := \mu p$.

Finally, note that $\mathfrak{E}(p)$ can be obtained from $\mathfrak{E}(p_0)$ by taking every μ-th element of each equivalence class. Hence, $\mathfrak{E}(p)$ is isomorphic to the substructure of $\mathfrak{E}(p_0)$ defined by the formula

$$\varphi(x) := \exists^{0,\mu} y[y \sim x \wedge y \leq_{\mathrm{llex}} x]\,.$$

We have obtained FOC-interpretations of $\mathfrak{E}(p)$ in $\mathfrak{E}(p_0)$ and of $\mathfrak{E}(p_0)$ in $\langle\omega,\leq\rangle$. By Theorem 3.2 (and the fact that FOC-interpretations are closed under composition), it follows that $\mathfrak{E}(p)$ is poly-growth automatic. □

For the other direction, we need some results about sets FO-definable in the structure $\langle\omega,<\rangle$.

Definition 6.8. (a) For a partial function $f : A \to B$, we denote by $\mathfrak{K}(f) := \langle A, \ker f\rangle$ the equivalence structure where

$$\ker f := \{\, \langle a, a'\rangle \mid a, a' \in \mathrm{dom}(f),\ f(a) = f(a')\,\}\,.$$

(b) For a relation $R \subseteq A \times B$, let $\mathrm{fib}_R : B \to \mathbb{N} \cup \{\infty\}$ be the function

$$\mathrm{fib}_R(b) := \big|\{\, a \in A \mid \langle a, b\rangle \in R\,\}\big|\,.$$

(c) A polynomial $p(\bar{x})$ is *basic* if it can be written as a sum of products of binomial coefficients of the form

$$\binom{a_0x_0 + \cdots + a_{n-1}x_{n-1} + b}{c} \quad \text{with } a_0, \ldots, a_{n-1}, b, c \in \mathbb{N}\,.$$

⌟

Remark. Note that $\mathfrak{K}(f) \cong \mathfrak{E}(\mathrm{fib}_f)$. Hence we can use the former if we want to construct structures of the form $\mathfrak{E}(p)$. ⌟

Similarly to how we can characterise the Presburger-definable relations by the notion of a semilinear set, we can describe relations definable in $\langle\omega,\leq\rangle$ in a purely combinatorial way. We will show below that, for every n-ary FO-definable relation R, there exists some number $s < \omega$ such that we can write R as a finite union of equivalence classes of the equivalence relation

$$\begin{aligned}
\bar{a} \sim_s \bar{b} \quad :\text{iff}\quad & \text{for all } i < n,\ \text{we have } a_i = b_i < s \text{ or } a_i, b_i \geq s\,,\ \text{and,}\\
& \text{for all } i, j < n,\ \text{one of the following conditions holds:}\\
& -\ \ a_i = a_j + k \text{ and } b_i = b_j + k\,, \quad \text{for some } 0 \leq k < s\,,\\
& -\ \ a_i \geq a_j + s \text{ and } b_i \geq b_j + s\,,\\
& -\ \ a_i \leq a_j - s \text{ and } b_i \leq b_j - s\,.
\end{aligned}$$

We call the equivalence classes of this relation *s-cells.* Each s-cell can uniquely be described by a permutation $\sigma : [n] \to [n]$ and a function $d : [n] \to [s] + \{\infty\}$ as follows.

Definition 6.9. Let $s, n < \omega$. Given a permutation $\sigma : [n] \to [n]$ and a function $d : [n] \to [s] + \{\infty\}$, we denote by $C(\sigma, d)$ the set of all tuples $\bar{a} \in \mathbb{N}^n$ such that

$$\begin{aligned}
d(0) < \infty \quad &\text{implies} \quad a_{\sigma(0)} = d(0)\,,\\
d(0) = \infty \quad &\text{implies} \quad a_{\sigma(0)} \geq s\,,\\
d(i) < \infty \quad &\text{implies} \quad a_{\sigma(i)} = a_{\sigma(i-1)} + d(i)\,, \quad \text{for } i > 0\,,\\
d(i) = \infty \quad &\text{implies} \quad a_{\sigma(i)} \geq a_{\sigma(i-1)} + s\,, \quad \text{for } i > 0\,.
\end{aligned}$$

Sets of this form are called *s-cells.* ⌟

Example. The 4-cell $C(\mathrm{id}, d)$ associated with the identity permutation and the function $d : [7] \to [4] + \{\infty\}$ given by

$$d(0) := \infty\,,\quad d(1) := 2\,,\quad d(2) := 0\,,\quad d(3) := \infty\,,$$
$$d(4) := 1\,,\quad d(5) := \infty\,,\quad d(6) := 0$$

contains all tuples $\bar{a} \in \mathbb{N}^7$ satisfying the following inequalities.

$$a_0 \geq 4\,,\quad a_2 = a_1 = a_0 + 2\,,\quad a_3 \geq a_2 + 4\,,$$
$$a_4 = a_3 + 1\,,\quad a_6 = a_5 \geq a_4 + 4\,.$$ ⌟

Lemma 6.10.

(a) *Two s-cells are either disjoint or equal.*

(b) *In the structure $\langle \omega, \leq \rangle$, every* FO*-definable relation R is a finite union of disjoint s-cells, for some $s < \omega$.*

(c) *For every s-cell $C(\sigma, d)$, there exists an injective affine function $g : \omega^m \to \omega^n$ with* $\operatorname{rng} g = C(\sigma, d)$.

(d) *For all $m, n, s < \omega$, there exist finitely many polynomials $p_0, \dots, p_{k-1} \in \mathbb{N}[\bar{x}]$ with the following properties. For every s-cell $C(\sigma, d) \subseteq \omega^m \times \omega^n$, there exists a quantifier-free formula $\theta(\bar{x})$ and some $i \leq k$ such that*

$$\mathrm{fib}_{C(\sigma,d)}(\bar{b}) = \begin{cases} p_i(\bar{b}) & \text{if } \langle \omega, \leq \rangle \models \theta(\bar{b})\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where, in case $i = k$, we use the definition $p_k(\bar{x}) := \infty$.

Furthermore, for every affine map φ whose range is included in the set defined by θ, the composition $p_i \circ \varphi$ is a basic polynomial.

Proof. (a) Consider two s-cells $C(\sigma, d)$ and $C(\sigma', d')$ that share a common element $\bar{a} \in C(\sigma, d) \cap C(\sigma', d')$. Then

$$a_{\sigma(0)} \leq \dots \leq a_{\sigma(n-1)} \quad \text{and} \quad a_{\sigma'(0)} \leq \dots \leq a_{\sigma'(n-1)}\,,$$

which implies that $(a_{\sigma(i)})_{i<n} = (a_{\sigma'(i)})_{i<n}$. Consequently,

$$\sigma' = \tau \circ \sigma \quad \text{and} \quad d = d'\,,$$

for some permutation τ such that $a_{\tau(i)} = a_i$, for all i. It follows that $C(\sigma, d) = C(\tau \circ \sigma, d) = C(\sigma', d')$.

(b) Since the structure $\langle \omega, \leq, \mathrm{suc}, 0 \rangle$ admits quantifier elimination (see, e.g., Section 3.2 of [End01]), the relation R is a finite union of relations definable by a conjunction of atomic formulae and their negations. Such a conjunction can be written as a conjunction of formulae of the form

$$x_i = x_j + c\,,\quad x_i \geq x_j + c\,,\quad x_i = c\,,\quad x_i \geq c\,,\quad \text{for } c \in \mathbb{N}\,.$$

In particular, it can be written as a finite union of s-cells, for some s. Hence, so can R. Disjointness follows by (a).

(c) It is sufficient to construct g for cells of the form $C(\mathrm{id}, d)$ since we then obtain the corresponding function for $C(\sigma, d)$ with an arbitrary permutation σ by permuting the coordinates of g in accordance to σ. We construct g by induction on the dimension n of $C(\mathrm{id}, d)$.

If $n = 1$ and $d(0) < s$, we have $C(\mathrm{id}, d) = \{d(0)\}$ and we can set $g : \mathbb{N}^0 \to \mathbb{N}$ with $g(0) := d(0)$. If $n = 1$ and $d(0) = \infty$, we have $C(\mathrm{id}, d) = d(0) + \mathbb{N}$ and we can set $g : \mathbb{N} \to \mathbb{N}$ with $g(x) := x + d(0)$.

For the inductive step, suppose that $n > 1$. By inductive hypothesis, there exists a function $g' : \mathbb{N}^m \to \mathbb{N}^{n-1}$ whose range is the projection of $C(\mathrm{id}, d)$ to the first $n-1$ coordinates. Suppose that the components of g' are $g_0, \dots, g_{n-2} : \mathbb{N}^m \to \mathbb{N}$. We set $g := \langle g_0, \dots, g_{n-2}, g_{n-1}\rangle$ where the additional component g_{n-1} is defined as follows. If $d(n-1) < s$, we set

$$g_{n-1}(\bar{x}) := g_{n-2}(\bar{x}) + d(n-1)\,.$$

If $d(n-1) = \infty$, we set

$$g_{n-1}(\bar{x}, y) := g_{n-2}(\bar{x}) + y + d(n-1)\,.$$

(d) It is sufficient to prove the claim for tuples $\bar{b}$ with $b_0 \leq \dots \leq b_{n-1}$. For other tuples we then obtain the desired polynomials p_i and formulae θ by permuting the variables. Hence, fix such a tuple $\bar{b}$. We claim that

$$\mathrm{fib}_{C(\sigma,d)}(\bar{b}) = \binom{b_0 - s_0}{t_0} \cdot \prod_{0<j<n} \binom{b_j - b_{j-1} - s_j}{t_j},$$

for suitable constants $s_j, t_j < \omega$. For $j < n-1$, let

$$\begin{aligned} I_0 &:= \{\, i < m \mid \sigma(i) \leq \sigma(m+0) \,\}\,, \\ I_{j+1} &:= \{\, i < m \mid \sigma(m+j) < \sigma(i) \leq \sigma(m+j+1) \,\}\,. \end{aligned}$$

Then $i \in I_j$ if $b_{j-1} \leq a_{\sigma(i)} \leq b_j$ for some/all tuples $\bar{a}$ with $\bar{a}\bar{b} \in C(\sigma, d)$. (To avoid case distinctions, we will use the convention that $b_{-1} := 0$ for the rest of the proof.) Furthermore, set

$$\begin{aligned} I_j^0 := \{\, i \in I_j \mid{} & d(i) = \infty \text{ and there is } i' \in I_j \text{ with} \\ & i' > i \text{ and } d(i') = \infty \,\}\,. \end{aligned}$$

Note that every $a_{\sigma(i')}$ with $i' \in I_j \setminus I_j^0$ is at a fixed distance from some $a_{\sigma(i)}$ with $i \in I_j^0$. Hence, to choose a tuple $\bar{a}$ with $\bar{a}\bar{b} \in C(\sigma, d)$ amounts to choosing the values for $a_{\sigma(i)}$ with $i \in I_0^0 \cup \dots \cup I_{n-1}^0$. There are

$$t_j := |I_j^0|$$

such elements $a_{\sigma(i)}$ with $i \in I_j^0$, and the number of choices for each of them is equal to $b_j - b_{j-1} - s_j$ where

$$s_j := \sum \{\, d(i) \mid i \in I_j^0,\ d(i) \neq \infty \,\}$$

is the number of choices that are inadmissible because they are too close to some other element. Consequently, there are

$$\binom{b_j - b_{j-1} - s_j}{t_j}$$

choices for the part of $\bar{a}$ between b_{j-1} and b_j.

Having established the above claim, it follows from its proof that we can use the formula

$$\theta(\bar{x}) := x_0 \geq s_0 \wedge \bigwedge_{j<n-1} x_{j+1} \geq x_j + s_j \\ \wedge \bigwedge \{ x_j = x_{j-1} + s_j \mid t_j = 0 \} .$$

Finally, let φ be an affine map whose range is included in the set S defined by θ and suppose that the associated coordinate maps are

$$\varphi_j(\bar{y}) = \sum_i a_{ji} y_i + c_j \quad \text{with coefficients } a_{ji}, c_j \in \mathbb{N} .$$

Since $\operatorname{rng} \varphi \subseteq S$, we have

$$\varphi_j(\bar{y}) - \varphi_{j-1}(\bar{y}) - s_j \geq 0 , \quad \text{for all } \bar{y} .$$

Consequently,

$$\sum_i (a_{j,i} - a_{j-1,i}) y_i + (c_j - c_{j-1}) \geq s_j , \quad \text{for all } \bar{y} ,$$

which implies that

$$\alpha_{j,i} := a_{j,i} - a_{j-1,i} \geq 0 , \quad \text{for all } i , \\ \beta_j := c_j - c_{j-1} - s_j \geq 0 .$$

Hence, we have

$$\begin{aligned} &\operatorname{fib}_{C(\sigma,d)}(\varphi(\bar{y})) \\ &\quad = \binom{\varphi_0(\bar{y}) - s_0}{t_0} \cdot \prod_{0<j<n} \binom{\varphi_j(\bar{y}) - \varphi_{j-1}(\bar{y}) - s_j}{t_j} \\ &\quad = \binom{\sum_i \alpha_{0,i} y_i + \beta_0}{t_0} \cdot \prod_{0<j<n} \binom{\sum_i \alpha_{j,i} y_i + \beta_j}{t_j} , \end{aligned}$$

which is basic. □

Lemma 6.11. *Let $\mathfrak{A}$ be an equivalence structure with no infinite classes. The following statements are equivalent.*

(1) *$\mathfrak{A}$ is poly-growth automatic.*
(2) *$\mathfrak{A} \cong \mathfrak{K}(f)$ for some partial function $f : \omega^m \to \omega^n$ that is* FO*-definable in $\langle \omega, \leq \rangle$.*
(3) *$\mathfrak{A}$ is a finite disjoint union of structures of the form $\mathfrak{E}(p)$, for some polynomial $p \in \mathbb{N}[\bar{x}]$.*

Proof. (3) ⇒ (1) We have seen in Lemma 6.7 that every structure $\mathfrak{E}(p)$ with $p \in \mathbb{N}[\bar{x}]$ is poly-growth automatic. Consequently, so is every finite disjoint union of such structures.

(1) ⇒ (2) Suppose that $\mathfrak{A} = \langle A, \sim \rangle$ is poly-growth automatic and let $f : A \to A$ be the function mapping each element $a \in A$ to the $\leq_{\text{llex}}$-minimal element of its $\sim$-class. Since $\mathfrak{A}$ is FO-interpretable in $\langle \omega, \leq \rangle$, we can regard f as a partial function $\omega^k \to \omega^k$, for some k. It follows that $\mathfrak{A} \cong \mathfrak{K}(f)$.

(2) ⇒ (3) Let $f : \omega^m \to \omega^n$ be a definable partial function. Note that

$$\mathfrak{K}(f) \cong \mathfrak{E}(\operatorname{fib}_f) .$$

For $s, n < \omega$, we denote by $\mathcal{C}_n^s$ the set of all s-cells of dimension n. We can use Lemma 6.10 (b) to find some constant s such that we can write (the graph of) f as a disjoint union of s-cells.

By Lemma 6.10 (d), there exist a finite set $\mathcal{P}$ of polynomials and a finite set Θ of FO-formulae such that, for every $C \in \mathcal{C}^s_{m+n}$, there is some $p \in \mathcal{P}$ and some $\theta \in \Theta$ such that

$$\mathrm{fib}_C(\bar{b}) = p(\bar{b}), \quad \text{for all } \bar{b} \in \omega^n \text{ satisfying } \theta.$$

Using Lemma 6.10 (b) again, we obtain a constant t such that the relations defined by the formulae in Θ are unions of t-cells. It follows that there exists a function $\pi_C : \mathcal{C}^t_m \to \mathcal{P}$ such that

$$\mathrm{fib}_C \upharpoonright D = \pi_C(D), \quad \text{for all } C \in \mathcal{C}^s_{m+n} \text{ and } D \in \mathcal{C}^t_m.$$

Consequently,

$$\mathrm{fib}_f \upharpoonright D = \sum \{\, \pi_C(D) \mid C \in \mathcal{C}^s_{m+n},\ C \subseteq f \,\}, \quad \text{for every } D \in \mathcal{C}^t_m,$$

which is a polynomial in $\mathbb{Q}[\bar{x}]$.

For each $D \in \mathcal{C}^t_m$, fix an affine map φ_D with $\mathrm{rng}\,\varphi_D = D$. Then

$$\mathfrak{E}(\mathrm{fib}_f) = \sum_{D \in \mathcal{C}^t_m} \mathfrak{E}(\mathrm{fib}_f \circ \varphi_D)$$

and it follows by Lemma 6.10 (d) that each map $g_D := \mathrm{fib}_f \circ \varphi_D$ is a basic polynomial.

Fix a number $c \in \mathbb{N}$ such that $c \geq k$, for every binomial coefficient $\binom{\cdots}{k}$ appearing in g_D and set

$$g'_D(x_0, \dots, x_{n-1}) := g_D(x_0 + c, \dots, x_{n-1} + c).$$

Then

$$\mathfrak{E}(g'_D) \cong \mathfrak{E}(g_D)$$

and every binomial coefficient appearing in g'_D is of the form

$$\binom{a_0x_0 + \cdots + a_{n-1}x_{n-1} + b}{k} = \frac{1}{k!}\prod_{i<k}(a_0x_0 + \cdots + a_{n-1}x_{n-1} + (b-i)) \quad \text{with } b \geq k.$$

In particular $g'_D \in \mathbb{Q}_{\geq 0}[\bar{x}]$.

Finally, fix a number d such that $d \cdot g'_D \in \mathbb{N}[\bar{x}]$ and set

$$g''_D(x_0, \dots, x_{n-1}) := g'_D(dx_0, \dots, dx_{n-1}).$$

Then

$$\mathfrak{E}(g''_D) \cong \mathfrak{E}(g'_D) \quad \text{and} \quad g''_D \in \mathbb{N}[\bar{x}].$$

Since

$$\mathfrak{K}(f) \cong \mathfrak{E}(\mathrm{fib}_f) \cong \sum_{D \in \mathcal{C}^t_m} \mathfrak{E}(\mathrm{fib}_f \circ \varphi_D) \cong \sum_{D \in \mathcal{C}^t_m} \mathfrak{E}(g''_D),$$

the claim follows. □

Proof of Theorem 6.3. (⇒) Let $\mathfrak{A}$ be poly-growth automatic. We decompose it as $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ where $\mathfrak{B}$ is an equivalence structures with only finite classes and $\mathfrak{C}$ is one with only infinite classes. Since B and C are FOC-definable, it follows that $\mathfrak{B}$ and $\mathfrak{C}$ are poly-growth automatic. Furthermore, we can use Lemma 6.11 to decompose $\mathfrak{B}$ into a disjoint union of structures of the form $\mathfrak{E}(p)$ with $p \in \mathbb{N}[\bar{x}]$.

($\Leftarrow$) We have seen in Lemma 6.7 that every structure of the form $\mathfrak{E}(p)$ with $p \in \mathbb{N}[\bar{x}]$ is poly-growth automatic. Furthermore, the equivalence structures $\mathfrak{A}_1 := \langle 0^*, E_1 \rangle$ and $\mathfrak{A}_\infty := \langle 0^*1^*, E_\infty \rangle$ with

$$E_1 := 0^* \times 0^* \quad \text{and} \quad E_\infty := \{ \langle 0^n 1^k, 0^n 1^l \rangle \mid n, k, l < \omega \}$$

are poly-growth automatic. ($\mathfrak{A}_1$ has a single infinite class and $\mathfrak{A}_\infty$ has countably infinitely many.) The claim follows since the class of poly-growth automatic structures is closed under finite disjoint unions. □

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