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THE EXPANSION PROBLEM FOR INFINITE TREES

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Abstract. We study Ramsey-like theorems for infinite trees and similar combinatorial tools. As an application we consider the expansion problem for tree algebras.

1. Introduction

While the theory of languages of infinite trees is well-established by now, it is far less developed than other formal language theories. Further progress in this direction is currently hampered by our lack of understanding of the combinatorial properties of infinite trees. In particular, for many purposes the currently known Ramsey type theorems for trees are simply not strong enough. What would be needed instead are, for instance, analogues of Simon factorisation trees for infinite trees. Such Ramsey arguments are ubiquitous in the study of languages of infinite objects. For instance, in the theory of ω-words they appear in the original complementation proof for Büchi automata (for a modern account see, e.g., [Tho90]), when expanding a Wilke algebra to an ω-semigroup (see, e.g., [PP04]), or in the more recent work on distance automata and boundedness problems (see, e.g., [Sim90, Sim94, AKY08, Col21]).

There are at least two persistent problems when trying to extend our repertoire of combinatorial tools to infinite trees. The first one concerns the step from arbitrary trees to regular ones: while many arguments only work if the considered trees are regular, the only known method of reducing a given tree to an equivalent regular one is based on automata, which are not applicable in all contexts. For instance, in [BI09] all proofs work exclusively with regular trees, and only at the very end the authors transfer their results to arbitrary trees (which was possible in this particular case since the languages under consideration were regular and therefore uniquely determined by which regular trees they contain).

The second problem concerns trees that are highly-branching. Many of the known tools from the theory of ω-semigroups can be generalised to trees that are thin, i.e., that have only countably many infinite branches. But all attempts to extend them to (at least some) non-thin trees have failed so far. For example, in [BIS13] the authors only consider languages of thin trees since their methods do not apply to non-thin ones. Later in [BS13], they then successfully adapted their approach to study unambiguous languages of non-thin trees, utilising the fact that trees in unambiguous languages are in a certain sense governed by their thin prefixes.

The only combinatorial methods known so far that work well even in light of the above issues are those based on automata and games since, in a certain sense, games provide a way to reduce a problem concerning the whole tree into one only involving a single branch.

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(In fact, one of the motivations for this paper stems from a wish to deeper understand how exactly this is achieved, in particular during the translation of a formula into an automaton. Theorem 5.27 below might be considered to give a partial answer.) Unfortunately, there are many questions that resist to being phrased in automata-theoretic or game-theoretic terms.

In this article we start by quickly reviewing the existing techniques to study combinatorial properties of infinite trees, presenting them in the unifying language of tree algebras from [Blu20]. We then take a look at several new approaches. We determine how far they carry and what the problems are that prevent us from continuing. Our contributions are mainly conceptual. We raise many open questions, but provide few answers. None of the results below are very deep and several remain partial. The main purpose of the article is to draw attention to a problem I consider central for further progress.

It seems that such progress will likely not come from abstract considerations but by working on concrete problems. The recently developed algebraic approaches to languages of infinite trees [BI09, Blu11, Blu13, BIS13, BS13, BK19, BCPS19, Blu20, Blu21, Blu23] seems to provide many opportunities to test our combinatorial tools. Our focus will therefore be on a particular application from this domain, one that we call the *Expansion Problem:* the problem of whether a given algebra whose product is defined only for some trees can be expanded to one whose product is defined everywhere, analogously to the expansion of a Wilke algebra (where the product is defined only for ultimately periodic words) to an ω-semigroup (where we can multiply arbitrary ω-words). This problem turns out to be a good test bed for the various approaches we consider. We solve it in some special cases, but none of our approaches is strong enough to solve the general case.

The overview of the article is as follows. We start in Section 2 with setting up the algebraic framework we will be working in. Section 3 contains a brief survey of the existing Ramsey Theorems for trees. The Expansion Problem is defined in Section 4, where we also recall some tools from [Blu21] to prove uniqueness of expansions. The main technical part of the article are Sections 5 and 6, which contain two tools to study expansions. The first one are so-called *evaluations,* which are a weak form of a Simon tree, the second one are *consistent labellings,* which are somewhat similar to automata. The final two sections (7 and 8) contain two applications. The first one recalls results of [BS13] about a characterisation of unambiguous languages in terms of consistent labellings, while the second one uses consistent labellings to define classes of tree algebras with unique expansions.

Finally, let us highlight the concrete contributes of this article. (All terminology will be defined in the respective section below.)

- We streamline and generalise the definitions of two combinatorial tools from the literature: evaluations [Pup10, CCP18] and consistent labellings [BS13].
- We prove the existence of expansions for MSO-definable $\mathbb{T}^{\mathrm{reg}}$-algebras in Theorem 4.6.
- We prove the existence of certain evaluations in Theorems 5.26 and 5.27.
- We solve the expansion problem for thin trees in Section 5.1, and the expansion problems for deterministic and (partially for) branch-continuous tree algebras in Section 8.

2. Tree algebras

We start with a brief introduction to the algebraic framework we will be working in. A more detailed account can be found in [Blu20, Blu21, Blua] (in increasing order of abstractness). Let us fix notation and conventions. For $n < \omega$, we set $[n] := \{0, \dots, n-1\}$. We denote tuples $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle$ with a bar. The empty tuple is $\langle\rangle$. The *range* of a function $f : A \to B$ is the set $\operatorname{rng} f := f[A]$. We denote the disjoint union of two sets by $A + B$, and we denote the union of two functions $f : A \to B$ and $f' : A' \to B$ by $f + f' : A + A' \to B$.

Let us quickly recall some material from the theory of ω-semigroups (see, e.g., [PP04] for an introduction). An *ω-semigroup* is a two-sorted structure $\langle S_1, S_\omega \rangle$ with three products

$$\cdot : S_1 \times S_1 \to S_1\,, \quad \cdot : S_1 \times S_\omega \to S_\omega\,, \quad \pi : (S_1)^\omega \to S_\omega$$

satisfying several associative laws. A *Wilke algebra* is a two-sorted structure $\langle S_1, S_\omega \rangle$ with two products and an ω-power operation

$$\cdot : S_1 \times S_1 \to S_1\,, \quad \cdot : S_1 \times S_\omega \to S_\omega\,, \quad -^\omega : S_1 \to S_\omega$$

again satisfying several associative laws. The laws for the ω-power are

$$(ab)^\omega = a(ba)^\omega \quad \text{and} \quad (a^n)^\omega = a^\omega, \quad \text{for all } a, b \in S_1 \text{ and } 0 < n < \omega\,.$$

One can show via a Ramsey argument that every finite Wilke algebra has a unique expansion to an ω-semigroup.

To make this article accessible to a wider audience, we have tried to keep the category-theoretic prerequisites at a minimum. Let us briefly recall some standard notions. A *functor* $\mathbb{F}$ is an operation that maps every sorted set A to some sorted set $\mathbb{F}A$, and every function $f : A \to B$ between such sets to a function $\mathbb{F}f : \mathbb{F}A \to \mathbb{F}B$ such that $\mathbb{F}$ preserves identity maps and composition of functions. A *natural transformation* $\tau : \mathbb{F} \Rightarrow \mathbb{G}$ between two functors $\mathbb{F}$ and $\mathbb{G}$ is a family $\tau = (\tau_A)_A$ (indexed by all sorted sets A) of functions $\tau_A : \mathbb{F}A \to \mathbb{G}A$ satisfying

$$\tau_A \circ \mathbb{F}f = \mathbb{G}f \circ \tau_B\,, \quad \text{for all } f : A \to B\,.$$

Usually, we omit the index A and simply write τ instead of τ_A.

To model ranked trees, we work in a many-sorted setting where the sort of a tree represents the set of *variables* or *holes* appearing in it. Hence, we fix a countably infinite set X of variables and use the set $\Xi := \mathscr{P}_{\mathrm{fin}}(X)$ of finite subsets of X as sorts. In addition, for technical reasons we equip the labels of our trees with an ordering. Hence, we will work with *partially ordered Ξ-sorted sets* which are families $A = (A_\xi)_{\xi \in \Xi}$ where each component A_ξ is partially ordered. A function $f : A \to B$ between two such sets is then a family $f = (f_\xi)_{\xi \in \Xi}$ of monotone functions $f_\xi : A_\xi \to B_\xi$. In the following we will for simplicity use the term *sorted set* for 'partially ordered Ξ-sorted set' and the term *function* for a function between such sets. Sometimes it is convenient to identify a sorted set $A = (A_\xi)_{\xi \in \Xi}$ with its disjoint union $A = \sum_{\xi \in \Xi} A_\xi$. Then a function $f : A \to B$ corresponds to a sort-preserving and order-preserving function between the corresponding disjoint unions.

Given a sorted set A an *A-labelled tree* is a possibly infinite tree t where the vertices are labelled by elements of A and the edges by variables from X in such a way that a vertex with a label $a \in A_\xi$ of sort ξ has exactly one outgoing edge labelled by x, for every $x \in \xi$ (and no other edges). We identify such a tree with a function $t : \operatorname{dom}(t) \to A$, where $\operatorname{dom}(t)$ is the set of vertices of t. (We consider $\operatorname{dom}(t)$ to be a sorted set where $v \in \operatorname{dom}(t)$ has the same sort as its label $t(v)$.) As usual, we identify the vertices of t with finite sequences of directions. Since, in our case, we can take the variables for directions, this turns $\operatorname{dom}(t)$ into a prefix-closed subset of X^*. Using this identification, we can write the root of t as $\langle\rangle$. If there is an x-labelled edge from a vertex u to v, we call v the *x-successor* of u. We denote it by $\operatorname{suc}_x(v)$. A *branch* of t is a maximal path starting at the root.

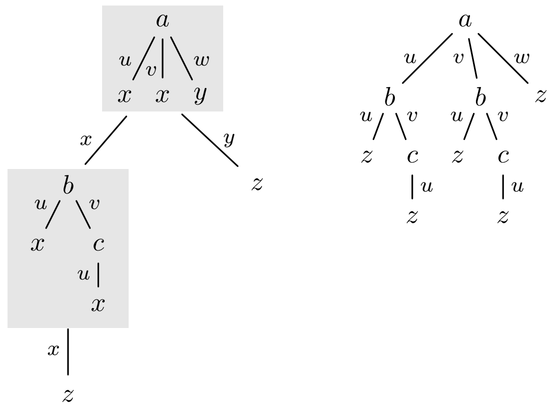


Figure 1: The flattening operation: t and $\mathrm{flat}(t)$

Definition 2.1. Let A be a sorted set.

(a) We set $\mathbb{T}^\times A := (\mathbb{T}^\times_\xi A)_{\xi \in \Xi}$ where $\mathbb{T}^\times_\xi A$ denotes the set of all $(A + \xi)$-labelled trees t (where the elements in ξ are assumed to have sort $\emptyset$) satisfying the following conditions.
- Every variable $x \in \xi$ appears at least once in t.
- The root of t is not labelled by a variable.

We order the elements of $\mathbb{T}^\times A$ by

$$s \leq t \quad :\text{iff} \quad \mathrm{dom}(s) = \mathrm{dom}(t) \quad \text{and} \quad s(v) \leq t(v) \text{ for all vertices } v\,.$$

(b) For a function $f : A \to B$, we denote by $\mathbb{T}^\times f : \mathbb{T}^\times A \to \mathbb{T}^\times B$ the function applying f to every label of the given tree (leaving the variables unchanged).

(c) For $t \in \mathbb{T}^\times_\xi A$, we denote by $\mathrm{dom}_0(t) \subseteq \mathrm{dom}(t)$ the set of all vertices that are not labelled by a variable. ⌟

We need the following two operations on trees.

Definition 2.2. Let A be a sorted set.

(a) The *singleton operation* $\mathrm{sing} : A \to \mathbb{T}^\times A$ maps every letter $a \in A_\xi$, to the tree $\mathrm{sing}(a)$ consisting of the root with label a attached to which is one leaf with label x, for every $x \in \xi$.

(b) The *flattening operation* $\mathrm{flat} : \mathbb{T}^\times \mathbb{T}^\times A \to \mathbb{T}^\times A$ is a generalisation of term substitution. It takes a tree t labelled by trees $t(v) \in \mathbb{T}^\times A$ and combines them into a single tree as follows (see Figure 1).

- We take the disjoint union of all trees $t(v)$, for $v \in \mathrm{dom}(t)$ (where, if $t(v) = x \in X$ is a variable and not a tree, we treat $t(v)$ as a 1-vertex tree whose root is labelled x);
- from each component $t(v)$ such that $t(v)$ is a proper tree and not just a variable, we delete every vertex labelled by a variable $x \in X$;
- we redirect every edge of $t(v)$ leading to such a deleted vertex to the root of $t(u_x)$, where u_x is the x-successor of v in t; and
- we unravel the resulting graph into a tree. ⌟

Remark. The triple $\langle \mathbb{T}^\times, \mathrm{flat}, \mathrm{sing} \rangle$ forms what is called a *monad* in category-theoretical language, which means that $\mathrm{flat} : \mathbb{T}^\times \mathbb{T}^\times \Rightarrow \mathbb{T}^\times$ and $\mathrm{sing} : \mathrm{Id} \Rightarrow \mathbb{T}^\times$ are natural transformations satisfying the following three equations.

$$\begin{aligned}\text{flat} \circ \text{sing} &= \text{id}\,,\\ \text{flat} \circ \mathbb{T}^\times \text{sing} &= \text{id}\,,\\ \text{flat} \circ \mathbb{T}^\times \text{flat} &= \text{flat} \circ \text{flat}\,.\end{aligned}$$ ⌟

Definition 2.3. Let $t \in \mathbb{T}^\times A$ be a tree.

(a) The *tree order* $\preceq$ is the ordering on $\text{dom}(t)$ defined by

$$u \preceq v \quad :\text{iff} \quad u \text{ lies on the path from the root to } v\,.$$

(b) A *factorisation* of t is a tree $T \in \mathbb{T}^\times\mathbb{T}^\times A$ with $\text{flat}(T) = t$. We call each tree $T(v)$, for $v \in \text{dom}_0(T)$, a *factor* of t.

(c) Given vertices u and $\bar{v} = (v_x)_{x\in\xi}$ of t such that $\bar{v}$ forms an antichain (with respect to $\preceq$) and $u \prec v_x$, for all x, we define

$$[u, \bar{v}) := \{\, w \in \text{dom}(t) \mid u \preceq w \text{ and } v_x \npreceq w\,,\ \text{for all } x \,\}\,.$$

We call ξ the *sort* of $[u, \bar{v})$.

We denote by $t[u, \bar{v})$ the restriction of $t : \text{dom}(t) \to A$ to the set $[u, \bar{v}) \cup \bar{v}$ (where v_x is labelled by the variable x while all other vertices have the same label as in t). We call $t[u, \bar{v})$ the *factor* of t *between* u and $\bar{v}$. In the special case where $\xi = \emptyset$, we obtain the *subtree* of t *rooted* at u, which we usually denote by $t|_u$. ⌟

Remark. (a) When identifying the vertices of a tree with words in X^*, the tree order $\preceq$ is just the prefix ordering.

(b) A factor $t[u, \bar{v})$ may contain additional variables besides those at the vertices $\bar{v}$. More precisely, we have $t[u, \bar{v}) \in \mathbb{T}^\times_{\xi\cup\zeta} A$ where ξ is the sort of $[u, \bar{v})$ and ζ is the set of those variables of t that appear at some vertex $w \in [u, \bar{v})$. ⌟

There are several special classes of trees we are interested in below.

Definition 2.4. (a) A *submonad* of $\mathbb{T}^\times$ is a functor $\mathbb{T}^0$ such that

- $\mathbb{T}^0 A \subseteq \mathbb{T}^\times A$, for every sorted set A,
- $\mathbb{T}^0 f = \mathbb{T}^\times f \restriction \mathbb{T}^0 A$, for every function $f : A \to B$, and
- $\mathbb{T}^0 A$ is closed under flat and sing, that is,

$$\begin{aligned}\text{flat}(t) &\in \mathbb{T}^0 A\,, \quad \text{for all } t \in \mathbb{T}^0\mathbb{T}^0 A\,,\\ \text{sing}(a) &\in \mathbb{T}^0 A\,, \quad \text{for all } a \in A\,.\end{aligned}$$

We write $\mathbb{T}^0 \subseteq \mathbb{T}^\times$ to denote this fact.

(b) Similarly, given two submonads $\mathbb{T}^0, \mathbb{T}^1 \subseteq \mathbb{T}^\times$, we write $\mathbb{T}^0 \subseteq \mathbb{T}^1$ if $\mathbb{T}^0 A \subseteq \mathbb{T}^1 A$, for all A.

(c) We are particularly interested in the following submonads. $\mathbb{T}$ denotes the subset of all *linear trees,* i.e., trees where each variable appears exactly once. $\mathbb{T}^{\text{fin}}$ denotes the subset of *finite linear trees,* $\mathbb{T}^{\text{reg}}$ the subset of *regular linear trees,* $\mathbb{T}^{\text{thin}}$ the subset of *thin linear trees,* i.e., trees with only countably many infinite branches, and $\mathbb{T}^{\text{wilke}} := \mathbb{T}^{\text{thin}} \cap \mathbb{T}^{\text{reg}}$ the subset of all trees that are thin and regular. The corresponding classes of non-linear trees are denoted by $\mathbb{T}^{\times\text{fin}}$, $\mathbb{T}^{\times\text{reg}}$, etc. ⌟

Remark. Note that $\mathbb{T}^{\times\text{thin}}$ and $\mathbb{T}^{\times\text{wilke}}$ do not form submonads of $\mathbb{T}^\times$, since they are not closed under flat. This is different for $\mathbb{T}^{\text{thin}}$ and $\mathbb{T}^{\text{wilke}}$, which are in fact submonads of $\mathbb{T}$. ⌟

In algebraic language theory one uses algebras (usually finite ones) to recognise languages. In our setting these algebras take the following form.

Definition 2.5. Let $\mathbb{T}^0 \subseteq \mathbb{T}^\times$.

(a) A $\mathbb{T}^0$-*algebra* $\mathfrak{A} = \langle A, \pi \rangle$ consists of a sorted set A and a *product* $\pi : \mathbb{T}^0 A \to A$ satisfying

$$\pi \circ \mathrm{sing} = \mathrm{id} \quad \text{and} \quad \pi \circ \mathbb{T}^0 \pi = \pi \circ \mathrm{flat}\,.$$

The first equation is called the *unit law,* the second one the *associative law.*

(b) A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *finitary* if it is finitely generated and every domain A_ξ is finite.

(c) A *morphism* between two $\mathbb{T}^0$-algebras $\mathfrak{A} = \langle A, \pi \rangle$ and $\mathfrak{B} = \langle B, \pi \rangle$ is a function $\varphi : A \to B$ commuting with the respective products in the sense that

$$\varphi \circ \pi = \pi \circ \mathbb{T}^0 \varphi\,.$$

(d) The *free* $\mathbb{T}^0$-*algebra* generated by a sorted set Σ is the algebra $\mathbb{T}^0\Sigma := \langle \mathbb{T}^0\Sigma, \mathrm{flat} \rangle$.

(e) A $\mathbb{T}^0$-algebra $\mathfrak{A}$ *recognises* a language $K \subseteq \mathbb{T}^0_\xi \Sigma$ if there exists a morphism $\eta : \mathbb{T}^0\Sigma \to \mathfrak{A}$ with $K = \eta^{-1}[P]$, for some $P \subseteq A_\xi$. ⌟

Example. The following algebra $\mathfrak{A} = \langle A, \pi \rangle$ recognises the language K of all trees $t \in \mathbb{T}^\times_\emptyset \{a, b\}$ that contain at least one letter a. For each sort ξ, we use two elements $0_\xi, 1_\xi$. Hence,

$$A_\xi := \{0_\xi, 1_\xi\}\,.$$

The product is defined by

$$\pi(t) := \begin{cases} 1 & \text{if } t \text{ contains the label } 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $K = \varphi^{-1}(1_\emptyset)$, where $\varphi : \mathbb{T}^\times\{a, b\} \to A$ is the morphism mapping $\mathrm{sing}(a)$ to 1 and $\mathrm{sing}(b)$ to 0. ⌟

We will often use the usual term notation for trees and elements in an algebra. That is, for $s \in \mathbb{T}^\times_\xi A$ and a ξ-tuple $\bar{r}$ of trees and/or variables, we denote by $s(\bar{r})$ the tree obtained from s by replacing every variable $x \in \xi$ by the tree r_x. Similarly, for an algebra $\mathfrak{A}$, an element $a \in A_\xi$, and a ξ-tuple $\bar{b}$ of elements and/or variables, we denote by $a(\bar{b})$ the product of the tree $s(\bar{r})$ where $s := \mathrm{sing}(a)$ and $r_x := \mathrm{sing}(b_x)$ (or $r_x := b_x$, if b_x is a variable).

A complication of the theory of infinite trees is the fact that some finitary $\mathbb{T}^\times$-algebras recognise non-regular languages [BK19]. For this reason we have to consider a smaller class of algebras.

Definition 2.6. (a) We denote *first-order logic* by FO and *monadic second-order logic* by MSO.

(b) Let $\mathbb{T}^0 \subseteq \mathbb{T}^\times$. A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is MSO-*definable* if it is finitary and there exists a finite set $C \subseteq A$ of generators of $\mathfrak{A}$ with the following property: for every $a \in A$, there exists an MSO-formula φ_a such that

$$t \models \varphi_a \quad :\text{iff} \quad \pi(t) \geq a\,, \quad \text{for all } t \in \mathbb{T}^0 C\,.$$

If all formulae φ_a belong to FO, we call $\mathfrak{A}$ FO-*definable.* ⌟

Example. The algebra from the previous example is FO-definable. (The formulae φ_0 and φ_1 only have to check whether or not the given tree contains the label 1.) ⌟

Using this notion we obtain the following characterisation (for proofs see Theorems 3.3 and 3.4 of [Blu20]; or more generally Theorem 9.4 and Corollary 9.7 of [Blu21]).

Theorem 2.7. *Let* $\mathbb{T}^0 \subseteq \mathbb{T}^\times$.

(a) *A finitary* $\mathbb{T}^0$-*algebra* $\mathfrak{A}$ *is* MSO-*definable if, and only if, every language recognised by* $\mathfrak{A}$ *is regular.*

(b) *A language $K \subseteq \mathbb{T}^0\Sigma$ is regular if, and only if, it is recognised by some* MSO*-definable* $\mathbb{T}^0$*-algebra.*

The definition above is not very enlightening as it is basically just a restatement of the preceding theorem. Although a more algebraic characterisation has been found in [Blu20], a simpler one would be appreciated. In particular, it would be nice to find a system of inequalities axiomatising the class of MSO-definable algebras.

Open Question. *Find a concrete description of a system of inequalities that axiomatises the class of* MSO*-definable* $\mathbb{T}^\times$*-algebras.*

By general arguments we know that such a system of inequalities exists, although it might be infinite and the terms in the inequalities are in general profinite (see [Blu21] for the details).

3. Partition theorems for trees

Let us start with a brief overview of the existing partition theorems for trees, followed by some remarks on how they might be extended and how they might not. The seminal partition theorem for trees is the one by Milliken.

Definition 3.1. Let $t \in \mathbb{T}_\emptyset\Sigma$ be a tree.

(a) The *level* of a vertex $v \in \mathrm{dom}(t)$ is the number of vertices u with $u \prec v$. We denote it by $|v|$.

(b) Let 1 be a set containing exactly one element of each sort. A *strong embedding* of a tree $s \in \mathbb{T}_\emptyset 1$ into t is a function $\varphi : \mathrm{dom}(s) \to \mathrm{dom}(t)$ such that, for all $u, v \in \mathrm{dom}(s)$,

$$\begin{aligned} u \preceq v \quad &\text{iff} \quad \varphi(u) \preceq \varphi(v)\,, \\ \mathrm{suc}_x(u) \preceq v \quad &\text{iff} \quad \mathrm{suc}_x(\varphi(u)) \preceq \varphi(v)\,, \\ |u| = |v| \quad &\text{implies} \quad |\varphi(u)| = |\varphi(v)|\,. \end{aligned}$$

⌟

Theorem 3.2 (Milliken [Mil79])**.** *Let $t \in \mathbb{T}A$ be an infinite tree without leaves, C a finite set of colours, $m < \omega$, and let $S_m \subseteq \mathbb{T}_\emptyset 1$ be the set of all finite trees such that every leaf has level m. Given a function λ mapping every strong embedding $s \to t$ with $s \in S_m$ to some colour in C, there exist a strong embedding $\varphi : h \to t$ and a colour $c \in C$ such that $h \in \mathbb{T}_\emptyset 1$ is a tree without leaves and*

$$\lambda(\varphi \circ \psi) = c\,, \quad \textit{for all strong embeddings } \psi : s \to h \textit{ with } s \in S_m\,.$$

For our purposes it is sufficient to consider embeddings $s \to t$ with $s \in S_1$, which correspond to strongly embedded factors $t[u, \bar v)$ of t. The main limitation of the theorem is that it does not give us any information about factors $[u, \bar v)$ whose end-points are not strongly embedded. For a stronger statement we need additional assumptions on the labelling. For instance, for labellings of finite words, there is the Factorisation Tree Theorem of Simon [Sim90] which states that, if the labelling is additive (i.e., the colours form a semigroup), we can recursively factorise the given word into homogeneous parts. This theorem has been adapted to trees by Colcombet [Col21] as follows.

Definition 3.3. Let $t \in \mathbb{T}A$ be a tree and $\mathfrak{S}$ a semigroup.

(a) An *additive $\mathfrak{S}$-labelling* of t is a function λ that maps every edge e of t to a semigroup element $\lambda(e) \in S$. Each such function can be extended to all non-empty finite paths $p = (v_i)_i$ by setting

$$\lambda(p) := \prod_i \lambda(v_i, v_{i+1})\,.$$

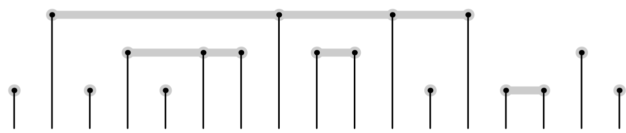


Figure 2: A function σ (along a single branch of t) and the corresponding relation $\sqsubseteq_\sigma$, indicated via the grey bars.

In case $\mathfrak{S}$ is an ω-semigroup, we can extend the notation $\lambda(p)$ to infinite paths using the same definition as above. Finally, for vertices $u \prec v$, we will also use the notation

$$\lambda(u,v) := \lambda(p)\,, \quad \text{where } p \text{ is the path from } u \text{ to } v\,.$$

(b) Given a function $\sigma : \mathrm{dom}(t) \to [k]$, we define a binary relation $\sqsubset_\sigma$ on $\mathrm{dom}(t)$ (see Figure 2) by

$$x \sqsubset_\sigma y \quad :\text{iff} \quad x \prec y\,, \quad \sigma(x) = \sigma(y)\,, \quad \text{and}$$
$$\sigma(z) \leq \sigma(x)\,, \quad \text{for all } x \preceq z \preceq y\,.$$

As usual, $\sqsubseteq_\sigma$ denotes the reflexive version of $\sqsubset_\sigma$.

(c) A *weak Ramseyan split* of an $\mathfrak{S}$-labelling λ is a function $\sigma : \mathrm{dom}(t) \to [k]$ such that

$$\lambda(x,y) = \lambda(x,y) \cdot \lambda(x',y')\,, \quad \text{for all } x \sqsubset_\sigma y \text{ and } x' \sqsubset_\sigma y'$$
$$\text{such that } y \sqsubseteq_\sigma y' \text{ or } y' \sqsubseteq_\sigma y\,.$$

⌟

Theorem 3.4 (Colcombet)**.** *Let $\mathfrak{S}$ be a finite semigroup. There exists a number $N < \omega$ such that every $\mathfrak{S}$-labelling λ of some tree t has a weak Ramseyan split $\sigma : \mathrm{dom}(t) \to [N]$ with $\sigma(\langle\rangle) = N-1$. Furthermore, this split is* MSO*-definable.*

As an example of how to apply this theorem, let us mention the following result from [Blu13] (Theorem 4.4) that can be used to turn arbitrary trees into regular ones while preserving an edge labelling.

Definition 3.5. Let $\mathfrak{S}$ be an ω-semigroup and λ an $\mathfrak{S}$-labelling of some tree t. We write

$$\lim \lambda := \{\, \lambda(\beta) \mid \beta \text{ a branch of } t \,\}\,.$$

⌟

Theorem 3.6. *Let $\mathfrak{S}$ be a finite ω-semigroup. For every $\mathfrak{S}$-labelling λ of a tree t, there exists a regular tree t_0 and a regular $\mathfrak{S}$-labelling λ_0 of t_0 such that* $\lim \lambda_0 = \lim \lambda$*.*

The proof consists in fixing a weak Ramseyan split of λ and using it to replace certain subtrees of t by back-edges. The unravelling of the resulting graph is the desired regular tree t_0.

As a second application let us see how to use Ramseyan splits to evaluate products in a $\mathbb{T}$-algebra.

Definition 3.7. Let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra, for some $\mathbb{T}^0 \subseteq \mathbb{T}^\times$.

(a) The *canonical edge labelling* λ of a tree $t \in \mathbb{T}^0 A$ is defined by

$$\lambda(u,v) := \pi(t[u,v))\,, \quad \text{for } u \prec v\,.$$

(b) Let L be either FO or MSO, and let λ be a given type of labelling, i.e., a function assigning to each tree t some labelling λ_t (which may be an edge labelling or a vertex labelling). We say that $\mathfrak{A}$ is L-definable *with respect to* λ if $\mathfrak{A}$ is finitary and, for every finite $C \subseteq A$ and every $a \in A_\xi$, there exists an L-formula φ_a such that

$$\langle t, \lambda_t \rangle \models \varphi_a \quad \text{iff} \quad \pi(t) \geq a\,, \quad \text{for all } t \in \mathbb{T}^0_\xi C\,,$$

where $\langle t, \lambda_t \rangle := \big\langle \mathrm{dom}(t), \preceq, (\mathrm{suc}_x)_x, (P_c)_{c \in C+\xi}, (R_s)_{s \in S} \big\rangle$ denotes the usual encoding of t as a relational structure with unary predicates P_c for the labels of t and additional relations R_s (either unary or binary) for the labelling λ_t. In case λ_t is an edge labelling, we assume that the relations R_s contain the labels for all pairs of vertices $u \prec v$, not only those where v is a successor of u. (This is important in the case where $L = \mathrm{FO}$.) ⌟

Remark. Note that the canonical edge labelling of a tree $t \in \mathbb{T}_\xi A$ is an $\mathfrak{S}$-labelling for the ω-semigroup $\mathfrak{S} = \langle S_1, S_\omega \rangle$ with elements

$$S_1 := \sum_{\eta \subseteq \xi} A_{\eta + \{z\}} \quad \text{and} \quad S_\omega := \sum_{\eta \subseteq \xi} A_\eta \,,$$

where $z \in X$ is some fixed variable with $z \notin \xi$ that we use to define the semigroup product. ⌟

Proposition 3.8. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^\times$. Every finitary $\mathbb{T}^0$-algebra is* FO*-definable with respect to the canonical edge labelling.*

Proof. Fix $a \in A_\xi$ and $C \subseteq A$. The formula φ_a checks whether the given tree t has a leaf. If this is the case it picks one, say v, and then checks whether

$$\lambda(\langle\rangle, v) \cdot t(v) \geq a \,.$$

Otherwise, for every sort ζ used by some element of C, we fix some variable $z \in \zeta$. If we start at the root of t and follow the successors labelled by one of these chosen variables, we obtain an infinite branch of t. This branch is FO-definable. The formula φ_a checks that the branch contains an infinite sequence $v_0 \prec v_1 \prec \dots$ of vertices such that

$$\lambda(v_i, v_k) = \lambda(v_0, v_1) \,, \quad \text{for all } i < k < \omega \,,$$
$$\lambda(\langle\rangle, v_0) \cdot \lambda(v_0, v_1)^\omega \geq a \,.$$

This fact can be expressed in first-order logic using a trick of Thomas [Tho81, Lemma 1.4]. □

We can improve this result by encoding the corresponding edge labelling by a vertex labelling.

Definition 3.9. Let L be either FO or MSO, and let $\tau : \mathrm{dom}(t) \to C$ be a vertex labelling of some tree $t \in \mathbb{T}^\times A$.

(a) We say that τ is L-definable if, for every $c \in \mathrm{rng}\,\tau$, there exists an L-formula $\varphi_c(x)$ such that

$$\tau(v) = c \quad \text{iff} \quad t \models \varphi_c(v) \,, \quad \text{for every } v \in \mathrm{dom}(t) \,.$$

The definition for labellings of edges is analogous.

(b) We say that the product π of an algebra $\mathfrak{A}$ is L-definable on t *in terms of* a vertex labelling $\tau : \mathrm{dom}(t) \to C$ if, for every $a \in A_\xi$ and every $\eta \subseteq \xi$, there exists an L-formula $\psi_{a,\eta}(x, \bar{y})$ (where $\bar{y}$ is an η-tuple) such that

$$\pi(t[u, \bar{v})) = a \quad \text{iff} \quad \langle t, \tau \rangle \models \psi_{a,\eta}(u, \bar{v}) \,,$$

for every factor $[u, \bar{v})$ of sort η. ⌟

The following result is based on a similar theorem by Colcombet [Col07].

Theorem 3.10. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}$-algebra. For every finite set $C \subseteq A$, there exists a finite (unsorted) set S such that every $t \in \mathbb{T}C$ has a labelling $\sigma : \mathrm{dom}(t) \to S$ such that the product on t is* FO*-definable in terms of σ. Furthermore, if $\mathfrak{A}$ is* MSO*-definable, then so is σ.*

Proof. We have shown in Proposition 3.8 that $\mathfrak{A}$ is FO-definable with respect to the canonical edge labelling λ. Consequently, it is sufficient to find a vertex labelling σ such that λ is FO-definable in terms of σ.

By Theorem 3.4, there exists an MSO-definable weak Ramseyan split $\sigma_0 : \mathrm{dom}(t) \to [k]$ for λ. Unfortunately, λ does not need to be FO-definable in terms of σ_0. Therefore, we define an extended labelling

$$\sigma : \mathrm{dom}(t) \to [k] \times A_{\{z\}}$$

(for some fixed but arbitrary variable z) as follows. We denote by $p(v)$ the predecessor of $v \in \mathrm{dom}(t)$ (if it exists). Fixing an arbitrary element $a_0 \in A_{\{z\}}$, we set

$$\sigma(v) := \begin{cases} \langle \sigma_0(v),\, \lambda(p(v), v) \rangle & \text{if } p(v) \text{ is defined,} \\ \langle \sigma_0(v),\, a_0 \rangle & \text{otherwise.} \end{cases}$$

Note that, if $\mathfrak{A}$ is MSO-definable, then so is σ. Hence, it remains to prove that λ is FO-definable in $\langle t, \sigma \rangle$.

We will construct an FO-formula computing $\lambda(u, v)$ by induction on the minimal number n such that

$$\sigma_0(w) \leq m\,, \quad \text{for all } u \prec w \prec v\,.$$

Let w_0 be the minimal vertex $u \prec w_0 \prec v$ with $\sigma_0(w_0) = m$, let w_2 be the maximal one, and let w_1 be the minimal vertex $w_0 \prec w_1 \preceq w_2$ with $\sigma_0(w_1) = m$. We distinguish several cases. If v is the successor of u, we can read off $\lambda(u, v)$ from $\sigma(v)$. If $w_0 = w_2$, we have

$$\lambda(u, v) = \lambda(u, w_0) \cdot \lambda(w_0, v)\,,$$

where both factors can be computed by inductive hypothesis. Finally, consider the case where $w_0 \prec w_2$. Then $w_0 \sqsubset_{\sigma_0} w_1 \sqsubseteq_{\sigma_0} w_2$ implies that

$$\begin{aligned} \lambda(u, v) &= \lambda(u, w_0) \cdot \lambda(w_0, w_2) \cdot \lambda(w_2, v) \\ &= \lambda(u, w_0) \cdot \lambda(w_0, w_1) \cdot \lambda(w_2, v)\,. \end{aligned}$$

All three factors can be computed by inductive hypothesis. □

The problem with the above theorems is that they require access to the canonical edge-labelling λ and, in order to obtain this labelling, we need to be able to compute products $\pi([u, v))$ where the factors $[u, v)$ are usually infinite. Unfortunately, in many applications we only know the products of *finite* factors. For instance, we cannot combine Theorem 3.6 and Proposition 3.8 to conclude that every tree $t \in \mathbb{T}A$ over a finitary $\mathbb{T}$-algebra $\mathfrak{A}$ can be replaced by a regular one with the same product since we do not know whether or not the regular labelling λ_0 constructed in Theorem 3.6 is a canonical edge-labelling.

We conclude this section with a counterexample showing that some natural ways to strengthen Theorem 3.10 do not work. It turns out that in general, if we want to compute $\pi(t[u, \bar{v}))$, we need to know how the factor $[u, \bar{v})$ is embedded in the tree. Just looking at the values $\sigma(u)$ and $\sigma(v_i)$ provided by some labelling σ is not enough. This is why the FO-formula constructed in Theorem 3.10 also inspects the value of σ for vertices on the paths between u and the v_i.

Definition 3.11. Let $t \in \mathbb{T}^{\times} A$ be a tree.

(a) For $u, v \in \mathrm{dom}(t)$, we denote by $u \sqcap v$ their infimum in the tree order $\preceq$.

(b) The *branching pattern* of a tuple $\bar{v}$ of vertices is the partial order consisting of the root of t and all vertices of the form $v_i \sqcap v_j$ where each edge $u \prec w$ is labelled by the variable x such that $\mathrm{suc}_x(u) \preceq w$.
(c) The *branching type* of a factor $[u, \bar{v})$ of t is the isomorphism type of the branching pattern of $\bar{v}$ in the subtree $t|_u$. Alternatively, we can define the branching type as the atomic type of $\bar{v}$ in the structure $\langle t|_u, (\prec_x)_x, \sqcap, u\rangle$ where

$$u \prec_x v \quad :\text{iff} \quad \mathrm{suc}_x(u) \preceq v\,.$$

The branching type of a tree $t \in \mathbb{T}_\xi A$ is the branching type of $(v_x)_{x\in\xi}$, where v_x is the vertex labelled by the variable x. ⌟

Example. We construct a finitary $\mathbb{T}$-algebra $\mathfrak{A}$ such that, for every $t \in \mathbb{T}A$ with factors $[u, \bar{v})$ and $[u', \bar{v}')$,

$$\pi(t[u, \bar{v})) = \pi(t[u', \bar{v}')) \quad \text{implies} \quad [u, \bar{v}) \text{ and } [u', \bar{v}') \text{ have the same branching type.}$$

For $\xi \in \Xi$, let A_ξ be the set of all branching types of finite trees $t \in \mathbb{T}_\xi \mathbf{1}$ such that
- every leaf of t is labelled by a variable and
- every sort ζ of a label in t is a finite initial segment of ω.

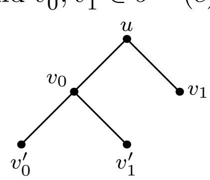
($\mathbf{1}$ is a set with exactly one element of each sort.) As the arity of all labels in t is bounded by the sort ξ of t, it follows that the set A_ξ is finite.

Note that, given a tree $s \in \mathbb{T}\mathbb{T}\Sigma$, we can compute the branching type of $\mathrm{flat}(s)$ from the branching types of $s(v)$, for $v \in \mathrm{dom}(s)$. Here the branching type of a tree is defined as the branching type of the factor $[\langle\rangle, \bar{v})$ where $\bar{v}$ are the vertices labelled by a variable. It follows that there exists a function $\pi : \mathbb{T}A \to A$ satisfying $\pi \circ \mathbb{T}\tau = \tau \circ \mathrm{flat}$, where $\tau : \mathbb{T}\Sigma \to A$ is the function mapping each tree to its branching type. It follows that $\mathfrak{A} := \langle A, \pi\rangle$ is a finitary $\mathbb{T}$-algebra with the desired property.

Let $t \in \mathbb{T}\Sigma$ be an infinite binary tree and let λ be the labelling mapping a factor $[u, \bar{v})$ to its branching type. We claim that there is no function $\sigma : \mathrm{dom}(t) \to C$ with a finite codomain C such that the label $\lambda([u, \bar{v}))$ only depends on the values $\sigma(u)$ and $\sigma(v_x)$. We fix a vertex $u \in \mathrm{dom}(t)$ such that the set

$$C_0 := \{\, \sigma(v) \mid v \succeq u \,\}$$

is minimal. Set $c := \sigma(u)$. By choice of u there are vertices $v_0, v_1 \in \sigma^{-1}(c)$ with $\mathrm{suc}_0(u) \preceq v_0$ and $\mathrm{suc}_1(u) \preceq v_1$. Similarly, we can find $v'_0, v'_1 \in \sigma^{-1}(c)$ with $\mathrm{suc}_0(v_0) \preceq v'_0$ and $\mathrm{suc}_1(v_0) \preceq v'_1$.



Then $\mathrm{suc}_0(u) \preceq v'_1$ implies that $\lambda([u, v'_0 v_1)) \neq \lambda([u, v'_0 v'_1))$, but all four vertices have the same colour c. ⌟

We conclude this section by briefly mentioning two alternative approaches.

Remark. When talking about partition theorems for trees, we also have to mention automata. Every automaton can be seen as a prescription producing labellings (runs) of trees. The advantage of automata is that they can be used even if we know very little about the underlying algebra. In particular, they can be used in cases where we can only evaluate finite trees. Their disadvantage is that runs are usually not unique and that every run only contains a limited amount of information. For instance, in general there is no automaton that allows us to evaluate every factor $\pi(t[u,\bar{v}])$ of a given tree t, only factors of a fixed arity. ⌟

Remark. The proof of Theorem 3.4 is based on semigroup-theoretic methods, in particular, it makes heavy use of Green's relations. It looks plausible that, in order to prove a stronger partition theorem for trees, we have to develop a similar theory of Green's relations for tree algebras. As it turns out, it is rather straightforward to generalise these relations to the setting of monoidal categories where all hom-sets are finite. (We omit the details since the statements and proofs are virtually identical to those for semigroups.) Furthermore, every $\mathbb{T}^{\text{fin}}$-algebra $\mathfrak{A}$ can be seen as such a category $\mathcal{A}$ where the objects are the sorts $\xi \in \Xi$ and the hom-sets $\mathcal{A}(\xi,\zeta)$ are given by $(A_\xi)^\zeta$. The question is how to apply the resulting theory to the case at hand. The main problem with doing so seems to be that, in general, a finitary $\mathbb{T}^{\text{fin}}$-algebra can have infinitely many J-classes. ⌟

4. Expansions and dense submonads

When looking for a strengthening of the results in the previous section it is always useful to have an application in mind that can serve as a test case and reality check. The following problem on expansions seems to be a good candidate for this purpose.

Definition 4.1. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ be submonads. We say that a $\mathbb{T}^0$-algebra $\mathfrak{A}_0$ is a *reduct* of a $\mathbb{T}^1$-algebra $\mathfrak{A}_1 = \langle A, \pi_1 \rangle$ if $\mathfrak{A}_0 = \langle A, \pi_0 \rangle$ where $\pi_0 : \mathbb{T}^0 A \to A$ is the restriction of $\pi_1 : \mathbb{T}^1 A \to A$. In this case, we call $\mathfrak{A}_1$ a $\mathbb{T}^1$-*expansion* of $\mathfrak{A}_0$. ⌟

Expansion Problem. *Given monads $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$, which $\mathbb{T}^0$-algebras have $\mathbb{T}^1$-expansions? And for which algebras are these expansions unique?*

The motivating example of an expansion problem is the result that every finite Wilke algebra has a unique expansion to an ω-semigroup. For trees, problems of this kind turn often out to be quite hard and seem to require advanced techniques from combinatorics. In this article, we develop tools that help answering such questions, with a focus on the tree monads we have defined above. For the inclusion $\mathbb{T}^{\text{wilke}} \subseteq \mathbb{T}^{\text{thin}}$, we will prove in Corollary 5.11 below that expansions (of finitary algebras) always exists. Unfortunately, the (more interesting) inclusions $\mathbb{T}^{\text{reg}} \subseteq \mathbb{T}$ and $\mathbb{T}^{\text{thin}} \subseteq \mathbb{T}$ turn out to be much more complicated and our current techniques seem to allow us to prove only partial results.

We start our investigation with recalling some results from [Blu21] that can be used to prove the uniqueness of expansions, if not their existence.

Definition 4.2. A submonad $\mathbb{T}^0 \subseteq \mathbb{T}^1$ is *dense* in $\mathbb{T}^1$ over a class $\mathcal{C}$ of $\mathbb{T}^1$-algebras if, for all algebras $\mathfrak{A} \in \mathcal{C}$, subsets $C \subseteq A$, and trees $s \in \mathbb{T}^1 C$, there exists a tree $s^0 \in \mathbb{T}^0 C$ with $\pi(s^0) = \pi(s)$. ⌟

Example. The notion of denseness is a generalisation of the fact from the theory of ω-semigroups that every infinite product has a factorisation of the form ab^ω. This translates to the fact that, over the class of all finite ω-semigroups, the monad for Wilke algebras is dense in the monad for ω-semigroups: for every infinite word u in a finite ω-semigroup there is an ultimately periodic word u^0 with the same product. Details can be found in [Blu21]. ⌟

We have shown in Lemma 4.13 (a) of [Blu21] that denseness implies the uniqueness of expansions (if they exist). One technical requirement of the proof is that we need to assume that the class in question is closed under binary products. Such products are defined in the usual way: the product $\mathfrak{A} \times \mathfrak{B}$ of two algebras has the universe $A \times B := (A_\xi \times B_\xi)_{\xi \in \Xi}$ and the product is defined component-wise, i.e., the product of a tree $t \in \mathbb{T}(A \times B)$ is defined by taking the projections of t to the two components, multiplying them separately, and returning the pair of values obtained in this way.

Proposition 4.3. *Let $\mathbb{T}^0$ be dense in $\mathbb{T}^1$ over some class $\mathcal{C}$ that is closed under binary products. Then every $\mathbb{T}^0$-algebra $\mathfrak{A}$ has at most one $\mathbb{T}^1$-expansion that belongs to $\mathcal{C}$.*

Remark. The class of MSO-definable $\mathbb{T}^\times$-algebras is closed under binary products [Blu20]. The same holds for MSO-definable $\mathbb{T}$-algebras. ⌟

The fact that a regular language is uniquely determined by which regular trees it contains can be generalised to the following theorem (which is a consequence of (the proof of) Theorem 10.1 of [Blu21]).

Theorem 4.4. *$\mathbb{T}^{\text{reg}}$ is dense in $\mathbb{T}$ over the class of all* MSO*-definable $\mathbb{T}$-algebras, and $\mathbb{T}^{\times\text{reg}}$ is dense in $\mathbb{T}^\times$ over the class of all* MSO*-definable $\mathbb{T}^\times$-algebras.*

It is currently unknown whether this property characterises the class of MSO-definable algebras.

Open Question. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}^\times$-algebra such that, for every $C \subseteq A$ and every $t \in \mathbb{T}C$, there is some $t^0 \in \mathbb{T}^{\times\text{reg}}C$ with $\pi(t^0) = \pi(t)$. Is $\mathfrak{A}$* MSO*-definable?*

For linear trees we can strengthen Theorem 4.4 to include existence. The proof requires some tools from automata theory (for details we refer the reader to [Tho97, L2̈1]). We only consider automata over trees of sort $\emptyset$. When we need an automaton to read a tree $t \in \mathbb{T}^\times_\xi \Sigma$, we will encode it as an element of $\mathbb{T}^\times_\emptyset(\Sigma + \xi)$. Therefore, we will use *(alternating) tree automata* of the form $\mathcal{A} = \langle Q, \Sigma + \xi, \Delta, q_0, \Omega \rangle$ where Q is the set of states, $\Sigma + \xi$ the input alphabet, $\Omega : Q \to \omega$ a priority function, and Δ the transition relation. The latter consists of triples of the form $\langle p, a, (P_z)_{z \in \zeta} \rangle$ for a state $p \in Q$, a letter $a \in (\Sigma + \xi)_\zeta$, and sets of states $P_z \subseteq Q$. The behaviour of such an automaton $\mathcal{A}$ on a given input tree $t \in \mathbb{T}^\times_\xi \Sigma$ is defined via a certain parity game $\mathcal{G}$, called the *Automaton-Pathfinder game* of $\mathcal{A}$ on t. The two players in this game are called *Automaton* and *Pathfinder*. The positions for the former are the pairs in $\text{dom}(t) \times Q$, while the positions for the latter are those in $\text{dom}(t) \times \Delta$. In a position $\langle v, q \rangle$, Automaton chooses a transition $\langle p, a, \bar{P} \rangle \in \Delta$ with $p = q$ and $a = t(v)$; Pathfinder replies with some z-successor u_z of v and some state $r \in P_z$; and the game continues in the position $\langle u_z, r \rangle$. Automaton wins a play in this game if he either manoevres Pathfinder into a position where the latter cannot make a move, or if the play is infinite and the corresponding sequence $(q_n)_{n<\omega}$ of states from the Automaton positions satisfies the parity condition:

$$\liminf_{n<\omega} \Omega(q_n) \quad \text{is even.}$$

We say that the automaton $\mathcal{A}$ *accepts* the tree $t \in \mathbb{T}^\times_\xi \Sigma$ if Player Automaton has a winning strategy for $\mathcal{G}$ when starting in the position $\langle \langle\rangle, q_0 \rangle$ that consists of the root $\langle\rangle$ of t and the initial state q_0 of $\mathcal{A}$. Finally, we call the automaton *non-deterministic* if, in every transition $\langle p, a, (P_z)_{z \in \zeta} \rangle \in \Delta$, all the sets P_z are singletons.

It is frequently useful to consider the restriction of an Automaton-Pathfinder game to some factor $[u, \bar{v})$ of the given input tree t. In such cases we can collect information about the possible strategies of Automaton on this restriction in a set of *profiles*. Such a profile is a pair $\langle p, \bar{U} \rangle$ consisting of a state $p \in Q$ and a tuple $\bar{U} = (U_z)_{z \in \zeta}$ of sets $U_z \subseteq \text{rng}\, \Omega \times Q$, where ζ is the sort of the factor $[u, \bar{v})$. We say that $\langle p, \bar{U} \rangle$ is a profile of the factor $[u, \bar{v})$ in t if there exists a strategy for Automaton such that, when starting the game in position $\langle u, p \rangle$, Pathfinder has, for each $z \in \zeta$ and every $\langle k, q \rangle \in U_z$, a strategy to reach the position $\langle v_z, q \rangle$ such that the least priority seen in between is equal to k.

Finally, given a tree $s \in \mathbb{T}^\times_\zeta \Sigma$ with an arbitrary sort ζ, we say that $\langle p, \bar{U}\rangle$ is a profile of s if there exists some tree $t \in \mathbb{T}^\times_\xi \Sigma$ containing a factor $[u, \bar{v})$ such that $s = t[u, \bar{v})$ and $\langle p, \bar{U}\rangle$ is a profile of $[u, \bar{v})$ in t. (Note that this fact does not depend on the choice of t.)

By definition, every MSO-definable $\mathbb{T}^\times$-algebra $\mathfrak{A}$ has a finite set of generators $C \subseteq A$ such that the preimages $\pi^{-1}(a) \cap \mathbb{T}^\times C$ are regular, that is, recognised by an automaton. For each given sort $\xi \in \Xi$, we can combine all these automata into a single automaton $\mathcal{A}$ that has, for every element $a \in A_\xi$, some state q_a such that, when using q_a as the initial state, the automaton accepts all trees in $\mathbb{T}^\times_\xi C$ with product a.

The fact that this combined automaton $\mathcal{A}$ does not work for all trees in $\mathbb{T}^\times_\xi A$, but only for those in $\mathbb{T}^\times_\xi C$, is frequently inconvenient. The problem is that the set A of possible labels is infinite. Given an automaton $\mathcal{A} = \langle Q, C + \xi, \Delta, q_0, \Omega\rangle$ over the alphabet $C + \xi$, we can extend it to the infinite alphabet $A + \xi$ as follows. Since C is a set of generators, there exists some function $\vartheta : A \to \mathbb{T}^\times C$ such that $\pi \circ \vartheta = \mathrm{id}$. As new set of states we use $D \times Q$, where $D := \mathrm{rng}\, \Omega$ is the set of priorities. The priority function is $\Omega^+(k, p) := k$ and the initial state $q_0^+ := \langle 0, q_0\rangle$. Finally, the extended transition relation Δ_+ consists of all triples $\langle\langle l, p\rangle,\, a,\, (U_z)_{z\in\zeta}\rangle$ such that, in the original automaton, the tree $\vartheta(a)$ has the profile $\langle p, (U_z)_{z\in\zeta}\rangle$. Let $\mathcal{A}_+ := \langle D \times Q, A + \xi, \Delta_+, q_0^+, \Omega^+\rangle$ be the automaton with this transition relation. (Note that $\mathcal{A}_+$ is not a finite automaton anymore, although it still has only finitely many states.) By construction, it follows that $\mathcal{A}_+$ accepts a tree $t \in \mathbb{T}^\times_\xi A$ if, and only if, $\mathcal{A}$ accepts the tree $\mathrm{flat}(\mathbb{T}^\times \vartheta(t))$. Since

$$(\pi \circ \mathrm{flat} \circ \mathbb{T}^\times \vartheta)(t) = (\pi \circ \mathbb{T}^\times \pi \circ \mathbb{T}^\times \vartheta)(t) = (\pi \circ \mathbb{T}^\times(\pi \circ \vartheta))(t) = \pi(t)\,,$$

it follows that $\mathcal{A}_+$ recognises the language $\pi^{-1}(a)$ (with no restriction on the labels allowed in the trees).

For linear trees, we are able to improve upon Theorem 4.4 by also establishing uniqueness. We start with a technical lemma which is based on an argument originally due to Bojańczyk and Klin (the first published version of which appears in [Blu20]).

Lemma 4.5. *Let Σ be a finite sorted set, $m < \omega$, and let $R \subseteq \mathbb{T}\Sigma \times \mathbb{T}\Sigma$ be a (sorted) binary relation between trees. For every tree $U \in \mathbb{T}_\xi R$, there exists a tree $U^0 \in \mathbb{T}^{\mathrm{reg}}_\xi R$ such that, for each $i < 2$, the trees*

$$\mathrm{flat}(\mathbb{T}p_i(U)) \quad \text{and} \quad \mathrm{flat}(\mathbb{T}p_i(U^0))$$

have identical MSO*-theories of quantifier-rank m, where $p_0, p_1 : \mathbb{T}\Sigma \times \mathbb{T}\Sigma \to \mathbb{T}\Sigma$ are the two projections.*

Proof. To construct U^0 we use a variant of the Automaton-Pathfinder Game from above. Given two non-deterministic tree automata $\mathcal{A}$ and $\mathcal{B}$, we define a game $\mathcal{G}_R(\mathcal{A}, \mathcal{B})$ where the first player wins if, and only if, there exist two trees $S, T \in \mathbb{T}\mathbb{T}\Sigma$ with the same domain such that

- $\langle S(v), T(v)\rangle \in R$, for all vertices v,
- $\mathcal{A}$ accepts $\mathrm{flat}(S)$,
- $\mathcal{B}$ accepts $\mathrm{flat}(T)$.

The difference to the usual Automaton-Pathfinder Game is that we simulate two automata at the same time and that, instead of playing single letters, we play larger trees in each step. The game has two players *Automaton* and *Pathfinder.* Each round starts in a position of the form $\langle p, q\rangle$, where p is a state of $\mathcal{A}$ and q one of $\mathcal{B}$. In the first round of the game, p and q

are the initial states of the respective automata. Given such a position $\langle p, q\rangle$, Automaton chooses

- a pair of trees $\langle s, t\rangle \in R_\xi$, for some $\xi \in \Xi$,
- a profile $\langle p, \bar{V}\rangle$ for $\mathcal{A}$ on s that starts in the state p, and
- a profile $\langle q, \bar{W}\rangle$ for $\mathcal{B}$ on t that starts in the state q.

Pathfinder responds by selecting a variable $x \in \xi$ and pairs $\langle k, p'\rangle \in V_x$ and $\langle l, q'\rangle \in W_x$. The *outcome* of this round is the pair $\langle\langle p, k, p'\rangle, \langle q, l, q'\rangle\rangle$ and the game will continue in the position $\langle p', q'\rangle$.

If at some point in the game one of the players cannot make his choice, that player loses the game. Otherwise, the players produce an infinite sequence $\langle\delta_0, \varepsilon_0\rangle, \langle\delta_1, \varepsilon_1\rangle, \dots$ of outcomes. Let k_i be the priority in δ_i and l_i the priority in ε_i. Player Automaton wins the game if each of the sequences $k_0, k_1, \dots$ and $l_0, l_1, \dots$ satisfies the parity condition. Otherwise, Pathfinder wins.

Clearly, if there are two trees $S, T \in \mathbb{T}\mathbb{T}\Sigma$ with the same domain such that

- $\langle S(v), T(v)\rangle \in R$, for all vertices v,
- $\mathcal{A}$ accepts $\mathrm{flat}(S)$, and
- $\mathcal{B}$ accepts $\mathrm{flat}(T)$,

then Automaton has the following winning strategy in $\mathcal{G}_R(\mathcal{A}, \mathcal{B})$. He fixes two winning strategies ϱ and σ for the games on, respectively, $\mathrm{flat}(S)$ and $\mathrm{flat}(T)$. During the game he descends through the trees S and T. When the game reaches a vertex v, Automaton chooses the trees $S(v)$ and $T(v)$ and the profiles on the corresponding factors of $\mathrm{flat}(S)$ and $\mathrm{flat}(T)$ that are associated with ϱ and σ, respectively.

Conversely, if Automaton has a winning strategy in the game, we can use it to construct

- two trees $S, T \in \mathbb{T}\mathbb{T}\Sigma$ such that $\langle S(v), T(v)\rangle \in R$, for all v, and
- winning strategies for Automaton in the games for $\mathrm{flat}(S)$ and $\mathrm{flat}(T)$, respectively.

Having defined the game $\mathcal{G}_R(\mathcal{A}, \mathcal{B})$, we prove the statement of the lemma as follows. Fix $U \in \mathbb{T}R$, let θ_i be the MSO-theory of quantifier-rank m of the tree

$$\mathrm{flat}(\mathbb{T}p_i(U)),$$

and let $\mathcal{A}_i$ be an automaton recognising the class of all trees with theory θ_i. It follows that Player Automaton has a winning strategy in the game $\mathcal{G}_R(\mathcal{A}_0, \mathcal{A}_1)$. As the winning condition of this game is regular, we can apply the Büchi-Landweber Theorem [BL69], which tells us that Automaton even has a finite-memory winning strategy. Since the choice of $S(v)$ and $T(v)$ by Automaton in the game only depends on the current position $\langle p, q\rangle$ and on the contents of the memory, it follows that the resulting trees S and T are regular, i.e., they belong to $\mathbb{T}^{\mathrm{reg}}\mathbb{T}\Sigma$. The tree $U^0 \in \mathbb{T}^{\mathrm{reg}}R$ with labels

$$U^0(v) := \langle S(v), T(v)\rangle$$

has the desired properties. □

Theorem 4.6. *Every* MSO-*definable* $\mathbb{T}^{\mathrm{reg}}$-*algebra can be expanded to a unique* MSO-*definable* $\mathbb{T}$-*algebra.*

Proof. Uniqueness follows by Theorem 4.4 and Proposition 4.3. For existence, consider an MSO-definable $\mathbb{T}^{\mathrm{reg}}$-algebra $\mathfrak{A}$ and let $C \subseteq A$ be a finite set of generators. For each $a \in A$, we fix an MSO-formula φ_a defining the set $\pi^{-1}(a) \cap \mathbb{T}^{\mathrm{reg}}C$. Intuitively, we obtain the desired expansion $\mathfrak{A}_+ = \langle A, \pi_+\rangle$ by taking for π_+ the function on all trees defined by these formulae.

To make this work we have to (I) find a way to use the formulae φ_a on trees containing labels from $A \setminus C$, (II) show that the resulting function is well-defined, (III) show that it extends π, and (IV) show that it satisfies the axioms of a $\mathbb{T}$-algebra.

(I) We define π_+ as follows. As C generates $\mathfrak{A}$, there exists a function $\sigma : A \to \mathbb{T}^{\mathrm{reg}}C$ such that $\pi(\sigma(a)) = a$, for all $a \in A$. We choose σ such that $\sigma(c) = \mathrm{sing}(c)$, for $c \in C$. Let $\hat\sigma := \mathrm{flat} \circ \mathbb{T}\sigma : \mathbb{T}A \to \mathbb{T}C$ be its extension to $\mathbb{T}A$. We set

$$\pi_+ := \pi_0 \circ \hat\sigma\,,$$

where

$$\pi_0(t) := a \quad :\text{iff} \quad t \models \varphi_a\,, \quad \text{for } t \in \mathbb{T}C\,.$$

(II) To see that π_+ is well-defined, we have to check that, for every tree $t \in \mathbb{T}_\xi C$, there is exactly one element $a \in A_\xi$ with $t \models \varphi_a$. For a contradiction, suppose otherwise. Then we can find a tree $t \in \mathbb{T}_\xi C$ such that

$$t \models \bigvee_{a \neq b} (\varphi_a \wedge \varphi_b) \vee \neg \bigwedge_{a \in A_\xi} \varphi_a\,.$$

Since every non-empty MSO-definable tree language contains a regular tree, it follows that we can choose $t \in \mathbb{T}^{\mathrm{reg}}C$. By choice of the formulae φ_a this means that $\pi(t)$ has either no value, or more than one. A contradiction.

(III) π_+ extends π since, for $t \in \mathbb{T}^{\mathrm{reg}}A$, we have

$$\pi_+(t) = \pi_0(\hat\sigma(t)) = \pi(\hat\sigma(t)) = \pi(t)\,,$$

where the second step follows since $\hat\sigma(t) \in \mathbb{T}^{\mathrm{reg}}C$, for $t \in \mathbb{T}^{\mathrm{reg}}C$.

(IV) It remains to check the axioms of a $\mathbb{T}$-algebra. First, note that

$$\begin{aligned}
\pi_+ \circ \hat\sigma &= \pi_0 \circ \hat\sigma \circ \hat\sigma \\
&= \pi_0 \circ \mathrm{flat} \circ \mathbb{T}\sigma \circ \mathrm{flat} \circ \mathbb{T}\sigma \\
&= \pi_0 \circ \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}(\mathbb{T}\sigma \circ \sigma) && \text{[flat nat. trans.]} \\
&= \pi_0 \circ \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}(\mathbb{T}\mathrm{sing} \circ \sigma) \\
&= \pi_0 \circ \mathrm{flat} \circ \mathbb{T}\mathrm{sing} \circ \mathrm{flat} \circ \mathbb{T}\sigma && \text{[flat nat. trans.]} \\
&= \pi_0 \circ \mathrm{flat} \circ \mathbb{T}\sigma && \text{[}\mathbb{T}\text{ monad]} \\
&= \pi_0 \circ \hat\sigma \\
&= \pi_+\,.
\end{aligned}$$

For the unit law, it therefore follows that

$$\pi_+ \circ \mathrm{sing} = \pi_+ \circ \hat\sigma \circ \mathrm{sing} = \pi_+ \circ \sigma = \pi \circ \sigma = \mathrm{id}\,.$$

It remains to check the associative law. For a set of sorts $\Delta \subseteq \Xi$ and a sorted set A, we denote by $A|_\Delta$ the sorted set obtained from A by removing all elements whose sort does not belong to Δ. Below we will establish the following two claims.

(a) The associative law holds for all trees $t \in \mathbb{T}\mathbb{T}^{\mathrm{reg}}A$.

(b) The associative law holds for all trees $t \in \mathbb{T}(\mathbb{T}C)|_\Delta$ with finite Δ satisfying $C \subseteq A|_\Delta$.

Then we can prove the general case as follows. Given $t \in \mathbb{T}\mathbb{T}A$, we consider the tree $s := \mathbb{T}\hat\sigma(t)$. Each tree $r \in \mathbb{T}C$ can be written as $r = p(\bar u, \bar x)$ for a finite tree p, variables $\bar x$, and infinite trees $\bar u$ that do not contain variables. This implies that each tree $r \in \mathbb{T}C$ can be written as $r = \mathrm{flat}(R)$, for some $R \in \mathbb{T}^{\mathrm{fin}}(\mathbb{T}C)|_\Delta$, where $\Delta \subseteq \Xi$ is the finite set consisting of the sort $\emptyset$ and the sorts of the elements of C. (Note that this argument does not work for $\mathbb{T}^{\times}C$.) Consequently, there exist a finite set $\Delta \subseteq \Xi$ and a tree $S \in \mathbb{T}\mathbb{T}^{\mathrm{fin}}(\mathbb{T}C)|_{\Delta}$ with $s = \mathbb{T}\mathrm{flat}(S)$. By the two claims above, it follows that

$$\begin{aligned}
(\pi_+ \circ \mathrm{flat})(t) &= (\pi_+ \circ \hat{\sigma} \circ \mathrm{flat})(t) && [\pi_+ \circ \hat{\sigma} = \pi_+] \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\sigma \circ \mathrm{flat})(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\sigma)(t) && [\text{flat nat. trans.}] \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\mathrm{flat} \circ \mathbb{T}\mathbb{T}\sigma)(t) && [\mathbb{T} \text{ monad}] \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\hat{\sigma})(t) \\
&= (\pi_+ \circ \mathrm{flat})(s) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\mathrm{flat})(S) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathrm{flat})(S) && [\mathbb{T} \text{ monad}] \\
&= (\pi_+ \circ \mathbb{T}\pi_+ \circ \mathrm{flat})(S) && [\text{by (b)}] \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\pi_+)(S) && [\text{flat nat. trans.}] \\
&= (\pi_+ \circ \mathbb{T}\pi_+ \circ \mathbb{T}\mathbb{T}\pi_+)(S) && [\text{by (a)}] \\
&= (\pi_+ \circ \mathbb{T}\pi_+ \circ \mathbb{T}\mathrm{flat})(S) && [\text{by (b)}] \\
&= (\pi_+ \circ \mathbb{T}\pi_+)(s) \\
&= (\pi_+ \circ \mathbb{T}\pi_+ \circ \mathbb{T}\hat{\sigma})(t) \\
&= (\pi_+ \circ \mathbb{T}\pi_+)(t)\,. && [\pi_+ \circ \hat{\sigma} = \pi_+]
\end{aligned}$$

Hence, it remains to prove the two claims.

(a) Fix a tree $t \in \mathbb{T}\mathbb{T}^{\mathrm{reg}}A$. We consider the relation $R \subseteq \mathbb{T}C \times \mathbb{T}C$ given by

$$\langle s, s' \rangle \in R \quad :\text{iff} \quad s, s' \in \mathbb{T}^{\mathrm{reg}}C \text{ and } \pi(s) = \pi(s')\,,$$

and the tree $U \in \mathbb{T}R$ with labels

$$U(v) := \big\langle \hat{\sigma}(t(v)),\, \sigma(\pi_+(t(v))) \big\rangle\,, \quad \text{for } v \in \mathrm{dom}(t)\,.$$

Let $p_0, p_1 : \mathbb{T}C \times \mathbb{T}C \to \mathbb{T}C$ be the two projections. Then

$$\begin{aligned}
a &:= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}p_0)(U) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\hat{\sigma})(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}(\mathrm{flat} \circ \mathbb{T}\sigma))(t) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathrm{flat} \circ \mathbb{T}\mathbb{T}\sigma)(t) && [\mathbb{T} \text{ monad}] \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}\sigma \circ \mathrm{flat})(t) && [\text{flat nat. trans.}] \\
&= (\pi_+ \circ \hat{\sigma} \circ \mathrm{flat})(t) \\
&= \pi_+(\mathrm{flat}(t))\,, && [\pi_+ \circ \hat{\sigma} = \pi_+] \\
b &:= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}p_1)(U) \\
&= (\pi_+ \circ \mathrm{flat} \circ \mathbb{T}(\sigma \circ \pi_+))(t) \\
&= (\pi_+ \circ \hat{\sigma} \circ \mathbb{T}\pi_+)(t) \\
&= \pi_+(\mathbb{T}\pi_+(t))\,. && [\pi_+ \circ \hat{\sigma} = \pi_+]
\end{aligned}$$

Hence, we have to prove that $a = b$. By Lemma 4.5, there exists a regular tree $U^0 \in \mathbb{T}^{\text{reg}}R$ with

$$\pi_0(\text{flat}(\mathbb{T}p_0(U^0))) = \pi_0(\text{flat}(\mathbb{T}p_0(U))) = a\,,$$
$$\pi_0(\text{flat}(\mathbb{T}p_1(U^0))) = \pi_0(\text{flat}(\mathbb{T}p_1(U))) = b\,.$$

Setting $s_i := \mathbb{T}p_i(U^0) \in \mathbb{T}^{\text{reg}}\mathbb{T}^{\text{reg}}C$, it follows by associativity of π and the fact that we have $\langle s_0(v), s_1(v)\rangle \in R$, for all v, that

$$a = \pi(\text{flat}(s_0)) = \pi(\mathbb{T}\pi(s_0)) = \pi(\mathbb{T}\pi(s_1)) = \pi(\text{flat}(s_1)) = b\,.$$

(b) Fix $t \in \mathbb{T}(\mathbb{T}C)|_\Delta$ where $\Delta \subseteq \Xi$ is a finite set such that $C \subseteq A|_\Delta$. Recall that φ_a is a formula defining the set $\pi^{-1}(a) \cap \mathbb{T}^{\text{reg}}C$. Since the MSO-definable tree languages are closed under inverse homomorphisms (see, e.g., Lemma A.1 of [Blu20]), we can construct an MSO-formula φ_a^Δ such that

$$s \models \varphi_a^\Delta \quad \text{iff} \quad \hat{\sigma}(s) \models \varphi_a\,, \quad \text{for } s \in \mathbb{T}(A|_\Delta)\,.$$

For $a \in A$, let ψ_a^Δ be a formula stating that

"For every factorisation of the given tree such that each factor has a sort in Δ, there exists a labelling of the factors by elements of $A|_\Delta$ such that
- each factor with label $b \in A|_\Delta$ satisfies the formula φ_b, and
- the tree consisting of the guessed labels satisfies φ_a^Δ."

As every factor (in $\mathbb{T}C$) of a regular tree in $\mathbb{T}^{\text{reg}}C$ is regular, it follows by (a) that

$$s \models \psi_a^\Delta \leftrightarrow \varphi_a\,, \quad \text{for every } s \in \mathbb{T}^{\text{reg}}C\,.$$

(We can choose $\mathbb{T}^{\text{reg}}\text{sing}(s)$ for the factorisation guessed by ψ_a^Δ.) It follows that the same is true for every $s \in \mathbb{T}C$. Consequently,

$$\text{flat}(t) \models \varphi_a \quad \text{implies} \quad \mathbb{T}\pi_+(t) \models \varphi_a^\Delta\,, \quad \text{for all } a\,.$$

Furthermore, by choice of φ_a^Δ, we have

$$\begin{aligned}\mathbb{T}\pi_+(t) \models \varphi_a^\Delta \quad &\text{iff} \quad (\hat{\sigma} \circ \mathbb{T}\pi_+)(t) \models \varphi_a \\ &\text{iff} \quad (\pi_0 \circ \hat{\sigma} \circ \mathbb{T}\pi_+)(t) = a \\ &\text{iff} \quad (\pi_+ \circ \mathbb{T}\pi_+)(t) = a\,.\end{aligned}$$

Consequently, it follows that

$$(\pi_+ \circ \text{flat})(t) = (\pi_+ \circ \mathbb{T}\pi_+)(t)\,.$$ □

Note that our uniqueness result in Proposition 4.3 only concerns expansions in the given class $\mathcal{C}$. It is possible that there exist additional expansions outside of $\mathcal{C}$.

Example. In [BK19] Bojańczyk and Klin have presented an example of a finitary $\mathbb{T}^\times$-algebra that is not MSO-definable. This algebra can be used to find an MSO-definable $\mathbb{T}^{\text{reg}}$-algebra with several $\mathbb{T}$-expansions, one of them MSO-definable. (By the preceding theorem, there can only be one of the latter.) The construction of this $\mathbb{T}^{\text{reg}}$-algebra $\mathfrak{A}^0 = \langle A, \pi\rangle$ and two of its $\mathbb{T}$-expansions $\mathfrak{A}^{\text{reg}} = \langle A, \pi_{\text{reg}}\rangle$ (MSO-definable) and $\mathfrak{A}^{\text{non}} = \langle A, \pi_{\text{non}}\rangle$ (not MSO-definable) is as follows.

For $\mathfrak{A}^{\text{non}}$, we take (a simplified version) of the algebra from [BK19]. Let $\Sigma := \{a, b\}$ where both elements have arity 2 and set $\Delta_\xi := \mathbb{T}^{\text{fin}}_\xi \Sigma$. As Σ contains only binary elements, every leaf of a tree $t \in \Delta_\xi$ must be labelled by a variable. Hence, t has at most $|\xi|$ leaves and, therefore, at most $|\xi| - 1$ internal vertices. This implies that Δ_ξ is a finite set.

We call a tree $t \in \mathbb{T}_\emptyset \Sigma$ *antiregular* if no two subtrees of t are isomorphic. It is *densely antiregular* if every subtree of t has an antiregular subtree.

The domains of all three algebras are

$$A_\xi := \Delta_\xi \cup \{\bot, *\}, \quad \text{for } \xi \in \Xi,$$

which we order such that $\bot$ is the least element and all other elements are incomparable. Intuitively, the elements Δ_ξ encode finite trees (by themselves), while the element $*$ encodes a densely antiregular tree and $\bot$ encodes an infinite tree that is not densely antiregular.

Let us call a tree $t \in \mathbb{T}_\emptyset A$ *good* if every subtree s of t contains some vertex v such that $s(v) = *$, or such that $s|_v \in \mathbb{T}\Delta$ and $\text{flat}(s|_v)$ is densely antiregular.

For $t \in \mathbb{T}_\xi A$, we define the product $\pi_{\text{non}}(t)$ of $\mathfrak{A}^{\text{non}}$ by the following case distinction. (We tacitly assume in each case that the conditions for the previous cases are not satisfied.)

- $\pi_{\text{non}}(t) = \bot$ if t contains the label $\bot$.
- $\pi_{\text{non}}(t) = \text{flat}(t)$ if $t \in \mathbb{T}^{\text{fin}}_\xi \Delta$.
- $\pi_{\text{non}}(t) = *$ if every subtree without variables is good.
- $\pi_{\text{non}}(t) = \bot$ if t has a subtree without variables that is not good.

The product $\pi_{\text{reg}}(t)$ of $\mathfrak{A}^{\text{reg}}$ is defined as follows.

- $\pi_{\text{reg}}(t) = \bot$ if t contains the label $\bot$.
- $\pi_{\text{reg}}(t) = \text{flat}(t)$ if $t \in \mathbb{T}^{\text{fin}}_\xi \Delta$.
- $\pi_{\text{reg}}(t) = *$ otherwise.

Note that this product is MSO-definable and that the restrictions of π_{reg} and π_{non} to $\mathbb{T}^{\text{reg}} A$ coincide. Thus $\mathfrak{A}^{\text{reg}}$ and $\mathfrak{A}^{\text{non}}$ are $\mathbb{T}$-expansions of the same $\mathbb{T}^{\text{reg}}$-algebra. ⌟

To conclude this section, let me mention one of the main open problems concerning the relation between regular trees and arbitrary ones (for details, see [Blu21]).

Open Question. *Does there exist a system of equations (in the algebra $\mathbb{T}^\times \Sigma$) modulo which every tree is equivalent to a regular one? If so, does it have an explicit description?*

Having such an equational characterisation would be invaluable for applications, where proofs frequently require a reduction to regular trees. For instance, there exists an equational characterisation of bisimulation-invariant languages of *regular* trees [BI09], but so far nobody was able to generalise it to languages of arbitrary trees.

5. Evaluations

The notion of denseness seems to be only useful if we already know that the algebra in question has an expansion. To actually prove existence we need different techniques. We start with the following simple idea. Given a submonad $\mathbb{T}^0 \subseteq \mathbb{T}$ and a $\mathbb{T}^0$-algebra $\mathfrak{A}$, we try to compute the product of a tree $t \in \mathbb{T}A$ inductively bottom-up using the given $\mathbb{T}^0$-product. That is, we factorise t into pieces that belong to $\mathbb{T}^0$, evaluate them, and then recursively evaluate the remaining tree. If we can show that,

- after a finite number of such steps, the tree t is reduced to a single vertex and
- that the final result does not depend on the choice of factorisation used in each step,

it follows that we can uniquely evaluate every tree in $\mathbb{T}A$ using only the $\mathbb{T}^0$-product. In particular, the product of $\mathfrak{A}$ can be uniquely extended to the set of all trees.

A well-known use of this technique is given by Simon's Factorisation Tree Theorem (see, e.g., [Col21]). Such a factorisation tree is just a hierarchical decomposition of a given semigroup-product using binary products and products of idempotents only. A second example of this approach was used in [CCP18] to prove, in our terminology, that a certain inclusion between monads of countable linear orders is dense. The aim of the current section is to make this idea of using an inductive approach work for trees. Below we will use recursive factorisations to settle the expansion problem for thin trees and we will derive partial results for general ones.

The definition below is a bit more general than the above intuitive description. We will need the added generality for the more powerful decompositions used further below. Suppose that we are given a $\mathbb{T}^0$-algebra which we want to expand to a $\mathbb{T}^1$-algebra, for some $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$, and suppose that we have already found some sorted set $S \supseteq \mathbb{T}^0 A$ such that we can extend the product $\pi : \mathbb{T}^0 A \to A$ to $\rho : S \to A$. To extend ρ further to a function $\mathbb{T}^1 A \to A$, consider a tree $t \in \mathbb{T}^1 A$. We choose a factorisation T of t where we have already inductively assigned some value $\mathrm{val}(T(v))$ to each factor. If the reduced tree $\mathbb{T}^1\mathrm{val}(T)$ belongs to S, then we can set $\mathrm{val}(t) := \rho(\mathbb{T}^1\mathrm{val}(T))$.

In the following formal definition, $\mathbb{E}_\alpha(\rho, \mathbb{T}^1)$ is the set of all recursive factorisations with α levels of recursion, $\mathrm{term}_\alpha : \mathbb{E}_\alpha(\rho, \mathbb{T}^1) \to \mathbb{T}^1 A$ maps each such factorisation to the tree it factorises, and $\mathrm{val}_\alpha : \mathbb{E}_\alpha(\rho, \mathbb{T}^1) \to A$ maps each factorisation to the value obtained by recursively evaluating every factor.

Definition 5.1. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ be submonads, $\mathfrak{A}$ a $\mathbb{T}^0$-algebra, and $\rho : S \to A$ a function with domain $S \supseteq \mathbb{T}^0 A$ with $\rho \restriction \mathbb{T}^0 A = \pi$.

(a) For each ordinal α, we inductively define a sorted set $\mathbb{E}_\alpha(\rho, \mathbb{T}^1)$ of *ρ-evaluations* and two functions

$$\mathrm{term}_\alpha : \mathbb{E}_\alpha(\rho, \mathbb{T}^1) \to \mathbb{T}^1 A \quad \text{and} \quad \mathrm{val}_\alpha : \mathbb{E}_\alpha(\rho, \mathbb{T}^1) \to A$$

by

$$\begin{aligned} \mathbb{E}_0(\rho, \mathbb{T}^1) &:= A\,, \\ \mathbb{E}_{\alpha+1}(\rho, \mathbb{T}^1) &:= \mathbb{E}_\alpha(\rho, \mathbb{T}^1) + \big\{\, \gamma \in \mathbb{T}^1\mathbb{E}_\alpha(\rho, \mathbb{T}^1 \cap \mathbb{T}) \;\big|\; \mathbb{T}^1\mathrm{val}_\alpha(\gamma) \in \mathrm{dom}(\rho) \,\big\}\,, \\ \mathbb{E}_\delta(\rho, \mathbb{T}^1) &:= \bigcup_{\alpha<\delta} \mathbb{E}_\alpha(\rho, \mathbb{T}^1)\,, \qquad \text{for limit ordinals } \delta\,, \end{aligned}$$

and

$$\begin{aligned} \mathrm{term}_0 &:= \mathrm{sing}\,, & \mathrm{val}_0 &:= \mathrm{id}\,, \\ \mathrm{term}_{\alpha+1} &:= \mathrm{term}_\alpha + \mathrm{flat} \circ \mathbb{T}^1\mathrm{term}_\alpha\,, & \mathrm{val}_{\alpha+1} &:= \mathrm{val}_\alpha + \rho \circ \mathbb{T}^1\mathrm{val}_\alpha\,, \\ \mathrm{term}_\delta &:= \bigcup_{\alpha<\delta} \mathrm{term}_\alpha\,. & \mathrm{val}_\delta &:= \bigcup_{\alpha<\delta} \mathrm{val}_\alpha\,. \end{aligned}$$

Finally, we set[1]

$$\mathbb{E}(\rho, \mathbb{T}^1) := \bigcup_\alpha \mathbb{E}_\alpha(\rho, \mathbb{T}^1)\,, \quad \mathrm{term} := \bigcup_\alpha \mathrm{term}_\alpha\,, \quad \text{and} \quad \mathrm{val} := \bigcup_\alpha \mathrm{val}_\alpha\,.$$

(b) We call $\mathrm{term}(\gamma)$ the *underlying term* of $\gamma \in \mathbb{E}(\rho, \mathbb{T}^1)$ and $\mathrm{val}(\gamma)$ its *value.* If $t = \mathrm{term}(\gamma)$, we say that γ is a *ρ-evaluation* of t.

(c) We say that the algebra $\mathfrak{A}$ *has ρ-evaluations for* $\mathbb{T}^1$ if $\mathrm{term} : \mathbb{E}(\rho, \mathbb{T}^1) \to \mathbb{T}^1 A$ is surjective, and we say that it has *essentially unique ρ-evaluations* if furthermore

$$\mathrm{term}(\gamma) = \mathrm{term}(\gamma') \quad \text{implies} \quad \mathrm{val}(\gamma) = \mathrm{val}(\gamma')\,.$$

[1]Note that these classes are sets since the unions stabilise after at most ω_1 steps.

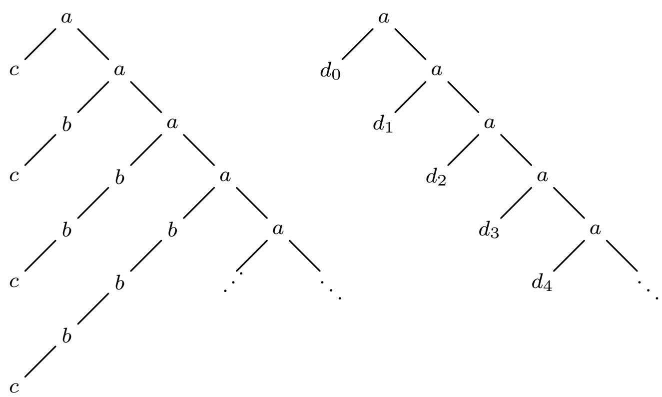


Figure 3: The trees t and t'.

(d) In the special case where $\rho = \pi$, we also write $\mathbb{E}(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}(\pi, \mathbb{T}^1)$ and we call the elements of $\mathbb{E}(\mathfrak{A}, \mathbb{T}^1)$ *simple* $\mathbb{T}^0$*-evaluations.* ⌟

Remark. Note that in the recursive definition of $\mathbb{E}_{\alpha+1}(\rho, \mathbb{T}^1)$, we only use decompositions into *linear* factors. The reason why we allow $\mathbb{T}^1 \subseteq \mathbb{T}^\times$ is to be able to have evaluations of non-linear trees. ⌟

Example. Let $\mathfrak{A}$ be a $\mathbb{T}^{\text{reg}}$-algebra, $a \in A_{\{x,y\}}$, $b \in A_{\{x\}}$, $c \in A_\emptyset$ elements, and t the tree consisting of an infinite branch labelled a attached to which are trees of the form $s_n := b^n(c)$, for every $n < \omega$ (see Figure 3). We construct a simple $\mathbb{T}^{\text{reg}}$-evaluation of t as follows. In the first step, we evaluate each of the subtrees $s_n \in \mathbb{T}^{\text{reg}} A$. The resulting tree has the form

$$t' := a(d_0, a(d_1, \dots)), \quad \text{where } d_n := \pi(s_n),$$

i.e., t' consists of an infinite branch labelled a to each vertex of which we have attached a leaf with label d_n. Since $A_{\{x\}}$ forms a finite semigroup and $d_{n+1} = b(d_n)$, the sequence $d_0, d_1, \dots$ is ultimately periodic. Fixing indices $k < l$ with $d_k = d_l$, we can write $t' = uv^\omega$ where

$$u := a(d_0, a(d_1, \cdots a(d_{k-1}, x) \cdots)),$$
$$v := a(d_k, a(d_{k+1}, \cdots a(d_{l-1}, x) \cdots)).$$

Consequently, t' is regular and we can take it as the second and final level of our evaluation. ⌟

As a first application of evaluations let us note that the existence of simple evaluations implies the denseness of the corresponding monads.

Lemma 5.2. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ and let $\mathcal{C}$ be a class of $\mathbb{T}^1$-algebras such that every $\mathfrak{A} \in \mathcal{C}$ has simple $\mathbb{T}^0$-evaluations for $\mathbb{T}^1$. Then $\mathbb{T}^0$ is dense in $\mathbb{T}^1$ over $\mathcal{C}$.*

Proof. We make use of a characterisation of denseness from Lemma 4.21 in [Blua] according to which $\mathbb{T}^0$ is dense in $\mathbb{T}^1$ over $\mathcal{C}$ if, and only if, for all $\mathfrak{A} \in \mathcal{C}$, every subalgebra of the $\mathbb{T}^0$-reduct of $\mathfrak{A}$ is also a subalgebra of $\mathfrak{A}$. Hence, fix $\mathfrak{A} \in \mathcal{C}$ and let $C \subseteq A$ be a set inducing a subalgebra of the $\mathbb{T}^0$-reduct of $\mathfrak{A}$. We have to show that C is closed under the product of $\mathfrak{A}$.

Let $t \in \mathbb{T}^1 C$. If $t \in \mathbb{T}^0 C$, we have $\pi(t) \in C$ since C is closed under $\pi \restriction \mathbb{T}^0 A$. Hence, suppose otherwise. By assumption, t has a simple $\mathbb{T}^0$-evaluation γ. Let α be the ordinal such that $\gamma \in \mathbb{E}_{\alpha+1}(\mathfrak{A}, \mathbb{T}^1) \setminus \mathbb{E}_\alpha(\mathfrak{A}, \mathbb{T}^1)$. By induction on α, we prove that

$$\pi(t) = \text{val}_{\alpha+1}(\gamma) \quad \text{and} \quad \text{val}_{\alpha+1}(\gamma) \in C.$$

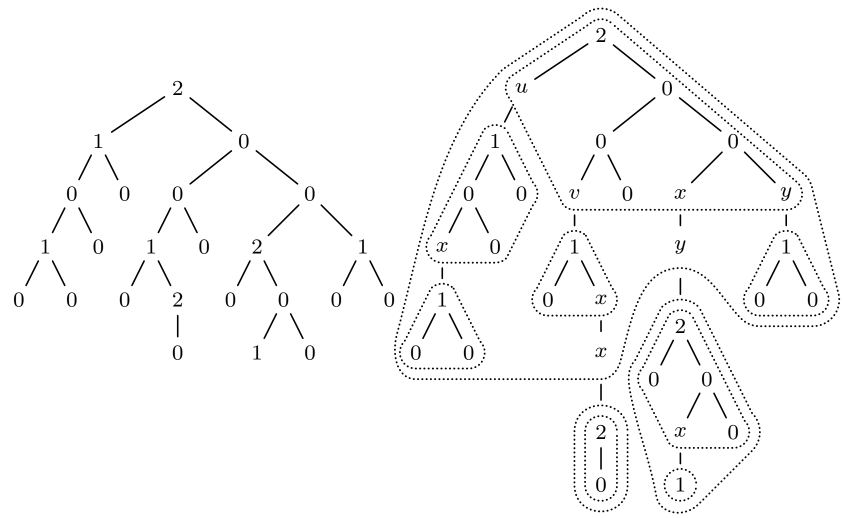


Figure 4: An evaluation (on the right) encoded by a function (on the left). For simplicity, we have omitted the labels from A and drawn just the τ-labelling.

For the first claim, associativity of π implies that

$$\begin{aligned}\pi(t) &= \pi(\mathrm{term}_{\alpha+1}(\gamma))\\ &= \pi(\mathrm{flat}(\mathbb{T}^1\mathrm{term}_\alpha(\gamma)))\\ &= \pi(\mathbb{T}^1\pi(\mathbb{T}^1\mathrm{term}_\alpha(\gamma)))\\ &= \pi(\mathbb{T}^1\mathrm{val}_\alpha(\gamma))\\ &= \mathrm{val}_{\alpha+1}(\gamma)\,.\end{aligned}$$

For the second claim, note that every subevaluation $\gamma(v)$ is an evaluation of some factor of t. Hence, we can use the inductive hypothesis to prove that $\mathbb{T}^1\mathrm{val}_\alpha(\gamma) \in \mathbb{T}^1 C$. Since C is closed under $\pi \restriction \mathbb{T}^0 A$, it follows that

$$\mathrm{val}_{\alpha+1}(\gamma) = \pi(\mathbb{T}^1\mathrm{val}_\alpha(\gamma)) \in C\,. \qquad \square$$

The only reason why we allow evaluations $\gamma \in \mathbb{E}_{\alpha+1}(\rho, \mathbb{T}^1)$ with components $\gamma(v)$ belonging to $\mathbb{E}_\beta(\rho, \mathbb{T}^1)$, for $\beta < \alpha$, is to define $\mathbb{E}_\alpha(\rho, \mathbb{T}^1)$ for all ordinals α. In all of our applications below, the case $\alpha < \omega$ will actually be sufficient. In this case, we can always assume that $\gamma(v) \in \mathbb{E}_\alpha(\rho, \mathbb{T}^1) \setminus \mathbb{E}_{\alpha-1}(\rho, \mathbb{T}^1)$ for all v. (If not, we can replace $\gamma(v)$ by $\mathrm{sing}(\gamma(v))$.)

Definition 5.3. We say that an evaluation $\gamma \in \mathbb{E}_\alpha(\rho, \mathbb{T}^1)$ has *uniform depth* α if $\alpha = 0$ or $\gamma \in \mathbb{T}^\times\mathbb{T}^{\alpha-1}(A)$. ⌟

As another example, let us show how to encode an evaluation of uniform depth n by a function $\tau : \mathrm{dom}_0(t) \to [n]$ similar to a Ramseyan split. By definition, such an evaluation is nothing but a nested factorisation $\gamma \in \mathbb{T}^\times\mathbb{T}\cdots\mathbb{T}A$. We can encode such a factorisation by assigning to every vertex $w \in \mathrm{dom}_0(t)$ the depth at which it appears in this nesting (see Figure 4 for an example).

Lemma 5.4. *Let $0 < n < \omega$ and let $t \in \mathbb{T}^\times A$. The construction in Figure 4 induces a bijection between all simple $\mathbb{T}^\times$-evaluations $\gamma \in \mathbb{E}_n(\mathfrak{A}, \mathbb{T}^\times)$ of t of uniform depth n and all functions $\tau : \mathrm{dom}_0(t) \to [n]$ with $\tau(\langle\rangle) = n - 1$.*

Proof. Given an evaluation γ of t of uniform depth n, we construct the corresponding function τ by induction on n. If $n = 1$, we set $\tau(w) := 0$, for all w. Otherwise, let $\tau_v : \mathrm{dom}_0(\mathrm{term}_{n-1}(\gamma(v))) \to [n-1]$, for $v \in \mathrm{dom}_0(\gamma)$, be the functions obtained from the inductive hypothesis for $\gamma(v) \in \mathbb{E}_{n-1}(\mathfrak{A}, \mathbb{T})$. Note that $t = \mathrm{term}_n(\gamma) = \mathrm{flat}(\mathbb{T}^\times \mathrm{term}_{n-1}(\gamma))$. Hence, every vertex $w \in \mathrm{dom}_0(t)$ corresponds to a pair $\langle v, u\rangle$ of vertices with $v \in \mathrm{dom}_0(\gamma)$ and $u \in \mathrm{dom}_0(\mathrm{term}_{n-1}(\gamma(v)))$. We set

$$\tau(w) := \begin{cases} n-1 & \text{if } u \text{ is the root of } \mathrm{term}_{n-1}(\gamma(v))\,, \\ \tau_v(u) & \text{otherwise}\,. \end{cases}$$

Conversely, let $\tau : \mathrm{dom}_0(t) \to [n]$ be a function with $\tau(\langle\rangle) = n - 1$. If $n = 1$, we set $\gamma := t$. Otherwise, let T be the factorisation of t induced by the set

$$H := \{\, w \in \mathrm{dom}_0(t) \mid \tau(w) = n-1 \,\}\,,$$

that is, H is the set of vertices corresponding to the roots of the factors $T(v)$. For each $v \in \mathrm{dom}_0(T)$, we can use the inductive hypothesis to find an evaluation $\gamma_v \in \mathbb{E}_{n-1}(\mathfrak{A}, \mathbb{T}^\times)$ of $T(v)$. (The corresponding function τ' is obtained from the restriction of τ to $T(v)$ by setting the value at the root to $n-2$.) Since $T(v) \in \mathbb{T}A$ is linear, we have $\gamma_v \in \mathbb{E}_{n-1}(\mathfrak{A}, \mathbb{T})$. Consequently, we can set

$$\gamma(v) := \gamma_v\,.$$

It is straightforward to check that the two functions defined above are inverse to each other. □

Let us next explain how to use ρ-evaluations to construct $\mathbb{T}^1$-expansions. The proof makes use of the following glueing operation for evaluations.

Lemma 5.5. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$ be monads, $\gamma \in \mathbb{T}^1\mathbb{E}(\rho, \mathbb{T}^1)$ a tree of evaluations, and let β be a ρ-evaluation of the tree $t := \mathbb{T}^1\mathrm{val}(\gamma)$. There exists a ρ-evaluation $\beta|\gamma$ of the tree $(\mathrm{flat} \circ \mathbb{T}^1\mathrm{term})(\gamma)$ such that $\mathrm{val}(\beta|\gamma) = \mathrm{val}(\beta)$.*

Proof. We define $\beta|\gamma$ by the following induction on the ordinal α with $\beta \in \mathbb{E}_\alpha(\rho, \mathbb{T}^1)$. If $\alpha = 0$, then $\beta = a \in A$ and $\gamma = \mathrm{sing}(\gamma_0)$, for some $\gamma_0 \in \mathbb{E}(\rho, \mathbb{T}^1)$ with $\mathrm{val}(\gamma_0) = a$. Setting $\beta|\gamma := \gamma_0$ we obtain $\mathrm{val}(\beta|\gamma) = \mathrm{val}(\gamma_0) = a = \mathrm{val}(\beta)$.

For the inductive step, suppose that $\beta \in \mathbb{E}_{\alpha+1}(\rho, \mathbb{T}^1) \setminus \mathbb{E}_\alpha(\rho, \mathbb{T}^1)$. Then $\beta \in \mathbb{T}^1\mathbb{E}_\alpha(\rho, \mathbb{T}^1)$. Let $\gamma' \in \mathbb{T}^1\mathbb{T}^1\mathbb{E}(\rho, \mathbb{T}^1)$ be the tree with the same domain as β where $\gamma'(v)$, for $v \in \mathrm{dom}_0(\beta)$, is (isomorphic to) the restriction of γ to $\mathrm{dom}_0(\mathrm{term}(\beta(v))) \subseteq \mathrm{dom}(\gamma)$ (plus vertices for the variables). We can use the inductive hypothesis to obtain evaluations $\beta(v)|\gamma'(v) \in \mathbb{E}(\rho, \mathbb{T}^1)$, for $v \in \mathrm{dom}_0(\beta)$. We choose for $\beta|\gamma$ the tree with the same domain as β and

$$(\beta|\gamma)(v) := \beta(v)|\gamma'(v)\,, \quad \text{for } v \in \mathrm{dom}_0(\beta)\,.$$

By inductive hypothesis it then follows that

$$\begin{aligned} \mathrm{term}((\beta|\gamma)(v)) &= \mathrm{term}(\beta(v)|\gamma'(v)) = (\mathrm{flat} \circ \mathbb{T}^1\mathrm{term})(\gamma'(v))\,, \\ \mathrm{val}((\beta|\gamma)(v)) &= \mathrm{val}(\beta(v)|\gamma'(v)) \quad = \mathrm{val}(\beta(v))\,. \end{aligned}$$

Consequently, we have

$$\begin{aligned}\mathrm{term}(\beta|\gamma) &= (\mathrm{flat}\circ\mathbb{T}^1\mathrm{term})(\beta|\gamma)\\ &= (\mathrm{flat}\circ\mathbb{T}^1(\mathrm{flat}\circ\mathbb{T}^1\mathrm{term}))(\gamma')\\ &= (\mathrm{flat}\circ\mathrm{flat}\circ\mathbb{T}^1\mathbb{T}^1\mathrm{term})(\gamma') && [\mathbb{T}^1 \text{ monad}]\\ &= (\mathrm{flat}\circ\mathbb{T}^1\mathrm{term}\circ\mathrm{flat})(\gamma') && [\text{flat nat. trans.}]\\ &= (\mathrm{flat}\circ\mathbb{T}^1\mathrm{term})(\gamma)\,,\\ \mathrm{val}(\beta|\gamma) &= (\rho\circ\mathbb{T}^1\mathrm{val})(\beta|\gamma)\\ &= (\rho\circ\mathbb{T}^1\mathrm{val})(\beta) = \mathrm{val}(\beta)\,.\end{aligned}$$

Furthermore, note that $\beta|\gamma$ really is an evaluation since

$$\mathbb{T}^1\mathrm{val}(\beta|\gamma) = \mathbb{T}^1\mathrm{val}(\beta) \in \mathrm{dom}(\rho)\,.$$

Finally, if $\beta \in \mathbb{E}_\delta(\rho, \mathbb{T}^1)$, for some limit ordinal δ, then there is some $\alpha < \delta$ with $\beta \in \mathbb{E}_\alpha(\rho, \mathbb{T}^1)$. Hence, the claim follows by inductive hypothesis. □

Theorem 5.6. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$ be submonads. Every $\mathbb{T}^0$-algebra $\mathfrak{A}$ with essentially unique ρ-evaluations has a $\mathbb{T}^1$-expansion $\langle A, \pi_+\rangle$ such that*

$$\pi_+ \circ \mathrm{term} = \mathrm{val}\,.$$

Proof. For $t \in \mathbb{T}^1 A$, we define

$$\pi_+(t) := \mathrm{val}(\gamma)\,, \quad \text{for some } \gamma \in \mathrm{term}^{-1}(t)\,.$$

As ρ-evaluations are essentially unique, it does not matter which evaluation γ we choose. Hence, this function is well-defined. We claim that $\mathfrak{A}_+ := \langle A, \pi_+\rangle$ is the desired $\mathbb{T}^1$-expansion of $\mathfrak{A}$.

The equation $\pi_+ \circ \mathrm{term} = \mathrm{val}$ follows immediately from the definition of π_+. Since every tree $t \in \mathbb{T}^0 A \subseteq \mathbb{E}_1(\rho, \mathbb{T}^1)$ is its own evaluation, we further have

$$\pi_+(t) = \mathrm{val}_1(t) = \rho(t) = \pi(t)\,, \quad \text{for } t \in \mathbb{T}^0 A\,.$$

Consequently, π_+ is an extension of π and it remains to check the axioms of a $\mathbb{T}^1$-algebra.

For the unit law, we have $\pi_+(\mathrm{sing}(a)) = \pi(\mathrm{sing}(a)) = a$ since $\mathrm{sing}(a) \in \mathbb{T}^0 A$. For associativity, let $t \in \mathbb{T}^1\mathbb{T}^1 A$. Then there exists a tree of ρ-evaluations $\gamma \in \mathbb{T}^1\mathbb{E}(\rho, \mathbb{T}^1)$ such that $t = \mathbb{T}^1\mathrm{term}(\gamma)$. Furthermore, we can fix an evaluation $\beta \in \mathbb{E}(\rho, \mathbb{T}^1)$ of the tree $\mathbb{T}^1\mathrm{val}(\gamma)$. Let $\beta|\gamma$ be the ρ-evaluation from Lemma 5.5. Then

$$\begin{aligned}(\pi_+\circ\mathrm{flat})(t) &= (\pi_+\circ\mathrm{flat}\circ\mathbb{T}^1\mathrm{term})(\gamma)\\ &= (\pi_+\circ\mathrm{term})(\beta|\gamma)\\ &= \mathrm{val}(\beta|\gamma)\\ &= \mathrm{val}(\beta)\\ &= (\pi_+\circ\mathrm{term})(\beta)\\ &= (\pi_+\circ\mathbb{T}^1\mathrm{val})(\gamma)\\ &= (\pi_+\circ\mathbb{T}^1\pi_+\circ\mathbb{T}^1\mathrm{term})(\gamma) = (\pi_+\circ\mathbb{T}^1\pi_+)(t)\,.\end{aligned}$$

□

The theorem tells us how to use evaluations to construct expansions. Let us see next where the limits of this method are.

Proposition 5.7. *Let $\mathfrak{A} = \langle A, \pi \rangle$ be a $\mathbb{T}^0$-algebra with a $\mathbb{T}^1$-expansion $\mathfrak{A}_+ = \langle A, \pi_+ \rangle$, and suppose that $\rho : S \to A$ is the restriction of π_+ to some set $S \supseteq \mathbb{T}^0 A$. Then*

$$\mathrm{val}(\gamma) = (\pi_+ \circ \mathrm{term})(\gamma)\,, \quad \text{for every } \rho\text{-evaluation } \gamma\,.$$

Proof. We prove that $\mathrm{val}_\alpha = \pi_+ \circ \mathrm{term}_\alpha$ by induction on α. For $\alpha = 0$, we have

$$\mathrm{val}_0(\gamma) = \gamma = \pi_+(\mathrm{sing}(\gamma)) = \pi_+(\mathrm{term}_0(\gamma))\,, \quad \text{for } \gamma \in \mathbb{E}_0(\rho, \mathbb{T}^1) = A\,.$$

For the successor step, suppose that the equation holds for α and consider some $\gamma \in \mathbb{E}_{\alpha+1}(\rho, \mathbb{T}^1) \setminus \mathbb{E}_\alpha(\rho, \mathbb{T}^1)$. Then

$$\begin{aligned}
\mathrm{val}_{\alpha+1}(\gamma) &= \rho(\mathbb{T}^1 \mathrm{val}_\alpha(\gamma)) \\
&= \rho(\mathbb{T}^1(\pi_+ \circ \mathrm{term}_\alpha)(\gamma)) \\
&= \pi_+(\mathbb{T}^1 \pi_+(\mathbb{T}^1 \mathrm{term}_\alpha(\gamma))) \\
&= \pi_+(\mathrm{flat}(\mathbb{T}^1 \mathrm{term}_\alpha(\gamma))) \\
&= \pi_+(\mathrm{term}_{\alpha+1}(\gamma))\,.
\end{aligned}$$

Finally, for a limit ordinal α, the claim follows immediately from the inductive hypothesis. □

Corollary 5.8. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}$ and let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra.*

(a) *If $\mathfrak{A}$ has several different $\mathbb{T}^1$-expansions, there exist trees $t \in \mathbb{T}^1 A$ without a simple $\mathbb{T}^0$-evaluation.*
(b) *If $\mathfrak{A}$ has simple $\mathbb{T}^0$-evaluations for $\mathbb{T}^1$ that are not essentially unique, it has no $\mathbb{T}^1$-expansion.*

Example. Before using these results to study thin trees, let us quickly recall the results of [CCP18] about countable linear orders. We denote by $\mathbb{C}A$ the (unsorted) set of all countable A-labelled linear orders and by $\mathbb{C}^{\mathrm{reg}} A \subseteq \mathbb{C}A$ the subset of all *regular* linear orders. By definition, a linear order is regular if it can be denoted by a finite term using the following operations: (I) constants for singletons, (II) binary ordered sums, (III) multiplication by ω and ω^{op} (ω with the reverse ordering), and (IV) dense shuffles. In [CCP18] (Propositions 3.8 and 3.9) it is shown that every finite $\mathbb{C}^{\mathrm{reg}}$-algebra has essentially unique $\mathbb{C}^{\mathrm{reg}}$-evaluations for $\mathbb{C}$. This fact can be used to prove the following results (for the proofs, see [CCP18, Boj20, Blua]).

- $\mathbb{C}^{\mathrm{reg}} \subseteq \mathbb{C}$ is dense over the class of all finite $\mathbb{C}$-algebras (Theorem 3.14 of [Boj20]).
- Every finite $\mathbb{C}^{\mathrm{reg}}$-algebra has a unique $\mathbb{C}$-expansion (Theorem 3.11 of [CCP18]).
- Every finite $\mathbb{C}$-algebra is MSO-definable (Theorem 5.1 of [CCP18]).
- A language $K \subseteq \mathbb{C}\Sigma$ of countable linear orders is MSO-definable if, and only if, it is recognised by some finite $\mathbb{C}$-algebra (Theorems 4.1 and 5.1 of [CCP18]).
- Every MSO-definable language $K \subseteq \mathbb{C}\Sigma$ is recognised by a finite $\mathbb{C}$-algebra (Theorem 3.12 of [Boj20]). ⌟

5.1. **Thin trees.** As an application of simple evaluations we consider thin trees, where we can use the Theorem of Ramsey and other tools from semigroup theory. Note that, with every $\mathbb{T}^{\mathrm{fin}}$-algebra $\mathfrak{A}$, we can associate the semigroup with universe $A_{\{z\}}$ (for some fixed variable z whose choice does not matter) and where the product is defined by

$$a \cdot b := a(b(z))\,, \quad \text{for } a, b \in A_{\{z\}}\,.$$

If $\mathfrak{A}$ is a $\mathbb{T}^{\text{wilke}}$-algebra, this semigroup can be expanded to a Wilke algebra $\langle A_{\{z\}}, A_\emptyset \rangle$ by setting

$$a \cdot c := a(c) \quad \text{and} \quad a^\omega := \pi(t_a)\,, \quad \text{for } a \in A_{\{z\}} \text{ and } c \in A_\emptyset\,,$$

where t_a is an infinite path each vertex of which is labelled by a. Finally, if $\mathfrak{A}$ is a $\mathbb{T}^{\text{thin}}$-algebra, we obtain an ω-semigroup with infinite product

$$\pi(a_0, a_1, \dots) := a_0(a_1(\cdots))\,, \quad \text{for } a_i \in A_{\{z\}}\,.$$

We start by generalising the fact that every finite Wilke algebra has a unique expansion to an ω-semigroup to the monads $\mathbb{T}^{\text{wilke}} \subseteq \mathbb{T}^{\text{thin}}$.

Proposition 5.9. *Every finitary $\mathbb{T}^{\text{wilke}}$-algebra has essentially unique simple $\mathbb{T}^{\text{wilke}}$-evaluations for $\mathbb{T}^{\text{thin}}$.*

Proof. Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\text{wilke}}$-algebra and $t \in \mathbb{T}^{\text{thin}} A$ a thin tree. We construct the desired simple evaluation of t by induction on the Cantor-Bendixson rank α of t. (Recall that the Cantor-Bendixson rank of a tree t is the least ordinal α such that $\partial^{\alpha+1}(t)$ is empty, where $\partial(t)$ denotes the tree obtained from t by removing every subtree with only finitely many infinite branches, and ∂^α is the α-th iteration of ∂. One can show that such an ordinal α exists if, and only if, the given tree t is thin.)

By inductive hypothesis, every subtree $t|_v$ of rank less than α has a simple evaluation $\gamma_v \in \mathbb{E}(\mathfrak{A}, \mathbb{T}^{\text{thin}})$. Let s be tree obtained from t by replacing every such subtree $t|_v$ by $\text{val}(\gamma_v)$. It is sufficient to find a simple evaluation of s. Then we can use the glueing operation from Lemma 5.5 to construct the desired evaluation of t.

By construction, s has only finitely many infinite branches. We distinguish three cases.

(I) If s is finite, it is its own evaluation.

(II) Next, suppose that s has a single infinite branch. By the Theorem of Ramsey, we can find a factorisation $s = p_0 p_1 p_2 \dots$ such that $\pi(p_i) = \pi(p_j)$, for all $i, j > 0$. As each factor p_i is finite, we obtain simple evaluations β_i of p_i by (I). The path $\rho := \pi(p_0), \pi(p_1), \pi(p_2), \dots$ is of the form ae^ω for $a := \pi(p_0)$ and $e := \pi(p_1)$. In particular, it is regular. Let β_* be the path $\beta_0, \beta_1, \beta_2, \dots$. Then

$$\mathbb{T}^{\text{thin}}\text{val}(\beta_*) = aeee\cdots \in \mathbb{T}^{\text{reg}} A \quad \text{and} \quad \text{term}(\beta_*) = p_0 p_1 p_2 \cdots = s\,.$$

Hence, β_* is the desired simple evaluation of s.

(III) Finally, suppose that s has at least two infinite branches. Then we can factorise s into a finite prefix and finitely many trees with a single infinite branch. By (I) and (II), each of these factors has a simple evaluation. Let β be the finite tree consisting of these evaluations. Then β is a simple evaluation of s. ☐

Using Lemma 5.2 we obtain the following corollary.

Corollary 5.10. $\mathbb{T}^{\text{wilke}} \subseteq \mathbb{T}^{\text{thin}}$ *is dense over the class of all finitary $\mathbb{T}^{\text{thin}}$-algebras.*

Corollary 5.11. *Every finitary $\mathbb{T}^{\text{wilke}}$-algebra has a unique $\mathbb{T}^{\text{thin}}$-expansion.*

It follows that the step from a $\mathbb{T}^{\text{wilke}}$-algebra to a $\mathbb{T}^{\text{thin}}$-expansion is fairly well understood. The inclusion $\mathbb{T}^{\text{fin}} \subseteq \mathbb{T}^{\text{wilke}}$ is slightly more complicated since expansions are no longer unique.

Definition 5.12. Let $\mathfrak{A}$ be a $\mathbb{T}^{\text{wilke}}$-algebra. The corresponding *ω-power operation* $-^\omega : A_{\{z\}} \to A_\emptyset$ is defined by

$$a^\omega := \pi(aaa\dots) \qquad \text{(an infinite path labelled } a)\,. \qquad ⌟$$

Proposition 5.13. *Let $\mathfrak{A} = \langle A, \pi \rangle$ be a finitary $\mathbb{T}^{\mathrm{fin}}$-algebra. The function mapping a $\mathbb{T}^{\mathrm{wilke}}$-expansions of $\mathfrak{A}$ to the corresponding ω-power operation provides a bijection between all $\mathbb{T}^{\mathrm{wilke}}$-expansions of $\mathfrak{A}$ and all functions $-^{\omega} : A_{\{z\}} \to A_{\emptyset}$ satisfying the equations*

$$(ab)^{\omega} = a(ba)^{\omega} \quad \text{and} \quad (a^n)^{\omega} = a^{\omega}, \quad \text{for all } a, b \in A_{\{z\}}\,.$$

Proof. Clearly the ω-power operation $a \mapsto a^{\omega}$ associated with a $\mathbb{T}^{\mathrm{wilke}}$-expansion $\mathfrak{A}_+ = \langle A, \pi_+ \rangle$ satisfies the two axioms above since the product π_+ is associative. It therefore remains to show that the correspondence is bijective.

Note that every tree $t \in \mathbb{T}^{\mathrm{wilke}}A$ is the unravelling of a finite graph all of which strongly connected components are either singletons or induced cycles (cycles without any additional edges).

For injectivity, suppose that there are two expansions $\mathfrak{A}_0 = \langle A, \pi_0 \rangle$ and $\mathfrak{A}_1 = \langle A, \pi_1 \rangle$ of $\mathfrak{A}$ with the same associated ω-power. Let $t \in \mathbb{T}^{\mathrm{wilke}}A$ be the unravelling of a graph $\mathfrak{G}$ with n strongly connected components. We prove that

$$\pi_0(t) = \pi_1(t)$$

by induction on n.

Since both products agree on finite trees and we can write every tree $s \in \mathbb{T}^{\mathrm{wilke}}_{\xi}A$ as $s = p(\bar{u}, \bar{x})$ where $\bar{x}$ is an enumeration of ξ, $p \in \mathbb{T}^{\mathrm{fin}}A$ is finite, and each $u_i \in \mathbb{T}^{\mathrm{wilke}}_{\emptyset}A$ is a tree without variables, it is sufficient to prove the claim for $t \in \mathbb{T}^{\mathrm{wilke}}_{\emptyset}A$.

Let C be the strongly connected component of $\mathfrak{G}$ containing the root of t. First, suppose that C is a singleton that is not a cycle. Then $t = a(s_0, \dots, s_{m-1})$, for some $a \in A$ and $s_i \in \mathbb{T}^{\mathrm{wilke}}A$. Hence, it follows by inductive hypothesis that

$$\pi_0(t) = a\big(\pi_0(s_0), \dots, \pi_0(s_{m-1})\big) = a\big(\pi_1(s_0), \dots, \pi_1(s_{m-1})\big) = \pi_1(t)\,.$$

Next, suppose that C is a cycle. For every vertex $v \in \mathrm{dom}_0(t) \setminus C$, it follows by inductive hypothesis that

$$\pi_0(t|_v) = \pi_1(t|_v)\,.$$

Replacing these subtrees by their product (and merging the obtained leaves into their predecessor) we may assume that $\mathfrak{G}$ has a single strongly connected component C. Since t is thin, this component forms a cycle and there exists a finite path p such that $t = p^{\omega}$. Consequently,

$$\pi_0(t) = \pi(p)^{\omega} = \pi_1(t)\,.$$

For surjectivity, suppose that $-^{\omega} : A_{\{z\}} \to A_{\emptyset}$ is an ω-power operation. To construct an expansion $\mathfrak{A}^+ = \langle A, \pi_+ \rangle$ of $\mathfrak{A}$, we define a function $\hat{\pi}$ mapping every finite graph $\mathfrak{G}$ whose unravelling belongs to $\mathbb{T}^{\mathrm{wilke}}A$ to some element $\hat{\pi}(\mathfrak{G}) \in A$. Then we set $\pi_+(t) := \hat{\pi}(\mathfrak{G})$ for some $\mathfrak{G}$ with unravelling t.

We define $\hat{\pi}(\mathfrak{G})$ by induction on the number of connected components of $\mathfrak{G}$. Let C be the strongly connected component of $\mathfrak{G}$ containing the root. For every vertex $v \notin C$ that is not a variable, we can compute $\hat{\pi}(\mathfrak{G}|_v)$ by inductive hypothesis, where $\mathfrak{G}|_v$ denotes the subgraph of $\mathfrak{G}$ consisting of all vertices reachable from v. Let $\mathfrak{G}'$ be the graph obtained from $\mathfrak{G}$ by replacing every such subgraph $\mathfrak{G}|_v$ by its product $\hat{\pi}(\mathfrak{G}|_v)$ and then merging the resulting leaves into their predecessor. If $\mathfrak{G}'$ is a tree with a root with label a attached to which are variables $\bar{x}$, we set

$$\hat{\pi}(\mathfrak{G}) := a(\bar{x})\,.$$

Otherwise, $\mathfrak{G}'$ consists of a cycle. (Since the unravelling of $\mathfrak{G}'$ is a linear tree, it follows that $\mathfrak{G}'$ cannot contain any variables.) Let $p \in \mathbb{T}^{\mathrm{fin}}A$ be the finite path obtained from $\mathfrak{G}'$ by removing the edge leading to the root and replacing it by a variable. We set

$$\hat{\pi}(\mathfrak{G}) := \pi(p)^{\omega}\,.$$

It remains to check that that the value of $\hat{\pi}(\mathfrak{G})$ only depends on the unravelling of $\mathfrak{G}$, that the resulting function π_+ satisfies the axioms of a $\mathbb{T}^{\mathrm{wilke}}$-algebra, and that the associated ω-operation coincides with the given one. We start with the latter. Let $a \in A_{\{z\}}$ and let $\mathfrak{G}$ be a graph with unravelling $aaa\dots$. Then $\mathfrak{G}$ is a path of length $m < \omega$ leading to a cycle of length $n < \omega$. It follows that

$$\hat{\pi}(\mathfrak{G}) = a^m \cdot \pi(a^n)^{\omega} = a^m \cdot (a^n)^{\omega} = a^{\omega},$$

as desired.

Next let us show that $\hat{\pi}(\mathfrak{G}) = \hat{\pi}(\mathfrak{H})$, for graphs $\mathfrak{G}$ and $\mathfrak{H}$ with the same unravelling. Given a graph $\mathfrak{G}$ and a vertex v, we define an equivalence relation $\sim_{\mathfrak{G}}$ on its set of vertices by

$$u \sim_{\mathfrak{G}} v \quad :\text{iff} \quad \text{the unravellings of } \mathfrak{G}|_u \text{ and } \mathfrak{G}|_v \text{ are isomorphic.}$$

Since two graphs $\mathfrak{G}$ and $\mathfrak{H}$ have the same unravelling if, and only if, their quotients by, respectively, $\sim_{\mathfrak{G}}$ and $\sim_{\mathfrak{H}}$ are isomorphic, it is sufficient to show that

$$\hat{\pi}(\mathfrak{G}) = \hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}})\,, \quad \text{for every graph } \mathfrak{G}\,.$$

We prove the claim by induction on the number of strongly connected components of $\mathfrak{G}$. Let C be the strongly connected component of $\mathfrak{G}$ containing the root. We distinguish several cases.

First, suppose that C is a union of $\sim_{\mathfrak{G}}$-classes. Then the image $C/{\sim_{\mathfrak{G}}}$ of C under the quotient map forms a strongly connected component of $\mathfrak{G}/{\sim_{\mathfrak{G}}}$.

If $C = \{v\}$ is a singleton that is not a cycle, we argue as follows. Let $u_0, \dots, u_{n-1}$ be the non-variable successors of v and let $y_0, \dots, y_{m-1}$ be the variables attached to v. By inductive hypothesis, we have

$$\hat{\pi}(\mathfrak{G}|_{u_i}) = \hat{\pi}((\mathfrak{G}/{\sim_{\mathfrak{G}}})|_{u_i})\,.$$

Let a be the label of v and set $c_i := \hat{\pi}(\mathfrak{G}|_{u_i})$, for $i < n$. It follows that

$$\hat{\pi}(\mathfrak{G}) = a(c_0, \dots, c_{n-1}, y_0, \dots, y_{m-1}) = \hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}})\,.$$

If C (and therefore $C/{\sim_{\mathfrak{G}}}$) form cycles, the product $\hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}})$ is computed by (I) evaluating all products $\hat{\pi}((\mathfrak{G}/{\sim_{\mathfrak{G}}})|_v)$ for $v \notin C/{\sim_{\mathfrak{G}}}$; (II) merging the resulting values into the label of the predecessor of v; and (III) computing p^{ω}, where p is the path obtained by (II). By inductive hypothesis it follows that, when performing the same procedure for $\mathfrak{G}$, we obtain a path of the form p^m, for some $0 < m < \omega$. Hence,

$$\hat{\pi}(\mathfrak{G}) = (p^m)^{\omega} = p^{\omega} = \hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}})\,.$$

It remains to consider the case where there are vertices outside of C $\sim_{\mathfrak{G}}$-equivalent to some vertex of C. Then the strongly connected component D of $\mathfrak{G}/{\sim_{\mathfrak{G}}}$ containing the root forms a cycle.

Suppose that $C = \{v\}$ is a singleton that is not a cycle. Let $u_0, \dots, u_{n-1}$ be the successors of v. (Note that none of them can be variables since there is some vertex $v' \sim_{\mathfrak{G}} v$, but each variable occurs only once in $\mathfrak{G}$.) Note that there can be only one successor u_i whose $\sim_{\mathfrak{G}}$-class belongs to D since, otherwise, the unravelling of $\mathfrak{G}$ would not be thin. We choose the ordering of $u_0, \dots, u_{n-1}$ such that this vertex is u_0. Let a be the label of v and set $c_i := \hat{\pi}(\mathfrak{G}|_{u_i})$. By inductive hypothesis, we have

$$\hat{\pi}((\mathfrak{G}/{\sim_{\mathfrak{G}}})|_{u_i}) = c_i\,.$$

Let p be the path such that

$$\hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}}) = p^\omega\,.$$

Then the first label of p is $b := a(z, c_1, \dots, c_{n-1})$. Hence, $p = bq$, for some path q, and it follows by inductive hypothesis that

$$c_0 = (qb)^\omega.$$

Consequently, we have

$$\begin{aligned}\hat{\pi}(\mathfrak{G}) &= a(c_0, \dots, c_{n-1})\\ &= a\big((qb)^\omega, c_1, \dots, c_{n-1}\big)\\ &= b(qb)^\omega\\ &= (bq)^\omega\\ &= \hat{\pi}(\mathfrak{G}/{\sim_{\mathfrak{G}}})\,.\end{aligned}$$

Finally, suppose that C is a cycle. We claim that this case cannot occur. For a contradiction, suppose otherwise. Then there is some vertex $v \in C$ with two different successors u, u' from that we can reach vertices w and w', respectively, with $w \sim_{\mathfrak{G}} w'$ and such that the corresponding $\sim_{\mathfrak{G}}$-class belongs to D. Since D is strongly connected, it follows that we can reach from w and w' vertices $\hat{v}$ and $\hat{v}'$ with $\hat{v} \sim_{\mathfrak{G}} v \sim_{\mathfrak{G}} \hat{v}'$. This implies that the unravelling of $\mathfrak{G}$ is not thin. A contradiction.

It remains to check the axioms for a $\mathbb{T}^{\mathrm{wilke}}$-algebra. It follows directly by definition that

$$\pi_+(\mathrm{sing}(a)) = a\,.$$

For associativity, fix $t \in \mathbb{T}^{\mathrm{wilke}}\mathbb{T}^{\mathrm{wilke}}A$ and let $\mathfrak{G}$ be a finite graph with unravelling t. For each vertex v of $\mathfrak{G}$, we fix a finite graph $\mathfrak{H}_v$ with unravelling $t(v)$. Then $\mathrm{flat}(t)$ is the unravelling of the graph obtained from the disjoint union of all $\mathfrak{H}_v$ by adding edges according to $\mathfrak{G}$. Let us call this graph $\mathfrak{K}$. We will prove that

$$\hat{\pi}(\mathfrak{K}) = \hat{\pi}(\mathfrak{G}')\,,$$

where $\mathfrak{G}'$ is the graph obtained from $\mathfrak{G}$ by replacing each label by the corresponding product $\hat{\pi}(\mathfrak{H}_v)$. Then it follows that

$$\pi_+(\mathrm{flat}(t)) = \pi_+(\mathbb{T}^{\mathrm{wilke}}\pi_+(t))\,,$$

as desired.

We proceed by induction on the number of vertices of $\mathfrak{K}$. Let C be the strongly connected component of $\mathfrak{G}$ containing the root. For every vertex $v \in \mathrm{dom}_0(\mathfrak{G}) \setminus C$, it follows by inductive hypothesis that

$$\hat{\pi}(\mathfrak{K}|_{\mu(v)}) = \hat{\pi}(\mathfrak{G}'|_v)\,,$$

where $\mu : \mathrm{dom}_0(\mathfrak{G}) \to \mathrm{dom}_0(\mathfrak{K})$ is the function mapping each vertex v of $\mathfrak{G}$ to the root of $\mathfrak{H}_v$. Note that the products $\hat{\pi}(\mathfrak{K})$ and $\hat{\pi}(\mathfrak{G}')$ can be computed by (I) replacing all the subgraphs $\mathfrak{K}|_{\mu(v)}$ and $\mathfrak{G}'|_v$, for $v \in \mathrm{dom}_0(\mathfrak{G}) \setminus C$, by their respective products, (II) merging the resulting leaves into their parents, and then (III) computing the products of the remaining graphs. (For the graph $\mathfrak{G}'$, this statement follows immediately from the definition of $\hat{\pi}$; for $\mathfrak{K}$, it follows by a straightforward induction on the size of $\mathfrak{K}$.) We may therefore assume that $\mathfrak{G}$ consists of a single strongly connected component C (plus possibly some variables in case C is a singleton). If $C = \{v\}$ is a singleton which is not a cycle, we have $\mathfrak{K} = \mathfrak{H}_v$ and

$$\hat{\pi}(\mathfrak{K}) = \hat{\pi}(\mathfrak{H}_v) = \hat{\pi}(\mathfrak{G}')\,.$$

Hence, suppose that C is a cycle. Each graph $\mathfrak{H}_v$ consists of a finite path p_v leading to a variable to which are possibly attached additional graphs without variables. By inductive hypothesis, associativity holds for these subgraphs. Again, replacing each such subgraph by its product, we may assume that $\mathfrak{H}_v$ is equal to p_v. Consequently, $\mathfrak{K}$ is a single path consisting of the concatenation of all p_v, while $\mathfrak{G}'$ is the path labelled by the products $\pi(p_v)$. The product of these two paths is the same. □

Combining this result with Corollary 5.11 we obtain the corresponding statement for $\mathbb{T}^{\text{thin}}$-expansions.

Corollary 5.14. *Let $\mathfrak{A} = \langle A, \pi\rangle$ be a finitary $\mathbb{T}^{\text{fin}}$-algebra. There exists a bijection between all $\mathbb{T}^{\text{thin}}$-expansions of $\mathfrak{A}$ and all functions $-^{\omega} : A_{\{z\}} \to A_{\emptyset}$ satisfying the axioms of a Wilke algebra.*

Corollary 5.15. *Every $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ is uniquely determined by* (I) *its $\mathbb{T}^{\text{fin}}$-reduct and* (II) *the associated ω-semigroup.*

5.2. **Evaluations with merging.** When we try to go beyond $\mathbb{T}^{\text{thin}}$ our machinery breaks down since we cannot use the results for semigroups anymore. The following counterexample shows that a naïve generalisation of our definitions does not work.

Lemma 5.16. *There exists an* MSO*-definable $\mathbb{T}^{\text{reg}}$-algebra $\mathfrak{A}$ and a tree $t \in \mathbb{T}A$ that has no simple $\mathbb{T}^{\text{reg}}$-evaluation.*

Proof. Let $\mathfrak{A}$ be the $\mathbb{T}^{\text{reg}}$-reduct of the Bojańczyk-Klin algebra from the example on page 18. Then the claim follows immediately from Corollary 5.8 (a). Nevertheless we give an explicit proof to see what exactly is going wrong. Set $\Delta := \mathbb{T}^{\text{fin}}\{a, b\}$ and recall that $\Delta \subseteq A$. We will prove by induction on α that

$$t \notin \operatorname{rng}\operatorname{term}_{\alpha}\,, \qquad \text{for all } t \in \mathbb{T}\Delta \text{ where every subtree has vertices of arbitrarily high arity.}$$

For a contradiction, suppose otherwise. Let α be the minimal ordinal such that there is some simple evaluation $\gamma \in \mathbb{E}_{\alpha}(\mathfrak{A}, \mathbb{T})$ where every subtree of $\operatorname{term}(\gamma)$ has vertices of arbitrarily high arity. If $\alpha = 0$, then $\operatorname{term}(\gamma) = \operatorname{sing}(a)$ in contradiction to our choice of γ. Hence, $\alpha = \beta + 1$, for some β. Fix $v \in \operatorname{dom}(\gamma)$. Note that every subtree s of $\operatorname{term}_{\beta}(\gamma(v))$ has a simple evaluation in $\mathbb{E}_{\beta}(\mathfrak{A}, \mathbb{T})$ which, by inductive hypothesis, means that s has a subtree where the arity of the vertices is bounded. We claim that this implies that $t_v := \operatorname{term}(\gamma(v))$ is finite. Suppose otherwise. Since t_v has only finitely many variables, it has some infinite subtree s without variables. But s is also a subtree of $\operatorname{term}(\gamma)$. By choice of γ this implies that the arities of the vertices of s are unbounded. A contradiction.

Hence, we have $\operatorname{term}(\gamma(v)) \in \mathbb{T}^{\text{fin}}\Delta$, which implies that

$$\operatorname{val}(\gamma(v)) = \pi(\operatorname{term}(\gamma(v))) = \operatorname{term}(\gamma(v))\,.$$

Furthermore, $\operatorname{term}(\gamma(v))$ being finite its arity is at least as high as the maximal arity of a vertex in $\operatorname{dom}(\gamma(v))$. It follows that, for every $n < \omega$, there is some $v \in \operatorname{dom}(\gamma)$ such that $\operatorname{val}(\gamma(v))$ has arity at least n. But $\gamma \in \mathbb{E}_{\alpha+1}(\mathfrak{A}, \mathbb{T})$ implies that $\mathbb{T}\operatorname{val}(\gamma) \in \mathbb{T}^{\text{reg}}A$. In particular, $\mathbb{T}\mathrm{val}(\gamma)$ uses only finitely many different labels. This implies that their arity is bounded. A contradiction. □

A closer look at the above proof reveals two possible reasons making simple evaluations impossible. Firstly, our counterexample uses a tree with infinitely many different labels. It still might be possible that trees with only finitely many different labels always have simple evaluations. Secondly, we made essential use of the fact that every factor of an infinite binary tree has a subtree that is itself an infinite binary tree. To be able to use factorisations of trees into pieces that are significantly simpler, we will probably have to allow more general factors, which then necessarily have infinitely many variables. Unfortunately, it is hard to combine these two modifications since factors with infinitely many variables usually give rise to infinitely many different elements of the algebra. What seems to be missing is some technique that, given a tree with infinitely many different labels, allows us to bound their arity by merging different variables (e.g., replacing $a(x, y, z)$ by, say, $a(x, x, z)$).

This observation leads to the following attempt to allow for evaluations where variables are merged. To make our definitions precise we need a bit of terminology. First, as we want to identify variables, we need to work in $\mathbb{T}^\times$ instead of $\mathbb{T}$. We also need a set of labels telling us which variables to identify.

Definition 5.17. (a) For a tree $t \in \mathbb{T}^\times_\zeta A$ and a function $\sigma : \zeta \to \xi$, we denote by ${}^\sigma t \in \mathbb{T}^\times_{\xi_0} A$ the tree obtained from t by replacing every variable z by $\sigma(z)$ (where $\xi_0 \subseteq \xi$ is the range of σ).

If $\mathbb{T}^0 \subseteq \mathbb{T}^\times$ is closed under the operation ${}^\sigma -$ we can extend this operation to $\mathbb{T}^0$-algebras $\mathfrak{A}$ by setting

$${}^\sigma a := \pi({}^\sigma \mathrm{sing}(a)), \quad \text{for } a \in A_\zeta .$$

(b) For a sort $\xi \in \Xi$, we set $\Gamma(\xi) := (\Gamma_\zeta(\xi))_{\zeta \in \Xi}$ where

$$\Gamma_\zeta(\xi) := \{ \sigma \mid \sigma : \zeta \to \xi \} .$$ ⌟

Given a tree $t \in \mathbb{T}^\times A$ we can choose some sort $\xi \in \Xi$ and functions $\sigma_v \in \Gamma(\xi)$, for every $v \in \mathrm{dom}(t)$, and then replace every label $t(v)$ by ${}^{\sigma_v} t(v)$. The problem is that the resulting tree is not well-formed since the sorts do not match anymore. For instance, in the tree $a(b, c)$ with $a = a(z_0, z_1)$ we can replace z_0 and z_1 by the same variable x. This produces the label $a' := a(x, x)$ of arity $\{x\}$. Consequently, we need to produce a tree where the corresponding vertex has a single successor instead of two. Given the tree $a(b, c)$ the only obvious choices for such a tree would be $a'(b)$ or $a'(c)$. This idea can be generalised as follows.

Definition 5.18. Let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra where $\mathbb{T}^0 \subseteq \mathbb{T}^\times$ is closed under the operations ${}^\sigma -$ and let $p : \Gamma(\xi) \times A \to \Gamma(\xi)$ and $q : \Gamma(\xi) \times A \to A$ be the two projections.

(a) A *condensation* of a tree $t \in \mathbb{T}^\times A$ is a tree $s \in \mathbb{T}^\times(\Gamma(\xi) \times A)$ such that

$$t = \mathbb{T}^\times q(s) .$$

(b) Let s be a condensation of t and set $\sigma_v := p(s(v))$, for $v \in \mathrm{dom}_0(s)$. A *choice function* for s is a family $\mu = (\mu_v)_{v \in \mathrm{dom}_0(s)}$ of functions $\mu_v : \mathrm{rng}\, \sigma_v \to \mathrm{dom}\, \sigma_v$ such that $\sigma_v \circ \mu_v = \mathrm{id}$.

(c) Let s be a condensation of t and $\mu = (\mu_v)_v$ a choice function for s. We define the tree $s \parallel \mu \in \mathbb{T}^\times A$ as follows. For every $v \in \mathrm{dom}_0(s)$,

- we delete from s all subtrees $s|_{\mathrm{suc}_x(v)}$ with $x \notin \mathrm{rng}\, \mu_v$,
- for $x \in \mathrm{rng}\, \mu_v$, we change the x-successor of v to a $\sigma_v(x)$-successor, where $\sigma_v := p(s(v))$, and

- we replace every label $s(v) = \langle \sigma, a \rangle$ by ${}^{\sigma}a$ (which we consider to be an element of sort $\operatorname{rng} \sigma$). If $s(v)$ is a variable, we leave it unchanged. ⌟

Example. Let $t \in \mathbb{T}_{\emptyset}A$ be an infinite tree where all labels on the same level are equal, and let $s \in \mathbb{T}(\Gamma(\{x\}) \times A)$ be the tree with the same domain as t such that

$$s(v) = \langle \sigma_n, a_n \rangle \,, \quad \text{for every vertex } v \text{ with } |v| = n \,,$$

where $a_n = t(v) \in A_{\zeta_n}$ is the label on level n of t and $\sigma_n : \zeta_n \to \{x\}$ is the function mapping all variables of a_n to the same variable x. For every choice function μ for s, we obtain a path

$$s \parallel \mu = b_0 b_1 \cdots \quad \text{with} \quad b_n(x) := a_n(x, \dots, x) \,.$$ ⌟

Remark. Note that, in a tree of the form $s \parallel \mu$, every vertex has a sort which is a subset of ξ. In particular, the number of sorts used is finite. ⌟

We can produce well-formed trees $s \parallel \mu$ using a choice function μ. But which one do we take? The easiest case is if all choice functions produce the same result (cf. [Pup10]), then it does not matter. (A more general construction will be presented further below.)

Definition 5.19. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^{\times}$ be submonads such that $\mathbb{T}^0$ is closed under all operations ${}^{\sigma}-$, and let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra.

(a) A condensation $s \in \mathbb{T}^{\times}(\Gamma(\xi) \times A)$ is *uniform* if

$$s \parallel \mu = s \parallel \mu' \,, \quad \text{for all choice functions } \mu, \mu' \,.$$

(b) A *uniform* $\mathbb{T}^0$*-condensation* of $t \in \mathbb{T}^1 A$ is a uniform condensation $s \in \mathbb{T}^1(\Gamma(\xi) \times A)$ of t such that

$$s \parallel \mu \in \mathbb{T}^0 A \,, \quad \text{for some/all choice functions } \mu \,.$$

(c) Let τ be a partial function mapping each tree t to some uniform $\mathbb{T}^0$-condensation of t (if such a condensation exists). We set

$$\pi^{\mathrm{u}}_{\tau}(t) := \pi(\tau(t) \parallel \mu) \,, \quad \text{where } \mu \text{ is an arbitrary choice function} \,.$$

If $\tau(t)$ is undefined, we let $\pi^{\mathrm{u}}_{\tau}(t)$ be undefined as well. We call π^{u}_{τ}-evaluations $\mathbb{T}^0$-*evaluations with uniform merging,* and we denote the corresponding set by

$$\mathbb{E}^{\mathrm{u},\tau}_{\alpha}(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}_{\alpha}(\pi^{\mathrm{u}}_{\tau}, \mathbb{T}^1) \,.$$ ⌟

Remark. Since $\mathbb{T}^0$-evaluations with uniform merging are π^{u}_{τ}-evaluations, Theorem 5.6 and Proposition 5.7 apply to them. Hence, we can use such more general evaluations to study $\mathbb{T}^1$-expansions. A similar statement holds for evaluations with consistent merging, which we will define below. ⌟

Example. The tree s from the preceding example is a uniform condensation of t with $s \parallel \mu \in \mathbb{T}^{\mathrm{thin}}A$. For technical reasons, it is not a uniform $\mathbb{T}^{\mathrm{thin}}$-condensation of t: the set $\mathbb{T}^{\mathrm{thin}}A$ is not closed under the operations $-^{\sigma}$. Instead, s is a uniform $\mathbb{T}^0$-condensation where $\mathbb{T}^0$ is the monad 'generated' by $\mathbb{T}^{\times\mathrm{thin}}$, that is, the set $\mathbb{T}^0 X$ is the closure of $\mathbb{T}^{\times\mathrm{thin}} X$ under flat. ⌟

Example. $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with uniform merging were introduced in [Pup10] where they were used to derive decidability results for trees. To do so Puppis considers trees $t \in \mathbb{T}^{\times} X$ such that (in our terminology), for every MSO-definable $\mathbb{T}^{\times\mathrm{reg}}$-algebra $\mathfrak{A}$ and every function $\beta : X \to A$, the image $\mathbb{T}^{\times}\beta(t)$ has an evaluation $\gamma \in \mathbb{E}^{\mathrm{u},\tau}_{n}(\mathfrak{A}, \mathbb{T}^{\times})$, for some number $n < \omega$ independent of β and $\mathfrak{A}$. The function τ chooses condensations based on the runs of an automaton recognising the product of $\mathfrak{A}$. (The details can be found in [Pup10]. A similar construction is used at the beginning of the proof of Lemma 5.25 below.) Let us call such trees *reducible.*

By induction on n, we can transform every reducible tree t into a regular tree t_0 with the same value as t. Puppis considers reducible trees t where this transformation is computable using a particular algorithm. (We again omit the details.) Let us call such trees *effectively reducible.* [Pup10] contains the following results.

- Every deterministic tree in the Caucal hierarchy is effectively reducible.
- The class of effectively reducible trees is closed under a number of natural operations.
- Every effectively reducible tree has a decidable MSO-theory. ⌟

The key technical result of [Pup10] is the following recipe of how to evaluate products in an MSO-definable $\mathbb{T}^\times$-algebra using evaluations with uniform merging (cf. Theorem 5 of [Pup10]). Intuitively, the following proposition states that every MSO-definable property of trees is also MSO-definable in every given uniform $\mathbb{T}^\times$-condensation of them.

Proposition 5.20 (Puppis)**.** *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^\times$-algebra and $\xi \in \Xi$ a sort. There exists an* MSO*-definable $\mathbb{T}^\times$-algebra $\mathfrak{B}$, a morphism $\rho : \mathbb{T}^\times A \to \mathfrak{B}$, and* MSO*-formulae φ_a, for $a \in A_\xi$, such that, given a tree $T \in \mathbb{T}^\times_\xi \mathbb{T}^\times A$ and a uniform $\mathbb{T}^\times$-condensation s of $\mathbb{T}^\times \rho(T)$, we have*

$$\pi(\operatorname{flat}(T)) = a \quad \text{iff} \quad s \parallel \mu \models \varphi_a\,, \quad \textit{for all } a \in A_\xi \textit{ and all choice functions } \mu\,.$$

The proof uses similar techniques as that of Lemma 5.25 below.

Since evaluations with uniform merging generalise simple evaluations, they allow us to decompose more trees. Unfortunately, there are still trees left without an evaluation. We can generalise our evaluations even further by not requiring that all choice functions lead to the same tree, but only to one that is 'equivalent'.

Definition 5.21. Let $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \mathbb{T}^\times$ be submonads such that $\mathbb{T}^0$ is closed under all operations ${}^\sigma-$. Let $\mathfrak{A} = \langle A, \pi\rangle$ be a $\mathbb{T}^0$-algebra, and $t \in \mathbb{T}^1 A$.

(a) A condensation $s \in \mathbb{T}^\times(\Gamma(\xi) \times A)$ is π-*consistent* if

$$s|_v \parallel \mu \in \mathbb{T}^0 A\,, \qquad \text{for every choice function } \mu \text{ of } s|_v \text{ and each vertex } v \in \operatorname{dom}(s)\,,$$

$$\pi(s|_v \parallel \mu) = \pi(s|_v \parallel \mu')\,, \qquad \text{for all choice functions } \mu, \mu' \text{ of } s|_v \text{ and each vertex } v \in \operatorname{dom}_0(s)\,.$$

(b) A π-consistent condensation $s \in \mathbb{T}^1(\Gamma(\xi) \times A)$ of t is also called a *consistent $\mathbb{T}^0$-condensation* of t.

(c) Let τ be a partial function mapping each tree t to some consistent $\mathbb{T}^0$-condensation of t (if such a condensation exists). We set

$$\pi^{\mathrm{c}}_\tau(t) := \pi(\tau(t) \parallel \mu)\,, \quad \text{where } \mu \text{ is an arbitrary choice function}\,.$$

If $\tau(t)$ is undefined, we let $\pi^{\mathrm{c}}_\tau(t)$ be undefined as well. We call π^{c}_τ-evaluations $\mathbb{T}^0$-*evaluations with consistent merging,* and we denote the corresponding set by

$$\mathbb{E}^{\mathrm{c};\tau}_\alpha(\mathfrak{A}, \mathbb{T}^1) := \mathbb{E}_\alpha(\pi^{\mathrm{c}}_\tau, \mathbb{T}^1)\,.$$

⌟

Clearly, consistent merging generalises uniform merging. While $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with uniform merging seem to exist only in special cases, our hope is that $\mathbb{T}^{\times\mathrm{reg}}$-evaluations with consistent merging always exist (at least for MSO-definable algebras). At the moment we are only able to obtain partial results. To do so we need a bit of terminology. First, it is convenient to work with arbitrary directed graphs instead of just trees.

Definition 5.22. Let A be a sorted set.

(a) An *A-labelled rooted graph* g is a countable directed graph with a distinguished vertex r, the *root* of g, where every vertex is labelled by some element of A and where every edge is labelled by some variable in such a way that, if a vertex v is labelled by $a \in A_\xi$, then v has exactly one out-going edge labelled x, for every $x \in \xi$. As usual, we treat an A-labelled rooted graph as a function $g : \mathrm{dom}(g) \to A$. We denote by $\mathbb{R}_\xi A$ the set of all $(A + \xi)$-labelled rooted graphs (where, as usual, the variables are considered to be elements of sort $\emptyset$), and we set $\mathbb{R}A := (\mathbb{R}_\xi A)_\xi$.

(b) Given a graph $g \in \mathbb{R}\mathbb{R}A$, we denote by $\mathrm{flat}(g)$ the graph obtained from the disjoint union $\sum_{v \in \mathrm{dom}(g)} g(v)$ of all component graphs by replacing, in every component $g(v)$, every x-labelled edge $u \to u'$ to a vertex labelled by some variable z to an x-labelled edge $u \to w$ where w is the root of the the component $g(v')$ for the z-successor v' of v. Then we delete all vertices labelled by a variable. (Again, if $g(v)$ is just a variable z, we consider it a 1-vertex graph with label z and we do not delete this vertex.)

(c) We denote by $\mathbb{R}^{\mathrm{thin}} A \subseteq \mathbb{R}A$ the set of all graphs whose unravelling is a thin tree. ⌟

We start with a technical lemma which is based on the following variant of a condensation. In a usual condensation we can only redirect edges to another successor of the same vertex. Below we will need a variant where we can also redirect edges to vertices that are farther away. We specify which destinations are allowed in such a redirection via a labelling σ of the tree by numbers.

Definition 5.23. Let $t \in \mathbb{T}^\times_\xi A$ be a tree and $\sigma : \mathrm{dom}(t) \to [N]$ a function.

(a) The *k-th σ-parent* $p^k_\sigma(v)$ of $v \in \mathrm{dom}(t)$ is the maximal vertex u satisfying

$$u \prec v \quad \text{and} \quad \sigma(u) \geq k\,.$$

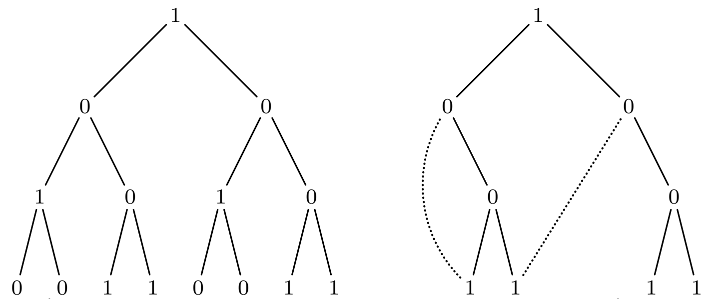
If no such vertex exists, we set $p^k_\sigma(v) := \bot$ and we say that v does not have a k-th σ-parent. For $k = \sigma(v)$, we omit the superscript and write just $p_\sigma(v) := p^{\sigma(v)}_\sigma(v)$. We define a relation $\approx_\sigma$ on $\mathrm{dom}(t)$ by

$$u \approx_\sigma v \quad :\text{iff} \quad \sigma(u) = \sigma(v) \quad \text{and} \quad p_\sigma(u) = p_\sigma(v)\,.$$

(b) We say that a tree $t' \in \mathbb{T}^\times A$ is a *σ-rewiring* of t if it is the unravelling of some graph $g \in \mathbb{R}_\xi A$ satisfying the following conditions.
- $\mathrm{dom}(g) \subseteq \mathrm{dom}(t)$ is prefix-closed and non-empty.
- If v is the x-successor of u in t and $u, v \in \mathrm{dom}(g)$, then v is also the x-successor of u in g.
- If v is the x-successor of u in g and v' is its x-successor in t, then $v \approx_\sigma v'$.

We say that g is the graph *inducing* the σ-rewiring t'. ⌟

Example. Let t be the tree on the left and σ the depicted labelling of t. Then the (unravelling of the) graph g on the right is a σ-rewiring of t.



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Lemma 5.24. *Let t' be a σ-rewiring of t and let ρ be a path in t'.*

(a) *If ρ starts at the root and it contains a vertex v, it also contains $p_\sigma(v)$ (if $p_\sigma(v) \neq \bot$).*

(b) *Suppose that ρ starts at u, ends in v, and that*

$$\bot \neq p_\sigma(x) \not\preceq p_\sigma(u) \neq \bot\,, \quad \text{for all vertices } x \text{ of } \rho \text{ different from } u\,.$$

Then $u \preceq v$.

(c) *There is no pair of vertices $x \preceq y$ such that x is after y along the path ρ.*

Proof. In the following we will always use the notation $\preceq$ and $p_\sigma(v)$ with respect to the tree t. We use the notation $u \to v$ to indicate that, in t', there is an edge from u to v.

(a) We prove the claim by induction on the length of ρ. Let $u \to v$ be the last edge of the path ρ. By inductive hypothesis, ρ contains the vertex $p_\sigma(u)$. Repeating this argument, it follows that it also contains all iterates $(p_\sigma)^i(u)$ (that are defined). It is therefore sufficient to show that $(p_\sigma)^i(u) = p_\sigma(v)$, for some i. We distinguish two cases.

If $u \prec v$, the claim is immediate. Hence, suppose that $u \to v$ is a redirected edge. Then there exists a successor v' of u (in t) such that $v' \approx_\sigma v$. It follows that there is some i with $(p_\sigma)^i(u) = p_\sigma(v') = p_\sigma(v)$.

(b) We proceed by induction on the length of ρ. If $v = u$, the claim is trivial. For the inductive step, consider the last edge $w \to v$ of ρ. By inductive hypothesis, we know that $u \preceq w$. If the edge is one of the original edges of t, we have $u \preceq w \prec v$. Hence, suppose that the edge is a redirected one. Let v' be the successor of w (in t) with $v' \approx_\sigma v$. If $u \preceq p_\sigma(v')$, we have $u \preceq p_\sigma(v') = p_\sigma(v) \prec v$, as desired. Hence, suppose otherwise. Then $p_\sigma(v') \prec u \prec v'$ implies that $\sigma(u) < \sigma(v')$, and it follows that $p_\sigma(v) = p_\sigma(v') \preceq p_\sigma(u)$. A contradiction.

(c) Without loss of generality we may assume that the path ρ starts at the root. In order to avoid case distinctions, we add a new root r to t that is mapped by σ to the maximal possible value. We also add r to the beginning of ρ, so that ρ starts at r. Then $p_\sigma(v) \neq \bot$, for all vertices v (except for the new root). For a contradiction, suppose that there are vertices $x \preceq y$ such that x is after y along the path ρ. We choose x $\preceq$-minimal and, given x, we choose y $\preceq$-minimal. Note that, since ρ contains the root r only once, it follows that $x \neq r$ and, hence, $y \neq r$.

It follows by (a) that the vertex $p_\sigma(y)$ lies on ρ before y and, hence, also before x. By minimality of y, it therefore follows that $p_\sigma(y) \prec x$. Let $u_0 \to v_0, \dots, u_m \to v_m$ be an enumeration (in order) of all edges of ρ that lie between y and x and that satisfy $p_\sigma(v_i) \preceq p_\sigma(x)$. Let v'_i be the successor of u_i (in t) such that $v'_i \approx_\sigma v_i$. (If the edge $u_i \to v_i$ is not redirected, we have $v'_i = v_i$.) Note that the part of ρ between v_i and u_{i+1} cannot contain a vertex z with $p_\sigma(z) \preceq p_\sigma(v_i)$ since this would imply $p_\sigma(z) \preceq p_\sigma(x)$, meaning z would be one of the v_j. Hence, we can apply (b) to the part of ρ between v_i and u_{i+1}

and we obtain $v_i \preceq u_{i+1} \prec v'_{i+1}$. Since $p_\sigma(x) \prec x$, it follows by minimality of x that $v_i \not\preceq p_\sigma(x)$. Hence, $p_\sigma(v'_{i+1}) = p_\sigma(v_{i+1}) \preceq p_\sigma(x)$ implies that $v_i \not\preceq p_\sigma(v'_{i+1})$. Therefore, we have $p_\sigma(v'_{i+1}) \prec v_i \prec v'_{i+1}$ and $\sigma(v_i) < \sigma(v'_{i+1}) = \sigma(v_{i+1})$. Similarly, applying (b) to the part of ρ between y and v_0, we obtain $p_\sigma(v'_0) = p_\sigma(v_0) \preceq p_\sigma(x) \prec y \preceq u_0 \prec v'_0$, which means that $\sigma(y) < \sigma(v'_0) = \sigma(v_0)$. Furthermore, $p_\sigma(y) \prec x \preceq y$ implies that $\sigma(x) \leq \sigma(y)$. Finally, we have $v_m = x$ since minimality of x implies that the edge of ρ leading to x is a redirected one and we trivially have $p_\sigma(x) \preceq p_\sigma(x)$. Altogether, we obtain

$$\sigma(x) = \sigma(v_m) \geq \sigma(v_0) > \sigma(y) \geq \sigma(x)\,.$$

A contradiction. □

After these preparations we can state and prove our key technical lemma.

Lemma 5.25. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^\times$-algebra and $\xi \in \Xi$ a sort. There exists a constant $N < \omega$ with the following property. For every tree $t \in \mathbb{T}^\times_\xi A$, we can find a function $\sigma : \mathrm{dom}(t) \to [N]$ such that*

$$\pi(t') = \pi(t)\,, \quad \textit{for every } \sigma\textit{-rewiring } t' \textit{ of } t\,,$$

and σ maps the root of t to $N-1$.

Proof. Fix a tree $t \in \mathbb{T}^\times_\xi A$, let $\mathcal{A} = \langle Q, A+\xi, \Delta, q_0, \Omega\rangle$ be the automaton checking that the product of a given tree in $\mathbb{T}^\times_\xi A$ is equal to $\pi(t)$ (extended to the infinite alphabet $A+\xi$ as explained in Section 4 (page 14)), and let $\mathcal{G}$ be the corresponding Automaton-Pathfinder game for $\mathcal{A}$ on the input tree t. Without loss of generality, we may assume that $\mathcal{A}$ is a non-deterministic automaton. Since $\mathcal{G}$ is a parity game, there exists a positional winning strategy τ for Automaton.

Given a vertex $v \in \mathrm{dom}(t)$, we denote by $\mu(v)$ the state q such that the unique play of $\mathcal{G}$ conforming to τ that reaches the vertex v does so in state q. Fix some bijective function $h : Q \to [N]$, for $N < \omega$, such that

$$\Omega(p) < \Omega(q) \quad \text{implies} \quad h(p) > h(q)\,,$$

and set

$$\sigma(v) := \begin{cases} N-1 & \text{if } v \text{ is the root,} \\ h(\mu(v)) & \text{otherwise.} \end{cases}$$

We claim that σ is the desired function.

Let t' be a σ-rewiring and let $\mathcal{G}'$ be the Automaton-Pathfinder game on t'. We have to show that $\pi(t') = \pi(t)$. Note that the Automaton-Pathfinder game $\mathcal{G}'$ for $\mathcal{A}$ on t' can be obtained from the game $\mathcal{G}$ for t by removing some positions and redirecting some of the edges. Consequently, the strategy τ for t induces a strategy τ' for the game on t'. For a path ν (in $\mathcal{G}$ or $\mathcal{G}'$), we denote by $\Omega(\nu)$ the least priority seen along ν. We start by proving the following claim.

Claim. *Let $\langle u_1, q_1\rangle$ be a position of Automaton in $\mathcal{G}'$ that is reachable by some play confoming to τ', and let ν, ν' be two plays of $\mathcal{G}'$ confoming to τ' that start at $\langle u_1, q_1\rangle$ and that end in two positions $\langle u_2, q_2\rangle$ and $\langle u_2, q'_2\rangle$, respectively, with the same vertex u_2. Suppose that*

$$u_1 \prec u_2 \quad \textit{and} \quad \sigma(u_1) \geq \sigma(x)\,, \quad \textit{for all } u_1 \preceq x \preceq u_2\,.$$

Then we have

$$q_2 = q'_2 \quad \textit{and} \quad \Omega(\nu) = \Omega(\nu')\,.$$

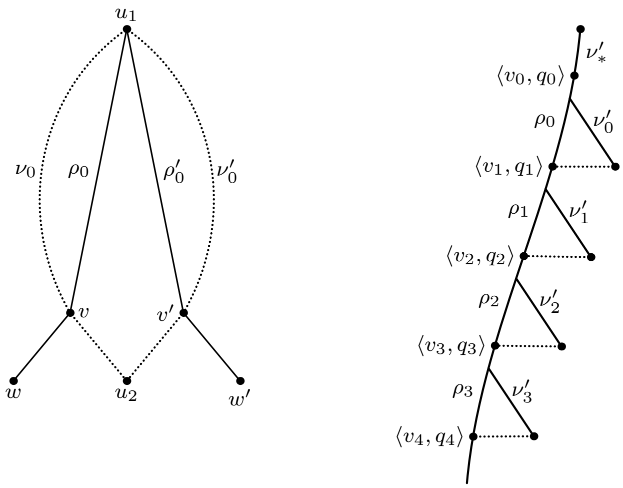


Figure 5: (a) Left: the plays ν and ν'. Dashed lines represent plays in $\mathcal{G}'$, solid ones plays in $\mathcal{G}$.
(b) Right: the plays ρ_i and ν'_i, projected to the game $\mathcal{G}$.

We prove the claim by an induction on the following two numbers (in order of decreasing priority):

- first on the total number of edges in ν or ν' that are not edges of $\mathcal{G}$,
- then on the sum of the lengths of ν and ν'.

We distinguish three cases.

(I) Suppose that there is some vertex $u_1 \prec u_3 \prec u_2$ such that

$$\sigma(u_3) \geq \sigma(x)\,, \quad \text{for all } u_3 \preceq x \preceq u_2\,,$$

ν contains the position $\langle u_3, q_3\rangle$ and ν' the position $\langle u_3, q'_3\rangle$. Then we can split the respective plays at these positions. Let $\nu_0, \nu_1, \nu'_0, \nu'_1$ be the corresponding parts. Applying the inductive hypothesis twice, it follows that

$$\Omega(\nu_0) = \Omega(\nu'_0)\,, \quad q_3 = q'_3\,, \quad \Omega(\nu_1) = \Omega(\nu'_1)\,, \quad \text{and} \quad q_2 = q'_2\,.$$

This implies that $\Omega(\nu) = \Omega(\nu')$.

(II) Suppose that the second but last positions of ν and ν' contain the same vertex u_3 with $u_1 \prec u_3 \prec u_2$. Let $\langle u_3, q_3\rangle$ and $\langle u_3, q'_3\rangle$ be these positions and let ν_0 and ν'_0 be the corresponding prefixes of the two plays. By inductive hypothesis, we have $q_3 = q'_3$ and $\Omega(\nu_0) = \Omega(\nu'_0)$. As the plays ν and ν' both conform to the same strategy τ', it follows that $q_2 = q'_2$ and $\Omega(\nu) = \Omega(\nu')$.

(III) Finally, suppose that cases (I) and (II) do not hold. Let $\langle v, s\rangle$ and $\langle v', s'\rangle$ be the second but last positions of Automaton in the plays ν and ν', respectively (see Figure 5 (a)). Since we are not in case (II), we have either $v \neq v'$ or $v = v' \not\prec u_2$. In both cases, at least one of the edges $v \to u_2$ and $v' \to u_2$ is an edge of t' but not one of t. Let w and w' be the successors of, respectively, v and v' in t that correspond to these two edges and let q_3 and q'_3 be the states at w and w'. Let ρ be the play that conforms to τ, starts at $\langle u_1, q_1\rangle$, ends in $\langle w, q_3\rangle$, and that uses only edges that are present in t. Similarly, let ρ' be the corresponding play to $\langle w', q'_3\rangle$.

Note that, by definition of a σ-rewiring, we have $w \approx_\sigma u_2 \approx_\sigma w'$. This implies that

$$\sigma(w) = \sigma(u_2) = \sigma(w') \quad \text{and} \quad p_\sigma(w) = p_\sigma(u_2) = p_\sigma(w')\,.$$

Since we are not in case (I), it further follows by Lemma 5.24 that $p_\sigma(u_2) = u_1$.

Let ν_0, ν'_0, ρ_0, and ρ'_0 be the prefixes of the plays ν, ν', ρ, and ρ' ending in the position $\langle v, s\rangle$ and $\langle v', s'\rangle$, respectively. Since ρ_0 and ρ'_0 do not contain redirected edges, $p_\sigma(w) = u_1 = p_\sigma(w')$, and $u_1 \preceq v \prec w$ and $u_1 \preceq v' \prec w'$, we can use the inductive hypothesis to show that

$$\Omega(\rho_0) = \Omega(\nu_0) \quad \text{and} \quad \Omega(\rho'_0) = \Omega(\nu'_0)\,.$$

Furthermore, $\sigma(w) = \sigma(u_2) = \sigma(w')$ implies that $q_3 = \mu(w) = \mu(u_2) = \mu(w') = q'_3$. Let $k := \Omega(q_3)$ be the priority of this state. Since

$$\Omega(\rho) = \min\{k, \Omega(\rho_0)\}\,,$$

and similarly for the other plays, it follows that

$$\Omega(\rho) = \Omega(\nu) \quad \text{and} \quad \Omega(\rho') = \Omega(\nu')\,.$$

Hence, it remains to show that $\Omega(\rho) = \Omega(\rho')$. We claim that

$$\Omega(\rho) = \Omega(\mu(u_1)) \quad \text{and} \quad \Omega(\rho') = \Omega(\mu(u_1))\,.$$

By symmetry, it is sufficient to prove the first equation. Since $w \approx_\sigma u_2$, we have $p_\sigma(w) = p_\sigma(u_2)$. By definition of p_σ, we have

$$\sigma(x) \leq \sigma(w) \leq \sigma(p_\sigma(w))\,, \quad \text{for all } p_\sigma(u_2) = p_\sigma(w) \preceq x \preceq w\,.$$

Since, as we have shown above, $p_\sigma(u_2) = u_1$, it follows that

$$\sigma(x) \leq \sigma(u_1)\,, \quad \text{for all } u_1 \preceq x \preceq w\,.$$

Consequently, we have $\Omega(\rho) = \Omega(\mu(u_1))$.

Having proved the above claim we can now show that $\pi(t') = \pi(t)$, i.e., that $\mathcal{A}$ accepts the tree t'. We claim that the strategy τ' defined above is winning. To do so it is sufficient to show that, every play ν' in $\mathcal{G}'$ conforming to τ' induces a play ν in $\mathcal{G}$ conforming to τ such that the minimal priorities seen infinitely often along ν and ν' are the same.

Let k be the maximal number such that ν' contains infinitely many positions $\langle v, q\rangle$ of Automaton with $\sigma(v) = k$. Then there exists an infinite sequence $\langle v_i, q_i\rangle_{i<\omega}$ of positions of ν' with $\sigma(v_i) = k$ and

$$v_0 \sqsubset_\sigma v_1 \sqsubset_\sigma \cdots\,.$$

We choose this sequence maximal (with respect to inclusion), which implies that $v_i = p_\sigma(v_{i+1})$. Let $\nu_0 = \nu'_* \nu'_0 \nu'_1 \dots$ be the factorisation of the play ν_0 where ν'_* is the prefix ending in $\langle v_0, q_0\rangle$ and ν'_i is the part between the positions $\langle v_i, q_i\rangle$ and $\langle v_{i+1}, q_{i+1}\rangle$.

For every $i < \omega$, let ρ_i be the partial play of $\mathcal{G}'$ from $\langle v_i, q_i\rangle$ to $\langle v_{i+1}, q'_{i+1}\rangle$, for some state q'_{i+1}, that conforms to τ' and that uses only edges belonging to $\mathcal{G}$, see Figure 5 (b). (To see that such a play exists note that a position of the form $\langle v_{i+1}, q'_{i+1}\rangle$ is reachable by some play conforming to τ' and this play contains the vertex $p_\sigma(v_{i+1}) = v_i$ by Lemma 5.24. Furthermore, it follows by the above claim that the state at this vertex is equal to q_i.) By the above claim, we have $q'_i = q_i$ and $\Omega(\rho_i) = \Omega(\nu'_i)$, for all i. Consequently, the composition $\nu := \nu'_* \rho_0 \rho_1 \rho_2 \dots$ forms a play in $\mathcal{G}$ conforming to τ and the least priority seen infinitely often in ν is the same as in ν'. $\square$

Theorem 5.26. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^\times$-algebra. Every tree $t \in \mathbb{T}^\times A$ has a consistent $\mathbb{T}^\times$-condensation s such that*

$$\pi(t) = \pi(s \parallel \mu)\,, \quad \textit{for all choice functions } \mu\,.$$

Proof. Let $\sigma : \mathrm{dom}(t) \to [N]$ be the function from Lemma 5.25. Fix a set of variables ξ with $|\xi| = N$ and let $\mu : [N] \to \xi$ be a bijection. We define the desired condensation s by

$$s(v) := \langle \tau_v, t(v) \rangle \quad \text{with} \quad \tau_v(x) := \mu(\sigma(u_x))\,,$$

where u_x is the x-successor of v. Then every tree of the form $s \parallel \mu$ is a σ-rewiring of t. By choice of σ this implies that $\pi(s \parallel \mu) = \pi(t)$. □

In particular, it follows that every MSO-definable $\mathbb{T}^\times$-algebra has $\mathbb{T}^\times$-evaluations with consistent merging. Note that this statement is not as trivial as it sounds since trees can contain labels of arbitrarily high arity, while every $\mathbb{T}^\times$-condensation produces a tree where these arities are bounded. In particular, the statement is false for $\mathbb{T}^\times$-evaluations with uniform merging.

Our hope is that a more elaborate version of the construction from the proof of Lemma 5.25 can be used to construct a $\mathbb{T}^{\times\mathrm{reg}}$-condensation instead of a $\mathbb{T}^\times$-one, or at least that we can iterate such a construction to obtain a $\mathbb{T}^{\times\mathrm{reg}}$-evaluation.

Conjecture. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^{\times\mathrm{reg}}$-algebra. Then every tree $t \in \mathbb{T}A$ has a $\mathbb{T}^{\times\mathrm{reg}}$-evaluation with consistent merging.*

If we relax our notion of an evaluation and of consistent merging a bit, we even obtain something similar to '$\mathbb{T}^{\times\mathrm{thin}}$-evaluations'. (Note that, formally, this notion does not exist, since $\mathbb{T}^{\times\mathrm{thin}}$ does not form a monad.)

Theorem 5.27. *Let $\mathfrak{A}$ be an* MSO*-definable $\mathbb{T}^\times$-algebra and $C \subseteq A$ a subset. For every $t \in \mathbb{T}^\times_\xi C$, there is some $s \in (\mathbb{T}^{\times\mathrm{thin}})^n C$ with $\pi(\mathrm{flat}^{n-1}(s)) = \pi(t)$, where the exponent $n < \omega$ only depends on $\mathfrak{A}$ and ξ.*

Proof. Let $\sigma : \mathrm{dom}(t) \to [n]$ be the function from Lemma 5.25 and let

$$D := \{\, {}^\tau c \mid c \in C_\zeta,\ \tau : \zeta \to \eta \text{ surjective}, \eta, \zeta \in \Xi \,\}$$

be the closure of C under variable substitutions. By induction on $i \leq n$, we construct graphs $s_i \in (\mathbb{T}^\times)^{n-i}(\mathbb{R}^{\times\mathrm{thin}})^i D$ satisfying the following conditions.

- The graph $\mathrm{flat}^{n-1}(s_i) \in \mathbb{R}D$ induces a σ-rewiring of t. In particular, $\mathrm{flat}^{n-1}(s_i)$ is obtained from a subgraph of t by redirecting all edges leaving this subgraph.
- Let s'_i be the tree obtained from s_i by deleting all redirected edges and let t'_i be the tree obtained from t by deleting all subtrees reachable by an edge that has been redirected in s_i. (Note that s'_i and t'_i are not elements of $(\mathbb{T}^\times)^n D$ and $\mathbb{T}^\times C$, respectively, since some vertices are missing some of their successors.) Then s'_i is the recursive factorisation of t'_i that corresponds via the construction of Lemma 5.4 and Figure 4 to the restriction of σ to $\mathrm{dom}(t'_i) \subseteq \mathrm{dom}(t)$.

Then we can produce the desired tree s as follows. As each $d \in D$ is of the form $d = \pi(r)$, for some $r \in \mathbb{T}^{\mathrm{fin}}C$, there exists a function $\vartheta : D \to \mathbb{T}^{\mathrm{fin}}C$ with $\pi \circ \vartheta = \mathrm{id}$. We chose for $s \in (\mathbb{T}^{\times\mathrm{thin}})^{n+1}C$ the unravelling (at all $n+1$ levels) of the graph $\mathbb{R}^n\vartheta(s_n) \in (\mathbb{R}^{\times\mathrm{thin}})^{n+1}C$. Since the unravelling s'_n of $\mathrm{flat}^{n-1}(s_n)$ is a σ-rewiring of t it follows that

$$\pi(\mathrm{flat}^n(s)) = \pi(s'_n) = \pi(t)\,,$$

as desired.

It remains to construct $s_0, \dots, s_n$. For s_0, we choose the $\mathbb{T}^\times$-evaluation of t induced by σ as in Lemma 5.4. For the inductive step, suppose that we have already defined s_i. For every component $s_i(u_0)\cdots(u_{n-i-2})$ there exists a canonical injection

$$\mathrm{dom}_0(s_i(u_0)\cdots(u_{n-i-2})) \to \mathrm{dom}_0(\mathrm{flat}^{n-1}(s_i))$$

mapping every $u_{n-i-1} \in \mathrm{dom}_0(s_i(u_0)\cdots(u_{n-i-2}))$ to the vertex of the flattening corresponding to the root of the graph $\mathrm{flat}^{i-1}(s_i(u_0)\cdots(u_{n-i-2})(u_{n-i-1}))$. Since $\mathrm{dom}(\mathrm{flat}^{n-1}(s_i))$ is contained in $\mathrm{dom}(t)$, the restriction of σ to the range of this injection induces a function

$$\sigma_i : \mathrm{dom}_0(s_i(u_0)\cdots(u_{n-i-2})) \to [n]\,.$$

By inductive hypothesis, we have $\sigma_i(v) \geq i$, for all v. Let s_{i+1} be obtained from s_i by replacing every component $s_i(u_0)\cdots(u_{n-i-2}) \in \mathbb{T}^\times(\mathbb{R}^{\times\mathrm{thin}})^i A$ by a path $r \in (\mathbb{R}^{\times\mathrm{thin}})^{i+1} A$ which we construct as follows. We define a branch $v_0, v_1, \dots$ of the tree $s_i(u_0)\cdots(u_{n-i-2})$ starting at the root v_0. This branch will form the domain of r. For the inductive step, suppose that we have already defined the vertex v_j and the labels $r(v_k)$, for all $k < j$. Suppose that

$$s_i(u_0)\cdots(u_{n-i-2})(v_j) = p(\bar{x}, \bar{y})\,, \quad \text{for } p \in (\mathbb{R}^{\times\mathrm{thin}})^i A\,,$$

where $\bar{x}$ are the variables such that the corresponding successors w of v_j are labelled by non-variables (which implies that $\sigma_i(w) = i$), while $\bar{y}$ are the remaining ones. Let $\bar{y}'$ be the variables labelling the successors corresponding to $\bar{y}$.

If $\bar{x}$ is empty, we set

$$r(v_j) := p(\bar{y}')$$

and the construction terminates. Otherwise, we pick one variable $x' \in \bar{x}$, we choose for v_{j+1} the x'-successor of v_j, and we set

$$r(v_j) := p(x' \dots x', \bar{y}')\,.$$

The flattening $\mathrm{flat}^{n-1}(s_{i+1})$ of the resulting graph s_{i+1} induces the desired σ-rewiring of t. □

As already mentioned above, we cannot conclude that $\mathbb{T}^{\times\mathrm{thin}}$ is dense in $\mathbb{T}^\times$ since the former does not form a monad. But we obtain the following, weaker statements.

Corollary 5.28. *Let $\mathbb{T}^0 \subseteq \mathbb{T}^\times$ be the closure of $\mathbb{T}^{\times\mathrm{thin}}$ under* flat. *Then $\mathbb{T}^0$ is dense in $\mathbb{T}^\times$ over the class of all* MSO-*definable $\mathbb{T}^\times$-algebras.*

Corollary 5.29. *Let $\mathfrak{A}$ be an* MSO-*definable $\mathbb{T}^\times$-algebra. A set $C \subseteq A$ induces a subalgebra of $\mathfrak{A}$ if, and only if,*

$$\pi(t) \in C\,, \quad \textit{for all } t \in \mathbb{T}^{\times\mathrm{thin}} C\,.$$

Corollary 5.30. *The product of every* MSO-*definable $\mathbb{T}^\times$-algebra $\mathfrak{A}$ is uniquely determined by its restriction to the set $\mathbb{T}^{\times\mathrm{thin}} A$.*

Proof. Suppose that there are two MSO-definable $\mathbb{T}^\times$-algebras $\mathfrak{A}_0 = \langle A, \pi_0 \rangle$ and $\mathfrak{A}_1 = \langle A, \pi_1 \rangle$ with the same universe A and whose products have the same restriction to $\mathbb{T}^{\times\mathrm{thin}} A$. Fix a tree $t \in \mathbb{T}^\times A$. We have to show that $\pi_0(t) = \pi_1(t)$.

Let $\delta : A \to A \times A$ be the diagonal map, let $\varDelta := \mathrm{rng}\,\delta \subseteq A \times A$ be its range, and set $s := \mathbb{T}^\times \delta(t) \in \mathbb{T}^\times \varDelta$. As the product $\mathfrak{A}_0 \times \mathfrak{A}_1$ is also MSO-definable, we can use Theorem 5.27

to find a tree $r \in (\mathbb{T}^{\times\mathrm{thin}})^n \Delta$ with $\pi(\mathrm{flat}^{n-1}(r)) = \pi(s)$. Since π_0 and π_1 agree on all trees in $\mathbb{T}^{\times\mathrm{thin}} A$, we have

$$\pi(u) \in \Delta\,, \quad \text{for all } u \in \mathbb{T}^{\times\mathrm{thin}} \Delta\,.$$

Consequently, we can evaluate

$$\pi(\mathrm{flat}^{n-1}(r)) = \pi(\mathbb{T}^{\times\mathrm{thin}}\pi(\mathbb{T}^{\times\mathrm{thin}}\mathbb{T}^{\times\mathrm{thin}}\pi(\cdots(\mathbb{T}^{\times\mathrm{thin}})^{n-1}\pi(r)\cdots)))$$

by recursion using only products of trees in $\mathbb{T}^{\times\mathrm{thin}}\Delta$. In particular, we have $\pi(\mathrm{flat}^{n-1}(r)) \in \Delta$ and it follows that

$$\langle \pi_0(t), \pi_1(t)\rangle = \pi(s) = \pi(\mathrm{flat}^{n-1}(r)) \in \Delta \quad \text{implies} \quad \pi_0(t) = \pi_1(t)\,. \qquad \square$$

Remark. Using a variant of Proposition 5.9 for non-linear trees, we can strenghten Theorem 5.27 to obtain a tree $s \in (\mathbb{T}^{\times\mathrm{wilke}})^n C$. Since the closure of $\mathbb{T}^{\times\mathrm{wilke}}$ under flat coincides with $\mathbb{T}^{\times\mathrm{reg}}$, the argument in Corollary 5.28 then provides an alternative proof of Theorem 4.4. ⌟

6. Consistent labellings

As we have seen in the previous section, we can construct expansions with the help of evaluations if the two monads in question are sufficiently well-behaved. What do we do if they are not? Let us turn to a second idea of how to prove that a $\mathbb{T}^0$-algebra $\mathfrak{A}$ has a $\mathbb{T}$-expansion: when we want to define the product of $t \in \mathbb{T}A$, we first annotate t with additional information that makes it easier to determine the value of the product. For instance, for each vertex v, we can guess the value $\pi(t|_v)$ of the corresponding subtree and then check that these guesses are correct.

Definition 6.1. Let $\mathbb{T}^{\mathrm{fin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}$ be a submonad, $\mathfrak{A} = \langle A, \pi\rangle$ a $\mathbb{T}^0$-algebra, and $t \in \mathbb{T}_\xi A$.

(a) A *labelling* of t is a function $\lambda : \mathrm{dom}_0(t) \to A$ (not necessarily arity-preserving) such that, for every vertex v,

$$\lambda(v) \in A_\zeta \quad \text{iff} \quad \zeta \text{ is the set of variables appearing in } t|_v\,.$$

(b) A labelling $\lambda : \mathrm{dom}_0(t) \to A$ is *weakly* $\mathbb{T}^0$*-consistent* if, for every factor $[u, \bar v)$ with $t[u, \bar v) \in \mathbb{T}^0 A$,

$$\lambda(u) = \pi\big(t[u, \bar v)(\lambda(v_0), \dots, \lambda(v_{n-1}))\big)\,.$$

⌟

Example. For every $\mathbb{T}$-algebra $\mathfrak{A}$ and every tree $t \in \mathbb{T}A$, we can define a labelling by

$$\lambda(v) := \pi(t|_v)\,.$$

This labelling is obviously weakly $\mathbb{T}$-consistent and, hence, weakly $\mathbb{T}^0$-consistent for every $\mathbb{T}^0 \subseteq \mathbb{T}$. In particular, if a $\mathbb{T}^0$-algebra has a $\mathbb{T}$-expansion, then every tree has at least one weakly $\mathbb{T}^0$-consistent labelling. ⌟

Weak consistency is based on factors with finitely many variables. In many situations this is not sufficient and we have to use the following stronger version of consistency where we also allow factors with infinitely many variables.

Definition 6.2. Let $\mathbb{T}^0 \subseteq \mathbb{T}$ be a submonad, $\mathfrak{A} = \langle A, \pi\rangle$ a $\mathbb{T}^0$-algebra, and $t \in \mathbb{T}_\xi A$.

(a) Given a factor $[u, \bar v)$ of t, possibly with infinitely many holes $\bar v$, we denote by $t[u, \bar v)(a_0, a_1, \dots)$ the tree obtained from $t[u, \bar v)$ by replacing each leaf labelled by a variable x_i by the tree $\mathrm{sing}(a_i)$.

(b) A labelling $\lambda : \mathrm{dom}_0(t) \to A$ is *strongly* $\mathbb{T}^0$*-consistent* if, for every factor $[u, \bar{v})$, possibly with infinitely many holes $\bar{v}$, with $t[u,\bar{v})(\lambda(v_0), \lambda(v_1), \dots) \in \mathbb{T}^0 A$, we have

$$\lambda(u) = \pi\big(t[u,\bar{v})(\lambda(v_0), \lambda(v_1), \dots)\big)\,.$$

⌟

We start with the following easy observation.

Lemma 6.3. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\mathrm{fin}}$-algebra. Every tree $t \in \mathbb{T}A$ has a strongly $\mathbb{T}^{\mathrm{fin}}$-consistent labelling.*

Proof. We call a labelling λ of some tree t *locally consistent* if

$$\lambda(v) = t(v)\big(\lambda(u_0), \dots, \lambda(u_{n-1})\big)\,,$$

for every vertex v with successors $u_0, \dots, u_{n-1}$. Fix an increasing sequence $P_0 \subset P_1 \subset \dots \subset \mathrm{dom}(t)$ of finite prefixes of t with $\bigcup_i P_i = \mathrm{dom}(t)$, and let Λ_i be the set of all locally consistent labellings of P_i, for $i < \omega$. Then $\Lambda := \bigcup_i \Lambda_i$ ordered by $\subset$ forms a finitely-branching tree. By Kőnig's Lemma, there exists an infinite branch $\lambda_0 \subset \lambda_1 \subset \dots$. Let λ be its limit. Then λ is locally consistent.

It therefore, remains to prove that every locally consistent labelling of t is strongly $\mathbb{T}^{\mathrm{fin}}$-consistent. Consider a finite factor H of t with root v and leaves $u_0, \dots, u_{m-1}$. By induction on $|H|$ it follows that

$$\lambda(v) = \pi\big((t \restriction H)(\lambda(u_0), \dots, \lambda(u_{m-1}))\big)\,.$$

□

Next, let us show how to use consistent labellings to characterise possible $\mathbb{T}$-expansions of a given $\mathbb{T}^0$-algebra. We need the following additional property.

Definition 6.4. Let $\mathbb{T}^0 \subseteq \mathbb{T}$.

(a) A *weak labelling scheme* for a $\mathbb{T}^0$-algebra $\mathfrak{A}$ is a function σ assigning to each tree $t \in \mathbb{T}A$ a weakly $\mathbb{T}^0$-consistent labelling $\sigma(t)$ of t. Similarly, a *strong labelling scheme* σ assigns to each tree $t \in \mathbb{T}A$ a strongly $\mathbb{T}^0$-consistent labelling $\sigma(t)$.

(b) A labelling scheme σ for $\mathfrak{A}$ is *associative* if, for every tree $T \in \mathbb{T}\mathbb{T}A$, we have

$$\sigma(t) = \sigma(\mathrm{flat}(T)) \circ \mu\,,$$

where $\mu : \mathrm{dom}_0(T) \to \mathrm{dom}_0(\mathrm{flat}(T))$ maps each vertex $v \in \mathrm{dom}_0(T)$ to the vertex of $\mathrm{flat}(T)$ corresponding to the root of $T(v)$, and $t \in \mathbb{T}A$ is the tree such that

$$t(v) := \sigma(T(v))(\langle\rangle)\,, \quad \text{for } v \in \mathrm{dom}_0(T)\,.$$

⌟

Example. There are algebras with several associative strong labelling schemes. Let $\mathfrak{A}$ be the $\mathbb{T}^{\mathrm{thin}}$-algebra with domains $A_\xi := [n]$, for some fixed number $n < \omega$, where the product is just the maximum

$$\pi(t) := \max\{\, t(v) \mid v \in \mathrm{dom}_0(t) \,\}\,.$$

For every $k < n$, we obtain an associative labelling scheme σ_k defined by

$$\sigma_k(t)(v) := \begin{cases} \pi(t|_v) & \text{if } t|_v \in \mathbb{T}^{\mathrm{thin}}A\,, \\ \max\{k\} \cup \{\, t(u) \mid u \succeq v \,\}\,, & \text{otherwise}\,. \end{cases}$$

⌟

There is a tight connection between $\mathbb{T}$-expansions and associative labelling schemes (weak or strong, it does not matter).

Proposition 6.5. *Let $\mathbb{T}^0 \subseteq \mathbb{T}$ and let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra.*

(a) *Every associative weak labelling scheme for $\mathfrak{A}$ is strong.*

(b) *There exists a bijective correspondence between associative labelling schemes σ and $\mathbb{T}$-expansions of $\mathfrak{A}$.*

Proof. We define two mutually-inverse functions mapping (I) every associative weak labelling scheme to a $\mathbb{T}$-expansion of $\mathfrak{A}$ and (II) every such expansion to an associative strong labelling scheme.

(I) Given a weak scheme σ we define the corresponding function π_+ by

$$\pi_+(t) := \sigma(t)(\langle\rangle)\,, \quad \text{for } t \in \mathbb{T}A\,.$$

Then π_+ extends π since weak $\mathbb{T}^0$-consistency of σ implies that

$$\pi(t) = \sigma(t)(\langle\rangle) = \pi_+(t)\,, \quad \text{for } t \in \mathbb{T}^0 A\,.$$

Hence, it remains to show that π_+ is associative. Fix $T \in \mathbb{T}\mathbb{T}A$. By the definition of associativity of σ, we have

$$\sigma(\mathbb{T}\pi_+(T)) = \sigma(\mathrm{flat}(T)) \circ \mu\,,$$

which in particular implies that

$$\begin{aligned}\pi_+(\mathbb{T}\pi_+(T)) &= \sigma(\mathbb{T}\pi_+(T))(\langle\rangle) \\ &= \sigma(\mathrm{flat}(T))(\langle\rangle) = \pi_+(\mathrm{flat}(T))\,.\end{aligned}$$

(II) Conversely, given a product $\pi_+ : \mathbb{T}A \to A$ we define a scheme σ by

$$\sigma(t)(v) := \pi_+(t|_v)\,, \quad \text{for } t \in \mathbb{T}A \text{ and } v \in \mathrm{dom}_0(t)\,.$$

To show that this function σ is a strong labelling scheme, fix a factor $[u, \bar{v})$ (possibly with infinitely many holes $\bar{v}$) of some tree $t \in \mathbb{T}A$. Let $T \in \mathbb{T}\mathbb{T}A$ be the tree with $\mathrm{dom}(T) = \mathrm{dom}(t[u, \bar{v}))$ (plus some leaves for the variables below some of the v_i) and labelling

$$T(w) := \begin{cases} \mathrm{sing}(t(w)) & \text{for } w \in [u, \bar{v})\,, \\ t|_{v_i} & \text{for } w = v_i\,. \end{cases}$$

Then it follows by associativity of π_+ that

$$\begin{aligned}\sigma(t)(u) &= \pi_+(t|_u) \\ &= \pi_+(\mathrm{flat}(T)) \\ &= \pi_+(\mathbb{T}\pi_+(T)) \\ &= \pi_+\big(t[u, \bar{v})(\pi_+(t|_{v_0}), \pi_+(T|_{v_1}), \dots)\big) \\ &= \pi_+\big(t[u, \bar{v})(\sigma(t)(v_0), \sigma(t)(v_1), \dots)\big)\,,\end{aligned}$$

as desired. To show that σ is associative, let $T \in \mathbb{T}\mathbb{T}A$, $v \in \mathrm{dom}_0(T)$, and let t be the tree from the definition of associativity. Then

$$t(x) = \sigma(T(x))(\langle\rangle) = \pi_+(T(x)|_{\langle\rangle}) = \pi_+(T(x))\,, \quad \text{for } x \in \mathrm{dom}_0(t)\,,$$

implies that

$$\begin{aligned}\sigma(t)(v) &= \pi_+(t|_v) \\ &= \pi_+(\mathbb{T}\pi_+(T|_v)) \\ &= \pi_+(\mathrm{flat}(T|_v)) \\ &= \pi_+(\mathrm{flat}(T)|_{\mu(v)}) = \sigma(\mathrm{flat}(T))(\mu(v))\,.\end{aligned}$$

It remains to show that the mappings $\sigma \mapsto \pi_+$ and $\pi_+ \mapsto \sigma$ are inverse to each other. Translating a product π_+ to a scheme and back, we obtain the product

$$\pi'_+(t) = \pi_+(t|_{\langle\rangle}) = \pi_+(t)\,.$$

Conversely, translating a scheme σ to a product and back, we obtain the scheme

$$\sigma'(t)(v) = \sigma(t|_v)(\langle\rangle)\,.$$

To see that this value is equal to $\sigma(t)(v)$, consider the tree $T \in \mathbb{T}\mathbb{T}A$ consiting of a root labelled by the tree $t[\langle\rangle, v)$ to which we attach the tree $\mathrm{sing}(t|_v)$. Let w be the successor of the root of T and let t' be tree with domain $\mathrm{dom}(t') = \mathrm{dom}(T)$ and labels

$$t'(u) := \sigma(T(u))(\langle\rangle)\,, \quad \text{for } u \in \mathrm{dom}_0(T)\,,$$

as in the definition of associativity. By associativity of σ, it follows that

$$\begin{aligned}\sigma(t|_v)(\langle\rangle) &= \sigma(T(w))(\langle\rangle)\\ &= t'(w)\\ &= \sigma(\mathrm{flat}(T))(\mu(w))\\ &= \sigma(t)(\mu(w))\\ &= \sigma(t)(v)\,.\end{aligned}$$

□

In particular, if labellings are unique, so is the expansion. In fact, the following proposition shows that we do not need to assume associativity of the labelling scheme here.

Proposition 6.6. *Let $\mathbb{T}^{\mathrm{fin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}$ and let $\mathfrak{A}$ be a $\mathbb{T}^0$-algebra satisfying at least one of the following two conditions.*

(i) *Every tree $t \in \mathbb{T}A$ has a unique weakly $\mathbb{T}^0$-consistent labelling.*
(ii) *Every tree $t \in \mathbb{T}A$ has a unique strongly $\mathbb{T}^0$-consistent labelling.*

Then $\mathfrak{A}$ has a unique $\mathbb{T}$-expansion.

Proof. Let σ be the unique labelling scheme (weak or strong). By Proposition 6.5, it is sufficient to prove that σ is associative. Hence, fix a tree $T \in \mathbb{T}\mathbb{T}A$ and let t and μ be as in the definition of associativity. We claim that the labelling

$$\lambda := \sigma(\mathrm{flat}(T)) \circ \mu$$

is a strongly $\mathbb{T}^0$-consistent labelling of t. Then uniqueness of labellings implies that

$$\sigma(t) = \lambda = \sigma(\mathrm{flat}(T)) \circ \mu\,,$$

as desired.

For the proof, fix a factor $[u, \bar{v})$ of t (possibly with infinitely many holes $\bar{v}$). We have to show that

$$\lambda(u) = \pi\big(t[u, \bar{v})(\lambda(v_0), \lambda(v_1), \dots)\big)\,.$$

Since every tree $T(w)$ (with w corresponding to some vertex in $[u, \bar{v})$) only contains finitely many variables, we can replace in $T(w)$ some subtrees $T(w)|_{w'}$ (without variables) by the corresponding constant $\sigma(T(w))(w')$. Let $P(w)$ be a finite tree obtained in this way from $T(w)$. Using the consistency of $\sigma(P(w))$ and $\sigma(T(w))$, we can show by induction on w' (starting at the leaves) that

$$\sigma(P(w))(w') = \sigma(T(w))(w')\,, \quad \text{for all } w' \in \mathrm{dom}_0(P(w))\,.$$

Consequently, consistency implies that

$$\pi(P(w)) = \sigma(P(w))(\langle\rangle) = \sigma(T(w))(\langle\rangle) = t(w)\,.$$

We regard the family $(P(w))_w$ as a tree $P \in \mathbb{T}\mathbb{T}A$ with domain

$$\mathrm{dom}(P) = [u, \bar{v}) \cup \{v_0, v_1, \dots\} \cup \bigcup_i L_i\,,$$

where L_i is a set of leaves attached to v_i corresponding to the variables appearing in the subtree $t|_{v_i}$, and the vertices v_i are labelled by

$$P(v_i) := \lambda(v_i)\,.$$

Then the domain of P is that of a tree in $\mathbb{T}^0$ (if we ignore the fact that the root is u and not $\langle\rangle$). Hence, we have $P \in \mathbb{T}^0\mathbb{T}^{\mathrm{fin}}A \subseteq \mathbb{T}^0\mathbb{T}^0 A$. This implies by consistency that $\mathrm{flat}(P) \in \mathbb{T}^0 A$ and

$$\begin{aligned}\sigma(\mathrm{flat}(P))(\langle\rangle) &= \pi(\mathrm{flat}(P))\\ &= \pi(\mathbb{T}^0\pi(P)) = \pi\big(t[u, \bar{v})(\lambda(v_0), \lambda(v_1), \dots)\big)\,,\end{aligned}$$

where the last step follows by the fact that

$$\pi(P(w)) = \begin{cases} t(w) & \text{if } w \in [u, \bar{v})\,,\\ \lambda(v_i) & \text{if } w = v_i\,.\end{cases}$$

Hence, consistency of σ implies that

$$\begin{aligned}\pi\big(t[u, \bar{v})(\lambda(v_0), \lambda(v_1), \dots)\big) &= \sigma(\mathrm{flat}(P))(\langle\rangle)\\ &= \sigma(\mathrm{flat}(T)|_{\mu(u)})(\langle\rangle)\\ &= \sigma(\mathrm{flat}(T))(\mu(u)) = \lambda(u)\,,\end{aligned}$$

where the second step follows by consistency of σ and the third one by uniqueness of the labellings. (The function $w \mapsto \sigma(\mathrm{flat}(T))(\mu(u)w)$ is a consistent labelling of $\mathrm{flat}(T)|_{\mu(u)}$.) □

As an application let us show how to use consistent labellings to prove that an algebra is definable.

Proposition 6.7. *Every finitary $\mathbb{T}^{\mathrm{thin}}$-algebra is* MSO*-definable.*

Proof. Let $\mathfrak{A}$ be a finitary $\mathbb{T}^{\mathrm{thin}}$-algebra, fix a finite set $C \subseteq A$ of generators, a sort ξ, and an element $a \in A_\xi$. Set $B := \bigcup_{\zeta\subseteq\xi} A_\zeta$. We construct a formula that, given a tree $t \in \mathbb{T}^{\mathrm{thin}}_\xi C$, guesses the labelling $\lambda : \mathrm{dom}(t) \to B$ induced by the product $\lambda(v) := \pi(t|_v)$ and then verifies the correctness of its guess by checking for each vertex v that

- $\lambda(v) = t(v)(\lambda(u_0), \dots, \lambda(u_{n-1}))$, where $u_0, \dots, u_{n-1}$ are the successors of v,
- $\lambda(v) = \pi(s_\beta)$, for every branch β starting at v, where s_β is the path obtained from $t|_v$ by replacing every subtree attached at a vertex u not belonging to β by the tree $\mathrm{sing}(\lambda(u))$.

The latter condition can be expressed in MSO since this logic can evaluate products in finite ω-semigroups and the ω-semigroup we are working with has the domains $\bigcup_{\zeta\subseteq\xi} A_{\zeta+\{x\}}$ and B, both of which are finite. By induction on the Cantor-Bendixson rank of t it follows that the above checks ensure that the guessed labelling coincides with the intended one. □

Corollary 6.8. *Let $\mathfrak{A}$ be a finitary $\mathbb{T}$-algebra where every tree $t \in \mathbb{T}A$ has exactly one weakly $\mathbb{T}^{\mathrm{thin}}$-consistent labelling. Then $\mathfrak{A}$ is* MSO*-definable.*

Proof. Fix a finite set $C \subseteq A$ of generators of $\mathfrak{A}$. We claim that $D := C \cup A_\emptyset$ is a set of generators of the $\mathbb{T}^{\mathrm{thin}}$-reduct $\mathfrak{A}^0$ of $\mathfrak{A}$. For the proof, fix $a \in A_\xi$. By assumption, there is some tree $t \in \mathbb{T}_\xi C$ with $\pi(t) = a$. Let s be the tree obtained from t by replacing every subtree without a variable by its product. Then $s \in \mathbb{T}^{\mathrm{thin}}D$ and $\pi(s) = \pi(t) = a$.

Consequently, $\mathfrak{A}^0$ is finitary and we can use Proposition 6.7 to show that $\mathfrak{A}^0$ is MSO-definable. In particular, there exists a finite set $C \subseteq A$ generating A via the product of $\mathfrak{A}^0$. It follows that C is also a set of generators for $\mathfrak{A}$. We claim that we can evaluate products of trees in $\mathbb{T}C$ in MSO.

Given a tree $t \in \mathbb{T}C$, we guess a labelling λ of t and check that it is weakly $\mathbb{T}^{\text{thin}}$-consistent. As the canonical labelling given by $v \mapsto \pi(t|_v)$ is weakly consistent, it then follows by uniqueness of such labellings that $\lambda(\langle\rangle) = \pi(t)$. To see that consistency of λ can be expressed in MSO note that, by MSO-definability of $\mathfrak{A}^0$, we can evaluate products of the form

$$\pi\big(t[u,\bar{v}](\lambda(v_0),\dots,\lambda(v_{n-1})\big) \quad \text{with} \quad t[u,\bar{v}] \in \mathbb{T}^{\text{thin}}C\,.$$

Furthermore, note that the class of all thin trees is MSO-definable. (We only have to say that there is no embedding of the infinite binary tree.) □

7. Unambiguous algebras

In Corollary 5.14, we have obtained a complete classification of all $\mathbb{T}^{\text{thin}}$-expansions of a $\mathbb{T}^{\text{fin}}$-algebra. In the present section, we use consistent labellings to study the inclusion $\mathbb{T}^{\text{thin}} \subseteq \mathbb{T}$. First let us remark that it is not dense.

Lemma 7.1. *There exists a* $\mathbb{T}^{\text{thin}}$*-algebra* $\mathfrak{A}$ *with two different* MSO*-definable* $\mathbb{T}$*-expansions.*

Proof. Let A be the set with two elements $0_\xi, 1_\xi$ for every sort ξ. We consider two different products $\pi_0, \pi_1 : \mathbb{T}A \to A$ on this set. The first one is just the minimum operation:

$$\pi_0(t) := \min\{\, t(v) \mid v \in \operatorname{dom}(t)\,\}\,, \quad \text{for } t \in \mathbb{T}A\,.$$

The second one is given by

$$\pi_1(t) := \begin{cases} 1 & \text{if } t \in \mathbb{T}^{\text{thin}}C\,, \\ 0 & \text{if } t \in \mathbb{T}A \setminus \mathbb{T}^{\text{thin}}C\,, \end{cases}$$

where $C \subseteq A$ is the subset consisting of the elements 1_ξ, $\xi \in \Xi$. Note that both functions coincide when restricted to thin trees. Since there exists an MSO-formula expressing that a given tree is thin, both products are MSO-definable. Furthermore, π_0 is clearly associative. To show that so is π_1, fix a tree $t \in \mathbb{T}\mathbb{T}A$. We distinguish three cases.

- If there is some $v \in \operatorname{dom}(t)$ with $t(v) \in \mathbb{T}A \setminus \mathbb{T}^{\text{thin}}C$, we have $\operatorname{flat}(t) \notin \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 0 = \pi_1(\operatorname{flat}(t))$.
- If $t \in \mathbb{T}^{\text{thin}}\mathbb{T}^{\text{thin}}C$, then $\operatorname{flat}(t) \in \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 1 = \pi_1(\operatorname{flat}(t))$.
- Finally, suppose that $t \in \mathbb{T}\mathbb{T}^{\text{thin}}C \setminus \mathbb{T}^{\text{thin}}\mathbb{T}^{\text{thin}}C$. Then we have $\operatorname{flat}(t) \in \mathbb{T}C \setminus \mathbb{T}^{\text{thin}}C$ and $\pi_1(\mathbb{T}\pi_1(t)) = 0 = \pi_1(\operatorname{flat}(t))$. □

Consistent labellings have been used in [BS13] to study unambiguous tree languages. Let us give a brief overview over these results. The central notion is the following one.

Definition 7.2. Let $\mathbb{T}^0 \subseteq \mathbb{T}$. A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *unambiguous* if every tree $t \in \mathbb{T}A$ has at most one strongly $\mathbb{T}^0$-consistent labelling. ⌟

Remarks.
(a) For $\mathbb{T}^0 = \mathbb{T}^{\text{thin}}$ these algebras were introduced in [BS13] under the name *prophetic thin algebras.*
(b) The fact that a given tree has a unique strongly $\mathbb{T}^{\text{thin}}$-consistent labelling is expressible in MSO. ⌟

First, note that there exist $\mathbb{T}^{\text{thin}}$-algebras which are not unambiguous.

Example. Let $\mathfrak{A}$ be the $\mathbb{T}^{\text{thin}}$-algebra generated by the elements $0, 1$ (of arity 0), b_0, b_1, c_0, c_1 (of arity 1), and a (of arity 2) subject to the following equations.

$$\begin{aligned} b_i(j) &= j\,, & a(x,i) &= b_i(x)\,,\\ b_i(b_j(x)) &= b_{\max\{i,j\}}(x)\,, & a(i,x) &= c_i(x)\,,\\ c_i(j) &= i\,, & b_i^\omega &= 1-i\,,\\ c_i(b_j(x)) &= c_i(x)\,, & c_i^\omega &= i\,,\\ c_i(c_j(x)) &= c_i(x)\,, \end{aligned}$$

for $i, j \in \{0,1\}$. This algebra is not unambiguous since the (unique) tree $t \in \mathbb{T}_\emptyset\{a\}$ has several consistent labellings, including

$$\lambda(w) := |w|_1 \bmod 2 \quad\text{and}\quad \mu(w) := (|w|_1 + 1) \bmod 2\,,$$

where $|w|_1$ denotes the number of letters 1 in $w \in \{0,1\}^*$. ⌟

The connection between unambiguous $\mathbb{T}^{\text{thin}}$-algebras and unambiguous tree languages is given by the following theorem.

Definition 7.3. (a) A tree automaton is *unambiguous* if it has at most one accepting run on each given input tree.

(b) A language $K \subseteq \mathbb{T}_\xi\Sigma$ is called *bi-unambiguous* if both K and $\mathbb{T}_\xi\Sigma \setminus K$ are recognised by unambiguous automata. ⌟

Theorem 7.4 (Bilkowski, Skrzypczak [BS13])**.** *A language $K \subseteq \mathbb{T}_\xi\Sigma$ is bi-unambiguous if, and only if, it is recognised by a morphism $\varphi : \mathbb{T}\Sigma \to \mathfrak{A}$ of $\mathbb{T}^{\text{thin}}$-algebras to a finitary unambiguous $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$.*

Unfortunately, the question of whether all trees have strongly $\mathbb{T}^{\text{thin}}$-consistent labellings is still an open problem, one which turns out to be equivalent to the existence of the following kind of choice functions.

Definition 7.5. The *Thin Choice Conjecture* states that there does *not* exist an MSO-formula $\varphi(x; Z)$ such that, for every thin (unlabelled) tree t and every non-empty set $P \subseteq \operatorname{dom}(t)$ of parameters, the formula $\varphi(x; P)$ defines a unique element of P. ⌟

Theorem 7.6 (Bilkowski, Skrzypczak [BS13])**.** *The following statements are equivalent.*

(1) *The Thin Choice Conjecture holds.*
(2) *All trees have strongly $\mathbb{T}^{\text{thin}}$-consistent labellings, for every finitary $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$.*
(3) *The unique tree in $\mathbb{T}_\emptyset\{a\}$ has a strongly $\mathbb{T}^{\text{thin}}$-consistent labelling, for every $\mathbb{T}^{\text{thin}}$-algebra $\mathfrak{A}$ and every $a \in A$.*
(4) *For every morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of $\mathbb{T}^{\text{thin}}$-algebras and every strongly $\mathbb{T}^{\text{thin}}$-consistent labelling β of some tree $t \in \mathbb{T}B$, there exists a strongly $\mathbb{T}^{\text{thin}}$-consistent labelling α with $\varphi \circ \alpha = \beta$.*

(A full proof can be found in Section 8.3 of [Skr16], in particular Theorem 8.74 and Proposition 8.22.)

The main applications of the Thin Choice Conjecture proved in [BS13] are as follows (cf. Theorem 5 and Section 4 of [BS13]). For the definition of a pseudo-variety and a syntactic algebra we refer the reader to standard accounts on algebraic language theory. In the context of the monadic framework, this theory is worked out in [Boj20, Blu21, Blua].

Theorem 7.7 (Bilkowski, Skrzypczak)**.** *Suppose that the Thin Choice Conjecture holds.*

(a) *The class of unambiguous $\mathbb{T}^{\text{thin}}$-algebras forms a pseudo-variety.*

(b) *A language $K \subseteq \mathbb{T}\Sigma$ is bi-unambiguous if, and only if, (the $\mathbb{T}^{\text{thin}}$-reduct of) its syntactic algebra is unambiguous.*
(c) *Bi-unambiguity of a language is decidable.*

8. Branch-continuous algebras

In this final section, we take a look at a few other natural classes of $\mathbb{T}^{\text{thin}}$-algebras where unique $\mathbb{T}$-expansions exist. Originally, these classes were introduced in [Blub, Blu20]. A unified account can be found in [Blua]. The simplest example consists of algebras that are constructed from an ω-semigroup as follows. For the definition it is convenient to introduce a variant $\mathbb{T}^?$ of the monad $\mathbb{T}$ for trees where we are allowed to omit variables.

Definition 8.1. We set $\mathbb{T}^? X := (\mathbb{T}^?_\xi X)_{\xi \in \Xi}$ where

$$\mathbb{T}^?_\xi X := \sum_{\zeta \subseteq \xi} \mathbb{T}_\zeta X\,.$$

The functor $\mathbb{T}^{?\text{thin}}$ is defined analogously from $\mathbb{T}^{\text{thin}}$.

The corresponding flattening operation $\text{flat} : \mathbb{T}^?\mathbb{T}^? \Rightarrow \mathbb{T}^?$ is defined in the obvious way: if the label at a vertex v is missing some variables, we omit the corresponding subtrees and then flatten the resulting tree. ⌟

It follows that a $\mathbb{T}^?$-algebra is just a $\mathbb{T}$-algebra that is equipped with additional functions $A_\zeta \to A_\xi$, for all $\zeta \subseteq \xi$ (satisfying some natural laws). We start with algebras arising from an ω-semigroup in the following way.

Definition 8.2. (a) Let $\mathbb{T}^{?\text{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$. A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *semigroup-like* if it is generated by $A_\emptyset \cup A_{\{z\}}$.

(b) Let $\mathfrak{S} = \langle S, S_\omega \rangle$ be an ω-semigroup. We denote by $\text{TA}(\mathfrak{S})$ the $\mathbb{T}^?$-algebra $\langle A, \pi \rangle$ with domains

$$A_\xi := S_\omega + S \times \xi\,, \quad \text{for } \xi \in \Xi\,.$$

For elements $\langle a, x \rangle \in S \times \xi$, we will use the more suggestive notation $a(x)$. The product is defined as follows. Given $t \in \mathbb{T}^?_\xi A$, let $\beta = (v_i)_i$ be the path defined as follows. We start with the root v_0 of t. Having chosen v_i, we take a look at its label $t(v_i)$. If $t(v_i) = a_i(z_i) \in S \times \zeta_i$, we choose $v_{i+1} := \text{suc}_{z_i}(v_i)$. Otherwise, the path ends at v_i. Let $(a_i)_i$ be the corresponding sequence of labels. (If the path is finite, the last label a_n is either an element of S_ω or a variable.) We set

$$\pi(t) := \prod_i a_i\,.$$

Note that this product can be of one the following forms:
- an infinite product $a_0 \cdot a_1 \cdot \dots \in S_\omega$ with $a_i \in S$,
- a finite product $a_0 \cdot \dots \cdot a_n \in S_\omega$ with $a_0, \dots, a_{n-1} \in S$ and $a_n \in S_\omega$,
- a finite product $\langle a_0 \cdot \dots \cdot a_{n-1}, a_n \rangle \in S \times \xi$ with $a_0, \dots, a_{n-1} \in S$ and $a_n \in \xi$ is a variable. ⌟

The following characterisation of semigroup-like algebras is from [Blub, Blua] (Proposition 4.21 of the former, Proposition VIII.2.7 of the latter; these propositions are only formulated for $\mathbb{T}^?$-algebras, but the proof also works for $\mathbb{T}^{?\text{thin}}$-algebras).

Proposition 8.3. *A $\mathbb{T}^?$-algebra $\mathfrak{A}$ is semigroup-like if, and only if, there exists a surjective $\mathbb{T}^?$-morphism* $\mathrm{TA}(\mathfrak{S}) \to \mathfrak{A}$*, for some ω-semigroup $\mathfrak{S}$.*

Similarly, a $\mathbb{T}^{?\mathrm{thin}}$-algebra $\mathfrak{A}$ is semigroup-like if, and only if, there exists a surjective $\mathbb{T}^{?\mathrm{thin}}$-morphism $\mathrm{TA}(\mathfrak{S}) \to \mathfrak{A}$*, for some ω-semigroup $\mathfrak{S}$.*

Lemma 8.4. *Every semigroup-like $\mathbb{T}^{?\mathrm{thin}}$-algebra is unambiguous and has a unique $\mathbb{T}^?$-expansion. This expansion is again semigroup-like.*

Proof. Let $\mathfrak{A}$ be a semigroup-like $\mathbb{T}^{?\mathrm{thin}}$-algebra. By Proposition 8.3, $\mathfrak{A}$ is a quotient of the $\mathbb{T}^{?\mathrm{thin}}$-reduct of $\mathrm{TA}(\mathfrak{S})$. The corresponding quotient of $\mathrm{TA}(\mathfrak{S})$ is a $\mathbb{T}^?$-expansion of $\mathfrak{A}$. In particular, $\mathfrak{A}$ has a semigroup-like $\mathbb{T}^?$-expansion.

For uniqueness, let us first prove that there exists at most one strong labelling scheme for $\mathfrak{A}$. Fix a tree $t \in \mathbb{T}^? A$ and let λ and μ be two strongly $\mathbb{T}^{?\mathrm{thin}}$-consistent labellings of t. Given a vertex $v \in \mathrm{dom}(t)$, let β be the path starting at v that we constructed in the definition of $\pi_+(t|_v)$. We choose a thin factor p of $t|_v$ containing this path. Then $\mathbb{T}^{?\mathrm{thin}}$-consistency implies that $\lambda(v) = \pi(p) = \mu(v)$. Hence, $\lambda = \mu$.

To conclude the proof, consider two $\mathbb{T}^?$-expansions $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $\mathfrak{A}$. By Proposition 6.5 (b) and the claim we have proved above, it follows that the $\mathbb{T}^{\mathrm{thin}}$-reduct of $\mathfrak{A}$ has at most one $\mathbb{T}$-expansion. Hence, the $\mathbb{T}$-reducts of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ coincide. Furthermore, the product of a $\mathbb{T}^?$-algebra is uniquely determined by its $\mathbb{T}$-reduct and the by functions $A_\zeta \to A_\xi$ it induces. Since the latter functions are given by products of finite trees and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same $\mathbb{T}^{?\mathrm{thin}}$-reduct, it follows that these functions are also the same for $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Consequently, the products of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ coincide. □

This lemma is hardly surprising, since the product of a semigroup-like algebra only depends on a single branch of the given tree. We can extend this result to more complicated classes of algebras as follows. So far, we have mostly ignored the fact that our algebras are ordered. (The ordering is needed to characterise logics that are not closed under negation, something we are not concerned with in the present article.) The next two classes of examples on the other hand make essential use of the ordering. We start by introducing some notation concerning meets and joins.

Definition 8.5. Let A be a sorted set.

(a) For $X \subseteq A$, we set

$$\Uparrow X := \{\, a \in A \mid a \geq x \text{ for some } x \in X \,\},$$
$$\Downarrow X := \{\, a \in A \mid a \leq x \text{ for some } x \in X \,\}.$$

The set X is *upwards closed* if $\Uparrow X = X$, and it is *downwards closed* if $\Downarrow X = X$.

(b) We define two functors $\mathbb{U}$ and $\mathbb{D}$ as follows. For sorted sets A, we set $\mathbb{U}A := (\mathbb{U}_\xi A)_\xi$ and $\mathbb{D}A := (\mathbb{D}_\xi A)_\xi$ where

$$\mathbb{U}_\xi A := \{\, I \subseteq A_\xi \mid I \text{ is upwards closed} \,\},$$
$$\mathbb{D}_\xi A := \{\, I \subseteq A_\xi \mid I \text{ is downwards closed} \,\}.$$

We order elements of $\mathbb{D}A$ by inclusion, and those of $\mathbb{U}A$ by inverse inclusion. For functions $f : A \to B$, we define

$$\mathbb{U}f(I) := \Uparrow\{\, f(a) \mid a \in I \,\},$$
$$\mathbb{D}f(I) := \Downarrow\{\, f(a) \mid a \in I \,\}.$$

(c) For $\mathbb{T}^{?\text{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$, $t \in \mathbb{T}^0 A$, and $T \in \mathbb{T}^0 \mathbb{U} A$ or $T \in \mathbb{T}^0 \mathbb{D} A$, we write

$$t \in^{\mathbb{T}^0} T \quad :\text{iff} \quad t \text{ and } T \text{ have the same domain and if}$$
$$t(v) \in T(v)\,, \quad \text{for all vertices } v \in \mathrm{dom}_0(t)\,,$$
$$t(v) = T(v)\,, \quad \text{for all vertices } v \text{ labelled by a variable}\,.$$

(d) Let $C \subseteq A$. We denote by $\langle\!\langle C \rangle\!\rangle_{\text{inf}}$ the closure of C under (possibly infinite) meets and by $\langle\!\langle C \rangle\!\rangle_{\text{sup}}$ its closure under (possibly infinite) joins. C is a set of *meet-generators* if $\langle\!\langle C \rangle\!\rangle_{\text{inf}} = A$ and a set of *join-generators* if $\langle\!\langle C \rangle\!\rangle_{\text{sup}} = A$.

(e) We say that A is *completely ordered* if every subset $C \subseteq A$ has an infimum and a supremum. ⌟

The next, more interesting class of algebras we take a look at is the class of *deterministic* algebras, which was introduced in [Blu20] to give an algebraic characterisation of the class of MSO-definable $\mathbb{T}^?$-algebras. Here, we are interested in the fact that their product is determined by its $\mathbb{T}^{?\text{thin}}$-reduct. The definition is as follows.

Definition 8.6. Let $\mathbb{T}^{?\text{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$.

(a) We define a function $\mathrm{dist} : \mathbb{T}^0 \mathbb{U} A \to \mathbb{U} \mathbb{T}^0 A$ by

$$\mathrm{dist}(t) := \{\, s \in \mathbb{T}^0 A \mid s \in^{\mathbb{T}^0} t \,\}\,.$$

(b) A function $g : \mathfrak{A} \to B$ from a $\mathbb{T}^0$-algebra $\mathfrak{A} = \langle A, \pi \rangle$ to a completely ordered sorted set B is *meet-distributive* if g preserves meets and there exists a function $\sigma : \mathbb{T}^0 \langle\!\langle \mathrm{rng}\, g \rangle\!\rangle_{\text{inf}} \to B$ such that

$$\sigma \circ \mathbb{T}^0(\inf \circ \mathbb{U} g) = \inf \circ \mathbb{U}(g \circ \pi) \circ \mathrm{dist}\,.$$

We call the function σ the *product of B induced* by g.

A completely ordered $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *meet-distributive* if the identity $\mathrm{id} : \mathfrak{A} \to \mathfrak{A}$ is meet-distributive. *Join-distributivity* is defined analogously with $\mathbb{D}$ and sup instead of $\mathbb{U}$ and inf.

(c) A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *deterministic* if it is meet-distributive and it has a semigroup-like subalgebra $\mathfrak{C}$ such that C forms a set of meet-generators of $\mathfrak{A}$. ⌟

We will show in the next lemma that, for a meet-distributive algebra, the induced product coincides with the actual product. Hence, meet-distributive algebras are those where the product commutes with meets. More generally, it will follow by the results below that an injective morphism $e : \mathfrak{A} \to \mathfrak{B}$ of $\mathbb{T}^?$-algebras is meet-distributive if the restriction of $\mathfrak{B}$ to $\mathrm{rng}\, e$ is meet-distributive. The name 'deterministic algebra' stems from the fact that such algebras correspond to deterministic automata. A typical example of a deterministic algebra is one where every element is of the form

$$a_0(x_0) \sqcap \cdots \sqcap a_{m-1}(x_{m-1}) \sqcap b_0 \sqcap \cdots \sqcap b_{n-1}\,,$$

where $a_0, \dots, a_{m-1} \in S$ and $b_0, \dots, b_{n-1} \in S_\omega$ are elements of some ω-semigroup $\mathfrak{S} = \langle S, S_\omega \rangle$ and the $x_0, \dots, x_{m-1}$ are variables.

We start with two technical lemmas. The first one is trivial.

Lemma 8.7. *Let $\mathbb{T}^{?\text{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$. A completely ordered $\mathbb{T}^0$-algebra $\mathfrak{A}$ is meet-distributive if, and only if,*

$$\pi \circ \mathbb{T}^0 \inf = \inf \circ \mathbb{U}\pi \circ \mathrm{dist}\,.$$

Proof. (⇐) In the definition of meet-distributivity, we can take $\sigma := \pi$.

(⇒) Let σ be the function from the definition of meet-distributivity. Given a tree $t \in \mathbb{T}^0 A$, let $T \in \mathbb{T}^0 \mathbb{U} A$ be the tree with labels $T(v) = \{\, a \in A \mid a \geq t(v) \,\}$. Then

$$\sigma(t) = \sigma(\mathbb{T}^0 \inf(T)) = \inf \Uparrow\{\, \pi(s) \mid s \in^{\mathbb{T}^0} T \,\} = \inf \Uparrow\{\pi(t)\} = \pi(t)\,.$$

Hence, we have

$$\begin{aligned} \pi \circ \mathbb{T}^0 \mathrm{inf} &= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\mathrm{id}) \\ &= \mathrm{inf} \circ \mathbb{U}(\mathrm{id} \circ \pi) \circ \mathrm{dist} = \mathrm{inf} \circ \mathbb{U}\pi \circ \mathrm{dist}\,. \end{aligned}$$ □

Meet-distributive functions can be used to transfer a $\mathbb{T}^?$-algebra product from their domain to their codomain.

Lemma 8.8. *Let $\mathbb{T}^{?\mathrm{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$. Let $\varphi : \mathfrak{C} \to A$ be a meet-distributive function such that* $\mathrm{rng}\,\varphi$ *is a set of meet-generators of A. There exists a unique function $\sigma : \mathbb{T}^0 A \to A$ such that $\langle A, \sigma\rangle$ is a meet-distributive $\mathbb{T}^0$-algebra and φ a morphism of $\mathbb{T}^0$-algebras.*

Proof. To make our proof more concise, we use some properties of the function $\mathrm{dist} : \mathbb{T}^0\mathbb{U} \Rightarrow \mathbb{U}\mathbb{T}^0$. We have shown in [Blu23] that dist is what is called a *distributive law,* which means it is a natural transformation satisfying the equations

$$\begin{aligned} \mathrm{dist} \circ \mathrm{flat} &= \mathbb{U}\mathrm{flat} \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{dist}\,, & \mathrm{dist} \circ \mathrm{sing} &= \mathbb{U}\mathrm{sing}\,, \\ \mathrm{dist} \circ \mathbb{T}^0\mathrm{union} &= \mathrm{union} \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist}\,, & \mathrm{dist} \circ \mathbb{T}^0\mathrm{pt} &= \mathrm{pt}\,, \end{aligned}$$

where $\mathrm{union} : \mathbb{U}\mathbb{U}A \to \mathbb{U}A$ maps a set of sets to its union and $\mathrm{pt} : A \to \mathbb{U}A$ is defined by $\mathrm{pt}(a) := \Uparrow\{a\}$.

Let $\sigma : \mathbb{T}^0 A \to A$ be the product induced by φ as in the definition of meet-distributivity. To see that $\langle A, \sigma\rangle$ is a $\mathbb{T}^0$-algebra, note that

$$\begin{aligned} &\sigma \circ \mathrm{sing} \circ (\mathrm{inf} \circ \mathbb{U}\varphi) \\ &= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathrm{pt}) \circ \mathrm{sing} \circ (\mathrm{inf} \circ \mathbb{U}\varphi) && [\mathrm{inf} \circ \mathrm{pt} = \mathrm{id}] \\ &= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathrm{pt}) \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi) \circ \mathrm{sing} && [\text{sing nat. trans.}] \\ &= \sigma \circ \mathbb{T}^0\mathrm{inf} \circ \mathbb{T}^0\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi) \circ \mathbb{T}^0\mathrm{pt} \circ \mathrm{sing} && [\text{pt nat. trans.}] \\ &= \sigma \circ \mathbb{T}^0\mathrm{inf} \circ \mathbb{T}^0(\mathrm{union} \circ \mathbb{U}\mathbb{U}\varphi) \circ \mathbb{T}^0\mathrm{pt} \circ \mathrm{sing} && [\mathrm{inf} \circ \mathbb{U}\mathrm{inf} = \mathrm{inf} \circ \mathrm{union}] \\ &= \sigma \circ \mathbb{T}^0\mathrm{inf} \circ \mathbb{T}^0(\mathbb{U}\varphi \circ \mathrm{union}) \circ \mathbb{T}^0\mathrm{pt} \circ \mathrm{sing} && [\text{union nat. trans.}] \\ &= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{union} \circ \mathbb{T}^0\mathrm{pt} \circ \mathrm{sing} && [\varphi \text{ meet-dist.}] \\ &= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathrm{sing} && [\mathrm{union} \circ \mathrm{pt} = \mathrm{id}] \\ &= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathbb{U}\mathrm{sing} && [\text{dist dist. law}] \\ &= \mathrm{inf} \circ \mathbb{U}\varphi\,, && [\text{unit law for } \pi] \end{aligned}$$

$$\begin{aligned}
&\sigma \circ \mathbb{T}^0\sigma \circ \mathbb{T}^0\mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist}) && [\varphi \text{ meet-dist.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}^0\mathbb{U}\pi \circ \mathbb{T}^0\mathrm{dist} && [\varphi \text{ meet-dist.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathbb{U}\mathbb{T}^0\pi \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{dist} && [\text{dist nat. trans.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathbb{U}\mathrm{flat} \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{dist} && [\mathfrak{C}\ \mathbb{T}^0\text{-algebra}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathrm{flat} && [\text{dist dist. law}] \\
&= \sigma \circ \mathbb{T}^0\mathrm{inf} \circ \mathbb{T}^0\mathbb{U}\varphi \circ \mathrm{flat} && [\varphi \text{ meet-dist.}] \\
&= \sigma \circ \mathrm{flat} \circ \mathbb{T}^0\mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi)\,. && [\text{flat nat. trans.}]
\end{aligned}$$

Since $\mathrm{inf} \circ \mathbb{U}\varphi$ is surjective and $\mathbb{T}^0$ preserves surjectivity, it follows that

$$\sigma \circ \mathrm{sing} = \mathrm{id} \quad \text{and} \quad \sigma \circ \mathbb{T}^0\sigma = \sigma \circ \mathrm{flat}\,.$$

To see that $\langle A, \sigma\rangle$ is meet-distributive, note that

$$\begin{aligned}
&\sigma \circ \mathbb{T}^0\mathrm{inf} \circ \mathbb{T}^0\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi) \\
&= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathrm{union} \circ \mathbb{U}\mathbb{U}\varphi) && [\mathrm{inf} \circ \mathbb{U}\mathrm{inf} = \mathrm{inf} \circ \mathrm{union}] \\
&= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi \circ \mathrm{union}) && [\text{union nat. trans.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{union} && [\varphi \text{ meet-dist.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{union} \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist} && [\text{dist dist. law}] \\
&= \mathrm{inf} \circ \mathrm{union} \circ \mathbb{U}\mathbb{U}(\varphi \circ \pi) \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist} && [\text{union nat. trans.}] \\
&= \mathrm{inf} \circ \mathbb{U}\mathrm{inf} \circ \mathbb{U}\mathbb{U}(\varphi \circ \pi) \circ \mathbb{U}\mathrm{dist} \circ \mathrm{dist} && [\mathrm{inf} \circ \mathbb{U}\mathrm{inf} = \mathrm{inf} \circ \mathrm{union}] \\
&= \mathrm{inf} \circ \mathbb{U}(\sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi)) \circ \mathrm{dist} && [\varphi \text{ meet-dist.}] \\
&= \mathrm{inf} \circ \mathbb{U}\sigma \circ \mathrm{dist} \circ \mathbb{T}^0\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi)\,. && [\text{dist nat. trans.}]
\end{aligned}$$

By surjectivity of $\mathbb{T}^0\mathbb{U}(\mathrm{inf} \circ \mathbb{U}\varphi)$, this implies that

$$\sigma \circ \mathbb{T}^0\mathrm{inf} = \mathrm{inf} \circ \mathbb{U}\sigma \circ \mathrm{dist}\,.$$

Hence, the claim follows by Lemma 8.7.

To see that φ is a morphism of $\mathbb{T}^0$-algebras, note that

$$\begin{aligned}
\sigma \circ \mathbb{T}^0\varphi &= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathrm{pt} \circ \varphi) && [\mathrm{inf} \circ \mathrm{pt} = \mathrm{id}] \\
&= \sigma \circ \mathbb{T}^0(\mathrm{inf} \circ \mathbb{U}\varphi \circ \mathrm{pt}) && [\text{pt nat. trans.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} \circ \mathbb{T}^0\mathrm{pt} && [\varphi \text{ meet-dist.}] \\
&= \mathrm{inf} \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{pt} && [\text{dist dist. law}] \\
&= \mathrm{inf} \circ \mathrm{pt} \circ \varphi \circ \pi && [\text{pt nat. trans.}] \\
&= \varphi \circ \pi\,. && [\mathrm{inf} \circ \mathrm{pt} = \mathrm{id}]
\end{aligned}$$

Finally, for uniqueness suppose that $\sigma' : \mathbb{T}^0 A \to A$ is another function such that $\langle A, \sigma' \rangle$ is meet-distributive and $\varphi : \mathfrak{C} \to \langle A, \sigma' \rangle$ is a morphism of $\mathbb{T}^0$-algebras. Then it follows that

$$\begin{aligned}
\sigma \circ \mathbb{T}^0(\inf \circ \mathbb{U}\varphi) &= \inf \circ \mathbb{U}(\varphi \circ \pi) \circ \mathrm{dist} && [\text{choice of } \sigma] \\
&= \inf \circ \mathbb{U}(\sigma' \circ \mathbb{T}^0\varphi) \circ \mathrm{dist} && [\varphi \text{ morphism}] \\
&= \inf \circ \mathbb{U}\sigma' \circ \mathrm{dist} \circ \mathbb{T}^0\mathbb{U}\varphi && [\text{dist nat. trans.}] \\
&= \sigma' \circ \mathbb{T}^0\inf \circ \mathbb{T}^0\mathbb{U}\varphi\,. && [\text{Lemma 8.7}]
\end{aligned}$$

Hence, the fact that $\mathbb{T}^0(\inf \circ \mathbb{U}\varphi)$ is surjective implies that $\sigma = \sigma'$. □

Lemma 8.9. *Let A be a sort-wise finite, completely ordered set and $\mathfrak{C}$ a semigroup-like $\mathbb{T}^?$-algera whose universe $C \subseteq A$ is contained in A. If the inclusion $\mathfrak{C}|_{\mathbb{T}^{?\text{thin}}} \to A$ is meet-distributive, so is the inclusion $\mathfrak{C} \to A$.*

Proof. Suppose that the inclusion $i_0 : \mathfrak{C}|_{\mathbb{T}^{?\text{thin}}} \to A$ is meet-distributive and let $\sigma_0 : \mathbb{T}^{?\text{thin}}\langle\!\langle C \rangle\!\rangle_{\inf} \to A$ be the corresponding function. To prove that $i : \mathfrak{C} \to A$ is also meet-distributive, it is sufficient to show that

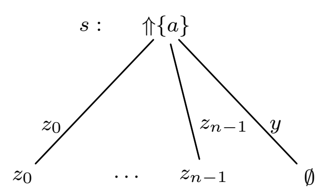
$$\mathbb{T}^? \inf(t) \leq \mathbb{T}^? \inf(t') \quad \Rightarrow \quad \inf\{\, \pi(s) \mid s \in^{\mathbb{T}^?} t \,\} \leq \inf\{\, \pi(s') \mid s' \in^{\mathbb{T}^?} t' \,\},$$

for all $t, t' \in \mathbb{T}^?\mathbb{U}C$, that is, the *kernel* of the function $\mathbb{T}^? \inf$ is included in the kernel of $\inf \circ \mathbb{U}\pi \circ \mathrm{dist}$. It then follows that there exists a unique function $\sigma : \mathbb{T}^? \langle\!\langle C \rangle\!\rangle_{\inf} \to A$ with $\sigma \circ \mathbb{T}^?\inf = \inf \circ \mathbb{U}\pi \circ \mathrm{dist}$.

Hence, fix two trees $t, t' \in \mathbb{T}^?_\xi\mathbb{U}C$ with $\mathbb{T}^? \inf(t) \leq \mathbb{T}^? \inf(t')$. We distinguish two cases. Let $\top_\zeta$ be the top element of A_ζ. First, suppose that $\top_\emptyset \in C_\emptyset$. We claim that, in this case, we have

$$C_\zeta = \{\top_\zeta\}, \quad \text{for all } \zeta \in \Xi\,.$$

Fix an element $a \in C_\zeta$ and let $z_0, \dots, z_{n-1}$ be an enumeration of ζ. We have to show that $a = \top_\zeta$. Since $\mathfrak{C}$ is semigroup like, the element a is of the form $a(\bar z) = a'(z_i)$, for some $a' \in C_{\{x\}}$ and some $i < n$. We consider the tree $s \in \mathbb{T}^{?\text{thin}}_\zeta\mathbb{U}C$ given by



(The vertex $\Uparrow\{a\}$ is considered to be an element of sort $\zeta + \{y\}$.) By meet-distributivity of $\mathfrak{C}|_{\mathbb{T}^{?\text{thin}}} \to A$, we have

$$\begin{aligned}
a = a'(z_i) = \pi(\mathbb{T}^{?\text{thin}} \inf(s)) &= \sigma_0(\mathbb{T}^{?\text{thin}} \inf(s)) \\
&= \inf\{\, \pi(r) \mid r \in^{\mathbb{T}^{?\text{thin}}} s \,\} = \inf \emptyset = \top_\zeta\,,
\end{aligned}$$

as desired. (By (the proof of) Lemma 8.7 the restriction of σ_0 to $\mathbb{T}^{?\text{thin}}C$ coincides with π.)

To prove meet-distributivity of $\mathfrak{C} \to A$, it is now sufficient to note that,

$$\begin{aligned}
&\inf\{\, \pi(r) \mid r \in^{\mathbb{T}^{?\text{thin}}} s \,\} = \inf \emptyset \quad\;\; = \top_\xi \\
\text{or} \quad &\inf\{\, \pi(r) \mid r \in^{\mathbb{T}^{?\text{thin}}} s \,\} = \inf\{\top_\xi\} = \top_\xi\,,
\end{aligned}$$

for every tree $s \in \mathbb{T}^{?\text{thin}}_\xi\mathbb{U}C$. In particular, this holds for t and t'.

It remains to consider the case where $\top_\emptyset \notin C_\emptyset$. Note that, by definition of the product of TA($\mathfrak{S}$) and Proposition 8.3, there exists, for every $s \in \mathbb{T}^? C$, some branch β such that

$$\pi(s|_\beta) = \pi(s)\,,$$

where $s|_\beta$ is the tree obtained from s by removing all vertices that do not belong to β. (This changes the sorts of the remaining vertices: every vertex of sort ξ has a label of the form $a(x)$ or c, for some $a \in C_{\{z\}}$, $c \in C_\emptyset$, and $x \in \xi$. We change the sort of such a vertex to $\{x\}$ while keeping the label. In the latter case, the variable x is arbitrary.) For each tree s, we pick one such branch β and call it the *main branch* of s.

As A is sort-wise finite, every infimum in A can be written as the infimum of a finite subset. We can therefore find a finite set B of branches of t such that

$$\inf\big\{\, \pi(s) \bigm| s \in^{\mathbb{T}^?} t \,\big\} = \inf\big\{\, \pi(s|_\beta) \bigm| \beta \in B,\ s \in^{\mathbb{T}^?} t,\ \beta \text{ main branch of } s \,\big\}\,,$$

$$\inf\big\{\, \pi(s') \bigm| s' \in^{\mathbb{T}^?} t' \,\big\} = \inf\big\{\, \pi(s'|_\beta) \bigm| \beta \in B,\ s' \in^{\mathbb{T}^?} t',\ \beta \text{ main branch of } s' \,\big\}\,.$$

(Note that t and t' have the same domain and, hence, the same set of branches.)

For each $v \in \mathrm{dom}(t)$, set

$$X_v := \big\{\, c \in C \bigm| c \geq \pi(s),\ s \in^{\mathbb{T}^?} t|_v \,\big\}\,.$$

Let r and r' be the trees obtained from, respectively, t and t' by replacing every subtree whose root v does not lie on a branch from B by a leaf with label X_v (the same label in r and r'). As B is finite, the trees r and r' are thin. By meet-distributivity of i_0, it therefore follows that

$$\sigma_0(\mathbb{T}^{?\mathrm{thin}} \inf(r)) = \inf\big\{\, \pi(s) \bigm| s \in^{\mathbb{T}^{?\mathrm{thin}}} r \,\big\}\,,$$

$$\sigma_0(\mathbb{T}^{?\mathrm{thin}} \inf(r')) = \inf\big\{\, \pi(s') \bigm| s' \in^{\mathbb{T}^{?\mathrm{thin}}} r' \,\big\}\,.$$

Furthermore, by construction of r and r',

$$\mathbb{T}^? \inf(t) \leq \mathbb{T}^? \inf(t') \quad \text{implies} \quad \mathbb{T}^{?\mathrm{thin}} \inf(r) \leq \mathbb{T}^{?\mathrm{thin}} \inf(r')\,.$$

It follows that

$$\begin{aligned}
&\inf\{\, \pi(s) \mid s \in^{\mathbb{T}^?} t \,\}\\
&\quad = \inf\{\, \pi(s) \mid s \in^{\mathbb{T}^{?\mathrm{thin}}} r \,\}\\
&\quad = \sigma_0(\mathbb{T}^{?\mathrm{thin}} \inf(r))\\
&\quad \leq \sigma_0(\mathbb{T}^{?\mathrm{thin}} \inf(r'))\\
&\quad = \inf\{\, \pi(s') \mid s' \in^{\mathbb{T}^{?\mathrm{thin}}} r' \,\}\\
&\quad \leq \inf\big\{\, \pi(s'|_\beta) \bigm| \beta \in B,\ s' \in^{\mathbb{T}^{?\mathrm{thin}}} r',\ \beta \text{ main branch of } s' \,\big\}\\
&\quad \leq \inf\big\{\, \pi(s'|_\beta) \bigm| \beta \in B,\ s' \in^{\mathbb{T}^?} t',\ \beta \text{ main branch of } s' \,\big\}\\
&\quad = \inf\{\, \pi(s') \mid s' \in^{\mathbb{T}^?} t' \,\}\,.
\end{aligned}$$

For the first step above, note that associativity of the product implies that

$$\begin{aligned}
&\inf\big\{\, \pi(s) \bigm| s \in^{\mathbb{T}^?} t,\ \text{the main branch of } s \text{ contains } v \,\big\}\\
=\ &\inf\big\{\, \pi(s) \bigm| s \in^{\mathbb{T}^?} r,\ \text{the main branch of } s \text{ contains } v \,\big\}\,,
\end{aligned}$$

for every vertex v that is replaced in r by a constant. For the sixth step, we have to show that every $s' \in^{\mathbb{T}^?} t'$ induces some $s'' \in^{\mathbb{T}^{?\mathrm{thin}}} r'$. For a contradiction, suppose otherwise. Then there must be some leaf v of r' with $X_v = \emptyset$. By definition of X_v, it follows that we can find a vertex $u \in \mathrm{dom}(t) \setminus \mathrm{dom}(r)$ in the subtree attached at v such that $t(u) = \emptyset$. Hence, $\inf t(u) \leq \inf t'(u)$ implies that $\inf t'(u) = \top$. Since $\top \notin C$, it follows that $t'(u) = \emptyset$. A contradiction to the fact that $s' \in^{\mathbb{T}^?} t'$. □

Theorem 8.10. *Every deterministic* $\mathbb{T}^{?\mathrm{thin}}$*-algebra has a unique meet-distributive* $\mathbb{T}^?$*-expansion.*

Proof. Let $\mathfrak{A} = \langle A, \pi \rangle$ be a deterministic $\mathbb{T}^{?\mathrm{thin}}$-algebra and let $\mathfrak{C} \subseteq \mathfrak{A}$ be the corresponding semigroup-like subalgebra. We can use Lemma 8.4 to find a unique $\mathbb{T}^?$-expansion $\mathfrak{C}_+$ of $\mathfrak{C}$. By Lemma 8.9, the inclusion $\mathfrak{C}_+ \to A$ is meet-distributive. Consequently, we can use Lemma 8.8 to find a unique meet-distributive algebra $\mathfrak{A}_+ = \langle A, \pi_+ \rangle$ with universe A that contains $\mathfrak{C}_+$ as a subalgebra.

It therefore remains to prove that $\mathfrak{A}$ is the $\mathbb{T}^{?\mathrm{thin}}$-reduct of this algebra $\mathfrak{A}_+$. Hence, let $t \in \mathbb{T}^{?\mathrm{thin}}A$ and fix a tree $T \in \mathbb{T}^{?\mathrm{thin}}\mathbb{U}C$ such that $t = \mathbb{T}^{?\mathrm{thin}} \inf(T)$. By meet-distributivity and the fact that the products π and π_+ agree on trees in $\mathbb{T}^{?\mathrm{thin}}C$, it follows that

$$\pi_+(t) = \inf {\Uparrow}\{\, \pi_+(s) \mid s \in^{\mathbb{T}^?} T \,\} = \inf {\Uparrow}\{\, \pi(s) \mid s \in^{\mathbb{T}^?} T \,\} = \pi(t)\,,$$

as desired. □

Corollary 8.11. *Every deterministic* $\mathbb{T}^?$*-algebra is uniquely determined by its* $\mathbb{T}^{?\mathrm{thin}}$*-reduct.*

We can generalise deterministic algebras by also allowing joins. The resulting algebras are called *branch-continuous.* They were introduced in [Blub] as an algebraic analogue to tree automata.

Definition 8.12. Let $\mathbb{T}^{?\mathrm{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$. A $\mathbb{T}^0$-algebra $\mathfrak{A}$ is *branch-continuous* if it is join-distributive and it has a deterministic subalgebra $\mathfrak{C}$ such that C forms a set of join-generators of $\mathfrak{A}$ and the inclusion $\mathfrak{C} \to A$ is meet-distributive. ⌟

Example. A typical example of a branch-continuous algebra consists of an algebra of (profiles of) certain games. A *regular game* is played on a directed graph the form $\mathfrak{G} = \langle V_{\mathrm{I}}, V_{\mathrm{II}}, E, \lambda, \mu, v_0 \rangle$ where the set $V = V_{\mathrm{I}} + V_{\mathrm{II}}$ of vertices is divided into two parts, one for each player, and the vertices and edges are labelled by elements of some finite ω-semigroup $\mathfrak{S} = \langle S, S_\omega \rangle$. This labelling is given by the functions

$$\lambda : E \to S \quad \text{and} \quad \mu : V \to \mathscr{P}(S_\omega)\,.$$

We assume that $\mu(v) \neq \emptyset$, for every leaf v. The vertex $v_0 \in V$ denotes the initial position of the game.

Since we want to compose games we allow some of the leaves of $\mathfrak{G}$ to be labelled by (distinct) variables. For such leaves v, we assume that $\mu(v) = \emptyset$.

The game is played between two players, Player I and Player II, and proceeds as follows. It starts in the position v_0. If the game has reached some position $v \in V$, the player to whom v belongs chooses either some semigroup element $c \in \mu(v)$ or some outgoing edge $v \to u$. In the first case the game terminates, otherwise it continues in position u. It follows that each play of the game produces a finite or infinite path starting at the root. The labelling of this path is of one of the following forms.

$$a_0 a_1 a_2 \cdots\,, \quad a_0 \cdots a_{n-1} c\,, \quad a_0 \cdots a_{n-1} x\,,$$

where $a_i \in S$, $c \in S_\omega$, and x is one of the variables. Each such sequence can be multiplied to either an element of S_ω or an element of the form $a(x)$ with $a \in S$ and x a variable. We call this product the *outcome* of the play.

The set of all games (over some fixed ω-semigroup $\mathfrak{S}$) forms a $\mathbb{T}^?$-algebra. But here we are interested in a finitary quotient of this algebra. When we are only interested in the possible outcomes of a game and not in its game graph, we can represent each game $\mathfrak{G}$ as a term of the form

$$\sup_{\sigma_{\mathrm{I}}} \inf_{\sigma_{\mathrm{II}}} a_{\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}}\,,$$

where σ_{I} and σ_{II} range over all strategies for the respective player and $a_{\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}}$ is the outcome of the game when both players play according to the indicated strategies. Let us call such a term the *profile* of the game $\mathfrak{G}$.

The set of all possible profiles (again, for some fixed ω-semigroup $\mathfrak{S}$), forms a $\mathbb{T}^?$-algebra $\mathfrak{A}$ which is branch-continuous. The elements of the form $a_{\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}}$ form a semigroup-like subalgebra, those of the form $\inf_{\sigma_{\mathrm{II}}} a_{\sigma_{\mathrm{I}},\sigma_{\mathrm{II}}}$ form a deterministic subalgebra, and every element of A is a join of such elements. ⌟

Using join-distributivity and meet-distributivity, one can show that a product $\pi(t)$ in a branch-continuous algebra can be computed by taking a join over meets over products along single branches of t (see [Blub] for details). In particular, a product of this form is MSO-definable. Together with the translation of automata into branch-continuous $\mathbb{T}^?$-algebras, this leads to the following two results from [Blub] (Proposition 4.37 and Theorem 4.42, respectively).

Proposition 8.13. *Every finitary branch-continuous $\mathbb{T}^?$-algebra is* MSO*-definable.*

Theorem 8.14. *A language $K \subseteq \mathbb{T}^?\Sigma$ is regular if, and only if, it is recognised by a morphism into a finitary branch-continuous $\mathbb{T}^?$-algebra.*

The class of branch-continuous algebras is a proper subclass of the one of MSO-definable algebras. Both classes can play a similar role in language theory. The reason we usually work with MSO-definable algebras instead of branch-continuous ones is that the latter do not form a pseudo-variety: the class of branch-continuous algebras is not closed under finitely-generated subalgebras. In particular, syntactic algebras are usually not branch-continuous. Here, we are more interested in the fact that branch-continuous algebras have unique branch-continuous expansions, although we can only prove uniqueness, not existance, since we are missing an analogue of Lemma 8.9. The proof makes use of the following observation.

Lemma 8.15. *Let $\mathbb{T}^{?\mathrm{thin}} \subseteq \mathbb{T}^0 \subseteq \mathbb{T}^?$. Every finitary branch-continuous $\mathbb{T}^0$-algebra $\mathfrak{A}$ has a least deterministic subalgebra $\mathfrak{C}$ such that C forms a set of join-generators of $\mathfrak{A}$ and the inclusion $\mathfrak{C} \to A$ is meet-distributive.*

Proof. Let $I \subseteq A$ be the set of all *join-irreducible* elements of A, i.e., elements that cannot be expressed as a (possibly infinite) join of strictly smaller elements. Then I is contained in every set of join-generators of A. Furthermore, since A is sort-wise finite, I forms a set of join-generators of A. (Given an element $a \in A$, one can show by induction on the number of elements strictly smaller than a that a is a join of elements of I.)

As $\mathfrak{A}$ is branch-continuous, there exists a deterministic subalgebra $\mathfrak{D} \subseteq \mathfrak{A}$ such that D forms a set of join-generators of A and the inclusion $D \to A$ is meet-distributive. Note that $I \subseteq D$ by the above remark. Let U be the closure of $I_{<2} := I_\emptyset \cup I_{\{z\}}$ under the product of $\mathfrak{A}$ and let C be the closure of U under meets. We claim that the set C induces the desired subalgebra of $\mathfrak{A}$.

We start by proving that C is a set of join-generators of $\mathfrak{A}$. To do so it is sufficient to show that $I \subseteq C$. Hence, let $a \in I$. Since $I \subseteq D$, it follows that $a \in D$. As $\mathfrak{D}$ is deterministic, we can write a as a meet of elements of some semigroup-like subalgebra of $\mathfrak{D}$. Furthermore, each element of this subalgebra can be written as a product of elements of arity at most 1. Hence,

$$a = \inf_{i<n} \pi(t_i)\,, \quad \text{for some } t_i \in \mathbb{T}^0 D_{<2}\,.$$

As $D_{<2} \subseteq \langle\!\langle I_{<2} \rangle\!\rangle_{\text{sup}}$, we can find trees $T_i \in \mathbb{T}^0 \mathscr{P}(I_{<2})$ such that

$$t_i = \mathbb{T}^0 \sup(T_i)\,.$$

Hence,

$$\begin{aligned} a &= \inf_{i<n} \pi(\mathbb{T}^0 \sup(T_i)) \\ &= \inf_{i<n} \sup\{\, \pi(s) \mid s \in^{\mathbb{T}^0} T_i \,\} \\ &= \sup_f \inf_{i<n} \pi(f(i))\,, \end{aligned}$$

where f ranges over all functions $f : [n] \to \mathbb{T}^0 I_{<2}$ such that $f(i) \in^{\mathbb{T}^0} T_i$, for all $i < n$. By join-irreducibility of $a \in I$, it follows that

$$\begin{aligned} a &= \inf_{i<n} \pi(f(i))\,, && \text{for some } f\,, \\ &= \inf_{i<n} \pi(s_i)\,, && \text{for some } s_i \in^{\mathbb{T}^0} T_i\,. \end{aligned}$$

Since $s_i \in \mathbb{T}^0 I_{<2}$ and C is closed under meets, it follows that $a \in C$.

Next, let us prove that C does indeed induce a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$. Hence, let $t \in \mathbb{T}^0 C$. Then there exists a tree $T \in \mathbb{T}^0 \mathscr{P}(U)$ such that

$$t = \mathbb{T}^0 \inf(T)\,.$$

Since $T \in \mathbb{T}^0 \mathscr{P}(D)$ and $\mathfrak{D}$ is meet-distributive, it follows that

$$\pi(t) = \pi(\mathbb{T}^0 \inf(T)) = \inf\{\, \pi(s) \mid s \in^{\mathbb{T}^0} T \,\}\,.$$

Since U induces a subalgebra, it follows that $\pi(s) \in U$, for all $s \in^{\mathbb{T}^0} T$. By closure of C under meets, we therefore have $\pi(t) \in C$.

We claim that the subalgebra $\mathfrak{C}$ is deterministic and that the inclusion $\mathfrak{C} \to A$ is meet-distributive. The latter holds since $C \subseteq D$ and the inclusion $\mathfrak{D} \to A$ is meet-distributive. For the former, note that $\mathfrak{C}$ has a semigroup-like subalgebra with domain U. Furthermore, U forms a set of meet-generators of $\mathfrak{C}$. Finally, meet-distributivity of $\mathfrak{C}$ follows from the facts that $\mathfrak{C} \subseteq \mathfrak{D}$ and that $\mathfrak{D}$ is meet-distributive.

It remains to show $\mathfrak{C}$ is the least subalgebra with the above properties. Since I is contained in every set of join-generators, it follows that U is contained in every subalgebra of $\mathfrak{A}$ that forms a set of join-generators. Hence, C is contained in every subalgebra of $\mathfrak{A}$ that is closed under meets and that forms a set of join-generators of $\mathfrak{A}$. □

Theorem 8.16. *Every finitary branch-continuous $\mathbb{T}^?$-algebra is uniquely determined by its $\mathbb{T}^{?\text{thin}}$-reduct.*

Proof. Let $\mathfrak{A} = \langle A, \pi \rangle$ and $\mathfrak{A}' = \langle A, \pi' \rangle$ be two branch-continuous $\mathbb{T}^?$-algebras with the same $\mathbb{T}^{?\text{thin}}$-reduct. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be the $\mathbb{T}^{?\text{thin}}$-reduct of the corresponding deterministic subalgebras. Then $\mathfrak{C}$ and $\mathfrak{C}'$ are deterministic subalgebras of $\mathfrak{A}|_{\mathbb{T}^{?\text{thin}}}$ such that the universes C and C' form sets of join-generators of A and the inclusions $\mathfrak{C} \to A$ and $\mathfrak{C}' \to A$ are meet-distributive. By Lemma 8.15, there exists a deterministic subalgebra $\mathfrak{C}_0 \subseteq \mathfrak{A}$ such that C_0 forms a set of join-generators of $\mathfrak{A}$, the inclusion $\mathfrak{C}_0 \to A$ is meet-distributive, and such that $C_0 \subseteq C, C'$. Note that, by Theorem 8.10, the restrictions of π and π' to $\mathbb{T}^? C_0$ coincide. For $t \in \mathbb{T}^? A$, it therefore follows by join-distributivity of $\mathfrak{A}$ and $\mathfrak{A}'$ that

$$\begin{aligned} \pi(t) &= \sup \{ \pi(s) \mid s \leq t,\ s \in \mathbb{T}^? C_0 \} \\ &= \sup \{ \pi'(s) \mid s \leq t,\ s \in \mathbb{T}^? C_0 \} = \pi'(t) . \end{aligned}$$

□

Example. In Lemma 7.1, we have constructed an MSO-definable $\mathbb{T}^{?\text{thin}}$-algebra $\mathfrak{A}$ with two different MSO-definable $\mathbb{T}$-expansions $\mathfrak{A}_0$ and $\mathfrak{A}_1$. It is straightforward to check that $\mathfrak{A}$ and $\mathfrak{A}_0$ are branch-continuous. The corresponding ω-semigroup is $\mathfrak{S} := \langle S, S_\omega \rangle$ where $S = \{0, 1\}$, $S_\omega = \{0, 1\}$ and every product is just the minimum. The algebra $\mathfrak{A}_1$ on the other hand is not branch-continuous. While it is meet-distributive and join-distributive, it is not generated (as in the definition of branch-continuity) by a semigroup-like subalgebra. ⌟

9. Conclusion

We have presented several approaches to the expansion problem for tree algebras. Our results suggest that there exists a dividing line between thin trees and non-thin ones. For classes of thin trees, we can use the existing combinatorial theory for ω-semigroups. As a consequence we were able to obtain the results in Section 5.1, which can be considered to completely solve the expansion problem for such classes. For non-thin trees on the other hand, our results are much more fragmentary. We were able to solve the problem for the inclusion $\mathbb{T}^{\text{reg}} \subseteq \mathbb{T}$ (at least for MSO-definable algebra), but the more important inclusions $\mathbb{T}^{\text{thin}} \subseteq \mathbb{T}$ and $\mathbb{T}^{\text{wilke}} \subseteq \mathbb{T}^{\text{reg}}$ had to be left open. Theorems 5.27 and 8.16 might be seen as an indication the MSO-definable algebras are, in a certain sense, ‘controlled’ by their $\mathbb{T}^{\text{thin}}$-reducts. But note that we have shown in Lemma 7.1 that, in general, MSO-definable algebras are not uniquely determined by their $\mathbb{T}^{\text{thin}}$-reduct. An important task that we have to leave open is a classification of all $\mathbb{T}$-extensions of a given MSO-definable $\mathbb{T}^{\text{thin}}$-algebra.

In general, the methods we have developed seem to work somewhat well if there exists a unique expansion (or at least a unique expansion with a certain property, like a unique MSO-definable expansion, or a unique branch-continuous one), but there is currently no approach to prove the existence of several expansions.

Promising next steps towards further progress seem to include

- trying to generalise some of our existing tools from thin trees to general ones; and/or
- finding counterexamples delineating the parameter space where such generalisations do not exist any more.

As a problem to work on let us mention a generalisation of Simon’s Factorisation Tree Theorem to trees. But this seems to be a very hard problem. There are two technical frameworks that might be of help here: (I) one can try to flesh out the theory of Green’s relations for tree algebras, and (II) one can try to make use of Tame Congruence Theory [HM88]. While developing these two theories for tree algebras is not that difficult, it is not at all obvious how to apply them to concrete problems, like the one mentioned above.

References

[AKY08] P. A. Abdulla, P. Krcál, and W. Yi. R-Automata. In F. van Breugel and M. Chechik, editors, *CONCUR 2008 – Concurrency Theory, 19th International Conference, CONCUR 2008, Toronto, Canada, August 19–22, 2008. Proceedings*, volume 5201 of *Lecture Notes in Computer Science*, pages 67–81. Springer, 2008.

[BCPS19] M. Bojańczyk, F. Cavallari, T. Place, and M. Skrzypczak. Regular Tree Languages in Low Levels of the Wadge Hierarchy. *Log. Methods Comput. Sci.*, 15:27:1–27:61, 2019.

[BI09] M. Bojańczyk and T. Idziaszek. Algebra for Infinite Forests with an Application to the Temporal Logic EF. In *Proc. 20th International Conference on Concurrency Theory, CONCUR, LNCS 5710*, pages 131–145, 2009.

[BIS13] M. Bojańczyk, T. Idziaszek, and M. Skrzypczak. Regular Languages of Thin Trees. In *Proc. 30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013*, pages 562–573, 2013.

[BK19] M. Bojańczyk and B. Klin. A Non-Regular Language of Infinite Trees That is Recognizable by a Finite Algebra. *Logical Methods in Computer Science*, 15, 2019.

[BL69] J. R. Büchi and L. H. Landweber. Solving Sequential Conditions by Finite-State Strategies. *Trans. Amer. Math. Soc.*, 139:367–378, 1969.

[Blua] A. Blumensath. Abstract Algebraic Language Theory. unpublished notes, `https://www.fi.muni.cz/~blumens/ALT.pdf`.

[Blub] A. Blumensath. Branch-Continuous Tree Algebras. arXiv:1807.04568, unpublished.

[Blu11] A. Blumensath. Recognisability for Algebras of Infinite Trees. *Theoretical Computer Science*, 412:3463–3486, 2011.

[Blu13] A. Blumensath. An Algebraic Proof of Rabin's Tree Theorem. *Theoretical Computer Science*, 478:1–21, 2013.

[Blu20] A. Blumensath. Regular Tree Algebras. *Logical Methods in Computer Science*, 16:16:1–16:25, 2020.

[Blu21] A. Blumensath. Algebraic Language Theory for Eilenberg–Moore Algebras. *Logical Methods in Computer Science*, 17:6:1–6:60, 2021.

[Blu23] A. Blumensath. The Power-Set Operation for Tree Algebras. *Logical Methods in Computer Science*, 19:9:1–9:47, 2023.

[Boj20] M. Bojańczyk. Languages Recognises by Finite Semigroups and their generalisations to objects such as Trees and Graphs with an emphasis on definability in Monadic Second-Order Logic. lecture notes, arXiv:2008.11635, 2020.

[BS13] M. Bilkowski and M. Skrzypczak. Unambiguity and Uniformization Problems on Infinite Trees. In *Computer Science Logic 2013 (CSL 2013), CSL 2013, September 2–5, 2013, Torino, Italy*, volume 23 of *LIPIcs*, pages 81–100. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2013.

[CCP18] O. Carton, T. Colcombet, and G. Puppis. An Algebraic Approach to MSO-Definability on Countable Linear Orderings. *J. Symb. Log.*, 83:1147–1189, 2018.

[Col07] T. Colcombet. A Combinatorial Theorem for Trees. In *Proc. 34th Int. Colloquium on Automata, Languages and Programming, ICALP*, pages 901–912, 2007.

[Col21] T. Colcombet. The Factorisation Forest Theorem. In J.-É. Pin, editor, *Handbook of Automata Theory*, pages 653–693. European Mathematical Society, 2021.

[HM88] D. Hobby and R McKenzie. *The Structure of Finite Algebras.* American Mathematical Society, 1988.

[L21] C. Löding. Automata on Infinite Trees. In J.-É. Pin, editor, *Handbook of Automata Theory*, pages 265–302. European Mathematical Society, 2021.

[Mil79] K. R. Milliken. A Ramsey Theorem for Trees. *Journal of Combinatorial Theory, Series A*, 26:215–237, 1979.

[PP04] D. Perrin and J.-É. Pin. *Infinite Words – Automata, Semigroups, Logic and Games.* Elsevier, 2004.

[Pup10] G. Puppis. *Automata for Branching and Layered Temporal Structures: An Investigation into Regularities of Infinite Transition Systems*, volume 5955 of *Lecture Notes in Computer Science.* Springer, 2010.

[Sim90] I. Simon. Factorization Forests of Finite Height. *Theoretical Computer Science*, 72:65–94, 1990.

[Sim94] I. Simon. On Semigroups of Matrices over the Tropical Semiring. *RAIRO Theor. Informatics Appl.*, 28:277–294, 1994.

[Skr16] M. Skrzypczak. *Descriptive Set Theoretic Methods in Automata Theory – Decidability and Topological Complexity*, volume 9802 of *Lecture Notes in Computer Science*. Springer, 2016.

[Tho81] W. Thomas. A Combinatorial Approach to the Theory of ω-Automata. *Information and Control*, 48:261–283, 1981.

[Tho90] W. Thomas. Automata on Infinite Objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, pages 135–191. Elsevier, Amsterdam, 1990.

[Tho97] W. Thomas. Languages, Automata, and Logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, volume 3, pages 389–455. Springer, New York, 1997.