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HOMOTOPY TYPE THEORY AS A LANGUAGE FOR DIAGRAMS OF ∞-LOGOSES

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Abstract. We show that certain diagrams of ∞-logoses are reconstructed in homotopy type theory extended with some lex, accessible modalities, which enables us to use plain homotopy type theory to reason about not only a single ∞-logos but also a diagram of ∞-logoses. This also provides a higher dimensional version of Sterling's synthetic Tait computability—a type theory for higher dimensional logical relations.

1. Introduction

An ∞-*logos*, also known as an ∞-*topos* [Lur09a, AJ21][1], is an $(\infty, 1)$-category that looks like the $(\infty, 1)$-category of spaces, among other aspects of it. An ∞-logos is a place where one can do homotopy theory just as an ordinary logos is a place where one can do set-level mathematics.

Homotopy type theory [The13] is another place to do homotopy theory. It is a type theory in the style of Martin-Löf [ML75] extended by the *univalence axiom* and *higher inductive types*. The former forces types to behave like spaces rather than sets, and the latter allow us to build types representing spaces such as spheres and tori.

∞-logoses are conjectured to admit interpretations of homotopy type theory so that theorems proved in homotopy type theory can be translated in an arbitrary ∞-logos. Although the conjecture has not yet been fully solved (see, for example, [Shu19] for substantial progress), homotopy type theory has brought insight to ∞-logos theory. For example, the proof of the Blakers-Massey connectivity theorem in homotopy type theory [HFLL16] has led to a new generalized Blakers-Massey theorem that holds in an arbitrary ∞-logos [ABFJ20].

An ∞-logos, however, does not live alone. ∞-logoses are often connected by functors which are also connected by natural transformations. Plain homotopy type theory is, at first sight, not sufficient to reason about a diagram of ∞-logoses, because the actions of the functors and natural transformations are not internalized to type theory. Even worse, it is impossible to naively internalize some diagrams: some internal adjunction leads a contradiction [LOPS18]; there are only trivial internal idempotent comonads [Shu18]. While

Key words and phrases: homotopy type theory, ∞-logos, ∞-topos, oplax limit, Artin gluing, modality, synthetic Tait computability, logical relation.

This article is an extended version of [Uem23].

[1] The term ∞-logos is Anel and Joyal's terminology [AJ21] for ∞-topos considered as an algebraic structure rather than a geometric object. A morphism of ∞-logoses is always considered in the direction of the inverse image functor. We use this terminology to clarify the direction of morphisms when speaking about (co)limits of ∞-logoses.

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there is no chance of naive internalization of such interesting but problematic diagrams to plain homotopy type theory, some other diagrams can be internalized in a clever way pointed out by Shulman[2]. A minimal non-trivial example is a diagram consisting of two ∞-logoses and a lex, accessible functor between them in one direction. The two ∞-logoses are *lex, accessible localizations* of another ∞-logos obtained by the *Artin gluing* for the functor, and the functor is reconstructed by composing the inclusion from one localization and the reflector to the other. Moreover, this reconstruction is *internal* to the glued ∞-logos, because lex, accessible localizations of an ∞-logos are expected to correspond to *lex, accessible modalities* in its internal language. Hence, plain homotopy type theory as a language for the glued ∞-logos is sufficient to reason about the original diagram.

In this paper, we propose a class of shapes of diagrams of ∞-logoses for which the internal reconstruction technique explained in the previous paragraph works. We call shapes in the proposed class *mode sketches*. Our main results are summarized as follows. Let $\mathfrak{M}$ be a mode sketch.

(1) We associate to $\mathfrak{M}$ certain axioms in type theory, one of which is to postulate some lex, accessible modalities from which one can construct a diagram of ∞-logoses internally to type theory (Sections 3.1 and 3.2).
(2) We show that the $(\infty, 1)$-category of models of the axioms associated to $\mathfrak{M}$ in ∞-logoses is equivalent to the $(\infty, 1)$-category of diagrams of ∞-logoses indexed over $\mathfrak{M}$ (Theorem 6.4), where $\mathfrak{M}$ is regarded as a presentation of an $(\infty, 2)$-category. The right to left construction is given by *oplax limits*, a generalization of the Artin gluing [Wra74, Shu15].

In Sections 1.1 to 1.3 below, we explain other new results (Propositions 2.17 and 5.58, Theorems 4.4 and 5.43, and Corollary 5.55).

This paper is an extended version of our conference paper [Uem23]. The conference version presents Proposition 2.17 and proof sketches of Theorems 4.4 and 6.4. The current version includes every detail of those results. The statement of Theorem 6.4 is modified not to rely on an interpretation of type theory in ∞-logoses.

1.1. Modalities in homotopy type theory.

A *modality* in homotopy type theory [RSS20, CORS20, CR22] is a subuniverse satisfying certain conditions. A modality that is moreover lex and accessible is expected to correspond to a sub-∞-topos or a localization of a ∞-logos [ABFJ22, Ver19]. The *fracture and gluing theorem* of Rijke, Shulman, and Spitters [RSS20, Theorem 3.50] gives a construction of the join $\mathfrak{m} \vee \mathfrak{n}$ of two lex modalities $\mathfrak{m}$ and $\mathfrak{n}$ under some assumption. The join obtained by this theorem satisfies that every type A in $\mathfrak{m} \vee \mathfrak{n}$ is canonically fractured into a type $A_{\mathfrak{n}}$ in $\mathfrak{n}$ and a type family $A_{\mathfrak{m}}$ on $A_{\mathfrak{n}}$ valued in $\mathfrak{m}$, and A is reconstructed as $A \simeq \sum_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x})$.

In this paper, we improve the fracture and gluing theorem. We show that the construction of joins of lex modalities preserves accessibility as well (Proposition 2.17).

1.2. Synthetic Tait computability.

The fracture and gluing theorem is used in Sterling's *synthetic Tait computability* [SH21, Ste21]. It is a technique of working with *logical relations*, which are used in the study of type theories and programming languages, in an internal language for the Artin gluing and has applications to, for example, normalization theorems for complex type theories [SA21, Gra22]. There the fracture and gluing theorem is instantiated

[2] `https://golem.ph.utexas.edu/category/2011/11/internalizing_the_external_or.html`

by the *closed* and *open* modalities associated to a proposition. Then every type in the internal language is canonically fractured into an open type and a closed (unary, proof-relevant) relation on it which are glued back together. The internal language for the Artin gluing is thus a type theory with an indeterminate proposition in which *types are relations* and provides a synthetic method of working with logical relations.

In this paper, we relate synthetic Tait computability and mode sketches. The core axiom for synthetic Tait computability is to postulate some indeterminate propositions. We show that part of the axioms associated to a mode sketch is equivalent to postulating a lattice of propositions (Theorem 4.4).

Mode sketches thus provide a synthetic method of working with logical relations that is alternative to and generalizes synthetic Tait computability. This is also natural from Shulman's point of view [Shu15] that interpretations of type theory in oplax limits are generalized logical relations. Since we work in homotopy type theory, what we get is actually *higher-dimensional logical relations*, and our primary application of mode sketches in upcoming paper(s) [Uem22] will be normalization for ∞-*type theories* introduced by Nguyen and Uemura [NU25] as a higher-dimensional generalization of type theories.

1.3. **Oplax limits of** $(\infty, 1)$**-categories.** *Oplax limits* are special $(\infty, 2)$-categorical limits and analogous to oplax limits in 2-category theory [Str76, Kel89, JY21]. Oplax limits indexed over $(\infty, 1)$-categories are studied by Gepner, Haugseng, and Nikolaus [GHN17] and generalized to arbitrary indexing $(\infty, 2)$-categories by Gagna, Harpaz, and Lanari [GHL21a].

Oplax limits of ∞-logoses are of our interest. Wraith [Wra74] shows that the oplax limit of a diagram of (elementary) logoses and lex functors is a logos. Lurie [Lur09a, Proposition 6.3.2.3] shows that the conical limit of a diagram of ∞-logoses is an ∞-logos.

The oplax limit of a diagram classifies *oplax natural transformations* from a constant diagram to the given diagram just as a (conical) limit classifies natural transformations from a constant diagram. It is known [Lur09b, Theorem 3.8.1] that natural transformations between diagrams of $(\infty, 1)$-categories correspond to fibered functors between the $(\infty, 2)$-categories of elements of the diagrams. Oplax natural transformations between diagrams of $(\infty, 1)$-categories are to correspond to not necessarily fibered functors between the $(\infty, 2)$-categories of elements. Some special cases of this have already been proved: Haugseng et al. [HHLN23, Theorem E] show the case when the diagrams are indexed over an $(\infty, 1)$-category; Gagna, Harpaz, and Lanari [GHL21a, Corollary 4.4.3] show the case when the domain is constant on the point.

The *mate correspondence* is a useful source of oplax natural transformations. In the 2-categorical case [Str72], it asserts that given two diagrams F and G of categories, oplax natural transformations $F \to G$ that are point-wise left adjoints correspond to lax natural transformations $G \to F$ that are point-wise right adjoints. An $(\infty, 2)$-categorical version is proved by Haugseng et al. [HHLN23, Corollary F] in the form of an equivalence between the $(\infty, 1)$-categories of oplax/lax natural transformations in the special case when the diagrams are indexed over an $(\infty, 1)$-category.

In this paper, we show some new results on oplax limits of $(\infty, 1)$-categories. All of them are consequences of results in the literature, but it is worth stating them explicitly. We prove an ∞-analogue of the result of Wraith [Wra74]: the oplax limit of a diagram of ∞-logoses and lex, accessible functors is an ∞-logos (Theorem 5.43). We show that oplax natural transformations between diagrams of $(\infty, 1)$-categories correspond to arbitrary functors between the $(\infty, 2)$-categories of elements (Corollary 5.55) by reducing it to the special cases proved by Haugseng et al. [HHLN23] and Gagna, Harpaz, and Lanari [GHL21a]. We show the mate correspondence for diagrams indexed over an arbitrary $(\infty, 2)$-category in the form of an equivalence between the spaces of oplax/lax natural transformations (Proposition 5.58).

1.4. **Organization.** The paper is split into two parts. The first part (Sections 2 to 4) provides the theory of mode sketches internally to type theory. The second part (Sections 5 and 6) is devoted to semantics of mode sketches in ∞-logoses. The two parts are independent of each other on a technical level.

In Section 2, we review the theory of modalities in homotopy type theory [RSS20]. Our focus is on the poset of lex, accessible modalities and on the open and closed modalities associated to propositions.

Sections 3 and 4 are the core of the paper. We introduce the notion of a *mode sketch* (Definition 3.5). For every mode sketch, we introduce two equivalent sets of axioms to encode a certain diagram of universes. One postulates some lex, accessible modalities while the other postulates a lattice of propositions. The open and closed modalities give a construction of the former from the latter which we show is an equivalence (Theorem 4.4). The latter is a higher dimensional analogue of Sterling's synthetic Tait computability [Ste21].

Section 5 is a preliminary section needed for the semantics of mode sketches.

Finally in Section 6, we show our main result (Theorem 6.4): for any mode sketch, the $(\infty, 1)$-category of models of the axioms associated to the mode sketch in ∞-logoses is equivalent to the $(\infty, 1)$-category of diagrams of ∞-logoses and lex, accessible functors indexed over the mode sketch.

1.5. **Related work.** An earlier version of *cohesive homotopy type theory* [SS12] uses modalities in plain homotopy type theory to internalize a series of adjunctions that arises in Lawvere's axiomatic cohesion [Law07]. However, because naive internalization of adjunctions do not work well [LOPS18, Shu18], the axiomatization is tricky and not ideal to work with. The newer version of cohesive homotopy type theory [Shu18] instead extends homotopy type theory by another layer of context and new modal operators. The resulting type theory works well for axiomatic cohesion but is complicated compared to plain homotopy type theory. It is also too optimized for axiomatic cohesion.

A more general framework for internal diagrams is *multimodal dependent type theory* [GKNB21]. It is roughly a family of type theories related to each other via modal operators and interpreted in a diagram of presheaf categories. The shape of diagram is specified directly by an arbitrary 2-category which is called a *mode theory* in this context. Our terminology "mode sketch" is chosen to mean a sketch of a mode theory. Multimodal dependent type theory is potentially an internal language for diagrams of ∞-logoses, but for this one would have to rectify not only ∞-logoses but also functors and natural transformations between them.

Our work brings back the ideas of earlier cohesive homotopy type theory. Although it might not be the best type theory, it has a lot of advantages: modalities are internal to plain homotopy type theory, and thus all results are ready to formalize in existing proof assistants; keeping type theory simple is also important in informal use of type theory in which the correctness of application of inference rules is not checked by computer; the semantics is no more complicated than the ∞-logos semantics of homotopy type theory; it also opens the door to internalization of more general diagrams in a uniform way, which is the motivation for the current work.

2. Modalities in homotopy type theory

We review the theory of *modalities* in homotopy type theory [RSS20]. In this section, we work in homotopy type theory. By *homotopy type theory* we mean dependent type theory with (dependent) function types, (dependent) pair types, a unit type, identity types (without equality reflection), at least two univalent universes $\mathcal{U} : \Uparrow\mathcal{U}$, an empty type, pushouts, and localizations [RSS20, Section 2.2]. Note that truncations are instances of localization. We mainly follow the HoTT Book [The13] for terminologies and notations in homotopy type theory.

A modality is in short a reflective subuniverse closed under pair types.

Definition 2.1. A *subuniverse* $\mathfrak{m}$ is a function $\mathsf{In}_{\mathfrak{m}} : \mathcal{U} \to \Uparrow\mathcal{U}$ such that $\mathsf{In}_{\mathfrak{m}}(A)$ is a proposition for all $A : \mathcal{U}$. A type A satisfying $\mathsf{In}_{\mathfrak{m}}(A)$ is called $\mathfrak{m}$-*modal*. We define a subtype $\mathcal{U}_{\mathfrak{m}} \subset \mathcal{U}$ to be $\{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\}$.

Definition 2.2. A subuniverse $\mathfrak{m}$ is *reflective* if it is equipped with functions $\bigcirc_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{m}}$ and $\eta_{\mathfrak{m}} : \prod_{A:\mathcal{U}} A \to \bigcirc_{\mathfrak{m}} A$ such that that the precomposition $\lambda f.f \circ \eta_{\mathfrak{m}}(A) : (\bigcirc_{\mathfrak{m}} A \to B) \to (A \to B)$ is an equivalence for any $B : \mathcal{U}_{\mathfrak{m}}$. Note that such a pair $(\bigcirc_{\mathfrak{m}}, \eta_{\mathfrak{m}})$ is unique.

Definition 2.3. A reflective subuniverse $\mathfrak{m}$ is a *modality* if $\mathsf{In}_{\mathfrak{m}}$ is closed under pair types, that is, for $A : \mathcal{U}$ and $B : A \to \mathcal{U}$, if $\mathsf{In}_{\mathfrak{m}}(A)$ and $\prod_{a:A} \mathsf{In}_{\mathfrak{m}}(B(a))$, then $\mathsf{In}_{\mathfrak{m}}(\sum_{a:A} B(a))$.

An important class of modalities is *accessible* modalities which are roughly modalities "presented by small data".

Definition 2.4. For types $A, B : \mathcal{U}$, we say A is *left orthogonal* to B or B is *right orthogonal* to A and write $A \perp B$ if the function $\lambda(b : B).\lambda(\_ : A).b : B \to (A \to B)$ is an equivalence. For a subuniverse $\mathfrak{m}$, we define subuniverses $\mathfrak{m}^{\perp}$ and ${}^{\perp}\mathfrak{m}$ by

$$\mathsf{In}_{\mathfrak{m}^{\perp}}(B) \equiv \prod_{A:\mathcal{U}_{\mathfrak{m}}} A \perp B$$
$$\mathsf{In}_{{}^{\perp}\mathfrak{m}}(A) \equiv \prod_{B:\mathcal{U}_{\mathfrak{m}}} A \perp B.$$

Definition 2.5. A *null generator* μ consists of $I_\mu : \mathcal{U}$ and $Z_\mu : I_\mu \to \mathcal{U}$. We write $\mathsf{NullGen}$ for the type of null generators. Given a null generator μ, we define a subuniverse $\mathfrak{Null}(\mu)$ by $\mathsf{In}_{\mathfrak{Null}(\mu)}(A) \equiv \prod_{i:I_\mu} Z_\mu(i) \perp A$. It is shown that $\mathfrak{Null}(\mu)$ is a modality using a higher inductive type [RSS20, Theorem 2.19]. A modality $\mathfrak{m}$ is *accessible* if it is in the image of $\mathfrak{Null}$, that is, $\| \sum_{\mu:\mathsf{NullGen}} \mathfrak{m} = \mathfrak{Null}(\mu)\|$.

Another important class of modalities is *lex* modalities.

Definition 2.6. For a modality $\mathfrak{m}$, a type $A : \mathcal{U}$ is $\mathfrak{m}$-*connected* if $\bigcirc_{\mathfrak{m}} A$ is contractible. This is equivalent to $\mathsf{In}_{{}^{\perp}\mathfrak{m}}(A)$ by [RSS20, Corollary 1.37].

Definition 2.7. A modality $\mathfrak{m}$ is *lex* if for any $\mathfrak{m}$-connected type $A : \mathcal{U}$, the identity type $a_1 = a_2$ is $\mathfrak{m}$-connected for any $a_1, a_2 : A$.

Modalities that are both lex and accessible are of particular importance because they correspond to subtoposes of an ∞-topos under the interpretation of types as sheaves on the ∞-topos. From now on, we are mostly interested lex, accessible modalities, so we give them a short name.

Terminology 2.8. LAM is an acronym for lex, accessible modality.

Fundamental examples of LAMs are *open* and *closed* modalities which correspond to open and closed, respectively, subtoposes.

Construction 2.9. Let P be a proposition. We define the *open modality* $\mathfrak{Op}(P)$ by $\bigcirc_{\mathfrak{Op}(P)} A \equiv (P \to A)$ and $\eta_{\mathfrak{Op}(P)}(A, a) \equiv \lambda\_.a$. It is lex and accessible by [RSS20, Example 2.24 and Example 3.10]. We also define the *closed modality* $\mathfrak{Cl}(P)$ by $\mathsf{In}_{\mathfrak{Cl}(P)}(A) \equiv (P \to \mathsf{IsContr}(A))$. It is lex and accessible by [RSS20, Example 2.25 and Example 3.14]. Note that $\mathfrak{Cl}(P) = {}^{\perp}\mathfrak{Op}(P)$ [RSS20, Example 1.31].

2.1. The poset of lex, accessible modalities.

Let SU denote the poset of subuniverses where $\mathfrak{m} \le \mathfrak{n}$ if $\mathsf{In}_{\mathfrak{m}}(A) \to \mathsf{In}_{\mathfrak{n}}(A)$ for every $A : \mathcal{U}$. We have the full subposets of SU

$$\mathsf{RSU} \supset \mathsf{Mdl} \supset \mathsf{AccMdl} \supset \mathsf{LAM}$$

consisting of reflective subuniverses, modalities, accessible modalities, and lex, accessible modalities, respectively. We also have the full subposet $\mathsf{Lex} \subset \mathsf{Mdl}$ of lex modalities. By definition, $\mathsf{LAM} = \mathsf{Lex} \cap \mathsf{AccMdl}$. We study the poset LAM in more detail.

Definition 2.10 [RSS20, Theorem 3.25]**.** Let $I : \mathcal{U}$ and $\mathfrak{m} : I \to \mathsf{LAM}$. A *canonical meet* $\bigwedge_{i:I} \mathfrak{m}(i)$ is a LAM that is the meet of $\mathfrak{m}(i)$'s in SU. A *canonical join* $\bigvee_{i:I} \mathfrak{m}(i)$ is a LAM satisfying that a type $A : \mathcal{U}$ is $(\bigvee_{i:I} \mathfrak{m}(i))$-connected if and only if it is $\mathfrak{m}(i)$-connected for all $i : I$. Note that a canonical join is the join in Mdl.

Example 2.11. The *top modality* $\mathfrak{Top}$, for which all the types are modal, is the canonical meet of the empty family. The *bottom modality* $\mathfrak{Bot}$, for which only the contractible types are modal, is the canonical join of the empty family.

The canonical meet of an arbitrary family of LAMs exists [RSS20, Theorem 3.29 and Remark 3.23]. Canonical joins are less understood than canonical meets. One important case when canonical joins exist and can be computed is the following.

Definition 2.12. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. $\mathfrak{n}$ is *strongly disjoint from* $\mathfrak{m}$ if any $\mathfrak{m}$-modal type is $\mathfrak{n}$-connected or equivalently if $\mathfrak{m} \le {}^{\perp}\mathfrak{n}$ in SU.

Proposition 2.13 (Fracture and gluing theorem)**.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \le {}^{\perp}\mathfrak{n}$.*
(1) *The canonical join $\mathfrak{m} \vee \mathfrak{n}$ exists.*
(2) *A type A is $(\mathfrak{m} \vee \mathfrak{n})$-modal if and only if the function $\eta_{\mathfrak{n}}(A) : A \to \bigcirc_{\mathfrak{n}} A$ has $\mathfrak{m}$-modal fibers.*
(3) $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \simeq \sum_{A:\mathcal{U}_{\mathfrak{m}}} \sum_{B:\mathcal{U}_{\mathfrak{n}}} A \to \bigcirc_{\mathfrak{m}}^{\mathfrak{n}} B$.

In the special case when $\mathfrak{m} = {}^{\perp}\mathfrak{n}$, we have $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Top}$.

Proof. All but the accessibility of $\mathfrak{m} \vee \mathfrak{n}$ are proved by Rijke, Shulman, and Spitters [RSS20, Theorem 3.50]. We will prove the accessibility of $\mathfrak{m} \vee \mathfrak{n}$ in Proposition 2.17 below using an open modality. □

The distributive law holds in some special cases.

Fact 2.14 [RSS20, Theorem 3.30]**.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. If $\bigcirc_{\mathfrak{m}}$ preserves $\mathfrak{n}$-modal types, then $\bigcirc_{\mathfrak{m}\wedge\mathfrak{n}} A = \bigcirc_{\mathfrak{m}} \bigcirc_{\mathfrak{n}} A$.

Proposition 2.15. *Let $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$ be LAMs. Suppose $\mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_3$ and $\mathfrak{m}_2 \leq {}^{\perp}\mathfrak{m}_3$. Then $(\mathfrak{m}_1 \vee \mathfrak{m}_3) \wedge (\mathfrak{m}_2 \vee \mathfrak{m}_3) = (\mathfrak{m}_1 \wedge \mathfrak{m}_2) \vee \mathfrak{m}_3$.*

Proof. Note that the right side exists as $\mathfrak{m}_1 \wedge \mathfrak{m}_2 \leq \mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_3$. Let $A : \mathcal{U}$. By Proposition 2.13, A is $((\mathfrak{m}_1 \vee \mathfrak{m}_3) \wedge (\mathfrak{m}_2 \vee \mathfrak{m}_3))$-modal if and only if fibers of $\eta_{\mathfrak{m}_3}(A)$ are both $\mathfrak{m}_1$-modal and $\mathfrak{m}_2$-modal, but this is equivalent to that A is $(\mathfrak{m}_1 \wedge \mathfrak{m}_2) \vee \mathfrak{m}_3$-modal again by Proposition 2.13. □

Proposition 2.16. *Let $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$ be LAMs. Suppose that $\mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_2$ and that $\bigcirc_{\mathfrak{m}_2}$ preserves $\mathfrak{m}_3$-modal types. Then $(\mathfrak{m}_1 \vee \mathfrak{m}_2) \wedge \mathfrak{m}_3 = (\mathfrak{m}_1 \wedge \mathfrak{m}_3) \vee (\mathfrak{m}_2 \wedge \mathfrak{m}_3)$.*

Proof. Note that the right side exists as $\mathfrak{m}_1 \wedge \mathfrak{m}_3 \leq \mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_2 \leq {}^{\perp}(\mathfrak{m}_2 \wedge \mathfrak{m}_3)$. Let A be a $((\mathfrak{m}_1 \vee \mathfrak{m}_2) \wedge \mathfrak{m}_3)$-modal type. We show that A is $((\mathfrak{m}_1 \wedge \mathfrak{m}_3) \vee (\mathfrak{m}_2 \wedge \mathfrak{m}_3))$-modal, which is by Proposition 2.13 equivalent to that $\eta_{\mathfrak{m}_2 \wedge \mathfrak{m}_3}(A) : A \to \bigcirc_{\mathfrak{m}_2 \wedge \mathfrak{m}_3} A$ has $(\mathfrak{m}_1 \wedge \mathfrak{m}_3)$-modal fibers. Since $\bigcirc_{\mathfrak{m}_2}$ preserves $\mathfrak{m}_3$-modal types and since A is $\mathfrak{m}_3$-modal, $\eta_{\mathfrak{m}_2 \wedge \mathfrak{m}_3}(A)$ is equivalent to $\eta_{\mathfrak{m}_2}(A) : A \to \bigcirc_{\mathfrak{m}_2} A$ by Fact 2.14. The fibers of $\eta_{\mathfrak{m}_2}(A)$ are $\mathfrak{m}_1$-modal by Proposition 2.13 and $\mathfrak{m}_3$-modal since both domain and codomain are $\mathfrak{m}_3$-modal. □

2.2. **Accessibility of the canonical join.**

Let us fill the gap in the proof of Proposition 2.13.

Proposition 2.17. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Then the canonical join $\mathfrak{m} \vee \mathfrak{n}$ (in Lex) is accessible.*

We have to find a null generator for $\mathfrak{m} \vee \mathfrak{n}$. A natural guess is the following.

Construction 2.18. Let μ and ν be null generators. We define a null generator $\mu \star \nu$ by $I_{\mu\star\nu} \equiv I_\mu \times I_\nu$ and $Z_{\mu\star\nu}(i, j) \equiv Z_\mu(i) \star Z_\nu(j) \equiv Z_\mu(i) +_{Z_\mu(i)\times Z_\nu(j)} Z_\nu(j)$.

Lemma 2.19. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs, and let μ and ν be null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Then $Z_{\mu\star\nu}(i, j)$ is both $\mathfrak{m}$-connected and $\mathfrak{n}$-connected for all $i : I_\mu$ and $j : I_\nu$.*

Proof. Recall that a function is $\mathfrak{m}$-*connected* if its fibers are $\mathfrak{m}$-connected and that the class of $\mathfrak{m}$-connected functions is the left class of a (stable) orthogonal factorization system [RSS20, Theorem 1.34]. Then the claim follows by the pushout stability and the right cancellability of $\mathfrak{m}$-connected and $\mathfrak{n}$-connected functions. □

Lemma 2.19 shows $\mathfrak{m} \vee \mathfrak{n} \leq \mathfrak{Null}(\mu \star \nu)$ for arbitrary accessible modalities $\mathfrak{m}$ and $\mathfrak{n}$ and for arbitrary choices of μ and ν. We know neither if the other direction holds in general for some choices of μ and ν nor if $\mathfrak{Null}(\mu \star \nu)$ is independent of μ and ν. Note that Finster [Fin] observed that $\mathfrak{Null}(\mu \star \nu)$ is lex whenever $\mathfrak{Null}(\mu)$ and $\mathfrak{Null}(\nu)$ are lex. In the special case when $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$, the idea of the proof of $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Null}(\mu \star \nu)$ is to show that $\mathfrak{n}$ is an *open modality* within the subuniverse of $\mathfrak{Null}(\mu \star \nu)$-modal types.

Lemma 2.20. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Then $\mathfrak{n} \leq \mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$.*

Proof. This is because $\bigcirc_{\mathfrak{m}} \mathbf{0}$ is $\mathfrak{n}$-connected by assumption. □

Lemma 2.21. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Suppose that μ and ν are null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively, and that μ admits a function $f : \bigcirc_{\mathfrak{m}} \mathbf{0} \to I_\mu$ such that $\mathbf{0} \simeq Z_\mu(f(i))$ for all $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$. Then $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A$ is $\mathfrak{n}$-modal for any $\mathfrak{Null}(\mu \star \nu)$-modal type A. Consequently, the canonical function $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A \to \bigcirc_{\mathfrak{n}} A$ induced by Lemma 2.20 is an equivalence for any $\mathfrak{Null}(\mu \star \nu)$-modal type A.*

Proof. We show that $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A \equiv (\bigcirc_{\mathfrak{m}} \mathbf{0} \to A)$ is $\mathfrak{n}$-modal. Since ν is a null generator for $\mathfrak{n}$, it suffices to show that $Z_\nu(j) \perp (\bigcirc_{\mathfrak{m}} \mathbf{0} \to A)$ for all $j : I_\nu$. This is equivalent to that $Z_\nu(j) \perp A$ under an assumption $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$. This holds since $Z_\nu(j) \simeq \mathbf{0} \star Z_\nu(j) \simeq Z_\mu(f(i)) \star Z_\nu(j)$ and since A is $\mathfrak{Null}(\mu \star \nu)$-modal. □

Lemma 2.22. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^\perp\mathfrak{n}$. Suppose that μ and ν are null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively, and that ν has an element $j : I_\nu$ such that $Z_\nu(j) \simeq \bigcirc_{\mathfrak{m}} \mathbf{0}$. Then, if a type A is $\mathfrak{Null}(\mu \star \nu)$-modal and $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected, then it is $\mathfrak{m}$-modal.*

Proof. We show that $Z_\mu(i) \perp A$ for all $i : I_\mu$. By the definition of $\star$, we have the following pullback square.

$$\begin{array}{ccc} (Z_\mu(i) \star \bigcirc_{\mathfrak{m}} \mathbf{0} \to A) & \xrightarrow{\simeq} & (Z_\mu(i) \to A) \\ \downarrow & \lrcorner & \downarrow \\ (\bigcirc_{\mathfrak{m}} \mathbf{0} \to A) & \xrightarrow[\simeq]{} & (Z_\mu(i) \to \bigcirc_{\mathfrak{m}} \mathbf{0} \to A) \end{array}$$

Since A is $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected, the domain and codomain of the bottom function are contractible, and thus the bottom function is an equivalence. It then follows that the top function is also an equivalence. Since A is $\mathfrak{Null}(\mu \star \nu)$-modal and since $Z_\nu(j) \simeq \bigcirc_{\mathfrak{m}} \mathbf{0}$, we have $A \simeq (Z_\mu(i) \star \bigcirc_{\mathfrak{m}} \mathbf{0} \to A) \simeq (Z_\mu(i) \to A)$, and thus $Z_\mu(i) \perp A$. □

Proof of Proposition 2.17. Let μ and ν be null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Note that the null generator obtained from a null generator for $\mathfrak{m}$ by adjoining a family of $\mathfrak{m}$-connected types yields the same modality $\mathfrak{m}$. Under an assumption $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$, the empty type $\mathbf{0}$ becomes $\mathfrak{m}$-connected, and thus we may assume that μ includes the type family $\lambda(\_ : \bigcirc_{\mathfrak{m}} \mathbf{0}).\mathbf{0}$. Since $\bigcirc_{\mathfrak{m}} \mathbf{0}$ is $\mathfrak{n}$-connected by assumption, we may assume that ν includes the type family $\lambda(\_ : \mathbf{1}).\bigcirc_{\mathfrak{m}} \mathbf{0}$.

We show that $\mathfrak{Null}(\mu \star \nu) = \mathfrak{m} \vee \mathfrak{n}$. By Lemma 2.19, $\mathfrak{m} \vee \mathfrak{n} \leq \mathfrak{Null}(\mu \star \nu)$. For the other direction, suppose that A is a $\mathfrak{Null}(\mu \star \nu)$-modal type. By [RSS20, Theorem 3.50], it suffices to show that $\eta_{\mathfrak{m}}(A) : A \to \bigcirc_{\mathfrak{n}} A$ has $\mathfrak{m}$-modal fibers. By Lemma 2.21, $\bigcirc_{\mathfrak{n}} A \simeq \bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A$. Then the fibers of $\eta_{\mathfrak{m}}(A)$ are $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected. Since both A and $\bigcirc_{\mathfrak{n}} A$ are $\mathfrak{Null}(\mu \star \nu)$-modal, the fibers of $\eta_{\mathfrak{m}}(A)$ are also $\mathfrak{Null}(\mu \star \nu)$-modal. Thus, by Lemma 2.22, $\eta_{\mathfrak{m}}(A)$ has $\mathfrak{m}$-modal fibers. □

As a by-product, we have the following.

Corollary 2.23. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^\perp\mathfrak{n}$. If $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Top}$, then $\mathfrak{m}$ and $\mathfrak{n}$ are the closed and open, respectively, modalities associated to the proposition $\bigcirc_{\mathfrak{m}} \mathbf{0}$.* □

2.3. Open and closed modalities.

We collect some properties of *open modalities* and *closed modalities*.

Proposition 2.24. *The function $\mathfrak{Cl} : \mathsf{Prop} \to \mathsf{LAM}$ is a contravariant full embedding of posets and takes arbitrary joins to canonical meets.*

Proof. $\mathfrak{Cl}$ takes joins to canonical meets by [RSS20, Example 3.27]. It then follows that $\mathfrak{Cl}$ contravariantly preserves ordering. To see that $\mathfrak{Cl}$ reflects ordering, let P and Q be propositions and suppose that $\mathfrak{Cl}(Q) \leq \mathfrak{Cl}(P)$. By definition, Q is $\mathfrak{Cl}(Q)$-modal and thus $\mathfrak{Cl}(P)$-modal, but this implies that $P \to Q$. □

Proposition 2.25. *The function* $\mathfrak{Op} : \mathsf{Prop} \to \mathsf{LAM}$ *is a covariant full embedding of posets and takes finite meets to canonical meets and arbitrary joins to canonical joins.*

Proof. Since $\mathfrak{Cl}(P) = {}^{\perp}\mathfrak{Op}(P)$, the function $\mathfrak{Op}$ is a covariant full embedding of posets and takes arbitrary joins to canonical joins by Proposition 2.24. $\mathfrak{Op}$ takes finite meets to canonical meets by [RSS20, Example 3.26]. □

Proposition 2.26. *Let* P *be a proposition and* $\mathfrak{m}$ *a LAM.* $\bigcirc_{\mathfrak{Op}(P)}$ *preserves* $\mathfrak{m}$*-modal types, and* $\bigcirc_{\mathfrak{m}}$ *preserves* $\mathfrak{Cl}(P)$*-types.*

Proof. Let $A : \mathcal{U}$. If A is $\mathfrak{m}$-modal, then $\bigcirc_{\mathfrak{Op}(P)} A \equiv P \to A$ is $\mathfrak{m}$-modal by [RSS20, Lemma 1.26]. If A is $\mathfrak{Cl}(P)$-modal, then $\bigcirc_{\mathfrak{m}} A$ is $\mathfrak{Cl}(P)$-modal by [RSS20, Lemma 1.27]. □

Proposition 2.27. $\mathfrak{m} = (\mathfrak{m} \wedge \mathfrak{Cl}(P)) \vee (\mathfrak{m} \wedge \mathfrak{Op}(P))$ *for any LAM* $\mathfrak{m}$ *and any proposition* P*.*

Proof. $\mathfrak{m} = \mathfrak{m} \wedge (\mathfrak{Cl}(P) \vee \mathfrak{Op}(P)) = (\mathfrak{m} \wedge \mathfrak{Cl}(P)) \vee (\mathfrak{m} \wedge \mathfrak{Op}(P))$, where the first identification is by Proposition 2.13 and the second is by Propositions 2.16 and 2.26. □

3. Mode sketches

We introduce *mode sketches* as shapes of diagrams of subuniverses definable internally to type theory. We work in homotopy type theory through the section.

3.1. Internal diagrams induced by modalities.

We consider postulating some LAMs to encode some diagram of subuniverses. The fundamental observation is that a pair of LAMs induces a canonical functor between them.

Construction 3.1. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. We define a function $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ to be the composite of the inclusion $\mathcal{U}_{\mathfrak{n}} \subset \mathcal{U}$ and $\bigcirc_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{m}}$.

Remark 3.2. We can say that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is a functor *externally*: we can construct a function $\prod_{A,B:\mathcal{U}_{\mathfrak{n}}}(A \to B) \to (\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} A \to \bigcirc^{\mathfrak{n}}_{\mathfrak{m}} B)$ and every instance of the coherence laws. However, it is not known how to state that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is a functor internally to type theory, because defining the type of $(\infty, 1)$-categories in plain homotopy type theory is still an open problem.

We have two functors $\bigcirc^{\mathfrak{m}}_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{m}} \to \mathcal{U}_{\mathfrak{n}}$ and $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ for every pair of LAMs $\mathfrak{m}$ and $\mathfrak{n}$, but we are often interested in only one direction. It is thus useful to cut off one direction by postulating that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$: by the definition of $\mathfrak{n}$-connectedness, $\bigcirc^{\mathfrak{m}}_{\mathfrak{n}}$ becomes constant at the unit type. The other direction $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ remains non-trivial. Therefore, a pair $(\mathfrak{m}, \mathfrak{n})$ of LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ encodes a functor $\mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$. When $\mathfrak{n} \leq {}^{\perp}\mathfrak{m}$ is also assumed, $\mathcal{U}_{\mathfrak{m}}$ and $\mathcal{U}_{\mathfrak{n}}$ are considered unrelated.

Given more than two LAMs, we have canonical natural transformations between the canonical functors.

Construction 3.3. Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2$ be LAMs. We define

$$\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1} : \prod_{A:\mathcal{U}_{\mathfrak{m}_2}} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} A \to \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} A$$

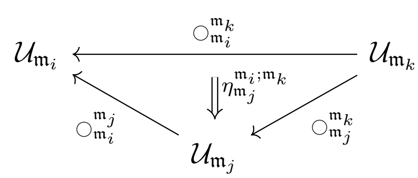
by $\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1}(A) \equiv \bigcirc_{\mathfrak{m}_0} \eta_{\mathfrak{m}_1}(A)$. This is well-typed as follows. $A : \mathcal{U}_{\mathfrak{m}_2}$ and $\bigcirc_{\mathfrak{m}_1} A : \mathcal{U}_{\mathfrak{m}_1}$ are implicitly coerced along the inclusions $\mathcal{U}_{\mathfrak{m}_2} \subset \mathcal{U}$ and $\mathcal{U}_{\mathfrak{m}_1} \subset \mathcal{U}$ respectively, and then $\bigcirc_{\mathfrak{m}_0} \eta_{\mathfrak{m}_1}(A)$ has type $\bigcirc_{\mathfrak{m}_0} A \to \bigcirc_{\mathfrak{m}_0} \bigcirc_{\mathfrak{m}_1} A$. By the definition of $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$, this type is definitionally equal to $\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} A \to \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} A$. The family of functions $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2}$ is natural in the sense that for any $A, B : \mathcal{U}_{\mathfrak{m}_2}$ and $f : A \to B$, we have a homotopy filling the following square.

$$\begin{array}{ccc}
\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} A & \xrightarrow{\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2}(A)} & \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} A \\
{\scriptstyle \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} f}\downarrow & & \downarrow{\scriptstyle \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} f} \\
\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} B & \xrightarrow[\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2}(B)]{} & \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} B
\end{array}$$

Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs. By naturality, the following diagram commutes.

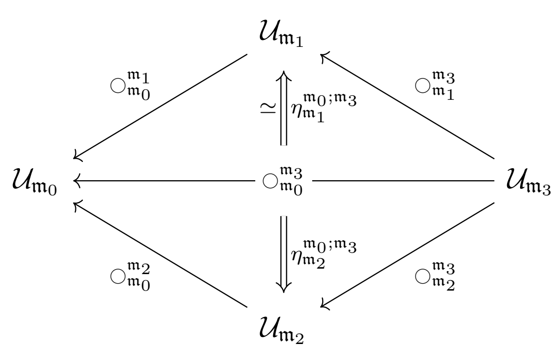
$$\begin{array}{ccc}
\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_3} & \xrightarrow{\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_3}} & \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_3} \\
{\scriptstyle \eta_{\mathfrak{m}_2}^{\mathfrak{m}_0;\mathfrak{m}_3}}\downarrow & & \downarrow{\scriptstyle \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \eta_{\mathfrak{m}_2}^{\mathfrak{m}_1;\mathfrak{m}_3}} \\
\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} \bigcirc_{\mathfrak{m}_2}^{\mathfrak{m}_3} & \xrightarrow[\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2} \bigcirc_{\mathfrak{m}_2}^{\mathfrak{m}_3}]{} & \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} \bigcirc_{\mathfrak{m}_2}^{\mathfrak{m}_3}
\end{array}$$

For more than four LAMs, higher coherence laws are also satisfied. Hence, a tuple $(\mathfrak{m}_0, \ldots, \mathfrak{m}_n)$ of LAMs such that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$ encodes an n-simplex with vertices $\mathcal{U}_{\mathfrak{m}_i}$, edges $\bigcirc_{\mathfrak{m}_i}^{\mathfrak{m}_j} : \mathcal{U}_{\mathfrak{m}_j} \to \mathcal{U}_{\mathfrak{m}_i}$ for $i < j$, triangles

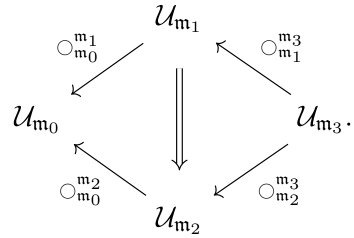


for $i < j < k$, and higher homotopies.

Shapes other than simplices are expressed by postulating invertibility of some of $\eta_{\mathfrak{m}_j}^{\mathfrak{m}_i;\mathfrak{m}_k}$'s. For example, let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs and suppose that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$, that $\mathfrak{m}_2 \leq {}^{\perp}\mathfrak{m}_1$, and that $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_3}$ is invertible. We have a diagram



which is equivalent to a diagram of the form



We cannot, however, naively postulate some properties of the functors $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$'s such as conservativity, fullness, faithfulness, adjointness, and invertibility. This is because the internal statements of these conditions are too strong due to stability under substitution, and indeed some "no-go" theorems on internalizing properties of functors are known [LOPS18, Theorem 5.1][Shu18, Theorem 4.1].

Remark 3.4. It is *possible* to postulate arbitrary properties of $\bigcirc^{\mathfrak{m}_j}_{\mathfrak{m}_i}$'s in the following way. We first postulate a "base" LAM $\mathfrak{Base}$ and assume $\mathfrak{Base} \leq {}^{\perp}\mathfrak{m}_i$ for all i. The universe $\mathcal{U}_{\mathfrak{Base}}$ is intended to be interpreted as the $(\infty, 1)$-category of spaces, so statements in $\mathcal{U}_{\mathfrak{Base}}$ will correspond to external statements. Since $\bigcirc_{\mathfrak{Base}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{Base}}$ preserves finite limits, it takes $(\infty, 1)$-categories to $(\infty, 1)$-categories and functors to functors. We can then postulate any property on the induced functor $\bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_j} \to \bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_i}$. In fact, cohesive homotopy type theory [SS12] was first formulated in a similar fashion where the $\sharp$ modality plays the role of $\mathfrak{Base}$. However, since we only know that $\bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_i}$ is an $(\infty, 1)$-category *externally*, this approach is not so convenient to work with especially for formalization in proof assistants. For this and some other reasons, the newer version of cohesive homotopy type theory [Shu18] is a proper extension of homotopy type theory. Nevertheless, this adding-base approach is attractive since it keeps type theory simple and works for any kind of diagram.

3.2. **Mode sketches.** We introduce *mode sketches* as shapes of diagrams definable by the methodology explained in Section 3.1.

Definition 3.5. A *mode sketch* $\mathfrak{M}$ consists of the following data:

- a decidable finite poset $I_{\mathfrak{M}}$;
- a subset $T_{\mathfrak{M}}$ of triangles in $I_{\mathfrak{M}}$.

Here, by a *decidable* poset we mean a poset whose ordering relation $\leq$ is decidable. A type is *finite* if it is merely equivalent to the coproduct of n copies of $\mathbf{1}$ for some $n : \mathbb{N}$ [Rij22, Definition 16.3.1]. The identity type on a finite type is decidable [Rij22, Remark 16.3.2]. The strict ordering relation $i < j$ defined as $(i \leq j) \wedge (i \neq j)$ is also decidable. By a *triangle* in $I_{\mathfrak{M}}$ we mean an ordered triple $(i_0 < i_1 < i_2)$ of elements of $I_{\mathfrak{M}}$. A triangle in $T_{\mathfrak{M}}$ is called *thin*.

Remark 3.6. The definition of mode sketches also makes sense in the metatheory. Every mode sketch $\mathfrak{M}$ in the metatheory can be encoded in type theory since it is finite.

Let $\mathfrak{M}$ be a mode sketch and $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ a function. We consider the following axioms.

Axiom A. $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $j \not\leq i$ in $\mathfrak{M}$.

Axiom B. For any thin triangle $(i_0 < i_1 < i_2)$ in $\mathfrak{M}$, the natural transformation $\eta^{\mathfrak{m}(i_0);\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)} : \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_0)} \Rightarrow \bigcirc^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_0)} \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)}$ is invertible.

Remark 3.7. Assuming Axiom A, if $i < j$, then $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$. The converse is not true: when neither $i \leq j$ nor $j \leq i$, we still get $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$.

Axioms A and B are motivated by the observation made in Section 3.1. That is, when $j \not\leq i$, the functor in the direction $\mathcal{U}_{\mathfrak{m}(i)} \to \mathcal{U}_{\mathfrak{m}(j)}$ is cut off.

Remark 3.8. A mode sketch $\mathfrak{M}$ is regarded as a presentation of an $(\infty, 2)$-category $|\mathfrak{M}|$. The strict ordering relation generates 1-cells $(i < j) : i \to j$, and the triangles $(i < j < k)$ generate 2-cells in the direction

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & \Downarrow & \searrow & \\ i & & \longrightarrow & & k. \end{array}$$

When the triangle is thin, the corresponding 2-cell is made invertible. Longer chains $(i_0 < i_1 < \cdots < i_n)$ present homotopies filling certain diagrams. A formal account is given in Construction 6.3. A function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and B is then considered as a diagram of subuniverses indexed over $|\mathfrak{M}|^{\mathrm{op}(1,2)}$, the $(\infty, 2)$-category obtained from $|\mathfrak{M}|$ by reversing the directions of 1-cells and 2-cells.

Example 3.9. Every decidable finite poset is a mode sketch where no triangle is thin. The $(\infty, 2)$-category presented by it is obtained from the left adjoint of the Duskin nerve [Dus02] by reversing 2-cells.

Example 3.10. The *mode sketch for functors* is drawn as

$$0 \longrightarrow 1.$$

Axiom A asserts $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(1)$. Axiom B is empty since there is no triangle. Thus, we get the following diagram.

$$\mathcal{U}_{\mathfrak{m}(0)} \xleftarrow{\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} \mathcal{U}_{\mathfrak{m}(1)}$$

Example 3.11. The *mode sketch for triangles* is drawn as

$$\begin{array}{ccccc} 0 & & \longrightarrow & & 2 \\ & \searrow & \simeq & \nearrow & \\ & & 1 & & \end{array}$$

where "$\simeq$" indicates that the triangle is thin. Axiom A asserts $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(1)$, $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(2)$, and $\mathfrak{m}(1) \leq {}^{\perp}\mathfrak{m}(2)$. Axiom B asserts that $\eta^{\mathfrak{m}(0);\mathfrak{m}(2)}_{\mathfrak{m}(1)}$ is invertible. Thus, we have the following commutative triangle.

$$\begin{array}{ccccc} \mathcal{U}_{\mathfrak{m}(0)} & & \xleftarrow{\bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(0)}} & & \mathcal{U}_{\mathfrak{m}(2)} \\ & \nwarrow^{\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} & & \swarrow_{\bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(1)}} & \\ & & \mathcal{U}_{\mathfrak{m}(1)} & & \end{array}$$

3.3. **Intended models, internally.** Let $\mathfrak{M}$ be a mode sketch. In this subsection, we see, internally to type theory, what kind of an ∞-logos is a model of $\mathfrak{M}$. Here, by a model of $\mathfrak{M}$ we mean an ∞-logos that admits an interpretation of a postulated function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and B and the following additional axiom.

Axiom C. The top modality is the canonical join $\bigvee_{\mathfrak{M}} \mathfrak{m}$.

Axiom C roughly asserts that the whole universe $\mathcal{U}$ is reconstructed from the subuniverses $\mathcal{U}_{\mathfrak{m}(i)}$'s. This is meant to exclude models other than intended models.

Example 3.12. Consider the case when $\mathfrak{M}$ is the mode sketch for functors (Example 3.10). Axiom C asserts $\mathfrak{Top} = \mathfrak{m}(0) \vee \mathfrak{m}(1)$. The equivalence $\mathcal{U} \simeq \sum_{A:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{B:\mathcal{U}_{\mathfrak{m}(1)}} A \to \bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} B$ (Proposition 2.13) suggests that $\mathcal{U}$ is the so-called *Artin gluing* for the functor $\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} : \mathcal{U}_{\mathfrak{m}(1)} \to \mathcal{U}_{\mathfrak{m}(0)}$. Therefore, our intended models of $\mathfrak{M}$ are ∞-logoses obtained by the Artin gluing.

A generalization of the Artin gluing is *oplax limits*. In the setting of Example 3.12, $\mathcal{U}$ fits into the following *universal oplax cone* over the diagram $\mathcal{U}_{\mathfrak{m}(0)} \xleftarrow{\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} \mathcal{U}_{\mathfrak{m}(1)}$.

$$\begin{array}{ccc} & \mathcal{U} & \\ \swarrow & \Longrightarrow & \searrow \\ \mathcal{U}_{\mathfrak{m}(0)} & \xleftarrow[\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}]{} & \mathcal{U}_{\mathfrak{m}(1)} \end{array} \tag{3.1}$$

An oplax cone over a diagram is a kind of cone but every triangle formed by two projections and a functor in the diagram is only filled by a not necessarily invertible natural transformation in the direction of Diagram (3.1). The universal oplax cone or oplax limit is the terminal object in the $(\infty, 1)$-category of oplax cones.

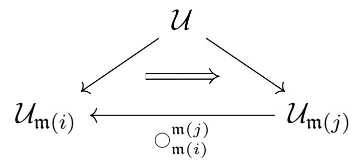
Example 3.13. Consider the case when $\mathfrak{M}$ is the mode sketch $\{0 \to 1 \to 2\}$ with no thin triangle. Axiom C asserts $\mathfrak{Top} = \mathfrak{m}(0) \vee \mathfrak{m}(1) \vee \mathfrak{m}(2)$. Iterating Proposition 2.13, we see that every type $A : \mathcal{U}$ is fractured into $A_0 : \mathcal{U}_{\mathfrak{m}(0)}$, $A_1 : \mathcal{U}_{\mathfrak{m}(1)}$, $A_2 : \mathcal{U}_{\mathfrak{m}(2)}$, $f_{01} : A_0 \to \bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} A_1$, $f_{02} : A_0 \to \bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(0)} A_2$, $f_{12} : A_1 \to \bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(1)} A_2$, and $p_{012} : \bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} f_{12} \circ f_{01} = \eta^{\mathfrak{m}(0);\mathfrak{m}(2)}_{\mathfrak{m}(1)}(A_2) \circ f_{02}$. Indeed, we have

$$\begin{aligned} & \mathcal{U} \\ \simeq \quad & \{\text{Proposition 2.13 for } \mathfrak{m}(0) \text{ and } \mathfrak{m}(1) \vee \mathfrak{m}(2)\} \\ & \textstyle\sum_{A_0:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_{12}:\mathcal{U}_{\mathfrak{m}(1)\vee\mathfrak{m}(2)}} A_0 \to \bigcirc_{\mathfrak{m}_0} A_{12} \\ \simeq \quad & \{\text{Proposition 2.13 for } \mathfrak{m}(1) \text{ and } \mathfrak{m}(2)\} \\ & \textstyle\sum_{A_0:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_1:\mathcal{U}_{\mathfrak{m}(1)}} \sum_{A_2:\mathcal{U}_{\mathfrak{m}(2)}} \sum_{f_{12}:A_1 \to \bigcirc_{\mathfrak{m}(1)} A_2} A_0 \to \bigcirc_{\mathfrak{m}(0)}(A_1 \times_{\bigcirc_{\mathfrak{m}(1)} A_2} A_2) \end{aligned}$$

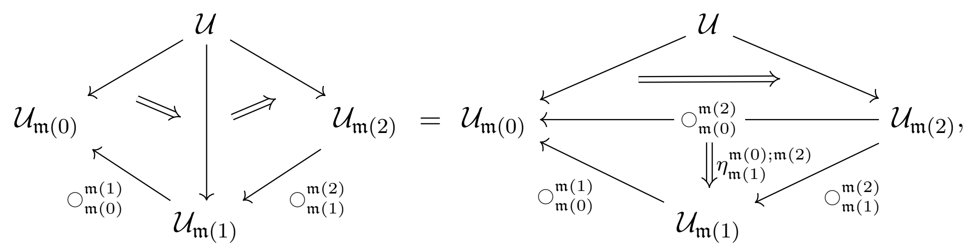
where the pullback is taken for $f_{12} : A_1 \to \bigcirc_{\mathfrak{m}(1)} A_2$ and $\eta_{\mathfrak{m}(1)}(A_2) : A_2 \to \bigcirc_{\mathfrak{m}(1)} A_2$. Since $\bigcirc_{\mathfrak{m}(0)}$ preserves pullbacks, the component $A_0 \to \bigcirc_{\mathfrak{m}(0)}(A_1 \times_{\bigcirc_{\mathfrak{m}(1)} A_2} A_2)$ corresponds to the components f_{01}, f_{02}, and p_{012}. Then $\mathcal{U}$ is the oplax limit of the diagram

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{m}(0)} & \xleftarrow{\bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(0)}} & \mathcal{U}_{\mathfrak{m}(2)}. \\ \nwarrow{\scriptstyle \bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} & \Downarrow \eta^{\mathfrak{m}(0);\mathfrak{m}(2)}_{\mathfrak{m}(1)} & \swarrow{\scriptstyle \bigcirc^{\mathfrak{m}(2)}_{\mathfrak{m}(1)}} \\ & \mathcal{U}_{\mathfrak{m}(1)} & \end{array} \tag{3.2}$$

That is, we have projections $\mathcal{U} \to \mathcal{U}_{\mathfrak{m}(i)}$ for all i, natural transformations



for all $i < j$, and a homotopy



and these data form a universal oplax cone over Diagram (3.2).

Let us make the triangle $(0 < 1 < 2)$ thin so that the natural transformation (3.2) becomes invertible. In this setting, $\mathcal{U}$ is still the oplax limit of Diagram (3.2), but the presentation can be simplified since the type of data (f_{02}, p_{012}) is contractible.

For a general mode sketch $\mathfrak{M}$, we apply Proposition 2.13 for a minimal element $\mathfrak{m}(i_0)$ and the rest $\bigvee_{i:\mathfrak{M}\setminus i_0} \mathfrak{m}(i)$ and repeat this for $\mathfrak{M} \setminus i_0$ to fracture types into modal types. Examples 3.12 and 3.13 suggest that $\mathcal{U}$ is the oplax limit of the diagram formed by $\mathcal{U}_{\mathfrak{m}(i)}$'s explained in Remark 3.8. Thus, our intended models of $\mathfrak{M}$ are oplax limits of ∞-logoses indexed over the $(\infty, 2)$-category presented by $\mathfrak{M}$. The formal account of this is described in Section 6.

4. Mode sketches and synthetic Tait computability

We give an alternative set of axioms for mode sketches and exhibit a connection between mode sketches and *synthetic Tait computability* of Sterling [Ste21]. The core axiom of synthetic Tait computability is to postulate a proposition. The proposition induces the open and closed modalities, and then every type is fractured into an open type equipped with a closed type family and behaves like a *logical relation*. In this story, the open and closed modalities seem more essential than the postulated proposition, so we aim to formulate synthetic Tait computability purely in terms of modalities. We work in homotopy type theory.

4.1. **Alternative mode sketch axioms.**

The ∞-logoses obtained by the Artin gluing can be characterized as ∞-logoses equipped with a subterminal object (cf. [Joh02, A4.5.6]). We generalize this from the Artin gluing to oplax limits indexed by mode sketches, internally to type theory: the type of functions $\mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and C is equivalent to the type of lattice morphisms $\mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ (Theorem 4.4).

Definition 4.1. A *cosieve* on a decidable poset I is an upward-closed decidable subset of it. Let $\mathsf{coSieve}(I)$ denote the poset of cosieves on I ordered by inclusion. Note that cosieves are closed under finite meets and joins, so $\mathsf{coSieve}(I)$ is a lattice.

Notation 4.2. For $i : \mathfrak{M}$, let $(i \downarrow \mathfrak{M})$ denote the cosieve $\{j : \mathfrak{M} \mid i \leq j\}$ and $\partial(i \downarrow \mathfrak{M})$ the cosieve $(i \downarrow \mathfrak{M}) \setminus \{i\}$.

Construction 4.3. Let $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ be a function. We define a function $\mathfrak{a}_P : \mathfrak{M} \to \mathsf{LAM}$ by $\mathfrak{a}_P(i) \equiv \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge \mathfrak{Cl}(P(\partial(i \downarrow \mathfrak{M})))$.

Theorem 4.4. *Construction 4.3 is restricted to an equivalence between the following types:*
(1) *the type of lattice morphisms* $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$*;*
(2) *the type of functions* $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ *satisfying Axioms A and C.*

We write $\varphi_{\mathfrak{m}} : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ *for the lattice morphism corresponding to* $\mathfrak{m}$*.*

Before giving a proof of Theorem 4.4, let us relate Theorem 4.4 to *synthetic Tait computability* [SH21, Ste21, SH22]. The core axiom of synthetic Tait computability is to postulate some propositions. One can work with those propositions directly but also with the induced open and closed modalities. Theorem 4.4 says that synthetic Tait computability can, in fact, be formulated completely in terms of modalities. The simplest version of synthetic Tait computability postulates a single proposition. The corresponding mode sketch is $\{0 \to 1\}$ as follows.

Example 4.5. Let $\mathfrak{M}$ be the mode sketch for functors (Example 3.10). Then $\mathsf{coSieve}(\mathfrak{M}) = \{\{\}, \{1\}, \{0, 1\}\}$ is the free lattice generated by the single element $\{1\}$. We thus have $\{\text{lattice morphisms } \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}\} \simeq \mathsf{Prop}$.

The rest of this subsection is devoted to the proof of Theorem 4.4. We first show that Construction 4.3 is restricted to a function $1 \to 2$.

Proposition 4.6. *Let* $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ *be a function.*
(1) *If* P *preserves binary meets, then* $\mathfrak{a}_P$ *satisfies Axiom A.*
(2) *If* P *preserves top elements and finite joins, then* $\mathfrak{a}_P$ *satisfies Axiom C.*

We prepare a couple of lemmas.

Lemma 4.7. *If a function* $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ *preserves finite joins, then* $\mathfrak{Op}(P(S)) = \bigvee_S \mathfrak{a}_P$ *for any cosieve* $S \subset \mathfrak{M}$*.*

Proof. By induction on the size of S. By assumption and by Proposition 2.25, we have $\mathfrak{Op}(P(S)) = \bigvee_{i:S} \mathfrak{Op}(P(i \downarrow \mathfrak{M}))$. Thus, it is enough to show the case when S is of the form $(i \downarrow \mathfrak{M})$. By Proposition 2.27, we have $\mathfrak{Op}(P(i \downarrow \mathfrak{M})) = (\mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge \mathfrak{Cl}(P(\partial(i \downarrow \mathfrak{M})))) \vee (\mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge \mathfrak{Op}(P(\partial(i \downarrow \mathfrak{M})))) = \mathfrak{a}_P(i) \vee \mathfrak{Op}(P(\partial(i \downarrow \mathfrak{M})))$. Then apply the induction hypothesis for $\partial(i \downarrow \mathfrak{M})$. □

Lemma 4.8. *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs. If* $\bigcirc_{\mathfrak{n}}$ *preserves* $\mathfrak{m}$*-modal types, then* $\mathfrak{m} \wedge {}^{\perp}(\mathfrak{n} \wedge \mathfrak{m}) = \mathfrak{m} \wedge {}^{\perp}\mathfrak{n}$*.*

Proof. By Fact 2.14. □

Proof of Proposition 4.6. If P preserves binary meets, then for any $j \not\leq i$ in $\mathfrak{M}$,

$$
\begin{aligned}
& \mathfrak{a}_P(i) \\
= \quad & \{\text{definition}\} \\
& \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge {}^{\perp}\mathfrak{Op}(P(\partial(i \downarrow \mathfrak{M}))) \\
\leq \quad & \{(j \downarrow \mathfrak{M}) \cap (i \downarrow \mathfrak{M}) \subset \partial(i \downarrow \mathfrak{M}) \text{ as } j \not\leq i\}
\end{aligned}
$$

$$
\begin{aligned}
& \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge {}^{\perp}\mathfrak{Op}(P((j \downarrow \mathfrak{M}) \cap (i \downarrow \mathfrak{M}))) \\
= \quad & \{\text{Proposition 2.25, and } P \text{ preserves binary meets}\} \\
& \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge {}^{\perp}(\mathfrak{Op}(P(j \downarrow \mathfrak{M})) \wedge \mathfrak{Op}(P(i \downarrow \mathfrak{M}))) \\
= \quad & \{\text{Lemma 4.8 and Proposition 2.26}\} \\
& \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge {}^{\perp}\mathfrak{Op}(P(j \downarrow \mathfrak{M})) \\
\leq \quad & \{\text{definition}\} \\
& {}^{\perp}\mathfrak{a}_P(j),
\end{aligned}
$$

and thus Axiom A is satisfied. If P preserves top elements and finite joins, then $\mathfrak{a}_P$ satisfies Axiom C by Lemma 4.7. □

We then construct the inverse function $2 \to 1$. The key observation is that canonical joins of $\mathfrak{m}(i)$'s exist and are well-behaved under Axiom A.

Proposition 4.9. *If a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfies Axiom A, then the canonical join $\bigvee_S \mathfrak{m}$ exists for any decidable subset $S \subset \mathfrak{M}$.*

Proof. By induction on the size of S. If S is empty, then $\bigvee_\emptyset \mathfrak{m}$ is the bottom modality. Suppose that S is non-empty. Since $\mathfrak{M}$ is finite, there is an element i_0 minimal in S. Then $S \setminus \{i_0\}$ admits a canonical join by the induction hypothesis. Since i_0 is minimal, $\mathfrak{m}(i_0) \leq {}^{\perp}\mathfrak{m}(i)$ for any $i : S \setminus \{i_0\}$ by Axiom A, and thus $\mathfrak{m}(i_0) \leq {}^{\perp}(\bigvee_{S\setminus\{i_0\}} \mathfrak{m})$. Then we have the canonical join $\bigvee_S \mathfrak{m} \equiv \mathfrak{m}(i_0) \vee (\bigvee_{(S\setminus\{i_0\})} \mathfrak{m})$ by Proposition 2.13. □

Lemma 4.10. *Let $\mathfrak{m}_0$, $\mathfrak{m}_1$, and $\mathfrak{m}_2$ be LAMs such that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for any $i < j$. Then $\mathfrak{m}_0 \vee \mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_2$.*

Proof. Let A be a $(\mathfrak{m}_0 \vee \mathfrak{m}_1)$-modal type. By Proposition 2.13, $\eta_{\mathfrak{m}_1}(A) : A \to \bigcirc_{\mathfrak{m}_1} A$ has $\mathfrak{m}_0$-modal fibers. Then, by assumption, $\bigcirc_{\mathfrak{m}_1} A$ and the fibers of $\eta_{\mathfrak{m}_1}(A)$ are made contractible by $\bigcirc_{\mathfrak{m}_2}$. Thus, $\bigcirc_{\mathfrak{m}_2} A$ is contractible. □

Proposition 4.11. *If a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfies Axiom A, then $\bigvee_{\mathfrak{M}\setminus S} \mathfrak{m} \leq {}^{\perp}(\bigvee_S \mathfrak{m})$ for any cosieve $S \subset \mathfrak{M}$.*

Proof. Since S is upward-closed, $j \not\leq i$ for any $i : \mathfrak{M} \setminus S$ and $j : S$. Thus, by Axiom A, $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $i : \mathfrak{M} \setminus S$ and $j : S$. The claim follows from Lemma 4.10 and the construction of the canonical join in Proposition 4.9. □

Fact 4.12 [RSS20, Example 3.32]**.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. If $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$, then $\mathfrak{m} \wedge \mathfrak{n} = \mathfrak{Bot}$.

Construction 4.13. Let $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ be a function satisfying Axioms A and C. We define a lattice morphism $\varphi_\mathfrak{m} : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ by $\varphi_\mathfrak{m}(S) \equiv \bigcirc_{\bigvee_{\mathfrak{M}\setminus S} \mathfrak{m}} \mathbf{0}$. By Corollary 2.23 and by Proposition 4.11, $\varphi_\mathfrak{m}(S)$ is the unique proposition such that $\mathfrak{Op}(\varphi_\mathfrak{m}(S)) = \bigvee_S \mathfrak{m}$. Because $S \mapsto \bigvee_S \mathfrak{m}$ preserves finite joins by definition, $\varphi_\mathfrak{m}$ preserves finite joins. By Axiom C, $\varphi_\mathfrak{m}$ preserves top elements. For preservation of binary meets, let S_1 and S_2 be cosieves on $\mathfrak{M}$. We have to show that $\bigvee_{S_1 \cap S_2} \mathfrak{m} = (\bigvee_{S_1} \mathfrak{m}) \wedge (\bigvee_{S_2} \mathfrak{m})$. Let $S_3 \equiv S_1 \cap S_2$, $S_1' \equiv S_1 \setminus S_3$, and $S_2' \equiv S_2 \setminus S_3$. By Proposition 4.11, $\bigvee_{S_1'} \mathfrak{m} \leq {}^{\perp}(\bigvee_{S_2} \mathfrak{m})$ and $\bigvee_{S_2'} \mathfrak{m} \leq {}^{\perp}(\bigvee_{S_1} \mathfrak{m})$. Then,

$$
\begin{aligned}
& (\textstyle\bigvee_{S_1} \mathfrak{m}) \wedge (\bigvee_{S_2} \mathfrak{m}) \\
= \quad & \{\text{definition}\}
\end{aligned}
$$

$$
\begin{aligned}
& ((\bigvee_{S_1'} \mathfrak{m}) \vee (\bigvee_{S_3} \mathfrak{m})) \wedge ((\bigvee_{S_2'} \mathfrak{m}) \vee (\bigvee_{S_3} \mathfrak{m})) \\
= \quad & \{\text{Proposition 2.15}\} \\
& ((\bigvee_{S_1'} \mathfrak{m}) \wedge (\bigvee_{S_2'} \mathfrak{m})) \vee (\bigvee_{S_3} \mathfrak{m}) \\
= \quad & \{\text{Fact 4.12}\} \\
& \bigvee_{S_3} \mathfrak{m}.
\end{aligned}
$$

Proof of Theorem 4.4. It remains to show that the constructions $P \mapsto \mathfrak{a}_P$ and $\mathfrak{m} \mapsto \varphi_{\mathfrak{m}}$ are mutually inverses. Lemma 4.7 implies that $\varphi_{\mathfrak{a}_P} = P$. For the other identification, we have

$$
\begin{aligned}
& \mathfrak{a}_{\varphi_{\mathfrak{m}}}(i) \\
= \quad & \{\text{definition}\} \\
& \mathfrak{Op}(\varphi_{\mathfrak{m}}(i \downarrow \mathfrak{M})) \wedge \mathfrak{Cl}(\varphi_{\mathfrak{m}}(\partial(i \downarrow \mathfrak{M}))) \\
= \quad & \{\text{Lemma 4.7}\} \\
& (\bigvee_{(i\downarrow\mathfrak{M})} \mathfrak{m}) \wedge \mathfrak{Cl}(\varphi_{\mathfrak{m}}(\partial(i \downarrow \mathfrak{M}))) \\
= \quad & \{\text{Propositions 2.16 and 2.26}\} \\
& \bigvee_{j:(i\downarrow\mathfrak{M})} \mathfrak{m}(j) \wedge \mathfrak{Cl}(\varphi_{\mathfrak{m}}(\partial(i \downarrow \mathfrak{M}))) \\
= \quad & \{\text{Lemma 4.7}\} \\
& \bigvee_{j:(i\downarrow\mathfrak{M})} \mathfrak{m}(j) \wedge (\bigwedge_{k:\partial(i\downarrow\mathfrak{M})} {}^{\perp}\mathfrak{m}(k)) \\
= \quad & \{\mathfrak{m}(k) \wedge {}^{\perp}\mathfrak{m}(k) = \mathfrak{Bot} \text{ for } k : \partial(i \downarrow \mathfrak{M}) \text{ by Fact 4.12}\} \\
& \mathfrak{m}(i) \wedge (\bigwedge_{k:\partial(i\downarrow\mathfrak{M})} {}^{\perp}\mathfrak{m}(k)) \\
= \quad & \{\text{Axiom A}\} \\
& \mathfrak{m}(i).
\end{aligned}
$$

□

4.2. **Logical relations as types.** We have seen in Section 4.1 that synthetic Tait computability is reformulated in terms of LAMs. The slogan of synthetic Tait computability is "logical relations as types" [SH21]. This is also formulated purely in terms of LAMs.

Fact 4.14 [RSS20, Theorem 3.11]**.** For any LAM $\mathfrak{m}$, the universe of $\mathfrak{m}$-modal types $\mathcal{U}_{\mathfrak{m}} \equiv \{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\}$ is $\mathfrak{m}$-modal.

Proposition 4.15 (Fracture and gluing)**.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs such that* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$*. Then we have an equivalence*

$$\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \simeq \sum_{B:\mathcal{U}_{\mathfrak{n}}} B \to \mathcal{U}_{\mathfrak{m}}$$

whose right-to-left function sends a (B, A) *to* $\sum_{\mathtt{x}:B} A(\mathtt{x})$*.*

Proof. For any $B : \mathcal{U}_{\mathfrak{n}}$, we have

$$
\begin{aligned}
& \sum_{A:\mathcal{U}_{\mathfrak{m}}} A \to \bigcirc_{\mathfrak{m}} B \\
\simeq \quad & \{\text{equivalence between fibrations and type families}\} \\
& \bigcirc_{\mathfrak{m}} B \to \mathcal{U}_{\mathfrak{m}} \\
\simeq \quad & \{\text{Fact 4.14}\} \\
& B \to \mathcal{U}_{\mathfrak{m}}.
\end{aligned}
$$

Then apply Proposition 2.13. □

Proposition 4.15 asserts that a type in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ is a $\mathfrak{n}$-modal type equipped with a $\mathfrak{m}$-modal unary (proof-relevant) relation on it, so *types (in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$) are relations.* More generally, for a mode sketch $\mathfrak{M}$ and a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axiom A, types in $\mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ are fractured into a sort of generalized relations by iterated applications of Proposition 4.15. Intuitively, the ordering on $\mathfrak{M}$ is understood as "dependency": every type $A : \mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ is fractured into a family of type families $\{A_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$ such that $A_{\mathfrak{m}(i)}$ depends on $A_{\mathfrak{m}(j)}$ for all $j > i$. One may also regard the underlying finite poset of $\mathfrak{M}$ as a FOLDS signature [Mak95].

Example 4.16. When $\mathfrak{M}$ is the mode sketch $\{0 \leftarrow 01 \to 1\}$, we have an equivalence

$$\mathcal{U}_{\mathfrak{m}(01)\vee\mathfrak{m}(1)\vee\mathfrak{m}(0)} \simeq \textstyle\sum_{A_0:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_1:\mathcal{U}_{\mathfrak{m}(1)}} A_0 \to A_1 \to \mathcal{U}_{\mathfrak{m}(01)}.$$

Example 4.17. When $\mathfrak{M}$ is the mode sketch $\{0 \to 1 \to 2\}$ (with no thin triangle), we have an equivalence

$$\mathcal{U}_{\mathfrak{m}(0)\vee\mathfrak{m}(1)\vee\mathfrak{m}(2)} \simeq \textstyle\sum_{A_2:\mathcal{U}_{\mathfrak{m}(2)}} \sum_{A_1:A_2\to\mathcal{U}_{\mathfrak{m}(1)}} \prod_{\mathtt{x}_2} A_1(\mathtt{x}_2) \to \mathcal{U}_{\mathfrak{m}(0)}.$$

The equivalence in Proposition 4.15 nicely interacts with type constructors, and we derive the *logical relation translation* (also called the parametricity translation) of dependent type theory [BJP12, Shu15, Uem17, Las14] as a *theorem* in type theory. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Type constructors in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ behave in the same way as the definition of the logical relation translation of type constructors [Las14, Section 3] as follows.

- $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} : \Uparrow\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair $(\mathcal{U}_{\mathfrak{n}}, \lambda B.B \to \mathcal{U}_{\mathfrak{m}})$;
- $\mathbf{1} : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair $(\mathbf{1}, \lambda\_.\mathbf{1})$;
- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$. Then $(A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}) : \Uparrow\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair
$$(A_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{n}}, \lambda B. \textstyle\prod_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x}) \to B(\mathtt{x}) \to \mathcal{U}_{\mathfrak{m}}).$$
Indeed,
$$\begin{aligned} & A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \\ \simeq \quad & \{\text{fracture and gluing}\} \\ & (\textstyle\sum_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x})) \to (\sum_{B:\mathcal{U}_{\mathfrak{n}}} B \to \mathcal{U}_{\mathfrak{m}}) \\ \simeq \quad & \{\textstyle\prod \text{ distributes over } \sum\} \\ & \textstyle\sum_{B:\prod_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x}) \to \mathcal{U}_{\mathfrak{n}}} \prod_{\mathtt{x}} \prod_{\mathtt{y}} B(\mathtt{x}, \mathtt{y}) \to \mathcal{U}_{\mathfrak{m}} \\ \simeq \quad & \{\mathcal{U}_{\mathfrak{n}} \simeq (A_{\mathfrak{m}}(\mathtt{x}) \to \mathcal{U}_{\mathfrak{n}}) \text{ since } \mathfrak{m} \leq {}^{\perp}\mathfrak{n}\} \\ & \textstyle\sum_{B:A_{\mathfrak{n}}\to\mathcal{U}_{\mathfrak{n}}} \prod_{\mathtt{x}} A_{\mathfrak{m}}(\mathtt{x}) \to B(\mathtt{x}) \to \mathcal{U}_{\mathfrak{m}}; \end{aligned}$$
- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$ and that $B : A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(B_{\mathfrak{n}}, B_{\mathfrak{m}})$. Then $\prod_{\mathtt{x}:A} B(\mathtt{x}) : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair
$$(\textstyle\prod_{\mathtt{x}_{\mathfrak{n}}:A_{\mathfrak{n}}} B_{\mathfrak{n}}(\mathtt{x}_{\mathfrak{n}}), \lambda f. \prod_{\mathtt{x}_{\mathfrak{n}}} \prod_{\mathtt{x}_{\mathfrak{m}}:A_{\mathfrak{m}}(\mathtt{x}_{\mathfrak{n}})} B_{\mathfrak{m}}(\mathtt{x}_{\mathfrak{n}}, \mathtt{x}_{\mathfrak{m}}, f(\mathtt{x}_{\mathfrak{n}})))$$
by a similar calculation to the previous clause. $\sum_{\mathtt{x}:A} B(\mathtt{x}) : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair
$$(\textstyle\sum_{\mathtt{x}_{\mathfrak{n}}:A_{\mathfrak{n}}} B_{\mathfrak{n}}(\mathtt{x}_{\mathfrak{n}}), \lambda(a_{\mathfrak{n}}, b_{\mathfrak{n}}). \sum_{\mathtt{x}_{\mathfrak{m}}:A_{\mathfrak{m}}(a_{\mathfrak{n}})} B_{\mathfrak{m}}(a_{\mathfrak{n}}, \mathtt{x}_{\mathfrak{m}}, b_{\mathfrak{n}}));$$
- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$, that $a : A$ corresponds to a pair $(a_{\mathfrak{n}}, a_{\mathfrak{m}})$, and that $a' : A$ corresponds to a pair $(a'_{\mathfrak{n}}, a'_{\mathfrak{m}})$. Then $a = a' : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair
$$(a_{\mathfrak{n}} = a'_{\mathfrak{n}}, \lambda p.a_{\mathfrak{m}} =^{A_{\mathfrak{m}}}_{p} a'_{\mathfrak{m}}).$$

Thus, any type $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ constructed using these type constructors is fractured into a type $A_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{n}}$ and a type family $A_{\mathfrak{m}} : A_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$, and $A_{\mathfrak{m}}$ is equivalent to the logical relation translation of $A_{\mathfrak{n}}$. In this sense, *types in* $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ *are logical relations.* The interaction of the equivalences in Examples 4.16 and 4.17 and type constructors is similarly calculated. We thus conclude that types in $\mathcal{U}_{\bigvee_{\mathfrak{M}}\mathfrak{m}}$ are generalized logical relations.

5. Higher category theory

We collect facts about higher categories needed to develop semantics of mode sketches in ∞-logoses.

We work with a model-independent language of $(\infty,1)$-category theory rather than choosing a specific model of $(\infty,1)$-categories. An $(\infty,1)$-*category* $\mathcal{C}$ consists of a space $\mathbf{Obj}(\mathcal{C})$ of *objects* of $\mathcal{C}$, a space $\mathbf{Map}_{\mathcal{C}}(x,y)$ of *morphisms from* x *to* y for any objects x and y, and composition operators unital and associative up to coherent homotopy. Concepts in category theory such as functors, natural transformations, equivalences, adjoints, (co)limits, and Kan extensions have $(\infty,1)$-categorical analogues. We refer the reader to [Lur09a, Cis19, RV22] for general $(\infty,1)$-category theory.

At least two Grothendieck universes $\mathfrak{U} \in \Uparrow\mathfrak{U}$ are assumed to exist. By *small* we mean $\mathfrak{U}$-small and by *large* we mean $\Uparrow\mathfrak{U}$-small. For an $(\infty,1)$-category $\mathcal{C}$ of small objects of some kind, we write $\Uparrow\mathcal{C}$ for the $(\infty,1)$-category of large objects of the same kind. Let $\mathbf{Spc}$ denote the $(\infty,1)$-category of small spaces. Let $\mathbf{Cat}$ denote the $(\infty,1)$-category of small $(\infty,1)$-categories. For $(\infty,1)$-categories $\mathcal{C}$ and $\mathcal{D}$, let $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ denote the $(\infty,1)$-category of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them. For a functor $x : I \to \mathcal{C}$, we write $\{\pi_i : \lim_{j\in I} x_j \to x_i\}_{i\in I}$ for the limit cone if it exists and $\{\iota_i : x_i \to \operatorname{colim}_{j\in I} x_j\}_{i\in I}$ for the colimit cocone if it exists.

Accessible and *presentable* $(\infty,1)$-categories [Lur09a, Chapter 5] are important classes of $(\infty,1)$-categories. We do not need precise definitions of them. Every presentable $(\infty,1)$-category is accessible and has small colimits and limits.

5.1. ∞-logoses.

We review the theory of ∞-*logoses*, also known as ∞-*toposes*. The standard reference is [Lur09a, Chapter 6].

Definition 5.1. An ∞-*logos* is a presentable $(\infty,1)$-category $\mathcal{L}$ such that, for any small $(\infty,1)$-category I, any natural transformation $f : B \Rightarrow A : I \to \mathcal{L}$ where all the naturality squares are pullbacks, and for any cocone over f of the form

$$\begin{array}{ccc} B_i & \xrightarrow{g_i} & B' \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f'} \\ A_i & \xrightarrow[\iota_i]{} & \operatorname{colim}_{i\in I} A_i, \end{array} \tag{5.1}$$

B' is the colimit of B_i's if and only if Diagram (5.1) is a pullback for every $i \in I$. A *morphism of* ∞-*logoses* is a functor preserving small colimits and finite limits. Note that any morphism of ∞-logoses has a right adjoint by the adjoint functor theorem [Lur09a, Corollary 5.5.2.9]. We write $\mathbf{Logos} \subset \Uparrow\mathbf{Cat}$ for the subcategory spanned by the ∞-logoses and the morphisms of ∞-logoses.

The following are immediate from the definition.

Proposition 5.2. *Let $\{F_i : \mathcal{L} \to \mathcal{K}_i\}_{i\in I}$ be a family of functors between presentable $(\infty,1)$-categories preserving small colimits and finite limits. If all the $\mathcal{K}_i$'s are ∞-logoses and if $\{F_i\}_{i\in I}$ is jointly conservative, then $\mathcal{L}$ is an ∞-logos.* □

Proposition 5.3. *Let $\mathcal{L}$ be an ∞-logos and let $A \in \mathcal{L}$ be an object. If there is a map $A \to \mathbf{0}$, then $A \simeq \mathbf{0}$.* □

LAMs in homotopy type theory are expected to correspond to (lex, accessible) localizations of ∞-logoses.

Definition 5.4. A morphism of ∞-logoses is a *localization* if its right adjoint is fully faithful. For an ∞-logos $\mathcal{L}$, let $\mathbf{LAM}(\mathcal{L})$ denote the full subcategory of $(\mathbf{Logos}_{\mathcal{L}/})^{\mathrm{op}}$ spanned by the localization morphisms $\mathcal{L} \to \mathcal{K}$. Note that $\mathbf{LAM}(\mathcal{L})$ is a poset. We call an object in $\mathbf{LAM}(\mathcal{L})$ a *lex, accessible modality (LAM) in $\mathcal{L}$*.

Notation 5.5. Let $\mathcal{L}$ be an ∞-logos. For a LAM $\mathfrak{m}$ in $\mathcal{L}$, we write $\bigcirc_{\mathfrak{m}} : \mathcal{L} \to \mathcal{L}_{\mathfrak{m}}$ for the localization corresponding to $\mathfrak{m}$ and $\eta_{\mathfrak{m}}$ for its unit. For two LAMs $\mathfrak{m}_0$ and $\mathfrak{m}_1$ in $\mathcal{L}$, let $\bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} : \mathcal{L}_{\mathfrak{m}_1} \to \mathcal{L}_{\mathfrak{m}_0}$ denote the restriction of $\bigcirc_{\mathfrak{m}_0}$ along the inclusion $\mathcal{L}_{\mathfrak{m}_1} \subset \mathcal{L}$. For three LAMs $\mathfrak{m}_0$, $\mathfrak{m}_1$, and $\mathfrak{m}_2$ in $\mathcal{L}$, let $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2} : \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_2} \Rightarrow \bigcirc_{\mathfrak{m}_0}^{\mathfrak{m}_1} \bigcirc_{\mathfrak{m}_1}^{\mathfrak{m}_2} : \mathcal{L}_{\mathfrak{m}_2} \to \mathcal{L}_{\mathfrak{m}_0}$ denote the natural transformation defined by $(\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_2})_A = \bigcirc_{\mathfrak{m}_0}(\eta_{\mathfrak{m}_1})_A$ for $A \in \mathcal{L}_{\mathfrak{m}_2}$.

$(\infty,1)$-categorical counterparts of open and closed modalities are open and closed, respectively, localizations (Construction 5.8).

Fact 5.6 [Lur09a, Proposition 6.3.5.1]**.** Let $\mathcal{L}$ be an ∞-logos. Then, for every object $A \in \mathcal{L}$, the slice $\mathcal{L}_{/A}$ is an ∞-logos. Moreover, the pullback functor $A^* : \mathcal{L} \to \mathcal{L}_{/A}$ is a morphism of ∞-logoses.

Fact 5.7 [ABFJ22, Propositions 4.3.6 and 4.3.7]**.** Let $\mathcal{L}$ be an ∞-logos and let Φ be a class of monomorphisms in $\mathcal{L}$. Let $\mathcal{K} \subset \mathcal{L}$ be the full subcategory spanned by those objects A such that $f^* : \mathbf{Map}(Y, A) \to \mathbf{Map}(X, A)$ is an equivalence for every pullback f of a morphism in Φ. Then $\mathcal{K}$ is an ∞-logos, and the inclusion $\mathcal{K} \to \mathcal{L}$ has a left adjoint which is a localization of ∞-logoses.

Construction 5.8. Let $\mathcal{L}$ be an ∞-logos and let $P \in \mathcal{L}$ be a (-1)-truncated object. The *open localization* associated to P is $P^* : \mathcal{L} \to \mathcal{L}_{/P}$. This is indeed a morphism of ∞-logoses by Fact 5.6, and its right adjoint is fully faithful since P is (-1)-truncated. The *closed localization* associated to P is the localization obtained by Fact 5.7 for the singleton class of monomorphisms $\{\mathbf{0} \to P\}$. Let $\mathcal{L} \to \mathcal{K}$ be the closed localization associated to P. Then an object $A \in \mathcal{L}$ belongs to $\mathcal{K}$ if and only if $\mathbf{Map}(X, A) \to \mathbf{Map}(f^*\mathbf{0}, A)$ is an equivalence for every $f : X \to P$. Since $f^*\mathbf{0} \simeq \mathbf{0}$ by Proposition 5.3, $\mathbf{Map}(f^*\mathbf{0}, A) \simeq \mathbf{1}$. Therefore, A belongs to $\mathcal{K}$ if and only if $P^*A \in \mathcal{L}_{/P}$ is the final object, which is equivalent to that the projection $P \times A \to P$ is an equivalence.

5.2. **The language of $(\infty,2)$-category theory.** We formulate concepts in $(\infty,2)$-category theory using the $(\infty,1)$-*category* 2-**Cat** *of* $(\infty,2)$-*categories* axiomatized and proved to be equivalent to various models of $(\infty,2)$-categories by Barwick and Schommer-Pries [BSP21].

Definition 5.9. A (strict) 2-category $\mathcal{C}$ is said to be *gaunt* if only invertible 1-cells and 2-cells are the identities.

Example 5.10. The *walking m-cell* $\mathbf{G}_m$ for $0 \le m \le 2$ is the 2-category freely generated by a single m-cell and is gaunt.

Among Barwick and Schommer-Pries's axioms, the following are important to us.

Fact 5.11 [BSP21, Basic Data]**.** 2-**Cat** is presentable and contains finitely presentable gaunt 2-categories as a full subcategory.

Fact 5.12 [BSP21, Axiom C.2]**.** $\{\mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2\}$ is a set of generators for 2-**Cat**.

Fact 5.13 [BSP21, Axiom C.3]**.** The $(\infty, 1)$-category 2-**Cat** (more generally $2\text{-}\mathbf{Cat}_{/\mathbf{G}_m}$ for $0 \le m \le 2$) is cartesian closed. For $\mathcal{C}, \mathcal{D} \in 2\text{-}\mathbf{Cat}$, let $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ denote the exponential in 2-**Cat**.

Fact 5.14 [BSP21, Axiom C.4]**.** $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$ satisfy certain pushout formulas. We will recall them when needed.

Let us fix terminology and notation.

Definition 5.15. Objects in 2-**Cat** are called $(\infty, 2)$*-categories*. For an $(\infty, 2)$-category $\mathcal{C}$, morphisms in 2-**Cat** to $\mathcal{C}$ from $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$ are called *objects* or 0*-cells*, *morphisms* or 1*-cells*, and 2*-morphisms* or 2*-cells*, respectively, in $\mathcal{C}$. Morphisms in 2-**Cat** are called *functors*. For $(\infty, 2)$-categories $\mathcal{C}$ and $\mathcal{D}$, 1-cells in the $(\infty, 2)$-category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ are called *natural transformations*.

Fact 5.12 is particularly useful and mostly used in the following form.

Proposition 5.16. *Let Φ be a class of small $(\infty, 2)$-categories. If Φ is closed under small colimits (that is, $\operatorname{colim}_{i \in I} \mathcal{C}_i \in \Phi$ whenever $\mathcal{C}_i \in \Phi$ for every $i \in I$, for any small diagram $\mathcal{C} : I \to 2\text{-}\mathbf{Cat}$) and if Φ contains $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$, then Φ contains all small $(\infty, 2)$-categories.* □

Constructions 5.17, 5.18 and 5.20 below are easily justified by using the 2-fold complete Segal spaces model [Bar05].

Construction 5.17. Let $\mathcal{C}$ be an $(\infty, 2)$-category. We define the space $\mathbf{Obj}(\mathcal{C})$ of objects in $\mathcal{C}$ to be $\mathbf{Map}_{2\text{-}\mathbf{Cat}}(\mathbf{G}_0, \mathcal{C})$. For a pair of objects (x, y), an $(\infty, 1)$-category $\mathbf{Map}_{\mathcal{C}}(x, y)$ called the *mapping* $(\infty, 1)$*-category* is constructed. It is defined by the pullbacks

$$\begin{array}{ccc}
\mathbf{Obj}(\mathbf{Map}_{\mathcal{C}}(x,y)) & \longrightarrow & \mathbf{Map}_{2\text{-}\mathbf{Cat}}(\mathbf{G}_1, \mathcal{C}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (\mathbf{dom}, \mathbf{cod})} \\
\mathbf{1} & \xrightarrow[(x,y)]{} & \mathbf{Obj}(\mathcal{C}) \times \mathbf{Obj}(\mathcal{C})
\end{array}$$

and

$$\begin{array}{ccc}
\mathbf{Map}_{\mathbf{Map}_{\mathcal{C}}(x,y)}(u,v) & \longrightarrow & \mathbf{Map}_{2\text{-}\mathbf{Cat}}(\mathbf{G}_2, \mathcal{C}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (\mathbf{dom}, \mathbf{cod})} \\
\mathbf{1} & \xrightarrow[(u,v)]{} & \mathbf{Obj}(\mathbf{Map}_{\mathcal{C}}(x,y)) \times \mathbf{Obj}(\mathbf{Map}_{\mathcal{C}}(x,y)).
\end{array}$$

$\mathcal{C}$ also has a functorial composition operator between its mapping $(\infty, 1)$-categories.

Construction 5.18. For an $(\infty, 2)$-category $\mathcal{C}$, we define an $(\infty, 1)$-category $\mathbf{Core}_{(1)}(\mathcal{C})$ called the $(\infty, 1)$*-core of C* by $\mathbf{Obj}(\mathbf{Core}_{(1)}(\mathcal{C})) = \mathbf{Obj}(\mathcal{C})$ and $\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathcal{C})}(x, y) =$

$\mathbf{Obj}(\mathbf{Map}_{\mathcal{C}}(x,y))$. This defines a functor $\mathbf{Core}_{(1)} : 2\text{-}\mathbf{Cat} \to \mathbf{Cat}$. It is shown that $\mathbf{Core}_{(1)}$ has a fully faithful left adjoint, and thus we regard $\mathbf{Cat}$ as a coreflective full subcategory of 2-$\mathbf{Cat}$. An $(\infty,2)$-category $\mathcal{C}$ is an $(\infty,1)$-category if and only if it is *locally discrete* in the sense that $\mathbf{Map}_{\mathcal{C}}(x,y)$ is an ∞-groupoid for any $x, y \in \mathcal{C}$.

Definition 5.19. Let $\mathcal{C}$ be an $(\infty,2)$-category. A *locally full subcategory* of $\mathcal{C}$ is an $(\infty,2)$-category $\mathcal{C}'$ equipped with a functor $F : \mathcal{C}' \to \mathcal{C}$ such that $\mathbf{Obj}(\mathcal{C}') \to \mathbf{Obj}(\mathcal{C})$ is mono and $\mathbf{Map}_{\mathcal{C}'}(x,y) \to \mathbf{Map}_{\mathcal{C}}(F(x),F(y))$ is fully faithful for any $x,y \in \mathcal{C}'$. A locally full subcategory of $\mathcal{C}$ is usually specified by a class $\mathcal{C}'_0$ of 0-cells in $\mathcal{C}$ and a class $\mathcal{C}'_1$ of 1-cells in $\mathcal{C}$ between objects in $\mathcal{C}'_0$ such that $\mathcal{C}'_1$ contains all the equivalences between objects in $\mathcal{C}'_0$ and is closed under composition.

The *unicity theorem* [BSP21, Theorem 7.3] asserts that there are exactly $(\mathbb{Z}/2\mathbb{Z})^2$ automorphisms on 2-$\mathbf{Cat}$. Those automorphisms are *opposite* constructions.

Construction 5.20. Let $\mathcal{C}$ be an $(\infty,2)$-category. $(\infty,2)$-categories $\mathcal{C}^{\mathrm{op}(1)}$ and $\mathcal{C}^{\mathrm{op}(2)}$ are defined by $\mathbf{Obj}(\mathcal{C}^{\mathrm{op}(1)}) = \mathbf{Obj}(\mathcal{C}^{\mathrm{op}(2)}) = \mathbf{Obj}(\mathcal{C})$ and

$$\begin{aligned}\mathbf{Map}_{\mathcal{C}^{\mathrm{op}(1)}}(x,y) &= \mathbf{Map}_{\mathcal{C}}(y,x)\\ \mathbf{Map}_{\mathcal{C}^{\mathrm{op}(2)}}(x,y) &= \mathbf{Map}_{\mathcal{C}}(x,y)^{\mathrm{op}}.\end{aligned}$$

We abbreviate $(\mathcal{C}^{\mathrm{op}(1)})^{\mathrm{op}(2)} \simeq (\mathcal{C}^{\mathrm{op}(2)})^{\mathrm{op}(1)}$ as $\mathcal{C}^{\mathrm{op}(1,2)}$.

The following is an $(\infty,2)$-categorical version of the fact that a natural transformation is invertible if it is point-wise invertible. It is proved without using a specific model.

Proposition 5.21. *For any $(\infty,2)$-categories $\mathcal{C}$ and $\mathcal{D}$, the restriction functor*

$$\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathcal{C},\mathcal{D})) \to \mathbf{Core}_{(1)}(\mathbf{Fun}(\mathbf{Obj}(\mathcal{C}),\mathcal{D}))$$

is conservative.

Proof. Let Φ be the class of $(\infty,2)$-categories $\mathcal{C}$ such that the functor $\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathcal{C},\mathcal{D})) \to \mathbf{Core}_{(1)}(\mathbf{Fun}(\mathbf{Obj}(\mathcal{C}),\mathcal{D}))$ is conservative for any $(\infty,2)$-category $\mathcal{D}$. We first show that Φ is closed under colimits. For a functor $\mathcal{C} : I \to 2\text{-}\mathbf{Cat}$, we have the following commutative diagram.

$$\begin{array}{ccc}\mathbf{Core}_{(1)}(\mathbf{Fun}(\operatorname{colim}_{i\in I}\mathcal{C}_i,\mathcal{D})) & \longrightarrow & \mathbf{Core}_{(1)}(\mathbf{Fun}(\mathbf{Obj}(\operatorname{colim}_{i\in I}\mathcal{C}_i),\mathcal{D})) \\ {\scriptstyle\simeq}\downarrow & & \downarrow \\ \lim_{i\in I}\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathcal{C}_i,\mathcal{D})) & \longrightarrow & \lim_{i\in I}\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathbf{Obj}(\mathcal{C}_i),\mathcal{D}))\end{array}$$

The left functor is an equivalence as $\mathbf{Core}_{(1)}$ preserves limits. If every $\mathcal{C}_i$ belongs to Φ, then the bottom functor is conservative, and so is the top.

It remains to show that Φ contains $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$. Note that when $\mathcal{C}$ is locally discrete, we have $\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathcal{C},\mathcal{D})) \simeq \mathbf{Fun}(\mathcal{C},\mathbf{Core}_{(1)}(\mathcal{D}))$, and $(\infty,1)$-category theory applies. The only non-trivial case is when $\mathcal{C} = \mathbf{G}_2$. We recall the following pushout in 2-$\mathbf{Cat}$ [BSP21, Axiom C.4].

$$\begin{array}{ccc}\mathbf{G}_2 & \longrightarrow & \mathbf{G}_1 +_{\mathbf{G}_0} \mathbf{G}_2 \\ \downarrow & \ulcorner & \downarrow \\ \mathbf{G}_2 +_{\mathbf{G}_0} \mathbf{G}_1 & \longrightarrow & \mathbf{G}_2 \times \mathbf{G}_1\end{array}$$

$\mathbf{G}_2 \times \mathbf{G}_1$ should be a cylinder

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \left(\Rightarrow\right) & & \left(\Rightarrow\right) \\ \bullet & \longrightarrow & \bullet, \end{array}$$

and the above pushout formula asserts that this is the case because the cylinder is obtained by gluing the lower-left part $\mathbf{G}_2 +_{\mathbf{G}_0} \mathbf{G}_1 = \left(\begin{array}{ccc} \bullet & & \\ \left(\Rightarrow\right) & & \\ \bullet & \longrightarrow & \bullet \end{array}\right)$ and the upper-right part $\mathbf{G}_1 +_{\mathbf{G}_0} \mathbf{G}_2 = \left(\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \left(\Rightarrow\right) \\ & & \bullet \end{array}\right)$. A functor $F \in \mathbf{Map}_{\mathbf{2\text{-}Cat}}(\mathbf{G}_1, \mathbf{Fun}(\mathbf{G}_2, \mathcal{D})) \simeq \mathbf{Map}_{\mathbf{2\text{-}Cat}}(\mathbf{G}_2 \times \mathbf{G}_1, \mathcal{D})$ is thus a diagram in $\mathcal{D}$ of the form

$$\begin{array}{ccc} F(0,0) & \xrightarrow{F(0,01)} & F(0,1) \\ \left(\Longrightarrow\right) & & \left(\Longrightarrow\right) \\ F(1,0) & \xrightarrow[F(1,01)]{} & F(1,1). \end{array}$$

When the morphisms $F(0,01)$ and $F(1,01)$ are invertible, then one can construct an inverse of F in $\mathbf{Fun}(\mathbf{G}_2, \mathcal{D})$. □

5.3. **Scaled simplicial sets.** We review one of models for $(\infty, 2)$-categories, *scaled simplicial sets* [Lur09b]. This model is convenient for presenting $(\infty, 2)$-categories by combinatorial data, provides computation of colimits of $(\infty, 2)$-categories, and gives the $(\infty, 2)$-category of $(\infty, 1)$-categories a useful universal property.

Definition 5.22. A *scaled simplicial set* is a simplicial set A equipped with a class of 2-simplices called *thin* 2-simplices such that all the degenerate 2-simplices are thin.

A scaled simplicial set A is thought of as a presentation of an $(\infty, 2)$-category: the 0-cells are the 0-simplices of A; the 1-cells are generated by 1-simplices of A; the 2-cells are generated by 2-simplices of A in the following direction;

$$\begin{array}{ccc} a_0 & \longrightarrow & a_2 \\ \searrow & \Downarrow & \nearrow \\ & a_1 & \end{array}$$

thin 2-simplices are made invertible 2-cells; higher simplices presents homotopies filling certain diagrams.

Proposition 5.23. *There is a model structure on the category of scaled simplicial sets that presents* 2-**Cat**.

Proof. See [Lur09b] for the model structure and comparison with other models for $(\infty, 2)$-categories. Barwick and Schommer-Pries [BSP21] show that the $(\infty, 1)$-category presented by this model structure indeed satisfies their axioms. □

Example 5.24 [Lur09a, Example 1.1.5.9]**.** Let I be a (decidable) poset and we regard it as a scaled simplicial set with no non-degenerate thin 2-simplex. It presents a gaunt 2-category $\langle I \rangle$ defined as follows. The objects of $\langle I \rangle$ are the elements of I. The mapping category $\mathbf{Map}_{\langle I \rangle}(i, j)$ for $i, j \in I$ is the poset of totally ordered subsets $J \subset I$ with least element i and largest element j. A morphism from i to j is thus a chain $i = k_0 < k_1 < \cdots < k_n = j$. It then follows that $\mathbf{Core}_{(1)}(\langle I \rangle)$ is the free category over the strict ordering relation on I. For any $i < j$ in I, the chain $(i < j)$ is the initial object in $\mathbf{Map}_{\langle I \rangle}(i, j)$.

We can calculate colimits in 2-**Cat** via homotopy colimits of scaled simplicial sets. We do not need much details, but a useful consequence is the following.

Corollary 5.25. *For any functor $\mathcal{C} : I \to 2\text{-}\mathbf{Cat}$, the map*

$$\operatorname*{colim}_{i \in I} \mathbf{Obj}(\mathcal{C}_i) \to \mathbf{Obj}(\operatorname*{colim}_{i \in I} \mathcal{C}_i)$$

is surjective. □

Lurie [Lur09b, Section 4.5] shows that the $(\infty, 2)$-category of $(\infty, 1)$-categories is presented by the scaled simplicial set classifying *locally cocartesian fibrations*.

Definition 5.26. Let A be a scaled simplicial set and B a simplicial set. A map $p : B \to A$ of simplicial sets is a *locally cocartesian fibration* if the following are satisfied:

(1) p is an inner fibration of simplicial sets;
(2) for every 1-simplex $a : \Delta^1 \to A$, the base change $a^* p : a^* B \to \Delta^1$ is a cocartesian fibration of simplicial sets;
(3) for every thin 2-simplex $a : \Delta^2 \to A$, the base change $a^* p$ is a cocartesian fibration of simplicial sets.

Fact 5.27 [Lur09b, Corollary 4.5.7]**.** There exists a *universal locally cocartesian fibration with small fibers* $\pi : \widetilde{\mathbf{Cat}}^{\mathrm{sc}} \to \mathbf{Cat}^{\mathrm{sc}}$ in the following sense:

(1) $\mathbf{Cat}^{\mathrm{sc}}$ is a large fibrant scaled simplicial set, $\widetilde{\mathbf{Cat}}^{\mathrm{sc}}$ is a large simplicial set, and π is a locally cocartesian fibration whose fibers are small;
(2) for any locally cocartesian fibration with small fibers p, there exists a unique, up to homotopy, homotopy pullback from p to π.

Construction 5.28. Let $\mathbf{Cat}^{(2)} \in {\Uparrow} 2\text{-}\mathbf{Cat}$ be the $(\infty, 2)$-category presented by $\mathbf{Cat}^{\mathrm{sc}}$. By the construction of $\mathbf{Cat}^{\mathrm{sc}}$ [Lur09b, Definition 4.5.1], we have $\mathbf{Obj}(\mathbf{Cat}^{(2)}) \simeq \mathbf{Obj}(\mathbf{Cat})$ and $\mathbf{Map}_{\mathbf{Cat}^{(2)}}(\mathcal{C}, \mathcal{D}) \simeq \mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

Notation 5.29. The constructions of $\mathbf{Cat}^{\mathrm{sc}}$ and $\mathbf{Cat}^{(2)}$ are parameterized by a universe. We use the notation ${\Uparrow} \mathbf{Cat}^{\mathrm{sc}}$ and ${\Uparrow} \mathbf{Cat}^{(2)}$ for those constructions with respect to the larger universe ${\Uparrow} \mathfrak{U}$.

The notion of locally cocartesian fibration is, however, specific to the scaled simplicial sets model. A more model-independent notion is as follows.

Definition 5.30. A functor $P : \mathcal{D} \to \mathcal{C}$ between $(\infty, 2)$-categories is a 1-*cocartesian* 2-*right fibration* if the following conditions are satisfied:

(1) P is *locally a right fibration* in the sense that for any objects $x, y \in \mathcal{D}$, the functor $P : \mathbf{Map}_{\mathcal{D}}(x, y) \to \mathbf{Map}_{\mathcal{C}}(P(x), P(y))$ is a right fibration;
(2) $\mathbf{Core}_{(1)}(P) : \mathbf{Core}_{(1)}(\mathcal{D}) \to \mathbf{Core}_{(1)}(\mathcal{C})$ is a cocartesian fibration.

By duality, 1-*(cocartesian/cartesian)* 2-*(right/left) fibrations* are defined. (These are called *inner/outer cocartesian/cartesian fibrations* for the scaled simplicial set model [GHL21a].)

Remark 5.31. Assuming Item 1 in Definition 5.30, any cocartesian morphism $u : x \to y$ in $\mathbf{Core}_{(1)}(\mathcal{D})$ is also cocartesian in the **Cat**-enriched sense: for any object $z \in \mathcal{D}$, the square

$$\begin{array}{ccc} \mathbf{Map}_{\mathcal{D}}(y,z) & \xrightarrow{\_\circ u} & \mathbf{Map}_{\mathcal{D}}(x,z) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbf{Map}_{\mathcal{C}}(P(y),P(z)) & \xrightarrow[\_\circ P(u)]{} & \mathbf{Map}_{\mathcal{C}}(P(x),P(z)) \end{array}$$

is a pullback in **Cat**, because $\mathbf{Obj} : \mathbf{Cat} \to \mathbf{Spc}$ reflects pullbacks of right fibrations.

Proposition 5.32. $\mathbf{Cat}^{(2)}$ *is part of a universal* 1*-cocartesian* 2*-right fibration with small fibers* $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}} \to \mathbf{Cat}^{(2)}$*. For a functor* $\mathcal{C} : I \to \mathbf{Cat}^{(2)}$*, let* $\mathbf{El}_I(\mathcal{C}) \to I$ *denote the corresponding* 1*-cocartesian* 2*-right fibration.*

Proof. This follows from the equivalence between locally cocartesian fibrations and inner cartesian fibrations of categories enriched over marked simplicial sets given by Gagna, Harpaz, and Lanari [GHL21a, Propositions 2.4.1 and 3.1.3]. □

An object in $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$ is a pair $(\mathcal{C}, x)$ consisting of an $(\infty,1)$-category $\mathcal{C}$ and an object $x \in \mathcal{C}$. A morphism $(\mathcal{C}, x) \to (\mathcal{D}, y)$ in $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$ is a pair (F, u) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a morphism $u : F(x) \to y$. A 2-morphism $(F, u) \Rightarrow (G, v) : (\mathcal{C}, x) \to (\mathcal{D}, y)$ in $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$ is a pair (σ, p) consisting of a natural transformation $\sigma : F \Rightarrow G$ and a path $p : u \sim v \circ \sigma_x$.

For the purpose of Section 5.4, we introduce a variant of $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$.

Construction 5.33. The equivalence of $(\infty,1)$-categories $\mathbf{Cat} \ni \mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}} \in \mathbf{Cat}$ extends to an equivalence of $(\infty,2)$-categories

$$(\_)^{\mathrm{op}} : \mathbf{Cat}^{(2)} \simeq (\mathbf{Cat}^{(2)})^{\mathrm{op}(2)}.$$

Let $\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}} = ((\_)^{\mathrm{op}})^*(\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}})^{\mathrm{op}(2)}$. By construction, the functor $(\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}})^{\mathrm{op}(1,2)} \to (\mathbf{Cat}^{(2)})^{\mathrm{op}(1,2)}$ classifies 1-cartesian 2-right fibrations with small fibers. For a functor $\mathcal{C} : I^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$, let $\mathbf{El}_I(\mathcal{C}) \to I$ denote the corresponding 1-cartesian 2-right fibration.

In other words, $\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}}$ is the fiberwise opposite of $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$. Thus, the objects of $\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}}$ are the same as $\widetilde{\mathbf{Cat}}^{(2),\mathrm{R}}$, but a morphism $(\mathcal{C}, x) \to (\mathcal{D}, y)$ in $\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}}$ is a pair (F, u) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a morphism $u : y \to F(x)$.

5.4. **Oplax limits.** *Oplax limits* in general $(\infty,2)$-categories are defined by Gagna, Harpaz, and Lanari [GHL21a]. In this paper, we only need oplax limits of $(\infty,1)$-categories which have the following simple construction [GHL21a, Example 5.3.12].

Construction 5.34. Let I be a small $(\infty, 2)$-category and $\mathcal{C} : I^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ a functor. The *oplax limit of* $\mathcal{C}$ is defined to be the pullback

$$\begin{array}{ccc} \mathrm{opLaxLim}_{i \in I} \mathcal{C}_i & \dashrightarrow & \mathbf{Fun}(I, (\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}})^{\mathrm{op}(1,2)}) \\ \vdots & \lrcorner & \downarrow \\ \mathbf{G}_0 & \xrightarrow[\mathcal{C}]{} & \mathbf{Fun}(I, (\mathbf{Cat}^{(2)})^{\mathrm{op}(1,2)}). \end{array}$$

Equivalently, it is the pullback

$$\begin{array}{ccc} \mathrm{opLaxLim}_{i \in I} \mathcal{C}_i & \dashrightarrow & \mathbf{Fun}(I, \mathbf{El}_I(\mathcal{C})) \\ \vdots & \lrcorner & \downarrow \\ \mathbf{G}_0 & \xrightarrow[\mathbf{id}_I]{} & \mathbf{Fun}(I, I). \end{array}$$

In other words, $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ is the $(\infty, 2)$-category of sections of $\mathbf{El}_I(\mathcal{C}) \to I$. Any functor $F : J \to I$ induces a functor $F^* : \mathrm{opLaxLim}_{i \in I} \mathcal{C}_i \to \mathrm{opLaxLim}_{j \in J} \mathcal{C}_{F(j)}$.

Remark 5.35. $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ is small and locally discrete. Indeed, the class of small $(\infty, 2)$-categories I such that $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ is small and locally discrete for any $\mathcal{C} : I^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ is closed under small colimits, and the cases when $I = \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2$ are directly calculated in Examples 5.37 to 5.39 below. Hence, we regard $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ as a small $(\infty, 1)$-category.

Concretely, an object x in $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ consists of: an object $x_i \in \mathcal{C}_i$ for any object $i \in I$; a morphism $x_\alpha : x_i \to \mathcal{C}_\alpha(x_j)$ for any morphism $\alpha : i \to j$ in I; some coherence data. A morphism $u : x \to y$ in $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ consists of: a morphism $u_i : x_i \to y_i$ for any object $i \in I$; a homotopy u_α filling the square

$$\begin{array}{ccc} x_i & \xrightarrow{u_i} & y_i \\ {\scriptstyle x_\alpha}\downarrow & & \downarrow{\scriptstyle y_\alpha} \\ \mathcal{C}_\alpha(x_j) & \xrightarrow[\mathcal{C}_\alpha(u_j)]{} & \mathcal{C}_\alpha(y_j) \end{array}$$

for any morphism $\alpha : i \to j$ in I; some coherence data.

Example 5.36. When I is an ∞-groupoid, we have $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i \simeq \prod_{i \in I} \mathcal{C}_i$. For a general I, we have the forgetful functor $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i \to \mathrm{opLaxLim}_{i \in \mathbf{Obj}(I)} \mathcal{C}_i \simeq \prod_{i \in I} \mathcal{C}_i$ which is conservative by Proposition 5.21.

Example 5.37. When $I = \mathbf{G}_0$, a functor $\mathbf{G}_0^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ corresponds to an $(\infty, 1)$-category $\mathcal{C}$, and its oplax limit is $\mathcal{C}$ itself.

Example 5.38. When $I = \mathbf{G}_1$, a functor $\mathcal{C} : \mathbf{G}_1^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ corresponds to a functor $F : \mathcal{C}_1 \to \mathcal{C}_0$. Its oplax limit is the $(\infty, 1)$-category of triples (x_0, x_1, u) consisting of objects $x_0 \in \mathcal{C}_0$ and $x_1 \in \mathcal{C}_1$ and a morphism $u : x_0 \to F(x_1)$. In other words, we have the following pullback.

$$\begin{array}{ccc} \mathrm{opLaxLim}_{i \in \mathbf{G}_1} \mathcal{C}_i & \longrightarrow & \mathcal{C}_0^{\to} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{cod}} \\ \mathcal{C}_1 & \xrightarrow[F]{} & \mathcal{C}_0 \end{array}$$

This oplax limit is called the *Artin gluing* for F and denoted by $\mathbf{Gl}(F)$.

Example 5.39. When $I = \mathbf{G}_2$, a functor $\mathcal{C} : \mathbf{G}_2^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ corresponds to a natural transformation $\sigma : F_1 \Rightarrow F_0 : \mathcal{C}_1 \to \mathcal{C}_0$. Since $(\widetilde{\mathbf{Cat}}^{(2),\mathrm{L}})^{\mathrm{op}(1,2)} \to (\mathbf{Cat}^{(2)})^{\mathrm{op}(1,2)}$ is locally a right fibration, a section of it over $\mathcal{C}$ is completely determined by the restriction along the codomain inclusion $\mathbf{G}_1 \to \mathbf{G}_2$. Therefore, $\mathrm{opLaxLim}_{i \in \mathbf{G}_2} \mathcal{C}_i \simeq \mathbf{Gl}(F_1)$.

Example 5.40. Let J be a small $(\infty,1)$-category and $I : J \to 2\text{-}\mathbf{Cat}$ a functor. For any functor $\mathcal{C} : (\mathrm{colim}_{j\in J} I_j)^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$, we have a canonical equivalence

$$\mathop{\mathrm{opLaxLim}}_{i \in \mathrm{colim}_{j\in J} I_j} \mathcal{C}_i \simeq \lim_{j \in J} \mathop{\mathrm{opLaxLim}}_{i \in I_j} \mathcal{C}_{\iota_j(i)}.$$

Hence, arbitrary oplax limits are constructed from Examples 5.37 to 5.39 using small limits.

We show that ∞-logoses are closed under oplax limits. For this, we consider a weaker notion of morphism of ∞-logoses.

Definition 5.41. A functor between accessible categories is *accessible* if it preserves small κ-filtered colimits for some regular cardinal κ.

Notation 5.42. We define $\mathbf{Logos}^{(2)}_{\mathrm{LexAcc}} \subset \Uparrow\mathbf{Cat}^{(2)}$ to be the locally full subcategory spanned by the ∞-logoses and the lex, accessible functors between ∞-logoses.

Theorem 5.43. *Let I be a small $(\infty,2)$-category and $\mathcal{L} : I^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ a functor. Then $\mathrm{opLaxLim}_{i\in I} \mathcal{L}_i$ is an ∞-logos. Moreover, the forgetful functor $\mathrm{opLaxLim}_{i\in I} \mathcal{L}_i \to \prod_{i\in I} \mathcal{L}_i$ preserves small colimits and finite limits.*

The rest of this subsection is devoted to the proof of Theorem 5.43.

Fact 5.44 [Lur09a, Proposition 6.3.2.3]**.** $\mathbf{Logos} \subset \Uparrow\mathbf{Cat}$ is closed under small limits.

Lemma 5.45. *Let $F : \mathcal{L}_1 \to \mathcal{L}_0$ be a lex, accessible functor between ∞-logoses. Then $\mathbf{Gl}(F)$ is an ∞-logos, and the projections $\mathbf{Gl}(F) \to \mathcal{L}_0$ and $\mathbf{Gl}(F) \to \mathcal{L}_1$ preserve small colimits and finite limits.*

Proof. We use a characterization of presentability: an $(\infty,1)$-category is presentable if and only if it is accessible and has small colimits [Lur09a, Definition 5.5.0.1]. It follows from [Lur09a, Propositions 5.4.4.3 and 5.4.6.6] that $\mathbf{Gl}(F)$ is accessible. By construction, $\mathbf{Gl}(F)$ is the $(\infty,1)$-category of triples (A_0, A_1, f) consisting of objects $A_0 \in \mathcal{L}_0$ and $A_1 \in \mathcal{L}_1$ and a map $f : A_0 \to F(A_1)$. It follows from this description that the projection $\mathbf{Gl}(F) \to \mathcal{L}_0 \times \mathcal{L}_1$ creates small colimits and finite limits. In particular, $\mathbf{Gl}(F)$ admits small colimits and thus is presentable. Since the projection $\mathbf{Gl}(F) \to \mathcal{L}_0 \times \mathcal{L}_1$ is conservative by Example 5.36, $\mathbf{Gl}(F)$ is an ∞-logos by Proposition 5.2. □

Proof of Theorem 5.43. Let Φ be the class of small $(\infty,2)$-categories I such that for any functor $\mathcal{L} : I^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$, the oplax limit $\mathrm{opLaxLim}_{i\in I} \mathcal{L}_i$ is an ∞-logos and the forgetful functor $\mathrm{opLaxLim}_{i\in I} \mathcal{L}_i \to \prod_{i\in I} \mathcal{L}_i$ preserves small colimits and finite limits. It is enough to show that Φ is closed under small colimits and contains $\mathbf{G}_0$, $\mathbf{G}_1$, and $\mathbf{G}_2$.

Let $I : J \to 2\text{-}\mathbf{Cat}$ be a functor from a small $(\infty,1)$-category J and suppose that every I_j belongs to Φ. Let $\mathcal{L} : (\mathrm{colim}_{j\in J} I_j)^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ be a functor. As in Example 5.40, we have

$$\mathop{\mathrm{opLaxLim}}_{i \in \mathrm{colim}_{j\in J} I_j} \mathcal{L}_i \simeq \lim_{j \in J} \mathop{\mathrm{opLaxLim}}_{i \in I_j} \mathcal{L}_{\iota_j(i)}.$$

Since $I_j \in \Phi$, small colimits and finite limits in $\mathrm{opLaxLim}_{i \in I_j} \mathcal{L}_{\iota_j(i)}$ are computed in $\prod_{i \in I_j} \mathcal{L}_{\iota_j(i)}$. Thus, for any morphism $j_1 \to j_2$ in J, the functor $\mathrm{opLaxLim}_{i \in I_{j_2}} \mathcal{L}_{\iota_{j_1}(i)} \to \mathrm{opLaxLim}_{i \in I_{j_1}} \mathcal{L}_{\iota_{j_2}(i)}$ preserves small colimits and finite limits. It then follows from Fact 5.44 that $\lim_{j \in J} \mathrm{opLaxLim}_{i \in I_j} \mathcal{L}_{\iota_j(i)}$ is an ∞-logos. Consider the following commutative square.

$$\begin{array}{ccc} \mathrm{opLaxLim}_{i \in \mathrm{colim}_{j \in J} I_j} \mathcal{L}_i & \xrightarrow{\simeq} & \lim_{j \in J} \mathrm{opLaxLim}_{i \in I_j} \mathcal{L}_{\iota_j(i)} \\ \downarrow & & \downarrow \\ \prod_{i \in \mathbf{Obj}(\mathrm{colim}_{j \in J} I_j)} \mathcal{L}_i & \longrightarrow & \lim_{j \in J} \prod_{i \in \mathbf{Obj}(I_j)} \mathcal{L}_{\iota_j(i)} \end{array}$$

We have seen that the top functor is an equivalence. The right functor preserves small colimits and finite limits as every I_j belongs to Φ. The bottom functor is equivalent to the restriction along $\mathrm{colim}_{j \in J} \mathbf{Obj}(I_j) \to \mathbf{Obj}(\mathrm{colim}_{j \in J} I_j)$ and thus preserves small colimits and finite limits. By Corollary 5.25, the bottom functor is conservative. We thus conclude that the left functor preserves small colimits and finite limits. Hence, $\mathrm{colim}_{j \in J} I_j$ belongs to Φ.

$\mathbf{G}_0$ belongs to Φ by Example 5.37. $\mathbf{G}_1$ belongs to Φ by Example 5.38 and Lemma 5.45. $\mathbf{G}_2$ belongs to Φ by Example 5.39 and by the case of $\mathbf{G}_1$. □

5.5. **Oplax natural transformations.** An alternative description of oplax limits is that they are $(\infty, 1)$-categories of *oplax natural transformations* (Proposition 5.57).

Construction 5.46 [GHL21b, Definition 2.1]**.** Let A and B be scaled simplicial sets. The *Gray product* $A \otimes B$ is the scaled simplicial set whose underlying simplicial set is the cartesian product of A and B and whose 2-simplex $(a, b) : \Delta^2 \to A \times B$ is thin if both a and b are thin and either a degenerates along $\Delta^{\{1,2\}}$ or b degenerates along $\Delta^{\{0,1\}}$.

Fact 5.47 [GHL21b, Theorem 2.17]**.** The Gray product is part of a left Quillen bifunctor on scaled simplicial sets. Consequently, it induces a functor

$$\_ \otimes \_ : 2\text{-}\mathbf{Cat} \times 2\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}$$

preserving small colimits on each variable.

Construction 5.48. By Fact 5.47 and by the adjoint functor theorem, for any $(\infty, 2)$-category $\mathcal{C}$, the functors $(\_ \otimes \mathcal{C})$ and $(\mathcal{C} \otimes \_)$ have right adjoints $\mathbf{Fun}(\mathcal{C}, \_)_{\mathrm{Lax}}$ and $\mathbf{Fun}(\mathcal{C}, \_)_{\mathrm{opLax}}$, respectively.

Example 5.49. $\mathcal{C} \otimes \mathbf{G}_0 \simeq \mathcal{C}$, and thus $\mathbf{Obj}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})_{\mathrm{opLax}}) \simeq \mathbf{Map}_{2\text{-}\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$. Dually, $\mathbf{Obj}(\mathbf{Fun}(\mathcal{C}, \mathcal{D})_{\mathrm{Lax}}) \simeq \mathbf{Map}_{2\text{-}\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$.

Example 5.50. Let $\mathcal{C}$ be a scaled simplicial set and $u : x \to y$ a 1-simplex in $\mathcal{C}$. Consider the following 2-simplices in $\mathcal{C} \otimes \Delta^1$.

$$\begin{array}{ccc} (x, 0) & \xrightarrow{(x, 0 \le 1)} & (x, 1) \\ {\scriptstyle (u,0)} \downarrow & \searrow^{(u, 0 \le 1)} & \downarrow {\scriptstyle (u,1)} \\ (y, 0) & \xrightarrow[(y, 0 \le 1)]{} & (y, 1) \end{array} \tag{5.2}$$

By definition, the lower 2-simplex is thin, but the upper one is not (unless u is degenerate). Hence, these 2-simplices compose and yields a 2-cell

$$\begin{array}{ccc} (x,0) & \xrightarrow{(x,0\leq 1)} & (x,1) \\ {\scriptstyle (u,0)}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle (u,1)} \\ (y,0) & \xrightarrow[(y,0\leq 1)]{} & (y,1) \end{array} \tag{5.3}$$

in the $(\infty,2)$-category presented by $\mathcal{C} \otimes \Delta^1$. Then, a 1-cell $\sigma : F \to G$ in $\mathbf{Fun}(\mathcal{C},\mathcal{D})_{\mathrm{opLax}}$ assigns: a 1-cell $\sigma_x : F(x) \to G(x)$ to every 0-cell $x \in \mathcal{C}$; a 2-cell

$$\begin{array}{ccc} F(x) & \xrightarrow{\sigma_x} & G(x) \\ {\scriptstyle F(u)}\big\downarrow & \overset{\sigma_u}{\nearrow} & \big\downarrow{\scriptstyle G(u)} \\ F(y) & \xrightarrow[\sigma_y]{} & G(y) \end{array}$$

to every 1-cell $u : x \to y$; and coherence data to higher cells. Such a structure is called an *oplax natural transformation from F to G*. Dually, 1-cells in $\mathbf{Fun}(\mathcal{C},\mathcal{D})_{\mathrm{Lax}}$ are called *lax natural transformations*. For a lax natural transformation $\sigma : F \to G$, the 2-cell σ_u is in the opposite direction $G(u) \circ \sigma_x \Rightarrow \sigma_y \circ F(u)$.

Remark 5.51. By definition, we have a map $A \otimes B \to A \times B$ of scaled simplicial sets which exhibits $A \times B$ as the one obtained from $A \otimes B$ by making the upper 2-simplex in the diagram of the form (5.2) thin. Thus, for $(\infty,2)$-categories $\mathcal{C}$ and $\mathcal{D}$, the cartesian product $\mathcal{C} \times \mathcal{D}$ is obtained from the Gray product $\mathcal{C} \otimes \mathcal{D}$ by making the 2-cell in the diagram of the form (5.3) invertible. By an adjoint argument, $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ is regarded as the locally full subcategory of $\mathbf{Fun}(\mathcal{C},\mathcal{D})_{\mathrm{Lax}}$ whose 1-cells are the lax natural transformations σ such that the 2-cell σ_u is invertible for any 1-cell u in $\mathcal{C}$. We may also regard $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ as a locally full subcategory of $\mathbf{Fun}(\mathcal{C},\mathcal{D})_{\mathrm{opLax}}$ in the same way.

Lax natural transformations correspond to functors between 1-cocartesian 2-right fibrations.

Proposition 5.52. *Let I be an $(\infty,2)$-category and let $\mathcal{C},\mathcal{D} : I \to \mathbf{Cat}^{(2)}$ be functors. We have an equivalence*

$$\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I,\mathbf{Cat}^{(2)})_{\mathrm{Lax}})}(\mathcal{C},\mathcal{D}) \simeq \mathbf{Map}_{\mathbf{2\text{-}Cat}_{/I}}(\mathbf{El}_I(\mathcal{C}),\mathbf{El}_I(\mathcal{D}))$$

natural in I.

Proposition 5.52 follows from the following special cases which are already known.

Fact 5.53 [GHL21a, Corollary 4.4.3]**.** Let I be an $(\infty,2)$-category. For any functor $\mathcal{D} : I \to \mathbf{Cat}^{(2)}$, we have an equivalence

$$\mathbf{Map}_{\mathbf{Fun}(I,\mathbf{Cat}^{(2)})_{\mathrm{Lax}}}(\lambda\_.\mathbf{1},\mathcal{D}) \simeq \mathbf{Map}_{\mathbf{2\text{-}Cat}^{(2)}_{/I}}(I,\mathbf{El}_I(\mathcal{D})).$$

Fact 5.54 [HHLN23, Theorem E]**.** Let I be an $(\infty,1)$-category. The map $(\mathcal{C} : I \to \mathbf{Cat}^{(2)}) \mapsto \mathbf{El}_I(\mathcal{C})$ induces an equivalence between $\mathbf{Fun}(I,\mathbf{Cat}^{(2)})_{\mathrm{Lax}}$ and the full subcategory of $\mathbf{Cat}^{(2)}_{/I}$ spanned by the cocartesian fibrations over I. Dually, $\mathbf{Fun}(I^{\mathrm{op}},\mathbf{Cat}^{(2)})_{\mathrm{opLax}}$ is equivalent to the full subcategory of $\mathbf{Cat}^{(2)}_{/I}$ spanned by the cartesian fibrations over I.

Proof of Proposition 5.52. We have a lax natural transformation $\eta : (\lambda\_.\mathbf{1}) \to \mathcal{C}|_{\mathbf{El}_I(\mathcal{C})}$ corresponding to the diagonal functor $\mathbf{El}_I(\mathcal{C}) \to \mathbf{El}_I(\mathcal{C}) \times_I \mathbf{El}_I(\mathcal{C})$ by Fact 5.53. The precomposition with η induces a map

$$\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I,\mathbf{Cat}^{(2)})_{\mathrm{Lax}})}(\mathcal{C},\mathcal{D}) \to \mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(\mathbf{El}_I(\mathcal{C}),\mathbf{Cat}^{(2)})_{\mathrm{Lax}})}(\lambda\_.\mathbf{1}, \mathcal{D}|_{\mathbf{El}_I(\mathcal{C})}), \quad (5.4)$$

and the codomain is by Fact 5.53 equivalent to $\mathbf{Map}_{2\text{-}\mathbf{Cat}_{/\mathbf{El}_I(\mathcal{C})}}(\mathbf{El}_I(\mathcal{C}), \mathbf{El}_I(\mathcal{C}) \times_I \mathbf{El}_I(\mathcal{D})) \simeq \mathbf{Map}_{2\text{-}\mathbf{Cat}_{/I}}(\mathbf{El}_I(\mathcal{C}), \mathbf{El}_I(\mathcal{D}))$. One can verify that the map (5.4) is an equivalence by reducing it to the cases when $\mathcal{C}$ is locally discrete (Fact 5.54) and when $\mathcal{C} = \mathbf{G}_2$. □

A dual argument shows the following.

Corollary 5.55. *Let I be an $(\infty,2)$-category and let $\mathcal{C},\mathcal{D} : I^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ be functors. We have an equivalence*

$$\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I^{\mathrm{op}(1,2)},\mathbf{Cat}^{(2)})_{\mathrm{opLax}})}(\mathcal{C},\mathcal{D}) \simeq \mathbf{Map}_{2\text{-}\mathbf{Cat}_{/I}}(\mathbf{El}_I(\mathcal{C}), \mathbf{El}_I(\mathcal{D}))$$

natural in I. □

Natural transformations correspond to functors preserving cocartesian morphisms.

Proposition 5.56. *The equivalence in Proposition 5.52 is restricted to an equivalence between $\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I,\mathbf{Cat}^{(2)}))}(\mathcal{C},\mathcal{D})$ and the space of functors $\mathbf{El}_I(\mathcal{C}) \to \mathbf{El}_I(\mathcal{D})$ over I preserving cocartesian 1-cells.*

Proof. The equivalence between these mapping spaces is due to Lurie [Lur09b, Theorem 3.8.1]. One can see that it coincides with the equivalence in Proposition 5.52. □

Oplax limits are $(\infty,1)$-categories of oplax natural transformations in the following sense.

Proposition 5.57. *Let $\mathcal{C} : I^{\mathrm{op}(1,2)} \to \mathbf{Cat}^{(2)}$ be a functor. For any $(\infty,1)$-category $\mathcal{D}$, we have a natural equivalence*

$$\mathbf{Map}_{\mathbf{Cat}}(\mathcal{D}, \operatorname*{opLaxLim}_{i\in I} \mathcal{C}_i) \simeq \mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I^{\mathrm{op}(1,2)},\mathbf{Cat}^{(2)})_{\mathrm{opLax}})}(\lambda\_.\mathcal{D},\mathcal{C}).$$

Proof.

$$\begin{aligned}
&\mathbf{Map}_{\mathbf{Cat}}(\mathcal{D}, \operatorname*{opLaxLim}_{i\in I} \mathcal{C}_i)\\
\simeq\quad &\{\text{definition}\}\\
&\mathbf{Map}_{2\text{-}\mathbf{Cat}_{/I}}(I \times \mathcal{D}, \mathbf{El}_I(\mathcal{C}))\\
\simeq\quad &\{\text{Corollary 5.55}\}\\
&\mathbf{Map}_{\mathbf{Core}_{(1)}(\mathbf{Fun}(I^{\mathrm{op}(1,2)},\mathbf{Cat}^{(2)})_{\mathrm{opLax}})}(\lambda_.\mathcal{D},\mathcal{C}).
\end{aligned}$$

□

A useful source of oplax natural transformations is the following *mate correspondence* whose special case when $\mathcal{C}$ is an $(\infty,1)$-category is shown by Haugseng et al. [HHLN23, Corollary F] in a stronger form of an equivalence between $(\infty,1)$-categories of (op)lax natural transformations.

Proposition 5.58. *Let $\mathcal{C}$ be an $(\infty,2)$-category and $F,G : \mathcal{C} \to \mathbf{Cat}^{(2)}$ a functor. We have an equivalence natural in $\mathcal{C}$ between the following space:*

- *the space of oplax natural transformations $\sigma : F \to G$ such that σ_x is a left adjoint for every 0-cell $x \in \mathcal{C}$;*

- *the space of lax natural transformations $\tau : G \to F$ such that τ_x is a right adjoint for every 0-cell $x \in \mathcal{C}$.*

Moreover, when an oplax natural transformation σ corresponds to a lax natural transformation τ via this equivalence, $\sigma_x \dashv \tau_x$ for any $x \in \mathcal{C}$.

Proof. By

$$\begin{aligned}
& \{\mathbf{G}_1 \to \mathbf{Fun}(\mathcal{C}, \mathbf{Cat}^{(2)})_{\mathrm{opLax}} \mid \text{left adjoint at every } x \in \mathcal{C}\} \\
\simeq & \quad \{\text{transpose}\} \\
& \{\mathcal{C} \to \mathbf{Fun}(\mathbf{G}_1, \mathbf{Cat}^{(2)})_{\mathrm{Lax}} \mid \text{valued in left adjoints}\} \\
\simeq & \quad \{\text{Fact 5.54}\} \\
& \{\mathcal{C} \to \mathbf{Cat}^{(2)}_{/\mathbf{G}_1} \mid \text{valued in } \textit{bicartesian fibrations}\} \\
\simeq & \quad \{\text{Fact 5.54}\} \\
& \{\mathcal{C} \to \mathbf{Fun}(\mathbf{G}_1^{\mathrm{op}}, \mathbf{Cat}^{(2)})_{\mathrm{opLax}} \mid \text{valued in right adjoints}\} \\
\simeq & \quad \{\text{transpose}\} \\
& \{\mathbf{G}_1^{\mathrm{op}} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Cat}^{(2)})_{\mathrm{Lax}} \mid \text{right adjoint at every } x \in \mathcal{C}\}.
\end{aligned}$$

Recall that a *bicartesian fibration* is a functor between $(\infty, 1)$-categories that is is both a cocartesian fibration and a cartesian fibration. Because adjunctions are bicartesian fibrations over $\mathbf{G}_1$ [Lur09a, Definition 5.2.2.1], the middle equivalences hold. □

6. Semantics of mode sketches

We show that models of a mode sketch $\mathfrak{M}$ are equivalent to diagrams of ∞-logoses indexed over $\mathfrak{M}$.

Definition 6.1. Let $\mathfrak{M}$ be a mode sketch. A *model of* $\mathfrak{M}$ is an ∞-logos $\mathcal{L}$ equipped with a function $\mathfrak{m} : \mathfrak{M} \to \mathbf{LAM}(\mathcal{L})$ satisfying semantic counterparts of Axioms A to C, that is:

A'. the functor $\bigcirc^{\mathfrak{m}(i)}_{\mathfrak{m}(j)} : \mathcal{L}_{\mathfrak{m}(i)} \to \mathcal{L}_{\mathfrak{m}(j)}$ is constant at $\mathbf{1}$ for any $j \not\leq i$ in $\mathfrak{M}$;

B'. for any thin triangle $(i_0 < i_1 < i_2)$ in $\mathfrak{M}$, the natural transformation $\eta^{\mathfrak{m}(i_0);\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)} : \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_0)} \Rightarrow \bigcirc^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_0)} \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)}$ is invertible;

C'. a map f in $\mathcal{L}$ is an equivalence whenever $\bigcirc_{\mathfrak{m}(i)} f$ is for every $i \in \mathfrak{M}$.

A *morphism* $(\mathcal{L}, \mathfrak{m}) \to (\mathcal{L}', \mathfrak{m}')$ *of models of* $\mathfrak{M}$ is a morphism of ∞-logoses $F : \mathcal{L} \to \mathcal{L}'$ such that, for every $i \in \mathfrak{M}$, there exists a morphism of ∞-logoses $F_i : \mathcal{L}_{\mathfrak{m}(i)} \to \mathcal{L}'_{\mathfrak{m}'(i)}$ making the following diagram commute.

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{F} & \mathcal{L}' \\
{\scriptstyle \bigcirc_{\mathfrak{m}(i)}}\downarrow & & \downarrow{\scriptstyle \bigcirc_{\mathfrak{m}'(i)}} \\
\mathcal{L}_{\mathfrak{m}(i)} & \underset{F_i}{\dashrightarrow} & \mathcal{L}'_{\mathfrak{m}'(i)}
\end{array}$$

Note that such a morphism F_i is unique since $\bigcirc_{\mathfrak{m}(i)}$ is a localization. The models of $\mathfrak{M}$ and their morphisms form an $(\infty, 1)$-category $\mathbf{Model}(\mathfrak{M})$ whose cells of dimension ≥ 2 are inherited from $\mathbf{Logos}$.

Remark 6.2. Axioms A' and B' are straightforward interpretations of Axioms A and B, but Axiom C' might look different from Axiom C. This is because an interpretation of a type-theoretic axiom must has the stability under base change, which corresponds to the stability under substitution in type theory. A naive interpretation of Axiom C would be that an object $A \in \mathcal{L}$ is contractible whenever $\bigcirc_{\mathfrak{m}(i)} A$ is for every $i \in \mathfrak{M}$, but this is not stable under base change in that it implies nothing about validity in slices $\mathcal{L}_{/X}$. In contrast, Axioms A' to C' are stable under base change in the following sense. Every LAM $\mathfrak{m}$ in $\mathcal{L}$ induces a LAM $\mathfrak{m}_X$ in each slice $\mathcal{L}_{/X}$ determined by $\bigcirc_{\mathfrak{m}_X} A \simeq (\eta_{\mathfrak{m}})^*_X \bigcirc_{\mathfrak{m}} A$ for $A \in \mathcal{L}_{/X}$. A function $\mathfrak{m} : \mathfrak{M} \to \mathbf{LAM}(\mathcal{L})$ then induces a function $\mathfrak{m}_X : \mathfrak{M} \to \mathbf{LAM}(\mathcal{L}_{/X})$ for every $X \in \mathcal{L}$ by $\mathfrak{m}_X(i) = \mathfrak{m}(i)_X$. One can verify that if $\mathfrak{m}$ satisfies Axioms A' to C', then so does $\mathfrak{m}_X$. In fact, Axiom C' is equivalent to that the naive interpretation of Axiom C holds in all the slices $\mathcal{L}_{/X}$.

Construction 6.3. Let $\mathfrak{M}$ be a mode sketch. We construct an $(\infty, 2)$-category $|\mathfrak{M}|$ as follows. We regard the underlying poset $I_{\mathfrak{M}}$ as a simplicial set by taking its nerve. The set $T_{\mathfrak{M}}$ of thin triangles makes $I_{\mathfrak{M}}$ a scaled simplicial set. Let $\langle I_{\mathfrak{M}}, T_{\mathfrak{M}} \rangle$ denote the $(\infty, 2)$-category presented by it. We set $|\mathfrak{M}| = \langle I_{\mathfrak{M}}, T_{\mathfrak{M}} \rangle^{\mathrm{op}(2)}$.

Theorem 6.4. *For any mode sketch $\mathfrak{M}$, we have an equivalence between the following $(\infty, 1)$-categories:*

- *the $(\infty, 1)$-category $\mathbf{Model}(\mathfrak{M})$ of models of $\mathfrak{M}$;*
- *the $(\infty, 1)$-category $\mathbf{D}(\mathfrak{M}) \subset \mathbf{Core}_{(1)}(\mathbf{Fun}(|\mathfrak{M}|^{\mathrm{op}(1,2)}, \Uparrow \mathbf{Cat}^{(2)})_{\mathrm{opLax}})$ whose objects are the functors $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ and morphisms $\mathcal{K} \to \mathcal{K}'$ are the oplax natural transformations $\sigma : \mathcal{K} \to \mathcal{K}'$ whose components $\sigma_i : \mathcal{K}_i \to \mathcal{K}'_i$ are all morphisms of ∞-logoses.*

Moreover, when a model $(\mathcal{L}, \mathfrak{m})$ of $\mathfrak{M}$ corresponds to a functor $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$, the following hold.

(1) $\mathcal{L} \simeq \mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$
(2) $\mathcal{K}_i \simeq \mathcal{L}_{\mathfrak{m}(i)}$ *for every $i \in \mathfrak{M}$.*

The rest of this section is devoted to the proof of Theorem 6.4. In Section 6.1, we give a construction of a model of $\mathfrak{M}$ from a functor $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$. In Section 6.2, we give an inverse construction.

6.1. Models of mode sketches in oplax limits.

We first show that the oplax limit of a functor $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ is part of a model of $\mathfrak{M}$. We fix a functor $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$.

Construction 6.5. For a cosieve S on $\mathfrak{M}$, we define an object $\psi(S) \in \mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$ by

$$\psi(S)_i = \begin{cases} \mathbf{1} & \text{if } i \in S \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The other components are uniquely determined by the universal properties of initial and final objects. This determines a lattice morphism ψ from cosieves on $\mathfrak{M}$ to (-1)-truncated objects in $\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$.

Notation 6.6. Let $S \subset \mathfrak{M}$ be a subset. We regard S as a mode sketch with the structure inherited from $\mathfrak{M}$. Let

$$\pi_S : \operatorname*{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i \to \operatorname*{opLaxLim}_{i \in |S|} \mathcal{K}_i$$

denote the restriction functor.

Lemma 6.7. *For any cosieve S on $\mathfrak{M}$, the restriction functor $\pi_{\mathfrak{M} \setminus S}$ is the closed localization associated to $\psi(S)$.*

Proof. Let $\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i \to \mathcal{M}$ denote the closed localization associated to $\psi(S)$. Recall that $\mathcal{M}$ is the full subcategory of $\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$ spanned by those objects A such that $\psi(S) \times A \simeq \psi(S)$. By the definition of $\psi(S)$, this condition is equivalent to that $A_i \simeq \mathbf{1}$ for all $i \in S$. Then $\pi_{\mathfrak{M} \setminus S}$ induces an equivalence $\mathcal{M} \simeq \mathrm{opLaxLim}_{i \in |\mathfrak{M} \setminus S|} \mathcal{K}_i$. □

Lemma 6.8. *For any cosieve S on $\mathfrak{M}$, the restriction functor π_S is the open localization associated to $\psi(S)$.*

Proof. An object $A \in (\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i)_{/\psi(S)}$ must satisfy that $A_i \simeq \mathbf{0}$ for all $i \in \mathfrak{M} \setminus S$ by the definition of $\psi(S)$ and by Proposition 5.3. Then π_S induces an equivalence $(\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i)_{/\psi(S)} \simeq (\mathrm{opLaxLim}_{i \in |S|} \mathcal{K}_i)_{/\pi_S(\psi(S))} \simeq \mathrm{opLaxLim}_{i \in |S|} \mathcal{K}_i$. □

Proposition 6.9. *For any $i \in \mathfrak{M}$, the projection $\pi_i : \mathrm{opLaxLim}_{j \in |\mathfrak{M}|} \mathcal{K}_j \to \mathcal{K}_i$ is a localization.*

Proof. π_i factors as

$$\operatorname*{opLaxLim}_{j \in |\mathfrak{M}|} \mathcal{K}_j \xrightarrow{\pi_{(i\downarrow\mathfrak{M})}} \operatorname*{opLaxLim}_{j \in |(i\downarrow\mathfrak{M})|} \mathcal{K}_j \xrightarrow{\pi_{(i\downarrow\mathfrak{M}) \setminus \partial(i\downarrow\mathfrak{M})}} \mathcal{K}_i,$$

Thus, it is a composite of localizations by Lemmas 6.7 and 6.8. □

Construction 6.10. For $i \in \mathfrak{M}$, we define $\mathfrak{b}_{\mathcal{K}}(i) \in \mathbf{LAM}(\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i)$ to be the LAM corresponding to the localization $\pi_i : \mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i \to \mathcal{K}_i$ (Proposition 6.9).

Proposition 6.11. *The pair $(\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i, \mathfrak{b}_{\mathcal{K}})$ is a model of $\mathfrak{M}$ for any functor $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$.*

Proposition 6.11 breaks into three parts (Propositions 6.12, 6.18 and 6.20). Axiom C' is immediate from the construction.

Proposition 6.12. *$\mathfrak{b}_{\mathcal{K}}$ satisfies Axiom C'.* □

For Axioms A' and B', we calculate $\bigcirc^{\mathfrak{b}_{\mathcal{K}}(i)}_{\mathfrak{b}_{\mathcal{K}}(j)}$ and $\eta^{\mathfrak{b}_{\mathcal{K}}(k);\mathfrak{b}_{\mathcal{K}}(i)}_{\mathfrak{b}_{\mathcal{K}}(j)}$.

Notation 6.13. For $j < i$ in $\mathfrak{M}$, let $(j < i)$ denote the associated generating 1-cell in $|\mathfrak{M}|$. For $k < j < i$ in $\mathfrak{M}$, let $(k < j < i)$ denote the associated generating 2-cell $(j < i) \circ (k < j) \Rightarrow (k < i)$ in $|\mathfrak{M}|$.

Lemma 6.14. *Let $\mathfrak{M}'$ be the mode sketch with the same underlying poset as $\mathfrak{M}$ but with no thin triangle. We have an equivalence $\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i \simeq \mathrm{opLaxLim}_{i \in |\mathfrak{M}'|} \mathcal{K}_i$*

Proof. This is because $\mathbf{El}_{|\mathfrak{M}|}(\mathcal{K}) \to |\mathfrak{M}|$ is locally a right fibration and thus locally conservative. □

Lemma 6.15. *Suppose that $\mathfrak{M}$ has no thin triangle. Then $\mathbf{Core}_{(1)}(|\mathfrak{M}|)$ is freely generated by the strict ordering relation, and $(j < i)$ is the final object in $\mathbf{Map}_{|\mathfrak{M}|}(j, i)$ for any $j < i$.*

Proof. By Example 5.24. □

Lemma 6.16. *For any $i \in \mathfrak{M}$, the right adjoint i_* is given by the following formula for $A \in \mathcal{K}_i$ and $j \in \mathfrak{M}$.*

$$i_*(A)_j \simeq \begin{cases} \mathcal{K}_{(j<i)}(A) & \text{if } j < i \\ A & \text{if } j = i \\ \mathbf{1} & \text{otherwise} \end{cases}$$

Proof. We first see that we may assume without loss of generality that i is the largest element of $\mathfrak{M}$. Otherwise, factor π_i as $\operatorname{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j \xrightarrow{\pi_{(\mathfrak{M}\downarrow i)}} \operatorname{opLaxLim}_{j\in|(\mathfrak{M}\downarrow i)|}\mathcal{K}_j \xrightarrow{\pi_{(i\downarrow\mathfrak{M})}} \mathcal{K}_i$, where $(\mathfrak{M}\downarrow i) = \{j \in \mathfrak{M} \mid j \le i\}$. The first functor is a closed localization by Lemma 6.7 because $\mathfrak{M}\setminus(\mathfrak{M}\downarrow i)$ is a cosieve, and the second functor is an open localization by Lemma 6.8. The right adjoint of $\pi_{(\mathfrak{M}\downarrow i)}$ is then defined by extending $A \in \operatorname{opLaxLim}_{j\in|(\mathfrak{M}\downarrow i)|}\mathcal{K}_j$ by the final objects at all $j \in \mathfrak{M}\setminus(\mathfrak{M}\downarrow i)$. Therefore, the problem is reduced to the calculation of the right adjoint of $\pi_{(i\downarrow\mathfrak{M})}$, and in this case i is the largest element of $(\mathfrak{M}\downarrow i)$.

Let $i'_*(A)_j$ be defined by the displayed formula. We turn $i'_*(A)$ into an object of $\operatorname{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j$. By Lemma 6.14, we assume that $\mathfrak{M}$ has no thin triangle. Since $\mathbf{El}_{|\mathfrak{M}|}(\mathcal{K}) \to |\mathfrak{M}|$ is locally a right fibration, it follows from Lemma 6.15 that an object $B \in \operatorname{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j$ is completely determined by $B_j \in \mathcal{K}_j$ for all $j \in \mathfrak{M}$ and $B_{(k<j)} : B_k \to \mathcal{K}_{(k<j)}(B_j)$ for all $k < j$ in $\mathfrak{M}$. We can then extend $i'_*(A)$ as follows.

$$i'_*(A)_{(k<j)} = \begin{cases} \mathcal{K}_{(k<j<i)}(A) : \mathcal{K}_{(k<i)}(A) \to \mathcal{K}_{(k<j)}(\mathcal{K}_{(j<i)}(A)) & \text{if } j < i \\ \mathbf{id} & \text{if } j = i \end{cases}$$

Since $i'_*(A)_i \simeq A$ by construction, we have a unique map $f : i'_*(A) \to i_*(A)$ whose i-th component is the identity on A. To see that f is invertible, it suffices to construct a retraction g of f. Indeed, if $g \circ f \simeq \mathbf{id}$, then the i-th component of g must be the identity, and thus $f \circ g \simeq \mathbf{id}$ follows by adjointness. Let $j \in \mathfrak{M}$. If $j = i$, then we must define $g_i = \mathbf{id}$. Suppose that $j < i$ and consider the following commutative diagram.

$$\begin{array}{ccc} i'_*(A)_j & \xrightarrow{\quad f_j \quad} & i_*(A)_j \\ {\scriptstyle i'_*(A)_{(j<i)}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle i_*(A)_{(j<i)}} \\ \mathcal{K}_{(j<i)}(i'_*(A)_i) & \xrightarrow[\mathcal{K}_{(j<i)}(f_i)]{\simeq} & \mathcal{K}_{(j<i)}(i_*(A)_i) \end{array}$$

The left and bottom maps are invertible by definition. Hence, we have a unique retraction g_j of f_j commuting with $i_*(A)_{j<i}$. This defines a retraction of f. □

Lemma 6.17. *For any $j < i$ in $\mathfrak{M}$, the functor $\bigcirc^{\mathfrak{b}_{\mathcal{K}}(i)}_{\mathfrak{b}_{\mathcal{K}}(j)} : \mathcal{K}_i \to \mathcal{K}_j$ is equivalent to $\mathcal{K}_{(j<i)}$.*

Proof. By Lemma 6.16. □

Proposition 6.18. *$\mathfrak{b}_{\mathcal{K}}$ satisfies Axiom A'.*

Proof. By Lemma 6.16. □

Lemma 6.19. *For any $k < j < i$ in $\mathfrak{M}$, the natural transformation $\eta^{\mathfrak{b}_{\mathcal{K}}(k);\mathfrak{b}_{\mathcal{K}}(i)}_{\mathfrak{b}_{\mathcal{K}}(j)}$ is equivalent to $\mathcal{K}_{(k<j<i)}$.*

Proof. This is a consequence of Lemma 6.16. □

Proposition 6.20. *$\mathfrak{b}_{\mathcal{K}}$ satisfies Axiom B'.*

Proof. By Lemma 6.19. ☐

Proof of Proposition 6.11. By Propositions 6.12, 6.18 and 6.20. ☐

Construction 6.21. We extend the construction $\mathcal{K} \mapsto (\mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i, \mathfrak{b}_{\mathcal{K}})$ to a functor $\mathfrak{b} : \mathbf{D}(\mathfrak{M}) \to \mathbf{Model}(\mathfrak{M})$ as follows. Characterized by the universal property (Proposition 5.57), the oplax limit construction extends to a functor

$$\mathbf{Core}_{(1)}(\mathbf{Fun}(I^{\mathrm{op}(1,2)}, \mathbf{Cat}^{(2)})_{\mathrm{opLax}}) \to \mathbf{Cat}^{(2)}.$$

It then restricts to a functor $\mathbf{D}(\mathfrak{M}) \to \mathbf{Logos}$ by Theorem 5.43. It further lifts to a functor $\mathbf{D}(\mathfrak{M}) \to \mathbf{Model}(\mathfrak{M})$ by the construction of $\mathfrak{b}_{\mathcal{K}}$ (Construction 6.10).

6.2. **Fracture and gluing.** We show that any model of a mode sketch $\mathfrak{M}$ induces a functor $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ and that this gives an inverse of the construction given in Section 6.1. This is an externalization and generalization of the *fracture and gluing theorem* (Proposition 2.13).

Construction 6.22. Let $(\mathcal{L}, \mathfrak{m})$ be a model of $\mathfrak{M}$. We define $\mathcal{Y}_{\mathfrak{m}}$ to be the full subcategory of $|\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathcal{L}$ spanned by those objects (i, A) such that A belongs to $\mathcal{L}_{\mathfrak{m}(i)}$.

Proposition 6.23. *For any model $(\mathcal{L}, \mathfrak{m})$ of $\mathfrak{M}$, the projection $\mathcal{Y}_{\mathfrak{m}} \to |\mathfrak{M}|^{\mathrm{op}(1,2)}$ is a 1-cocartesian 2-right fibration.*

Proof. We work with the scaled simplicial sets model. It suffices to show that the pullback $\mathcal{Y}'_{\mathfrak{m}}$ of $\mathcal{Y}_{\mathfrak{m}}$ along the fibrant replacement $I^{\mathrm{op}}_{\mathfrak{M}} \to |\mathfrak{M}|^{\mathrm{op}(1,2)}$ is a locally cocartesian fibration.

Let $(i_0 \le i_1) : \Delta^1 \to I^{\mathrm{op}}_{\mathfrak{M}}$ be a map which corresponds to an ordered pair $(i_0 \le i_1)$ in $I^{\mathrm{op}}_{\mathfrak{M}}$. A morphism $(i_0, A_0) \to (i_1, A_1)$ in $\mathcal{Y}'_{\mathfrak{m}}$ over $(i_0 \le i_1)$ is a map $A_0 \to A_1$ in $\mathcal{L}$, but it corresponds to a map $\bigcirc_{\mathfrak{m}(i_1)} A_0 \to A_1$ in $\mathcal{L}_{\mathfrak{m}(i_1)}$. Hence, $(i_0 \le i_1)^* \mathcal{Y}'_{\mathfrak{m}} \to \Delta^1$ is the Grothendieck construction for the diagram $\mathcal{L}_{\mathfrak{m}(i_0)} \xrightarrow{\bigcirc^{\mathfrak{m}(i_0)}_{\mathfrak{m}(i_1)}} \mathcal{L}_{\mathfrak{m}(i_1)}$ and thus a cocartesian fibration.

Let $(i_0 \le i_1 \le i_2) : \Delta^2 \to I^{\mathrm{op}}_{\mathfrak{M}}$ be a thin 2-simplex. By Axiom B, the canonical natural transformation

$$\begin{array}{ccc} \mathcal{L}_{\mathfrak{m}(i_0)} & \xrightarrow{\bigcirc^{\mathfrak{m}(i_0)}_{\mathfrak{m}(i_2)}} & \mathcal{L}_{\mathfrak{m}(i_2)} \\ {\scriptstyle \bigcirc^{\mathfrak{m}(i_0)}_{\mathfrak{m}(i_1)}} \searrow & \Downarrow \simeq & \nearrow {\scriptstyle \bigcirc^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_2)}} \\ & \mathcal{L}_{\mathfrak{m}(i_1)} & \end{array} \tag{6.1}$$

is invertible, and $(i_0 \le i_1 \le i_2)^* \mathcal{Y}'_{\mathfrak{m}} \to \Delta^2$ is the Grothendieck construction for the diagram (6.1) and thus a cocartesian fibration. ☐

Construction 6.24. Let $(\mathcal{L}, \mathfrak{m})$ be a model of $\mathfrak{M}$. By Propositions 6.23 and 5.32, the projection $\mathcal{Y}_{\mathfrak{m}} \to |\mathfrak{M}|^{\mathrm{op}(1,2)}$ is classified by a functor

$$\mathcal{X}_{\mathfrak{m}} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \Uparrow \mathbf{Cat}^{(2)}.$$

By construction, $\mathcal{X}_{\mathfrak{m}}(i) \simeq \mathcal{L}_{\mathfrak{m}(i)}$. As we have seen in the proof of Proposition 6.23, $\mathcal{X}_{\mathfrak{m}}$ maps a 1-cell $i_0 \le i_1$ in $|\mathfrak{M}|$ to $\bigcirc^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_0)} : \mathcal{L}_{\mathfrak{m}(i_1)} \to \mathcal{L}_{\mathfrak{m}(i_0)}$ which is lex and accessible (but need not preserve all colimits). Therefore, $\mathcal{X}_{\mathfrak{m}}$ factors through $\mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$.

We have constructed back and forth constructions between the $(\infty,1)$-category of models of $\mathfrak{M}$ and the $(\infty,1)$-category of functors $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$. We turn these constructions into an adjunction (Construction 6.25) and then show that its unit and counit are invertible (Propositions 6.27 and 6.28).

Construction 6.25. Let $(\mathcal{L},\mathfrak{m})$ be a model of $\mathfrak{M}$ and $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ a functor. We construct an equivalence natural in $\mathcal{K} \in \mathbf{D}(\mathfrak{M})$

$$\mathbf{Map}_{\mathbf{D}(\mathfrak{M})}(\mathcal{X}_{\mathfrak{m}},\mathcal{K}) \simeq \mathbf{Map}_{\mathbf{Model}(\mathfrak{M})}((\mathcal{L},\mathfrak{m}),(\operatorname*{opLaxLim}_{i\in|\mathfrak{M}|}\mathcal{K}_i,\mathfrak{b}_{\mathcal{K}})) \tag{6.2}$$

as follows. By Proposition 5.58, a morphism $\mathcal{X}_{\mathfrak{m}} \to \mathcal{K}$ corresponds to a lax natural transformation $\mathcal{K} \to \mathcal{X}_{\mathfrak{m}}$ whose components are right adjoints of morphisms of ∞-logoses. It corresponds by Proposition 5.52 to a map $\mathbf{El}_{|\mathfrak{M}|^{\mathrm{op}(1,2)}}(\mathcal{K}) \to \mathcal{Y}_{\mathfrak{m}}$ over $|\mathfrak{M}|^{\mathrm{op}(1,2)}$ whose fibers are right adjoints of morphisms of ∞-logoses. By the definition of $\mathcal{Y}_{\mathfrak{m}}$, it corresponds to a map $\mathbf{El}_{|\mathfrak{M}|^{\mathrm{op}(1,2)}}(\mathcal{K}) \to |\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathcal{L}$ over $|\mathfrak{M}|^{\mathrm{op}(1,2)}$ whose fiber over $i \in |\mathfrak{M}|$ is a right adjoint of a morphism of ∞-logos that factors through $\mathcal{L}_{\mathfrak{m}(i)}$. Again by Propositions 5.52 and 5.58, it corresponds to an oplax natural transformation $(\lambda\_.\mathcal{L}) \to \mathcal{K}$ whose component at $i \in |\mathfrak{M}|$ is a morphism of ∞-logoses that extends along $\bigcirc_{\mathfrak{m}(i)} : \mathcal{L} \to \mathcal{L}_{\mathfrak{m}(i)}$. By Proposition 5.57, it corresponds to a morphism $(\mathcal{L},\mathfrak{m}) \to (\operatorname{opLaxLim}_{i\in|\mathfrak{M}|}\mathcal{K}_i,\mathfrak{b}_{\mathcal{K}})$ in $\mathbf{Model}(\mathfrak{M})$. All of these correspondences are stated in the form of equivalence of spaces natural in $\mathcal{K} \in \mathbf{D}(\mathfrak{M})$, and thus we obtain Eq. (6.2).

By Eq. (6.2), the functor $\mathfrak{b} : \mathbf{D}(\mathfrak{M}) \to \mathbf{Model}(\mathfrak{M})$ has the left adjoint $(\mathcal{L},\mathfrak{m}) \mapsto \mathcal{X}_{\mathfrak{m}}$. Let $H_{\mathfrak{m}} : (\mathcal{L},\mathfrak{m}) \to (\operatorname{opLaxLim}_{i\in|\mathfrak{M}|}\mathcal{X}_{\mathfrak{m}}(i),\mathfrak{b}_{\mathcal{X}_{\mathfrak{m}}})$ and $\sigma_{\mathcal{K}} : \mathcal{X}_{\mathfrak{b}_{\mathcal{K}}} \to \mathcal{K}$ be the unit and counit, respectively, of the adjunction.

Lemma 6.26. *Let $\mathcal{L}$ be an ∞-logos and let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs in $\mathcal{L}$. Suppose that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is constant at $\mathbf{1}$. Then the functor $\mathcal{L} \to \mathbf{Gl}(\bigcirc^{\mathfrak{m}}_{\mathfrak{n}})$ that sends $A \in \mathcal{L}$ to $(\bigcirc_{\mathfrak{n}} A, \bigcirc_{\mathfrak{m}} A, \bigcirc_{\mathfrak{n}}(\eta_{\mathfrak{m}})_A) \in \mathbf{Gl}(\bigcirc^{\mathfrak{m}}_{\mathfrak{n}})$ is a localization.*

Proof. Observe that the right adjoint of the functor $\mathcal{L} \to \mathbf{Gl}(\bigcirc^{\mathfrak{m}}_{\mathfrak{n}})$ sends $(B', B, g) \in \mathbf{Gl}(\bigcirc^{\mathfrak{m}}_{\mathfrak{n}})$ to the pullback

$$\begin{array}{ccc} \eta_{\mathfrak{n}}^* B' & \longrightarrow & B' \\ \big\downarrow & \lrcorner & \big\downarrow g \\ B & \xrightarrow[\eta_{\mathfrak{n}}]{} & \bigcirc^{\mathfrak{m}}_{\mathfrak{n}} B. \end{array}$$

$\bigcirc_{\mathfrak{n}}$ inverts $\eta_{\mathfrak{n}}$, and thus $\bigcirc_{\mathfrak{n}} \eta_{\mathfrak{n}}^* B' \simeq B'$. Since $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is constant at $\mathbf{1}$, it sends g to the identity on $\mathbf{1}$, and thus $\bigcirc_{\mathfrak{m}} \eta_{\mathfrak{n}}^* B' \simeq B$. Therefore, the counit for this adjunction is invertible. □

Proposition 6.27. *The unit $H = H_{\mathfrak{m}} : (\mathcal{L},\mathfrak{m}) \to (\operatorname{opLaxLim}_{i\in|\mathfrak{M}|}\mathcal{X}_{\mathfrak{m}}(i),\mathfrak{b}_{\mathcal{X}_{\mathfrak{m}}})$ is an equivalence in $\mathbf{Model}(\mathfrak{M})$ for any model $(\mathcal{L},\mathfrak{m})$ of $\mathfrak{M}$.*

Proof. Since $\mathcal{L}_{\mathfrak{m}(i)} \simeq \mathcal{X}_{\mathfrak{m}}(i)$, it remains to show that the underlying functor of H is an equivalence. We show that, for any cosieve S on $\mathfrak{M}$, the composite

$$H_S : \mathcal{L} \xrightarrow{H} \operatorname*{opLaxLim}_{i\in|\mathfrak{M}|}\mathcal{X}_{\mathfrak{m}}(i) \xrightarrow{\pi_S} \operatorname*{opLaxLim}_{i\in|S|}\mathcal{X}_{\mathfrak{m}}(i)$$

is a localization. In particular, H itself is a localization. Then, H is an equivalence because it is conservative by Axiom C'. We proceed by induction on the size of S. The case when S is empty is trivial. Suppose that S is inhabited. There is an element $i_0 \in S$ minimal in S.

By induction hypothesis, $H_{S\setminus\{i_0\}}$ is a localization, and let $\mathfrak{n}$ be the corresponding LAM. By Axiom A', it follows that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}(i_0)}$ is constant at $\mathbf{1}$. By Lemma 6.26, $\mathbf{Gl}(\bigcirc^{\mathfrak{n}}_{\mathfrak{m}(i_0)})$ is a localization of $\mathcal{L}$. Again by Lemma 6.26, we have the localization $\mathrm{opLaxLim}_{i\in|S|}\mathcal{X}_{\mathfrak{m}}(i) \to \mathbf{Gl}(\bigcirc^{\mathfrak{n}}_{\mathfrak{m}(i_0)})$, but this is also conservative and thus an equivalence. Therefore, H_S is a localization. □

Proposition 6.28. *For any functor $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$, the unit $\sigma = \sigma_{\mathcal{K}} : \mathcal{X}_{\mathfrak{b}_{\mathcal{K}}} \to \mathcal{K}$ is an equivalence in $\mathbf{D}(\mathfrak{M})$.*

Proof. It suffices to show that σ is an invertible natural transformation. It suffices to show that the corresponding lax natural transformation $\sigma' : \mathcal{K} \to \mathcal{X}_{\mathfrak{b}_{\mathcal{K}}}$ by Proposition 5.58 is an invertible natural transformation. To see that σ' is a natural transformation, by Proposition 5.56, it suffices to check that the corresponding map $F : \mathbf{El}_{|\mathfrak{M}|^{\mathrm{op}(1,2)}}(\mathcal{K}) \to \mathcal{Y}_{\mathfrak{b}_{\mathcal{K}}}$ over $|\mathfrak{M}|^{\mathrm{op}(1,2)}$ preserves cocartesian morphisms. Unfolding the definition (Construction 6.25), $F : \mathbf{El}_{|\mathfrak{M}|^{\mathrm{op}(1,2)}}(\mathcal{K}) \to \mathcal{Y}_{\mathfrak{b}_{\mathcal{K}}} \subset |\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathrm{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j$ sends an object (i, A) to $(i, A) \in |\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathcal{K}_i \subset |\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathrm{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j$ and a morphism $(i \geq i', f) : (i, A) \to (i', A')$ in $\mathbf{El}_{|\mathfrak{M}|^{\mathrm{op}(1,2)}}(\mathcal{K})$ to $(i \geq i', f') : (i, A) \to (i', A')$ in $|\mathfrak{M}|^{\mathrm{op}(1,2)} \times \mathrm{opLaxLim}_{j\in|\mathfrak{M}|}\mathcal{K}_j$, where f' is the composite $A \xrightarrow{\eta_{\mathfrak{b}_{\mathcal{K}}(i')}} \bigcirc_{\mathfrak{b}_{\mathcal{K}}(i')} A \simeq \mathcal{K}_{(i'\leq i)}(A) \xrightarrow{f} A'$. When $(i \geq i', f)$ is cocartesian, f is invertible, and then $(i \geq i', f')$ is a cocartesian morphism in $\mathcal{Y}_{\mathfrak{b}_{\mathcal{K}}}$. By construction, σ' is point-wise invertible and thus invertible by Proposition 5.21. □

Proof of Theorem 6.4. The functor $\mathfrak{b} : \mathbf{D}(\mathfrak{M}) \to \mathbf{Model}(\mathfrak{M})$ gives an equivalence by Construction 6.25 and Propositions 6.27 and 6.28. □

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