

POSITIONALITY IN Σ_2^0 AND A COMPLETENESS RESULT

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ABSTRACT. We study the existence of positional strategies for the protagonist in infinite duration games over arbitrary game graphs. We prove that prefix-independent objectives in Σ_2^0 which are positional and admit a (strongly) neutral letter are exactly those that are recognised by history-deterministic monotone co-Büchi automata over countable ordinals. This generalises a criterion proposed by [Kopczyński, ICALP 2006] and gives an alternative proof of closure under union for these objectives, which was known from [Ohlmann, TheoretCS 2023].

We then give two applications of our result. First, we prove that the mean-payoff objective is positional over arbitrary game graphs. Second, we establish the following completeness result: for any objective W which is prefix-independent, admits a (weakly) neutral letter, and is positional over finite game graphs, there is an objective W' which is equivalent to W over finite game graphs and positional over arbitrary game graphs.

1. INTRODUCTION

1.1. Context. Before presenting our contributions, we introduce the necessary background.

Games. We study infinite duration games on graphs. In such a game, two players, Eve and Adam, alternate forever in moving a token along the edges of a directed, possibly infinite graph (called *arena*), whose edges are labelled with elements of some set C . An *objective* $W \subseteq C^\omega$ is specified in advance; Eve wins the game if the label of the produced infinite path belongs to W . A *strategy* in such a game is called *positional* if it depends only on the current vertex occupied by the token, regardless of the history of the play.

We are interested in *positional objectives*: those for which existence of a winning strategy for Eve entails existence of a winning positional strategy for Eve, on a arbitrary arena. Sometimes we also consider a weaker property: an objective is *positional over finite arenas* if the above implication holds on any finite arena.

Key words and phrases: infinite duration games, positionality, Borel class Σ_2^0 , history determinism.

We thank Antonio Casares and Lorenzo Clemente for discussions on and around the topic.

The first author was supported by the European Research Council (grant agreement No 948057 — BOBR). The second author was supported by the National Science Centre, Poland (grant no. 2021/41/B/ST6/03914).

Early results. Although the notion of positionality is already present in Shapley’s seminal work [Sha53], the first positionality result for infinite duration games was established by Ehrenfeucht and Mycielsky [EM79], and it concerns the mean-payoff objective

$$\text{Mean-Payoff}_{\leq 0} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i \leq 0 \right\},$$

over finite arenas. Nowadays, many proofs are known that establish positionality of mean-payoff games over finite arenas.

Later, and in a different context, Emerson and Jutla [EJ91] as well as Mostowski [Mos91] independently established positionality of the parity objective

$$\text{Parity}_d = \left\{ p_0 p_1 \cdots \in \{0, 1, \dots, d\}^\omega \mid \limsup_k p_k \text{ is even} \right\}$$

over arbitrary arenas. This result was used to give a direct proof of the possibility of complementing automata over infinite trees, which is the key step in modern proofs of Rabin’s theorem on decidability of S2S [Rab69]. By now, several proofs are known for positionality of parity games, some of which apply to arbitrary arenas.

Both parity games and mean-payoff games have been the object of considerable attention over the past three decades; we refer to [FBBD⁺] for a thorough exposition. By symmetry, these games are positional not only for Eve but also for the opponent, a property we call *bi-positionality*. Parity and mean-payoff objectives, as well as the vast majority of objectives that are considered in this context, are *prefix-independent*, that is, invariant under adding or removing finite prefixes.

Bi-positionality. Many efforts were devoted to understanding positionality in the early 2000’s. These culminated in Gimbert and Zielonka’s work [GZ05] establishing a general characterisation of bi-positional objectives over finite arenas, from which it follows that an objective is bi-positional over finite arenas if and only if it is the case for 1-player games. On the other hand, Colcombet and Niwiński [CN06] established that bi-positionality over arbitrary arenas is very restrictive: any prefix-independent objective which is bi-positional over arbitrary arenas can be recast as a parity objective.

Together, these two results give a good understanding of bi-positional objectives, both over finite and arbitrary arenas.

Positionality for Eve. In contrast, less is known about those objectives which are positional for Eve, regardless of the opponent (this is sometimes called half-positionality). This is somewhat surprising, considering that positionality is more in-line with the primary application in synthesis of reactive systems, where the opponent, who models an antagonistic environment, need not have structured strategies. The thesis of Kopczyński [Kop09] proposes a number of results on positionality, but no characterisation. Kopczyński proposed two classes of prefix-independent objectives, *concave objectives* and *monotone objectives*, which are positional respectively over finite and over arbitrary arenas. Both classes are closed under unions, which motivated the following conjecture.

Conjecture 1.1 (Kopczyński’s conjecture [Kop09, Kop06]). *Prefix-independent positional objectives are closed under unions.*

This conjecture was disproved by Kozachinskiy in the case of finite arenas [Koz22], however, it remains open for arbitrary ones (even in the case of countable unions instead of unions).

Neutral letters. Many of the considered objectives contain a *neutral letter*, that is an element $\varepsilon \in C$ such that W is invariant under removing arbitrary many occurrences of the letter ε from any infinite word. For instance, $\varepsilon = 0$ is a neutral letter of the parity objective Parity_d . There are two variants of this definition, *strongly neutral letter* and *weakly neutral letter*, which are formally introduced in the preliminaries. It is an open question whether adding a neutral letter to a given objective may affect its positionality [Kop09, Ohl23a], although Casares and Ohlmann established that this is not the case for ω -regular objectives [CO24].

Neutral letters are typically used when one wants to modify a given game arena, by allowing players to make some additional decisions. This requires to create intermediate edges in such a way that their labels do not affect the overall outcome of the play. For this reason, a number of results (including those in this paper) are conditional on the presence of a (strong or weak) neutral letter. For more discussion on neutral letters and their role in the study of memory, see also [CO25].

Borel classes. To stratify the complexity of the considered objectives we use the Borel hierarchy [Kec95]. This follows the classical approach to Gale-Stewart games [GS53], where the determinacy theorem was gradually proved for more and more complex Borel classes: Σ_2^0 in [Wol55] and Σ_3^0 in [Dav64]. This finally led to Martin’s celebrated result on all Borel objectives [Mar75].

To apply this technique, we assume for the rest of the paper that C is at most countable. Thus, C^ω is a Polish topological space, with open sets of the form $L \cdot C^\omega$ where $L \subseteq C^*$ is arbitrary. Closed sets are those whose complement is open. The class Σ_2^0 contains all sets which can be obtained as a countable union of some closed sets.

Recent developments. A step forward in the study of positionality (for Eve) was recently made by Ohlmann [Ohl23a] who established that an objective admitting a (strongly) neutral letter is positional over arbitrary arenas if and only if it admits well-ordered monotone universal graphs. Note that this characterisation concerns only positionality over arbitrary arenas. This allowed Ohlmann to prove closure of prefix-independent positional objectives (over arbitrary arenas) admitting a (strongly) neutral letter under finite lexicographic products, and, further assuming membership in Σ_2^0 , under finite unions¹.

Bouyer, Casares, Randour, and Vandenhove [BCRV22] also used universal graphs to characterise positionality for objectives recognised by deterministic Büchi automata. They observed that for such an objective W finiteness of the arena does not impact positionality: W is positional over arbitrary arenas if and only if it is positional over finite ones.

Going further, Casares [Cas23] recently proposed a characterisation of positionality for all ω -regular objectives. As a by-product, it follows that Conjecture 1.1 holds for ω -regular objectives², and that again finiteness of the arena does not impact positionality.

¹In [Ohl23a], an assumption called “non-healing” is used. This assumption is in fact implied by membership in Σ_2^0 .

²In fact, Casares proved a strengthening of the conjecture when only one objective is required to be prefix-independent.

1.2. Contributions. We now present our contributions.

Positionality in Σ_2^0 . As mentioned above, Kopczyński introduced the class of *monotonic objectives*, defined as those of the form $C^\omega \setminus L^\omega$, where L is a language recognised by a finite linearly-ordered automaton with certain monotonicity properties on transitions. He then proved that monotonic objectives are positional over arbitrary arenas. Such objectives are prefix-independent and belong to Σ_2^0 ; our first contribution is to extend Kopczyński's result to a complete characterisation (up to neutral letters) of positional objectives in Σ_2^0 .

Theorem 1.2. *Let $W \subseteq C^\omega$ be a prefix-independent Σ_2^0 objective admitting a strongly neutral letter. Then W is positional over arbitrary arenas if and only if it is recognised by a countable history-deterministic well-founded monotone co-Büchi automaton.*

The proof of Theorem 1.2 is based on Ohlmann's *structuration* technique which is the key ingredient to the proof of [Ohl23a]. As an easy by-product of the above characterisation, we reobtain the result that Kopczyński's conjecture holds for countable unions of Σ_2^0 objectives (assuming that the given objectives all have strongly neutral letters).

If W_0, W_1, \dots are all positional prefix-independent Σ_2^0 objectives, each admitting a strongly neutral letter (possibly a different one), then the union $\bigcup_{i \in \mathbb{N}} W_i$ is also positional.

From finite to arbitrary arenas. The most important natural example of an objective which is positional over finite arenas but not on infinite ones is Mean-Payoff $_{<0}$, as defined above. As a straightforward consequence of its positionality [BCD⁺11, Theorem 3], it holds that over finite arenas, Mean-Payoff $_{\leq 0}$ coincides with the energy condition

$$\text{Bounded} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \sup_k \sum_{i=0}^{k-1} w_i \text{ is finite} \right\},$$

which turns out to be positional even over arbitrary arenas [Ohl23a].

Applying Corollary 1.2, we establish that with strict threshold, the mean-payoff objective

$$\text{Mean-Payoff}_{<0} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i < 0 \right\}$$

is in fact positional over arbitrary arenas.

Now say that two prefix-independent objectives are *finitely equivalent*, written $W \equiv W'$, if they are won by Eve over the same finite arenas. As observed above, Mean-Payoff $_{\leq 0} \equiv \text{Bounded}$, which is positional over arbitrary arenas. Likewise, its complement

$$\mathbb{Z}^\omega \setminus \text{Mean-Payoff}_{\leq 0} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i > 0 \right\}$$

is, up to changing each weight $w \in \mathbb{Z}$ by the opposite one $-w \in \mathbb{Z}$, isomorphic to

$$\left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \liminf_k \frac{1}{k} \sum_{i=0}^{k-1} w_i < 0 \right\}.$$

The latter condition is finitely equivalent to Mean-Payoff $_{<0}$ (where the liminf is replaced with a limsup), which, as explained above, turns out to be positional over arbitrary arenas.

Thus, both Mean-Payoff $_{\leq 0}$ and its complement are finitely equivalent to objectives that are positional over arbitrary arenas. This brings us to our main contribution, which

generalises the above observation to any prefix-independent objective admitting a (weakly) neutral letter which is positional over finite arenas.

Theorem 1.3. *Let $W \subseteq C^\omega$ be a prefix-independent objective which is positional over finite arenas and admits a weakly neutral letter. Then there exists a prefix-independent objective $W' \subseteq W$ such that $W' \equiv W$ and W' is positional over arbitrary arenas.*

Structure of the paper. Section 2 introduces all necessary notions, including Ohlmann's structurations results. Section 3 proves our characterisation result Theorem 1.2 and its consequence Corollary 1.2, and provides a few examples. Then we proceed in Section 4 with establishing positionality of Mean-Payoff $_{<0}$ over arbitrary arenas, and proving Theorem 1.3.

2. PRELIMINARIES

Graphs. We fix a set of letters C , which we assume to be at most countable. A C -graph G is comprised of a (potentially infinite) set of *vertices* $V(G)$ together with a set of *edges* $E(G) \subseteq V(G) \times C \times V(G)$. An edge $e = (v, c, v') \in E(G)$ is written $v \xrightarrow{c} v'$, with c being the *label* of this edge. We say that e is *outgoing* from v , that it is *incoming* to v' , and that it is *adjacent* to both v and to v' . We assume that each vertex $v \in V(G)$ has at least one outgoing edge (we call this condition being *sinkless*, with a *sink* understood as a vertex where no outgoing edge is available).

We say that G is *finite* (resp. *countable*) if both $V(G)$ and $E(G)$ are finite (resp. countable). The *size* of a graph is defined to be $|G| = |V(G)|$. We write \mathbb{N} and ω for the set of natural numbers $\{0, 1, \dots\}$. The symbol ω is used to emphasise the order-type of this set, while \mathbb{N} emphasises the arithmetic structure. \mathbb{Z} stands for the set of all integers $\{\dots, -1, 0, 1, \dots\}$.

A (finite) *path* is a (finite) sequence of edges with matching endpoints, meaning of the form $v_0 \xrightarrow{c_0} v_1, v_1 \xrightarrow{c_1} v_2, \dots$, which we conveniently write as $v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots$. We say that π is a *path from v_0 in G* , and that vertices v_0, v_1, v_2, \dots appearing on the path are *reachable* from v_0 . We use $G[v_0]$ to denote the restriction of G to vertices reachable from v_0 . The *label* of a path π is the sequence $c_0 c_1 \dots$ of labels of its edges; it belongs to C^ω if π is infinite and to C^* otherwise. We sometimes write $v \overset{w}{\rightsquigarrow}$ to say that w labels an infinite path from v , or $v \overset{w}{\rightsquigarrow} v'$ to say that w labels a finite path from v to v' . We write $L(G, v_0) \subseteq C^\omega$ for the set of labels of all infinite paths from v_0 in G , and $L(G) \subseteq C^\omega$ for the set of labels of all infinite paths in G , that is the union of $L(G, v_0)$ over all $v_0 \in V(G)$.

A *graph morphism* from G to G' is a map $\phi: V(G) \rightarrow V(G')$ such that for every edge $v \xrightarrow{c} v' \in E(G)$, it holds that $\phi(v) \xrightarrow{c} \phi(v') \in E(G')$. We write $G \xrightarrow{\phi} G'$. We sometimes say that G *embeds* in G' or that G' *embeds* G , and we write $G \rightarrow G'$, to say that there exists a morphism from G to G' . Note that $G \rightarrow G'$ implies $L(G) \subseteq L(G')$.

A graph G is v_0 -*rooted* if it has a distinguished vertex $v_0 \in V(G)$ called the *root*. A *tree* T is a t_0 -rooted graph such that all vertices in T admit a unique finite path from the root t_0 .

Games. A C -arena is given by a C -graph A together with a partition of its vertices $V(A) = V_{\text{Eve}} \sqcup V_{\text{Adam}}$ into those controlled by Eve V_{Eve} and those controlled by Adam V_{Adam} . A *strategy* (for Eve) (S, π) in an arena A is a graph S together with a surjective morphism $\pi: S \rightarrow A$ satisfying that for every vertex $v \in V_{\text{Adam}}$, every outgoing edge $v \xrightarrow{c} v' \in E(A)$, and every $s \in \pi^{-1}(v)$, there is an outgoing edge $s \xrightarrow{c} s' \in E(S)$ with $\pi(s') = v'$. Recall that under our assumptions every vertex needs to have at least one outgoing edge, thus for every $v \in V_{\text{Eve}}$ and every $s \in \pi^{-1}(v)$ there must be at least one outgoing edge from s in S .

The example arenas in this work are drawn following a standard notation, where circles (resp. squares) denote vertices controlled by Eve (resp. Adam). Vertices with a single outgoing edge are denoted by a simple dot, it does not matter who controls them.

A strategy is *positional* if π is injective. In this case, we can assume that $V(S) = V(A)$ and $E(S) \subseteq E(A)$, with π being identity.

An *objective* is a set $W \subseteq C^\omega$ of infinite sequences of elements of C . In this paper, we will always work with *prefix-independent* objectives, meaning objectives which satisfy $cW = W$ for all $c \in C$ (that is $cw \in W \Leftrightarrow w \in W$ for any $w \in C^\omega$); this allows us to simplify many of the definitions.

A graph G *satisfies* an objective W if $L(G) \subseteq W$. A *game* is given by a C -arena A together with an objective W . It is *winning* (for Eve) if there is a strategy (S, π) such that S satisfies W . In this case, we also say that Eve *wins* the game (A, W) with the strategy (S, π) . We say that an objective W is *positional* (over finite arenas or over arbitrary arenas) if for any (finite or arbitrary) arena A , whenever Eve wins the game (A, W) , she wins (A, W) with a positional strategy.

Neutral letters. A letter $\varepsilon \in C$ is said to be *weakly neutral* for an objective $W \subseteq C^\omega$ if for any word $w \in C^\omega$ decomposed into $w = w_0 w_1 \dots$ with non-empty words $w_i \in C^+$,

$$w \in W \iff \varepsilon w_0 \varepsilon w_1 \varepsilon \dots \in W.$$

A weakly neutral letter $\varepsilon \in C$ is *strongly neutral* if in the above, the w_i can be chosen empty, and moreover, $\varepsilon^\omega \in W$. Thus, for prefix-independent objectives, the difference between the two notions relies on the membership of ε^ω in W .

A few examples: for the parity objective, the priority 0 is strongly neutral; for Bounded, the weight 0 is strongly neutral; for Mean-Payoff $_{\leq 0}$, the letter 0 is only weakly neutral (because $1^\omega \notin \text{Mean-Payoff}_{\leq 0}$ however $010010001 \dots \in \text{Mean-Payoff}_{\leq 0}$), and likewise for Mean-Payoff $_{< 0}$ because $0^\omega \notin \text{Mean-Payoff}_{< 0}$.

Monotone and universal graphs. An *ordered graph* is a graph G equipped with a total order \geq on its set of vertices $V(G)$. We say that it is *monotone* if

$$v \geq u \xrightarrow{c} u' \geq v' \text{ in } G \quad \text{implies} \quad v \xrightarrow{c} v' \in E(G).$$

Such a graph is *well founded* if the order \geq on $V(G)$ is well founded.

We will use a variant of universality called (uniform) *almost-universality* (for trees), which is convenient when working with prefix-independent objectives. A C -graph U is *almost W -universal*, if U satisfies W , and for any tree T satisfying W , there is a vertex $t \in V(T)$ such that $T[t] \rightarrow U$. We will rely on the following inductive result from [Ohl23a].

Theorem 2.1 (Follows from Theorem 3.2 and Lemma 4.5 in [Ohl23a]). *Let $W \subseteq C^\omega$ be a prefix-independent objective such that there is a graph which is almost W -universal. Then W is positional over arbitrary arenas.*

Structuration results. The following results were proved in Ohlmann’s PhD thesis (Theorems 3.1 and 3.2 in [Ohl21]); the two incomparable variants stem from two different techniques.

Lemma 2.2 (Finite structuration). *Let W be a prefix-independent objective which is positional over finite arenas and admits a weakly neutral letter, and let G be a finite graph satisfying W . Then there is a monotone graph G' satisfying W such that $G \rightarrow G'$.*

Lemma 2.3 (Infinite structuration). *Let W be a prefix-independent objective which is positional over arbitrary arenas and admits a strongly neutral letter, and let G be any graph satisfying W . Then there is a well-founded monotone graph G' satisfying W such that $G \rightarrow G'$.*

Note that in both results, we may assume that $|G'| \leq |G|$, simply by restricting to the image of G . Details of the proof of Lemma 2.3 can be found in [Ohl23a, Theorem 3]; Lemma 2.2 appears only in Ohlmann’s PhD thesis [Ohl21], we give details in Appendix A for completeness.

Automata. A *co-Büchi automaton* over C is a q_0 -rooted $C \times \{\mathcal{N}, \mathcal{F}\}$ -graph A . In this context, vertices $V(A)$ are called *states*, edges $E(A)$ are called *transitions*, and the root q_0 is called the *initial state*. Moreover, transitions of the form $q \xrightarrow{(c, \mathcal{N})} q'$ are called *normal transitions* and simply denoted $q \xrightarrow{c} q'$, while transitions of the form $q \xrightarrow{(c, \mathcal{F})} q'$ are called *co-Büchi transitions* and denoted $q \xrightarrow{c} q'$. For simplicity, we assume automata to be *complete* (for any state q and any letter c , there is at least one outgoing transition labelled c from q) and *reachable* (for any state q there is some path from q_0 to q in A).

A path $q_0 \xrightarrow{(c_0, a_0)} q_1 \xrightarrow{(c_1, a_1)} \dots$ in A is *accepting* if it contains only finitely many co-Büchi transitions, meaning that only finitely many of a_i equal \mathcal{F} . If $q \in V(A)$ is a state, then define the *language* $L(A, q) \subseteq C^\omega$ of a co-Büchi automaton *from a state* $q \in V(A)$ as the set of infinite words which label accepting paths from q in A . The *language* of A denoted $L(A)$ is $L(A, q_0)$. Note that in this paper, automata are not assumed to be finite.

We say that an automaton is *monotone* if it is monotone as a $C \times \{\mathcal{N}, \mathcal{F}\}$ -graph. Likewise, morphisms between automata are just morphisms of the corresponding $C \times \{\mathcal{N}, \mathcal{F}\}$ -graphs that moreover preserve the initial state. Note that $A \rightarrow A'$ implies $L(A) \subseteq L(A')$. A co-Büchi automaton is *deterministic* if for each state $q \in V(A)$ and each letter $c \in C$ there is exactly one transition labelled by c outgoing from q .

A *resolver* for an automaton A is a deterministic automaton R with a morphism $R \rightarrow A$. Note that the existence of this morphism implies that $L(R) \subseteq L(A)$. Such a resolver is *sound* if additionally $L(R) \supseteq L(A)$ (and thus $L(R) = L(A)$). A co-Büchi automaton is *history-deterministic* if there exists a sound resolver R . Our definition of history-determinism is slightly non-standard, but it fits well with our overall use of morphisms and of possibly infinite automata. This point of view was also adopted by Colcombet (see [Col12, Definition 13]). For more details on history determinism of co-Büchi automata, we refer to [HP06, KS15, BKS17, RK22].

We often make use of the following simple lemma, which follows directly from the definitions and the fact that composing morphisms results in a morphism.

Lemma 2.4. *Let A, A' be automata such that $A \rightarrow A'$, A is history-deterministic, and $L(A) = L(A')$. Then A' is history-deterministic.*

An automaton A is *saturated* if it has all possible co-Büchi transitions: $V(A) \times (C \times \{\mathcal{F}\}) \times V(A) \subseteq E(A)$. The *saturation* of an automaton A is obtained from A by adding all possible co-Büchi transitions. Similar techniques of saturating co-Büchi automata have been previously used to study their structure [KS15, IK19, RK22].

Note that languages of saturated automata are always prefix-independent. The lemma below states that co-Büchi transitions are somewhat irrelevant in history-deterministic automata recognising prefix-independent languages.

Lemma 2.5. *Let A be a history-deterministic automaton recognising a prefix-independent language and let A' be its saturation. Then $L(A) = L(A')$ and A' is history-deterministic. Moreover, $L(A') = L(A', q)$ for any $q \in V(A')$.*

Proof. Clearly $A \rightarrow A'$ thus $L(A) \subseteq L(A')$; it suffices to prove $L(A') \subseteq L(A)$ and conclude by Lemma 2.4. Let $w_0w_1 \cdots \in L(A')$ and let $q_0 \xrightarrow{(w_0, a_0)} q_1 \xrightarrow{(w_1, a_1)} \dots$ be an accepting path for w in A' . Then for some i , $q_i \xrightarrow{(w_i, a_i)} q_{i+1} \xrightarrow{(w_{i+1}, a_{i+1})} \dots$ is comprised only of normal transitions. Thus, this suffix of the path does not use edges added during the saturation process, which means this suffix is an accepting path in A . We conclude that $w_iw_{i+1} \cdots \in L(A, q_i)$. Due to the assumption that our automata are reachable, there is a finite word u such that $q_0 \xrightarrow{u} q_i$ which implies that $uw \in L(A')$ but because of prefix independence it means that $w \in L(A')$.

The claim that $L(A', q)$ is independent of q follows directly from the fact that A' is saturated: if $w_0w_1 \cdots \in L(A', q)$ as witnessed by a path starting with $q \xrightarrow{(w_0, a_0)} q'$ and q'' is some other state then we can also accept w from q' by first using a co-Büchi transition $q'' \xrightarrow{(w_0, \mathcal{F})} q'$ to reach q' and then continue as the original path does. \square

3. POSITIONAL PREFIX-INDEPENDENT Σ_2^0 OBJECTIVES

3.1. A characterisation. Recall that Σ_2^0 objectives are countable unions of closed objectives; for the purpose of this paper it is convenient to observe that these are exactly those objectives recognised by (countable) deterministic co-Büchi automata (see for instance [Skr13]).

The goal of the section is to prove Theorem 1.2 which we now restate for convenience.

Theorem 1.2. *Let $W \subseteq C^\omega$ be a prefix-independent Σ_2^0 objective admitting a strongly neutral letter. Then W is positional over arbitrary arenas if and only if it is recognised by a countable history-deterministic well-founded monotone co-Büchi automaton.*

Before moving on to the proof, we proceed with a quick technical statement that allows us to put automata in a slightly more convenient form.

Lemma 3.1. *Let A be a history-deterministic automaton recognising a non-empty prefix-independent language. There exists a history-deterministic automaton A' with $L(A') = L(A)$ and such that from every state $q' \in V(A')$, there is an infinite path comprised only of normal transitions. Moreover, if A is countable, well founded, and monotone, then so is A' .*

Proof. Let $V \subseteq V(A)$ be the set of states $q \in V(A)$ from which there is an infinite path of normal transitions. Note that $V \neq \emptyset$ since $L(A)$ is non-empty. Due to prefix-independence of A we can assume that q_0 belongs to V (if not, modify the initial state using Lemma 2.5).

Since every path from $V(A) \setminus V$ visits at least one co-Büchi transition, we turn all normal transitions adjacent to states in $V(A) \setminus V$ into co-Büchi ones; moreover we saturate A . These operations do not affect $L(A)$ or history-determinism. Thus, without loss of generality assume that A is already the modified automaton, with the initial state in V .

Call A' the restriction of A to states in V . It is clear that restricting A to some subset of states preserves being countable, well founded, and monotone.

We claim that $L(A) = L(A')$. The inclusion $L(A') \subseteq L(A)$ is obvious as A' is a restriction of A to a subset of states. We focus on the converse: let $w = w_0w_1 \cdots \in L(A)$ and take an accepting path π for w . Then there is a suffix of π which remains in V and therefore defines a path in A' ; we conclude thanks to prefix-independence of $L(A')$ (which is a consequence of it being saturated).

It remains to see that A' is history-deterministic. For this, we observe that any transition adjacent to states in $V(A) \setminus V$ is a co-Büchi transition; therefore the map $\phi : V(A) \rightarrow V(A') = V$ which is identity on V and sends $V(A) \setminus V$ to the initial state of A' defines a morphism $A \rightarrow A'$. We conclude by Lemma 2.4. \square

To prove Theorem 1.2, we separate both directions so as to provide more precise hypotheses.

Lemma 3.2. *Let W be a prefix-independent Σ_2^0 objective admitting a strongly neutral letter. If W is positional over arbitrary arenas then W is recognised by a countable history-deterministic monotone well-founded co-Büchi automaton.*

Proof. Note that W cannot be empty because it contains ϵ^ω . Let A be a history-deterministic co-Büchi automaton recognising W with initial state q_0 ; thanks to Lemma 3.1 we assume that every state in A participates in an infinite path of normal transitions. Clearly A is countable. Let G be the C -graph obtained from A by removing all the co-Büchi transitions. The fact that G is sinkless (and therefore, G is indeed a graph) follows from the assumption on A . Since W is prefix-independent, it holds that G satisfies W .

Apply the infinite structuration result (Lemma 2.3, which requires the strongly neutral letter) to G to obtain a well-founded monotone graph G' satisfying W and such that $G \xrightarrow{\phi} G'$. Note that we may restrict $V(G')$ to the image of ϕ . Due to the fact that C is countable, this guarantees that G' is countable.

Now let A' be the co-Büchi automaton obtained from G' by turning every edge into a normal transition, setting the initial state to be $q'_0 = \phi(q_0)$, and saturating. Note that A' is countable monotone and well-founded. Note moreover that the morphism $G \xrightarrow{\phi} G'$ is in fact also a morphism witnessing that $A \rightarrow A'$. Thus, since A is history deterministic, Lemma 2.4 implies that so is A' . It remains to see that A' recognises W .

Let $w \in L(A')$. Then $w = uw'$ where $w' \in L(G') \subseteq W$. It follows from prefix-independence that $w \in W$. Conversely, let $w_0w_1 \cdots \in W$ as witnessed by an accepting path $\pi = q_0 \xrightarrow{(w_0, a_0)} q_1 \xrightarrow{(w_1, a_1)} \dots$ from q_0 in A . This path has only finitely many co-Büchi transitions.

Then consider the path $\pi' = \phi(q_0) \xrightarrow{w_0} \phi(q_1) \xrightarrow{w_1} \dots$ in A' , where we use co-Büchi transitions only when necessary, meaning when there is no normal transition $\phi(q_i) \xrightarrow{w_i} \phi(q_{i+1})$ in A' . Since π visits only finitely many co-Büchi transitions, it is eventually a path in G , and thus since ϕ is a morphism, π' is eventually a path in G' , and hence it sees only finitely many co-Büchi transitions in A' . Hence $L(A') = W$. \square

For the converse direction, we do not require a neutral letter.

Lemma 3.3. *If W is a prefix-independent objective recognised by a countable history-deterministic monotone well-founded co-Büchi automaton then W is positional over arbitrary arenas.*

Proof. As previously, if W is empty then it is trivially positional, so we assume that W is non-empty, and we take an automaton A satisfying the hypotheses above and apply Lemma 3.1 so that every state participates in an infinite path of normal transitions. Let U be the C -graph obtained from A by removing all co-Büchi transitions and turning normal transitions into edges; thanks to Lemma 3.1, U is sinkless so it is indeed a graph. We prove that U is almost W -universal for trees. Let T be a tree satisfying W and let t_0 be its root.

Since A is history-deterministic, there is a mapping $\phi : V(T) \rightarrow V(A)$ such that for each edge $t \xrightarrow{c} t' \in E(T)$, there is a transition $\phi(t) \xrightarrow{(c,a)} \phi(t')$ in A with some $a \in \{\mathcal{N}, \mathcal{F}\}$, and such that for all infinite paths $t_0 \xrightarrow{w_0} t_1 \xrightarrow{w_1} \dots$ in T , there are only finitely many co-Büchi transitions on the path $\phi(t_0) \xrightarrow{(w_0,a_0)} \phi(t_1) \xrightarrow{(w_1,a_1)} \dots$ in A .

Claim 3.4. *There is a vertex $t'_0 \in V(T)$ such that for all infinite paths $t'_0 \xrightarrow{w_0} t'_1 \xrightarrow{w_1} \dots$ from t'_0 in T , there is no co-Büchi transition on the path $\phi(t'_0) \xrightarrow{w_0} \phi(t'_1) \xrightarrow{w_1} \dots$ in A .*

Proof of Claim 3.4. Assume towards contradiction that no such vertex exists. Then starting from the root t_0 , we build an infinite path $t_0 \xrightarrow{w_0} t_1 \xrightarrow{w_1} \dots$ in T such that $\phi(t_0) \xrightarrow{w_0} \phi(t_1) \xrightarrow{w_1} \dots$ has infinitely many co-Büchi transitions in A . Indeed, assuming the path built up to t_i , we simply pick $t_i \xrightarrow{w_i} t_{i+1}$ such that there is a co-Büchi transition in A on the corresponding path $\phi(t_i) \xrightarrow{w_i} \phi(t_{i+1})$. Thus, we constructed a path contradicting the observation below: this path has infinitely many co-Büchi transitions in A . \square

There remains to observe that ϕ maps $T[t'_0]$ to U , and thus U is almost W -universal for trees. We conclude by applying Lemma 2.1. \square

The above two lemmata constitute a proof of Theorem 1.2.

3.2. A few examples. We now elaborate on some examples.

Kopczyński-monotonic objectives. In our terminology, Kopczyński's monotonic objectives correspond to the prefix-independent languages that are recognised by finite monotone co-Büchi automata. Note that such automata are of course well-founded, but also they are history-deterministic (even determinisable by pruning): one should always follow a transition to a maximal state. Therefore our result proves that such objectives are positional over arbitrary arenas. A very easy example is the co-Büchi objective

$$\text{co-Büchi} = \{w \in \{\mathcal{N}, \mathcal{F}\}^\omega \mid w \text{ has finitely many occurrences of } \mathcal{F}\},$$

which is recognised by a (monotone) automaton with a single state. Some more advanced examples are given in Figure 1.



Figure 1: Two finite monotone co-Büchi automata recognising prefix-independent languages. For clarity, the co-Büchi transitions are not depicted but connect every pair of states; likewise, edges following from monotonicity (such as the dashed ones for example), are omitted. The automaton on the left recognises words with finitely many aab infixes. The automaton on the right recognises words with finitely many infixes in $c(a^*cb^*)^+c$.

Finite support. The finite support objective is defined over \mathbb{N} by

$$\text{Finite} = \{w \in \mathbb{N}^\omega \mid \text{finitely many distinct letters appear in } w\}$$

Consider the automaton A over $V(A) = \mathbb{N}$ with

$$v \xrightarrow{w} v' \in E(A) \iff w, v' \leq v,$$

co-Büchi transitions everywhere, and initial state 0 (see Figure 2).

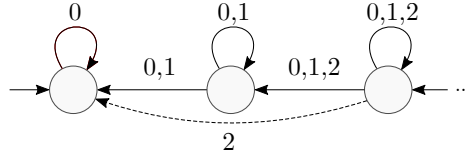


Figure 2: An automaton A for objective Finite. Co-Büchi edges, as well as some edges following from monotonicity (such as the dashed one) are omitted for clarity.

It is countable, history-deterministic, well-founded, and monotone and recognises $L(A) = \text{Finite}$. Details of the proof are easy and left to the reader. Positionality of Finite can also be established by Corollary 1.2, as it is a countable union of the safety languages $F^\omega \subseteq \mathbb{N}^\omega$, where F ranges over finite subsets of \mathbb{N} . As far as we are aware, this result is novel.³

Energy objectives. Recall the energy objective

$$\text{Bounded} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \sup_k \sum_{i=0}^{k-1} w_i \text{ is finite} \right\},$$

which is prefix-independent and belongs to Σ_2^0 . Consider the automaton A whose set of states is \mathbb{N} , with the initial state 0 and with all possible co-Büchi transitions, and normal transitions of the form $v \xrightarrow{w} v'$ where $w \leq v - v'$. Note that A is well-founded and monotone, so we should prove that it is history-deterministic and recognises Bounded.

³A similar positionality result is proved in [Gim04], but it assumes finite degree of the arena, vertex-labels (which is more restrictive), and injectivity of the colouring of the arena.

Note that any infinite path of normal edges $v_0 \xrightarrow{w_0} v_1 \xrightarrow{w_1} \dots$ in A is such that for all i , $w_i \leq v_i - v_{i+1}$, and therefore

$$\sum_{i=0}^{k-1} w_i \leq v_0 - v_k \leq v_0$$

and thus $L(A) \subseteq \text{Bounded}$.

A resolver for A works as follows: keep a counter c (initialised to zero), and from a vertex v and when reading an edge w ,

- if $v \geq w$, then take the normal transition $v \xrightarrow{w} v - w$;
- otherwise, take the co-Büchi transition $v \xrightarrow{w} c$ and increment the counter.

Formally, R is defined by $V(R) = V(A) \times \mathbb{N}$, initial state $r_0 = (0, 0)$ and

$$\begin{aligned} (v, c) \xrightarrow{w} (v', c') \in E(R) &\iff v' = v - w \geq 0 \text{ and } c' = c \\ (v, c) \xrightarrow{w} (v', c') \in E(R) &\iff v - w < 0 \text{ and } c' = v' = c + 1. \end{aligned}$$

Clearly $(v, c) \mapsto v$ defines a morphism from R to A which sends r_0 to q_0 , so there remains to see that $\text{Bounded} \subseteq L(R)$.

Consider a word $w_0 w_1 \dots \in \text{Bounded}$. By definition, there exists N such that

$$\sup_k \sum_{i=0}^{k-1} w_i \leq N. \quad (*)$$

Given a finite word $u \in \mathbb{Z}^*$, we let $s(u) \in \mathbb{Z}$ denote the sum of its letters. Let π be the unique path from r_0 in R labelled by w . Note that the counter (second coordinate) in states appearing in π always grows, and that a co-Büchi transition is read precisely when it is incremented; we show that it cannot exceed N which proves that the path is accepting. Assume towards a contradiction that the counter exceeds N , therefore π is of the form

$$(0, 0) \xrightarrow{u_0} (v_0, 0) \xrightarrow{u'_0} (1, 1) \xrightarrow{u_1} (v_1, 1) \xrightarrow{u'_1} \dots \xrightarrow{u'_{N-1}} (N, N) \xrightarrow{u_N} (v_N, N) \xrightarrow{u'_N} (N+1, N+1) \xrightarrow{w'}$$

where the u_i 's are finite words, the u'_i 's are letters in \mathbb{Z} , they concatenate to $u_0 u'_0 \dots u_N u'_N w' = w$, and for each $i \leq N$, it holds that $s(u_i) = i - v_i$ and $v_i - u'_i < 0$. Therefore we have

$$s(u_0 u'_0 \dots u_N u'_N) = \underbrace{s(u_0)}_{=0-v_0} + \underbrace{u'_0}_{>v_0} + \dots + \underbrace{s(u_N)}_{=N-v_N} + \underbrace{u'_N}_{>v_N} \geq 0 + 1 + \dots + N > N,$$

contradicting (*). Hence R is a sound resolver for A , so A is history-deterministic.

We conclude that Bounded is positional over arbitrary arenas.

Eventually non-increasing objective. Over the alphabet \mathbb{N} , consider the objective

$$\text{ENI} = \{w_0 w_1 \dots \in \mathbb{N}^\omega \mid \text{there are finitely many } i \text{ such that } w_{i+1} > w_i\}.$$

Note that since \mathbb{N} is well founded, a sequence belongs to ENI if and only if it is eventually constant. Consider the automaton A over \mathbb{N} with the initial state 0, with all possible co-Büchi transitions, and with normal transitions $v \xrightarrow{w} v'$ if and only if $v \geq w \geq v'$. Note that A is countable, well-founded, and monotone, so we should prove that it recognises ENI and is history-deterministic.

First, note that any infinite path of normal edges $v_0 \xrightarrow{w_0} v_1 \xrightarrow{w_1} \dots$ in A is such that $v_0 \geq w_0 \geq v_1 \geq w_1 \geq \dots$, and therefore $L(A) \subseteq \text{ENI}$. A sound resolver for A simply goes to the state w when reading a letter w , using a normal transition if possible, and a co-Büchi transition otherwise. We leave the formal definition to the reader.

Eventually non-decreasing objective. In contrast, the objective

$$\text{END} = \{w_0w_1 \cdots \in \mathbb{N}^\omega \mid \text{there are finitely many } i \text{ such that } w_{i+1} < w_i\}$$

is not positional over arbitrary arenas, as witnessed by Figure 3.

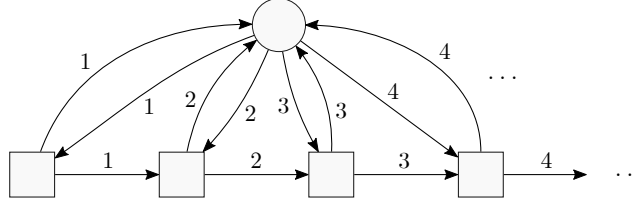


Figure 3: An arena over which Eve requires a non-positional strategy in order to produce a sequence which is eventually non-decreasing.

3.3. Closure under countable unions. We now move on to Corollary 1.2, which answers Kopczyński’s conjecture in the affirmative in the case of Σ_2^0 objectives.

If W_0, W_1, \dots are all positional prefix-independent Σ_2^0 objectives, each admitting a strongly neutral letter (possibly a different one), then the union $\bigcup_{i \in \mathbb{N}} W_i$ is also positional.

Proof. Let W_0, W_1, \dots be a family of countably many prefix-independent Σ_2^0 objectives admitting strongly neutral letters. Using Theorem 1.2 we get countable history-deterministic well-founded monotone co-Büchi automata A_0, A_1, \dots for the respective objectives; without loss of generality we assume that they are saturated (Lemma 2.5).

Then consider the automaton A obtained from the disjoint union of the A_i ’s by adding all possible co-Büchi transitions, and all normal transitions from A_i to A_j with $i > j$. The initial state in A can be chosen arbitrarily. Note that A is well-founded, monotone, and countable, so we should prove that it recognises $W = \bigcup_i W_i$ and is history deterministic.

Note that any infinite path in A which visits finitely many co-Büchi transitions eventually remains in some A_i , and thus by prefix-independence, $L(A) \subseteq W$.

It remains to prove history-determinism of A . Let R_0, R_1, \dots be resolvers for A_0, A_1, \dots witnessing that these automata are history deterministic. Consider a resolver which stores a sequence of states (r_0, r_1, \dots) , with r_i being a state of R_i . Initially these are all initial states of the respective resolvers and the transitions follow the transitions of all the resolvers synchronously. Additionally, we store a round-robin counter, which indicates one of the resolvers, following the sequence $R_0; R_0, R_1; R_0, R_1, R_2; R_0, R_1, R_2, R_3; \dots$. If we see a normal transition in the currently indicated resolver, then we also see a normal transition in R , and otherwise, we update the counter to the next resolver and see a co-Büchi transition in R .

Let $\ell_0\ell_1 \cdots = 0010120123 \dots$. We formally define the resolver R as follows: $V(R) = \prod_{i=0}^\infty V(R_i) \times \mathbb{N}$, with transitions

$$\begin{aligned} (r, j) \xrightarrow{(c, \mathcal{N})} (r', j') &\iff [\forall i. \exists a_i. r_i \xrightarrow{(c, a_i)} r'_i \in E(R_i)] \\ &\quad \text{and } r_{\ell_j} \xrightarrow{(c, \mathcal{N})} r'_{\ell_j} \in E(R_{\ell_j}) \text{ and } j' = j \\ (r, j) \xrightarrow{(c, \mathcal{F})} (r', j') &\iff [\forall i. \exists a_i. r_i \xrightarrow{(c, a_i)} r'_i \in E(R_i)] \\ &\quad \text{and } r_{\ell_j} \xrightarrow{(c, \mathcal{F})} r'_{\ell_j} \in E(R_{\ell_j}) \text{ and } j' = j+1 \end{aligned}$$

and morphism $\phi : (r, j) \mapsto \phi_j(r_j) \in V(A)$, where $\phi_0 : R_0 \rightarrow A_0, \phi_1 : R_1 \rightarrow A_1, \dots$ are the respective morphisms.

We now prove that $W \subseteq L(A)$ and the above resolver is sound. For that, consider a word w which belongs to $L(A_n)$ for some n . Assume for the sake of contradiction that the path in A constructed by the above resolver reading w contains infinitely many co-Büchi transitions. It means that infinitely many times the resolver R_n reached a co-Büchi transition in A_n . But this contradicts the assumption that R_n is sound. We conclude that W is positional by applying Lemma 3.3. \square

4. FROM FINITE TO ARBITRARY ARENAS

In this section we study the difference between positionality over finite and arbitrary arenas.

4.1. Mean-payoff games. There are, in fact, four non-isomorphic variants of the mean-payoff objective. Three of them fail to be positional over arbitrary arenas (even over bounded degree arenas), as expressed by the following facts.

Proposition 4.1. *The mean-payoff objective $\text{Mean-Payoff}_{\leq 0}$ over $w_0w_1 \dots \in \mathbb{Z}^\omega$ with the condition $\limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i \leq 0$ is not positional over arbitrary arenas.*

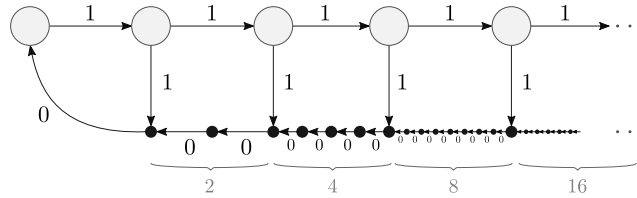


Figure 4: The arena used in the proof of Proposition 4.1.

Proof. Consider the arena depicted on Fig. 4. Eve can win by following bigger and bigger loops which reach arbitrarily far to the right. This strategy brings the average of the weights closer and closer to 0.

Nevertheless, each positional strategy of Eve either moves infinitely far to the right (resulting in $\lim_k \frac{1}{k} \sum_{i=0}^{k-1} w_i = 1$) or repeats some finite loop which results in a fixed positive limit $\lim_k \frac{1}{k} \sum_{i=0}^{k-1} w_i > 0$. In both cases it violates $\text{Mean-Payoff}_{\leq 0}$. \square

Proposition 4.2. *Consider two lim inf variants of the mean-payoff objective over $w_0w_1 \dots \in \mathbb{Z}^\omega$: one where we require that $\liminf_k \frac{1}{k} \sum_{i=0}^{k-1} w_i \leq 0$, and the other where that same quantity is < 0 . Both these objectives are not positional over arbitrary arenas.*

Proof. Consider the arena depicted on Fig. 5. Again, Eve has a winning strategy for both these objectives by always going sufficiently far to the left, to ensure that the average drops below for instance $-\frac{1}{2}$.

Nevertheless, each positional strategy of Eve either moves infinitely far to the left (resulting again in $\lim_k \frac{1}{k} \sum_{i=0}^{k-1} w_i = 1$), or repeats some finite loop, reaching a minimal negative weight -2^n for some $n > 0$. Now, Adam can win against this strategy by repeating

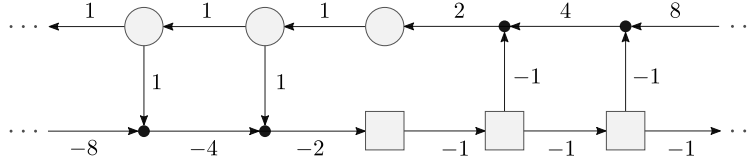


Figure 5: The arena used in the proof of Proposition 4.2.

a loop going to the right, in such a way to reach a weight 2^{n+1} . The label of such a path satisfies $\lim_k \frac{1}{k} \sum_{i=0}^{k-1} w_i = \frac{2^{n+1}-1}{4n+4} > 0$, violating both objectives. \square

The remaining fourth type of a mean-payoff objective is „lim sup < 0 ”:

$$\text{Mean-Payoff}_{<0} = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i < 0 \right\}.$$

Proposition 4.3. *The objective Mean-Payoff $_{<0}$ is positional over arbitrary arenas.*

Proof. Consider the tilted boundedness objective with parameter $n \geq 1$, defined as

$$\text{Tilted-Bounded}_n = \left\{ w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \sup_k \sum_{i=0}^{k-1} \left(w_i + \frac{1}{n} \right) \text{ is finite} \right\}$$

Note that renaming weights by $w \mapsto (1+nw)$ reduces Tilted-Bounded $_n$ to Bounded. More precisely, for every $w_0 w_1 \cdots \in \mathbb{Z}^\omega$ we have

$$w_0 w_1 \cdots \in \text{Tilted-Bounded}_n \iff (1+nw_0)(1+nw_1) \cdots \in \text{Bounded},$$

because the bound is just multiplied by the constant n .

Therefore, since Bounded is positional over arbitrary arenas, it follows that Tilted-Bounded $_n$ is also positional over arbitrary arenas: rename the labels, find a positional strategy and then recall original labels. Note also that for every n the objective Tilted-Bounded $_n$ belongs to Σ_2^0 , as a union ranging over $N \in \mathbb{N}$ of closed (in other words safety) objectives $\{w_0 w_1 \cdots \in \mathbb{Z}^\omega \mid \forall k \in \mathbb{N} \sum_{i=0}^{k-1} (w_i + \frac{1}{n}) \leq N\}$.

Claim 4.4. *It holds that Mean-Payoff $_{<0} = \bigcup_{n \geq 1} \text{Tilted-Bounded}_n$.*

Proof of Claim 4.4. Write $\text{mp}(w) = \limsup_k \frac{1}{k} \sum_{i=0}^{k-1} w_i$. If

$$w = w_0 w_1 \cdots \in \text{Tilted-Bounded}_n$$

then there is a bound N such that for all k , $\sum_{i=0}^{k-1} (w_i + \frac{1}{n}) \leq N$, therefore $\frac{1}{k} \sum_{i=0}^{k-1} w_i \leq \frac{N}{k} - \frac{1}{n}$ and thus $\text{mp}(w) \leq -\frac{1}{n} < 0$, so $w \in \text{Mean-Payoff}_{<0}$. Conversely, if $w \in \text{Mean-Payoff}_{<0}$ and n is large enough so that $\frac{1}{n} \leq \text{mp}(w)$, then $w \in \text{Tilted-Bounded}_n$. \square

Now, positionality of Mean-Payoff $_{<0}$ follows from the claim together with Corollary 1.2, as all Tilted-Bounded $_n$ are prefix-independent, admit a strongly neutral letter, are positional, and belong to Σ_2^0 .⁴ \square

⁴We thank Lorenzo Clemente for suggesting to use closure under union. A direct proof (constructing a universal graph) is available in the unpublished preprint [Ohl23b].

4.2. A completeness result. The aim of this section is to prove Theorem 1.3 below, which roughly states that any objective which is positional over finite arenas is equivalent to an objective which is positional over arbitrary arenas. First, we need to define what we mean by “equivalent”.

Equivalence over finite arenas. Recall that two prefix-independent objectives $W, W' \subseteq C^\omega$ are said to be *finitely equivalent*, written $W \equiv W'$, if for all finite C -arenas A ,

$$\text{Eve wins } (A, W) \iff \text{Eve wins } (A, W').$$

Since one may view strategies as games controlled by Adam, we obtain the following motivating result.

Lemma 4.5. *If $W \equiv W'$ and W is positional over finite arenas then so is W' .*

Proof. Let A be a finite C -arena such that Eve wins (A, W') . Then Eve wins (A, W) , so she wins with a positional strategy S . Looking at S as a finite C -arena controlled by Adam yields that Eve wins (S, W') , thus S satisfies W' . \square

We now move on to the proof of our completeness result.

Theorem 1.3. *Let $W \subseteq C^\omega$ be a prefix-independent objective which is positional over finite arenas and admits a weakly neutral letter. Then there exists a prefix-independent objective $W' \subseteq W$ such that $W' \equiv W$ and W' is positional over arbitrary arenas.*

We start with the following observation, which is a standard topological argument based on König’s lemma. Note that the assumption of finiteness of G is essential here.

Lemma 4.6. *Let G be a finite C -graph and $v \in G$. Then $L(G, v)$ is a closed subset of C^ω .*

Proof. Consider the language $K \subseteq C^*$ of finite words w such that there is no path from v labelled by w . By definition, KC^ω is open. Our goal is to show that its complement coincides with $L(G, v)$. Clearly $L(G, v) \cap K \cdot C^\omega = \emptyset$. For the remaining inclusion assume that $w \in C^\omega$ is an infinite word such that $w \notin K \cdot C^\omega$. We want to show that $w \in L(G, v)$. The assumption that $w \notin K \cdot C^\omega$ means that every prefix of w labels a finite path from v in G . By applying König’s lemma, relying on the fact that $E(G)$ is finite, we see that w must label an infinite path in G from v , thus $w \in L(G, v)$. \square

We may now give the crucial definition. Given a prefix-independent objective $W \subseteq C^\omega$, we define its finitary substitute to be

$$W_{\text{fin}} = \{w \in C^\omega \mid w \text{ labels a path in some finite graph } G \text{ which satisfies } W\}.$$

Note that $W_{\text{fin}} \subseteq W$ and W_{fin} is prefix-independent (we can prepend an additional finite path to any graph). Now observe that

$$W_{\text{fin}} = \bigcup_{\substack{G \text{ finite graph} \\ G \text{ satisfies } W}} L(G) = \bigcup_{\substack{G \text{ finite graph} \\ G \text{ satisfies } W \\ v \in V(G)}} L(G, v),$$

and since there are (up to isomorphism) only countably many finite graphs, it follows from Lemma 4.6 that $W_{\text{fin}} \in \Sigma_2^0$.

Lemma 4.7. *Let $W \subseteq C^\omega$ be a prefix-independent objective which is positional over finite arenas. Then $W_{\text{fin}} \equiv W$.*

Proof. Let A be a finite C -arena. Since $W_{\text{fin}} \subseteq W$, it is clear that if Eve wins (A, W_{fin}) , then she wins (A, W) . Conversely, assume Eve wins (A, W) . Then she has a positional strategy S in A which is winning for W . Since S is a finite graph, it is also winning for W_{fin} and therefore Eve wins (A, W_{fin}) . \square

We should make the following sanity check.

Lemma 4.8. *If W is prefix-independent, then W_{fin} as well.*

Proof. Take a letter $c \in C$, we aim to show that $cW_{\text{fin}} = W_{\text{fin}}$. Let $w \in cW_{\text{fin}}$, and let G be a finite graph satisfying W such that cw labels a path from $v \in V(G)$ in G . Then w labels a path from a c -successor of v in G , thus $w \in W_{\text{fin}}$.

Conversely, let $w \in W_{\text{fin}}$, and let G be a finite graph satisfying W such that w labels a path from $v \in V(G)$ in G . Let G' be the graph obtained from G by adding a fresh vertex v' with a unique outgoing c -edge towards v . Since W is prefix-independent, G' satisfies W . Since cw labels a path from v' in G' , it follows that $cw \in W_{\text{fin}}$. \square

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let W be a prefix-independent objective which is positional over finite arenas and admits a weakly neutral letter ε . We show that W_{fin} satisfies the assumptions of Lemma 3.3 which guarantees that it is positional over arbitrary arenas. Since Lemma 4.7 implies that $W_{\text{fin}} \equiv W$, this concludes the proof of Theorem 1.3.

Thanks to Lemma 2.2, any finite graph H satisfying W can be embedded into a monotone finite graph G which also satisfies W ; note that $L(H) \subseteq L(G)$. Therefore

$$W_{\text{fin}} = \bigcup_{\substack{H \text{ finite graph} \\ H \text{ satisfies } W}} L(H) = \bigcup_{\substack{G \text{ finite monotone graph} \\ G \text{ satisfies } W}} L(G).$$

Let G_0, G_1, \dots be an enumeration (up to isomorphism) of all finite monotone graphs satisfying W . Then consider the automaton A obtained from the disjoint union of the G_i 's by adding all normal transitions from G_i to G_j for $i > j$, and saturating with co-Büchi transitions. The initial state q_0 is chosen to be $\max V(G_0)$, the maximal state in G_0 . Note that A is countable, monotone, and well founded, so there remains to prove that $L(A) = W_{\text{fin}}$ and that A is history-deterministic.

Clearly for any monotone graph G satisfying W , it holds that $L(G) \subseteq L(A)$, and thus $W_{\text{fin}} \subseteq L(A)$. Conversely, let $w \in L(A)$, and consider an accepting path π for W . Then eventually, π visits only normal edges, and therefore eventually, π remains in some G_i . Thus $w = uw'$ with $w' \in L(G_i) \subseteq W_{\text{fin}}$, we conclude by prefix-independence of W_{fin} (Lemma 4.8).

To prove that A is history deterministic we now build a resolver: intuitively, we deterministically try to read in G_0 , then if we fail, go to G_1 , then G_2 and so on. The fact that reading in each G_i can be done deterministically follows from monotonicity: for each $v \in V(G_i)$ and each $c \in C$, the set $\{v' \in V(G_i) \mid v \xrightarrow{c} v' \in E(G_i)\}$ of c -successors of v is downward closed. We let $\delta_i(v, c)$ denote the maximal c -successor of v in G_i if it exists, and $\delta_i(v, c) = \perp$ if v does not have a c -successor. It is easy to see that in a monotone graph G , $v \leq v'$ implies $L(G, v) \subseteq L(G, v')$; in words, more continuations are available from bigger states.

Now we define the resolver R by $V(R) = V(A)$, $r_0 = q_0 = \max V(G_0)$, and for any $q, q' \in V(A)$ and $c \in C$,

$$\begin{aligned} q \xrightarrow{c} q' \in E(R) &\iff \exists i, q, q' \in V(G_i) \text{ and } q' = \delta_i(q) \neq \perp \\ q \xrightarrow{c} q' \in E(R) &\iff \exists i, q \in V(G_i) \text{ and } \delta_i(q, c) = \perp \text{ and } q' = \max V(G_{i+1}). \end{aligned}$$

Clearly, R is deterministic and $R \rightarrow A$ so R is a resolver; it remains to prove soundness. Take $w \in L(A)$ and let i such that $w \in L(G_i)$ (from some moment on we need to stop using transitions which change components and $L(A) = W_{\text{fin}}$). Let π be the unique path from $r_0 = \max V(G_0)$ in R labelled by w . We claim that π remains in $\bigcup_{j \leq i} V(G_j)$ and thus it can only visit at most i co-Büchi transitions, so it is accepting. Assume for contradiction that π reaches $V(G_{i+1})$.

Then it is of the form $\pi = \pi_0 \pi_1 \dots \pi_i \pi'$ where each π_j is a path from $\max(V(G_j))$ in G_j and π' starts from $\max(G_{i+1})$. Let w_0, w_1, \dots, w_i and w' be the words labelling the paths, so that $w = w_0 w_1 \dots w_i w'$. Denote $q = \max(V(G_i))$. Then w_i is not a label of a finite path from q in G_i , therefore $w_i w' \notin L(G_i, q) = L(G_i)$. At the same time $w \in L(G_i)$ thus $q \xrightarrow{w_0 \dots w_{i-1}} q' \xrightarrow{w_i w'}$ for some $q' \in V(G_i)$. But then $w_i w' \in L(G_i, q') \subseteq L(G_i, q)$, a contradiction. \square

5. CONCLUSION

We gave a characterisation of prefix-independent Σ_2^0 objectives which are positional over arbitrary arenas as being those recognised by countable history-deterministic well-founded monotone co-Büchi automata. We moreover deduced that this class is closed by unions. We proved that, with a proper definition, mean-payoff games are positional over arbitrary arenas. Finally, we showed that any prefix-independent objective which is positional over finite arenas is finitely equivalent to an objective which is positional over arbitrary arenas.

Open questions. There are many open questions on positionality. Regarding Σ_2^0 objectives, the remaining step would be to lift the prefix-independence assumptions; this requires some new techniques as the proofs presented here do not immediately adapt to this case.

As mentioned in the introduction, Casares [Cas23] obtained a characterisation of positional ω -regular objectives, while we characterised (prefix-independent) Σ_2^0 positional objectives. A common generalisation, which we see as a far reaching open question would be to characterise positionality within Δ_3^0 ; hopefully establishing closure under union for this class.

Another interesting direction would be to understand finite memory for prefix-independent Σ_2^0 objectives; useful tools (such as structuration results) are already available [CO25]. A related (but independent) path is to develop a better understanding of (non-prefix-independent) closed objectives, which so far has remained elusive.

REFERENCES

- [BCD⁺11] Lubos Brim, Jakub Chaloupka, Laurent Doyen, Raffaella Gentilini, and Jean-François Raskin. Faster algorithms for mean-payoff games. *Formal Methods Syst. Des.*, 38(2):97–118, 2011. doi:10.1007/s10703-010-0105-x.
- [BCRV22] Patricia Bouyer, Antonio Casares, Mickael Randour, and Pierre Vandenhover. Half-positional objectives recognized by deterministic büchi automata. In Bartek Klin, Slawomir Lasota, and Anca Muscholl, editors, *33rd International Conference on Concurrency Theory, CONCUR 2022, September 12-16, 2022, Warsaw, Poland*, volume 243 of *LIPICs*, pages 20:1–20:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPICs.CONCUR.2022.20.
- [BKS17] Udi Boker, Orna Kupferman, and Michał Skrzypczak. How deterministic are good-for-games automata? In *FSTTCS*, volume 93 of *LIPICs*, pages 18:1–18:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017. doi:10.4230/LIPICs.FSTTCS.2017.18.
- [Cas23] Antonio Casares. *Structural properties of ω -automata and strategy complexity in infinite duration games*. PhD thesis, Université de Bordeaux, 2023.
- [CFGO22] Thomas Colcombet, Nathanaël Fijalkow, Paweł Gawrychowski, and Pierre Ohlmann. The theory of universal graphs for infinite duration games. *Log. Methods Comput. Sci.*, 18(3), 2022. doi:10.46298/lmcs-18(3:29)2022.
- [CN06] Thomas Colcombet and Damian Niwiński. On the positional determinacy of edge-labeled games. *Theor. Comput. Sci.*, 352(1-3):190–196, 2006. doi:10.1016/j.tcs.2005.10.046.
- [CO24] Antonio Casares and Pierre Ohlmann. Positional ω -regular languages. In *LICS*, pages 21:1–21:14. ACM, 2024. doi:10.1145/3661814.3662087.
- [CO25] Antonio Casares and Pierre Ohlmann. Characterising memory in infinite games. *Log. Methods Comput. Sci.*, 21(1), 2025. URL: [https://doi.org/10.46298/lmcs-21\(1:28\)2025](https://doi.org/10.46298/lmcs-21(1:28)2025), doi:10.46298/LMCS-21(1:28)2025.
- [Col12] Thomas Colcombet. Forms of determinism for automata (invited talk). In *STACS*, volume 14 of *LIPICs*, pages 1–23. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2012. doi:10.4230/LIPICs.STACS.2012.1.
- [Dav64] Morton Davis. *Infinite games of perfect information*. Princeton Univ. Press, Princeton, N.J., 1964.
- [EJ91] E. Allen Emerson and Charanjit S. Jutla. Tree automata, mu-calculus and determinacy (extended abstract). In *32nd Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico, 1-4 October 1991*, pages 368–377. IEEE Computer Society, 1991. doi:10.1109/SFCS.1991.185392.
- [EM79] Andrzej Ehrenfeucht and Jan Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 109(8):109–113, 1979. doi:10.1007/BF01768705.
- [FBBD⁺] Nathanaël Fijalkow, Nathalie Bertrand, Patricia Bouyer-Decitre, Romain Brenguier, Arnaud Carayol, John Fearnley, Hugo Gimbert, Florian Horn, Rasmus Ibsen-Jensen, Nicolas Markey, Benjamin Monmege, Petr Novotný, Mickael Randour, Ocan Sankur, Sylvain Schmitz, Olivier Serre, and Mateusz Skomra. *Games on Graphs*. Online.
- [Gim04] Hugo Gimbert. Parity and exploration games on infinite graphs. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, 18th International Workshop, CSL 2004, 13th Annual Conference of the EACSL, Karpacz, Poland, September 20-24, 2004, Proceedings*, volume 3210 of *Lecture Notes in Computer Science*, pages 56–70. Springer, 2004. doi:10.1007/978-3-540-30124-0_8.
- [GS53] David Gale and Frank M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games*, volume 2 of *Annals of Mathematics Studies*, no. 28, pages 245–266. Princeton University Press, 1953.
- [GZ05] Hugo Gimbert and Wiesław Zielonka. Games where you can play optimally without any memory. In *CONCUR*, volume 3653 of *Lecture Notes in Computer Science*, pages 428–442. Springer, 2005. doi:10.1007/11539452_33.
- [HP06] Thomas A. Henzinger and Nir Piterman. Solving games without determinization. In Zoltán Ésik, editor, *Computer Science Logic, 20th International Workshop, CSL 2006, 15th Annual Conference of the EACSL, Szeged, Hungary, September 25-29, 2006, Proceedings*, volume 4207 of *Lecture Notes in Computer Science*, pages 395–410. Springer, 2006. doi:10.1007/11874683_26.
- [IK19] Simon Iosti and Denis Kuperberg. Eventually safe languages. In Piotrek Hofman and Michał Skrzypczak, editors, *Developments in Language Theory - 23rd International Conference, DLT*

- 2019, Warsaw, Poland, August 5-9, 2019, *Proceedings*, volume 11647 of *Lecture Notes in Computer Science*, pages 192–205. Springer, 2019. doi:10.1007/978-3-030-24886-4_14.
- [Kec95] Alexander Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.
- [Kop06] Eryk Kopczyński. Half-positional determinacy of infinite games. In *ICALP*, volume 4052 of *Lecture Notes in Computer Science*, pages 336–347. Springer, 2006. doi:10.1007/11787006_29.
- [Kop09] Eryk Kopczyński. *Half-positional determinacy of infinite games*. PhD thesis, University of Warsaw, 2009. URL: <https://www.mimuw.edu.pl/~erykk/papers/hpwc.pdf>.
- [Koz22] Alexander Kozachinskiy. Energy games over totally ordered groups. *CoRR*, abs/2205.04508, 2022. arXiv:2205.04508, doi:10.48550/arXiv.2205.04508.
- [KS15] Denis Kuperberg and Michał Skrzypczak. On determinisation of good-for-games automata. In *ICALP*, volume 9135 of *Lecture Notes in Computer Science*, pages 299–310. Springer, 2015. doi:10.1007/978-3-662-47666-6_24.
- [Mar75] Donald A. Martin. Borel determinacy. *Annals of Mathematics*, 102(2):363–371, 1975.
- [Mos91] Andrzej W. Mostowski. Games with forbidden positions. Technical Report 78, University of Gdansk, 1991.
- [Ohl21] Pierre Ohlmann. *Monotone graphs for solving parity and mean-payoff games*. PhD thesis, Université de Paris, 2021.
- [Ohl23a] Pierre Ohlmann. Characterizing Positionality in Games of Infinite Duration over Infinite Graphs. *TheoretCS*, Volume 2, January 2023. URL: <https://theoretics.episciences.org/10878>, doi:10.46298/theoretics.23.3.
- [Ohl23b] Pierre Ohlmann. Positionality of mean-payoff games on infinite graphs, 2023. arXiv:2305.00347.
- [Rab69] Michael O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969. URL: <http://www.jstor.org/stable/1995086>.
- [RK22] Bader Abu Radi and Orna Kupferman. Minimization and canonization of GFG transition-based automata. *Log. Methods Comput. Sci.*, 18(3), 2022. doi:10.46298/lmcs-18(3:16)2022.
- [Sha53] Lloyd S. Shapley. Stochastic games. 39(10):1095–1100, 1953. doi:10.1073/pnas.39.10.1095.
- [Skr13] Michał Skrzypczak. Topological extension of parity automata. *Information and Computation*, 228:16–27, 2013.
- [Wol55] Philip Wolfe. The strict determinateness of certain infinite games. *Pacific Journal of Mathematics*, 5:841–847, 1955.

APPENDIX A. PROOF OF THE FINITE STRUCTURATION RESULT

We include a proof of Lemma 2.2; it is identical to the one in [Ohl21]; a similar result (with the same proof) also appears in [CFG022, Theorem 4.8].

Proof of Lemma 2.2. Let $W \subseteq C^\omega$ be a prefix-independent objective admitting a weakly neutral letter ε and which is positional over finite arenas, and let G be a finite graph satisfying W . We let G^ε be a graph obtained from G by saturating it with ε -edges: we successively add arbitrary ε -edges until obtaining a graph satisfying W but such that adding any ε -edge would create a path whose label does not belong to W . Note that $G \rightarrow G^\varepsilon$. We claim that the relation $>$ defined by

$$v > v' \iff v \neq v' \text{ and } v \xrightarrow{\varepsilon} v' \in E(G^\varepsilon)$$

defines a strict total pre-order over $V(G)$. Transitivity is easy to prove: if $v \xrightarrow{\varepsilon} v' \xrightarrow{\varepsilon} v''$ in G^ε then adding the edge $v \xrightarrow{\varepsilon} v''$ cannot create a path whose label does not belong to W .

The difficulty lies in establishing totality. Let $v_0 \neq v_1$ be such that neither $v_0 \xrightarrow{\varepsilon} v_1$ nor $v_1 \xrightarrow{\varepsilon} v_0$ belong to $E(G^\varepsilon)$. Then consider the arena A defined by $V_{\text{Adam}} = V(G) = V(G^\varepsilon)$, $V_{\text{Eve}} = \{\bullet\}$, where $\bullet \notin V(G)$, and

$$E(A) = E(G^\varepsilon) \cup \{v \xrightarrow{c} \bullet \mid \exists p \in \{0, 1\}, v \xrightarrow{c} v_p \in E(G^\varepsilon)\} \cup \{\bullet \xrightarrow{\varepsilon} v_p \mid s \in \{0, p\}\}.$$

Observe that Eve wins the game (A, W) , simply by applying the strategy that heads to v_s after entering \bullet via an edge $v \xrightarrow{c} \bullet$ such that $v \xrightarrow{c} v_s \in E(G^\varepsilon)$ (we abstain from giving a more formal definition). Therefore, Eve wins with a positional strategy (S, π) ; without loss of generality we assume that it chooses v_0 , meaning that v_0 is an ε -successor of $\pi^{-1}(\bullet)$ in S .

Call G' the graph obtained from adding the edge $v_1 \xrightarrow{\varepsilon} v_0$ to G^ε , we claim that G' satisfies W , which contradicts the fact that G^ε is saturated with ε -edges. Indeed, for any path in G' , we observe that there is a path in S with the same labels: it is enough to replace each subpath of the form $v \xrightarrow{c} v_1 \xrightarrow{\varepsilon} v_0$ by $v \xrightarrow{c} \bullet \xrightarrow{\varepsilon} v_0$ (and possibly remove the first edge if it is labelled by ε). Since S satisfies W (it is a winning strategy), and ε is neutral, we conclude that G' satisfies W and thus the edge $v_1 \xrightarrow{\varepsilon} v_0$ should belong to $E(G^\varepsilon)$. Therefore, $>$ is a strict pre-order.

We now let \bar{G} be the graph defined over $V(\bar{G}) = V(G) = V(G^\varepsilon)$ by

$$E(\bar{G}) = \{v \xrightarrow{c} v' \mid \exists u, u', v \xrightarrow{\varepsilon^*} u \xrightarrow{c} u' \xrightarrow{\varepsilon^*} v' \text{ in } G^\varepsilon\}.$$

It is a direct check that \bar{G} satisfies W (by neutrality of ε) and that it is monotone with respect to the strict pre-order $>$ (by definition). Note that $G^\varepsilon \rightarrow \bar{G}$.

Now we would like to turn $>$ into a linear order by combining it's equivalence classes. For that, observe that vertices $v \neq v'$ such that $v > v'$ and $v' > v$, have identical incoming and outgoing edges in \bar{G} . Therefore, the graph G' defined over $V(G') = V(\bar{G}) / \sim$ where $v \sim v' \iff v > v' > v$ by

$$E(G') = \{[v] \xrightarrow{c} [v'] \mid v \xrightarrow{c} v' \text{ in } E(\bar{G})\},$$

makes sense, $v \mapsto [v]$ defines a morphism $\bar{G} \rightarrow G'$, and $>$ induces a total strict order over $V(G')$ which makes it monotone. This concludes the proof. \square