# BLOCK STRUCTURE VS SCOPE EXTRUSION: BETWEEN INNOCENCE AND OMNISCIENCE\*

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ABSTRACT. We study the semantic meaning of block structure using game semantics. To that end, we introduce the notion of block-innocent strategies and characterise call-by-value computation with block-allocated storage through soundness, finite definability and universality results. This puts us in a good position to conduct a comparative study of purely functional computation, computation with block storage as well as that with dynamic memory allocation. For example, we can show that dynamic variable allocation can be replaced with block-allocated variables exactly when the term involved (open or closed) is of base type and that block-allocated storage can be replaced with purely functional computation when types of order two are involved. To illustrate the restrictive nature of block structure further, we prove a decidability result for a finitary fragment of call-by-value Idealized Algol for which it is known that allowing for dynamic memory allocation leads to undecidability.

#### 1. Introduction

Most programming languages manage memory by employing a stack for local variables and heap storage for data that are supposed to live beyond their initial context. A prototypical example of the former mechanism is Reynolds's Idealized Algol [23], in which local variables can only be introduced inside blocks of ground type. Memory is then allocated on entry to the block and deallocated on exit. In contrast, languages such as ML permit variables to escape from their current context under the guise of pointers or references. In this case, after memory is allocated at the point of reference creation, the variable must be allowed to persist indefinitely (in practice, garbage collection or explicit deallocation can be used to put an end to its life).

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In this paper we would like to compare the expressivity of the two paradigms. As a simple example of heap-based memory allocation we consider the language RML, introduced by Abramsky and McCusker in [2], which is a fragment of ML featuring integer-valued references. In op. cit. the authors also construct a fully abstract game model of RML based on strategies (referred to as knowing strategies) that allow the Proponent to base his decisions on the full history of play. On the other hand, at around the same time Honda and Yoshida [8] showed that the purely functional core of RML, better known as call-by-value PCF [21], corresponds to innocent strategies [10], i.e. those that can only rely on a restricted view of the play when deciding on the next move. Since block-structured storage of Idealized Algol seems less expressive than dynamic memory allocation of ML and more expressive than PCF, it is natural to ask about its exact position in the spectrum of strategies between innocence and omniscience. Our first result is an answer to this question. We introduce the family of block-innocent strategies, situated strictly between innocent and knowing strategies, and exhibit a series of results relating such strategies to a call-by-value variant IA<sub>cbv</sub> of Idealized Algol.

Block-innocence captures the particular kind of uniformity exhibited by strategies originating from block-structured programs, akin to innocence yet strictly weaker. In fact, we shall define block-innocence through innocence in a setting enriched with explicit store annotations added to standard moves. For instance, in the play shown below<sup>1</sup>, if P follows a block-innocent strategy, P is free to use different moves as the fourth (1) and sixth (2) moves, but the tenth one (0) and the twelfth one (0) have to be the same.

The above play is present in the strategy representing the term

$$\begin{array}{ll} f: (\mathsf{unit} \to \mathsf{int}) \to (\mathsf{unit} \to \mathsf{int}) & \vdash & \left(\mathsf{new}\,x\,\mathsf{in}\,(\mathsf{let}\,\,g = f(\lambda y^{\mathsf{unit}}.(x \mathop{\mathop:}= \mathop{!}x + 1);\mathop{!}x)\,\,\mathsf{in}\,\,())\right); \\ & \quad \quad \lambda y^{\mathsf{unit}}.0: \mathsf{unit} \to \mathsf{int} \end{array}$$

and the necessity to play the same value (0 in this case) in the twelfth move once 0 has been played in the tenth one stems from the fact that variables in  $\mathsf{IA}_{\mathsf{cbv}}$  can only be allocated in blocks of ground type. For example, the block in which x was allocated cannot extend over  $\lambda y^{\mathsf{unit}}.0$ .

Additionally, our framework can detect "storage violations" resulting from an attempt to access a variable from outside of its block. For instance, no  $IA_{cbv}$ -term will ever produce the following play.

The last move is the offending one: for the term given above, it would amount to trying to use g after deallocation of the block for x. Note, though, that the very similar play drawn below does originate from an  $\mathsf{IA}_{\mathsf{cbv}}$ -term.

 $\mathrm{Take}, \ \mathrm{e.g.} \ f: (\mathsf{unit} \to \mathsf{int}) \to (\mathsf{unit} \to \mathsf{int}) \vdash \mathsf{let} \ g = f(\lambda y^{\mathsf{unit}}.1) \ \mathsf{in} \ \lambda y^{\mathsf{unit}}.g() : \mathsf{unit} \to \mathsf{int}.$ 

The notion of block-innocence provides us with a systematic methodology to address expressivity questions related to block structure such as "Does a given strategy originate"

<sup>&</sup>lt;sup>1</sup>For the sake of clarity, we only include pointers pointing more than one move ahead.

from a stack-based memory discipline?" or "Can a given program using dynamic memory allocation be replaced with an equivalent program featuring stack-based storage?". To illustrate the approach we conduct a complete study of the relationship between the three classes of strategies (innocent, block-innocent and knowing respectively) according to the underpinning type shape. We find that knowingness implies block-innocence when terms of base types (open or closed) are involved, that block-innocence implies innocence exactly for types of order at most two, and that knowingness implies innocence if the term is of base type and its free identifiers are of order 1. The fact that knowingness and innocence coincide at terms of base types implies, in particular, that RML and IA<sub>cbv</sub> contexts have the same expressive power: two RML-terms can be distinguished by an RML-context if, and only if, they can be distinguished by an IA<sub>cbv</sub>-context.

As a further confirmation of the restrictive nature of the stack discipline of  $IA_{cbv}$ , we prove that program equivalence is decidable for a finitary variant of  $IA_{cbv}$  which properly contains all second-order types as well as some third-order types (interestingly, this type discipline covers the available higher-order types in PASCAL). In contrast, the corresponding restriction of RML is known to be undecidable [15].

**Related work.** The stack discipline has always been regarded as part of the essence of Algol [23]. The first languages introduced in that lineage, i.e. Algol 58 and 60, featured both call-by-name and call-by-value parameters. Call-by-name was abandoned in Algol 68, though, and was absent from subsequent designs, such as Pascal and C.

On the semantic front, finding models embodying stack-oriented storage management has always been an important goal of research into Algol-like languages. In this spirit, in the early 1980s, Reynolds [23] and Oles [18] devised a semantic model of (call-by-name) Algol-like languages using a category of functors from a category of store shapes to the category of predomains. Perhaps surprisingly, in the 1990s, Pitts and Stark [20, 24] managed to adapt the techniques to (call-by-value) languages with dynamic allocation. This would appear to create a common platform suitable for a comparative study such as ours. However, despite the valuable structural insights, the relative imprecision of the functor category semantics (failure of definability and full abstraction) makes it unlikely that the results obtained by us can be proved via this route. The semantics of local effects has also been investigated from the category-theoretic point of view in [22].

As for the game semantics literature, Ong's work [19] based on strategies-with-state is the work closest to ours. His paper defines a compositional framework that is proved sound for the third-order fragment of call-by-name Idealized Algol. Adapting the results to call-by-value and all types is far from immediate, though. For a start, to handle higher-order types, we note that the state of O-moves is no longer determined by their justifier and the preceding move. Instead, the right state has to be computed globally using the whole history of play. However, the obvious adaptation of this idea to call-by-value does not capture the block structure of IA<sub>cbv</sub>. Quite the opposite: it seems to be more compatible with RML. Consequently, further changes are needed to characterize IA<sub>cbv</sub>. Firstly, to restore definability, the explicit stores have to become lists instead of sets. Secondly, conditions controlling state changes must be tightened. In particular, P must be forbidden from introducing fresh variables at any step and, in a similar vein, must be forced to drop some variables from his moves in certain circumstances.

Another related paper is [4], in which Abramsky and McCusker introduce a model of Idealized Algol with passive (side-effect-free) expressions [4]. Their framework is based on a

Figure 1: Syntax of  $\mathcal{L}$ 

distinction between active and passive moves, which correspond to active and passive types respectively. Legal plays must then satisfy a novel correctness condition, called activity, and strategies must be a/p-innocent. In contrast, our setting does not feature any type support for discovering the presence of storage. Moreover, as the sequences discussed in the Introduction demonstrate, in order to understand legality in our setting, it is sometimes necessary to scrutinise values used in plays: changing 1,1 to 1,2 may entail loss of correctness! This is different from the activity condition (and other conditions used in game semantics), where it suffices to consider the kind of moves involved (question/answer) or pointer patterns. Consequently, in order to capture the desired shape of plays and associated notion of innocence in our setting, we felt it was necessary to introduce moves explicitly decorated with stores.

Our paper is also related to the efforts of finding decidable fragments of (finitary) RML as far as contextual equivalence is concerned. Despite several papers in the area [7, 15, 9, 5], no full classification based on type shapes has emerged yet, even though the corresponding call-by-name case has been fully mapped out [16]. We show that, for certain types, moving from RML to  $IA_{cbv}$  (thus weakening storage capabilities) can help to regain decidability.

#### 2. Syntax

To set a common ground for our investigations, we introduce a higher-order programming language that features syntactic constructs for both block and dynamic memory allocation.

**Definition 2.1** (The language  $\mathcal{L}$ ). We define types as generated by the grammar below, where  $\beta$  ranges over the ground types unit and int.

$$\theta \ ::= \ \beta \ | \ \operatorname{var} \ | \ \theta \to \theta$$

The syntax of  $\mathcal{L}$  is given in Figure 1.

Note in particular the first two rules concerning variables and the rule for the mkvar constructor: the latter allows us to build "bad variables" in accordance with Idealized Algol. The order of a type is defined as follows:

$$\begin{array}{rcl} \operatorname{ord}(\beta) & = & 0 \\ \operatorname{ord}(\operatorname{var}) & = & 1 \\ \operatorname{ord}(\theta_1 \to \theta_2) & = & \max(\operatorname{ord}(\theta_1) + 1, \operatorname{ord}(\theta_2)). \end{array}$$

$$\frac{V \text{ is a value}}{s, V \Downarrow s, V} \qquad \frac{M \Downarrow 0 \quad N_0 \Downarrow V}{\text{if $M$ then $N_1$ else $N_0$} \Downarrow V} \qquad \frac{i \neq 0 \quad M \Downarrow i \quad N_1 \Downarrow V}{\text{if $M$ then $N_1$ else $N_0$} \Downarrow V}$$
 
$$\frac{M_1 \Downarrow i_1 \quad M_2 \Downarrow i_2}{M_1 \oplus M_2 \Downarrow i_1 \oplus i_2} \qquad \frac{M \Downarrow \lambda x. M' \quad N \Downarrow V' \quad M'[V'/x] \Downarrow V}{MN \Downarrow V}$$
 
$$\frac{s, M \Downarrow s', \alpha \quad s'(\alpha) = i}{s, !M \Downarrow s', i} \qquad \frac{s, M \Downarrow s', \alpha \quad s', N \Downarrow s'', i}{s, M := N \Downarrow s''(\alpha \mapsto i), ()}$$
 
$$\frac{M \Downarrow \mathsf{mkvar}(V_1, V_2) \quad V_1() \Downarrow i}{!M \Downarrow i} \qquad \frac{M \Downarrow \mathsf{mkvar}(V_1, V_2) \quad N \Downarrow i \quad V_2 i \Downarrow ()}{M := N \Downarrow ()}$$
 
$$\frac{M \Downarrow V_1 \quad N \Downarrow V_2}{\mathsf{mkvar}(M, N) \Downarrow \mathsf{mkvar}(V_1, V_2)} \qquad \frac{M \Downarrow V}{\mathsf{Y}(M) \Downarrow \lambda x^{\theta}.(V(\mathsf{Y}(V)))x}$$
 
$$\frac{s \cup (\alpha \mapsto 0), M[\alpha/x] \Downarrow s', V}{s, \mathsf{new} \, x \, \mathsf{in} \, M \Downarrow s' \setminus \alpha, V} \quad \alpha \not \in \mathsf{dom} \, s$$
 
$$\frac{s \cup (\alpha \mapsto 0), M[\alpha/x] \Downarrow s', V}{s, \mathsf{new} \, x \, \mathsf{in} \, M \Downarrow s' \setminus \alpha, V} \quad \alpha \not \in \mathsf{dom} \, s$$
 
$$\frac{s, \mathsf{ref} \, \Downarrow s \cup (\alpha \mapsto 0), \alpha}{s, \mathsf{ref} \, \Downarrow s \cup (\alpha \mapsto 0), \alpha} \quad \alpha \not \in \mathsf{dom} \, s$$

Figure 2: Operational semantics of  $\mathcal{L}$ 

For any  $i \geq 0$ , terms that are typable using exclusively judgments of the form

$$x_1:\theta_1,\cdots,x_n:\theta_n\vdash M:\theta$$

where  $\operatorname{ord}(\theta_i) < i \ (1 \le j \le n)$  and  $\operatorname{ord}(\theta) \le i$ , are said to form the *i*th-order fragment.

To spell out the operational semantics of  $\mathcal{L}$ , we need to assume a countable set Loc of *locations*, which are added to the syntax as auxiliary constants of type var. We shall write  $\alpha$  to range over them. The semantics then takes the form of judgments  $s, M \Downarrow s', V$ , where s, s' are finite partial functions from Loc to integers, M is a closed term and V is a value. Terms of the following shapes are values: (), integer constants, elements of Loc,  $\lambda$ -abstractions or terms of the form  $\mathsf{mkvar}(\lambda x^{\mathsf{unit}}.M, \lambda y^{\mathsf{int}}.N)$ .

The operational semantics is given via the large-step rules in Figure 2. Most of them take the form

$$\frac{M_1 \Downarrow V_1 \quad M_2 \Downarrow V_2 \quad \cdots \quad M_n \Downarrow V_n}{M \Downarrow V}$$

which is meant to abbreviate:

$$\frac{s_1, M_1 \Downarrow s_2, V_1 \quad s_2, M_2 \Downarrow s_3, V_2 \quad \cdots \quad s_n, M_n \Downarrow s_{n+1}, V_n}{s_1, M_1 \Downarrow s_{n+1}, V}$$

This is a common semantic convention, introduced in the Definition of Standard ML [11]. In particular, it means that the ordering of the hypotheses is significant. The penultimate rule in the figure encapsulates the state within the newly created block, while the last one creates a reference to a new memory cell that can be passed around without restrictions on its scope. Note that  $s' \setminus \alpha$  is the restriction of s' to  $\mathsf{dom} \, s' \setminus \{\alpha\}$ .

Given a closed term  $\vdash M$ : unit, we write  $M \Downarrow$  if there exists s such that  $\emptyset, M \Downarrow s, ()$ . We shall call two programs equivalent if they behave identically in every context. This is captured by the following definition, parameterised by the kind of contexts that are considered, to allow for testing of terms with contexts originating from a designated subset of the language.

**Definition 2.2.** Suppose  $\mathcal{L}'$  is a subset of  $\mathcal{L}$ . We say that the terms-in-context  $\Gamma \vdash M_1, M_2 : \theta$  are  $\mathcal{L}'$ -equivalent (written  $\Gamma \vdash M_1 \cong_{\mathcal{L}'} M_2$ ) if, for any  $\mathcal{L}'$ -context C such that  $\vdash C[M_1], C[M_2] :$  unit,  $C[M_1] \Downarrow$  if and only if  $C[M_2] \Downarrow$ .

We shall study three sublanguages of  $\mathcal{L}$ , called  $PCF^+$ ,  $IA_{cbv}$  and RML respectively. The latter two have appeared in the literature as paradigmatic examples of programming languages with stack discipline and dynamic memory allocation respectively.

- $\mathsf{PCF}^+$  is a purely functional language obtained from  $\mathcal L$  by removing  $\mathsf{new}\,x$  in M and  $\mathsf{ref}$ . It extends the language  $\mathsf{PCF}$  [21] with primitives for variable access, but not for memory allocation.
- $\mathsf{IA}_{\mathsf{cbv}}$  is  $\mathcal{L}$  without the ref constant. It can be viewed as a call-by-value variant of Idealized Algol [23]. Only block-allocated storage is available in  $\mathsf{IA}_{\mathsf{cbv}}$ .
- RML is  $\mathcal{L}$  save the construct new x in M. It is exactly the language introduced in [2] as a prototypical language for ML-like integer references.<sup>2</sup>

We shall often use let x = M in N as shorthand for  $(\lambda x.N)M$ . Moreover, let x = M in N, where x does not occur in N, will be abbreviated to M; N. Note also that new x in M is equivalent to let x = ref in M.

**Example 2.3.** The term  $\vdash$  let  $v = \text{ref in } \lambda x^{\text{unit}}$ .(if !v then  $\Omega$  else v := !v + 1): unit  $\rightarrow$  unit is an example of an RML-term that is not RML-equivalent to any term from  $\mathsf{IA}_{\mathsf{cbv}}$ . On the other hand,

 $\vdash \lambda f^{(\text{unit} \to \text{unit}) \to \text{unit}}$ .new v in  $f(\lambda y^{\text{unit}}$ .if !v then  $\Omega$  else  $v := !v + 1) : ((\text{unit} \to \text{unit}) \to \text{unit}) \to \text{unit}$  is an  $\mathsf{IA}_{\mathsf{cbv}}$ -term that has no RML-equivalent in PCF<sup>+</sup>. All of the inequivalence claims will

**Lemma 2.4.** Given any base type  $\mathcal{L}$ -term  $\Gamma, x : \mathsf{var} \vdash M : \beta$ , we have  $\Gamma \vdash \mathsf{new} \, x \, \mathsf{in} \, M \cong_{\mathcal{L}} \mathsf{let} \, x = \mathsf{ref} \, \mathsf{in} \, M : \beta$ .

*Proof.* The proof is based on the following two claims:

- if  $s, M \downarrow s', V$  and  $\alpha \in \mathsf{dom}(s)$  does not appear in M, then  $s \setminus \alpha, M \downarrow s' \setminus \alpha, V$ ;
- for any closed context C, value V and s, s', if  $s, C[\text{let } x = \text{ref in } M] \Downarrow s', V$  then there is a set of locations  $S \subseteq \text{dom } (s')$  such that  $s, C[\text{new } x \text{ in } M] \Downarrow s' \setminus S, V$  and S contains no locations from s or V;

which are proven by straightforward induction.

follow immediately from our results.

Hence, RML and  $\mathcal{L}$  merely differ on a syntactic level in that  $\mathcal{L}$  contains "syntactic sugar" for blocks. In the opposite direction, our results will show that ref cannot in general be replaced with an equivalent term that uses  $\operatorname{new} x \operatorname{in} M$ . Indeed, our paper provides a general methodology for identifying and studying scenarios in which this expressivity gap is real.

#### 3. Game semantics

We next introduce the game models used throughout the paper, which are based on the Honda-Yoshida approach to modelling call-by-value computation [8].

**Definition 3.1.** An arena  $A = (M_A, I_A, \vdash_A, \lambda_A)$  is given by

• a set  $M_A$  of moves, and a subset  $I_A \subseteq M_A$  of initial moves,

<sup>&</sup>lt;sup>2</sup>In other words, RML is Reduced ML [24] with the addition of the mkvar construct.

- a justification relation  $\vdash_A \subseteq M_A \times (M_A \setminus I_A)$ , and
- a labelling function  $\lambda_A: M_A \to \{O, P\} \times \{Q, A\}$

such that  $\lambda_A(I_A) \subseteq \{PA\}$  and, whenever  $m \vdash_A m'$ , we have  $(\pi_1 \lambda_A)(m) \neq (\pi_2 \lambda_A)(m')$  and  $(\pi_2 \lambda_A)(m') = A \implies (\pi_2 \lambda_A)(m) = Q$ .

The role of  $\lambda_A$  is to label moves as *Opponent* or *Proponent* moves and as *Questions* or *Answers*. We typically write them as  $m, n, \ldots$ , or  $o, p, q, a, q_P, q_O, \ldots$  when we want to be specific about their kind. Note that we abbreviate elements of the codomain of  $\lambda_A$ , e.g. (P, A) above is written as PA.

The simplest arena is  $0 = (\emptyset, \emptyset, \emptyset, \emptyset)$ . Other "flat" arenas are 1 and  $\mathbb{Z}$ , defined by:

$$M_1 = I_1 = \{*\}, \quad M_{\mathbb{Z}} = I_{\mathbb{Z}} = \mathbb{Z}.$$

Below we recall two standard constructions on arenas, where  $\bar{I}_A$  stands for  $M_A \setminus I_A$ , the OP-complement of  $\lambda_A$  is written as  $\bar{\lambda}_A$ , and  $i_A, i_B$  range over initial moves in the respective arenas.

$$\begin{split} M_{A\Rightarrow B} &= I_{A\Rightarrow B} \uplus I_A \uplus \overline{I}_A \uplus M_B \\ I_{A\Rightarrow B} &= \{*\} \\ \lambda_{A\Rightarrow B} &= [(*, PA), (i_A, OQ), \overline{\lambda}_A \upharpoonright \overline{I}_A, \lambda_B] \\ \vdash_{A\Rightarrow B} &= \{(*, i_A), (i_A, i_B)\} \cup \vdash_A \cup \vdash_B \\ \\ M_{A\otimes B} &= I_{A\otimes B} \uplus \overline{I}_A \uplus \overline{I}_B \\ I_{A\otimes B} &= I_A \times I_B \\ \lambda_{A\otimes B} &= [((i_A, i_B), PA), \lambda_A \upharpoonright \overline{I}_A, \lambda_B \upharpoonright \overline{I}_B] \\ \vdash_{A\otimes B} &= \{((i_A, i_B), m) \mid i_A \vdash_A m \vee i_B \vdash_B m\} \\ \cup (\vdash_A \upharpoonright \overline{I}_A^2) \cup (\vdash_B \upharpoonright \overline{I}_B^2) \end{split}$$

Types of  $\mathcal{L}$  can now be interpreted with arenas in the following way.

$$\begin{split} &\llbracket \mathsf{unit} \rrbracket = 1 \\ &\llbracket \mathsf{int} \rrbracket = \mathbb{Z} \\ &\llbracket \mathsf{var} \rrbracket = (1 \Rightarrow \mathbb{Z}) \otimes (\mathbb{Z} \Rightarrow 1) \\ &\llbracket \theta_1 \to \theta_2 \rrbracket = \llbracket \theta_1 \rrbracket \Rightarrow \llbracket \theta_2 \rrbracket \end{split}$$

Note that the type var is translated as a product arena the components of which represent its read and write methods.

Although arenas model types, the actual games will be played in **prearenas**, which are defined in the same way as arenas with the exception that initial moves must be O-questions. Given arenas A and B, we can construct the prearena  $A \to B$  by setting:

$$\begin{split} M_{A \to B} &= M_A \uplus M_B \\ I_{A \to B} &= I_A \\ \lambda_{A \to B} &= [(i_A, OQ) \, \cup \, (\bar{\lambda}_A \upharpoonright \overline{I}_A) \;, \; \lambda_B] \\ \vdash_{A \to B} &= \{(i_A, i_B)\} \cup \vdash_A \, \cup \vdash_B \;. \end{split}$$

For  $\Gamma = \{x_1 : \theta_1, \dots, x_n : \theta_n\}$ , typing judgments  $\Gamma \vdash \theta$  will eventually be interpreted by strategies for the prearena  $\llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_n \rrbracket \to \llbracket \theta \rrbracket$  (if n = 0 we take the left-hand side to be 1), which we shall denote by  $\llbracket \Gamma \vdash \theta \rrbracket$  or  $\llbracket \theta_1, \dots, \theta_n \vdash \theta \rrbracket$ .

A justified sequence in a prearena A is a finite sequence s of moves of A satisfying the following condition: the first move must be initial, but all other moves m must be equipped with a pointer to an earlier occurrence of a move m' such that  $m' \vdash_A m$ . We then say that m' justifies m. If m is an answer, we may also say that m answers m'. If a question remains unanswered in s, it is open; and the rightmost open question in s is its pending question.

Given a justified sequence s, we define its O-view  $\lfloor s \rfloor$  and its P-view  $\lceil s \rceil$  inductively as follows.

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\bullet \ \bot \epsilon \bot = \epsilon, \ \bot s o \bot = \bot s \bot o, \ \bot s o \overbrace{\cdots} p \bot = \bot s \bot o p;
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 $\bullet$   $\lceil \epsilon \rceil = \epsilon$ ,  $\lceil s p \rceil = \lceil s \rceil p$ ,  $\lceil s p \sim o \rceil = \lceil s \rceil p o$ .

Above, recall that o ranges over O-moves (i.e. moves m such that  $(\pi_1 \lambda_A)(m) = O$ ), and p ranges over P-moves.

**Definition 3.2.** A play in a prearena A is a justified sequence s satisfying the following conditions.

- If  $s = \cdots m n \cdots$  then  $\lambda_A^{OP}(m) = \bar{\lambda}_A^{OP}(n)$ . (Alternation) If  $s = s_1 q s_2 a \cdots$  then q is the pending question in  $s_1 q s_2$ . (Well-Bracketing)
- If  $s = s_1 o s_2 p \cdots$  then o appears in  $\lceil s_1 o s_2 \rceil$ ; if  $s = s_1 p s_2 o \cdots$  then p appears in  $\lfloor s_1 p s_2 \rfloor$ . (Visibility)

We write  $P_A$  to denote the set of plays in A.

We are going to model terms-in-context  $\Gamma \vdash M : \theta$  as sets of plays in  $\llbracket \Gamma \vdash \theta \rrbracket$  subject to specific conditions.

**Definition 3.3.** A (knowing) strategy  $\sigma$  on a prearena A is a non-empty prefix-closed set of plays from A satisfying the first two conditions below. A strategy is *innocent* if, in addition, the third condition holds.

- If even-length  $s \in \sigma$  and  $sm \in P_A$  then  $sm \in \sigma$ . (O-Closure)
- If even-length  $sm_1, sm_2 \in \sigma$  then  $m_1 = m_2$ . (Determinacy)
- If  $s_1m, s_2 \in \sigma$  with odd-length  $s_1, s_2$  and  $\lceil s_1 \rceil = \lceil s_2 \rceil$  then  $s_2m \in \sigma$ . (Innocence)

We write  $\sigma: A$  to denote that  $\sigma$  is a strategy on A.

Note, in particular, that every strategy  $\sigma: A$  contains the empty sequence  $\epsilon$  as well as the elements of  $I_A$ , the latter being the 1-move plays in A. Moreover, in the last condition above, the move m in  $s_2m$  points at the same move it points inside  $s_1m$ : by visibility and the fact that  $s_1$  and  $s_2$  have the same view, this is always possible.

In previous work it has been shown that knowing strategies yield a fully abstract semantics for RML in the following sense<sup>3</sup>.

**Theorem 3.4** ([2]). Two RML-terms are RML-equivalent if and only if their interpretations contain the same complete plays<sup>4</sup>.

Moreover, as an immediate consequence of the full abstraction result of [8], we have that innocent strategies (quotiented by the intrinsic preorder) yield full abstraction for PCF<sup>+</sup>. What remains open is the model for the intermediate language, IA<sub>cbv</sub>, which requires one to identify a family of strategies between the innocent and knowing ones. This is the problem examined in the next section. We address it in two steps.

<sup>&</sup>lt;sup>3</sup>Perhaps it is worth noting that the presentation of the model in [2] is in a different setting (we follow Honda-Yoshida call-by-value games, while [2] applies the family construction on call-by-name games) which, nonetheless, is equivalent to the one presented above.

<sup>&</sup>lt;sup>4</sup>A play is called *complete* if each question in that play has been answered.

- First we introduce a category of strategies that are equipped with explicit stores for registering private variables. We show that this category, of so-called innocent S-strategies, indeed models block allocation: terms of IA<sub>cbv</sub> translate into innocent S-strategies (Proposition 4.25) and, moreover, in a complete manner (Propositions 5.3 and 5.7).
- The strategies capturing IA<sub>cbv</sub>, called *block-innocent* strategies, are then defined by deleting stores from innocent S-strategies.

# 4. Games with stores and the model of IAchy

We shall now extend the framework to allow moves to be decorated with stores that contain name-integer pairs. This extension will be necessary for capturing block-allocated storage. The names should be viewed as semantic analogues of locations. The stores will be used for carrying the values of private, block-allocated variables.

4.1. Names and stores in games. When employing such moves-with-store, we are not interested in what exactly the names are, but we would like to know how they relate to names that have already been in play. Hence, the objects of study are rather the induced equivalence classes with respect to name-invariance, and all ensuing constructions and reasoning need to be compatible with it. This overhead can be dealt with robustly using the language of nominal set theory [6]. Let us fix a countably infinite set  $\mathbb{A}$ , the set of names, the elements of which we shall denote by  $\alpha, \beta$  and variants. Consider the group PERM( $\mathbb{A}$ ) of finite permutations of  $\mathbb{A}$ .

**Definition 4.1.** A *strong nominal set* [6, 25] is a set equipped with a group action<sup>5</sup> of PERM( $\mathbb{A}$ ) such that each of its elements has *finite strong support*. That is to say, for any  $x \in X$ , there exists a finite set  $\nu(x) \subseteq \mathbb{A}$ , called *the support of* x, such that, for all permutations  $\pi$ ,  $(\forall \alpha \in \nu(x). \pi(\alpha) = \alpha) \iff \pi \cdot x = x$ .

Intuitively,  $\nu(x)$  is the set of names "involved" in x. For example, the set  $\mathbb{A}^{\#}$  of finite lists of distinct atoms with permutations acting elementwise is a strong nominal set. If X and Y are strong nominal sets, then so is their cartesian product  $X \times Y$  (with permutations acting componentwise) and their disjoint union  $X \uplus Y$ . Name-invariance in a strong nominal set X is represented by the relation:  $x \sim x'$  if there exists  $\pi$  such that  $x = \pi \cdot x'$ .

We define a strong nominal set of **stores**, the elements of which are finite sequences of name-integer pairs. Formally,

$$\Sigma, T ::= \epsilon \mid (\alpha, i) :: \Sigma$$

where  $i \in \mathbb{Z}$  and  $\alpha \in \mathbb{A} \setminus \nu(\Sigma)$ . We view stores as finite functions from names to integers, though their domains are lists rather than sets. Thus, we define the **domain** of a store to be the *list* of names obtained by applying the first projection to all of its elements. In particular,  $\nu(\text{dom}(\Sigma)) = \nu(\Sigma)$ . If  $\alpha \in \nu(\Sigma)$  then we write  $\Sigma(\alpha)$  for the unique i such that  $(\alpha, i)$  is an element of  $\Sigma$ . For stores  $\Sigma, T$  we write:

$$\Sigma \leq T \quad \text{for dom } (\Sigma) \sqsubseteq \text{dom } (T),$$
  
  $\Sigma \leq_X T \quad \text{for dom } (\Sigma) \sqsubseteq_X \text{dom } (T),$ 

<sup>&</sup>lt;sup>5</sup>A group action of PERM(A) on X is a function  $\cdot \cdot \cdot \cdot : \text{PERM}(\mathbb{A}) \times X \to X$  such that, for all  $x \in X$  and  $\pi, \pi' \in \text{PERM}(\mathbb{A}), \pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$  and  $\text{id} \cdot x = x$ , where id is the identity permutation.

where  $X \in \{p, s\}$ , and  $\sqsubseteq, \sqsubseteq_p, \sqsubseteq_s$  denote the subsequence, prefix and suffix relations respectively. Note that  $\Sigma \leq_X T \leq_X \Sigma$  implies  $\mathsf{dom}(\Sigma) = \mathsf{dom}(T)$  but not  $\Sigma = T$ . Finally, let us write  $\Sigma \setminus T$  for  $\Sigma$  restricted to  $\nu(\Sigma) \setminus \nu(T)$ .

An **S-move** (or move-with-store) in a prearena A is a pair consisting of a move and a store. We typically write S-moves as  $m^{\Sigma}, n^{T}, o^{\Sigma}, p^{T}, q^{\Sigma}, a^{T}$ . The first projection function is viewed as store erasure and denoted by erase(\_). Note that moves contain no names and therefore, for any  $m^{\Sigma}$ ,  $\nu(m^{\Sigma}) = \nu(\Sigma) = \nu(\mathsf{dom}(\Sigma))$ . A justified S-sequence in A is a sequence of S-moves equipped with justifiers, so that its erasure is a justified sequence. The notions of O-view and P-view are extended to S-sequences in the obvious manner. We say that a name  $\alpha$  is closed in s if there are no open questions in s containing  $\alpha$ .

**Definition 4.2.** A justified S-sequence s in a prearena A is called an S-play if it satisfies the following conditions, for all  $\alpha \in \mathbb{A}$ .

```
• If s = m^{\Sigma} \cdots then \Sigma = \epsilon. (Init)
```

- If  $s = \cdots o^{\Sigma} \cdots p^{T} \cdots$  then  $\Sigma \leq_{p} T$ . If  $\lambda_{A}(p) = PA$  then  $T \leq_{p} \Sigma$  too. (Just-P) If  $s = \cdots p^{\Sigma} \cdots o^{T} \cdots$  then  $\Sigma \leq_{p} T \leq_{p} \Sigma$ . (Just-O)
- If  $s = s_1 o^{\Sigma} q_P^T \cdots$  then  $\Sigma \setminus T \leq_s \Sigma$  and  $\Sigma \setminus (\Sigma \setminus T) \leq_p T$  and (a) if  $\alpha \in \nu(T \setminus \Sigma)$  then  $\alpha \notin \nu(s_1 o^{\Sigma})$ ,
  - (b) if  $\alpha \in \nu(\Sigma \setminus T)$  then  $\alpha$  is closed in  $s_1 o^{\Sigma}$ . (Prev-PQ)
- If  $s = \cdots p^{\Sigma} s' o^T \cdots$  and  $\alpha \in (\nu(T) \cap \nu(\Sigma)) \setminus \nu(s')$  then  $T(\alpha) = \Sigma(\alpha)$ . (Val-O)

We write  $SP_A$  for the set of S-plays in A.

Let us remark that, as stores have strong support, the set of S-plays  $SP_A$  is a strong nominal set. The conditions we impose on S-plays reflect the restrictions pertaining to block-allocation of variables. In particular, given a move m, all block-allocated variables present at m are carried over to every move n justified by m. In addition, only P is allowed to allocate/deallocate such variables, or change their values.

**Just-P.** All variables allocated at o survive in p. If p is an answer then, in fact, the subsequence from o to p represents a whole sub-block, with p closing the sub-block. Thus, o and p must have the same private variables.

Just-O. Each O-move inherits its private variables from its justifier move. Put otherwise, a block does not extend beyond the current P-view and we only store variables that are created by and are private to P (so T cannot be larger that  $\Sigma$ ).

**Prev-PQ.** P-questions can open or close private variables and thus alter the domain of the store. This process must obey the nesting of variables, as reflected in the order of names in stores. Therefore, variables are closed by removing their corresponding names from the right end of the store:  $\Sigma \setminus T \leq_s \Sigma$ . On the other hand, variables/names that survive comprise the left end of the new store:  $\Sigma \setminus (\Sigma \setminus T) \leq_p T$ .

In addition, any names that are added in the store must be fresh for the whole sequence (they represent fresh private variables). A final condition disallows variables to be closed if their block still contains open questions.

Val-O. Since variables are private to P, it is not possible for O to change their value: at each O-move  $o^T$ , the value of each  $\alpha \in \mathsf{dom}(T)$  is the same as that of the last P-move  $p^{\Sigma}$ such that  $\alpha \in \mathsf{dom}(\Sigma)$ .

The above is a minimal collection of rules that we need to impose for block allocation. From them we can extract further properties for S-plays, whose proofs are delegated to Appendix A.

**Lemma 4.3.** The following properties hold for S-plays s.

- If  $s = \cdots m^{\Sigma} a_P^T \cdots$  then  $\Sigma \setminus T \leq_s \Sigma$  and  $\Sigma \setminus (\Sigma \setminus T) \leq_p T$  and
- (a) if  $\alpha \in \nu(T)$  then  $\alpha \in \nu(\Sigma)$ ,
- (b) if  $\alpha \in \nu(\Sigma \setminus T)$  then  $\alpha$  is closed in  $s_{\leq a_n^T}$ . (Prev-PA)
- For any  $\alpha$ , we have  $\lceil s \rceil = s_1 \underline{s}_2 s_3$ , where
  - $-\alpha \notin \nu(s_1) \cup \nu(s_3)$  and  $\forall m^{\Sigma} \in s_2$ .  $\alpha \in \nu(\Sigma)$ ,
  - if  $s_2 \neq \epsilon$  then its first element is the move introducing  $\alpha$  in s. (Block form)
- If  $s = s_1 o^{\Sigma} p^T s_2$  with  $\alpha \in \nu(\Sigma) \setminus \nu(T)$  then  $\alpha \notin \nu(s_2)$ . (Close)

We now move on to strategies for block allocation.

**Definition 4.4.** An S-strategy  $\sigma$  on an arena A is a non-empty prefix-closed set of S-plays from A satisfying the first three of the following conditions. An S-strategy is *innocent* if it also satisfies the last condition.

- If  $s' \sim s \in \sigma$  then  $s' \in \sigma$ . (Nominal Closure)

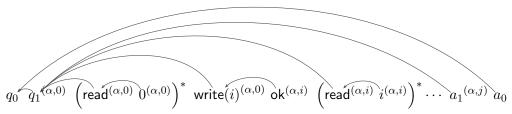
- If even-length  $s \in \sigma$  and  $sm^{\Sigma} \in SP_A$  then  $sm^{\Sigma} \in \sigma$ . (O-Closure)
   If even-length  $sm_1^{\Sigma_1}, sm_2^{\Sigma_2} \in \sigma$  then  $sm_1^{\Sigma_1} \sim sm_2^{\Sigma_2}$ . (Determinacy)
   If  $s_1m^{\Sigma_1}, s_2 \in \sigma$  with  $s_1, s_2$  odd-length and  $\lceil s_1 \rceil = \lceil s_2 \rceil$  then there exists  $s_2m^{\Sigma_2} \in \sigma$  with  $\lceil s_1m^{\Sigma_1} \rceil \sim \lceil s_2m^{\Sigma_2} \rceil$ . (Innocence)

We write  $\sigma: A$  to denote that  $\sigma$  is an S-strategy on A.

Observe how S-strategies are defined in the same manner as ordinary strategies but follow some additional conditions due to their involving of stores and names.

**Example 4.5.** For any base type  $\beta$ , consider the prearena  $\llbracket \mathsf{var} \to \beta \rrbracket \to \llbracket \beta \rrbracket$  given below, where we have indexed moves and type constructors to indicate provenance. We use read and write(i)  $(i \in \mathbb{Z})$  to refer to the question-moves from [var], and  $i (i \in \mathbb{Z})$  and ok for the corresponding answers.

Let us define the S-strategy  $\operatorname{cell}_{\beta} : \llbracket \operatorname{var} \to \beta \rrbracket \to \llbracket \beta \rrbracket$  as the S-strategy containing all evenlength prefixes of S-plays of the following form (where we use the Kleene star for move repetition).



Note that, although the cell strategy is typically non-innocent, it is innocent in our framework, where private variables are explicit in game moves (in their stores).

**Example 4.6.** Let us consider the prearena  $[\![unit \rightarrow int]\!] \rightarrow unit$ , depicted as on the left below. Had we used sets instead of lists for representing stores, the following "S-strategy" (right below), which represents incorrect overlap of scopes ( $\alpha$  and  $\beta$  are in scope of one another, but at the same time have different scopes), would have been valid (and innocent).

$$\begin{array}{c} q_0 \\ \downarrow \\ q_1 \end{array} \\ i \\ q_0 \overset{(\alpha,0),(\beta,0)}{q_1} \underbrace{0_1(\alpha,0),(\beta,0)}_{q_1}\underbrace{0_1(\alpha,0),(\beta,0$$

However, the above is not a valid S-strategy in our language setting. Intuitively, it would correspond to a term that determines the scope of its variables on the fly, depending on the value (0 or 1) received after memory allocation.

4.2. Composing S-plays. We next define composition of S-plays, following the approach of [8, 25]. Let us introduce some notation on stores. For a sequence of S-moves s and stores  $\Sigma, T$ , we write  $\Sigma[s], \Sigma[T]$  and  $\Sigma + T$  for the stores defined by:  $\epsilon[s] = \epsilon[T] \triangleq \epsilon, \ \epsilon + T \triangleq T$  and

$$((\alpha, i) :: \Sigma)[s] \triangleq \begin{cases} (\alpha, T(\alpha)) :: (\Sigma[s]) & \text{if } s = s_1 m^T s_2 \land \alpha \in \nu(T) \setminus \nu(s_2) \\ (\alpha, i) :: (\Sigma[s]) & \text{otherwise} \end{cases}$$

$$((\alpha, i) :: \Sigma)[T] \triangleq \begin{cases} (\alpha, T(\alpha)) :: (\Sigma[T]) & \text{if } \alpha \in \nu(T) \\ (\alpha, i) :: (\Sigma[T]) & \text{otherwise} \end{cases}$$

$$((\alpha, i) :: \Sigma) + T \triangleq \begin{cases} (\alpha, i) :: (\Sigma + T) & \text{if } \alpha \notin \nu(T) \\ \text{undefined} & \text{otherwise} \end{cases}$$

Moreover, we write  $\operatorname{st}(s)$  for the store of the final S-move in s. For instance, by  $\operatorname{Prev-PQ}$  and  $\operatorname{Prev-PA}$ , if  $o^{\Sigma}p^{T}$  are consecutive inside an S-play then  $T = \Sigma[T] \setminus (\Sigma \setminus T) + (T \setminus \Sigma)$ .

It will also be convenient to introduce the following store-constructor. For stores  $\Sigma_0, \Sigma_1, \Sigma_2$  we define

$$\Phi(\Sigma_0, \Sigma_1, \Sigma_2) \triangleq \Sigma_0[\Sigma_2] \setminus (\Sigma_1 \setminus \Sigma_2) + (\Sigma_2 \setminus \Sigma_1).$$

Considering  $\Sigma_1, \Sigma_2$  as consecutive, the constructor first updates  $\Sigma_0$  with values from  $\Sigma_2$ , removes those names that have been dropped in  $\Sigma_2$  and then adds those that have been introduced in it.

**Definition 4.7.** Let A, B, C be arenas and  $s \in SP_{A \to B}, t \in SP_{B \to C}$ . We say that s, t are compatible, written  $s \asymp t$ , if  $erase(s) \upharpoonright B = erase(t) \upharpoonright B$  and  $\nu(s) \cap \nu(t) = \varnothing$ . In such a case, we define their interaction,  $s \parallel t$ , and their mix,  $s \cdot t$ , recursively as follows,

$$\epsilon \parallel \epsilon \triangleq \epsilon 
sm_A^{\Sigma} \parallel t \triangleq (s \parallel t) m_A^{sm_A^{\Sigma} \cdot t} 
sm_A^{\Sigma} \parallel t \triangleq (s \parallel t) m_A^{sm_A^{\Sigma} \cdot t} 
sm_A^{\Sigma}(O) \cdot t \triangleq \Phi(sn^T \cdot t, T, \Sigma) 
sm_A^{\Sigma}(O) \cdot t \triangleq \tilde{\Sigma}[s \parallel t] 
sm_B^{\Sigma} \parallel t m_B^{\Sigma'} \triangleq (s \parallel t) m_B^{sm_B^{\Sigma} \cdot t m_B^{\Sigma'}} 
sm_B^{\Sigma}(O) \cdot t m_B^{\Sigma}(O) \triangleq \Phi(sn^T \cdot t, T, \Sigma) 
sm_B^{\Sigma}(O) \cdot t m_B^{\Sigma}(O) \triangleq \Phi(s \cdot t n^T, T, \Sigma) 
s \cdot t m_C^{\Sigma}(O) \triangleq \tilde{\Sigma}[s \parallel t]$$

where justification pointers in  $s \parallel t$  are inherited from s and t, and  $\tilde{\Sigma}$  is the store of  $m_{A/C(O)}$ 's justifier in  $s \parallel t$ . Note that  $s \cdot t$  is the store of the last S-move in  $s \parallel t$ . The *composite* of s and t is

$$s; t \triangleq (s \parallel t) \upharpoonright AC$$
.

We moreover let

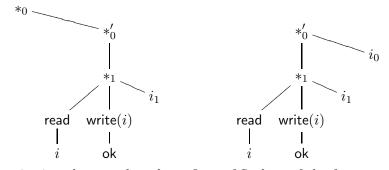
$$SInt(A, B, C) \triangleq \{s \mid |t| \mid s \in SP_{A \to B} \land t \in SP_{B \to C} \land s \asymp t\}$$

be the set of **S-interaction sequences** of A, B, C.

**Example 4.8.** Let us demonstrate how to compose S-plays from the following strategies:

$$\begin{split} \sigma &\triangleq \llbracket \vdash \lambda x^{\mathsf{var}}.\, x := !x + 1; !x : \mathsf{var} \to \mathsf{int} \rrbracket : 1 \to (\llbracket \mathsf{var} \rrbracket \Rightarrow \mathbb{Z}) \\ \tau &\triangleq \llbracket f : \mathsf{var} \to \mathsf{int} \vdash (\mathsf{new}\, x \, \mathsf{in} \, fx) + \mathsf{new} \, x \, \mathsf{in} \, fx : \mathsf{int} \rrbracket : (\llbracket \mathsf{var} \rrbracket \Rightarrow \mathbb{Z}) \to \mathbb{Z} \end{split}$$

We depict the two prearenss below (compared to Example 4.5, in the prearens on the right we have replaced q and a with concrete moves \* and  $i \in \mathbb{Z}$ ).



The S-strategy  $\sigma$  is given by even-length prefixes of S-plays of the form:

$$*_0 *'_0 *_1 \text{ read } i \text{ write}(i+1) \text{ ok read } j j_1 *_1 \text{ read } i' \text{ write}(i'+1) \text{ ok read } j' j'_1 \cdots$$

where the read's and write's point to the last  $*_1$  on their left (and each other missing pointer is assumed to point to the next move on the left). On the other hand, the elements of  $\tau$  are

even-length prefixes of S-plays with the following pattern:

$$*_0' *_1^{(\alpha,0)} \left( \mathsf{read}^{(\alpha,0)} \ 0^{(\alpha,0)} \right)^* \ \mathsf{write}(i)^{(\alpha,0)} \ \mathsf{ok}^{(\alpha,i)} \ \left( \mathsf{read}^{(\alpha,i)} \ i^{(\alpha,i)} \right)^* \cdots \ j_1^{(\alpha,k)} \\ *_1^{(\alpha',0)} \left( \mathsf{read}^{(\alpha',0)} \ 0^{(\alpha',0)} \right)^* \ \mathsf{write}(i')^{(\alpha',0)} \ \mathsf{ok}^{(\alpha',i')} \ \left( \mathsf{read}^{(\alpha',i')} \ i^{(\alpha',i')} \right)^* \cdots \ j_1'^{(\alpha',k')} \ (j+j')_0$$

Observe that an S-play  $s \in \sigma$  can only be composed with a  $t \in \tau$  if it satisfies the read-write discipline of variables: each read move must be answered by the last write value, apart of the initial read that should be answered by 0. Moreover, s must feature at most two moves s1. We therefore consider the following S-plays  $s \approx t$ .

$$\begin{split} s &= *_0 *_0' *_1 \text{ read } 0 \text{ write}(1) \text{ ok read } 1 \ 1_1 *_1 \text{ read } 0 \text{ write}(1) \text{ ok read } 1 \ 1_1 \\ t &= *_0' *_1^{(\alpha,0)} \text{ read}^{(\alpha,0)} \ 0^{(\alpha,0)} \text{ write}(1)^{(\alpha,0)} \text{ ok}^{(\alpha,1)} \text{ read}^{(\alpha,1)} \ 1_1^{(\alpha,1)} \ 1_1^{(\alpha',1)} \\ &*_1^{(\alpha',0)} \text{ read}^{(\alpha',0)} \ 0^{(\alpha',0)} \text{ write}(1)^{(\alpha',0)} \text{ ok}^{(\alpha',1)} \text{ read}^{(\alpha',1)} \ 1_1^{(\alpha',1)} \ 1_1^{(\alpha',1)} \ 2_0 \end{split}$$

By composing, we obtain:

$$s\|t=*_0*'_0*_1^{(\alpha,0)}\operatorname{read}^{(\alpha,0)}0^{(\alpha,0)}\operatorname{write}(1)^{(\alpha,0)}\operatorname{ok}^{(\alpha,1)}\operatorname{read}^{(\alpha,1)}1^{(\alpha,1)}1_1^{(\alpha,1)}\\ *_1^{(\alpha',0)}\operatorname{read}^{(\alpha',0)}0^{(\alpha',0)}\operatorname{write}(1)^{(\alpha',0)}\operatorname{ok}^{(\alpha',1)}\operatorname{read}^{(\alpha',1)}1^{(\alpha',1)}1_1^{(\alpha',1)}2_0$$

and  $s; t = *_0 2_0$ .

As a side-note, let us observe that, had we considered a slightly different strategy to compose S-plays from  $\sigma$  with:

$$\tau' \triangleq \llbracket f : \mathsf{var} \to \mathsf{int} \vdash \mathsf{new} \, x \, \mathsf{in} \, fx + fx : \mathsf{int} \rrbracket : (\llbracket \mathsf{var} \rrbracket \Rightarrow \mathbb{Z}) \to \mathbb{Z}$$

we would only be able to compose variants of s and t:

$$\begin{split} s' &= *_0 *_0' *_1 \text{ read } 0 \text{ write}(1) \text{ ok read } 1 \ 1_1 *_1 \text{ read } 1 \text{ write}(2) \text{ ok read } 2 \ 2_1 \\ t' &= *_0' *_1^{(\alpha,0)} \text{ read}^{(\alpha,0)} \ 0^{(\alpha,0)} \text{ write}(1)^{(\alpha,0)} \text{ ok}^{(\alpha,1)} \text{ read}^{(\alpha,1)} \ 1_1^{(\alpha,1)} \ 1_1^{(\alpha,1)} \\ &*_1^{(\alpha,1)} \text{ read}^{(\alpha,1)} \ 0^{(\alpha,1)} \text{ write}(2)^{(\alpha,1)} \text{ ok}^{(\alpha,2)} \text{ read}^{(\alpha,2)} \ 2^{(\alpha,2)} \ 2_1^{(\alpha,2)} \ 3_0 \end{split}$$

and thus obtain s';  $t' = *_0 3_0$ .

Given an interaction sequence u and a move m of u, we call m a **generalised P-move** if it is a P-move in AB or in BC (by which we mean a P-move in  $A \to B$  or  $B \to C$  respectively). We call m an **external O-move** if it is an O-move in AC. The notions of P-view and O-view extend to interaction sequences as follows. Let p be a generalised P-move and o be an external O-move.

Observe that if u ends in a P-move in AB (resp. BC) then  $\lfloor u \rfloor = \lfloor u \upharpoonright AB \rfloor (\lfloor u \upharpoonright BC \rfloor)$ .

We now show that play-composition is well-defined. The result follows from the next two lemmas. The first one is standard, while the proof of the second one is given in Appendix A.

**Lemma 4.9** (Zipper Lemma [8]). If  $s \in SP_{A \to B}$ ,  $t \in SP_{B \to C}$  and  $s \times t$  then either  $s \upharpoonright B = t = \epsilon$ , or s ends in A and t in B (with a P-move in BC), or s ends in B (with a P-move in AB) and t in C, or both s and t end in B (with the same move).

**Lemma 4.10.** Suppose  $s \in SP_{A \to B}, t \in SP_{B \to C}, s \times t$  and p a generalised P-move.

- (1) If  $s \parallel t = un^T p^{\Sigma}$  then  $\nu(\Sigma \setminus T) \cap \nu(un^T) = \emptyset$ .
- (2) For any  $\alpha$ ,  $\lceil s \parallel t \rceil$  is in block-form:  $\lceil s \parallel t \rceil = u_1 u_2 u_3$  where  $\alpha$  appears in every move of  $u_2, \alpha \notin \nu(u_1) \cup \nu(u_3)$  and if  $u_2 \neq \epsilon$  then its first move introduces  $\alpha$  in  $s \parallel t$ .
- (3) If  $s \parallel t = un^T p^{\Sigma}$  and  $\alpha \in \nu(T \setminus \Sigma)$  then  $\alpha$  is closed in  $un^T$ . (4) If  $s \parallel t = \cdots n^T \cdots m^{\Sigma}$  then  $T \leq_p \Sigma$ . If m is an answer then  $\Sigma \leq_p T$ .
- (5) If  $s \parallel t = u_1 n^T p^{\Sigma} u_2$  and  $\alpha \in \nu(T \setminus \Sigma)$  then  $\alpha \notin \nu(u_2)$ .
- (6) If  $s \parallel t = um^{\Sigma}$  with m an O-move in AC then, for any  $\alpha \in \nu(\Sigma)$ , the last appearance of  $\alpha$  in u occurs in AC.
- (7) If  $s \parallel t = um^{\Sigma}$  and  $s = s'm^{\Sigma'}$  or  $t = t'm^{\Sigma'}$  then  $\Sigma' \leq \Sigma$  and  $\Sigma[\Sigma'] = \Sigma$ . Moreover,  $\Sigma[\mathsf{st}(s)] = \Sigma[\mathsf{st}(t)] = \Sigma.$ (8) If  $s \parallel t = un^T p^{\Sigma}$  then  $T \setminus (T \setminus \Sigma) \leq_p \Sigma$  and  $T \setminus \Sigma \leq_s T$ .

**Proposition 4.11** (Compositionality). S-play composition is well defined, that is, if  $s \in$  $SP_{A\to B}, t \in SP_{B\to C}$  and  $s \approx t$  then  $s; t \in SP_{A\to C}$ .

*Proof.* We need to verify the 5 conditions of Definition 4.2. Init is straightforward. Just-P follows from part (d) of the Lemma 4.10. Just-O follows directly from the definition of interaction sequences. Prev-PQ follows from parts (a,c,h) Lemma 4.10. Finally, Val-O follows from part (f) of Lemma 4.10 and the definition of interaction sequences. 

4.3. Associativity. We next show that composition of S-plays is associative. We first extend interactions to triples of S-plays. For  $s \in SP_{A \to B}, t \in SP_{B \to C}, r \in SP_{C \to D}$  with  $(s;t) \times r$ ,  $s \times (t;r)$  and  $\nu(s) \cap \nu(r) = \emptyset$ , we define  $s \parallel t \parallel r$  and  $s \cdot t \cdot r$  as follows,

$$\epsilon \parallel \epsilon \parallel \epsilon \triangleq \epsilon \qquad \qquad \epsilon \cdot \epsilon \cdot \epsilon \triangleq \epsilon$$

$$sm_A^{\Sigma} \parallel t \parallel r \triangleq (s \parallel t \parallel r) m_A^{sm_A^{\Sigma} \cdot t \cdot r} \qquad sn^T m_{A(P)}^{\Sigma} \cdot t \cdot r \triangleq \Phi(sn^T \cdot t \cdot r, T, \Sigma)$$

$$sm_{B(D)}^{\Sigma} \cdot t \cdot r \triangleq \tilde{\Sigma}[s \cdot t \cdot r]$$

$$sm_B^{\Sigma} \parallel t m_B^{\Sigma'} \parallel r \triangleq (s \parallel t \parallel r) m_B^{sm_B^{\Sigma} \cdot t m_B^{\Sigma'} \cdot r} \qquad sn^T m_{B(P)}^{\Sigma} \cdot t m_{B(O)}^{\Sigma'} \cdot r \triangleq \Phi(sn^T \cdot t \cdot r, T, \Sigma)$$

$$sm_{B(O)}^{\Sigma} \cdot t n^T m_{B(P)}^{\Sigma} \cdot r \triangleq \Phi(s \cdot t n^T \cdot r, T, \Sigma)$$

$$s \parallel t m_C^{\Sigma} \parallel r m_C^{\Sigma'} \triangleq (s \parallel t \parallel r) m_C^{s \cdot t m_C^{\Sigma} \cdot r m_C^{\Sigma'}} \qquad s \cdot t n^T m_{C(P)}^{\Sigma} \cdot r m_{C(O)}^{\Sigma'} \triangleq \Phi(s \cdot t \cdot r n^T, T, \Sigma)$$

$$s \parallel t \parallel r m_D^{\Sigma} \triangleq (s \parallel t \parallel r) m_D^{s \cdot t \cdot r m_D^{\Sigma}} \qquad s \cdot t \cdot r n^T m_{D(P)}^{\Sigma} \triangleq \Phi(s \cdot t \cdot r n^T, T, \Sigma)$$

$$s \cdot t \cdot r m_{D(O)}^{\Sigma} \triangleq \tilde{\Sigma}[s \cdot t \cdot r]$$

where  $\tilde{\Sigma}$  is the store of the move justifying  $m_{A(O)}^{\Sigma}$  and  $m_{D(O)}^{\Sigma}$  respectively.

The next lemma proves a form of associativity in  $\Phi$  that will be used in the proof of the next proposition. Its proof is delegated to Appendix A.

**Lemma 4.12.** Let  $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$  be stores such that, for any  $\alpha$ ,

- (1)  $\alpha \in \nu(\Sigma_5 \setminus \Sigma_4) \implies \alpha \notin \nu(\Sigma_1) \cup \nu(\Sigma_2) \cup \nu(\Sigma_3)$ ,
- (2)  $\alpha \in \nu(\Sigma_1) \cap \nu(\Sigma_4) \implies \alpha \in \nu(\Sigma_2)$ .

Then,  $\Phi(\Sigma_1, \Sigma_2, \Phi(\Sigma_3, \Sigma_4, \Sigma_5)) = \Phi(\Phi(\Sigma_1, \Sigma_2, \Sigma_3), \Sigma_4, \Sigma_5)$ .

**Proposition 4.13** (Associativity). If  $s_1 \in SP_{A_1 \to A_2}$ ,  $s_2 \in SP_{A_2 \to A_3}$  and  $s_3 \in SP_{A_3 \to A_4}$  with  $s_1; s_2 \times s_3$  and  $s_1 \times s_2; s_3$  then:

- $(s_1; s_2); s_3 = (s_1 || s_2 || s_3) \upharpoonright A_1 A_4 = s_1; (s_2; s_3),$
- $\Phi((s_1; s_2) \cdot s_3, \mathsf{st}(s_1; s_2), s_1 \cdot s_2) = s_1 \cdot s_2 \cdot s_3 = \Phi(s_1 \cdot (s_2; s_3), \mathsf{st}(s_2; s_3), s_2 \cdot s_3).$

*Proof.* By induction on  $|s_1| |s_2| |s_3|$ . The base case is trivial. We examine the following inductive cases; the rest are similar.

-  $(s_1 m_{A_1}^{\Sigma}; s_2); s_3 = ((s_1; s_2); s_3) m_{A_1}^{\Sigma'} \stackrel{\text{IH}}{=} (s_1 || s_2 || s_3) m_{A_1}^{\Sigma'} \upharpoonright A_1 A_4 \text{ with } \Sigma' = (s_1 m_{A_1}^{\Sigma}; s_2) \cdot s_3.$ Moreover, as  $\mathsf{st}(s_1 m_{A_1}^{\Sigma}; s_2) = s_1 m_{A_1}^{\Sigma} \cdot s_2$  and using Lemma 4.10(g),

$$\Phi((s_1; s_2 m_{A_1}^{\Sigma}); s_3, \operatorname{st}(s_1 m_{A_1}^{\Sigma}; s_2), s_1 m_{A_1}^{\Sigma} \cdot s_2) = \Sigma'[s_1 m_{A_1}^{\Sigma} \cdot s_2] = \Sigma'.$$

We still need to show that  $\Sigma' = s_1 m_{A_1}^{\Sigma} \cdot s_2 \cdot s_3$ . If  $m_{A_1}$  is an O-move this is straightforward. If a P-move and, say,  $s_1 = s_1' n^T$  then  $\Sigma' = \Phi((s_1; s_2) \cdot s_3, \mathsf{st}(s_1; s_2), \Phi(s_1 \cdot s_2, T, \Sigma))$  and  $s_1 m_{A_1}^{\Sigma} \cdot s_2 \cdot s_3 = \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma) \stackrel{\text{IH}}{=} \Phi(\Phi((s_1; s_2) \cdot s_3, \mathsf{st}(s_1; s_2), s_1 \cdot s_2), T, \Sigma)$ . These are equal since they satisfy the hypotheses of Lemma 4.12. Moreover,

$$s_1 m_{A_1}^{\Sigma}; (s_2; s_3) = (s_1; (s_2; s_3)) m_{A_1}^{\Sigma''} \stackrel{\text{IH}}{=} (s_1 \parallel s_2 \parallel s_3) m_{A_1}^{\Sigma''} \upharpoonright A_1 A_4$$

with  $\Sigma'' = s_1 m_{A_1}^{\Sigma} \cdot (s_2; s_3)$ . Note that  $\mathsf{st}(s_2; s_3) = s_2 \cdot s_3$ , so it suffices to show that  $\Sigma'' = s_1 m_{A_1}^{\Sigma} \cdot s_2 \cdot s_3$ . If  $m_{A_1}$  an O-move then this is straightforward, otherwise

$$\Sigma'' = \Phi(s_1 \cdot (s_2; s_3), T, \Sigma) \stackrel{\text{IH}}{=} \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma) = s_1 m_{A_1}^{\Sigma} \cdot s_2 \cdot s_3.$$

 $-(s_{1}m_{A_{2}}^{\Sigma_{1}};s_{2}m_{A_{2}}^{\Sigma_{2}});s_{3}=(s_{1};s_{2});s_{3}\overset{\mathrm{IH}}{=}(s_{1}\parallel s_{2}\parallel s_{3})\upharpoonright A_{1}A_{4}=(s_{1}m_{A_{2}}^{\Sigma_{1}}\parallel s_{2}m_{A_{2}}^{\Sigma_{2}}\parallel s_{3})\upharpoonright A_{1}A_{4}.$  Assume WLOG that  $m_{A_{2}}$  is a P-move in  $A_{1}A_{2}$ , so  $\mathsf{st}(s_{2};s_{3})=s_{2} \cdot s_{3}$ , and suppose  $s_{1}=s_{1}'n^{T}$ . Then,

$$\begin{split} & \Phi((s_1 m_{A_2}^{\Sigma_1}; s_2 m_{A_2}^{\Sigma_2}) \bullet s_3, \mathsf{st}(s_1 m_{A_2}^{\Sigma_1}; s_2 m_{A_2}^{\Sigma_2}), s_1 m_{A_2}^{\Sigma_1} \bullet s_2 m_{A_2}^{\Sigma_2}) \\ & = \Phi((s_1; s_2) \bullet s_3, \mathsf{st}(s_1; s_2), \Phi(s_1 \bullet s_2, T, \Sigma_1)) \end{split}$$

 $s_1 m_{A_2}^{\Sigma_1} \cdot s_2 m_{A_2}^{\Sigma_2} \cdot s_3 = \mathbf{\Phi}(s_1 \cdot s_2 \cdot s_3, T, \Sigma_1) \stackrel{\text{IH}}{=} \mathbf{\Phi}(\mathbf{\Phi}((s_1; s_2) \cdot s_3, \mathsf{st}(s_1; s_2), s_1 \cdot s_2), T, \Sigma_1)$  and equality follows from Lemma 4.12. Moreover,

$$\begin{split} & \Phi(s_1 m_{A_2}^{\Sigma_1} \boldsymbol{\cdot} (s_2 m_{A_2}^{\Sigma_2}; s_3), \mathsf{st}(s_2 m_{A_2}^{\Sigma_2}; s_3), s_2 m_{A_2}^{\Sigma_2} \boldsymbol{\cdot} s_3) = (s_1 m_{A_2}^{\Sigma_1} \boldsymbol{\cdot} (s_2 m_{A_2}^{\Sigma_2}; s_3)) [s_2 m_{A_2}^{\Sigma_2} \boldsymbol{\cdot} s_3] \\ & \stackrel{\text{lm. 4.10(g)}}{=} s_1 m_{A_2}^{\Sigma_1} \boldsymbol{\cdot} (s_2 m_{A_2}^{\Sigma_2}; s_3) = \Phi(s_1 \boldsymbol{\cdot} (s_2; s_3), T, \Sigma_1) \end{split}$$

$$s_1 m_{A_2}^{\Sigma_1} \cdot s_2 m_{A_2}^{\Sigma_2} \cdot s_3 = \mathbf{\Phi}(s_1 \cdot s_2 \cdot s_3, T, \Sigma_1) \stackrel{\text{IH}}{=} \mathbf{\Phi}(\mathbf{\Phi}(s_1 \cdot (s_2; s_3), \mathsf{st}(s_2; s_3), s_2 \cdot s_3), T, \Sigma_1)$$

$$= \Phi((s_1 \cdot (s_2; s_3))[s_2 \cdot s_3], T, \Sigma_1) \stackrel{\text{lm. 4.10(g)}}{=} \Phi(s_1 \cdot (s_2; s_3), T, \Sigma_1)$$

as required.  $\Box$ 

4.4. The categories S and  $S_{inn}$ . We next show that S-strategies and arenas form a category, which we call  $\mathcal{S}$ , while innocent S-strategies form a wide subcategory of  $\mathcal{S}$ . We start this section with a lemma on strong nominal sets. Recall that, for a nominal set X and  $x, x' \in X$ , we write  $x \sim x'$  if there exists a permutation  $\pi$  such that  $x = \pi \cdot x'$ .

**Lemma 4.14** (Strong Support Lemma [25]). Let X be a strong nominal set and let  $x_i, y_i, z_i \in X$  with  $\nu(y_i) \cap \nu(z_i) \subseteq \nu(x_i)$ , for i = 1, 2. Then,  $(x_1, y_1) \sim (x_2, y_2)$  and  $(x_1, z_1) \sim (x_2, z_2)$  imply  $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ .

We proceed to show compositionality of S-strategies. The *interaction* of S-strategies  $\sigma: A \to B$  and  $\tau: B \to C$  is defined by:

$$\sigma \parallel \tau \triangleq \{s \parallel t \mid s \in \sigma \land t \in \tau \land s \asymp t\}$$

First, some lemmas for determinacy.

**Lemma 4.15.** If  $s_1 \| t_1, s_2 \| t_2 \in SInt(A, B, C)$  then  $s_1 \| t_1 = s_2 \| t_2$  implies  $s_1 = s_2$  and  $t_1 = t_2$ . Consequently, if  $s_1 \| t_1 \sim s_2 \| t_2$  then  $(s_1, t_1) \sim (s_2, t_2)$ .

*Proof.* The former part of the claim is shown by straightforward induction on the length of the interactions. For the latter part, if  $s_1 \parallel t_1 \sim s_2 \parallel t_2$  then there is some  $\pi$  such that  $s_1 \| t_1 = \pi \cdot (s_2 \| t_2) = (\pi \cdot s_2) \| (\pi \cdot t_2)$ . Hence, by the former part,  $(s_1, t_1) = (\pi \cdot s_2, \pi \cdot t_2)$ , from which we obtain  $(s_1, t_1) \sim (s_2, t_2)$ .

**Lemma 4.16.** If  $\sigma: A \to B$ ,  $\tau: B \to C$  are S-strategies and  $u_1 m_1^{\Sigma_1}, u_2 m_2^{\Sigma_2} \in \sigma \parallel \tau$  then  $u_1 \sim u_2$  and  $m_1$  a generalised P-move imply  $u_1 m_1^{\Sigma_1} \sim u_2 m_2^{\Sigma_2}$ .

*Proof.* Let us assume that  $u_i m_i^{\Sigma_i} = s_i \| t_i = (s_i' \| t_i') m_i^{\Sigma_i}$ . Then, by the previous lemma,  $(s_1', t_1') \sim (s_2', t_2')$ . If, say,  $m_1$  is a P-move in AB then  $m_2$  is also a P-move in AB and  $s_1 \sim s_2$ , by Zipper Lemma and determinacy of  $\sigma$ , so  $(s'_1, m_1^{\Sigma'_1}) \sim (s'_2, m_2^{\Sigma'_2})$ , where  $s_i = s'_i m_i^{\Sigma'_i}$ . Since  $\nu(t'_i) \cap \nu(m_i^{\Sigma'_i}) = \varnothing$ , we can apply the Strong Support Lemma to obtain  $u_1 m_1^{\Sigma_1} \sim u_2 m_2^{\Sigma_2}$ .  $\square$ 

**Lemma 4.17.** Let  $\sigma: A \to B$ ,  $\tau: B \to C$  be S-strategies and  $u_1, u_2 \in \sigma \parallel \tau$  with  $|u_1| \leq |u_2|$ . Then,  $u_1 \upharpoonright AC = u_2 \upharpoonright AC$  implies  $u_1 \sim \sqsubseteq_p u_2$ .

*Proof.* By induction on  $|u_1|$ . The base case is trivial. Now suppose  $u_1 = u_1' m_1^{\Sigma_1}$ , and let  $u_2 = u_2' u_2''$  with  $u_2'$  being the greatest prefix of  $u_2$  such that  $u_1' \upharpoonright AC = u_2' \upharpoonright AC$ . Then, by IH,  $u'_1 \sim \sqsubseteq_p u'_2$ . If  $m_1$  a generalised P-move then, by previous lemma,  $u_1 \sim \sqsubseteq_p u_2$ . If  $m_1$  an external O-move then  $u_1'$  ends in a P-move in AC and, by  $u_1' \upharpoonright AC = u_2' \upharpoonright AC$  and Zipper Lemma,  $u_2'$  must end in the same move. Thus,  $u_1 \upharpoonright AC \sim u_2 \upharpoonright AC$  implies  $u_1 \sim \sqsubseteq_p u_2$ .  $\square$ 

Now some lemmas for innocence. We say that a move m in an interaction sequence  $s \in SInt(A, B, C)$  is a generalised O-move if it is an O-move in AB or BC.

**Lemma 4.18.** If  $s_1, s_2 \in SP_{A \to B}$ ,  $t_1, t_2 \in SP_{B \to C}$  and  $s_i \parallel t_i$  end in a generalised O-move in component X,

- (1) if X = AB then  $\lceil (s_1 \parallel t_1) \upharpoonright AB \rceil = \lceil (s_2 \parallel t_2) \upharpoonright AB \rceil \Longrightarrow \lceil s_1 \rceil = \lceil s_2 \rceil$ , (2) if X = BC then  $\lceil (s_1 \parallel t_1) \upharpoonright BC \rceil = \lceil (s_2 \parallel t_2) \upharpoonright BC \rceil \Longrightarrow \lceil t_1 \rceil = \lceil t_2 \rceil$ .

*Proof.* We show (a) by induction on  $|s_1| \geq 1$ , and (b) is proved similarly. The base case is obvious. If  $s_1 = s_1' n^{T_1} s_1'' m^{\Sigma_1}$  with m an O-move in AB justified by n then  $s_2 = s_2' n^{T_2} s_2'' m^{\Sigma_2}$ and, by IH,  $\lceil s_1' \rceil = \lceil s_2' \rceil$ . We need to show that  $T_1 = T_2$  and  $\Sigma_1 = \Sigma_2$ , while we know that the stores of the corresponding moves in  $s_i \parallel t_i$  are equal, say  $\Sigma'_1 = \Sigma'_2$  and  $T'_1 = T'_2$ . We have that  $T_i' = \Phi(\mathsf{st}((s_i \parallel t_i)_{< n^{T_i'}}), \mathsf{st}(s_i'), T_i)$ , hence  $T_1 \setminus \mathsf{st}(s_1') = T_2 \setminus \mathsf{st}(s_2')$ . By IH we have that  $\mathsf{st}(s_1') = \mathsf{st}(s_2')$ , so if  $\alpha \in \nu(T_1) \cap \nu(\mathsf{st}(s_1'))$  then, by Lemma 4.10(g),  $\alpha \in \nu(T_1') \cap \nu(\mathsf{st}(s_1'))$  so  $\alpha \in \nu(T_2') \cap \nu(\mathsf{st}(s_2'))$ ,  $\alpha \in \nu(T_2) \cap \nu(\mathsf{st}(s_2'))$ , and viceversa. Moreover,  $T_i(\alpha) = T_i'(\alpha)$  for each such  $\alpha$ , thus  $T_1 = T_2$ . This also implies that  $\nu(\Sigma_1) = \nu(\Sigma_2)$  and so, by Lemma 4.10(g),  $\Sigma_1 = \Sigma_2$ .

**Lemma 4.19.** If  $\sigma: A \to B$ ,  $\tau: B \to C$  are innocent S-strategies then, if  $u_1 m^{\Sigma_1}$ ,  $u_2 \in \sigma \parallel \tau$  with  $u_i$  ending in a generalised O-move and  $\lceil u_1 \rceil \sim \lceil u_2 \rceil$  then there exists  $u_2 m^{\Sigma_2} \in \sigma \parallel \tau$  such that  $\lceil u_1 m^{\Sigma_1 \rceil} \sim \lceil u_2 m^{\Sigma_2 \rceil}$ .

Proof. Suppose  $u_1$  ends in an O-move in A—the other cases are shown similarly. Then,  $u_1m^{\Sigma_1}=s_1m^{\Sigma_1'}\parallel t_1$  and  $u_2=s_2\parallel t_2$  for some relevant S-plays of  $\sigma,\tau$ . Moreover,  $\lceil u_1\rceil\sim \lceil u_2\rceil$  implies, by Lemma 4.18, that  $\lceil s_1\rceil\sim \lceil s_2\rceil$  and thus, by innocence, there exists  $s_2m^{\Sigma_2'}\in\sigma$  such that  $s_1m^{\Sigma_1'}\sim s_2m^{\Sigma_2'}$ . In fact, we can pick a  $\Sigma_2'$  such that  $s_2m^{\Sigma_2'}\asymp t_2$ . Let  $s_2m^{\Sigma_2'}\parallel t_2=u_2m^{\Sigma_2}\in\sigma\parallel\tau$ . We have that  $(\lceil u_1\rceil,\lceil s_1\rceil)\sim (\lceil u_2\rceil,\lceil s_2\rceil)$  and  $(\lceil s_1\rceil,m^{\Sigma_1'})\sim (\lceil s_2\rceil,m^{\Sigma_2'})$  and, moreover,  $\nu(\lceil u_i\rceil)\cap\nu(m^{\Sigma_1'})\subseteq\nu(\lceil s_i\rceil)$  for i=1,2 thus, by Strong Support Lemma,  $(\lceil u_1\rceil,\lceil s_1\rceil,m^{\Sigma_1'})\sim (\lceil u_2\rceil,\lceil s_2\rceil,m^{\Sigma_2'})$ . This implies that  $\lceil u_1m^{\Sigma_1}\rceil\sim \lceil u_2m^{\Sigma_2}\rceil$ .  $\square$ 

**Lemma 4.20.** Let  $\sigma: A \to B$ ,  $\tau: B \to C$  be innocent S-strategies and let  $u_1, u_2 \in \sigma \parallel \tau$  with  $\lceil u_1 \rceil \rceil \leq \lceil u_2 \rceil$ . Then,  $\lceil u_1 \upharpoonright AC \rceil = \lceil u_2 \upharpoonright AC \rceil$  implies  $\lceil u_1 \rceil \sim \sqsubseteq_p \lceil u_2 \rceil$ .

Proof. Let us write  $u_i$  for  $s_i \parallel t_i$ . We argue by induction on  $|\lceil u_1 \rceil| + |\lceil u_2 \rceil|$ . The base cases are obvious. Now let  $u_1 = u_1' m^{\Sigma_1}$  with m a B-move. We have that  $|u_1'| \leq |u_2|$  and  $\lceil u_1' \rceil \upharpoonright AC \rceil = \lceil u_2 \rceil \upharpoonright AC \rceil$  so, by IH,  $\lceil u_1' \rceil \sim \sqsubseteq_p \lceil u_2 \rceil$ . In particular,  $u_2 = u_2' m_2^{\Sigma_2} u_2''$  with  $\lceil u_2' \rceil \sim \lceil u_1' \rceil$  so, by Lemma 4.19, there exists  $u_2' m_1^{\Sigma_2'} \in \sigma \parallel \tau$  with  $\lceil u_1 \rceil \sim \lceil u_2' m_1^{\Sigma_2'} \rceil$ . Using Lemma 4.16,  $\lceil u_2' \rceil m_2^{\Sigma_2} \sim \lceil u_2' \rceil m_1^{\Sigma_2'} \sim \lceil u_1 \rceil$ . The case of m being a P-move in AC is proved along the same lines. On the other hand,

The case of m being a P-move in AC is proved along the same lines. On the other hand, if m is an O-move in AC justified by n then  $u_1 = v_1 n^T v_1' m^{\Sigma}$  and  $u_2 = v_2 n^T v_2' m^{\Sigma} v_2''$  and, by IH,  $\lceil v_1 n^{T} \rceil = \pi \cdot \lceil v_2 n^{T} \rceil$ . In particular,  $\pi$  fixes dom (T) and therefore  $\pi \cdot m^{\Sigma} = m^{\Sigma}$ ,  $\therefore \lceil v_1 n^{T} \rceil m^{\Sigma} = \pi \cdot (\lceil v_2 n^{T} \rceil m^{\Sigma})$ .

**Proposition 4.21.** If  $\sigma: A \to B$ ,  $\tau: B \to C$  are S-strategies then  $\sigma; \tau: A \to C$  is an S-strategy. If  $\sigma$  and  $\tau$  are innocent, so is  $\sigma; \tau$ .

*Proof.* Prefix closure and nominal closure are easy to establish by noting that they hold at the level of interactions, for  $\sigma \parallel \tau$ . For O-closure, suppose  $v \in \sigma$ ;  $\tau$  and  $vm^{\Sigma} \in SP_{A \to C}$  with, say, m an O-move in A. Then, v = s; t for some  $s \in \sigma$ ,  $t \in \tau$  with s ending in a P-move in A. Moreover, we can construct a (unique) store  $\Sigma'$  such that  $sm^{\Sigma'} \in SP_{A \to B}$ , by means of the O-Just and O-Val conditions. Thus,  $sm^{\Sigma'} \in \sigma$  and  $vm^{\Sigma} \in \sigma$ ;  $\tau$ .

For determinacy, suppose even-length  $vm_i^{\Sigma_i} \in \sigma; \tau$  and  $vm_i^{\Sigma_i} = s_i; t_i$  with  $s_i, t_i$  not both ending in B, for i = 1, 2. Let  $s_i \parallel t_i = (s_i' \parallel t_i') m_i^{\Sigma_i}$  and suppose WLOG that  $|s_1' \parallel t_1'| \leq |s_2' \parallel t_2'|$ . Then, by Lemma 4.17,  $s_1' \parallel t_1' \sim \sqsubseteq_p s_2' \parallel t_2'$  and therefore, by Lemma 4.16,  $s_1 \parallel t_1 \sim \sqsubseteq_p s_2 \parallel t_2$ . In particular,  $vm_1^{\Sigma_1} \sim vm_2^{\Sigma_2}$ .

For innocence, let  $v_1 m_1^{\Sigma_1}$ ,  $v_2 \in \sigma$ ;  $\tau$  with  $\lceil v_1 \rceil = \lceil v_2 \rceil$  being of odd length, and  $u_1 m_1^{\Sigma_1}$ ,  $u_2 \in \sigma \parallel \tau$  such that  $v_1 m_1^{\Sigma_1} = u_1 m_1^{\Sigma_1} \upharpoonright AC$ ,  $v_2 = u_2 \upharpoonright AC$ . By Lemma 4.20, either  $\lceil u_2 \rceil \sim \sqsubseteq_p \lceil u_1 \rceil$  or  $\lceil u_1 \rceil \sim \sqsubseteq_p \lceil u_2 \rceil$  and  $\lceil u_1 \rceil \not\sim \lceil u_2 \rceil$ . In the latter case, by Lemmata 4.19 and 4.16 we obtain  $\lceil u_1 m_1^{\Sigma_1} \rceil \sim \sqsubseteq_p \lceil u_2 \rceil$ , contradicting  $\lceil v_1 \rceil = \lceil v_2 \rceil$ . In the former case, let us assume  $u_2$  is of maximum length such that  $u_2 \upharpoonright AC = v_2$  and  $\lceil u_2 \rceil \leq \lceil u_1 \rceil$ . Then, by Lemma 4.19,

there exists  $u_2 m_2^{\Sigma_2} \in \sigma \parallel \tau$  such that either  $\lceil u_2 m_2^{\Sigma_2} \rceil \sim \sqsubseteq_p \lceil u_1 \rceil$  or  $\lceil u_2 m_2^{\Sigma_2} \rceil \sim \lceil u_1 m_1^{\Sigma_1} \rceil$ . The former case contradicts maximality of  $u_2$ , while the latter implies  $v_2 m_2^{\Sigma_2} \in \sigma; \tau$  and  $\lceil v_1 m_1^{\Sigma_1} \rceil \sim \lceil v_2 m_2^{\Sigma_2} \rceil$ .

We proceed to construct a category of arenas and S-strategies. For each arena A, the identity S-strategy  $id_A : A \to A$  is the *copycat* S-strategy:

$$\mathsf{id}_A = \{ s \in SP_{A \to A} \mid \exists s'. s \sqsubseteq_p s' \land s' \upharpoonright A_l = s' \upharpoonright A_r \}$$

where by  $A_l$  and  $A_r$  we denote the LHS and RHS arena A of  $A \to A$  respectively.

**Proposition 4.22.** Let  $\sigma_i: A_i \to A_{i+1}$  for i = 1, 2, 3. Then,  $\sigma_i; \mathsf{id}_{A_{i+1}} = \mathsf{id}_{A_i}; \sigma_i = \sigma_i$  and  $\sigma_1; (\sigma_2; \sigma_3) = (\sigma_1; \sigma_2); \sigma_3$ .

Proof. Composition with identities is standard. For associativity, if  $s_1$ ;  $(s_2; s_3) \in \sigma_1$ ;  $(\sigma_2; \sigma_3)$  then there are  $s_i' \sim s_i$  such that  $\nu(s_{i_1}') \cap (\nu(s_{i_2}) \cup \nu(s_{i_3}) \cup \nu(s_{i_2}') \cup \nu(s_{i_3}')) = \varnothing$ , for any distinct  $i_1, i_2, i_3$ . These S-plays satisfy the hypotheses of Proposition 4.13, hence  $s_1'$ ;  $(s_2'; s_3') = (s_1'; s_2')$ ;  $s_3' \in (\sigma_1; \sigma_2)$ ;  $\sigma_3$ . Moreover, since  $\nu(s_2, s_2') \cap \nu(s_3, s_3') = \varnothing$ ,  $s_i \sim s_i'$  imply  $(s_2, s_3) \sim (s_2', s_3')$  and therefore  $s_2$ ;  $s_3 \sim s_2'$ ;  $s_3'$ . Since  $\nu(s_1, s_1') \cap \nu(s_2; s_3, s_2'; s_3') = \emptyset$ , we have  $s_1$ ;  $(s_2; s_3) \sim s_1'$ ;  $(s_2'; s_3')$ , hence  $s_1$ ;  $(s_2; s_3) \in (\sigma_1; \sigma_2)$ ;  $\sigma_3$ . Other direction proved dually.

We can now define our categories of games with stores. In the following definition we also make use of the observation that identity S-strategies are innocent.

**Definition 4.23.** Let S be the category whose objects are arenas and, for each pair of arenas A, B, the morphisms are given by  $S(A, B) = \{ \sigma : A \to B \mid \sigma \text{ an S-strategy} \}$ . Let  $S_{\text{inn}}$  be the wide subcategory of S of innocent S-strategies.

4.5. The model of  $A_{cbv}$ . We next construct the model of  $A_{cbv}$  in  $S_{inn}$ . An innocent S-strategy  $\sigma$  is specified by its *view-function*, viewf( $\sigma$ ), defined as follows.

$$\mathsf{viewf}(\sigma) \triangleq \{ \lceil s \rceil \mid s \in \sigma \land \mathsf{even}(|s|) \land s \neq \epsilon \}$$

Conversely, a **preplay** is defined exactly like a (non-empty) S-play only that it does not necessarily satisfy the Val-O condition. Let us write  $PP_A$  for the set of preplays of A. Obviously,  $SP_A \subseteq PP_A$ . Moreover, if  $s \in SP_A$  then  $\lceil s \rceil \in PP_A$ .

For instance, the cell strategy of Example 4.5 can be described as the least innocent S-strategy whose view-function contains the following preplays.

$$q_0 = q_1 \stackrel{(\alpha,0)}{=} a_1 \stackrel{(\alpha,i)}{=} a_0 \qquad \qquad q_0 = q_1 \stackrel{(\alpha,0)}{=} \operatorname{read}^{(\alpha,i)} i^{(\alpha,i)} \qquad \qquad q_0 = q_1 \stackrel{(\alpha,0)}{=} \operatorname{write}(j) \stackrel{(\alpha,i)}{=} \operatorname{ok}^{(\alpha,j)} = q_0 \stackrel{(\alpha,j)}{=} \operatorname{ok}^{(\alpha,j)$$

We next make formal the connection between view-functions and innocent S-strategies. A *view-function* f on A is a subset of  $PP_A$  satisfying:

- If  $s \in f$  then |s| is even and  $\lceil s \rceil = s$ . (View)
- If  $sn^Tm^{\Sigma} \in f$  and  $s \neq \epsilon$  then  $s \in f$ . (Even-Prefix Closure)
- If  $s' \sim s \in f$  then  $s' \in f$ . (Nominal Closure)
- If  $sm_1^{\Sigma_1}, sm_2^{\Sigma_2} \in f$  then  $sm_1^{\Sigma_1} \sim sm_2^{\Sigma_2}$ . (Determinacy)

From a view-function f we can derive an innocent S-strategy  $\mathsf{strat}(f)$  by the following procedure. We set  $\mathsf{strat}(f) \triangleq \bigcup_{i \in \omega} \mathsf{strat}_i(f)$ , where

$$\begin{split} & \mathsf{strat}_{2i+1}(f) \triangleq \{sm^{\Sigma} \in SP_A \mid s \in \mathsf{strat}_{2i}(f)\} \\ & \mathsf{strat}_{2i+2}(f) \triangleq \{sm^{\Sigma} \in SP_A \mid s \in \mathsf{strat}_{2i+1}(f) \wedge \lceil sm^{\Sigma} \rceil \in f\} \end{split}$$

and  $\operatorname{strat}_0(f) \triangleq \{\epsilon\}.$ 

**Lemma 4.24.** If  $\sigma$ , f are an innocent S-strategy and a view-function respectively then  $\mathsf{viewf}(\sigma), \mathsf{strat}(\sigma)$  are a view-function and an innocent S-strategy respectively. Moreover,  $\mathsf{strat}(\mathsf{viewf}(\sigma)) = \sigma$  and  $\mathsf{viewf}(\mathsf{strat}(f)) = f$ .

We can show that  $S_{\text{inn}}$  exhibits the same kind of categorical structure as that obtained in [8] (in the context of call-by-value PCF), which can be employed to model call-by-value higher-order computation with recursion. In particular, let us call an S-strategy  $\sigma: A \to B$  total if for all  $i_A \in I_A$  there is  $i_A i_B \in \sigma$ . We write  $S_{\text{inn}}^t$  for the wide subcategory of  $S_{\text{inn}}$  containing total innocent S-strategies.

For innocent S-strategies  $\sigma: A \to B$  and  $\tau: A \to C$ , we define their *left pairing* to be  $\langle \sigma, \tau \rangle_l = \mathsf{strat}(f)$ , where f is the view-function:

$$\begin{split} f &= \big\{ \, s \in PP_{A \to B \otimes C} \mid s \in \mathsf{viewf}(\sigma) \, \wedge \, s \upharpoonright (B \otimes C) = \epsilon \, \big\} \\ &\quad \cup \big\{ \, i_A s_1 s_2 \in PP_{A \to B \otimes C} \mid \exists i_B. i_A s_1 i_B \in \mathsf{viewf}(\sigma) \, \wedge \, i_A s_2 \in \mathsf{viewf}(\tau) \, \big\} \\ &\quad \cup \big\{ \, i_A s_1 s_2 (i_B, i_C) s \in PP_{A \to B \otimes C} \mid i_A s_1 i_B s \in \mathsf{viewf}(\sigma) \, \wedge \, i_A s_2 i_C \in \mathsf{viewf}(\tau) \, \big\} \\ &\quad \cup \big\{ \, i_A s_1 s_2 (i_B, i_C) s \in PP_{A \to B \otimes C} \mid i_A s_1 i_B \in \mathsf{viewf}(\sigma) \, \wedge \, i_A s_2 i_C s \in \mathsf{viewf}(\tau) \, \big\} \end{split}$$

We can show that left pairing yields a product in  $\mathcal{S}_{\text{inn}}^t$  with the usual projections:

 $\pi_1: A \otimes B \to A = \{ s \in SP_{A \otimes B \to A} \mid |s| \leq 1 \} \cup \{ (i_A, i_B)i_A s \in SP_{A \otimes B \to A} \mid i_A i_A s \in \mathsf{id}_A \}$  and dually for  $\pi_2$ . Moreover, for every A, B, C, there is a bijection

$$\Lambda: \mathcal{S}_{\text{inn}}(A \otimes B, C) \stackrel{\cong}{\to} \mathcal{S}_{\text{inn}}^t(A, B \Rightarrow C)$$

natural in A, C. In particular, for each innocent  $\sigma: A \otimes B \to C$ ,  $\Lambda(\sigma) = \mathsf{strat}(f)$ , where

$$f = \{ i_A * i_B s \in PP_{A \to B \Rightarrow C} \mid (i_A, i_B) s \in \mathsf{viewf}(\sigma) \}.$$

The inverse of  $\Lambda$  is defined is an analogous manner. We set  $ev_{A,B} = \Lambda^{-1}(id_{A\Rightarrow B})$ .

Thus, the functional part of  $|A_{cbv}|$  can be interpreted in  $S_{inn}$  using the same constructions as in [8]. Assignment, dereferencing and mkvar can in turn be modelled using the relevant (store-free) innocent strategies of [2]. Finally, the denotation of new x in M is obtained by using  $cell_{\beta}$  of Example 4.5. Let us write  $\|\cdot\cdot\cdot\|_{S}$  for the resultant semantic map.

**Proposition 4.25.** For any  $\mathsf{IA}_{\mathsf{cbv}}$ -term  $\Gamma \vdash M : \theta$ ,  $\llbracket \Gamma \vdash M : \theta \rrbracket_{\mathsf{S}}$  is an innocent S-strategy.

*Proof.* We present here the (inductive) constructions pertaining to variables,

- $\bullet \ \ \llbracket \Gamma \vdash \mathsf{new} \, x \, \mathsf{in} \, M : \beta \rrbracket_{\mathbf{S}} = \llbracket \Gamma \rrbracket \xrightarrow{\Lambda(\llbracket M \rrbracket_{\mathbf{S}})} \llbracket \mathsf{var} \rrbracket \Rightarrow \llbracket \beta \rrbracket \xrightarrow{\mathsf{cell}_{\beta}} \llbracket \beta \rrbracket$
- $\bullet \ \llbracket \Gamma \vdash M := N : \mathsf{unit} \rrbracket_{\mathbf{S}} = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket_{\mathbf{S}} ; \pi_2, \llbracket N \rrbracket_{\mathbf{S}} \rangle_l} (\mathbb{Z} \Rightarrow 1) \otimes \mathbb{Z} \xrightarrow{\mathsf{ev}} 1$
- $\bullet \ \ \llbracket\Gamma \vdash !M: \mathsf{int}\rrbracket_{\mathcal{S}} = \llbracket\Gamma\rrbracket \xrightarrow{\llbracket M\rrbracket_{\mathcal{S}}; \pi_1} 1 \Rightarrow \mathbb{Z} \xrightarrow{\cong} (1 \Rightarrow \mathbb{Z}) \otimes 1 \xrightarrow{\mathsf{ev}} \mathbb{Z}$
- $\bullet \ \ \llbracket\Gamma \vdash \mathsf{mkvar}(M,N) : \mathsf{var}\rrbracket_{\mathcal{S}} = \llbracket\Gamma\rrbracket \xrightarrow{\langle \llbracket M \rrbracket_{\mathcal{S}}, \llbracket N \rrbracket_{\mathcal{S}} \rangle_{l}} \llbracket\mathsf{var}\rrbracket$

and refer to [8] for the functional constructions and the treatment of fixpoints.

Our model of  $\mathsf{IA}_{\mathsf{cbv}}$ , based on innocent S-strategies, is closely related to one based on knowing strategies. First observe that by erasing storage annotations in an innocent S-strategy  $\sigma$  one obtains a knowing strategy (determinacy follows from the fact that stores in O-moves are uniquely determined). We shall refer to that knowing strategy by  $\mathsf{erase}(\sigma)$ . Next note that the (simpler) fully abstract model of RML from [2], based on knowing strategies, also yields a model of  $\mathsf{IA}_{\mathsf{cbv}}$ . Let us write  $\llbracket \cdots \rrbracket$  for this knowing-strategy semantics (cast in the Honda-Yoshida setting). Then we have:

**Lemma 4.26.** For any  $\mathsf{IA}_{\mathsf{cbv}}$ -term  $\Gamma \vdash M : \theta$ ,  $\llbracket \Gamma \vdash M : \theta \rrbracket = \mathsf{erase}(\llbracket \Gamma \vdash M : \theta \rrbracket_{\mathsf{S}})$ .

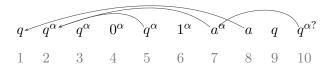
# 4.6. Block-innocent strategies.

**Definition 4.27.** A (knowing) strategy  $\sigma$ : A is **block-innocent** if  $\sigma = \text{erase}(\sigma')$  for some innocent S-strategy  $\sigma'$ .

**Example 4.28.** Let us revisit the two plays from the Introduction. The first one indeed comes from an innocent S-strategy (we reveal the stores below).

$$q = q^{(\alpha,0)} = q^{(\alpha,0)} = 1^{(\alpha,1)} = q^{(\alpha,1)} = 2^{(\alpha,2)} = a^{(\alpha,2)} = a^{(\alpha,2)} = a^{(\alpha,2)} = 0$$

For the second one to become innocent (in the setting with stores), a store with variable  $\alpha$ , say, would need to be introduced in the second move, to justify the different responses in moves 4 and 6. Then  $\alpha$  must also occur in the seventh move by JUST-O, but it must not occur in the eighth move by JUST-P (the PA clause). Hence, it will not be present in the ninth move by JUST-O. Consequently, the last move is bound to break either PREV-PQ(a) (if it contains  $\alpha$ ) or JUST-P (if it does not).



The knowledge that strategies determined by  $IA_{cbv}$  are block-innocent will be crucial in establishing a series of results in the following sections, where we shall galvanise the correspondence by investigating full abstraction (Corollary 6.3) and universality (Proposition 5.7).

# 5. Finitary definability and universality

In this section we demonstrate that the game model of  $IA_{cbv}$  is complete when restricted to finitary or recursively presentable innocent S-strategies. That is, every appropriately typed strategy of that kind is the denotation of some  $IA_{cbv}$  term. Although finitary innocent S-strategies are subsumed by recursively presentable ones, the method of proving completeness in the latter case is much more involved and we therefore prove the two results separately. The two completeness results are called finitary definability and universality respectively.

5.1. Finitary definability. We first formulate a decomposition lemma for innocent S-strategies which subsequently allows us to show the two results. The decomposition of innocent S-strategies follows the argument for call-by-value PCF [8] except for the case in which the strategy replies to Opponent's (unique) initial move with a question that introduces a new name (case 8 in the lemma below). Let us examine this case more closely, assuming  $\alpha$  to be the first variable from the non-empty store. In order to decompose the S-strategy, say  $\sigma$ , consider any P-view s in which  $\alpha$  occurs in the second move  $q_{\alpha}^{\Sigma_{\alpha}}$ . It turns out that s must be of the form  $q q_{\alpha}^{\Sigma_{\alpha}} s_{\alpha} s'$ , where a move  $m^{\Sigma}$  from s contains  $\alpha$  if, and only if, it is  $q_{\alpha}^{\Sigma_{\alpha}}$  or in  $s_{\alpha}$ . In addition, no justification pointers connect s' to  $q_{\alpha}^{\Sigma_{\alpha}} s_{\alpha}$ , because of the (Just-O) and (Close) conditions. This separation can be applied to decompose the view-function of  $\sigma$ . The  $s_{\alpha}$  segments, put together as a single S-strategy, can subsequently be dealt with in the style of factorisation arguments, which remove  $\alpha$  from moves at the cost of an additional var-component. Finally, to relate  $s_{\alpha}$ 's to the suitable s' one can use numerical codes for  $q_{\alpha}^{\Sigma_{\alpha}} s_{\alpha}$ . These ideas lie at the heart of the following result.

We fix a generic notation  $\lceil \_ \rceil$  for coding functions from enumerable sets to  $\omega$ . For example,  $\lceil i,j \rceil$  encodes the pair (i,j) as a number. There is an inherent abuse of notation in our coding notation which, nevertheless, we overlook for typographical economy. Moreover, we denote sequences  $\theta_1, \dots, \theta_n$  (where n may be left implicit) as  $\bar{\theta}$ . In such cases, we may write  $\bar{\theta}_i^j$   $(i \leq j)$  for subsequences  $\theta_i, \theta_{i+1}, \dots, \theta_j$ .

**Lemma 5.1** (Decomposition Lemma (DL)). Let  $\theta_1, \ldots, \theta_m, \delta$  be types of  $\mathsf{IA}_{\mathsf{cbv}}$ . Each innocent S-strategy  $\sigma : \llbracket \theta_1 \rrbracket \otimes \cdots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$  can be decomposed as follows.

1. If  $\theta_1, \ldots, \theta_m = \overline{\theta}_1^{m'}$ , int,  $\overline{\theta}_{m'+2}^m$  with none of  $\theta_1, \ldots, \theta_{m'}$  being int then:

$$\begin{split} \sigma &= \{ (\overline{*}_1^{m'}, i, \overline{q}_{m'+2}^m) \, s \mid (\overline{*}_1^{m'}, \overline{q}_{m'+2}^m) \, s \in \tau_i \} \\ \text{where } \tau_i &\triangleq [\![ \overline{\theta}_1^{m'} ]\!] \otimes [\![ \overline{\theta}_{m'+2}^m ]\!] \stackrel{\cong \, ; \, \mathrm{id} \otimes i \otimes \mathrm{id}}{\longrightarrow} [\![ \overline{\theta}_1^{m'} ]\!] \otimes \mathbb{Z} \otimes [\![ \overline{\theta}_{m'+2}^m ]\!] \stackrel{\sigma}{\to} [\![ \delta ]\!] \end{split}$$

- If none of  $\theta_1, \dots, \theta_m$  is int then one of the following is the case.
  - $-\overline{*}a \in \sigma$ , in which case either:
    - **2.**  $\delta = \text{unit and } \sigma = [\![\overline{\theta}]\!] \stackrel{!}{\to} 1$  (the unique total S-strategy into 1),
    - **3.**  $\delta = \text{int}, i \in \mathbb{Z} \text{ and } \sigma = \llbracket \overline{\theta} \rrbracket \xrightarrow{i} \mathbb{Z} \text{ (the unique total S-strategy into } \mathbb{Z} \text{ playing } i),$
    - **4.**  $\delta = \text{var} \text{ and } \sigma = \langle \sigma_1, \sigma_2 \rangle \text{ where } \sigma_i \stackrel{\triangle}{=} \sigma; \pi_i,$
    - **5.**  $\delta = \delta' \to \delta''$  and  $\sigma = \Lambda(\sigma')$  where  $\sigma' = \Lambda^{-1}(\sigma)$ , that is:

$$\sigma': \llbracket \overline{\theta} \rrbracket \otimes \llbracket \delta' \rrbracket \to \llbracket \delta'' \rrbracket \triangleq \mathsf{strat} \{ (\overline{*}, q_{\delta'}) \, s \mid \overline{*} \, a \, q_{\delta'} s \in \mathsf{viewf}(\sigma) \}$$

**6.**  $\overline{*}q \in \sigma$  with q played in some  $\theta_l = \text{var}$ , in which case  $\sigma \cong \sigma'$  where

$$\sigma': \llbracket \overline{\theta}_1^{l-1} \rrbracket \otimes \llbracket \overline{\theta}_{l+1}^m \rrbracket \otimes (1 \Rightarrow \mathbb{Z}) \otimes (\mathbb{Z} \Rightarrow 1) \longrightarrow \llbracket \delta \rrbracket$$

is obtained from  $\sigma$  by simply internally permuting and re-associating its initial moves.

7.  $\overline{*}q \in \sigma$  with q played in some  $\theta_l = \theta'_l \to \theta''_l$ , in which case

$$\sigma = \llbracket \overline{\theta} \rrbracket \xrightarrow{\langle \Lambda(\sigma''), \pi_l, \sigma' \rangle} (\llbracket \theta_l'' \rrbracket \Rightarrow \llbracket \delta \rrbracket) \otimes (\llbracket \theta_l' \rrbracket \Rightarrow \llbracket \theta_l'' \rrbracket) \otimes \llbracket \theta_l' \rrbracket \xrightarrow{\mathsf{id} \otimes \mathsf{ev}; \, \mathsf{ev}} \llbracket \delta \rrbracket$$

where, taking a to be q (seen as an answer):

$$\sigma': \llbracket \overline{\theta} \rrbracket \to \llbracket \theta'_l \rrbracket \triangleq \mathsf{strat}(\{\overline{*}\,a\} \cup \{\overline{*}\,a\,q'_l\,s \mid \overline{*}\,q\,q'_l\,s \in \mathsf{viewf}(\sigma) \land q'_l \in M_{\llbracket \theta'_l \rrbracket}\})$$

$$\sigma'': \llbracket \overline{\theta} \rrbracket \otimes \llbracket \theta''_l \rrbracket \to \llbracket \delta \rrbracket \triangleq \mathsf{strat}\{(\overline{*},q''_l)s \mid \overline{*}\,q\,a''_l\,s \in \mathsf{viewf}(\sigma) \land q''_l = a''_l\}$$

8. 
$$\overline{*}q^{\Sigma} \in \sigma$$
 with  $\operatorname{dom}(\Sigma) = \alpha \cdots$ , in which case 
$$\sigma = \llbracket \overline{\theta} \rrbracket \xrightarrow{\langle \Lambda(\sigma'), \operatorname{id} \rangle} (\llbracket \operatorname{var} \rrbracket \to \mathbb{Z}) \otimes \llbracket \overline{\theta} \rrbracket \xrightarrow{\operatorname{cell} \otimes \operatorname{id}} \mathbb{Z} \otimes \llbracket \overline{\theta} \rrbracket \xrightarrow{\sigma''} \llbracket \delta \rrbracket$$
 where: 
$$\sigma' : \llbracket \operatorname{var} \rrbracket \otimes \llbracket \overline{\theta} \rrbracket \to \mathbb{Z} \triangleq \operatorname{strat}(\{\psi(sm^T) \mid sm^T \in \operatorname{viewf}(\sigma) \wedge \alpha \in \nu(T)\}$$
 
$$\cup \{\psi(s)\lceil s\rceil \mid sm^T \in \operatorname{viewf}(\sigma) \wedge \alpha \in \nu(\operatorname{st}(s) \setminus T)\})$$
 
$$\psi(o^{(\alpha,i)::T}s) \triangleq o^T \operatorname{read}^T i^T \psi(s)$$
 
$$\psi(p^{(\alpha,i)::T}s) \triangleq \operatorname{write}(i)^T \operatorname{ok}^T p^T \psi(s)$$

*Proof.* Cases 1-7 are the standard ones that also occur for call-by-value PCF [8]. Case 8 is the most interesting one. Here we exploit the fact that, once  $\alpha$  occurs in the second move of a P-view, it appears continuously (in the P-view) until it is dropped by Proponent. Moreover, after  $\alpha$  has been dropped, no move will ever have a justification pointer to a move containing  $\alpha$  (because of *Just-O* and *Close*). The  $\sigma'$  strategy tracks the behaviour of  $\sigma$  until  $\alpha$  is dropped, at which point it returns the code of the current P-view.  $\sigma''$  in turn will take a code of such a P-view and will continue the play, as  $\sigma$  would. Additionally,  $\alpha$  is factored out in  $\sigma'$  through an extra  $\llbracket \text{var} \rrbracket$  arena, as in the factorisation argument of [3].

 $\sigma'': \mathbb{Z} \otimes \llbracket \overline{\theta} \rrbracket \to \llbracket \delta \rrbracket \triangleq \operatorname{strat} \{ (\lceil s \rceil, \overline{*}) \, m^T t \mid s \, m^T t \in \operatorname{viewf}(\sigma) \land \alpha \in \nu(\operatorname{st}(s) \setminus T) \}$ 

**Definition 5.2.** We call an innocent S-strategy  $\sigma$  *finitary* if its view-function is finite modulo name-permutation, that is, if the set

$$O(\mathsf{viewf}(\sigma)) = \{ \{ \pi \cdot s \mid \pi \in \mathsf{PERM} \} \mid s \in \mathsf{viewf}(s) \}$$

is finite. Accordingly, we call a block-innocent strategy finitary if the underlying innocent S-strategy is finitary.

**Proposition 5.3** (Finitary definability). Let  $\theta_1, \ldots, \theta_m, \delta$  be types of  $\mathsf{IA}_{\mathsf{cbv}}$ . For any finitary innocent S-strategy  $\sigma : \llbracket \theta_1 \rrbracket \otimes \cdots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$  there exists a term  $x_1 : \theta_1, \ldots, x_m : \theta_m \vdash M : \delta$  such that  $\sigma = \llbracket x_1 : \theta_1, \ldots, x_m : \theta_m \vdash M \rrbracket_{\mathsf{S}}$ .

*Proof.* We rely on the decomposition lemma to reduce the suitably calculated size of the strategy. The right measure is obtained by combining the size of the view-function quotiented by name-permutation and the maximum number of names occurring in a single P-view. It then suffices to establish that in each case the reconstruction of the original strategy can be supported by the syntax. For the first seven cases we can proceed as in [8]. For the eighth case, let  $y : \mathsf{var}, \Gamma \vdash M' : \mathsf{int}$  and  $x : \mathsf{int}, \Gamma \vdash M'' : \delta$  be the terms obtained by IH for  $\sigma'$  and  $\sigma''$  respectively. Then, in order to account for  $\sigma$ , one can take let  $x = (\mathsf{new}\,y\,\mathsf{in}\,M')$  in M''.

5.2. Universality. We now proceed with the universality result. In the rest of this section we closely follow the presentation of [1]; the reader is referred thereto for a more detailed exposition of the background material. Let us fix an enumeration of partial recursive functions such that  $\phi_n$  is the *n*-th partial recursive function.

The universality result concerns innocent S-strategies. Recall that S-strategies and their view-functions are saturated under name-permutations and, in fact, view-functions only become functions after nominal quotienting. To represent them we introduce an encoding

scheme that is not dependent on names. Let us define a function eff which converts S-plays to plays in which moves are attached with lists of integers:

$$\operatorname{eff}(so^{\varSigma}) \triangleq \operatorname{eff}(s)o^{\pi_2(\varSigma)}\,, \quad \operatorname{eff}(sm^{\varSigma}p^T) \triangleq \operatorname{eff}(sm^{\varSigma})p^{\pi_2(T),|T \setminus \varSigma|}\,.$$

Thus, from an O-move  $o^{\Sigma}$  we only keep the values stored in  $\Sigma$ , whereas in a P-move  $p^T$  we keep the values of T and a number indicating how many of the names of T are freshly introduced. Because of the conditions on stores that S-plays satisfy, eff maps two S-plays to the same encoding if, and only if, they are nominally equivalent. In the sequel we assume that S-plays are given using the encoding above.

For the rest of the section we assume that  $PP_A$  is recursively enumerable, which is clearly the case for denotable prearenas.

**Definition 5.4.** A subset of  $PP_A$  (for instance, a strategy or a view-function) will be called **recursively presentable** if it is a recursively enumerable subset of  $PP_A$ . A block-innocent strategy will be called recursively presentable if the underlying innocent S-strategy is recursively presentable.

It follows that an innocent S-strategy  $\sigma$  is recursively presentable if, and only if, its view-function is. We therefore encode an innocent S-strategy  $\sigma$  by  $\lceil \sigma \rceil$ , where the latter is the index n such that  $\mathsf{viewf}(\sigma) = \phi_n$  (with  $\phi_n$  seen as a partial function from codes of P-views of S-plays to codes of P-moves).

We want to show that any recursively presentable innocent S-strategy is definable by an IA<sub>cbv</sub>-term. The result will be proved by constructing a term that accepts the code of a given strategy, examines its initial behaviour, mimics it and, after a subsequent O-move, is ready to explore the relevant component of the decomposition. Observe that if we start from a strategy on  $\llbracket \theta_1, \cdots, \theta_k \vdash \theta \rrbracket$  the decomposition will lead us to consider strategies on  $\llbracket \theta'_1, \cdots, \theta'_l \vdash \theta' \rrbracket$ , where each  $\theta'_i$  (as well as  $\theta'$ ) is a subtype of some  $\theta_j$  or  $\theta$ . Since the given strategy will in general be infinite, repeated applications of the Decomposition Lemma will mean that l is unbounded. To keep track of the current component we will thus need to be able to represent unbounded lists of variables whose types are subtypes of  $\theta_1, \cdots, \theta_k, \theta$ . This issue is tackled next.

**List contexts.** We say that a set of types T is **closed** if whenever  $\theta \in T$  and  $\delta$  is a subtype of  $\theta$  then  $\delta \in T$ . For the rest of this section let us fix a closed finite set of types T and an ordering of T, say  $T = T_0, T_1, \ldots, T_n$ , such that  $T_0 = \text{unit}$ ,  $T_1 = \text{int}$ ,  $T_2 = \text{var}$ ,  $T_3 = \text{unit} \rightarrow \text{int}$  and  $T_4 = \text{int} \rightarrow \text{unit}$ .

For each i, we encode lists of type  $T_i$  as products int  $\times$  (int  $\to T_i$ ). In particular, we use the notation

$$z: \mathsf{List}(T_i), \; \Gamma \vdash M: \delta$$

as a shorthand for

$$z^L:\mathsf{int},\,z^R:\mathsf{int}\to T_i,\,\Gamma\vdash M:\delta$$

Thus,  $z^L$  represents the length of the represented list. For each  $1 \le i \le z^L$ , the value of the *i*-th element in the list is represented by  $z^Ri$ . The list can be shortened by simply 'reducing'  $z^L$ . For example, for a term  $z : \text{List}(T_i)$ ,  $\Gamma \vdash M : \delta$  we can form

$$z: \mathsf{List}(T_i), \ \Gamma \vdash \mathsf{let} \ z^L = z^L - 1 \ \mathsf{in} \ M: \delta.$$

Note that, although the notation seems to suggest differently, the above is unrelated to variable assignment: it stands for  $(\lambda z^L.M)(z^L-1)$ . A finer removal of a list element is

executed as follows. For a term  $z : \mathsf{List}(T_i), \Gamma \vdash M : \delta$  and an index j, we define the term

$$z: \mathsf{List}(T_i), \, \Gamma \vdash \mathsf{remove} \, (z,j) \, \mathsf{in} \, \, M: \delta$$

to be

$$z : \mathsf{List}(T_i), \ \Gamma \vdash (\lambda z^L \cdot \lambda z^R \cdot M)(z^L - 1)(\lambda x. \ \text{if} \ x < j \ \text{then} \ z^R x \ \text{else} \ z^R(x + 1)) : \delta.$$

A list can be extended as follows. For terms  $z : \mathsf{List}(T_i), \Gamma \vdash M : \delta, N : T_i$  and an index j we define the term

$$z: \mathsf{List}(T_i), \; \Gamma \vdash \mathsf{insert} \; (z, j, N) \; \mathsf{in} \; M: \delta$$

to be:

 $z: \operatorname{List}(T_i), \ \Gamma \vdash (\lambda z^L.\lambda z^R.M)(z^L+1)(\lambda x. \ \text{if} \ x < j \ \text{then} \ z^Rx \ \text{else} \ \text{if} \ x = j \ \text{then} \ N \ \text{else} \ z^R(x-1)): \delta$  Let us use the shorthands

$$let z = cons N z in M let z = snoc z N in M$$

for insert (z, 1, N) in M and insert  $(z, z_L + 1, N)$  in M respectively (that is, Hextend inserts at the head of lists and Textend at the tail).

We can define an (effective) indexing function indx which, for any sequence (possibly with repetitions)  $\theta_1, \ldots, \theta_m$  of types from T, returns a pair of numbers (i, j) such that  $\theta_m = T_i$  and there are j occurrences of  $T_i$  in  $\theta_1, \ldots, \theta_m$ .

Suppose now we have such a sequence  $\overline{\theta}$  and a term  $z_0: \mathsf{List}(T_0), z_1: \mathsf{List}(T_1), \ldots, z_n: \mathsf{List}(T_n) \vdash M: \delta.$  We can  $de\text{-}index\ M$  with respect to  $\overline{\theta}$ , obtaining the term  $x_1: \theta_1, \ldots, x_m: \theta_m \vdash \mathsf{deindx}\ M: \delta$ , defined as

$$\operatorname{deindx}\, M \triangleq \operatorname{let}\, \overline{z=\bot} \,\operatorname{in}\, (\operatorname{let}\, z_{l_m} = \operatorname{cons}\, x_m\, z_{l_m} \,\operatorname{in}\, (\ldots \,(\operatorname{let}\, z_{l_1} = \operatorname{cons}\, x_1\, z_{l_1} \,\operatorname{in}\, M))),$$

where

let 
$$\overline{z=\perp}$$
 in  $N \triangleq \text{let } z_0^L = 0, z_0^R = \lambda x.\Omega$  in  $(\dots (\text{let } z_n^L = 0, z_n^R = \lambda x.\Omega \text{ in } N))$ 

and, for each  $1 \le i \le m$ ,  $\theta_i = T_{l_i}$ . Note that the extensions above are executed from left to right so, in particular,  $x_1$  will be related to  $z_{l_1}^R 1$ .

**Universal terms.** Given a closed set of types T, the way we prove universality is by constructing for each  $\delta \in T$  a universal term  $z_0 : \mathsf{List}(T_0), \ldots, z_n : \mathsf{List}(T_n) \vdash F_\delta : \mathsf{int} \to \delta$  such that, for every sequence  $\theta_1, \ldots, \theta_m$  from T and recursively presentable S-strategy  $\sigma : \llbracket \theta_1 \rrbracket \otimes \cdots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$ ,

$$\sigma = [\operatorname{deindx} (F_{\delta}[\sigma])]_{S}.$$

We first need to make sure that we can move inside the Decomposition Lemma effectively, i.e. that the passage from the code of the original strategy to the code of the components is effective and that the case which applies can also be computed from the index of the original strategy. Here is such a recursive version of the Decomposition Lemma (for types in T).

# **Lemma 5.5.** There are partial recursive functions

$$D, H : \omega \rightharpoonup \omega$$
 and  $B : \omega \times \omega \rightharpoonup \omega$ 

such that, for any  $\theta_1, \dots, \theta_m, \delta \in T$  and recursively presentable S-strategy  $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$ ,

$$\begin{split} \mathcal{G}_m \rrbracket \to \llbracket \delta \rrbracket, \\ D \lceil \sigma \rceil &= \begin{cases} i & \text{if } \sigma \text{ falls within the } i\text{-th case of DL} \\ \bot & \text{otherwise} \end{cases} \\ B (\lceil \sigma \rceil, i) &= \begin{cases} \lceil \tau_i \rceil & \text{if } \sigma \text{ and } \tau_i \text{ are related as in first case of DL} \\ \bot & \text{otherwise} \end{cases} \\ \\ H \lceil \sigma \rceil &= \begin{cases} i & \text{if } \sigma, i \text{ are related as in third case of DL} \\ \lceil \sigma_1 \rceil, \lceil \sigma_2 \rceil \rceil & \text{if } \sigma, \sigma_1, \sigma_2 \text{ are related as in fourth case of DL} \\ \lceil \sigma' \rceil & \text{if } \sigma, \sigma' \text{ are related as in fifth case of DL} \\ \lceil i, \lceil \sigma' \rceil \rceil & \text{if } \sigma, \llbracket \theta_l \rrbracket, \sigma' \text{ are related as in sixth case of DL} \\ & \text{and indx}(\theta_1, \dots, \theta_l) = (2, i) \\ \lceil i_1, i_2, \lceil \sigma' \rceil, \lceil \sigma'' \rceil \rceil & \text{if } \sigma, \llbracket \theta_l \rrbracket, \sigma', \sigma'' \text{ are related as in seventh case of DL} \\ & \text{and indx}(\theta_1, \dots, \theta_l) = (i_1, i_2) \\ \lceil \lceil \sigma' \rceil, \lceil \sigma'' \rceil \rceil & \text{if } \sigma, \sigma', \sigma'' \text{ are related as in eighth case of DL} \end{cases} \end{split}$$

*Proof.* We assume that the type of  $\sigma$  is represented in  $\lceil \sigma \rceil$  and can be effectively decoded by D, B and H.  $D\lceil \sigma \rceil$  returns 1 if any of the  $\theta_i$ 's is int, otherwise it applies  $\phi_{\lceil \sigma \rceil}$  to the unique initial move of  $\lceil \overline{\theta} \rceil$  and returns the number corresponding to the result. For B, given  $\lceil \sigma \rceil, i$ , membership in  $\tau_i$  is checked as follows. For any (P-view) S-play s, we add i to its initial move and check whether the resulting S-play is a member of  $\sigma$ . Thus we obtain  $\phi_n$  such that  $s \mapsto \phi_n(s, \lceil \sigma \rceil, i)$  is the characteristic function of  $\tau_i$ . By an application of the S-m-n theorem we obtain  $\lceil \tau_i \rceil$ . For H we argue along the same lines.

Since (call-by-value) PCF is Turing complete, there are closed PCF-terms  $\tilde{D}, \tilde{H}: \operatorname{int} \to \operatorname{int}$  and  $\tilde{B}: \operatorname{int} \to \operatorname{int} \to \operatorname{int}$  that represent each of the above functions with plays of the form q\*nf(n) or q\*m\*nf(m,n). The terms will be used inside the universal term, which will be constructed by mutual recursion (there are standard techniques to recast such definitions in PCF). Let us write  $\theta = T_{l(\theta)} \to T_{r(\theta)}$  whenever  $\theta \in T$  is of arrow type (so  $l, r: T \to \{0, \ldots, n\}$ ).

**Definition 5.6.** For each  $\delta \in T$  we define terms

$$z_0: \mathsf{List}(T_0), \ldots, z_n: \mathsf{List}(T_n) \vdash F_\delta: \mathsf{int} \to \delta$$

by mutual recursion as follows.

```
F_{\delta} \triangleq \lambda k^{\text{int}}. if z_1^L \neq 0 then let x = z_1^R 1 in remove (z_1, 1) in F_{\delta}(\tilde{B} k x)
                          else case (\tilde{D} k) of
                                    2: skip
                                    3: \tilde{H}k
                                    4: let [k_1, k_2] = \tilde{H}k in \mathsf{mkvar}(F_{\mathsf{unit} \to \mathsf{int}} k_1, F_{\mathsf{int} \to \mathsf{unit}} k_2)
                                    5: \ \lambda y^{T_{l(\delta)}}. \, \mathrm{let} \ z_{l(\delta)} = \mathrm{snoc} \, z_{l(\delta)} \, y \, \, \mathrm{in} \, \, F_{T_{r(\delta)}}(\tilde{H} \, k)
                                    6: \operatorname{let} [i, k] = \tilde{H}k \operatorname{in}
                                              let z_3 = \operatorname{snoc} z_3 \lambda x^{\operatorname{unit}} . !(z_2^R i) in
                                                let z_4 = \operatorname{snoc} z_4 \lambda x^{\operatorname{int}} \cdot (z_2^R i) := x in remove (z_2, i) in F_{\delta} k
                                    7: let \lceil i_1, i_2, k_1, k_2 \rceil = \tilde{H}k in case i_1 of
                                              1: ...
                                             j: \text{ let } z_{r(T_i)} = \operatorname{snoc} z_{r(T_i)} (z_i^R i_2) (F_{T_{l(T_i)}} k_1) \text{ in } F_{\delta} k_2
                                             n: \ldots
                                    8: let \lceil k_1, k_2 \rceil = \tilde{H}k in
                                             let z_1 = \cos(\text{new } x \text{ in let } z_4 = \cos x z_4 \text{ in } F_{\text{int}} k_1) z_1 \text{ in } F_{\delta} k_2
                                    otherwise : \Omega
```

The construction of  $F_{\delta}$  follows closely the decomposition of  $\sigma$  according to the Decomposition Lemma. In particular, on receiving  $\lceil \sigma \rceil$ , the term decides, using the functions B and D of Lemma 5.5, to which branch of DL  $\sigma$  can be matched. Some branches decompose  $\sigma$  into further strategies, in which case  $F_{\delta}$  will recursively call some  $F_{\delta'}$  to simulate the rest of the strategy. The use of lists in contexts guarantees that such a call is indeed recursive: F is only parameterised by the output type  $\delta'$ , and each such  $\delta'$  is in T.

**Proposition 5.7** (Universality). For every  $\theta_1, \ldots, \theta_m, \delta \in T$  and recursively presentable innocent S-strategy  $\sigma : \llbracket \theta_1 \rrbracket \otimes \cdots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$ ,  $\sigma = \llbracket \operatorname{deindx} (F_{\delta} \lceil \sigma \rceil) \rrbracket_{S}$ .

*Proof.* Suppose 
$$F_{\delta}$$
 receives  $\lceil \sigma \rceil$  in its input  $k$ , where  $\sigma : \llbracket \theta_1 \rrbracket \otimes \cdots \otimes \llbracket \theta_m \rrbracket \to \llbracket \delta \rrbracket$ . Then if  $z_1^L \neq 0$  then let  $x = z_1^R 1$  in remove  $(z_1, 1)$  in  $F_{\delta}(\tilde{B} k x)$ 

recognizes the first branch of the DL. Recall that  $T_1 = \text{int}$ , so  $z_1 : \text{List(int)}$ , and therefore  $z_1^L \neq 0$  holds iff there is some  $\theta_i = \text{int}$ . If this is so, then  $F_\delta$  needs to return a term corresponding to the strategy instantiated with the leftmost element in the list  $z_1$ . This is achieved by first applying  $\tilde{B}$  to k,  $(z_1^R 1)$  to obtain  $\lceil \tau_i \rceil$  and applying  $F_\delta$  to it.

If  $z_1^L = 0$ ,  $F_{\delta}$  will call  $\tilde{D}$  on  $\lceil \sigma \rceil$ , which will return the number of the case from DL (2-8) that applies to  $\sigma$ . Subsequently,  $F_{\delta}$  will proceed to a case analysis. Below we examine two cases in detail.

- 5:  $\sigma$  is the currying of  $\sigma'$ , so  $F_{\delta}$  should return ' $\lambda y. F[\sigma'(y)]$ '. Now,  $\sigma'$  is  $F_{T_{r(\delta)}}[\sigma']$ , i.e.  $F_{T_{r(\delta)}}(\tilde{H}\,k)$ , where  $\delta = T_{l(\delta)} \Rightarrow T_{r(\delta)}$ . In order to preserve typability, we need to add the abstracted variable to the context of  $F_{T_{r(\delta)}}(\tilde{H}\,k)$ , which is what Textend  $z_{l(\delta)}$  with y achieves.
- 8:  $\sigma$  introduces some fresh name and decomposes to  $\sigma'$ ,  $\sigma''$  as in the Decomposition Lemma. Hence,  $F_{\delta}$  should return 'let  $y = (\text{new } x \text{ in } F\lceil \sigma'(x) \rceil)$  in  $F\lceil \sigma''(y) \rceil$ ', which is exactly what the code achieves.

The other cases are similar.

**Remark 5.8.** It is worth noting that the universality result for innocent S-strategies implies an analogous result for innocent strategies and PCF. Thanks to call-by-value, the result is actually sharper than the universality results of [1, 10], which had to be proved "up to observational equivalence". This was due to the fact that partial recursive functions could not always be represented in the canonical way (i.e. by terms for which the corresponding strategy contained plays of the form qqnf(n)). This is no longer the case under the call-by-value regime, where each partially recursive function f can be coded by a term whose denotation will be the strategy based on plays of the shape nf(n).

#### 6. From omniscience to innocence

In Section 2 we introduced the three languages:  $PCF^+$ ,  $IA_{cbv}$  and RML, interpreted respectively by innocent, block-innocent and knowing strategies. Let A be a prearena. We write  $\mathcal{I}_A$ ,  $\mathcal{B}_A$  and  $\mathcal{K}_A$  for the corresponding classes of (store-free) strategies in A. Obviously,  $\mathcal{I}_A \subseteq \mathcal{B}_A \subseteq \mathcal{K}_A$ . Next we shall study type-theoretic conditions under which one kind of strategy collapses to another. Thanks to universality results, this corresponds to the existence of an equivalent program in a weaker language.

**Theorem 6.1.** Let  $A = [\theta_1, \dots, \theta_n \vdash \theta \rightarrow \theta']$ . Then  $\mathcal{B}_A \subsetneq \mathcal{K}_A$ .

*Proof.* Observe that there exist moves  $q_0, a_0, q_1, a_1$  such that  $q_0 \vdash_A a_0 \vdash_A q_1 \vdash_A a_1$  and consider  $\sigma = \{\epsilon, q_0 a_0, q_0 a_0 q_1 a_1\}$ , i.e.  $\sigma$  has no response at  $q_0 a_0 q_1 a_1 q_1$ . Then  $\sigma \in \mathcal{K}_A \setminus \mathcal{B}_A$ . It is worth remarking that a strategy of the above kind denotes the RML-term  $\vdash$  let  $v = \text{ref in } \lambda x^{\text{unit}}$ . (if !v then  $\Omega$  else v := !v + 1): unit  $\to$  unit.

Theorem 6.1 confirms that, in general, block structure restricts expressivity. However, the next result shows this not to be the case for open terms of base type.

**Theorem 6.2.** Let 
$$A = [\theta_1, \dots, \theta_n \vdash \beta]$$
. Then  $\mathcal{B}_A = \mathcal{K}_A$ .

*Proof.* Observe that any knowing strategy for A becomes block-innocent if in the second-move P introduces a store with one variable that keeps track of the history of play (this is reminiscent of the factorization arguments in game semantics). The variable should be removed from the store by P only when he plays an answer to the initial question.

By universality, we can conclude that each RML-term of base type is equivalent to an  $IA_{cbv}$ -term. Since contexts used for testing equivalence are exactly of this kind, we obtain the following corollaries. The first one amounts to saying that RML is a conservative extension of  $IA_{cbv}$ . The second one states that block-structured contexts suffice to distinguish terms that might use scope extrusion.

Corollary 6.3. For any  $\mathsf{IA}_{\mathsf{cbv}}$ -terms  $\Gamma \vdash M_1, M_2 : \theta$  and  $\mathsf{RML}$ -terms  $\Gamma \vdash N_1, N_2 : \theta$ :

- $\Gamma \vdash M_1 \cong_{\mathsf{RML}} M_2$  if, and only if,  $\Gamma \vdash M_1 \cong_{\mathsf{IA}_{\mathsf{cbv}}} M_2$ ;
- $\Gamma \vdash N_1 \cong_{\mathsf{RML}} N_2$  if, and only if,  $\Gamma \vdash N_1 \cong_{\mathsf{IA}_{\mathsf{cbv}}} N_2$ .

Now we investigate the boundary between block structure and lack of state.

**Lemma 6.4.** Let A be a prearena such that each question enables an answer  $^6$ . The following conditions are equivalent.

- (1)  $\mathcal{B}_A \subseteq \mathcal{I}_A$ .
- (2) No O-question is enabled by a P-question:  $m \vdash_A q_O$  implies  $\lambda_A(m) = PA$ .
- (3) Store content of O-questions is trivial:  $sq_O^{\Sigma} \in SP_A$  implies dom  $\Sigma = \emptyset$ .

Proof.

 $(1 \Rightarrow 2)$  We prove the contrapositive. Assume that there exists a P-question  $q_P$  and an O-question  $q_O$  such that  $q_P \vdash_A q_O$ . Let s be a chain of hereditary enablers of  $q_P$  (starting from an initial move) augmented with pointers from non-initial moves to the respective preceding moves. Then

$$s \stackrel{\checkmark}{q_P} \stackrel{(x,0)}{q_O} \stackrel{(x,0)}{q_O} \stackrel{(x,0)}{q_P} \stackrel{(x,1)}{q_O} \stackrel{(x,1)}{q_P}$$

defines a block-innocent strategy that is not innocent.

- $(2 \Rightarrow 3)$  Suppose no P-question enables an O-question in A and let  $sq_O^X \in SP_A$ . Then the sequence of hereditary justifiers of  $q_O$  in s, in order of their occurrence in s, must have the form  $(q_O a_P)^*$ . Consequently, none of the stores involved can be non-empty, so X must be empty too.
- $(3 \Rightarrow 1)$  We observe that  $s_1 p^{X_p} s_2 o^{X_o} \in SP_A$ , where p justifies o, implies  $X_o = X_p$ . Note that, in presence of block innocence, this implies innocence because the store content of O-moves can be reconstructed uniquely from the P-view. Thus, it suffices to prove our observation correct.
  - If o is a question, we simply use our assumption: then we must have  $X_o = \emptyset$  and, because  $\operatorname{\mathsf{dom}} X_o = \operatorname{\mathsf{dom}} X_p$ , we can conclude  $X_p = \emptyset$ .
  - If o is an answer then p must be a question. We claim that no store in  $s_2$  can contain variables from  $\operatorname{dom} X_o = \operatorname{dom} X_p$ . Suppose this is not the case and there is such an occurrence. Then the earliest such occurrence must be part of an O-move. This move cannot be a question due to our assumption, so it is an answer move. By the bracketing condition, this must be an answer that an earlier P-question played after p. Moreover, the store accompanying that question must also contain a variable from  $\operatorname{dom} X_o = \operatorname{dom} X_p$  contradicting our choice of the earliest occurrence.

<sup>&</sup>lt;sup>6</sup>All denotable prearenas enjoy this property.

Thanks to the following lemma we will be able to determine precisely at which types block-innocence implies innocence.

**Lemma 6.5.**  $\llbracket \theta_1, \dots, \theta_n \vdash \theta \rrbracket$  satisfies condition 2. of Lemma 6.4 iff  $\operatorname{ord}(\theta_i) \leq 1$   $(i = 1, \dots, n)$  and  $\operatorname{ord}(\theta) \leq 2$ .

Consequently, second-order  $\mathsf{IA}_{\mathsf{cbv}}$ -terms always have purely functional equivalents. Finally, we can pinpoint the types at which strategies are bound to be innocent: it suffices to combine the previous findings.

**Theorem 6.6.** Let 
$$A = [\![\theta_1, \cdots, \theta_n \vdash \theta]\!]$$
. Then  $\mathcal{K}_A = \mathcal{I}_A$  iff  $\operatorname{ord}(\theta_i) \leq 1$   $(i = 1, \cdots, n)$  and  $\operatorname{ord}(\theta) = 0$ .

In the next section we demonstrate that the gap in expressivity between  $\mathcal{K}_A$  and  $\mathcal{B}_A$  also bears practical consequences. The undecidable equivalence problem for second-order finitary RML becomes decidable in second-order finitary IA<sub>cbv</sub> (as well as at some third-order types).

# 7. Decidability of a finitary fragment of IA<sub>cbv</sub>

In order to prove program equivalence decidable, we restrict the base datatype of integers to the finite segment  $\{0, \dots, N\}$  (N > 0) and replace recursive definitions (Y(M)) with looping (while M do N). Let us call the resultant language  $IA_{\circlearrowright}$ . Our decidability result will hold for a subset  $IA_{\circlearrowright}^{2+}$  of  $IA_{\circlearrowright}$ , in which type order is restricted.  $IA_{\circlearrowright}^{2+}$  will reside inside the third-order fragment of  $IA_{\circlearrowright}$  and contain its second-order fragment. Note that the second-order fragment of similarly restricted RML is known be undecidable (even without loops) [15].

The decidability of program equivalence in  $IA_{\bigcirc}^{2+}$  will be shown by translating terms to regular languages representing the corresponding *block-innocent* strategies. We stress that we are *not* going to work with the induced S-plays. Nevertheless, the translation will rely crucially on insights gleaned from the semantics with explicit stores. In particular, we shall take advantage of the uniformity inherent in block innocence to represent only subsets of the strategies in order to overcome technical problems presented by pointers. We discuss the issue next.

7.1. **Pointer-related issues.** Pointers from answer-moves need not be represented at all, because they are uniquely reconstructible through the well-bracketing condition. However, this need not apply to pointers from questions. The most obvious way to represent them is to decorate moves with integers that encode the distance from the target in some way. Unfortunately, there are scenarios in which the distance can grow arbitrarily.

Consider, for instance, the prearena  $A = \llbracket \theta \vdash \theta_1 \to \ldots \to \theta_k \to \beta \rrbracket$ . Due to the presence of the k arrows on the right-hand side we obtain chains of enablers  $q_0 \vdash a_0 \vdash \cdots \vdash q_k \vdash a_k$ , where  $q_0$  is initial and each  $q_i$   $(i = 1, \cdots, k)$  is initial in  $\llbracket \theta_i \rrbracket$ . We shall call the moves *spinal*. Observe that plays in A can have the following shape

$$q_0 \cdots a_0 q_1 \cdots a_1 q_1 \cdots a_1 q_1 \cdots a_1 q_2$$

and any of the occurrences of  $a_1$  could be used to justify  $q_2$ , thus creating several different options for justifying  $q_2$ . If we consider S-plays for A, Definition 4.2 implies that none of the moves  $q_i$ ,  $a_i$  will ever carry a non-empty store. Consequently, whenever a play of the above

kind comes from a block-innocent strategy, its behaviour in the  $q_1 \cdots a_1$  segments will not depend on that in the other  $q_1 \cdots a_1$  segments. Thus, in order to explore exhaustively the range of behaviours offered by a block-innocent strategy (so as to compare them reliably), it suffices to restrict the number of  $q_1$ 's to 1. Next, under the assumption that  $q_1$  occurs only once, one can repeat the same argument for  $q_2$  to conclude that a single occurrence of  $q_2$  will suffice, and so on. Altogether this yields the following lemma. Note that, due to Visibility, insisting on the presence of a unique copy of  $q_1, \cdots, q_k$  in a play amounts to asking that each  $q_i$  be preceded by  $a_{i-1}$ .

**Lemma 7.1.** Call a play *spinal* if each spinal question  $q_i$   $(0 < i \le k)$  occurring in it is the immediate successor of  $a_{i-1}$ . Let  $P_A^{sp}$  be the set of spinal plays of A. Let  $\sigma, \tau : A$  be block-innocent strategies. Then  $\sigma \cap P_A^{sp} = \tau \cap P_A^{sp}$  implies  $\sigma = \tau$ .

Hence, for the purpose of checking program equivalence, it suffices to compare the induced sets of *spinal* complete plays. Moreover, the pointer-related problems discussed above will not arise.

Now that we have dealt with one challenge, let us introduce another one, which cannot be overcome so easily. Consider the prearena  $[(\theta_1 \to \theta_2 \to \theta_3) \to \theta_4 \vdash \theta]$  and the enabling sequence  $q_0 \vdash q_1 \vdash q_2 \vdash a_2 \vdash q_3$  it contains. Now consider the plays  $q_0q_1(q_2a_2)^jq_3$ , where  $j \geq 0$ . Again, to represent the pointer from  $q_3$  to one of the j occurrences of  $a_2$ , one would need an unbounded number of indices. This time it is not sufficient to restrict j to 1, because the behaviour need not be uniform after each  $q_2$  (this is because in the setting with stores a non-empty store can be introduced as soon as in the second move  $q_1$ ). To see that the concern is real, consider the term  $f: (\text{unit} \to \text{unit} \to \text{unit}) \to \text{unit} \vdash \text{new } x \text{ in } f(\lambda y^{\text{unit}}, \dots, \lambda z^{\text{unit}}, \dots): \text{unit}, \text{ where } (\dots) \text{ contain some code inspecting and changing the value of } x.$ 

This leads us to introduce  $\mathsf{IA}^{2+}_{\circlearrowright}$  via a type system that will not generate the configuration just discussed. Another restriction is to omit third-order types in the context, as they lead beyond the realm of regular languages (cf.  $f:((\mathsf{unit}\to\mathsf{unit})\to\mathsf{unit})\to\mathsf{unit})\to\mathsf{unit}$ )  $f(\lambda g^{\mathsf{unit}\to\mathsf{unit}}.g())$ . Since var leads to identical problems as  $\mathsf{unit}\to\mathsf{unit}$ , we restrict its use accordingly.

7.2. 
$$IA_{\circlearrowright}^{2+}$$
.

**Definition 7.2.**  $\mathsf{IA}^{2+}_{\bigcirc}$  consists of  $\mathsf{IA}_{\bigcirc}$ -terms whose typing derivations rely solely on typing judgments of the shape  $x_1: ctype_1, \cdots, x_n: ctype_n \vdash M: ttype$ , where ctype and ttype are defined by the grammar below.

A lot of pointers from questions become uniquely determined in strategies representing  $\mathsf{IA}^{2+}_{\circlearrowleft}$  terms, namely, all pointers from any O-questions and all pointers from P-questions to O-questions.

**Lemma 7.3.** Let  $A = [[ctype_1, \cdots, ctype_n \vdash ttype]]$  and  $s_1, s_2$  be spinal plays of A that are equal after all pointers from O-questions and all pointers from P-questions to O-questions have been erased. Then  $s_1 = s_2$ .

*Proof.* Observe that whenever a P-question is enabled by an O-question in the prearenas under consideration, the O-question must be spinal. Hence, because both  $s_1$  and  $s_2$  are spinal, all such O-questions will occur only once, so pointers from P-questions to O-questions are uniquely reconstructible.

Now let us consider O-questions. Observe that, due to restrictions on the type system of  $\mathsf{IA}^{2+}_{\circlearrowright}$ , whenever an O-question is justified by a P-answer, both will be spinal. Hence only one copy of each can occur in a spinal position, making pointer reconstruction unambiguous. Finally, we tackle the case of O-questions justified by P-questions.

• If q comes from a type of the context then, due to the shape of types involved, any sequence of hereditary enablers of q must be of the form  $q'q'_1a'_1\cdots q'_ja'_jq'_{j+1}q$ , where q' is initial and each of the moves listed enables the following one. If a move m enables q hereditarily, let us define its degree as the distance from the initial move in the sequence above (this definition is independent of the actual choice of the chain of enablers; the degree of  $q'_i$  is 2i - 1, that of  $a'_i$  is 2i).

By induction on move-degree we show that in any O-view only one move of a given degree can be present, if at all. By visibility this makes pointer reconstruction unambiguous.

- By definition of a play q' can occur in an O-view only once. Whenever  $q'_1$  is present in an O-view, it must preceded by an initial move, so its position in an O-view is uniquely determined (always second).
- Since  $q'_i$  occurs only once in an O-view, so does  $a'_i$  (questions can be answered only once). Hence,  $q'_{i+1}$  can also occur only once, because each occurrence must be preceded by  $a'_i$ .

Consequently, any move of degree 2j + 1  $(q'_{j+1})$  can only occur once in an O-view and thus the pointer from q can be reconstructed uniquely.

• If q originated from the type on the right-hand side of the typing judgment, we can repeat the reasoning above. The only difference is that the sequences of enablers are now of the form

$$q_0 a_0 \cdots q_k a_k q' q_1' a_1' \cdots q_j' a_j' q_{j+1}' q$$

where  $q_i, a_i \ (i = 0, \dots, k)$  are spinal. Then the base case of the induction (q') follows from the fact that we are dealing with spinal plays.

Thus, the only pointers that need to be accounted for are those from P-questions to O-answers. Here is the simplest scenario illustrating that they can be ambiguous. Consider the terms

$$f: \mathsf{unit} \to \mathsf{unit} \to \mathsf{unit} \vdash \mathsf{let} \ q_1 = f() \ \mathsf{in} \ (\mathsf{let} \ q_2 = f() \ \mathsf{in} \ q_i()) : \mathsf{unit}$$

where i = 1, 2. They lead to the following plays, respectively for i = 1 and i = 2, which are equal up to pointers from P-questions to O-answers.

$$q_0$$
  $q_1$   $q_1$   $q_1$   $q_2$   $q_0$   $q_1$   $q_1$   $q_1$   $q_1$   $q_2$ 

Remark 7.4. In the conference version of the paper [17] we suggested that justification pointers of the above kind be represented with numerical indices encoding the target of the pointer inside the current P-view. More precisely, one could enumerate (starting from 0) all question-enabling O-answers in the P-view. Then pointers from P-questions to O-answers could be encoded by decorating the P-question with the index of the O-answer. The plays

above could then be coded as  $q_0q_1a_1q_1a_1q_2^0$  and  $q_0q_1a_1q_1a_1q_2^1$  respectively. Unfortunately, there exists terms that generate plays where such indices would be unbounded, such as

let 
$$g_1 = f()$$
 in (while  $h()$  do let  $g_2 = f()$  in  $g_2()$ );  $g_1()$ : unit

where  $f: \mathsf{unit} \to \mathsf{unit} \to \mathsf{unit}$  and  $h: \mathsf{unit} \to \mathsf{int}$ . Because the number of loop iterations is unrestricted, the number needed to represent the justification pointer corresponding to the rightmost occurrence of  $g_1()$  cannot be bounded. Consequently, the representation scheme proposed in [17] would lead to an infinite alphabet. Next we show that this problem can be overcome, though.

The above-mentioned defect can be patched with the help of a different representation scheme based on annotating targets and sources of justification pointers, with  $\circ$  and  $\bullet$  respectively. We shall use the same two symbols for each pointer. This can lead to ambiguities if many pointers are represented at the same time in a single string. However, to avoid that, we are going to use multiple strings to represent a single play. More precisely, these will be strings corresponding to the underlying sequence of moves in which each pointer may or may not be represented, i.e. plays featuring n pointers from P-questions to O-answers will be represented by  $2^n$  encoded strings. For example, to represent

$$q_0$$
  $q_1$   $q_1$   $q_1$   $q_2$   $q_2$   $q_2$ 

we shall use the following four strings

Note that the last string represents pointers ambiguously, though the second and third strings identify them uniquely. We could have achieved the same effect using strings in which only one pointer is represented [9] but, from the technical point of view, it is easier to include all possible pointer/no-pointer combinations.

7.3. **Regular-language interpretation.** In order to translate  $IA_{\bigcirc}$ -terms into regular languages representing their game semantics, we we restrict our translation to terms in a canonical shape, to be defined next. Any  $IA_{\bigcirc}$ -term can be converted effectively to such a form and the conversion preserves denotation.

The canonical forms are defined by the following grammar. We use types as superscripts, whenever we want to highlight the type of an identifier (u, v, x, y, z) range over identifier names). Note that the only identifiers in canonical form are those of base type, represented by  $x^{\beta}$  below.

**Lemma 7.5.** Let  $\Gamma \vdash M : \theta$  be an  $\mathsf{IA}_{\circlearrowleft}$ -term. There is an  $\mathsf{IA}_{\circlearrowleft}$ -term  $\Gamma \vdash N : \theta$  in canonical form, effectively constructible from M, such that  $\llbracket \Gamma \vdash M \rrbracket = \llbracket \Gamma \vdash N \rrbracket$ .

*Proof.* N can be obtained via a series of  $\eta$ -expansions,  $\beta$ -reductions and commuting conversions involving let and if. We present a detailed argument in Appendix B.

A useful feature of the canonical form is that the problems with pointers can be related to the syntactic shape: they concern references to let-bound identifiers  $x^{\theta}$  such that  $\theta$  is not a base type (i.e.  $\theta = \text{var} \text{ or } \theta$  is a function type). Below we state our representability theorem for  $\mathsf{IA}^{2+}_{\circlearrowright}$ -terms. The definition of  $\mathcal{A}_M$  is actually too generous, as we shall only need ∘, • to decorate P-questions enabled by O-answers.

**Proposition 7.6.** Suppose  $\Gamma \vdash M : \theta$  is an  $\mathsf{IA}^{2+}_{\circlearrowright}$ -term. Let  $\mathcal{A}_M = M_A + (M_A \times \{\circ, \bullet\})$ , where  $A = \llbracket \Gamma \vdash \theta \rrbracket$ . Let  $\mathcal{C}_{\Gamma \vdash M}$  be the set of non-empty spinal complete plays from  $\llbracket \Gamma \vdash M : \theta \rrbracket$ . Then  $\mathcal{C}_{\Gamma\vdash M}$  can be represented as a regular language over a *finite* subset of  $\mathcal{A}_M$ .

*Proof.*  $\mathcal{C}_{\Gamma \vdash M}$  can be decomposed as  $\sum_{i \in I_A} (i \ \mathcal{C}_{\Gamma \vdash M}^i)$ . Obviously  $\mathcal{C}_{\Gamma \vdash M}$  is regular if, and only if, any  $\mathcal{C}^i_{\Gamma \vdash M}$  is regular  $(i \in I_A)$ . Hence, it suffices to show that  $\mathcal{C}^i_{\Gamma \vdash M}$  is regular for any

As already discussed, the regular-language representations, which we shall also refer to by  $\mathcal{C}^{\imath}_{\Gamma \vdash M}$ , will consist of plays in which individual pointers may or may not be represented. However, because all possibilities are covered, this will yield a faithful representation of the induced complete plays. We proceed by induction on the structure of canonical forms. Let  $i_{\Gamma}$  range over  $I_{\Gamma\Gamma}$ .

- $C^{i_{\Gamma}}_{\Gamma \vdash ()} = \star$

- $\begin{array}{l} \circ \Gamma \vdash () = \lambda \\ \circ \mathcal{C}_{\Gamma \vdash j}^{i_{\Gamma}} = j \\ \circ \mathcal{C}_{\Gamma, x; \beta \vdash x; \beta}^{i_{\Gamma}} = j \\ \circ \mathcal{C}_{\Gamma, x; \beta \vdash x; \beta}^{(i_{\Gamma}, j_{x})} = j \\ \circ \mathcal{C}_{\Gamma, x; \text{int}, y; \text{int} \vdash x \oplus y}^{(i_{\Gamma}, j_{x}, k_{y})} = j \oplus k \\ \circ \mathcal{C}_{\Gamma, x; \text{int} \vdash \text{if } x \text{ then } M_{1} \text{ else } M_{0}} = \mathcal{C}_{\Gamma, x \vdash M_{h(j)}}^{(i_{\Gamma}, j_{x})}, \text{ where } h(j) = \left\{ \begin{array}{cc} 0 & j = 0 \\ 1 & j > 0 \end{array} \right. \\ \circ \mathcal{C}_{\Gamma, x; \text{var}, y; \text{int} \vdash x}^{(i_{\Gamma}, \star_{x}, j_{y})} = \text{write}(j)_{x} \text{ok}_{x} \star \\ \circ \mathcal{C}_{\Gamma, x; \text{var} \vdash !x}^{(i_{\Gamma}, \star_{x})} = \text{read}_{x}(\sum_{j=0}^{N} j_{x}j) \end{array}$

- $\bullet \ \ \mathcal{C}^{i_{\Gamma}}_{\Gamma \vdash \mathsf{mkvar}(\lambda x^{\mathsf{unit}}.M_1,\lambda y^{\mathsf{int}}.M_2)}^{i_{\Gamma}} = \star (\epsilon + \mathsf{read} \ \mathcal{C}^{(i_{\Gamma},\star_x)}_{\Gamma,x \vdash M_1} + \sum_{j=0}^{N} \mathsf{write}(j) \ \mathcal{C}^{(i_{\Gamma},j_y)}_{\Gamma,y \vdash M_2}[\mathsf{ok}/\star])$
- $C_{\Gamma \vdash \lambda x^{\theta}.M}^{i_{\Gamma}} = \star (\epsilon + \sum_{i \in I_{\llbracket \theta \rrbracket}} i \ C_{\Gamma,x \vdash M}^{(i_{\Gamma},i_{x})}[m'_{x}/m_{x}])$
- $C_{\Gamma \vdash \text{new } x \text{ in } M}^{i_{\Gamma}} = (C_{\Gamma, x \vdash M}^{(i_{\Gamma}, \star_x)} \cap C')[\epsilon/m_x]$  where, writing  $\parallel$  for the shuffle operator on strings,

$$\mathcal{C}' = (\mathcal{A}' \setminus (M_{\llbracket \mathsf{var} \rrbracket})_x)^* \quad || \quad ((\mathsf{read}_x \, 0_x)^* (\sum_{j=0}^N \mathsf{write}(j)_x \, \mathsf{ok}_x \, (\mathsf{read}_x \, j_x)^*)^*)$$

and  $\mathcal{A}'$  is the finite alphabet used to represent  $\Gamma, x : \mathsf{var} \vdash M$ .

The substitution  $[m'_x/m_x]$  highlights the fact that the moves associated with x have to be bijectively relabelled, because the copy of  $\theta$  moved from the left- to the right-hand side of the context.  $[\epsilon/m_x]$  stands for erasure of all moves associated with x. Obviously, these (homomorphic) operations preserve regularity. Note that the clause for  $\lambda x^{\theta}.M$  is correct because we consider spinal plays only.

For  $\Gamma \vdash M : \beta$  we sometimes refer to components determined by the following decomposition.

$$\mathcal{C}^{i_{\Gamma}}_{\Gamma \vdash M:\beta} = \sum_{j \in I_{\mathbb{T}\beta\mathbb{I}}} (\mathcal{C}^{i_{\Gamma},j}_{\Gamma \vdash M} \ j)$$

The components  $\mathcal{C}^{i_{\Gamma},j}_{\Gamma \vdash M}$  can be extracted from  $\mathcal{C}^{i_{\Gamma}}_{\Gamma \vdash M:\beta}$  by applying operations preserving regularity (intersection, erasure), so the latter is regular iff each of the former is.

- $\bullet \ \mathcal{C}^{i_{\Gamma}}_{\Gamma\vdash \mathsf{while}\, M \, \mathsf{do}\, N} = (\textstyle\sum_{j=1}^{N} \mathcal{C}^{i_{\Gamma},j}_{\Gamma\vdash M} \, \mathcal{C}^{i_{\Gamma},\star}_{\Gamma\vdash N})^* \, \mathcal{C}^{i_{\Gamma},0}_{\Gamma\vdash M}$
- $\bullet \ \mathcal{C}^{i_{\Gamma}}_{\Gamma \vdash \mathsf{let} \ x^{\beta} = M \ \mathsf{in} \ N} = \sum\nolimits_{j \in I_{\llbracket \beta \rrbracket}} \mathcal{C}^{i_{\Gamma},j}_{\Gamma \vdash M} \mathcal{C}^{(i_{\Gamma},j_{x})}_{\Gamma,x \vdash N}$

The remaining cases are those of let-bindings of the form let  $x=z(\cdots)$  in  $\cdots$ . First we explain some notation used throughout. Consider the following context  $\Gamma, z:\theta'\to\theta, x:\theta$ . We shall refer to moves contributed by  $x:\theta$  with  $m_x$ . If we want to range solely over O- or P-moves from the component, we use  $o_x$  and  $o_x$  respectively. Moreover, we use  $o_x$ ,  $o_x$ 

First we tackle cases where the bound value is of function type, i.e. those related to possibly ambiguous pointer reconstruction. Note that we include plays with and without pointer representations.

• 
$$\Gamma, z: \beta \to (\theta_1 \to \theta_2), y: \beta \vdash \text{let } x = zy \text{ in } N \text{ given } \Gamma, z, y, x: \theta_1 \to \theta_2 \vdash N$$

$$\begin{array}{lll} \mathcal{C}_{\Gamma,z,y\vdash \mathsf{let}\; \ldots \; \mathsf{in}\; \ldots}^{(i_{\Gamma},\star_{z},j_{y})} & = & j_{z} \star_{z,x}^{\circ} \, \mathcal{C}_{\Gamma,z,y,x\vdash N}^{(i_{\Gamma},\star_{z},j_{y},\star_{x})}[q_{z,x}^{\bullet}/q_{x}, \; m_{z,x}/m_{x}] \\ & + & j_{z} \star_{z,x} \, \mathcal{C}_{\Gamma,z,y,x\vdash N}^{(i_{\Gamma},\star_{z},j_{y},\star_{x})}[q_{z,x}/q_{x}, \; m_{z,x}/m_{x}]) \end{array}$$

•  $\Gamma, z: (\beta_1 \to \beta_2) \to (\theta_1 \to \theta_2) \vdash \text{let } x = z(\lambda y^{\beta_1}.M) \text{ in } N \text{ given } \Gamma, z, y: \beta_1 \vdash M: \beta_2 \text{ and } \Gamma, z, x: \theta_1 \to \theta_2 \vdash N$ 

$$\begin{array}{lcl} \mathcal{C}_{\Gamma,z\vdash \mathsf{let}\ \cdots\ \mathsf{in}\ \cdots}^{(i_{\Gamma},\star_{z})} &=& q_{z}\,\mathcal{C}'\star_{z,x}^{\circ}\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_{\Gamma},\star_{z},\star_{x})}[q_{z,x}^{\bullet}\mathcal{C}'/q_{x},\ p_{z,x}\mathcal{C}'/p_{x},\ o_{z,x}/o_{x}] \\ &+& q_{z}\,\mathcal{C}'\star_{z,x}\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_{\Gamma},\star_{z},\star_{x})}[q_{z,x}\mathcal{C}'/q_{x},\ p_{z,x}\mathcal{C}'/p_{x},\ o_{z,x}/o_{x}] \end{array}$$

where 
$$\mathcal{C}' = (\sum_{i \in I_{\llbracket \beta_1 \rrbracket}} i_z (\sum_{j \in I_{\llbracket \beta_2 \rrbracket}} \mathcal{C}_{\Gamma,z,y \vdash M}^{(i_{\Gamma},\star_z,i_y),j} j_z))^*$$

•  $\Gamma, z : \mathsf{var} \to (\theta_1 \to \theta_2) \vdash \mathsf{let} \ x = z \ \mathsf{mkvar}(\lambda u^{\mathsf{unit}}.M_1, \lambda v^{\mathsf{int}}.M_2) \ \mathsf{in} \ N \ \mathsf{given} \ \Gamma, z, u \vdash M_1, \ \mathsf{as} \ \mathsf{well} \ \mathsf{as} \ \Gamma, z, v \vdash M_2 \ \mathsf{and} \ \Gamma, z, x : \theta_1 \to \theta_2 \vdash N$ 

$$\begin{array}{lll} \mathcal{C}_{\Gamma,z\vdash \mathsf{let}\ \dots\ \mathsf{in}\ \dots}^{(i_{\Gamma},\star_{z})} &=& q_{z}\,\mathcal{C}'\star_{z,x}^{\circ}\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_{\Gamma},\star_{z},\star_{x})}[q_{z,x}^{\bullet}\mathcal{C}'/q_{x},\ p_{z,x}\mathcal{C}'/p_{x},\ o_{z,x}/o_{x}] \\ &+& q_{z}\,\mathcal{C}'\star_{z,x}\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_{\Gamma},\star_{z},\star_{x})}[q_{z,x}\mathcal{C}'/q_{x},\ p_{z,x}\mathcal{C}'/p_{x},\ o_{z,x}/o_{x}], \end{array}$$

where 
$$\mathcal{C}' = (\operatorname{read}_z (\sum_{j=0}^N \mathcal{C}_{\Gamma,z,u\vdash M_1}^{(i_\Gamma,\star_z,\star_u),j} \ j_z) + \sum_{j=0}^N \operatorname{write}(j)_z \ \mathcal{C}_{\Gamma,z,v\vdash M_2}^{(i_\Gamma,\star_z,j_v),\star} \ \operatorname{ok}_z)^*$$

To understand the second formula (the third case is analogous) observe that, after  $\star_{z,x}$  has been played, plays for let  $\cdots$  in  $\cdots$  are plays from N interleaved with possible detours to  $\lambda x^{\beta_1}.M$ : such a detour can be triggered by  $i_z$  from  $\beta_1$  each time the second move  $(q_z)$  is O-visible. Moreover, provided  $q_z$  is O-visible, such a detour can also take place between  $q_z$  and  $\star_{z,x}$ . The following auxiliary lemma will help us analyze when detours can occur.

**Lemma 7.7.** Let A be a prearena,  $s \in P_A$  be non-empty and  $P_i$  — the set of P-moves enabled by the initial move of s. Let s' be a prefix of s containing at least two moves. Then the O-view of s' contains exactly one move from  $P_i$ .

*Proof.* Any non-initial move must be either in  $P_i$  or hereditarily enabled by a move from  $P_i$ . By visibility the O-view of s' any prefix of s must thus contain a move from  $P_i$ . Because moves from  $P_i$  are enabled by the initial move, they are always the second moves in O-views. Hence, no two moves from  $P_i$  can occur in the same O-view.

Let us apply the lemma to the denotation of M. Because  $\beta_2$  is a base type it follows that at any time non-trivial O-views will contain a move from  $\Gamma$  enabled by the initial move or the answer from  $\beta_2$  (which completes the play). Returning to the play for let  $\cdots$  in  $\cdots$ , this means that during a detour the second move  $q_z$  will be hidden from O-view until the detour is completed, i.e. a single detour has to be completed before the next one begins. Hence,  $\mathcal{C}'$  has the form  $(\cdots)^*$ .

Also by the lemma above, once the play after  $\star_{z,x}$  progresses, the second move  $q_z$  will be O-visible if, and only if,  $q_{z,x}$  (which  $\star_{z,x}$  enables) is. Thus detours will be possible exactly after P plays a P-move hereditarily justified by  $\star_{z,x}$ , which corresponds to P playing a P-move hereditarily justified by  $q_x$  in N (hence the substitutions  $p_{z,x}\mathcal{C}'/p_x$  restricted to P-moves). A special case is then that of  $q_{z,x}$  which, as a question enabled by an answer, should be represented both with and without a pointer.

The above three cases cover all scenarios (that can arise in  $\mathsf{IA}^{2+}_{\circlearrowleft}$ ) in which z's type is of the form  $\theta \to (\theta_1 \to \theta_2)$ . The cases where  $z: \theta \to \mathsf{var}$  are analogous except that one needs to use  $q_x$  to range over  $\mathsf{read}_x$ ,  $\mathsf{write}(0)_x, \cdots, \mathsf{write}(N)_x$  rather than the single move enabled by  $\star_x$ . It remains to consider cases of  $z: \theta \to \beta$ . The bound values are of base type, so no new pointer indices need to be introduced.

•  $\Gamma, z: \beta' \to \beta, y: \beta' \vdash \text{let } x = zy \text{ in } N \text{ given } \Gamma, z, y, x: \beta \vdash N$ 

$$\mathcal{C}_{\Gamma,z,y\vdash \mathsf{let}}^{(i_{\Gamma},\star_{z},j_{y})} \ _{\mathsf{in}} \ = j_{z} \, (\sum_{k \in I_{\parallel\beta \parallel}} k_{z} \, \mathcal{C}_{\Gamma,z,x,y\vdash N}^{(i_{\Gamma},\star_{z},j_{y},k_{x})})$$

•  $\Gamma, z: (\beta_1 \to \beta_2) \to \beta \vdash \text{let } x = z(\lambda y^{\beta_1}.M) \text{ in } N \text{ given } \Gamma, z, y: \beta_1 \vdash M: \beta_2 \text{ and } \Gamma, z, x: \beta \vdash N$ 

$$\mathcal{C}_{\Gamma,z\vdash \mathsf{let} \quad \mathsf{in}}^{(i_{\Gamma},\star_z)} \ = q_z\,\mathcal{C}'(\sum_{k\in I_{\llbracket\beta\rrbracket}} k_z\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_{\Gamma},\star_z,k_x)})$$

where  $\mathcal{C}' = (\sum_{i \in I_{\llbracket \beta_1 \rrbracket}} i_z (\sum_{j \in I_{\llbracket \beta_2 \rrbracket}} \mathcal{C}_{\Gamma,z,y \vdash M}^{(i_{\Gamma},\star_z,i_y),j} j_z))^*$ 

•  $\Gamma, z: \mathsf{var} \to \beta \vdash \mathsf{let} \ x = z \ \mathsf{mkvar}(\lambda u^{\mathsf{unit}}.M_1, \lambda v^{\mathsf{int}}.M_2) \ \mathsf{in} \ N \ \mathsf{given} \ \Gamma, z, u \vdash M_1, \ \Gamma, z, v \vdash M_2 \ \mathsf{and} \ \Gamma, z, x \vdash N$ 

$$\mathcal{C}_{\Gamma,z\vdash \mathsf{let} \quad \mathsf{in}}^{(i_\Gamma,\star_z)} \ = q_z\,\mathcal{C}'(\sum_{k\in I_{\llbracket\beta\rrbracket}} k_z\,\mathcal{C}_{\Gamma,z,x\vdash N}^{(i_\Gamma,\star_z,k_x)})$$

where 
$$\mathcal{C}' = (\operatorname{read}_z (\sum_{j=0}^N \mathcal{C}_{\Gamma,z,u\vdash M_1}^{(i_{\Gamma},\star_z,\star_u),j} j_z) + \sum_{j=0}^N \operatorname{write}(j)_z \mathcal{C}_{\Gamma,z,v\vdash M_2}^{(i_{\Gamma},\star_z,j_v),\star} \operatorname{ok}_z)^*.$$

**Theorem 7.8.** Program equivalence of  $\mathsf{IA}^{2+}_{\circlearrowright}$ -terms is decidable.

We remark that adding dynamic memory allocation in the form of ref to  $\mathsf{IA}^{2+}_{\circlearrowright}$ , or its second-order sublanguage, results in undecidability [15]. Hence, at second order, block structure is "strictly weaker" than scope extrusion.

## 8. Summary

In this paper we have introduced the notion of block-innocence that has been linked with call-by-value Idealized Algol in a sequence of results. Thanks to the faithfulness of block-innocence, we could investigate the interplay between type theory, functional computation and stateful computation with block structure and dynamic allocation respectively. We have also shown a new decidability result for a carefully designed fragment of  $IA_{cbv}$ . Its extension to product types poses no particular difficulty. In fact, it suffices to follow the way we have tackled the var type, which is itself a product type. The result thus extends those from [7] and is a step forward towards a full classification of decidable fragments of  $IA_{cbv}$ : the language  $IA_{C}^{2+}$  we considered features all second-order types and some third-order types, while finitary  $IA_{cbv}$  is known to be undecidable at order 5 [14]. Interestingly,  $IA_{C}^{2+}$  features restrictions that are compatible with the use of higher-order types in PASCAL [12], in which procedure parameters cannot be procedures with procedure parameters. An interesting topic for future work would be to characterise the uniformity inherent in block-innocence in more abstract, possibly category-theoretic, terms.

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## APPENDIX

## APPENDIX A. S-PLAYS

In this section we use the term "move" and "S-move" interchangeably.

**Lemma A.1** (Prev-PA). If  $s = \cdots m^{\Sigma} a_P^T \cdots$  is an S-play then, for any  $\alpha$ ,

- (a) if  $\alpha \in \nu(T)$  then  $\alpha \in \nu(\Sigma)$ ,
- (b) if  $\alpha \in \nu(\varSigma \setminus T)$  then  $\alpha$  is closed in  $s_{< a_P^T}.$

Moreover,  $T \leq_p \Sigma$  and therefore  $\Sigma \setminus (\Sigma \setminus T) \leq_p T$  and  $\Sigma \setminus T \leq_s \Sigma$ .

*Proof.* Let  $s = s_1 q_0^{\Sigma_0} s_2 a^T \cdots$ . As  $q_0^{\Sigma_0}$  is the pending-Q in  $s_1 q_0^{\Sigma_0} s_2$ , we have that s is in fact of the form:

$$s_1 q_0^{\Sigma_0} \widetilde{q_1^{T_1} \cdots a_1^{\Sigma_1}} q_2^{T_2} \cdots a_2^{\Sigma_2} \cdots q_j^{T_j} \cdots a_j^{\Sigma_j} a^T$$

For (a), by Just-P we have that  $\alpha \in \nu(\Sigma_0)$  and therefore  $\alpha$  is not closed in  $s_1 m_0^{\Sigma_0} s_2$ . Hence, by Prev-PQ(b),  $\alpha \in \nu(T_1)$  and thus, by Just-O,  $\alpha \in \nu(\Sigma_1)$ . Repeating this argument j times we obtain  $\alpha \in \nu(\Sigma_j)$ , i.e.  $\alpha \in \nu(\Sigma)$ .

For (b), let  $\alpha \in \nu(\Sigma) = \nu(\Sigma_j)$  be open in  $s_{< a^T}$ . We claim that then  $\alpha \in \nu(\Sigma_0)$  and therefore  $\alpha \in \nu(T)$  by Just-P. We have that  $\alpha \in \nu(T_j)$ , by Just-O. Moreover, since there are no open questions in  $q_j^{T_j} \cdots q_j^{\Sigma_j}$ , we have that  $\alpha$  is open in  $s_{< q_j^{T_j}}$ , so  $\alpha \in \nu(s_{< q_j^{T_j}})$  and thus, by Prev-PQ(a),  $\alpha \in \text{dom}(\Sigma_{j-1})$ . Applying this argument j times we obtain  $\alpha \in \nu(\Sigma_0)$ .

Finally, we show by induction that  $\Sigma_0 \leq_p \Sigma_i$ , for all  $0 \leq i \leq j$ . For the inductive step, we have that  $\Sigma_i \setminus (\Sigma_i \setminus T_{i+1}) \leq_p T_{i+1} \leq_p \Sigma_{i+1}$ . By IH,  $\Sigma_0 \leq_p \Sigma_i$ . Moreover, if  $\alpha \in \nu(\Sigma_i \setminus T_{i+1})$  then  $\alpha \notin \nu(\Sigma_0)$ , by Prev-PQ(b), so  $\Sigma_0 \leq_p \Sigma_i \setminus (\Sigma_i \setminus T_{i+1})$ ,  $\therefore \Sigma_0 \leq_p \Sigma_{i+1}$ . Thus,  $T \leq_p \Sigma_0 \leq_p \Sigma_j = \Sigma$ .

**Lemma A.2** (Block Form). If s is an S-play then  $\lceil s \rceil$  is in *block-form*: for any  $\alpha$ , we have  $\lceil s \rceil = s_1 s_2 s_3$ , where

- $\alpha \notin \nu(s_1) \cup \nu(s_3)$  and  $\forall m^{\Sigma} \in s_2$ .  $\alpha \in \nu(\Sigma)$ ,
- if  $s_2 \neq \epsilon$  then its first element is the move introducing  $\alpha$  in s.

*Proof.* We do induction on |s|, with the cases of  $|s| \leq 1$  being trivial. If  $s = s'o^{\Sigma}$  then, by IH,  $\lceil s \rceil = s''p^To^{\Sigma}$ , some  $s''p^T$  in block-form, and  $\nu(T) = \nu(\Sigma)$ , which imply that  $\lceil s \rceil$  is in block-form. If  $s = s'p^{\Sigma}$  then  $\lceil s \rceil = \lceil s' \rceil p^{\Sigma}$  with  $\lceil s' \rceil = s_1s_2s_3$  in block-form by IH. If  $\alpha \notin \text{dom}(\Sigma)$  then OK. Otherwise, if  $\alpha$  does not appear in the last move in  $s_1s_2s_3$  then, by Prev,  $p^{\Sigma}$  in fact introduces  $\alpha$  in s and therefore  $\lceil s \rceil$  has block-form.

**Lemma A.3.** Let  $s = s_1 o^{\Sigma} p^T s_2$  be an S-play with  $\alpha \in \nu(\Sigma) \setminus \nu(T)$ . Then, for any  $s_2' \sqsubseteq_p s_2$ ,  $\alpha \notin \nu(\lfloor s_1 o^{\Sigma} p^T s_2' \rfloor)$ .

Proof. We do induction on  $|s_2'|$ . For the base case, by the previous lemma we have that  $\lceil s_1 o^{\Sigma} \rceil$  has block-form; in particular, it ends in a block of moves which contains the move introducing  $\alpha$  in s, and all moves in the block contain  $\alpha$  in their stores. Hence, since the justifier of  $p^T$ , say  $o'^{\Sigma'}$ , occurs in  $\lceil s_1 o^{\Sigma} \rceil$  and  $\alpha \notin \nu(\Sigma')$  (by Just and the fact that  $\alpha \notin \nu(T)$ ), we have that  $o'^{\Sigma'}$  occurs in s before the move introducing  $\alpha$  in it and therefore  $\alpha \notin \nu(\lfloor s_1 o^{\Sigma} p^T s_2'' \rfloor)$ . Now, if  $s_2' = s_2'' o'^{\Sigma'}$  then we need to show that  $\alpha \notin \nu(\lfloor s_1 o^{\Sigma} p^T s_2'' \rfloor o'^{\Sigma'})$  given by IH that  $\alpha \notin \nu(\lfloor s_1 o^{\Sigma} p^T s_2'' \rfloor)$ , which immediately follows from Just and Visibility. Finally, if  $s_2' = s_2'' p'^{T'}$  then, by IH,  $\alpha \notin \nu(s_2'')$  and therefore, by Prev,  $\alpha \notin \nu(T')$ . Let  $p'^{T'}$  be justified by some  $o'^{\Sigma'}$  in s. If  $o'^{\Sigma'}$  occurs in  $s_2''$  then our argument follows directly from the IH. Otherwise, arguing as before,  $o'^{\Sigma'}$  occurs in s before the introduction of  $\alpha$  and therefore  $\alpha \notin \nu(\lfloor s_1 o^{\Sigma} p^T s_2'' p'^{T'} \rfloor)$ .

Corollary A.4 (Close). If  $s = s_1 o^{\Sigma} p^T s_2$  is an S-play with  $\alpha \in \nu(\Sigma) \setminus \nu(T)$  then  $\alpha \notin \nu(s_2)$ .

Proof of Lemma 4.10. For (a), assuming WLOG p is a P-move in AB and taking  $s = s'n^{T'}p^{\Sigma'}$ , by definition of the interaction and Prev we have that if  $\alpha \in \nu(\Sigma \setminus T)$  then  $\alpha \in \nu(\Sigma' \setminus T')$ , hence  $\alpha \notin \nu(s'n^{T'})$  and therefore  $\alpha \notin \nu(un^T)$ .

For (b), we do induction on |s||t|, base case is trivial. Now, if  $s||t| = up^{\Sigma}$  with p a generalised P-move then  $\lceil s||t\rceil = \lceil u\rceil p^{\Sigma}$  and, by IH,  $\lceil u\rceil$  has block form, say  $u_1u_2u_3$ . If  $\alpha \notin \nu(\Sigma)$  then OK. Otherwise, if  $\alpha$  does not appear in the last move of  $\lceil u\rceil$  then  $p^{\Sigma}$  is in fact the move introducing  $\alpha$  in s||t, so OK. Otherwise,  $u_3 = \epsilon$  and therefore  $\lceil s||t\rceil$  in block form. If  $s||t| = uo^T$  with o an O-move in AC then  $\lceil s||t\rceil = u'o^T$  for some u' in block-form, and the last move in u' has domain dom(T). This implies that s||t is in block-form.

For (c), assuming WLOG p is a P-move in AB and taking  $s = s'n^{T'}p^{\Sigma'}$ , by definition of the interaction we have  $\alpha \in \nu(T' \setminus \Sigma')$  so, by Prev,  $\alpha$  is closed in  $s'n^{T'}$ . Now suppose  $\alpha$  is open in  $un^T$ , that is, there is an open question  $q_1^{\Sigma_1}$  in  $un^T$  with  $\alpha \in \nu(\Sigma_1)$ . Then, if  $q_0^{\Sigma_0}$  is the pending question of  $un^T$  then  $\alpha \in \nu(\Sigma_0)$ :  $\lceil un^T \rceil$  has block-form, by (b), and  $q_0^{\Sigma_0}$  appears in it, thus if  $\alpha \notin \nu(\Sigma_0)$  then  $q_0^{\Sigma_0}$  would precede the move introducing  $\alpha$  in  $un^T$  and hence it would precede  $q_1^{\Sigma_1}$  too. As the interaction ends in an O-move in AB,  $q_0^{\Sigma_0}$  is the pending

question of  $un^T \upharpoonright AB$ . Let  $q_0^{\Sigma_0'}$  be the move in s corresponding to  $q_0^{\Sigma_0}$ .  $q_0^{\Sigma_0'}$  is the pending question in  $s'n^{T'}$  and therefore it appears in  $\lceil s'n^{T'} \rceil$ . Moreover, since  $\alpha$  is closed in  $s'n^{T'}$ ,  $\alpha \notin \nu(\Sigma_0')$ , which means that  $q_0^{\Sigma_0'}$  occurs in  $\lceil s'n^{T'} \rceil$  before the move introducing  $\alpha$  in s. But then  $\alpha \notin \nu(\Sigma_0)$ , a contradiction.

For (d), we do induction on |s||t|, the base case being trivial. If  $m^{\Sigma}$  is an O-move in AC then the claim is obvious. So assume WLOG  $m^{\Sigma}$  is a P-move in AB and consider  $\lceil s \parallel t \mid AB \rceil_{\geq n^T}$ , and take two consecutive moves  $m_1^{\Sigma_1} m_2^{\Sigma_2}$  in it, and assume  $m_2^{\Sigma_2}$  be a Pmove in AB. Then these are consecutive also in  $s \parallel t \mid AB$  and hence, by switching, also in  $s \parallel t$ . Let  $\alpha \in \nu(\Sigma_1 \setminus \Sigma_2)$ . By definition of interaction, this dropping of  $\alpha$  happens in s too, at the respective consecutive moves  $m_1^{\Sigma_1'}m_2^{\Sigma_2'}$ , assuming  $s=\cdots n^{T'}\cdots m_1^{\Sigma_1'}m_2^{\Sigma_2'}\cdots m^{\Sigma'}$ . We can see that, as  $\lceil s \rceil$  is in block-form,  $\alpha \notin \nu(\Sigma')$  and therefore, by Just,  $\alpha \notin \nu(T')$ . More than that,  $n^{T'}$  occurs in s before the move introducing  $\alpha$  in s, thus  $\alpha \notin \nu(T)$ . We therefore have  $\Sigma_1 \setminus (\Sigma_1 \setminus \Sigma_2) \leq_p \Sigma_2$  and  $\nu(\Sigma_1 \setminus \Sigma_2) \cap \nu(T) = \emptyset$ . On the other hand, if  $m_2^{\Sigma_2}$  is an O-move in AB then  $m_1^{\Sigma_1}$  is its justifier and therefore, by IH,  $\Sigma_1 \leq_p \Sigma_2$ . Thus, for every move  $m''^{\Sigma''}$  in  $\lceil s \parallel t \upharpoonright AB \rceil_{\geq n^T}$ , we have  $T \leq_p \Sigma''$  and hence  $T \leq_p \Sigma$ . Finally, if  $m^{\Sigma}$  is an answer then, since  $\lceil s \parallel t \rceil AB \rceil$  satisfies well-bracketing, we have that  $\lceil s \parallel t \upharpoonright AB \rceil_{>n^T} = n^{T_0} q_1^{\Sigma_1} a_1^{T_1} \cdots q_i^{\Sigma_j} a_i^{T_j} m^{\Sigma_{j+1}} \text{ with dom } (\Sigma_i) = \text{dom } (T_i) \text{ for each } 1 \leq i \leq j$ by IH. Assuming  $s_{\geq nT'} = n^{T_0'} q_1^{\Sigma_1'} \cdots q_1^{T_1'} \cdots q_j^{\Sigma_j'} \cdots a_j^{T_j'} m^{\Sigma_{j+1}'}$ , if  $\alpha \in \nu(\Sigma_{i+1} \setminus T_i)$  for some i then  $\alpha \in \nu(\Sigma_{i+1}' \setminus T_i')$  and therefore  $\alpha \notin \nu(T_0') = \nu(\Sigma_{j+1}')$ . But the latter implies that  $\alpha \in \nu(T_{i'}' \setminus \Sigma_{i'+1}')$  for some  $i < i' \leq j$  and therefore  $\alpha \notin \nu(\Sigma_{i'+1})$ . Thus,  $\nu(\Sigma \setminus T) = \emptyset$ . For (e), we replay the proof of lemma A.3, that is, we show that, for every  $u_2' \sqsubseteq_p u_2$ ,  $\alpha \notin \nu(\lfloor u_1 n^T p^{\Sigma} u_2' \rfloor)$ . We do induction on  $|u_2'|$ . For the base case, by (b) we have that  $\lceil u_1 n^T \rceil$  has block-form; in particular, it ends in a block of moves which contains the move introducing  $\alpha$  in u, and all moves in the block contain  $\alpha$  in their stores. Hence, since  $\alpha \notin \nu(\lfloor u_1 n^T p^{\Sigma} \rfloor)$ . Now, if  $u_2' = u_2'' o'^{T'}$  (an O-move in AC) then we need to show that  $\alpha \notin \nu(\lfloor u_1 m^T p^{\Sigma} u_2'' \rfloor o'^{T'})$  given by IH that  $\alpha \notin \nu(\lfloor u_1 m^T p^{\Sigma} u_2'' \rfloor)$ , which immediately follows from Visibility. Finally, if  $u_2' = u_2'' p'^{\Sigma'}$  (a generalised P-move) then, by IH,  $\alpha \notin \nu(u_2'')$  and therefore, by (a),  $\alpha \notin \nu(\Sigma')$ . Let  $p'^{\Sigma'}$  be justified by some  $m'^{T'}$  in u. If  $m'^{T'}$  occurs in  $u_2''$ then our argument follows directly from the IH. Otherwise, arguing as before,  $m^{\prime T^\prime}$  occurs in u before the introduction of  $\alpha$  and therefore  $\alpha \notin \nu(\lfloor u_1 m^T p^{\Sigma} u_2'' p^{\prime \Sigma'} \rfloor)$ .

For (f), suppose  $\alpha$  appears in a *B*-move  $n^T$  of u. By (e), and because  $\alpha$  reappears in  $m^{\Sigma}$ , we have that  $\alpha$  also appears in the move following  $n^T$  in u. Applying this reasoning repeatedly, we obtain that  $\alpha$  appears in  $u \upharpoonright AC$  after  $n^T$ .

For (g), we do induction on |s||t|, the base case being trivial. Now assume  $s = s'm^{\Sigma'}$ . If m is an O-move then  $\Sigma' \leq \Sigma$  follows from (d) and from the IH applied to the subsequence ending in the justifier of  $m^{\Sigma}$ . Moreover, if  $\alpha \in \nu(\Sigma')$  then  $\Sigma'(\alpha)$  is determined by the last appearance of  $\alpha$  in s', say  $n^{T'}$ . By IH, the corresponding move of  $s' \parallel t$  has store T, with  $T(\alpha) = T'(\alpha)$ . But then, by inspection of the definition of interaction, any move in  $s \parallel t$  occurring after  $n^T$  does not change the value of  $\alpha$ , hence  $\Sigma(\alpha) = T(\alpha) = \Sigma'(\alpha)$ . If m is a P-move preceded by  $n^{T'}$  then, using the IH, we have

$$\Sigma' = T'[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T') \le (sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T') = \Sigma.$$

From the above, and using again the IH, we obtain also that  $\Sigma[\Sigma'] = \Sigma$ . Moreover, if  $\alpha \in \nu(\mathsf{st}(t))$  then, by IH, if the last B-move of  $s \parallel t$  has store  $\Sigma_B$  then  $\Sigma_B(\alpha) = \mathsf{st}(t)(\alpha)$ and the value of  $\alpha$  cannot be changed by subsequent A-moves. Hence, if  $\alpha \in \nu(\Sigma)$  then  $\Sigma(\alpha) = \mathsf{st}(t)(\alpha)$ . The case of  $t = t'm^{\Sigma'}$  is treated dually.

For (h), the last two moves come from the same sequence, say s, so let  $s = s'n^{T'}p^{\Sigma'}$ . We have that

$$T \setminus \Sigma = (sn^{T'} \cdot t) \setminus ((sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T'))$$
$$= (sn^{T'} \cdot t) \setminus ((sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma')) = (T' \setminus \Sigma')[sn^{T'} \cdot t]$$

where the last equality holds because  $T' \setminus \Sigma' \leq T' \leq T = sn^{T'} \cdot t$ , by (g). Hence,  $T \setminus (T \setminus \Sigma) = t$  $T\setminus (T'\setminus \Sigma')\leq_p \Sigma$ . We still need to show that  $T'\setminus \Sigma'\leq_s T$ . Let  $\alpha_1,\alpha_2$  be consecutive names in dom (T) such that  $\alpha_1 \in \nu(T' \setminus \Sigma')$ . Then, the point of introduction of  $\alpha_2$  in  $un^T$  does not precede that of  $\alpha_1$ , say  $q_1^{\Sigma_1}$ . Now, by (c),  $\alpha_1$  is closed in  $un^T$  so, using also (b), the latter has the form  $u'q_1^{\Sigma_1}\cdots a_1^{T_1}q_2^{\Sigma_2}\cdots a_2^{T_2}\cdots q_j^{\Sigma_j}\cdots a_j^{T_j}$ . Thus, by (d), the point of introduction of  $\alpha_2$  has to be one of the  $q_i^{\Sigma_i}$ 's. This implies that  $\alpha_1, \alpha_2 \in \nu(s)$  and therefore  $\alpha_1, \alpha_2$  are consecutive also in T'. But then  $\alpha_1 \in \nu(T' \setminus \Sigma')$  implies  $\alpha_2 \in \nu(T' \setminus \Sigma')$  too.

*Proof of lemma 4.12.* Let us write  $\Sigma_{ij}$  for  $\Sigma_i \setminus \Sigma_j$ . Then, the LHS is:

$$L = \Sigma_{1}[\mathbf{\Phi}(\Sigma_{3}, \Sigma_{4}, \Sigma_{5})] \setminus (\Sigma_{2} \setminus \mathbf{\Phi}(\Sigma_{3}, \Sigma_{4}, \Sigma_{5})) + \mathbf{\Phi}(\Sigma_{3}, \Sigma_{4}, \Sigma_{5}) \setminus \Sigma_{2}$$

$$= \Sigma_{1}[\mathbf{\Phi}(\Sigma_{3}, \Sigma_{4}, \Sigma_{5})] \setminus (\Sigma_{2} \setminus (\Sigma_{3}[\Sigma_{5}] \setminus \Sigma_{45} + \Sigma_{54})) + (\Sigma_{3}[\Sigma_{5}] \setminus \Sigma_{45} + \Sigma_{54}) \setminus \Sigma_{2}$$

$$= \Sigma_{1}[\mathbf{\Phi}(\Sigma_{3}, \Sigma_{4}, \Sigma_{5})] \setminus (\Sigma_{2} \setminus (\Sigma_{3} \setminus \Sigma_{45})) + (\Sigma_{3}[\Sigma_{5}] \setminus \Sigma_{45}) \setminus \Sigma_{2} + \Sigma_{54}$$

where the last equality holds because of (a). The first constituent above is:

$$L_1 = \Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54}] \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45})) = \Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45}))$$
On the other hand, the RHS is:

On the other hand, the RHS is:

$$R = \Phi(\Sigma_1, \Sigma_2, \Sigma_3)[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54} = (\Sigma_1[\Sigma_3] \setminus \Sigma_{23} + \Sigma_{32})[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54}$$
$$= ((\Sigma_1[\Sigma_3] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] + (\Sigma_{32} \setminus \Sigma_{45})[\Sigma_5] + \Sigma_{54}$$
$$= ((\Sigma_1[\Sigma_3] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] + (\Sigma_3[\Sigma_5] \setminus \Sigma_{45}) \setminus \Sigma_2 + \Sigma_{54}$$

The first constituent above is:

$$R_1 = ((\Sigma_1[\Sigma_3 \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] = ((\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5]$$
  
=  $(\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45}$ 

For the last equality, let  $\alpha$  by in the domain of the resulting store. Then  $\alpha \in \nu(\Sigma_5) \cap \nu(\Sigma_1)$ ,  $\therefore \alpha \in \nu(\Sigma_4) \cap \nu(\Sigma_1)$ , so  $\alpha \in \nu(\Sigma_2)$  by (b). Moreover,  $\alpha \notin \nu(\Sigma_{23})$ ,  $\alpha \in \nu(\Sigma_3)$ . Now, let us write  $\Sigma$  for  $\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}]$ . We need to show that

$$\varSigma \setminus (\varSigma_2 \setminus (\varSigma_3 \setminus \varSigma_{45})) = (\varSigma \setminus (\varSigma_2 \setminus \varSigma_3)) \setminus \varSigma_{45}$$

and, in fact, it suffices to show that these stores, say  $\Sigma_L$  and  $\Sigma^R$ , have the same domain. By elementary computation,

- $\alpha \in \nu(\Sigma_L)$  iff  $(\alpha \in \nu(\Sigma) \land \alpha \notin \nu(\Sigma_2)) \lor (\alpha \in \nu(\Sigma) \land \alpha \in \nu(\Sigma_3) \land \alpha \notin \nu(\Sigma_{45}))$ ,
- $\alpha \in \nu(\Sigma^R)$  iff  $(\alpha \in \nu(\Sigma) \land \alpha \notin \nu(\Sigma_2) \land \alpha \notin \nu(\Sigma_{45})) \lor (\alpha \in \nu(\Sigma) \land \alpha \in \nu(\Sigma_3) \land \alpha \notin \nu(\Sigma_{45}))$ .

But note now that  $\alpha \in \nu(\Sigma) \land \alpha \notin \nu(\Sigma_2)$  implies that  $\alpha \notin \nu(\Sigma_4)$ , by (b), so  $\alpha \notin \nu(\Sigma_{45})$ .

## APPENDIX B. PROOF OF LEMMA 7.5

We prove two auxiliary results first, which are special cases of Lemma 7.5.

**Lemma B.1.** Any identifier  $x^{\theta}$  satisfies Lemma 7.5. Moreover, the canonical form is of the form  $\lambda y_1^{\theta}$ .  $\mathbb{C}$  when  $\theta \equiv \theta_1 \to \theta_2$  and of the shape  $\mathsf{mkvar}(\lambda y^{\mathsf{unit}}.\mathbb{C}, \lambda z^{\mathsf{int}}.\mathbb{C})$  if  $\theta \equiv \mathsf{var}$ .

*Proof.* Induction with respect to type structure. If  $\theta$  is a base type,  $x^{\theta}$  is already in canonical form.  $x^{\mathsf{var}}$  can be converted to one using the rule

$$x^{\mathsf{var}} \longrightarrow \mathsf{mkvar}(\lambda u^{\mathsf{int}}.x := u, \lambda v^{\mathsf{unit}}.!x)$$

For  $\theta \equiv \theta_1 \rightarrow \theta_2$  we use the rule

$$x^{\theta_1 \to \theta_2} \longrightarrow \lambda z^{\theta_1}$$
.let  $v^{\theta_2} = xz^{\theta_1}$  in  $v$ 

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and appeal to the inductive hypothesis for  $z^{\theta_1}$  and  $v^{\theta_2}$ .

**Lemma B.2.** Suppose  $C_1, C_2$  are canonical forms. Then let  $y^{\theta} = C_1$  in  $C_2$ , if typable, satisfies Lemma 7.5.

*Proof.* Induction with respect to type structure. If  $\theta$  is a base type, the term is already in canonical form. If  $\theta$  is not a base type,  $C_1$  can take one of the following three shapes:  $\mathsf{mkvar}(\lambda x^{\mathsf{unit}}.\mathbb{C}, \lambda y^{\mathsf{int}}.\mathbb{C}), \, \lambda x_1^{\theta_1}.\mathbb{C}$ , if  $x^{\beta}$  then  $\mathbb{C}$  else  $\mathbb{C}$  or let  $\cdots$  in  $\mathbb{C}$ .

We first focus on the first two of them to which the remaining two cases will be reduced later.

• Suppose  $C_1 \equiv \mathsf{mkvar}(\lambda x_1^{\mathsf{unit}}.C_{11}, \lambda x_2^{\mathsf{int}}.C_{12})$ . Then  $\theta \equiv \mathsf{var}$ . Since  $C_2$  is in canonical form, y can only occur in it as part of a canonical subterm of the form  $y^{\mathsf{var}} := z^{\mathsf{int}}$  or !y. Hence, after substitution for y, we will obtain non-canonical subterms of the shape  $\mathsf{mkvar}(\lambda x_1^{\mathsf{unit}}.C_{11}, \lambda x_2^{\mathsf{int}}.C_{12}) := z$  and  $!(\mathsf{mkvar}(\lambda x_1^{\mathsf{unit}}.C_{11}, \lambda x_2^{\mathsf{int}}.C_{12}))$ . Using the rules

$$\begin{array}{l} !\mathsf{mkvar}(\lambda u^{\mathsf{unit}}.D_1,\lambda v^{\mathsf{int}}.D_2) \,\longrightarrow\, D_1[()/u] \\ \mathsf{mkvar}(\lambda u^{\mathsf{unit}}.D_1,\lambda v^{\mathsf{int}}.D_2) := z \,\longrightarrow\, D_2[z/v] \end{array}$$

we can easily convert them (and thus the whole term) to canonical form.

• Suppose  $C_1 \equiv \lambda x_1^{\theta_1}.C_3$  and  $\theta \equiv \theta_1 \rightarrow \theta_2$ . Let us substitute  $C_1$  for the rightmost occurrence of y in  $C_2$ . This will create a non-canonical subterm in  $C_2$  of the form let  $x^{\theta_2} = (\lambda x_1^{\theta_1}.C_3)C_4$  in  $C_5 \equiv \text{let } x^{\theta_2} = (\text{let } x_1^{\theta_1} = C_4 \text{ in } C_3)$  in  $C_5$ . By inductive hypothesis for  $\theta_1$ , let  $x_1^{\theta_1} = C_4$  in  $C_3$  can be converted to canonical form, say,  $C_6$ . Consequently, the non-canonical subterm let  $x^{\theta_2} = (\lambda x_1^{\theta_1}.C_3)C_4$  in  $C_5$  can be transformed into the form let  $x^{\theta_2} = C_6$  in  $C_5$ , which — by inductive hypothesis for  $\theta_2$  — can also be converted to canonical form. Thus, we have shown how to recover canonical forms after substitution for the rightmost occurrence of y. Because of the choice of the rightmost occurrence, the transformation does not involve terms containing other occurrences of y, so it will also decrease their overall number in  $C_2$  by one. Consequently, by repeated substitution for rightmost occurrences one can eventually arrive at a canonical form for let  $y^{\theta} = (\lambda x_1^{\theta_1}.C_3)$  in  $C_2$ .

For the remaining two cases it suffices to take advantage of the following conversions before referring to the two cases above.

$$\text{let } y = (\text{if } x \text{ then } D_1 \text{ else } D_0) \text{ in } E \longrightarrow \text{if } x \text{ then } (\text{let } y = D_1 \text{ in } E) \text{ else } (\text{let } y = D_0 \text{ in } E) \\ \text{let } y = (\text{let } x = D \text{ in } E) \text{ in } F \longrightarrow \text{let } x = D \text{ in } (\text{let } y = E \text{ in } F)$$

Now we are ready to prove Lemma 7.5 by induction on term structure. The base cases of (), i are trivial. That of  $x^{\theta}$  follows from Lemma B.1.

The following inductive cases follow directly from the inductive hypothesis:  $\operatorname{new} x \operatorname{in} M$ ,  $\lambda x^{\theta}.M$ , while M do M. The cases of  $M_1 \oplus M_2$  and if M then  $N_1$  else  $N_0$  are only slightly more difficult. After invoking the inductive hypothesis one needs to apply the rules given below. Note that in this case all the let-bindings are of base type.

$$D_1\oplus D_2 \longrightarrow \text{let } x=D_1 \text{ in } (\text{let } y=D_2 \text{ in } (x\oplus y))$$
 if  $D$  then  $D_1$  else  $D_0 \longrightarrow \text{let } x=D$  in  $(\text{if } x \text{ then } D_1 \text{ else } D_0)$ 

For !M and M := N we take advantage of the fact that a canonical form of type var can only take three shapes:  $\mathsf{mkvar}(\lambda x^{\mathsf{unit}}.\mathbb{C}, \lambda y^{\mathsf{int}}.\mathbb{C})$ , if  $x^{\beta}$  then  $\mathbb{C}$  else  $\mathbb{C}$  or let  $\cdots$  in  $\mathbb{C}$ . An appeal to the inductive hypothesis for M and N and the conversions given below will then yield the canonical forms for !M and M := N.

```
\begin{array}{l} !\mathsf{mkvar}(\lambda u^{\mathsf{unit}}.D_1,\lambda v^{\mathsf{int}}.D_2) \,\longrightarrow\, D_1[()/u] \\ \mathsf{mkvar}(\lambda u^{\mathsf{unit}}.D_1,\lambda v^{\mathsf{int}}.D_2) := E \,\longrightarrow\, \mathsf{let}\,\, x^{\mathsf{int}} = E\,\,\mathsf{in}\,\, D_2[x/v] \\ !(\mathsf{if}\,x\,\mathsf{then}\,D_1\,\mathsf{else}\,D_0) \,\longrightarrow\, \mathsf{if}\,x\,\mathsf{then}\,!D_1\,\mathsf{else}\,!D_0 \\ (\mathsf{if}\,x\,\mathsf{then}\,D_1\,\mathsf{else}\,D_0) := D \,\longrightarrow\, \mathsf{if}\,x\,\mathsf{then}\,(D_1 := D)\,\mathsf{else}\,(D_0 := D) \\ !(\mathsf{let}\,\,x = D\,\,\mathsf{in}\,\,E) \,\longrightarrow\, \mathsf{let}\,\,x = D\,\,\mathsf{in}\,\,!E \\ (\mathsf{let}\,\,x = D\,\,\mathsf{in}\,\,E) := F \,\longrightarrow\, \mathsf{let}\,\,x = D\,\,\mathsf{in}\,\,(E := F) \end{array}
```

To convert  $\mathsf{mkvar}(M,N)$  to canonical form we observe that a canonical form of function type can take the following shapes:  $\lambda x^{\theta}.\mathbb{C}$ , if  $x^{\beta}$  then  $\mathbb{C}$  else  $\mathbb{C}$  or let  $\cdots$  in  $\mathbb{C}$ . Hence, by appealing to the inductive hypothesis and then repeately applying the rules below we will arrive at a canonical form.

```
\begin{array}{c} \operatorname{mkvar}(\operatorname{if} x \operatorname{then} D_1 \operatorname{else} D_0, E) \longrightarrow \operatorname{if} x \operatorname{then} \operatorname{mkvar}(D_1, E) \operatorname{else} \operatorname{mkvar}(D_0, E) \\ \operatorname{mkvar}(\operatorname{let} x = M \operatorname{in} D, E) \longrightarrow \operatorname{let} x = M \operatorname{in} \operatorname{mkvar}(D, E) \\ \operatorname{mkvar}(\lambda u^{\operatorname{unit}}.D, \operatorname{if} x \operatorname{then} E_1 \operatorname{else} E_0) \longrightarrow \operatorname{if} x \operatorname{then} \operatorname{mkvar}(\lambda u^{\operatorname{unit}}.D, E_1) \operatorname{else} \operatorname{mkvar}(\lambda u^{\operatorname{unit}}.D, E_0) \\ \operatorname{mkvar}(\lambda u^{\operatorname{unit}}.D, \operatorname{let} x = M \operatorname{in} E) \longrightarrow \operatorname{let} x = M \operatorname{in} \operatorname{mkvar}(\lambda u^{\operatorname{unit}}.D, E) \end{array}
```

Finally, we handle application MN. First we apply the inductive hypothesis to both terms. Then we use the rules below to reveal the  $\lambda$ -abstraction inside the canonical form of M.

(if 
$$x$$
 then  $D_1$  else  $D_0$ ) $E \longrightarrow \text{if } x$  then  $(D_1E)$  else  $(D_2E)$   
(let  $x = D$  in  $E$ ) $F \longrightarrow \text{let } x = D$  in  $(EF)$ 

Now it suffices to be able to deal with terms of the form  $(\lambda x^{\theta}.C_1)C_2 \equiv \text{let } x = C_2 \text{ in } C_1$ , and this is exactly what Lemma B.2 does.

All our transformation preserve denotations: the proofs are simple exercises in the use of Moggi's monadic approach [13] to modelling call-by-value languages (the store-free game model is an instance of the monadic framework).