

BLOCK STRUCTURE VS SCOPE EXTRUSION: BETWEEN INNOCENCE AND OMNISCIENCE*

ANDRZEJ S. MURAWSKI^a AND NIKOS TZEVELEKOS^b

^a University of Warwick
e-mail address: A.Murawski@warwick.ac.uk

^b Queen Mary University of London
e-mail address: nikos.tzevelekos@qmul.ac.uk

ABSTRACT. We study the semantic meaning of block structure using game semantics. To that end, we introduce the notion of block-innocent strategies and characterise call-by-value computation with block-allocated storage through soundness, finite definability and universality results. This puts us in a good position to conduct a comparative study of purely functional computation, computation with block storage as well as that with dynamic memory allocation. For example, we can show that dynamic variable allocation can be replaced with block-allocated variables exactly when the term involved (open or closed) is of base type and that block-allocated storage can be replaced with purely functional computation when types of order two are involved. To illustrate the restrictive nature of block structure further, we prove a decidability result for a finitary fragment of call-by-value Idealized Algol for which it is known that allowing for dynamic memory allocation leads to undecidability.

1. INTRODUCTION

Most programming languages manage memory by employing a stack for local variables and heap storage for data that are supposed to live beyond their initial context. A prototypical example of the former mechanism is Reynolds’s Idealized Algol [23], in which local variables can only be introduced inside blocks of ground type. Memory is then allocated on entry to the block and deallocated on exit. In contrast, languages such as ML permit variables to escape from their current context under the guise of pointers or references. In this case, after memory is allocated at the point of reference creation, the variable must be allowed to persist indefinitely (in practice, garbage collection or explicit deallocation can be used to put an end to its life).

2012 ACM CCS: [Theory of computation]: Models of computation; Semantics and reasoning—Program semantics.

Key words and phrases: Game semantics, references, contextual equivalence.

* Extended abstract appeared in FOSSACS [17].

^a Supported by the Engineering and Physical Sciences Research Council (EP/C539753/1).

^b Supported by the Engineering and Physical Sciences Research Council (EP/F067607/1) and the Royal Academy of Engineering.

In this paper we would like to compare the expressivity of the two paradigms. As a simple example of heap-based memory allocation we consider the language RML, introduced by Abramsky and McCusker in [2], which is a fragment of ML featuring integer-valued references. In op. cit. the authors also construct a fully abstract game model of RML based on strategies (referred to as *knowing* strategies) that allow the Proponent to base his decisions on the full history of play. On the other hand, at around the same time Honda and Yoshida [8] showed that the purely functional core of RML, better known as call-by-value PCF [21], corresponds to *innocent* strategies [10], i.e. those that can only rely on a restricted view of the play when deciding on the next move. Since block-structured storage of Idealized Algol seems less expressive than dynamic memory allocation of ML and more expressive than PCF, it is natural to ask about its exact position in the spectrum of strategies between innocence and omniscience. Our first result is an answer to this question. We introduce the family of *block-innocent* strategies, situated strictly between innocent and knowing strategies, and exhibit a series of results relating such strategies to a call-by-value variant \mathbf{IA}_{cbv} of Idealized Algol.

Block-innocence captures the particular kind of uniformity exhibited by strategies originating from block-structured programs, akin to innocence yet strictly weaker. In fact, we shall define block-innocence through innocence in a setting enriched with explicit store annotations added to standard moves. For instance, in the play shown below¹, if P follows a block-innocent strategy, P is free to use different moves as the fourth (1) and sixth (2) moves, but the tenth one (0) and the twelfth one (0) have to be the same.

The above play is present in the strategy representing the term

$$f : (\text{unit} \rightarrow \text{int}) \rightarrow (\text{unit} \rightarrow \text{int}) \vdash (\text{new } x \text{ in } (\text{let } g = f(\lambda y^{\text{unit}}.(x := !x + 1); !x) \text{ in } ())) ; \\ \lambda y^{\text{unit}}.0 : \text{unit} \rightarrow \text{int}$$

and the necessity to play the same value (0 in this case) in the twelfth move once 0 has been played in the tenth one stems from the fact that variables in \mathbf{IA}_{cbv} can only be allocated in blocks of ground type. For example, the block in which x was allocated cannot extend over $\lambda y^{\text{unit}}.0$.

Additionally, our framework can detect “storage violations” resulting from an attempt to access a variable from outside of its block. For instance, no \mathbf{IA}_{cbv} -term will ever produce the following play.

The last move is the offending one: for the term given above, it would amount to trying to use g after deallocation of the block for x . Note, though, that the very similar play drawn below does originate from an \mathbf{IA}_{cbv} -term.

Take, e.g. $f : (\text{unit} \rightarrow \text{int}) \rightarrow (\text{unit} \rightarrow \text{int}) \vdash \text{let } g = f(\lambda y^{\text{unit}}.1) \text{ in } \lambda y^{\text{unit}}.g() : \text{unit} \rightarrow \text{int}$.

The notion of block-innocence provides us with a systematic methodology to address expressivity questions related to block structure such as “Does a given strategy originate

¹For the sake of clarity, we only include pointers pointing more than one move ahead.

from a stack-based memory discipline?” or “Can a given program using dynamic memory allocation be replaced with an equivalent program featuring stack-based storage?”. To illustrate the approach we conduct a complete study of the relationship between the three classes of strategies (innocent, block-innocent and knowing respectively) according to the underpinning type shape. We find that knowingness implies block-innocence when terms of base types (open or closed) are involved, that block-innocence implies innocence exactly for types of order at most two, and that knowingness implies innocence if the term is of base type and its free identifiers are of order 1. The fact that knowingness and innocence coincide at terms of base types implies, in particular, that RML and IA_{cbv} contexts have the same expressive power: two RML-terms can be distinguished by an RML-context if, and only if, they can be distinguished by an IA_{cbv} -context.

As a further confirmation of the restrictive nature of the stack discipline of IA_{cbv} , we prove that program equivalence is decidable for a finitary variant of IA_{cbv} which properly contains all second-order types as well as some third-order types (interestingly, this type discipline covers the available higher-order types in PASCAL). In contrast, the corresponding restriction of RML is known to be undecidable [15].

Related work. The stack discipline has always been regarded as part of the essence of Algol [23]. The first languages introduced in that lineage, i.e. Algol 58 and 60, featured both call-by-name and call-by-value parameters. Call-by-name was abandoned in Algol 68, though, and was absent from subsequent designs, such as Pascal and C.

On the semantic front, finding models embodying stack-oriented storage management has always been an important goal of research into Algol-like languages. In this spirit, in the early 1980s, Reynolds [23] and Oles [18] devised a semantic model of (call-by-name) Algol-like languages using a category of functors from a category of store shapes to the category of predomains. Perhaps surprisingly, in the 1990s, Pitts and Stark [20, 24] managed to adapt the techniques to (call-by-value) languages with dynamic allocation. This would appear to create a common platform suitable for a comparative study such as ours. However, despite the valuable structural insights, the relative imprecision of the functor category semantics (failure of definability and full abstraction) makes it unlikely that the results obtained by us can be proved via this route. The semantics of local effects has also been investigated from the category-theoretic point of view in [22].

As for the game semantics literature, Ong’s work [19] based on strategies-with-state is the work closest to ours. His paper defines a compositional framework that is proved sound for the third-order fragment of *call-by-name* Idealized Algol. Adapting the results to call-by-value and all types is far from immediate, though. For a start, to handle higher-order types, we note that the state of O-moves is no longer determined by their justifier and the preceding move. Instead, the right state has to be computed globally using the whole history of play. However, the obvious adaptation of this idea to call-by-value does not capture the block structure of IA_{cbv} . Quite the opposite: it seems to be more compatible with RML. Consequently, further changes are needed to characterize IA_{cbv} . Firstly, to restore definability, the explicit stores have to become lists instead of sets. Secondly, conditions controlling state changes must be tightened. In particular, P must be forbidden from introducing fresh variables at any step and, in a similar vein, must be forced to drop some variables from his moves in certain circumstances.

Another related paper is [4], in which Abramsky and McCusker introduce a model of Idealized Algol with passive (side-effect-free) expressions [4]. Their framework is based on a

$$\begin{array}{c}
\frac{\Gamma, x : \text{var} \vdash M : \beta}{\Gamma \vdash \text{new } x \text{ in } M : \beta} \quad \frac{}{\Gamma \vdash \text{ref} : \text{var}} \quad \frac{}{\Gamma \vdash () : \text{unit}} \quad \frac{i \in \mathbb{Z}}{\Gamma \vdash i : \text{int}} \quad \frac{(x : \theta) \in \Gamma}{\Gamma \vdash x : \theta} \\
\\
\frac{\Gamma \vdash M_1 : \text{int} \quad \Gamma \vdash M_2 : \text{int}}{\Gamma \vdash M_1 \oplus M_2 : \text{int}} \quad \frac{\Gamma \vdash M : \text{int} \quad \Gamma \vdash N_0 : \theta \quad \Gamma \vdash N_1 : \theta}{\Gamma \vdash \text{if } M \text{ then } N_1 \text{ else } N_0 : \theta} \\
\\
\frac{\Gamma \vdash M : \text{var}}{\Gamma \vdash !M : \text{int}} \quad \frac{\Gamma \vdash M : \text{var} \quad \Gamma \vdash N : \text{int}}{\Gamma \vdash M := N : \text{unit}} \quad \frac{\Gamma \vdash M : \text{unit} \rightarrow \text{int} \quad \Gamma \vdash N : \text{int} \rightarrow \text{unit}}{\Gamma \vdash \text{mkvar}(M, N) : \text{var}} \\
\\
\frac{\Gamma \vdash M : \theta \rightarrow \theta' \quad \Gamma \vdash N : \theta}{\Gamma \vdash MN : \theta'} \quad \frac{\Gamma, x : \theta \vdash M : \theta'}{\Gamma \vdash \lambda x^\theta. M : \theta \rightarrow \theta'} \quad \frac{\Gamma \vdash M : (\theta \rightarrow \theta') \rightarrow (\theta \rightarrow \theta')}{\Gamma \vdash Y(M) : \theta \rightarrow \theta'}
\end{array}$$

Figure 1: Syntax of \mathcal{L}

distinction between *active* and *passive* moves, which correspond to active and passive types respectively. Legal plays must then satisfy a novel correctness condition, called *activity*, and strategies must be *a/p-innocent*. In contrast, our setting does not feature any type support for discovering the presence of storage. Moreover, as the sequences discussed in the Introduction demonstrate, in order to understand legality in our setting, it is sometimes necessary to scrutinise values used in plays: changing 1, 1 to 1, 2 may entail loss of correctness! This is different from the activity condition (and other conditions used in game semantics), where it suffices to consider the kind of moves involved (question/answer) or pointer patterns. Consequently, in order to capture the desired shape of plays and associated notion of innocence in our setting, we felt it was necessary to introduce moves explicitly decorated with stores.

Our paper is also related to the efforts of finding decidable fragments of (finitary) RML as far as contextual equivalence is concerned. Despite several papers in the area [7, 15, 9, 5], no full classification based on type shapes has emerged yet, even though the corresponding call-by-name case has been fully mapped out [16]. We show that, for certain types, moving from RML to IA_{cbv} (thus weakening storage capabilities) can help to regain decidability.

2. SYNTAX

To set a common ground for our investigations, we introduce a higher-order programming language that features syntactic constructs for both block and dynamic memory allocation.

Definition 2.1 (The language \mathcal{L}). We define types as generated by the grammar below, where β ranges over the ground types `unit` and `int`.

$$\theta ::= \beta \mid \text{var} \mid \theta \rightarrow \theta$$

The syntax of \mathcal{L} is given in Figure 1.

Note in particular the first two rules concerning variables and the rule for the `mkvar` constructor: the latter allows us to build “bad variables” in accordance with Idealized Algol. The order of a type is defined as follows:

$$\begin{aligned}
\text{ord}(\beta) &= 0 \\
\text{ord}(\text{var}) &= 1 \\
\text{ord}(\theta_1 \rightarrow \theta_2) &= \max(\text{ord}(\theta_1) + 1, \text{ord}(\theta_2)).
\end{aligned}$$

$$\begin{array}{c}
\frac{V \text{ is a value}}{s, V \Downarrow s, V} \quad \frac{M \Downarrow 0 \quad N_0 \Downarrow V}{\text{if } M \text{ then } N_1 \text{ else } N_0 \Downarrow V} \quad \frac{i \neq 0 \quad M \Downarrow i \quad N_1 \Downarrow V}{\text{if } M \text{ then } N_1 \text{ else } N_0 \Downarrow V} \\
\\
\frac{M_1 \Downarrow i_1 \quad M_2 \Downarrow i_2}{M_1 \oplus M_2 \Downarrow i_1 \oplus i_2} \quad \frac{M \Downarrow \lambda x.M' \quad N \Downarrow V' \quad M'[V'/x] \Downarrow V}{MN \Downarrow V} \\
\\
\frac{s, M \Downarrow s', \alpha \quad s'(\alpha) = i}{s, !M \Downarrow s', i} \quad \frac{s, M \Downarrow s', \alpha \quad s', N \Downarrow s'', i}{s, M := N \Downarrow s''(\alpha \mapsto i), ()} \\
\\
\frac{M \Downarrow \text{mkvar}(V_1, V_2) \quad V_1() \Downarrow i}{!M \Downarrow i} \quad \frac{M \Downarrow \text{mkvar}(V_1, V_2) \quad N \Downarrow i \quad V_2 i \Downarrow ()}{M := N \Downarrow ()} \\
\\
\frac{M \Downarrow V_1 \quad N \Downarrow V_2}{\text{mkvar}(M, N) \Downarrow \text{mkvar}(V_1, V_2)} \quad \frac{M \Downarrow V}{Y(M) \Downarrow \lambda x^\theta.(V(Y(V)))x} \\
\\
\frac{s \cup (\alpha \mapsto 0), M[\alpha/x] \Downarrow s', V}{s, \text{new } x \text{ in } M \Downarrow s' \setminus \alpha, V} \quad \alpha \notin \text{dom } s \quad \frac{}{s, \text{ref} \Downarrow s \cup (\alpha \mapsto 0), \alpha} \quad \alpha \notin \text{dom } s
\end{array}$$

Figure 2: Operational semantics of \mathcal{L}

For any $i \geq 0$, terms that are typable using exclusively judgments of the form

$$x_1 : \theta_1, \dots, x_n : \theta_n \vdash M : \theta$$

where $\text{ord}(\theta_j) < i$ ($1 \leq j \leq n$) and $\text{ord}(\theta) \leq i$, are said to form the i th-order fragment.

To spell out the operational semantics of \mathcal{L} , we need to assume a countable set Loc of *locations*, which are added to the syntax as auxiliary constants of type `var`. We shall write α to range over them. The semantics then takes the form of judgments $s, M \Downarrow s', V$, where s, s' are finite partial functions from Loc to integers, M is a closed term and V is a value. Terms of the following shapes are values: $()$, integer constants, elements of Loc , λ -abstractions or terms of the form $\text{mkvar}(\lambda x^{\text{unit}}.M, \lambda y^{\text{int}}.N)$.

The operational semantics is given via the large-step rules in Figure 2. Most of them take the form

$$\frac{M_1 \Downarrow V_1 \quad M_2 \Downarrow V_2 \quad \dots \quad M_n \Downarrow V_n}{M \Downarrow V}$$

which is meant to abbreviate:

$$\frac{s_1, M_1 \Downarrow s_2, V_1 \quad s_2, M_2 \Downarrow s_3, V_2 \quad \dots \quad s_n, M_n \Downarrow s_{n+1}, V_n}{s_1, M_1 \Downarrow s_{n+1}, V}$$

This is a common semantic convention, introduced in the Definition of Standard ML [11]. In particular, it means that the ordering of the hypotheses is significant. The penultimate rule in the figure encapsulates the state within the newly created block, while the last one creates a reference to a new memory cell that can be passed around without restrictions on its scope. Note that $s' \setminus \alpha$ is the restriction of s' to $\text{dom } s' \setminus \{\alpha\}$.

Given a closed term $\vdash M : \text{unit}$, we write $M \Downarrow$ if there exists s such that $\emptyset, M \Downarrow s, ()$. We shall call two programs equivalent if they behave identically in every context. This is captured by the following definition, parameterised by the kind of contexts that are considered, to allow for testing of terms with contexts originating from a designated subset of the language.

Definition 2.2. Suppose \mathcal{L}' is a subset of \mathcal{L} . We say that the terms-in-context $\Gamma \vdash M_1, M_2 : \theta$ are \mathcal{L}' -equivalent (written $\Gamma \vdash M_1 \cong_{\mathcal{L}'} M_2$) if, for any \mathcal{L}' -context C such that $\vdash C[M_1], C[M_2] : \text{unit}$, $C[M_1] \Downarrow$ if and only if $C[M_2] \Downarrow$.

We shall study three sublanguages of \mathcal{L} , called PCF^+ , IA_{cbv} and RML respectively. The latter two have appeared in the literature as paradigmatic examples of programming languages with stack discipline and dynamic memory allocation respectively.

- PCF^+ is a purely functional language obtained from \mathcal{L} by removing `new x in M` and `ref`. It extends the language PCF [21] with primitives for variable access, but not for memory allocation.
- IA_{cbv} is \mathcal{L} without the `ref` constant. It can be viewed as a call-by-value variant of Idealized Algol [23]. Only block-allocated storage is available in IA_{cbv} .
- RML is \mathcal{L} save the construct `new x in M` . It is exactly the language introduced in [2] as a prototypical language for ML-like integer references.²

We shall often use `let $x = M$ in N` as shorthand for $(\lambda x.N)M$. Moreover, `let $x = M$ in N` , where x does not occur in N , will be abbreviated to $M;N$. Note also that `new x in M` is equivalent to `let $x = \text{ref}$ in M` .

Example 2.3. The term $\vdash \text{let } v = \text{ref in } \lambda x^{\text{unit}}.(\text{if } !v \text{ then } \Omega \text{ else } v := !v + 1) : \text{unit} \rightarrow \text{unit}$ is an example of an RML -term that is not RML -equivalent to any term from IA_{cbv} . On the other hand,

$\vdash \lambda f^{(\text{unit} \rightarrow \text{unit}) \rightarrow \text{unit}}. \text{new } v \text{ in } f(\lambda y^{\text{unit}}. \text{if } !v \text{ then } \Omega \text{ else } v := !v + 1) : ((\text{unit} \rightarrow \text{unit}) \rightarrow \text{unit}) \rightarrow \text{unit}$

is an IA_{cbv} -term that has no RML -equivalent in PCF^+ . All of the inequivalence claims will follow immediately from our results.

Lemma 2.4. Given any base type \mathcal{L} -term $\Gamma, x : \text{var} \vdash M : \beta$, we have $\Gamma \vdash \text{new } x \text{ in } M \cong_{\mathcal{L}} \text{let } x = \text{ref in } M : \beta$.

Proof. The proof is based on the following two claims:

- if $s, M \Downarrow s', V$ and $\alpha \in \text{dom}(s)$ does not appear in M , then $s \setminus \alpha, M \Downarrow s' \setminus \alpha, V$;
- for any closed context C , value V and s, s' , if $s, C[\text{let } x = \text{ref in } M] \Downarrow s', V$ then there is a set of locations $S \subseteq \text{dom}(s')$ such that $s, C[\text{new } x \text{ in } M] \Downarrow s' \setminus S, V$ and S contains no locations from s or V ;

which are proven by straightforward induction. □

Hence, RML and \mathcal{L} merely differ on a syntactic level in that \mathcal{L} contains “syntactic sugar” for blocks. In the opposite direction, our results will show that `ref` cannot in general be replaced with an equivalent term that uses `new x in M` . Indeed, our paper provides a general methodology for identifying and studying scenarios in which this expressivity gap is real.

3. GAME SEMANTICS

We next introduce the game models used throughout the paper, which are based on the Honda-Yoshida approach to modelling call-by-value computation [8].

Definition 3.1. An *arena* $A = (M_A, I_A, \vdash_A, \lambda_A)$ is given by

- a set M_A of moves, and a subset $I_A \subseteq M_A$ of *initial* moves,

²In other words, RML is Reduced ML [24] with the addition of the `mkvar` construct.

- a justification relation $\vdash_A \subseteq M_A \times (M_A \setminus I_A)$, and
- a labelling function $\lambda_A : M_A \rightarrow \{O, P\} \times \{Q, A\}$

such that $\lambda_A(I_A) \subseteq \{PA\}$ and, whenever $m \vdash_A m'$, we have $(\pi_1 \lambda_A)(m) \neq (\pi_2 \lambda_A)(m')$ and $(\pi_2 \lambda_A)(m') = A \implies (\pi_2 \lambda_A)(m) = Q$.

The role of λ_A is to label moves as *Opponent* or *Proponent* moves and as *Questions* or *Answers*. We typically write them as m, n, \dots , or $o, p, q, a, q_P, q_O, \dots$ when we want to be specific about their kind. Note that we abbreviate elements of the codomain of λ_A , e.g. (P, A) above is written as PA .

The simplest arena is $0 = (\emptyset, \emptyset, \emptyset, \emptyset)$. Other “flat” arenas are 1 and \mathbb{Z} , defined by:

$$M_1 = I_1 = \{*\}, \quad M_{\mathbb{Z}} = I_{\mathbb{Z}} = \mathbb{Z}.$$

Below we recall two standard constructions on arenas, where \bar{I}_A stands for $M_A \setminus I_A$, the OP -complement of λ_A is written as $\bar{\lambda}_A$, and i_A, i_B range over initial moves in the respective arenas.

$$\begin{aligned} M_{A \Rightarrow B} &= I_{A \Rightarrow B} \uplus I_A \uplus \bar{I}_A \uplus M_B \\ I_{A \Rightarrow B} &= \{*\} \\ \lambda_{A \Rightarrow B} &= [(*, PA), (i_A, OQ), \bar{\lambda}_A \upharpoonright \bar{I}_A, \lambda_B] \\ \vdash_{A \Rightarrow B} &= \{(*, i_A), (i_A, i_B)\} \cup \vdash_A \cup \vdash_B \end{aligned}$$

$$\begin{aligned} M_{A \otimes B} &= I_{A \otimes B} \uplus \bar{I}_A \uplus \bar{I}_B \\ I_{A \otimes B} &= I_A \times I_B \\ \lambda_{A \otimes B} &= [((i_A, i_B), PA), \lambda_A \upharpoonright \bar{I}_A, \lambda_B \upharpoonright \bar{I}_B] \\ \vdash_{A \otimes B} &= \{((i_A, i_B), m) \mid i_A \vdash_A m \vee i_B \vdash_B m\} \\ &\quad \cup (\vdash_A \upharpoonright \bar{I}_A^2) \cup (\vdash_B \upharpoonright \bar{I}_B^2) \end{aligned}$$

Types of \mathcal{L} can now be interpreted with arenas in the following way.

$$\begin{aligned} \llbracket \text{unit} \rrbracket &= 1 \\ \llbracket \text{int} \rrbracket &= \mathbb{Z} \\ \llbracket \text{var} \rrbracket &= (1 \Rightarrow \mathbb{Z}) \otimes (\mathbb{Z} \Rightarrow 1) \\ \llbracket \theta_1 \rightarrow \theta_2 \rrbracket &= \llbracket \theta_1 \rrbracket \Rightarrow \llbracket \theta_2 \rrbracket \end{aligned}$$

Note that the type **var** is translated as a product arena the components of which represent its read and write methods.

Although arenas model types, the actual games will be played in *prearenas*, which are defined in the same way as arenas with the exception that initial moves must be O -questions. Given arenas A and B , we can construct the prearena $A \rightarrow B$ by setting:

$$\begin{aligned} M_{A \rightarrow B} &= M_A \uplus M_B \\ I_{A \rightarrow B} &= I_A \\ \lambda_{A \rightarrow B} &= [(i_A, OQ) \cup (\bar{\lambda}_A \upharpoonright \bar{I}_A), \lambda_B] \\ \vdash_{A \rightarrow B} &= \{(i_A, i_B)\} \cup \vdash_A \cup \vdash_B. \end{aligned}$$

For $\Gamma = \{x_1 : \theta_1, \dots, x_n : \theta_n\}$, typing judgments $\Gamma \vdash \theta$ will eventually be interpreted by strategies for the prearena $\llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_n \rrbracket \rightarrow \llbracket \theta \rrbracket$ (if $n = 0$ we take the left-hand side to be 1), which we shall denote by $\llbracket \Gamma \vdash \theta \rrbracket$ or $\llbracket \theta_1, \dots, \theta_n \vdash \theta \rrbracket$.

A **justified sequence** in a prearena A is a finite sequence s of moves of A satisfying the following condition: the first move must be initial, but all other moves m must be equipped with a pointer to an earlier occurrence of a move m' such that $m' \vdash_A m$. We then say that m' *justifies* m . If m is an answer, we may also say that m *answers* m' . If a question remains unanswered in s , it is *open*; and the rightmost open question in s is its *pending question*.

Given a justified sequence s , we define its *O-view* $\lfloor s \rfloor$ and its *P-view* $\lceil s \rceil$ inductively as follows.

- $\lfloor \epsilon \rfloor = \epsilon$, $\lfloor s o \rfloor = \lfloor s \rfloor o$, $\lfloor s \widehat{o \cdots p} \rfloor = \lfloor s \rfloor o p$;
- $\lceil \epsilon \rceil = \epsilon$, $\lceil s p \rceil = \lceil s \rceil p$, $\lceil s \widehat{p \cdots o} \rceil = \lceil s \rceil p o$.

Above, recall that o ranges over O-moves (i.e. moves m such that $(\pi_1 \lambda_A)(m) = O$), and p ranges over P-moves.

Definition 3.2. A **play** in a prearena A is a justified sequence s satisfying the following conditions.

- If $s = \cdots m n \cdots$ then $\lambda_A^{OP}(m) = \bar{\lambda}_A^{OP}(n)$. (*Alternation*)
- If $s = s_1 \widehat{q s_2} a \cdots$ then q is the pending question in $s_1 q s_2$. (*Well-Bracketing*)
- If $s = s_1 \widehat{o s_2} p \cdots$ then o appears in $\lceil s_1 o s_2 \rceil$;
if $s = s_1 \widehat{p s_2} o \cdots$ then p appears in $\lfloor s_1 p s_2 \rfloor$. (*Visibility*)

We write P_A to denote the set of plays in A .

We are going to model terms-in-context $\Gamma \vdash M : \theta$ as sets of plays in $\llbracket \Gamma \vdash \theta \rrbracket$ subject to specific conditions.

Definition 3.3. A (**knowing**) **strategy** σ on a prearena A is a non-empty prefix-closed set of plays from A satisfying the first two conditions below. A strategy is **innocent** if, in addition, the third condition holds.

- If even-length $s \in \sigma$ and $sm \in P_A$ then $sm \in \sigma$. (*O-Closure*)
- If even-length $sm_1, sm_2 \in \sigma$ then $m_1 = m_2$. (*Determinacy*)
- If $s_1 m, s_2 \in \sigma$ with odd-length s_1, s_2 and $\lceil s_1 \rceil = \lceil s_2 \rceil$ then $s_2 m \in \sigma$. (*Innocence*)

We write $\sigma : A$ to denote that σ is a strategy on A .

Note, in particular, that every strategy $\sigma : A$ contains the empty sequence ϵ as well as the elements of I_A , the latter being the 1-move plays in A . Moreover, in the last condition above, the move m in $s_2 m$ points at the same move it points inside $s_1 m$: by visibility and the fact that s_1 and s_2 have the same view, this is always possible.

In previous work it has been shown that knowing strategies yield a fully abstract semantics for RML in the following sense³.

Theorem 3.4 ([2]). Two RML-terms are RML-equivalent if and only if their interpretations contain the same complete plays⁴.

Moreover, as an immediate consequence of the full abstraction result of [8], we have that innocent strategies (quotiented by the intrinsic preorder) yield full abstraction for PCF⁺. What remains open is the model for the intermediate language, \mathbf{IA}_{cbv} , which requires one to identify a family of strategies between the innocent and knowing ones. This is the problem examined in the next section. We address it in two steps.

³Perhaps it is worth noting that the presentation of the model in [2] is in a different setting (we follow Honda-Yoshida call-by-value games, while [2] applies the *family construction* on call-by-name games) which, nonetheless, is equivalent to the one presented above.

⁴A play is called *complete* if each question in that play has been answered.

- First we introduce a category of strategies that are equipped with explicit stores for registering private variables. We show that this category, of so-called innocent *S-strategies*, indeed models block allocation: terms of \mathbf{IA}_{cbv} translate into innocent S-strategies (Proposition 4.25) and, moreover, in a complete manner (Propositions 5.3 and 5.7).
- The strategies capturing \mathbf{IA}_{cbv} , called *block-innocent* strategies, are then defined by deleting stores from innocent S-strategies.

4. GAMES WITH STORES AND THE MODEL OF \mathbf{IA}_{cbv}

We shall now extend the framework to allow moves to be decorated with stores that contain name-integer pairs. This extension will be necessary for capturing block-allocated storage. The names should be viewed as semantic analogues of locations. The stores will be used for carrying the values of private, block-allocated variables.

4.1. Names and stores in games. When employing such moves-with-store, we are not interested in what exactly the names are, but we would like to know how they relate to names that have already been in play. Hence, the objects of study are rather the induced equivalence classes with respect to name-invariance, and all ensuing constructions and reasoning need to be compatible with it. This overhead can be dealt with robustly using the language of nominal set theory [6]. Let us fix a countably infinite set \mathbb{A} , the set of *names*, the elements of which we shall denote by α, β and variants. Consider the group $\text{PERM}(\mathbb{A})$ of finite permutations of \mathbb{A} .

Definition 4.1. A *strong nominal set* [6, 25] is a set equipped with a group action⁵ of $\text{PERM}(\mathbb{A})$ such that each of its elements has *finite strong support*. That is to say, for any $x \in X$, there exists a finite set $\nu(x) \subseteq \mathbb{A}$, called *the support of x* , such that, for all permutations π , $(\forall \alpha \in \nu(x). \pi(\alpha) = \alpha) \iff \pi \cdot x = x$.

Intuitively, $\nu(x)$ is the set of names “involved” in x . For example, the set $\mathbb{A}^\#$ of finite lists of distinct atoms with permutations acting elementwise is a strong nominal set. If X and Y are strong nominal sets, then so is their cartesian product $X \times Y$ (with permutations acting componentwise) and their disjoint union $X \uplus Y$. Name-invariance in a strong nominal set X is represented by the relation: $x \sim x'$ if there exists π such that $x = \pi \cdot x'$.

We define a strong nominal set of *stores*, the elements of which are finite sequences of name-integer pairs. Formally,

$$\Sigma, T ::= \epsilon \mid (\alpha, i) :: \Sigma$$

where $i \in \mathbb{Z}$ and $\alpha \in \mathbb{A} \setminus \nu(\Sigma)$. We view stores as finite functions from names to integers, though their domains are lists rather than sets. Thus, we define the *domain* of a store to be the *list* of names obtained by applying the first projection to all of its elements. In particular, $\nu(\text{dom}(\Sigma)) = \nu(\Sigma)$. If $\alpha \in \nu(\Sigma)$ then we write $\Sigma(\alpha)$ for the unique i such that (α, i) is an element of Σ . For stores Σ, T we write:

$$\begin{aligned} \Sigma \leq T & \text{ for } \text{dom}(\Sigma) \sqsubseteq \text{dom}(T), \\ \Sigma \leq_X T & \text{ for } \text{dom}(\Sigma) \sqsubseteq_X \text{dom}(T), \end{aligned}$$

⁵A group action of $\text{PERM}(\mathbb{A})$ on X is a function $\cdot : \text{PERM}(\mathbb{A}) \times X \rightarrow X$ such that, for all $x \in X$ and $\pi, \pi' \in \text{PERM}(\mathbb{A})$, $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ and $\text{id} \cdot x = x$, where id is the identity permutation.

where $X \in \{p, s\}$, and $\sqsubseteq, \sqsubseteq_p, \sqsubseteq_s$ denote the subsequence, prefix and suffix relations respectively. Note that $\Sigma \leq_X T \leq_X \Sigma$ implies $\text{dom}(\Sigma) = \text{dom}(T)$ but not $\Sigma = T$. Finally, let us write $\Sigma \setminus T$ for Σ restricted to $\nu(\Sigma) \setminus \nu(T)$.

An **S-move** (or *move-with-store*) in a prearena A is a pair consisting of a move and a store. We typically write S-moves as $m^\Sigma, n^T, o^\Sigma, p^T, q^\Sigma, a^T$. The first projection function is viewed as *store erasure* and denoted by $\text{erase}(-)$. Note that moves contain no names and therefore, for any m^Σ , $\nu(m^\Sigma) = \nu(\Sigma) = \nu(\text{dom}(\Sigma))$. A **justified S-sequence** in A is a sequence of S-moves equipped with justifiers, so that its erasure is a justified sequence. The notions of *O-view* and *P-view* are extended to S-sequences in the obvious manner. We say that a name α **is closed** in s if there are no open questions in s containing α .

Definition 4.2. A justified S-sequence s in a prearena A is called an **S-play** if it satisfies the following conditions, for all $\alpha \in \mathbb{A}$.

- If $s = m^\Sigma \dots$ then $\Sigma = \epsilon$. (*Init*)
- If $s = \dots o^\Sigma \overleftarrow{\dots} p^T \dots$ then $\Sigma \leq_p T$. If $\lambda_A(p) = PA$ then $T \leq_p \Sigma$ too. (*Just-P*)
- If $s = \dots p^\Sigma \overrightarrow{\dots} o^T \dots$ then $\Sigma \leq_p T \leq_p \Sigma$. (*Just-O*)
- If $s = s_1 o^\Sigma q_p^T \dots$ then $\Sigma \setminus T \leq_s \Sigma$ and $\Sigma \setminus (\Sigma \setminus T) \leq_p T$ and
 - (a) if $\alpha \in \nu(T \setminus \Sigma)$ then $\alpha \notin \nu(s_1 o^\Sigma)$,
 - (b) if $\alpha \in \nu(\Sigma \setminus T)$ then α is closed in $s_1 o^\Sigma$.
 (*Prev-PQ*)
- If $s = \dots p^\Sigma s' o^T \dots$ and $\alpha \in (\nu(T) \cap \nu(\Sigma)) \setminus \nu(s')$ then $T(\alpha) = \Sigma(\alpha)$. (*Val-O*)

We write SP_A for the set of S-plays in A .

Let us remark that, as stores have strong support, the set of S-plays SP_A is a strong nominal set. The conditions we impose on S-plays reflect the restrictions pertaining to block-allocation of variables. In particular, given a move m , all block-allocated variables present at m are carried over to every move n justified by m . In addition, only P is allowed to allocate/deallocate such variables, or change their values.

Just-P. All variables allocated at o survive in p . If p is an answer then, in fact, the subsequence from o to p represents a whole sub-block, with p closing the sub-block. Thus, o and p must have the same private variables.

Just-O. Each O-move inherits its private variables from its justifier move. Put otherwise, a block does not extend beyond the current P-view and we only store variables that are created by and are private to P (so T cannot be larger than Σ).

Prev-PQ. P-questions can open or close private variables and thus alter the domain of the store. This process must obey the nesting of variables, as reflected in the order of names in stores. Therefore, variables are closed by removing their corresponding names from the right end of the store: $\Sigma \setminus T \leq_s \Sigma$. On the other hand, variables/names that survive comprise the left end of the new store: $\Sigma \setminus (\Sigma \setminus T) \leq_p T$.

In addition, any names that are added in the store must be fresh for the whole sequence (they represent fresh private variables). A final condition disallows variables to be closed if their block still contains open questions.

Val-O. Since variables are private to P, it is not possible for O to change their value: at each O-move o^T , the value of each $\alpha \in \text{dom}(T)$ is the same as that of the last P-move p^Σ such that $\alpha \in \text{dom}(\Sigma)$.

The above is a minimal collection of rules that we need to impose for block allocation. From them we can extract further properties for S-plays, whose proofs are delegated to Appendix A.

Lemma 4.3. The following properties hold for S-plays s .

- If $s = \dots m^\Sigma a_P^T \dots$ then $\Sigma \setminus T \leq_s \Sigma$ and $\Sigma \setminus (\Sigma \setminus T) \leq_p T$ and
 - (a) if $\alpha \in \nu(T)$ then $\alpha \in \nu(\Sigma)$,
 - (b) if $\alpha \in \nu(\Sigma \setminus T)$ then α is closed in $s_{<a_P^T}$.
(*Prev-PA*)
- For any α , we have $\ulcorner s \urcorner = s_1 s_2 s_3$, where
 - $\alpha \notin \nu(s_1) \cup \nu(s_3)$ and $\forall m^\Sigma \in s_2. \alpha \in \nu(\Sigma)$,
 - if $s_2 \neq \epsilon$ then its first element is the move introducing α in s .
(*Block form*)
- If $s = s_1 o^\Sigma p^T s_2$ with $\alpha \in \nu(\Sigma) \setminus \nu(T)$ then $\alpha \notin \nu(s_2)$. (*Close*) □

We now move on to strategies for block allocation.

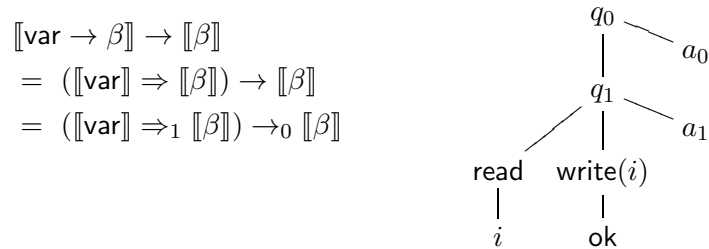
Definition 4.4. An **S-strategy** σ on an arena A is a non-empty prefix-closed set of S-plays from A satisfying the first three of the following conditions. An S-strategy is **innocent** if it also satisfies the last condition.

- If $s' \sim s \in \sigma$ then $s' \in \sigma$. (*Nominal Closure*)
- If even-length $s \in \sigma$ and $sm^\Sigma \in SP_A$ then $sm^\Sigma \in \sigma$. (*O-Closure*)
- If even-length $sm_1^{\Sigma_1}, sm_2^{\Sigma_2} \in \sigma$ then $sm_1^{\Sigma_1} \sim sm_2^{\Sigma_2}$. (*Determinacy*)
- If $s_1 m^{\Sigma_1}, s_2 \in \sigma$ with s_1, s_2 odd-length and $\ulcorner s_1 \urcorner = \ulcorner s_2 \urcorner$ then there exists $s_2 m^{\Sigma_2} \in \sigma$ with $\ulcorner s_1 m^{\Sigma_1} \urcorner \sim \ulcorner s_2 m^{\Sigma_2} \urcorner$. (*Innocence*)

We write $\sigma : A$ to denote that σ is an S-strategy on A .

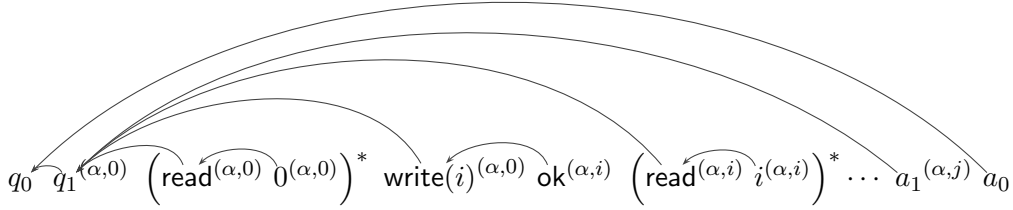
Observe how S-strategies are defined in the same manner as ordinary strategies but follow some additional conditions due to their involving of stores and names.

Example 4.5. For any base type β , consider the prearena $\llbracket \text{var} \rightarrow \beta \rrbracket \rightarrow \llbracket \beta \rrbracket$ given below, where we have indexed moves and type constructors to indicate provenance. We use **read** and **write(i)** ($i \in \mathbb{Z}$) to refer to the question-moves from $\llbracket \text{var} \rrbracket$, and i ($i \in \mathbb{Z}$) and **ok** for the corresponding answers.



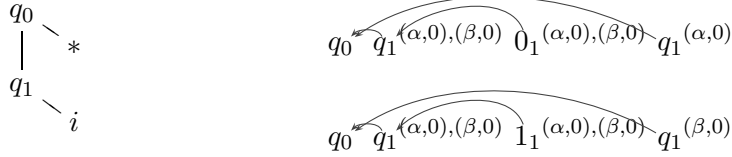
Let us define the S-strategy $\text{cell}_\beta : \llbracket \text{var} \rightarrow \beta \rrbracket \rightarrow \llbracket \beta \rrbracket$ as the S-strategy containing all even-length prefixes of S-plays of the following form (where we use the Kleene star for move

repetition).



Note that, although the cell strategy is typically non-innocent, it is innocent in our framework, where private variables are explicit in game moves (in their stores).

Example 4.6. Let us consider the prearena $\llbracket \text{unit} \rightarrow \text{int} \rrbracket \rightarrow \text{unit}$, depicted as on the left below. Had we used sets instead of lists for representing stores, the following ‘‘S-strategy’’ (right below), which represents incorrect overlap of scopes (α and β are in scope of one another, but at the same time have different scopes), would have been valid (and innocent).



However, the above is not a valid S-strategy in our language setting. Intuitively, it would correspond to a term that determines the scope of its variables on the fly, depending on the value (0 or 1) received after memory allocation.

4.2. Composing S-plays. We next define composition of S-plays, following the approach of [8, 25]. Let us introduce some notation on stores. For a sequence of S-moves s and stores Σ, T , we write $\Sigma[s]$, $\Sigma[T]$ and $\Sigma + T$ for the stores defined by: $\epsilon[s] = \epsilon[T] \triangleq \epsilon$, $\epsilon + T \triangleq T$ and

$$\begin{aligned} ((\alpha, i) :: \Sigma)[s] &\triangleq \begin{cases} (\alpha, T(\alpha)) :: (\Sigma[s]) & \text{if } s = s_1 m^T s_2 \wedge \alpha \in \nu(T) \setminus \nu(s_2) \\ (\alpha, i) :: (\Sigma[s]) & \text{otherwise} \end{cases} \\ ((\alpha, i) :: \Sigma)[T] &\triangleq \begin{cases} (\alpha, T(\alpha)) :: (\Sigma[T]) & \text{if } \alpha \in \nu(T) \\ (\alpha, i) :: (\Sigma[T]) & \text{otherwise} \end{cases} \\ ((\alpha, i) :: \Sigma) + T &\triangleq \begin{cases} (\alpha, i) :: (\Sigma + T) & \text{if } \alpha \notin \nu(T) \\ \text{undefined} & \text{otherwise} \end{cases} \end{aligned}$$

Moreover, we write $\text{st}(s)$ for the store of the final S-move in s . For instance, by *Prev-PQ* and *Prev-PA*, if $o^{\Sigma} p^T$ are consecutive inside an S-play then $T = \Sigma[T] \setminus (\Sigma \setminus T) + (T \setminus \Sigma)$.

It will also be convenient to introduce the following store-constructor. For stores $\Sigma_0, \Sigma_1, \Sigma_2$ we define

$$\Phi(\Sigma_0, \Sigma_1, \Sigma_2) \triangleq \Sigma_0[\Sigma_2] \setminus (\Sigma_1 \setminus \Sigma_2) + (\Sigma_2 \setminus \Sigma_1).$$

Considering Σ_1, Σ_2 as *consecutive*, the constructor first updates Σ_0 with values from Σ_2 , removes those names that have been *dropped* in Σ_2 and then adds those that have been *introduced* in it.

Definition 4.7. Let A, B, C be arenas and $s \in SP_{A \rightarrow B}, t \in SP_{B \rightarrow C}$. We say that s, t are *compatible*, written $s \asymp t$, if $\text{erase}(s) \upharpoonright B = \text{erase}(t) \upharpoonright B$ and $\nu(s) \cap \nu(t) = \emptyset$. In such a case, we define their *interaction*, $s \parallel t$, and their *mix*, $s \cdot t$, recursively as follows,

$$\begin{array}{ll}
\epsilon \parallel \epsilon \triangleq \epsilon & \epsilon \cdot \epsilon \triangleq \epsilon \\
sm_A^\Sigma \parallel t \triangleq (s \parallel t)m_A^{sm_A^\Sigma \cdot t} & sn^T m_{A(P)}^\Sigma \cdot t \triangleq \Phi(sn^T \cdot t, T, \Sigma) \\
& sm_{A(O)}^\Sigma \cdot t \triangleq \tilde{\Sigma}[s \parallel t] \\
sm_B^\Sigma \parallel tm_B^{\Sigma'} \triangleq (s \parallel t)m_B^{sm_B^\Sigma \cdot tm_B^{\Sigma'}} & sn^T m_{B(P)}^\Sigma \cdot tm_{B(O)}^{\Sigma'} \triangleq \Phi(sn^T \cdot t, T, \Sigma) \\
& sm_{B(O)}^{\Sigma'} \cdot tn^T m_{B(P)}^\Sigma \triangleq \Phi(s \cdot tn^T, T, \Sigma) \\
s \parallel tm_C^\Sigma \triangleq (s \parallel t)m_C^{s \cdot tm_C^\Sigma} & s \cdot tn^T m_{C(P)}^\Sigma \triangleq \Phi(s \cdot tn^T, T, \Sigma) \\
& s \cdot tm_{C(O)}^\Sigma \triangleq \tilde{\Sigma}[s \parallel t]
\end{array}$$

where justification pointers in $s \parallel t$ are inherited from s and t , and $\tilde{\Sigma}$ is the store of $m_{A/C(O)}$'s justifier in $s \parallel t$. Note that $s \cdot t$ is the store of the last S-move in $s \parallel t$. The *composite* of s and t is

$$s; t \triangleq (s \parallel t) \upharpoonright AC.$$

We moreover let

$$SInt(A, B, C) \triangleq \{s \parallel t \mid s \in SP_{A \rightarrow B} \wedge t \in SP_{B \rightarrow C} \wedge s \asymp t\}$$

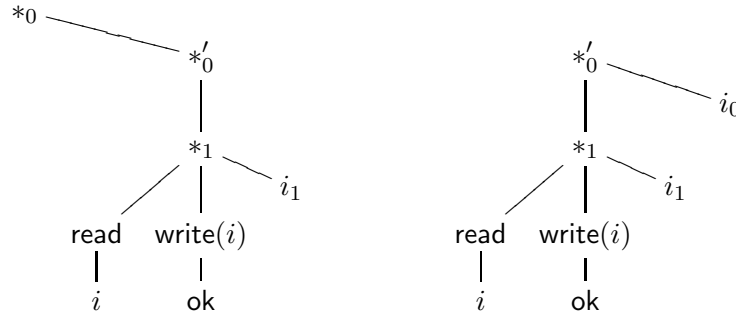
be the set of **S-interaction sequences** of A, B, C .

Example 4.8. Let us demonstrate how to compose S-plays from the following strategies:

$$\sigma \triangleq \llbracket \vdash \lambda x^{\text{var}}. x := !x + 1; !x : \text{var} \rightarrow \text{int} \rrbracket : 1 \rightarrow (\llbracket \text{var} \rrbracket \Rightarrow \mathbb{Z})$$

$$\tau \triangleq \llbracket f : \text{var} \rightarrow \text{int} \vdash (\text{new } x \text{ in } fx) + \text{new } x \text{ in } fx : \text{int} \rrbracket : (\llbracket \text{var} \rrbracket \Rightarrow \mathbb{Z}) \rightarrow \mathbb{Z}$$

We depict the two prearenas below (compared to Example 4.5, in the prearena on the right we have replaced q and a with concrete moves $*$ and $i \in \mathbb{Z}$).



The S-strategy σ is given by even-length prefixes of S-plays of the form:

$$*_0 \ *'_0 \ *_1 \ \text{read } i \ \text{write}(i+1) \ \text{ok} \ \text{read } j \ j_1 \ *_1 \ \text{read } i' \ \text{write}(i'+1) \ \text{ok} \ \text{read } j' \ j'_1 \ \dots$$

where the *read*'s and *write*'s point to the last $*_1$ on their left (and each other missing pointer is assumed to point to the next move on the left). On the other hand, the elements of τ are

even-length prefixes of S-plays with the following pattern:

$$\begin{aligned} *'_0 *'_1^{(\alpha,0)} & \left(\text{read}^{(\alpha,0)} 0^{(\alpha,0)} \right)^* \text{write}(i)^{(\alpha,0)} \text{ok}^{(\alpha,i)} \left(\text{read}^{(\alpha,i)} i^{(\alpha,i)} \right)^* \dots j_1^{(\alpha,k)} \\ & *'_1^{(\alpha',0)} \left(\text{read}^{(\alpha',0)} 0^{(\alpha',0)} \right)^* \text{write}(i')^{(\alpha',0)} \text{ok}^{(\alpha',i')} \left(\text{read}^{(\alpha',i')} i^{(\alpha',i')} \right)^* \dots j'_1^{(\alpha',k')} (j+j')_0 \end{aligned}$$

Observe that an S-play $s \in \sigma$ can only be composed with a $t \in \tau$ if it satisfies the read-write discipline of variables: each read move must be answered by the last write value, apart of the initial read that should be answered by 0. Moreover, s must feature at most two moves $*_1$. We therefore consider the following S-plays $s \asymp t$.

$$\begin{aligned} s &= *_0 *'_0 *'_1 \text{read } 0 \text{ write}(1) \text{ok read } 1 \ 1_1 *'_1 \text{read } 0 \text{ write}(1) \text{ok read } 1 \ 1_1 \\ t &= *_0 *'_1^{(\alpha,0)} \text{read}^{(\alpha,0)} 0^{(\alpha,0)} \text{write}(1)^{(\alpha,0)} \text{ok}^{(\alpha,1)} \text{read}^{(\alpha,1)} 1^{(\alpha,1)} 1_1^{(\alpha,1)} \\ & \quad *_1^{(\alpha',0)} \text{read}^{(\alpha',0)} 0^{(\alpha',0)} \text{write}(1)^{(\alpha',0)} \text{ok}^{(\alpha',1)} \text{read}^{(\alpha',1)} 1^{(\alpha',1)} 1_1^{(\alpha',1)} 2_0 \end{aligned}$$

By composing, we obtain:

$$\begin{aligned} s \parallel t &= *_0 *'_0 *'_1^{(\alpha,0)} \text{read}^{(\alpha,0)} 0^{(\alpha,0)} \text{write}(1)^{(\alpha,0)} \text{ok}^{(\alpha,1)} \text{read}^{(\alpha,1)} 1^{(\alpha,1)} 1_1^{(\alpha,1)} \\ & \quad *_1^{(\alpha',0)} \text{read}^{(\alpha',0)} 0^{(\alpha',0)} \text{write}(1)^{(\alpha',0)} \text{ok}^{(\alpha',1)} \text{read}^{(\alpha',1)} 1^{(\alpha',1)} 1_1^{(\alpha',1)} 2_0 \end{aligned}$$

and $s; t = *_0 2_0$.

As a side-note, let us observe that, had we considered a slightly different strategy to compose S-plays from σ with:

$$\tau' \triangleq \llbracket f : \text{var} \rightarrow \text{int} \vdash \text{new } x \text{ in } fx + fx : \text{int} \rrbracket : (\llbracket \text{var} \rrbracket \Rightarrow \mathbb{Z}) \rightarrow \mathbb{Z}$$

we would only be able to compose variants of s and t :

$$\begin{aligned} s' &= *_0 *'_0 *'_1 \text{read } 0 \text{ write}(1) \text{ok read } 1 \ 1_1 *'_1 \text{read } 1 \text{ write}(2) \text{ok read } 2 \ 2_1 \\ t' &= *_0 *'_1^{(\alpha,0)} \text{read}^{(\alpha,0)} 0^{(\alpha,0)} \text{write}(1)^{(\alpha,0)} \text{ok}^{(\alpha,1)} \text{read}^{(\alpha,1)} 1^{(\alpha,1)} 1_1^{(\alpha,1)} \\ & \quad *_1^{(\alpha,1)} \text{read}^{(\alpha,1)} 0^{(\alpha,1)} \text{write}(2)^{(\alpha,1)} \text{ok}^{(\alpha,2)} \text{read}^{(\alpha,2)} 2^{(\alpha,2)} 2_1^{(\alpha,2)} 3_0 \end{aligned}$$

and thus obtain $s'; t' = *_0 3_0$.

Given an interaction sequence u and a move m of u , we call m a **generalised P-move** if it is a P-move in AB or in BC (by which we mean a P-move in $A \rightarrow B$ or $B \rightarrow C$ respectively). We call m an **external O-move** if it is an O-move in AC . The notions of P-view and O-view extend to interaction sequences as follows. Let p be a generalised P-move and o be an external O-move.

$$\begin{aligned} \ulcorner o \urcorner &\triangleq o & \llcorner \epsilon \llcorner &\triangleq \epsilon \\ \ulcorner up^\Sigma \urcorner &\triangleq \ulcorner u \urcorner p^\Sigma & \llcorner uo^\Sigma \llcorner &\triangleq \llcorner u \llcorner o^\Sigma \\ \ulcorner um^T \overleftarrow{\dots} o^\Sigma \urcorner &\triangleq \ulcorner u \urcorner m^T o^\Sigma & \llcorner um^T \overleftarrow{\dots} p^\Sigma \llcorner &\triangleq \llcorner u \llcorner m^T p^\Sigma \end{aligned}$$

Observe that if u ends in a P-move in AB (resp. BC) then $\llcorner u \llcorner = \llcorner u \upharpoonright AB \llcorner$ ($\llcorner u \upharpoonright BC \llcorner$).

We now show that play-composition is well-defined. The result follows from the next two lemmas. The first one is standard, while the proof of the second one is given in Appendix A.

Lemma 4.9 (Zipper Lemma [8]). If $s \in SP_{A \rightarrow B}$, $t \in SP_{B \rightarrow C}$ and $s \asymp t$ then either $s \upharpoonright B = t = \epsilon$, or s ends in A and t in B (with a P-move in BC), or s ends in B (with a P-move in AB) and t in C , or both s and t end in B (with the same move).

Lemma 4.10. Suppose $s \in SP_{A \rightarrow B}, t \in SP_{B \rightarrow C}, s \asymp t$ and p a generalised P-move.

- (1) If $s \parallel t = un^T p^\Sigma$ then $\nu(\Sigma \setminus T) \cap \nu(un^T) = \emptyset$.
- (2) For any $\alpha, \lceil s \parallel t \rceil$ is in block-form: $\lceil s \parallel t \rceil = u_1 u_2 u_3$ where α appears in every move of $u_2, \alpha \notin \nu(u_1) \cup \nu(u_3)$ and if $u_2 \neq \epsilon$ then its first move introduces α in $s \parallel t$.
- (3) If $s \parallel t = un^T p^\Sigma$ and $\alpha \in \nu(T \setminus \Sigma)$ then α is closed in un^T .
- (4) If $s \parallel t = \dots n^T \overleftarrow{\dots} m^\Sigma$ then $T \leq_p \Sigma$. If m is an answer then $\Sigma \leq_p T$.
- (5) If $s \parallel t = u_1 n^T p^\Sigma u_2$ and $\alpha \in \nu(T \setminus \Sigma)$ then $\alpha \notin \nu(u_2)$.
- (6) If $s \parallel t = um^\Sigma$ with m an O-move in AC then, for any $\alpha \in \nu(\Sigma)$, the last appearance of α in u occurs in AC .
- (7) If $s \parallel t = um^\Sigma$ and $s = s'm^{\Sigma'}$ or $t = t'm^{\Sigma'}$ then $\Sigma' \leq \Sigma$ and $\Sigma[\Sigma'] = \Sigma$. Moreover, $\Sigma[\text{st}(s)] = \Sigma[\text{st}(t)] = \Sigma$.
- (8) If $s \parallel t = un^T p^\Sigma$ then $T \setminus (T \setminus \Sigma) \leq_p \Sigma$ and $T \setminus \Sigma \leq_s T$.

Proposition 4.11 (Compositionality). S-play composition is well defined, that is, if $s \in SP_{A \rightarrow B}, t \in SP_{B \rightarrow C}$ and $s \asymp t$ then $s; t \in SP_{A \rightarrow C}$.

Proof. We need to verify the 5 conditions of Definition 4.2. Init is straightforward. Just-P follows from part (d) of the Lemma 4.10. Just-O follows directly from the definition of interaction sequences. Prev-PQ follows from parts (a,c,h) Lemma 4.10. Finally, Val-O follows from part (f) of Lemma 4.10 and the definition of interaction sequences. \square

4.3. Associativity. We next show that composition of S-plays is associative. We first extend interactions to triples of S-plays. For $s \in SP_{A \rightarrow B}, t \in SP_{B \rightarrow C}, r \in SP_{C \rightarrow D}$ with $(s; t) \asymp r, s \asymp (t; r)$ and $\nu(s) \cap \nu(r) = \emptyset$, we define $s \parallel t \parallel r$ and $s \cdot t \cdot r$ as follows,

$$\begin{array}{ll}
\epsilon \parallel \epsilon \parallel \epsilon \triangleq \epsilon & \epsilon \cdot \epsilon \cdot \epsilon \triangleq \epsilon \\
sm_A^\Sigma \parallel t \parallel r \triangleq (s \parallel t \parallel r) m_A^{sm_A^\Sigma \cdot t \cdot r} & sn^T m_{A(P)}^\Sigma \cdot t \cdot r \triangleq \Phi(sn^T \cdot t \cdot r, T, \Sigma) \\
& sm_{A(O)}^\Sigma \cdot t \cdot r \triangleq \tilde{\Sigma}[s \cdot t \cdot r] \\
sm_B^\Sigma \parallel tm_B^{\Sigma'} \parallel r \triangleq (s \parallel t \parallel r) m_B^{sm_B^\Sigma \cdot tm_B^{\Sigma'} \cdot r} & sn^T m_{B(P)}^\Sigma \cdot tm_{B(O)}^{\Sigma'} \cdot r \triangleq \Phi(sn^T \cdot t \cdot r, T, \Sigma) \\
& sm_{B(O)}^{\Sigma'} \cdot tn^T m_{B(P)}^\Sigma \cdot r \triangleq \Phi(s \cdot tn^T \cdot r, T, \Sigma) \\
s \parallel tm_C^\Sigma \parallel rm_C^{\Sigma'} \triangleq (s \parallel t \parallel r) m_C^{s \cdot tm_C^\Sigma \cdot rm_C^{\Sigma'}} & s \cdot tn^T m_{C(P)}^\Sigma \cdot rm_{C(O)}^{\Sigma'} \triangleq \Phi(s \cdot tn^T \cdot r, T, \Sigma) \\
& s \cdot tm_{C(O)}^{\Sigma'} \cdot rn^T m_{C(P)}^\Sigma \triangleq \Phi(s \cdot t \cdot rn^T, T, \Sigma) \\
s \parallel t \parallel rm_D^\Sigma \triangleq (s \parallel t \parallel r) m_D^{s \cdot t \cdot rm_D^\Sigma} & s \cdot t \cdot rn^T m_{D(P)}^\Sigma \triangleq \Phi(s \cdot t \cdot rn^T, T, \Sigma) \\
& s \cdot t \cdot rm_{D(O)}^\Sigma \triangleq \tilde{\Sigma}[s \cdot t \cdot r]
\end{array}$$

where $\tilde{\Sigma}$ is the store of the move justifying $m_{A(O)}^\Sigma$ and $m_{D(O)}^\Sigma$ respectively.

The next lemma proves a form of associativity in Φ that will be used in the proof of the next proposition. Its proof is delegated to Appendix A.

Lemma 4.12. Let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$ be stores such that, for any α ,

- (1) $\alpha \in \nu(\Sigma_5 \setminus \Sigma_4) \implies \alpha \notin \nu(\Sigma_1) \cup \nu(\Sigma_2) \cup \nu(\Sigma_3)$,
- (2) $\alpha \in \nu(\Sigma_1) \cap \nu(\Sigma_4) \implies \alpha \in \nu(\Sigma_2)$.

Then, $\Phi(\Sigma_1, \Sigma_2, \Phi(\Sigma_3, \Sigma_4, \Sigma_5)) = \Phi(\Phi(\Sigma_1, \Sigma_2, \Sigma_3), \Sigma_4, \Sigma_5)$.

Proposition 4.13 (Associativity). If $s_1 \in SP_{A_1 \rightarrow A_2}$, $s_2 \in SP_{A_2 \rightarrow A_3}$ and $s_3 \in SP_{A_3 \rightarrow A_4}$ with $s_1; s_2 \asymp s_3$ and $s_1 \asymp s_2; s_3$ then:

- $(s_1; s_2); s_3 = (s_1 \parallel s_2 \parallel s_3) \upharpoonright A_1 A_4 = s_1; (s_2; s_3)$,
- $\Phi((s_1; s_2) \cdot s_3, \text{st}(s_1; s_2), s_1 \cdot s_2) = s_1 \cdot s_2 \cdot s_3 = \Phi(s_1 \cdot (s_2; s_3), \text{st}(s_2; s_3), s_2 \cdot s_3)$.

Proof. By induction on $|s_1 \parallel s_2 \parallel s_3|$. The base case is trivial. We examine the following inductive cases; the rest are similar.

– $(s_1 m_{A_1}^\Sigma; s_2); s_3 = ((s_1; s_2); s_3) m_{A_1}^{\Sigma'} \stackrel{\text{IH}}{=} (s_1 \parallel s_2 \parallel s_3) m_{A_1}^{\Sigma'} \upharpoonright A_1 A_4$ with $\Sigma' = (s_1 m_{A_1}^\Sigma; s_2) \cdot s_3$. Moreover, as $\text{st}(s_1 m_{A_1}^\Sigma; s_2) = s_1 m_{A_1}^\Sigma \cdot s_2$ and using Lemma 4.10(g),

$$\Phi((s_1; s_2 m_{A_1}^\Sigma); s_3, \text{st}(s_1 m_{A_1}^\Sigma; s_2), s_1 m_{A_1}^\Sigma \cdot s_2) = \Sigma' [s_1 m_{A_1}^\Sigma \cdot s_2] = \Sigma'.$$

We still need to show that $\Sigma' = s_1 m_{A_1}^\Sigma \cdot s_2 \cdot s_3$. If m_{A_1} is an O-move this is straightforward. If a P-move and, say, $s_1 = s'_1 n^T$ then $\Sigma' = \Phi((s_1; s_2) \cdot s_3, \text{st}(s_1; s_2), \Phi(s_1 \cdot s_2, T, \Sigma))$ and $s_1 m_{A_1}^\Sigma \cdot s_2 \cdot s_3 = \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma) \stackrel{\text{IH}}{=} \Phi(\Phi((s_1; s_2) \cdot s_3, \text{st}(s_1; s_2), s_1 \cdot s_2), T, \Sigma)$. These are equal since they satisfy the hypotheses of Lemma 4.12. Moreover,

$$s_1 m_{A_1}^\Sigma; (s_2; s_3) = (s_1; (s_2; s_3)) m_{A_1}^{\Sigma''} \stackrel{\text{IH}}{=} (s_1 \parallel s_2 \parallel s_3) m_{A_1}^{\Sigma''} \upharpoonright A_1 A_4$$

with $\Sigma'' = s_1 m_{A_1}^\Sigma \cdot (s_2; s_3)$. Note that $\text{st}(s_2; s_3) = s_2 \cdot s_3$, so it suffices to show that $\Sigma'' = s_1 m_{A_1}^\Sigma \cdot s_2 \cdot s_3$. If m_{A_1} an O-move then this is straightforward, otherwise

$$\Sigma'' = \Phi(s_1 \cdot (s_2; s_3), T, \Sigma) \stackrel{\text{IH}}{=} \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma) = s_1 m_{A_1}^\Sigma \cdot s_2 \cdot s_3.$$

– $(s_1 m_{A_2}^{\Sigma_1}; s_2 m_{A_2}^{\Sigma_2}); s_3 = (s_1; s_2); s_3 \stackrel{\text{IH}}{=} (s_1 \parallel s_2 \parallel s_3) \upharpoonright A_1 A_4 = (s_1 m_{A_2}^{\Sigma_1} \parallel s_2 m_{A_2}^{\Sigma_2} \parallel s_3) \upharpoonright A_1 A_4$. Assume WLOG that m_{A_2} is a P-move in $A_1 A_2$, so $\text{st}(s_2; s_3) = s_2 \cdot s_3$, and suppose $s_1 = s'_1 n^T$. Then,

$$\begin{aligned} & \Phi((s_1 m_{A_2}^{\Sigma_1}; s_2 m_{A_2}^{\Sigma_2}) \cdot s_3, \text{st}(s_1 m_{A_2}^{\Sigma_1}; s_2 m_{A_2}^{\Sigma_2}), s_1 m_{A_2}^{\Sigma_1} \cdot s_2 m_{A_2}^{\Sigma_2}) \\ &= \Phi((s_1; s_2) \cdot s_3, \text{st}(s_1; s_2), \Phi(s_1 \cdot s_2, T, \Sigma_1)) \end{aligned}$$

$$s_1 m_{A_2}^{\Sigma_1} \cdot s_2 m_{A_2}^{\Sigma_2} \cdot s_3 = \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma_1) \stackrel{\text{IH}}{=} \Phi(\Phi((s_1; s_2) \cdot s_3, \text{st}(s_1; s_2), s_1 \cdot s_2), T, \Sigma_1)$$

and equality follows from Lemma 4.12. Moreover,

$$\begin{aligned} & \Phi(s_1 m_{A_2}^{\Sigma_1} \cdot (s_2 m_{A_2}^{\Sigma_2}; s_3), \text{st}(s_2 m_{A_2}^{\Sigma_2}; s_3), s_2 m_{A_2}^{\Sigma_2} \cdot s_3) = (s_1 m_{A_2}^{\Sigma_1} \cdot (s_2 m_{A_2}^{\Sigma_2}; s_3)) [s_2 m_{A_2}^{\Sigma_2} \cdot s_3] \\ & \stackrel{\text{lm. 4.10(g)}}{=} s_1 m_{A_2}^{\Sigma_1} \cdot (s_2 m_{A_2}^{\Sigma_2}; s_3) = \Phi(s_1 \cdot (s_2; s_3), T, \Sigma_1) \end{aligned}$$

$$\begin{aligned} & s_1 m_{A_2}^{\Sigma_1} \cdot s_2 m_{A_2}^{\Sigma_2} \cdot s_3 = \Phi(s_1 \cdot s_2 \cdot s_3, T, \Sigma_1) \stackrel{\text{IH}}{=} \Phi(\Phi(s_1 \cdot (s_2; s_3), \text{st}(s_2; s_3), s_2 \cdot s_3), T, \Sigma_1) \\ &= \Phi((s_1 \cdot (s_2; s_3)) [s_2 \cdot s_3], T, \Sigma_1) \stackrel{\text{lm. 4.10(g)}}{=} \Phi(s_1 \cdot (s_2; s_3), T, \Sigma_1) \end{aligned}$$

as required. \square

4.4. **The categories \mathcal{S} and \mathcal{S}_{inn} .** We next show that S-strategies and arenas form a category, which we call \mathcal{S} , while innocent S-strategies form a wide subcategory of \mathcal{S} . We start this section with a lemma on strong nominal sets. Recall that, for a nominal set X and $x, x' \in X$, we write $x \sim x'$ if there exists a permutation π such that $x = \pi \cdot x'$.

Lemma 4.14 (Strong Support Lemma [25]). Let X be a strong nominal set and let $x_i, y_i, z_i \in X$ with $\nu(y_i) \cap \nu(z_i) \subseteq \nu(x_i)$, for $i = 1, 2$. Then, $(x_1, y_1) \sim (x_2, y_2)$ and $(x_1, z_1) \sim (x_2, z_2)$ imply $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$. \square

We proceed to show compositionality of S-strategies. The *interaction* of S-strategies $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ is defined by:

$$\sigma \parallel \tau \triangleq \{s \parallel t \mid s \in \sigma \wedge t \in \tau \wedge s \asymp t\}$$

First, some lemmas for determinacy.

Lemma 4.15. If $s_1 \parallel t_1, s_2 \parallel t_2 \in \text{SInt}(A, B, C)$ then $s_1 \parallel t_1 = s_2 \parallel t_2$ implies $s_1 = s_2$ and $t_1 = t_2$. Consequently, if $s_1 \parallel t_1 \sim s_2 \parallel t_2$ then $(s_1, t_1) \sim (s_2, t_2)$.

Proof. The former part of the claim is shown by straightforward induction on the length of the interactions. For the latter part, if $s_1 \parallel t_1 \sim s_2 \parallel t_2$ then there is some π such that $s_1 \parallel t_1 = \pi \cdot (s_2 \parallel t_2) = (\pi \cdot s_2) \parallel (\pi \cdot t_2)$. Hence, by the former part, $(s_1, t_1) = (\pi \cdot s_2, \pi \cdot t_2)$, from which we obtain $(s_1, t_1) \sim (s_2, t_2)$. \square

Lemma 4.16. If $\sigma : A \rightarrow B, \tau : B \rightarrow C$ are S-strategies and $u_1 m_1^{\Sigma_1}, u_2 m_2^{\Sigma_2} \in \sigma \parallel \tau$ then $u_1 \sim u_2$ and m_1 a generalised P-move imply $u_1 m_1^{\Sigma_1} \sim u_2 m_2^{\Sigma_2}$.

Proof. Let us assume that $u_i m_i^{\Sigma_i} = s_i \parallel t_i = (s'_i \parallel t'_i) m_i^{\Sigma'_i}$. Then, by the previous lemma, $(s'_1, t'_1) \sim (s'_2, t'_2)$. If, say, m_1 is a P-move in AB then m_2 is also a P-move in AB and $s_1 \sim s_2$, by Zipper Lemma and determinacy of σ , so $(s'_1, m_1^{\Sigma'_1}) \sim (s'_2, m_2^{\Sigma'_2})$, where $s_i = s'_i m_i^{\Sigma'_i}$. Since $\nu(t'_i) \cap \nu(m_i^{\Sigma'_i}) = \emptyset$, we can apply the Strong Support Lemma to obtain $u_1 m_1^{\Sigma_1} \sim u_2 m_2^{\Sigma_2}$. \square

Lemma 4.17. Let $\sigma : A \rightarrow B, \tau : B \rightarrow C$ be S-strategies and $u_1, u_2 \in \sigma \parallel \tau$ with $|u_1| \leq |u_2|$. Then, $u_1 \upharpoonright AC = u_2 \upharpoonright AC$ implies $u_1 \sim_{\sqsubseteq_p} u_2$.

Proof. By induction on $|u_1|$. The base case is trivial. Now suppose $u_1 = u'_1 m_1^{\Sigma_1}$, and let $u_2 = u'_2 u''_2$ with u'_2 being the greatest prefix of u_2 such that $u'_1 \upharpoonright AC = u'_2 \upharpoonright AC$. Then, by IH, $u'_1 \sim_{\sqsubseteq_p} u'_2$. If m_1 a generalised P-move then, by previous lemma, $u_1 \sim_{\sqsubseteq_p} u_2$. If m_1 an external O-move then u'_1 ends in a P-move in AC and, by $u'_1 \upharpoonright AC = u'_2 \upharpoonright AC$ and Zipper Lemma, u'_2 must end in the same move. Thus, $u_1 \upharpoonright AC \sim u_2 \upharpoonright AC$ implies $u_1 \sim_{\sqsubseteq_p} u_2$. \square

Now some lemmas for innocence. We say that a move m in an interaction sequence $s \in \text{SInt}(A, B, C)$ is a *generalised O-move* if it is an O-move in AB or BC .

Lemma 4.18. If $s_1, s_2 \in \text{SP}_{A \rightarrow B}, t_1, t_2 \in \text{SP}_{B \rightarrow C}$ and $s_i \parallel t_i$ end in a generalised O-move in component X ,

- (1) if $X = AB$ then $\ulcorner (s_1 \parallel t_1) \upharpoonright AB \urcorner = \ulcorner (s_2 \parallel t_2) \upharpoonright AB \urcorner \implies \ulcorner s_1 \urcorner = \ulcorner s_2 \urcorner$,
- (2) if $X = BC$ then $\ulcorner (s_1 \parallel t_1) \upharpoonright BC \urcorner = \ulcorner (s_2 \parallel t_2) \upharpoonright BC \urcorner \implies \ulcorner t_1 \urcorner = \ulcorner t_2 \urcorner$.

Proof. We show (a) by induction on $|s_1| \geq 1$, and (b) is proved similarly. The base case is obvious. If $s_1 = s'_1 n^{T_1} s''_1 m^{\Sigma_1}$ with m an O-move in AB justified by n then $s_2 = s'_2 n^{T_2} s''_2 m^{\Sigma_2}$ and, by IH, $\ulcorner s'_1 \urcorner = \ulcorner s'_2 \urcorner$. We need to show that $T_1 = T_2$ and $\Sigma_1 = \Sigma_2$, while we know that the stores of the corresponding moves in $s_i \parallel t_i$ are equal, say $\Sigma'_1 = \Sigma'_2$ and $T'_1 = T'_2$. We

have that $T'_i = \Phi(\text{st}((s_i \parallel t_i)_{\prec_n T'_i}), \text{st}(s'_i), T_i)$, hence $T_1 \setminus \text{st}(s'_1) = T_2 \setminus \text{st}(s'_2)$. By IH we have that $\text{st}(s'_1) = \text{st}(s'_2)$, so if $\alpha \in \nu(T_1) \cap \nu(\text{st}(s'_1))$ then, by Lemma 4.10(g), $\alpha \in \nu(T'_1) \cap \nu(\text{st}(s'_1))$ so $\alpha \in \nu(T'_2) \cap \nu(\text{st}(s'_2))$, $\therefore \alpha \in \nu(T_2) \cap \nu(\text{st}(s'_2))$, and viceversa. Moreover, $T_i(\alpha) = T'_i(\alpha)$ for each such α , thus $T_1 = T_2$. This also implies that $\nu(\Sigma_1) = \nu(\Sigma_2)$ and so, by Lemma 4.10(g), $\Sigma_1 = \Sigma_2$. \square

Lemma 4.19. If $\sigma : A \rightarrow B$, $\tau : B \rightarrow C$ are innocent S-strategies then, if $u_1 m^{\Sigma_1}, u_2 \in \sigma \parallel \tau$ with u_i ending in a generalised O-move and $\ulcorner u_1 \urcorner \sim \ulcorner u_2 \urcorner$ then there exists $u_2 m^{\Sigma_2} \in \sigma \parallel \tau$ such that $\ulcorner u_1 m^{\Sigma_1} \urcorner \sim \ulcorner u_2 m^{\Sigma_2} \urcorner$.

Proof. Suppose u_1 ends in an O-move in A —the other cases are shown similarly. Then, $u_1 m^{\Sigma_1} = s_1 m^{\Sigma'_1} \parallel t_1$ and $u_2 = s_2 \parallel t_2$ for some relevant S-plays of σ, τ . Moreover, $\ulcorner u_1 \urcorner \sim \ulcorner u_2 \urcorner$ implies, by Lemma 4.18, that $\ulcorner s_1 \urcorner \sim \ulcorner s_2 \urcorner$ and thus, by innocence, there exists $s_2 m^{\Sigma'_2} \in \sigma$ such that $s_1 m^{\Sigma'_1} \sim s_2 m^{\Sigma'_2}$. In fact, we can pick a Σ'_2 such that $s_2 m^{\Sigma'_2} \asymp t_2$. Let $s_2 m^{\Sigma'_2} \parallel t_2 = u_2 m^{\Sigma_2} \in \sigma \parallel \tau$. We have that $(\ulcorner u_1 \urcorner, \ulcorner s_1 \urcorner) \sim (\ulcorner u_2 \urcorner, \ulcorner s_2 \urcorner)$ and $(\ulcorner s_1 \urcorner, m^{\Sigma'_1}) \sim (\ulcorner s_2 \urcorner, m^{\Sigma'_2})$ and, moreover, $\nu(\ulcorner u_i \urcorner) \cap \nu(m^{\Sigma'_i}) \subseteq \nu(\ulcorner s_i \urcorner)$ for $i = 1, 2$ thus, by Strong Support Lemma, $(\ulcorner u_1 \urcorner, \ulcorner s_1 \urcorner, m^{\Sigma'_1}) \sim (\ulcorner u_2 \urcorner, \ulcorner s_2 \urcorner, m^{\Sigma'_2})$. This implies that $\ulcorner u_1 m^{\Sigma_1} \urcorner \sim \ulcorner u_2 m^{\Sigma_2} \urcorner$. \square

Lemma 4.20. Let $\sigma : A \rightarrow B$, $\tau : B \rightarrow C$ be innocent S-strategies and let $u_1, u_2 \in \sigma \parallel \tau$ with $|\ulcorner u_1 \urcorner| \leq |\ulcorner u_2 \urcorner|$. Then, $\ulcorner u_1 \urcorner \upharpoonright AC \urcorner = \ulcorner u_2 \urcorner \upharpoonright AC \urcorner$ implies $\ulcorner u_1 \urcorner \sim_{\sqsubseteq_p} \ulcorner u_2 \urcorner$.

Proof. Let us write u_i for $s_i \parallel t_i$. We argue by induction on $|\ulcorner u_1 \urcorner| + |\ulcorner u_2 \urcorner|$. The base cases are obvious. Now let $u_1 = u'_1 m^{\Sigma_1}$ with m a B-move. We have that $|u'_1| \leq |u_2|$ and $\ulcorner u'_1 \urcorner \upharpoonright AC \urcorner = \ulcorner u_2 \urcorner \upharpoonright AC \urcorner$ so, by IH, $\ulcorner u'_1 \urcorner \sim_{\sqsubseteq_p} \ulcorner u_2 \urcorner$. In particular, $u_2 = u'_2 m^{\Sigma'_2} u''_2$ with $\ulcorner u'_2 \urcorner \sim \ulcorner u'_1 \urcorner$ so, by Lemma 4.19, there exists $u'_2 m^{\Sigma'_2} \in \sigma \parallel \tau$ with $\ulcorner u_1 \urcorner \sim \ulcorner u'_2 m^{\Sigma'_2} \urcorner$. Using Lemma 4.16, $\ulcorner u'_2 m^{\Sigma'_2} \urcorner \sim \ulcorner u'_2 \urcorner m^{\Sigma'_2} \urcorner \sim \ulcorner u_1 \urcorner$.

The case of m being a P-move in AC is proved along the same lines. On the other hand, if m is an O-move in AC justified by n then $u_1 = v_1 n^T v'_1 m^{\Sigma}$ and $u_2 = v_2 n^T v'_2 m^{\Sigma} v''_2$ and, by IH, $\ulcorner v_1 n^T \urcorner = \pi \cdot \ulcorner v_2 n^T \urcorner$. In particular, π fixes $\text{dom}(T)$ and therefore $\pi \cdot m^{\Sigma} = m^{\Sigma}$, $\therefore \ulcorner v_1 n^T m^{\Sigma} \urcorner = \pi \cdot (\ulcorner v_2 n^T m^{\Sigma} \urcorner)$. \square

Proposition 4.21. If $\sigma : A \rightarrow B$, $\tau : B \rightarrow C$ are S-strategies then $\sigma; \tau : A \rightarrow C$ is an S-strategy. If σ and τ are innocent, so is $\sigma; \tau$.

Proof. Prefix closure and nominal closure are easy to establish by noting that they hold at the level of interactions, for $\sigma \parallel \tau$. For O-closure, suppose $v \in \sigma; \tau$ and $vm^{\Sigma} \in SP_{A \rightarrow C}$ with, say, m an O-move in A . Then, $v = s; t$ for some $s \in \sigma$, $t \in \tau$ with s ending in a P-move in A . Moreover, we can construct a (unique) store Σ' such that $sm^{\Sigma'} \in SP_{A \rightarrow B}$, by means of the O-Just and O-Val conditions. Thus, $sm^{\Sigma'} \in \sigma$ and $vm^{\Sigma} \in \sigma; \tau$.

For determinacy, suppose even-length $vm^{\Sigma_i} \in \sigma; \tau$ and $vm^{\Sigma_i} = s_i; t_i$ with s_i, t_i not both ending in B , for $i = 1, 2$. Let $s_i \parallel t_i = (s'_i \parallel t'_i) m^{\Sigma_i}$ and suppose WLOG that $|s'_1 \parallel t'_1| \leq |s'_2 \parallel t'_2|$. Then, by Lemma 4.17, $s'_1 \parallel t'_1 \sim_{\sqsubseteq_p} s'_2 \parallel t'_2$ and therefore, by Lemma 4.16, $s_1 \parallel t_1 \sim_{\sqsubseteq_p} s_2 \parallel t_2$. In particular, $vm^{\Sigma_1} \sim vm^{\Sigma_2}$.

For innocence, let $v_1 m^{\Sigma_1}, v_2 \in \sigma; \tau$ with $\ulcorner v_1 \urcorner = \ulcorner v_2 \urcorner$ being of odd length, and $u_1 m^{\Sigma_1}, u_2 \in \sigma \parallel \tau$ such that $v_1 m^{\Sigma_1} = u_1 m^{\Sigma_1} \upharpoonright AC$, $v_2 = u_2 \upharpoonright AC$. By Lemma 4.20, either $\ulcorner u_2 \urcorner \sim_{\sqsubseteq_p} \ulcorner u_1 \urcorner$ or $\ulcorner u_1 \urcorner \sim_{\sqsubseteq_p} \ulcorner u_2 \urcorner$ and $\ulcorner u_1 \urcorner \not\sim \ulcorner u_2 \urcorner$. In the latter case, by Lemmata 4.19 and 4.16 we obtain $\ulcorner u_1 m^{\Sigma_1} \urcorner \sim_{\sqsubseteq_p} \ulcorner u_2 \urcorner$, contradicting $\ulcorner v_1 \urcorner = \ulcorner v_2 \urcorner$. In the former case, let us assume u_2 is of maximum length such that $u_2 \upharpoonright AC = v_2$ and $|\ulcorner u_2 \urcorner| \leq |\ulcorner u_1 \urcorner|$. Then, by Lemma 4.19,

there exists $u_2 m_2^{\Sigma_2} \in \sigma \parallel \tau$ such that either $\ulcorner u_2 m_2^{\Sigma_2} \urcorner \sim_{\sqsubseteq_p} \ulcorner u_1 \urcorner$ or $\ulcorner u_2 m_2^{\Sigma_2} \urcorner \sim \ulcorner u_1 m_1^{\Sigma_1} \urcorner$. The former case contradicts maximality of u_2 , while the latter implies $v_2 m_2^{\Sigma_2} \in \sigma; \tau$ and $\ulcorner v_1 m_1^{\Sigma_1} \urcorner \sim \ulcorner v_2 m_2^{\Sigma_2} \urcorner$. \square

We proceed to construct a category of arenas and S-strategies. For each arena A , the identity S-strategy $\text{id}_A : A \rightarrow A$ is the *copycat* S-strategy:

$$\text{id}_A = \{ s \in SP_{A \rightarrow A} \mid \exists s'. s \sqsubseteq_p s' \wedge s' \upharpoonright A_l = s' \upharpoonright A_r \}$$

where by A_l and A_r we denote the LHS and RHS arena A of $A \rightarrow A$ respectively.

Proposition 4.22. Let $\sigma_i : A_i \rightarrow A_{i+1}$ for $i = 1, 2, 3$. Then, $\sigma_i; \text{id}_{A_{i+1}} = \text{id}_{A_i}; \sigma_i = \sigma_i$ and $\sigma_1; (\sigma_2; \sigma_3) = (\sigma_1; \sigma_2); \sigma_3$.

Proof. Composition with identities is standard. For associativity, if $s_1; (s_2; s_3) \in \sigma_1; (\sigma_2; \sigma_3)$ then there are $s'_i \sim s_i$ such that $\nu(s'_{i_1}) \cap (\nu(s_{i_2}) \cup \nu(s_{i_3}) \cup \nu(s'_{i_2}) \cup \nu(s'_{i_3})) = \emptyset$, for any distinct i_1, i_2, i_3 . These S-plays satisfy the hypotheses of Proposition 4.13, hence $s'_1; (s'_2; s'_3) = (s'_1; s'_2); s'_3 \in (\sigma_1; \sigma_2); \sigma_3$. Moreover, since $\nu(s_2, s'_2) \cap \nu(s_3, s'_3) = \emptyset$, $s_i \sim s'_i$ imply $(s_2, s_3) \sim (s'_2, s'_3)$ and therefore $s_2; s_3 \sim s'_2; s'_3$. Since $\nu(s_1, s'_1) \cap \nu(s_2, s_3, s'_2; s'_3) = \emptyset$, we have $s_1; (s_2; s_3) \sim s'_1; (s'_2; s'_3)$, hence $s_1; (s_2; s_3) \in (\sigma_1; \sigma_2); \sigma_3$. Other direction proved dually. \square

We can now define our categories of games with stores. In the following definition we also make use of the observation that identity S-strategies are innocent.

Definition 4.23. Let \mathcal{S} be the category whose objects are arenas and, for each pair of arenas A, B , the morphisms are given by $\mathcal{S}(A, B) = \{ \sigma : A \rightarrow B \mid \sigma \text{ an S-strategy} \}$. Let \mathcal{S}_{inn} be the wide subcategory of \mathcal{S} of innocent S-strategies.

4.5. The model of \mathbf{IA}_{cbv} . We next construct the model of \mathbf{IA}_{cbv} in \mathcal{S}_{inn} . An innocent S-strategy σ is specified by its *view-function*, $\text{viewf}(\sigma)$, defined as follows.

$$\text{viewf}(\sigma) \triangleq \{ \ulcorner s \urcorner \mid s \in \sigma \wedge \text{even}(|s|) \wedge s \neq \epsilon \}$$

Conversely, a **preplay** is defined exactly like a (non-empty) S-play only that it does not necessarily satisfy the Val-O condition. Let us write PP_A for the set of preplays of A . Obviously, $SP_A \subseteq PP_A$. Moreover, if $s \in SP_A$ then $\ulcorner s \urcorner \in PP_A$.

For instance, the cell strategy of Example 4.5 can be described as the least innocent S-strategy whose view-function contains the following preplays.

$$\begin{array}{ccc} \overset{\curvearrowright}{q_0} \xleftarrow{(\alpha,0)} q_1 \xrightarrow{(\alpha,i)} a_1 \xrightarrow{(\alpha,i)} a_0 & \overset{\curvearrowright}{q_0} \xleftarrow{(\alpha,0)} q_1 \xrightarrow{\text{read}^{(\alpha,i)}} i^{(\alpha,i)} & \overset{\curvearrowright}{q_0} \xleftarrow{(\alpha,0)} q_1 \xrightarrow{\text{write}(j)^{(\alpha,i)}} \text{ok}^{(\alpha,j)} \end{array}$$

We next make formal the connection between view-functions and innocent S-strategies.

A **view-function** f on A is a subset of PP_A satisfying:

- If $s \in f$ then $|s|$ is even and $\ulcorner s \urcorner = s$. (*View*)
- If $sn^T m^\Sigma \in f$ and $s \neq \epsilon$ then $s \in f$. (*Even-Prefix Closure*)
- If $s' \sim s \in f$ then $s' \in f$. (*Nominal Closure*)
- If $sm_1^{\Sigma_1}, sm_2^{\Sigma_2} \in f$ then $sm_1^{\Sigma_1} \sim sm_2^{\Sigma_2}$. (*Determinacy*)

From a view-function f we can derive an innocent S-strategy $\text{strat}(f)$ by the following procedure. We set $\text{strat}(f) \triangleq \bigcup_{i \in \omega} \text{strat}_i(f)$, where

$$\begin{aligned} \text{strat}_{2i+1}(f) &\triangleq \{sm^\Sigma \in SP_A \mid s \in \text{strat}_{2i}(f)\} \\ \text{strat}_{2i+2}(f) &\triangleq \{sm^\Sigma \in SP_A \mid s \in \text{strat}_{2i+1}(f) \wedge \ulcorner sm^\Sigma \urcorner \in f\} \end{aligned}$$

and $\text{strat}_0(f) \triangleq \{\epsilon\}$.

Lemma 4.24. If σ, f are an innocent S-strategy and a view-function respectively then $\text{viewf}(\sigma), \text{strat}(\sigma)$ are a view-function and an innocent S-strategy respectively. Moreover, $\text{strat}(\text{viewf}(\sigma)) = \sigma$ and $\text{viewf}(\text{strat}(f)) = f$. \square

We can show that \mathcal{S}_{inn} exhibits the same kind of categorical structure as that obtained in [8] (in the context of call-by-value PCF), which can be employed to model call-by-value higher-order computation with recursion. In particular, let us call an S-strategy $\sigma : A \rightarrow B$ **total** if for all $i_A \in I_A$ there is $i_A i_B \in \sigma$. We write $\mathcal{S}_{\text{inn}}^t$ for the wide subcategory of \mathcal{S}_{inn} containing total innocent S-strategies.

For innocent S-strategies $\sigma : A \rightarrow B$ and $\tau : A \rightarrow C$, we define their *left pairing* to be $\langle \sigma, \tau \rangle_l = \text{strat}(f)$, where f is the view-function:

$$\begin{aligned} f &= \{s \in PP_{A \rightarrow B \otimes C} \mid s \in \text{viewf}(\sigma) \wedge s \upharpoonright (B \otimes C) = \epsilon\} \\ &\cup \{i_A s_1 s_2 \in PP_{A \rightarrow B \otimes C} \mid \exists i_B. i_A s_1 i_B \in \text{viewf}(\sigma) \wedge i_A s_2 \in \text{viewf}(\tau)\} \\ &\cup \{i_A s_1 s_2(i_B, i_C) s \in PP_{A \rightarrow B \otimes C} \mid i_A s_1 i_B s \in \text{viewf}(\sigma) \wedge i_A s_2 i_C \in \text{viewf}(\tau)\} \\ &\cup \{i_A s_1 s_2(i_B, i_C) s \in PP_{A \rightarrow B \otimes C} \mid i_A s_1 i_B \in \text{viewf}(\sigma) \wedge i_A s_2 i_C s \in \text{viewf}(\tau)\} \end{aligned}$$

We can show that left pairing yields a product in $\mathcal{S}_{\text{inn}}^t$ with the usual projections:

$$\pi_1 : A \otimes B \rightarrow A = \{s \in SP_{A \otimes B \rightarrow A} \mid |s| \leq 1\} \cup \{(i_A, i_B) i_A s \in SP_{A \otimes B \rightarrow A} \mid i_A i_A s \in \text{id}_A\}$$

and dually for π_2 . Moreover, for every A, B, C , there is a bijection

$$\Lambda : \mathcal{S}_{\text{inn}}(A \otimes B, C) \xrightarrow{\cong} \mathcal{S}_{\text{inn}}^t(A, B \Rightarrow C)$$

natural in A, C . In particular, for each innocent $\sigma : A \otimes B \rightarrow C$, $\Lambda(\sigma) = \text{strat}(f)$, where

$$f = \{i_A * i_B s \in PP_{A \rightarrow B \Rightarrow C} \mid (i_A, i_B) s \in \text{viewf}(\sigma)\}.$$

The inverse of Λ is defined in an analogous manner. We set $\text{ev}_{A,B} = \Lambda^{-1}(\text{id}_{A \Rightarrow B})$.

Thus, the functional part of \mathbf{IA}_{cbv} can be interpreted in \mathcal{S}_{inn} using the same constructions as in [8]. Assignment, dereferencing and `mkvar` can in turn be modelled using the relevant (store-free) innocent strategies of [2]. Finally, the denotation of `new x in M` is obtained by using cell_β of Example 4.5. Let us write $\llbracket \cdot \cdot \cdot \rrbracket_S$ for the resultant semantic map.

Proposition 4.25. For any \mathbf{IA}_{cbv} -term $\Gamma \vdash M : \theta$, $\llbracket \Gamma \vdash M : \theta \rrbracket_S$ is an innocent S-strategy.

Proof. We present here the (inductive) constructions pertaining to variables,

- $\llbracket \Gamma \vdash \text{new } x \text{ in } M : \beta \rrbracket_S = \llbracket \Gamma \rrbracket \xrightarrow{\Lambda(\llbracket M \rrbracket_S)} \llbracket \text{var} \rrbracket \Rightarrow \llbracket \beta \rrbracket \xrightarrow{\text{cell}_\beta} \llbracket \beta \rrbracket$
- $\llbracket \Gamma \vdash M := N : \text{unit} \rrbracket_S = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket_S; \pi_2, \llbracket N \rrbracket_S \rangle_l} (\mathbb{Z} \Rightarrow 1) \otimes \mathbb{Z} \xrightarrow{\text{ev}} 1$
- $\llbracket \Gamma \vdash !M : \text{int} \rrbracket_S = \llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket_S; \pi_1} 1 \Rightarrow \mathbb{Z} \xrightarrow{\cong} (1 \Rightarrow \mathbb{Z}) \otimes 1 \xrightarrow{\text{ev}} \mathbb{Z}$
- $\llbracket \Gamma \vdash \text{mkvar}(M, N) : \text{var} \rrbracket_S = \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket M \rrbracket_S, \llbracket N \rrbracket_S \rangle_l} \llbracket \text{var} \rrbracket$

and refer to [8] for the functional constructions and the treatment of fixpoints. \square

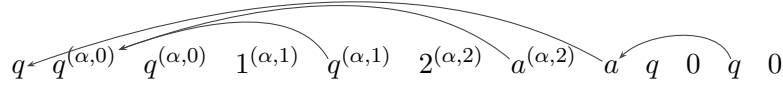
Our model of \mathbf{IA}_{cbv} , based on innocent S -strategies, is closely related to one based on knowing strategies. First observe that by erasing storage annotations in an innocent S -strategy σ one obtains a knowing strategy (determinacy follows from the fact that stores in O -moves are uniquely determined). We shall refer to that knowing strategy by $\text{erase}(\sigma)$. Next note that the (simpler) fully abstract model of RML from [2], based on knowing strategies, also yields a model of \mathbf{IA}_{cbv} . Let us write $\llbracket \cdot \cdot \cdot \rrbracket$ for this knowing-strategy semantics (cast in the Honda-Yoshida setting). Then we have:

Lemma 4.26. For any \mathbf{IA}_{cbv} -term $\Gamma \vdash M : \theta$, $\llbracket \Gamma \vdash M : \theta \rrbracket = \text{erase}(\llbracket \Gamma \vdash M : \theta \rrbracket_S)$.

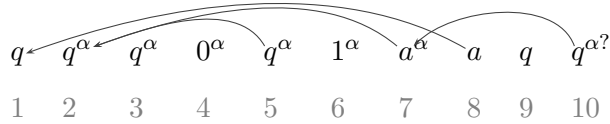
4.6. Block-innocent strategies.

Definition 4.27. A (knowing) strategy $\sigma : A$ is *block-innocent* if $\sigma = \text{erase}(\sigma')$ for some innocent S -strategy σ' .

Example 4.28. Let us revisit the two plays from the Introduction. The first one indeed comes from an innocent S -strategy (we reveal the stores below).



For the second one to become innocent (in the setting with stores), a store with variable α , say, would need to be introduced in the second move, to justify the different responses in moves 4 and 6. Then α must also occur in the seventh move by JUST-O, but it must not occur in the eighth move by JUST-P (the PA clause). Hence, it will not be present in the ninth move by JUST-O. Consequently, the last move is bound to break either PREV-PQ(a) (if it contains α) or JUST-P (if it does not).



The knowledge that strategies determined by \mathbf{IA}_{cbv} are block-innocent will be crucial in establishing a series of results in the following sections, where we shall galvanise the correspondence by investigating full abstraction (Corollary 6.3) and universality (Proposition 5.7).

5. FINITARY DEFINABILITY AND UNIVERSALITY

In this section we demonstrate that the game model of \mathbf{IA}_{cbv} is complete when restricted to *finitary* or *recursively presentable* innocent S -strategies. That is, every appropriately typed strategy of that kind is the denotation of some \mathbf{IA}_{cbv} term. Although finitary innocent S -strategies are subsumed by recursively presentable ones, the method of proving completeness in the latter case is much more involved and we therefore prove the two results separately. The two completeness results are called *finitary definability* and *universality* respectively.

5.1. Finitary definability. We first formulate a decomposition lemma for innocent S-strategies which subsequently allows us to show the two results. The decomposition of innocent S-strategies follows the argument for call-by-value PCF [8] except for the case in which the strategy replies to Opponent's (unique) initial move with a question that introduces a new name (case 8 in the lemma below). Let us examine this case more closely, assuming α to be the first variable from the non-empty store. In order to decompose the S-strategy, say σ , consider any P-view s in which α occurs in the second move $q_\alpha^{\Sigma\alpha}$. It turns out that s must be of the form $q q_\alpha^{\Sigma\alpha} s_\alpha s'$, where a move m^Σ from s contains α if, and only if, it is $q_\alpha^{\Sigma\alpha}$ or in s_α . In addition, no justification pointers connect s' to $q_\alpha^{\Sigma\alpha} s_\alpha$, because of the (*Just-O*) and (*Close*) conditions. This separation can be applied to decompose the view-function of σ . The s_α segments, put together as a single S-strategy, can subsequently be dealt with in the style of factorisation arguments, which remove α from moves at the cost of an additional var-component. Finally, to relate s_α 's to the suitable s' one can use numerical codes for $q_\alpha^{\Sigma\alpha} s_\alpha$. These ideas lie at the heart of the following result.

We fix a generic notation $\llbracket - \rrbracket$ for coding functions from enumerable sets to ω . For example, $\llbracket i, j \rrbracket$ encodes the pair (i, j) as a number. There is an inherent abuse of notation in our coding notation which, nevertheless, we overlook for typographical economy. Moreover, we denote sequences $\theta_1, \dots, \theta_n$ (where n may be left implicit) as $\bar{\theta}$. In such cases, we may write $\bar{\theta}_i^j$ ($i \leq j$) for subsequences $\theta_i, \theta_{i+1}, \dots, \theta_j$.

Lemma 5.1 (Decomposition Lemma (DL)). Let $\theta_1, \dots, \theta_m, \delta$ be types of \mathbf{IA}_{cbv} . Each innocent S-strategy $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$ can be decomposed as follows.

1. If $\theta_1, \dots, \theta_m = \bar{\theta}_1^{m'}$, $\text{int}, \bar{\theta}_{m'+2}^m$ with none of $\theta_1, \dots, \theta_{m'}$ being int then:

$$\sigma = \{(\bar{\ast}_1^{m'}, i, \bar{q}_{m'+2}^m) s \mid (\bar{\ast}_1^{m'}, \bar{q}_{m'+2}^m) s \in \tau_i\}$$

$$\text{where } \tau_i \triangleq \llbracket \bar{\theta}_1^{m'} \rrbracket \otimes \llbracket \bar{\theta}_{m'+2}^m \rrbracket \xrightarrow{\cong; \text{id} \otimes i \otimes \text{id}} \llbracket \bar{\theta}_1^{m'} \rrbracket \otimes \mathbb{Z} \otimes \llbracket \bar{\theta}_{m'+2}^m \rrbracket \xrightarrow{\sigma} \llbracket \delta \rrbracket$$

- If none of $\theta_1, \dots, \theta_m$ is int then one of the following is the case.
 - $\bar{\ast} a \in \sigma$, in which case either:
 2. $\delta = \text{unit}$ and $\sigma = \llbracket \bar{\theta} \rrbracket \xrightarrow{1} 1$ (the unique total S-strategy into 1),
 3. $\delta = \text{int}$, $i \in \mathbb{Z}$ and $\sigma = \llbracket \bar{\theta} \rrbracket \xrightarrow{i} \mathbb{Z}$ (the unique total S-strategy into \mathbb{Z} playing i),
 4. $\delta = \text{var}$ and $\sigma = \langle \sigma_1, \sigma_2 \rangle$ where $\sigma_i \triangleq \sigma; \pi_i$,
 5. $\delta = \delta' \rightarrow \delta''$ and $\sigma = \Lambda(\sigma')$ where $\sigma' = \Lambda^{-1}(\sigma)$, that is:

$$\sigma' : \llbracket \bar{\theta} \rrbracket \otimes \llbracket \delta' \rrbracket \rightarrow \llbracket \delta'' \rrbracket \triangleq \text{strat}\{(\bar{\ast}, q_{\delta'}) s \mid \bar{\ast} a q_{\delta'} s \in \text{viewf}(\sigma)\}$$

6. $\bar{\ast} q \in \sigma$ with q played in some $\theta_l = \text{var}$, in which case $\sigma \cong \sigma'$ where

$$\sigma' : \llbracket \bar{\theta}_1^{l-1} \rrbracket \otimes \llbracket \bar{\theta}_{l+1}^m \rrbracket \otimes (1 \Rightarrow \mathbb{Z}) \otimes (\mathbb{Z} \Rightarrow 1) \rightarrow \llbracket \delta \rrbracket$$

is obtained from σ by simply internally permuting and re-associating its initial moves.

7. $\bar{\ast} q \in \sigma$ with q played in some $\theta_l = \theta'_l \rightarrow \theta''_l$, in which case

$$\sigma = \llbracket \bar{\theta} \rrbracket \xrightarrow{\langle \Lambda(\sigma''), \pi_l, \sigma' \rangle} (\llbracket \theta'_l \rrbracket \Rightarrow \llbracket \delta \rrbracket) \otimes (\llbracket \theta'_l \rrbracket \Rightarrow \llbracket \theta''_l \rrbracket) \otimes \llbracket \theta'_l \rrbracket \xrightarrow{\text{id} \otimes \text{ev}; \text{ev}} \llbracket \delta \rrbracket$$

where, taking a to be q (seen as an answer):

$$\sigma' : \llbracket \bar{\theta} \rrbracket \rightarrow \llbracket \theta'_l \rrbracket \triangleq \text{strat}\{\{\bar{\ast} a\} \cup \{\bar{\ast} a q'_l s \mid \bar{\ast} q q'_l s \in \text{viewf}(\sigma) \wedge q'_l \in M_{\llbracket \theta'_l \rrbracket}\}\}$$

$$\sigma'' : \llbracket \bar{\theta} \rrbracket \otimes \llbracket \theta''_l \rrbracket \rightarrow \llbracket \delta \rrbracket \triangleq \text{strat}\{(\bar{\ast}, q''_l) s \mid \bar{\ast} q a''_l s \in \text{viewf}(\sigma) \wedge q''_l = a''_l\}$$

8. $\bar{*}q^\Sigma \in \sigma$ with $\text{dom}(\Sigma) = \alpha \cdots$, in which case

$$\sigma = \llbracket \bar{\theta} \rrbracket \xrightarrow{\langle \Lambda(\sigma'), \text{id} \rangle} (\llbracket \text{var} \rrbracket \rightarrow \mathbb{Z}) \otimes \llbracket \bar{\theta} \rrbracket \xrightarrow{\text{cell} \otimes \text{id}} \mathbb{Z} \otimes \llbracket \bar{\theta} \rrbracket \xrightarrow{\sigma''} \llbracket \delta \rrbracket$$

where:

$$\begin{aligned} \sigma' : \llbracket \text{var} \rrbracket \otimes \llbracket \bar{\theta} \rrbracket &\rightarrow \mathbb{Z} \triangleq \text{strat}(\{ \psi(sm^T) \mid sm^T \in \text{viewf}(\sigma) \wedge \alpha \in \nu(T) \} \\ &\quad \cup \{ \psi(s)[s] \mid sm^T \in \text{viewf}(\sigma) \wedge \alpha \in \nu(\text{st}(s) \setminus T) \}) \\ \psi(o^{(\alpha, i)::T} s) &\triangleq o^T \text{read}^T i^T \psi(s) \\ \psi(p^{(\alpha, i)::T} s) &\triangleq \text{write}(i)^T \text{ok}^T p^T \psi(s) \\ \sigma'' : \mathbb{Z} \otimes \llbracket \bar{\theta} \rrbracket &\rightarrow \llbracket \delta \rrbracket \triangleq \text{strat}\{([s], \bar{*}) m^T t \mid sm^T t \in \text{viewf}(\sigma) \wedge \alpha \in \nu(\text{st}(s) \setminus T)\} \end{aligned}$$

Proof. Cases **1-7** are the standard ones that also occur for call-by-value PCF [8]. Case **8** is the most interesting one. Here we exploit the fact that, once α occurs in the second move of a P-view, it appears continuously (in the P-view) until it is dropped by Proponent. Moreover, after α has been dropped, no move will ever have a justification pointer to a move containing α (because of *Just-O* and *Close*). The σ' strategy tracks the behaviour of σ until α is dropped, at which point it returns the code of the current P-view. σ'' in turn will take a code of such a P-view and will continue the play, as σ would. Additionally, α is factored out in σ' through an extra $\llbracket \text{var} \rrbracket$ arena, as in the factorisation argument of [3]. \square

Definition 5.2. We call an innocent S-strategy σ *finitary* if its view-function is finite modulo name-permutation, that is, if the set

$$O(\text{viewf}(\sigma)) = \{ \{ \pi \cdot s \mid \pi \in \text{PERM} \} \mid s \in \text{viewf}(s) \}$$

is finite. Accordingly, we call a block-innocent strategy finitary if the underlying innocent S-strategy is finitary.

Proposition 5.3 (Finitary definability). Let $\theta_1, \dots, \theta_m, \delta$ be types of \mathbf{IA}_{cbv} . For any finitary innocent S-strategy $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$ there exists a term $x_1 : \theta_1, \dots, x_m : \theta_m \vdash M : \delta$ such that $\sigma = \llbracket x_1 : \theta_1, \dots, x_m : \theta_m \vdash M \rrbracket_{\text{S}}$.

Proof. We rely on the decomposition lemma to reduce the suitably calculated size of the strategy. The right measure is obtained by combining the size of the view-function quotiented by name-permutation and the maximum number of names occurring in a single P-view. It then suffices to establish that in each case the reconstruction of the original strategy can be supported by the syntax. For the first seven cases we can proceed as in [8]. For the eighth case, let $y : \text{var}, \Gamma \vdash M' : \text{int}$ and $x : \text{int}, \Gamma \vdash M'' : \delta$ be the terms obtained by IH for σ' and σ'' respectively. Then, in order to account for σ , one can take let $x = (\text{new } y \text{ in } M')$ in M'' . \square

5.2. Universality. We now proceed with the universality result. In the rest of this section we closely follow the presentation of [1]; the reader is referred thereto for a more detailed exposition of the background material. Let us fix an enumeration of partial recursive functions such that ϕ_n is the n -th partial recursive function.

The universality result concerns innocent S-strategies. Recall that S-strategies and their view-functions are saturated under name-permutations and, in fact, view-functions only become functions after nominal quotienting. To represent them we introduce an encoding

scheme that is not dependent on names. Let us define a function eff which converts S-plays to plays in which moves are attached with lists of integers:

$$\text{eff}(so^\Sigma) \triangleq \text{eff}(s)o^{\pi_2(\Sigma)}, \quad \text{eff}(sm^\Sigma p^T) \triangleq \text{eff}(sm^\Sigma)p^{\pi_2(T), |T \setminus \Sigma|}.$$

Thus, from an O-move o^Σ we only keep the values stored in Σ , whereas in a P-move p^T we keep the values of T and a number indicating how many of the names of T are freshly introduced. Because of the conditions on stores that S-plays satisfy, eff maps two S-plays to the same encoding if, and only if, they are nominally equivalent. In the sequel we assume that S-plays are given using the encoding above.

For the rest of the section we assume that PP_A is recursively enumerable, which is clearly the case for denotable prearenas.

Definition 5.4. A subset of PP_A (for instance, a strategy or a view-function) will be called *recursively presentable* if it is a recursively enumerable subset of PP_A . A block-innocent strategy will be called recursively presentable if the underlying innocent S-strategy is recursively presentable.

It follows that an innocent S-strategy σ is recursively presentable if, and only if, its view-function is. We therefore encode an innocent S-strategy σ by $[\sigma]$, where the latter is the index n such that $\text{viewf}(\sigma) = \phi_n$ (with ϕ_n seen as a partial function from codes of P-views of S-plays to codes of P-moves).

We want to show that any recursively presentable innocent S-strategy is definable by an IA_{cbv} -term. The result will be proved by constructing a term that accepts the code of a given strategy, examines its initial behaviour, mimics it and, after a subsequent O-move, is ready to explore the relevant component of the decomposition. Observe that if we start from a strategy on $[\theta_1, \dots, \theta_k \vdash \theta]$ the decomposition will lead us to consider strategies on $[\theta'_1, \dots, \theta'_l \vdash \theta']$, where each θ'_i (as well as θ') is a subtype of some θ_j or θ . Since the given strategy will in general be infinite, repeated applications of the Decomposition Lemma will mean that l is unbounded. To keep track of the current component we will thus need to be able to represent unbounded lists of variables whose types are subtypes of $\theta_1, \dots, \theta_k, \theta$. This issue is tackled next.

List contexts. We say that a set of types T is *closed* if whenever $\theta \in T$ and δ is a subtype of θ then $\delta \in T$. For the rest of this section let us fix a closed finite set of types T and an ordering of T , say $T = T_0, T_1, \dots, T_n$, such that $T_0 = \text{unit}$, $T_1 = \text{int}$, $T_2 = \text{var}$, $T_3 = \text{unit} \rightarrow \text{int}$ and $T_4 = \text{int} \rightarrow \text{unit}$.

For each i , we encode lists of type T_i as products $\text{int} \times (\text{int} \rightarrow T_i)$. In particular, we use the notation

$$z : \text{List}(T_i), \Gamma \vdash M : \delta$$

as a shorthand for

$$z^L : \text{int}, z^R : \text{int} \rightarrow T_i, \Gamma \vdash M : \delta$$

Thus, z^L represents the length of the represented list. For each $1 \leq i \leq z^L$, the value of the i -th element in the list is represented by $z^R i$. The list can be shortened by simply ‘reducing’ z^L . For example, for a term $z : \text{List}(T_i), \Gamma \vdash M : \delta$ we can form

$$z : \text{List}(T_i), \Gamma \vdash \text{let } z^L = z^L - 1 \text{ in } M : \delta.$$

Note that, although the notation seems to suggest differently, the above is unrelated to variable assignment: it stands for $(\lambda z^L.M)(z^L - 1)$. A finer removal of a list element is

executed as follows. For a term $z : \text{List}(T_i)$, $\Gamma \vdash M : \delta$ and an index j , we define the term

$$z : \text{List}(T_i), \Gamma \vdash \text{remove}(z, j) \text{ in } M : \delta$$

to be

$$z : \text{List}(T_i), \Gamma \vdash (\lambda z^L. \lambda z^R. M)(z^L - 1)(\lambda x. \text{if } x < j \text{ then } z^R x \text{ else } z^R(x+1)) : \delta.$$

A list can be extended as follows. For terms $z : \text{List}(T_i)$, $\Gamma \vdash M : \delta$, $N : T_i$ and an index j we define the term

$$z : \text{List}(T_i), \Gamma \vdash \text{insert}(z, j, N) \text{ in } M : \delta$$

to be:

$$z : \text{List}(T_i), \Gamma \vdash (\lambda z^L. \lambda z^R. M)(z^L + 1)(\lambda x. \text{if } x < j \text{ then } z^R x \text{ else if } x = j \text{ then } N \text{ else } z^R(x-1)) : \delta$$

Let us use the shorthands

$$\text{let } z = \text{cons } N \text{ in } M \quad \text{let } z = \text{snoc } z \ N \text{ in } M$$

for $\text{insert}(z, 1, N)$ in M and $\text{insert}(z, z_L + 1, N)$ in M respectively (that is, Hextend inserts at the head of lists and Textend at the tail).

We can define an (effective) indexing function indx which, for any sequence (possibly with repetitions) $\theta_1, \dots, \theta_m$ of types from T , returns a pair of numbers (i, j) such that $\theta_m = T_i$ and there are j occurrences of T_i in $\theta_1, \dots, \theta_m$.

Suppose now we have such a sequence $\bar{\theta}$ and a term $z_0 : \text{List}(T_0)$, $z_1 : \text{List}(T_1)$, \dots , $z_n : \text{List}(T_n) \vdash M : \delta$. We can *de-index* M with respect to $\bar{\theta}$, obtaining the term $x_1 : \theta_1, \dots, x_m : \theta_m \vdash \text{deindx } M : \delta$, defined as

$$\text{deindx } M \triangleq \overline{\text{let } z = \perp \text{ in } (\text{let } z_{l_m} = \text{cons } x_m \ z_{l_m} \text{ in } (\dots (\text{let } z_{l_1} = \text{cons } x_1 \ z_{l_1} \text{ in } M)))},$$

where

$$\overline{\text{let } z = \perp \text{ in } N} \triangleq \text{let } z_0^L = 0, z_0^R = \lambda x. \Omega \text{ in } (\dots (\text{let } z_n^L = 0, z_n^R = \lambda x. \Omega \text{ in } N))$$

and, for each $1 \leq i \leq m$, $\theta_i = T_{l_i}$. Note that the extensions above are executed from left to right so, in particular, x_1 will be related to $z_{l_1}^R 1$.

Universal terms. Given a closed set of types T , the way we prove universality is by constructing for each $\delta \in T$ a *universal term* $z_0 : \text{List}(T_0), \dots, z_n : \text{List}(T_n) \vdash F_\delta : \text{int} \rightarrow \delta$ such that, for every sequence $\theta_1, \dots, \theta_m$ from T and recursively presentable S-strategy $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$,

$$\sigma = \llbracket \text{deindx}(F_\delta[\sigma]) \rrbracket_S.$$

We first need to make sure that we can move inside the Decomposition Lemma effectively, i.e. that the passage from the code of the original strategy to the code of the components is effective and that the case which applies can also be computed from the index of the original strategy. Here is such a recursive version of the Decomposition Lemma (for types in T).

Lemma 5.5. There are partial recursive functions

$$D, H : \omega \rightarrow \omega \text{ and } B : \omega \times \omega \rightarrow \omega$$

such that, for any $\theta_1, \dots, \theta_m, \delta \in T$ and recursively presentable S-strategy $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$,

$$D[\sigma] = \begin{cases} i & \text{if } \sigma \text{ falls within the } i\text{-th case of DL} \\ \perp & \text{otherwise} \end{cases}$$

$$B([\sigma], i) = \begin{cases} [\tau_i] & \text{if } \sigma \text{ and } \tau_i \text{ are related as in first case of DL} \\ \perp & \text{otherwise} \end{cases}$$

$$H[\sigma] = \begin{cases} i & \text{if } \sigma, i \text{ are related as in third case of DL} \\ [[\sigma_1], [\sigma_2]] & \text{if } \sigma, \sigma_1, \sigma_2 \text{ are related as in fourth case of DL} \\ [\sigma'] & \text{if } \sigma, \sigma' \text{ are related as in fifth case of DL} \\ [i, [\sigma']] & \text{if } \sigma, \llbracket \theta_l \rrbracket, \sigma' \text{ are related as in sixth case of DL} \\ & \text{and } \text{indx}(\theta_1, \dots, \theta_l) = (2, i) \\ [i_1, i_2, [\sigma'], [\sigma'']] & \text{if } \sigma, \llbracket \theta_l \rrbracket, \sigma', \sigma'' \text{ are related as in seventh case of DL} \\ & \text{and } \text{indx}(\theta_1, \dots, \theta_l) = (i_1, i_2) \\ [[\sigma'], [\sigma'']] & \text{if } \sigma, \sigma', \sigma'' \text{ are related as in eighth case of DL} \end{cases}$$

Proof. We assume that the type of σ is represented in $[\sigma]$ and can be effectively decoded by D, B and H . $D[\sigma]$ returns 1 if any of the θ_i 's is int, otherwise it applies $\phi_{[\sigma]}$ to the unique initial move of $\llbracket \bar{\theta} \rrbracket$ and returns the number corresponding to the result. For B , given $[\sigma], i$, membership in τ_i is checked as follows. For any (P-view) S-play s , we add i to its initial move and check whether the resulting S-play is a member of σ . Thus we obtain ϕ_n such that $s \mapsto \phi_n(s, [\sigma], i)$ is the characteristic function of τ_i . By an application of the S-m-n theorem we obtain $[\tau_i]$. For H we argue along the same lines. \square

Since (call-by-value) PCF is Turing complete, there are closed PCF-terms $\tilde{D}, \tilde{H} : \text{int} \rightarrow \text{int}$ and $\tilde{B} : \text{int} \rightarrow \text{int} \rightarrow \text{int}$ that represent each of the above functions with plays of the form $q * n f(n)$ or $q * m * n f(m, n)$. The terms will be used inside the universal term, which will be constructed by mutual recursion (there are standard techniques to recast such definitions in PCF). Let us write $\theta = T_{l(\theta)} \rightarrow T_{r(\theta)}$ whenever $\theta \in T$ is of arrow type (so $l, r : T \rightarrow \{0, \dots, n\}$).

Definition 5.6. For each $\delta \in T$ we define terms

$$z_0 : \text{List}(T_0), \dots, z_n : \text{List}(T_n) \vdash F_\delta : \text{int} \rightarrow \delta$$

by mutual recursion as follows.

$$\begin{aligned}
F_\delta \triangleq & \lambda k^{\text{int}}. \text{ if } z_1^L \neq 0 \text{ then let } x = z_1^R 1 \text{ in remove } (z_1, 1) \text{ in } F_\delta(\tilde{B} k x) \\
& \text{ else case } (\tilde{D} k) \text{ of} \\
& \quad 2 : \text{ skip} \\
& \quad 3 : \tilde{H} k \\
& \quad 4 : \text{ let } [k_1, k_2] = \tilde{H} k \text{ in mkvar}(F_{\text{unit} \rightarrow \text{int}} k_1, F_{\text{int} \rightarrow \text{unit}} k_2) \\
& \quad 5 : \lambda y^{T_l(\delta)}. \text{ let } z_{l(\delta)} = \text{snoc } z_{l(\delta)} y \text{ in } F_{T_r(\delta)}(\tilde{H} k) \\
& \quad 6 : \text{ let } [i, k] = \tilde{H} k \text{ in} \\
& \quad \quad \text{let } z_3 = \text{snoc } z_3 \lambda x^{\text{unit}}.!(z_2^R i) \text{ in} \\
& \quad \quad \text{let } z_4 = \text{snoc } z_4 \lambda x^{\text{int}}. (z_2^R i) := x \text{ in remove } (z_2, i) \text{ in } F_\delta k \\
& \quad 7 : \text{ let } [i_1, i_2, k_1, k_2] = \tilde{H} k \text{ in case } i_1 \text{ of} \\
& \quad \quad 1 : \dots \\
& \quad \quad \vdots \\
& \quad \quad j : \text{ let } z_{r(T_j)} = \text{snoc } z_{r(T_j)} (z_j^R i_2)(F_{T_l(T_j)} k_1) \text{ in } F_\delta k_2 \\
& \quad \quad \vdots \\
& \quad \quad n : \dots \\
& \quad 8 : \text{ let } [k_1, k_2] = \tilde{H} k \text{ in} \\
& \quad \quad \text{let } z_1 = \text{cons}(\text{new } x \text{ in let } z_4 = \text{cons } x z_4 \text{ in } F_{\text{int}} k_1) z_1 \text{ in } F_\delta k_2 \\
& \quad \text{ otherwise : } \Omega
\end{aligned}$$

The construction of F_δ follows closely the decomposition of σ according to the Decomposition Lemma. In particular, on receiving $[\sigma]$, the term decides, using the functions B and D of Lemma 5.5, to which branch of DL σ can be matched. Some branches decompose σ into further strategies, in which case F_δ will recursively call some $F_{\delta'}$ to simulate the rest of the strategy. The use of lists in contexts guarantees that such a call is indeed recursive: F is only parameterised by the output type δ' , and each such δ' is in T .

Proposition 5.7 (Universality). For every $\theta_1, \dots, \theta_m, \delta \in T$ and recursively presentable innocent S-strategy $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$, $\sigma = \llbracket \text{deindx}(F_\delta[\sigma]) \rrbracket_S$.

Proof. Suppose F_δ receives $[\sigma]$ in its input k , where $\sigma : \llbracket \theta_1 \rrbracket \otimes \dots \otimes \llbracket \theta_m \rrbracket \rightarrow \llbracket \delta \rrbracket$. Then

$$\text{ if } z_1^L \neq 0 \text{ then let } x = z_1^R 1 \text{ in remove } (z_1, 1) \text{ in } F_\delta(\tilde{B} k x)$$

recognizes the first branch of the DL. Recall that $T_1 = \text{int}$, so $z_1 : \text{List}(\text{int})$, and therefore $z_1^L \neq 0$ holds iff there is some $\theta_i = \text{int}$. If this is so, then F_δ needs to return a term corresponding to the strategy instantiated with the leftmost element in the list z_1 . This is achieved by first applying \tilde{B} to $k, (z_1^R 1)$ to obtain $[\tau_i]$ and applying F_δ to it.

If $z_1^L = 0$, F_δ will call \tilde{D} on $[\sigma]$, which will return the number of the case from DL (2-8) that applies to σ . Subsequently, F_δ will proceed to a case analysis. Below we examine two cases in detail.

- 5:** σ is the currying of σ' , so F_δ should return $\lambda y. F[\sigma'(y)]$. Now, σ' is $F_{T_r(\delta)}[\sigma']$, i.e. $F_{T_r(\delta)}(\tilde{H} k)$, where $\delta = T_l(\delta) \Rightarrow T_r(\delta)$. In order to preserve typability, we need to add the abstracted variable to the context of $F_{T_r(\delta)}(\tilde{H} k)$, which is what $\text{Textend } z_{l(\delta)}$ with y achieves.
- 8:** σ introduces some fresh name and decomposes to σ', σ'' as in the Decomposition Lemma. Hence, F_δ should return $\text{let } y = (\text{new } x \text{ in } F[\sigma'(x)]) \text{ in } F[\sigma''(y)]$, which is exactly what the code achieves.

The other cases are similar. \square

Remark 5.8. It is worth noting that the universality result for innocent S-strategies implies an analogous result for innocent strategies and PCF. Thanks to call-by-value, the result is actually sharper than the universality results of [1, 10], which had to be proved “up to observational equivalence”. This was due to the fact that partial recursive functions could not always be represented in the canonical way (i.e. by terms for which the corresponding strategy contained plays of the form $q q n f(n)$). This is no longer the case under the call-by-value regime, where each partially recursive function f can be coded by a term whose denotation will be the strategy based on plays of the shape $n f(n)$.

6. FROM OMNISCIENCE TO INNOCENCE

In Section 2 we introduced the three languages: PCF^+ , IA_{cbv} and RML , interpreted respectively by innocent, block-innocent and knowing strategies. Let A be a prearena. We write \mathcal{I}_A , \mathcal{B}_A and \mathcal{K}_A for the corresponding classes of (store-free) strategies in A . Obviously, $\mathcal{I}_A \subseteq \mathcal{B}_A \subseteq \mathcal{K}_A$. Next we shall study type-theoretic conditions under which one kind of strategy collapses to another. Thanks to universality results, this corresponds to the existence of an equivalent program in a weaker language.

Theorem 6.1. *Let $A = \llbracket \theta_1, \dots, \theta_n \vdash \theta \rightarrow \theta' \rrbracket$. Then $\mathcal{B}_A \subsetneq \mathcal{K}_A$.*

Proof. Observe that there exist moves q_0, a_0, q_1, a_1 such that $q_0 \vdash_A a_0 \vdash_A q_1 \vdash_A a_1$ and consider $\sigma = \{\epsilon, q_0 a_0, q_0 a_0 q_1 a_1\}$, i.e. σ has no response at $q_0 a_0 q_1 a_1 q_1$. Then $\sigma \in \mathcal{K}_A \setminus \mathcal{B}_A$. It is worth remarking that a strategy of the above kind denotes the RML -term $\text{let } v = \text{ref in } \lambda x^{\text{unit}}. (\text{if } !v \text{ then } \Omega \text{ else } v := !v + 1) : \text{unit} \rightarrow \text{unit}$. \square

Theorem 6.1 confirms that, in general, block structure restricts expressivity. However, the next result shows this not to be the case for open terms of base type.

Theorem 6.2. *Let $A = \llbracket \theta_1, \dots, \theta_n \vdash \beta \rrbracket$. Then $\mathcal{B}_A = \mathcal{K}_A$.*

Proof. Observe that any knowing strategy for A becomes block-innocent if in the second-move P introduces a store with one variable that keeps track of the history of play (this is reminiscent of the factorization arguments in game semantics). The variable should be removed from the store by P only when he plays an answer to the initial question. \square

By universality, we can conclude that each RML-term of base type is equivalent to an IA_{cbv} -term. Since contexts used for testing equivalence are exactly of this kind, we obtain the following corollaries. The first one amounts to saying that RML is a conservative extension of IA_{cbv} . The second one states that block-structured contexts suffice to distinguish terms that might use scope extrusion.

Corollary 6.3. For any IA_{cbv} -terms $\Gamma \vdash M_1, M_2 : \theta$ and RML-terms $\Gamma \vdash N_1, N_2 : \theta$:

- $\Gamma \vdash M_1 \cong_{\text{RML}} M_2$ if, and only if, $\Gamma \vdash M_1 \cong_{\text{IA}_{\text{cbv}}} M_2$;
- $\Gamma \vdash N_1 \cong_{\text{RML}} N_2$ if, and only if, $\Gamma \vdash N_1 \cong_{\text{IA}_{\text{cbv}}} N_2$.

□

Now we investigate the boundary between block structure and lack of state.

Lemma 6.4. Let A be a prearena such that each question enables an answer⁶. The following conditions are equivalent.

- (1) $\mathcal{B}_A \subseteq \mathcal{I}_A$.
- (2) No O-question is enabled by a P-question: $m \vdash_A q_O$ implies $\lambda_A(m) = PA$.
- (3) Store content of O-questions is trivial: $sq_O^\Sigma \in SP_A$ implies $\text{dom } \Sigma = \emptyset$.

Proof.

- (1 \Rightarrow 2) We prove the contrapositive. Assume that there exists a P-question q_P and an O-question q_O such that $q_P \vdash_A q_O$. Let s be a chain of hereditary enablers of q_P (starting from an initial move) augmented with pointers from non-initial moves to the respective preceding moves. Then

$$s \overset{(x,0)}{\curvearrowright} q_P \overset{(x,0)}{\curvearrowright} q_O \overset{(x,0)}{\curvearrowright} a_P \overset{(x,1)}{\curvearrowright} q_O \overset{(x,1)}{\curvearrowright} q_P$$

defines a block-innocent strategy that is not innocent.

- (2 \Rightarrow 3) Suppose no P-question enables an O-question in A and let $sq_O^X \in SP_A$. Then the sequence of hereditary justifiers of q_O in s , in order of their occurrence in s , must have the form $(q_O a_P)^*$. Consequently, none of the stores involved can be non-empty, so X must be empty too.
- (3 \Rightarrow 1) We observe that $s_1 p^{X_p} s_2 o^{X_o} \in SP_A$, where p justifies o , implies $X_o = X_p$. Note that, in presence of block innocence, this implies innocence because the store content of O-moves can be reconstructed uniquely from the P-view. Thus, it suffices to prove our observation correct.
- If o is a question, we simply use our assumption: then we must have $X_o = \emptyset$ and, because $\text{dom } X_o = \text{dom } X_p$, we can conclude $X_p = \emptyset$.
 - If o is an answer then p must be a question. We claim that no store in s_2 can contain variables from $\text{dom } X_o = \text{dom } X_p$. Suppose this is not the case and there is such an occurrence. Then the earliest such occurrence must be part of an O-move. This move cannot be a question due to our assumption, so it is an answer move. By the bracketing condition, this must be an answer that an earlier P-question played after p . Moreover, the store accompanying that question must also contain a variable from $\text{dom } X_o = \text{dom } X_p$ contradicting our choice of the earliest occurrence. □

⁶All denotable prearenas enjoy this property.

Thanks to the following lemma we will be able to determine precisely at which types block-innocence implies innocence.

Lemma 6.5. $\llbracket \theta_1, \dots, \theta_n \vdash \theta \rrbracket$ satisfies condition 2. of Lemma 6.4 iff $\text{ord}(\theta_i) \leq 1$ ($i = 1, \dots, n$) and $\text{ord}(\theta) \leq 2$.

Consequently, second-order IA_{cbv} -terms always have purely functional equivalents. Finally, we can pinpoint the types at which strategies are bound to be innocent: it suffices to combine the previous findings.

Theorem 6.6. Let $A = \llbracket \theta_1, \dots, \theta_n \vdash \theta \rrbracket$. Then $\mathcal{K}_A = \mathcal{I}_A$ iff $\text{ord}(\theta_i) \leq 1$ ($i = 1, \dots, n$) and $\text{ord}(\theta) = 0$.

In the next section we demonstrate that the gap in expressivity between \mathcal{K}_A and \mathcal{B}_A also bears practical consequences. The undecidable equivalence problem for second-order finitary RML becomes decidable in second-order finitary IA_{cbv} (as well as at some third-order types).

7. DECIDABILITY OF A FINITARY FRAGMENT OF IA_{cbv}

In order to prove program equivalence decidable, we restrict the base datatype of integers to the finite segment $\{0, \dots, N\}$ ($N > 0$) and replace recursive definitions ($Y(M)$) with looping ($\text{while } M \text{ do } N$). Let us call the resultant language IA_{\circ} . Our decidability result will hold for a subset IA_{\circ}^{2+} of IA_{\circ} , in which type order is restricted. IA_{\circ}^{2+} will reside inside the third-order fragment of IA_{\circ} and contain its second-order fragment. Note that the second-order fragment of similarly restricted RML is known to be undecidable (even without loops) [15].

The decidability of program equivalence in IA_{\circ}^{2+} will be shown by translating terms to regular languages representing the corresponding *block-innocent* strategies. We stress that we are *not* going to work with the induced S-plays. Nevertheless, the translation will rely crucially on insights gleaned from the semantics with explicit stores. In particular, we shall take advantage of the uniformity inherent in block innocence to represent only subsets of the strategies in order to overcome technical problems presented by pointers. We discuss the issue next.

7.1. Pointer-related issues. Pointers from answer-moves need not be represented at all, because they are uniquely reconstructible through the well-bracketing condition. However, this need not apply to pointers from questions. The most obvious way to represent them is to decorate moves with integers that encode the distance from the target in some way. Unfortunately, there are scenarios in which the distance can grow arbitrarily.

Consider, for instance, the prearena $A = \llbracket \theta \vdash \theta_1 \rightarrow \dots \rightarrow \theta_k \rightarrow \beta \rrbracket$. Due to the presence of the k arrows on the right-hand side we obtain chains of enablers $q_0 \vdash a_0 \vdash \dots \vdash q_k \vdash a_k$, where q_0 is initial and each q_i ($i = 1, \dots, k$) is initial in $\llbracket \theta_i \rrbracket$. We shall call the moves *spinal*. Observe that plays in A can have the following shape

$$q_0 \cdots a_0 q_1 \cdots a_1 q_1 \cdots a_1 q_1 \cdots a_1 q_2$$

and any of the occurrences of a_1 could be used to justify q_2 , thus creating several different options for justifying q_2 . If we consider S-plays for A , Definition 4.2 implies that none of the moves q_i, a_i will ever carry a non-empty store. Consequently, whenever a play of the above

kind comes from a block-innocent strategy, its behaviour in the $q_1 \cdots a_1$ segments will not depend on that in the other $q_1 \cdots a_1$ segments. Thus, in order to explore exhaustively the range of behaviours offered by a block-innocent strategy (so as to compare them reliably), it suffices to restrict the number of q_1 's to 1. Next, under the assumption that q_1 occurs only once, one can repeat the same argument for q_2 to conclude that a single occurrence of q_2 will suffice, and so on. Altogether this yields the following lemma. Note that, due to Visibility, insisting on the presence of a unique copy of q_1, \dots, q_k in a play amounts to asking that each q_i be preceded by a_{i-1} .

Lemma 7.1. Call a play *spinal* if each spinal question q_i ($0 < i \leq k$) occurring in it is the immediate successor of a_{i-1} . Let P_A^{sp} be the set of spinal plays of A . Let $\sigma, \tau : A$ be block-innocent strategies. Then $\sigma \cap P_A^{sp} = \tau \cap P_A^{sp}$ implies $\sigma = \tau$. \square

Hence, for the purpose of checking program equivalence, it suffices to compare the induced sets of *spinal* complete plays. Moreover, the pointer-related problems discussed above will not arise.

Now that we have dealt with one challenge, let us introduce another one, which cannot be overcome so easily. Consider the prearena $\llbracket (\theta_1 \rightarrow \theta_2 \rightarrow \theta_3) \rightarrow \theta_4 \vdash \theta \rrbracket$ and the enabling sequence $q_0 \vdash q_1 \vdash q_2 \vdash a_2 \vdash q_3$ it contains. Now consider the plays $q_0 q_1 (q_2 a_2)^j q_3$, where $j \geq 0$. Again, to represent the pointer from q_3 to one of the j occurrences of a_2 , one would need an unbounded number of indices. This time it is not sufficient to restrict j to 1, because the behaviour need not be uniform after each q_2 (this is because in the setting with stores a non-empty store can be introduced as soon as in the second move q_1). To see that the concern is real, consider the term $f : (\text{unit} \rightarrow \text{unit} \rightarrow \text{unit}) \rightarrow \text{unit} \vdash \text{new } x \text{ in } f(\lambda y^{\text{unit}}. \dots \lambda z^{\text{unit}}. \dots) : \text{unit}$, where (\dots) contain some code inspecting and changing the value of x .

This leads us to introduce $\text{IA}_{\circlearrowleft}^{2+}$ via a type system that will not generate the configuration just discussed. Another restriction is to omit third-order types in the context, as they lead beyond the realm of regular languages (cf. $f : ((\text{unit} \rightarrow \text{unit}) \rightarrow \text{unit}) \rightarrow \text{unit} \vdash f(\lambda g^{\text{unit} \rightarrow \text{unit}}. g())$). Since **var** leads to identical problems as $\text{unit} \rightarrow \text{unit}$, we restrict its use accordingly.

7.2. $\text{IA}_{\circlearrowleft}^{2+}$.

Definition 7.2. $\text{IA}_{\circlearrowleft}^{2+}$ consists of $\text{IA}_{\circlearrowleft}$ -terms whose typing derivations rely solely on typing judgments of the shape $x_1 : \text{ctype}_1, \dots, x_n : \text{ctype}_n \vdash M : \text{ttype}$, where *ctype* and *ttype* are defined by the grammar below.

$$\begin{array}{l} \text{ctype} ::= \beta \mid \text{var} \mid \beta \rightarrow \text{ctype} \mid \text{var} \rightarrow \text{ctype} \mid (\beta \rightarrow \beta) \rightarrow \text{ctype} \\ \text{ttype} ::= \beta \mid \text{var} \mid \text{ctype} \rightarrow \text{ttype} \end{array}$$

A lot of pointers from questions become uniquely determined in strategies representing $\text{IA}_{\circlearrowleft}^{2+}$ terms, namely, all pointers from any O-questions and all pointers from P-questions to O-questions.

Lemma 7.3. Let $A = \llbracket \text{ctype}_1, \dots, \text{ctype}_n \vdash \text{ttype} \rrbracket$ and s_1, s_2 be spinal plays of A that are equal after all pointers from O-questions and all pointers from P-questions to O-questions have been erased. Then $s_1 = s_2$.

Proof. Observe that whenever a P-question is enabled by an O-question in the prearenas under consideration, the O-question must be spinal. Hence, because both s_1 and s_2 are spinal, all such O-questions will occur only once, so pointers from P-questions to O-questions are uniquely reconstructible.

Now let us consider O-questions. Observe that, due to restrictions on the type system of $\mathbf{IA}_{\circlearrowleft}^{2+}$, whenever an O-question is justified by a P-answer, both will be spinal. Hence only one copy of each can occur in a spinal position, making pointer reconstruction unambiguous. Finally, we tackle the case of O-questions justified by P-questions.

- If q comes from a type of the context then, due to the shape of types involved, any sequence of hereditary enablers of q must be of the form $q'q'_1a'_1 \cdots q'_ja'_jq'_{j+1}q$, where q' is initial and each of the moves listed enables the following one. If a move m enables q hereditarily, let us define its degree as the distance from the initial move in the sequence above (this definition is independent of the actual choice of the chain of enablers; the degree of q'_i is $2i - 1$, that of a'_i is $2i$).

By induction on move-degree we show that in any O-view only one move of a given degree can be present, if at all. By visibility this makes pointer reconstruction unambiguous.

- By definition of a play q' can occur in an O-view only once. Whenever q'_1 is present in an O-view, it must be preceded by an initial move, so its position in an O-view is uniquely determined (always second).
- Since q'_i occurs only once in an O-view, so does a'_i (questions can be answered only once). Hence, q'_{i+1} can also occur only once, because each occurrence must be preceded by a'_i .

Consequently, any move of degree $2j + 1$ (q'_{j+1}) can only occur once in an O-view and thus the pointer from q can be reconstructed uniquely.

- If q originated from the type on the right-hand side of the typing judgment, we can repeat the reasoning above. The only difference is that the sequences of enablers are now of the form

$$q_0a_0 \cdots q_ka_ka'q'_1a'_1 \cdots q'_ja'_jq'_{j+1}q$$

where q_i, a_i ($i = 0, \dots, k$) are spinal. Then the base case of the induction (q') follows from the fact that we are dealing with spinal plays. \square

Thus, the only pointers that need to be accounted for are those from P-questions to O-answers. Here is the simplest scenario illustrating that they can be ambiguous. Consider the terms

$$f : \text{unit} \rightarrow \text{unit} \rightarrow \text{unit} \vdash \text{let } g_1 = f() \text{ in } (\text{let } g_2 = f() \text{ in } g_i()) : \text{unit}$$

where $i = 1, 2$. They lead to the following plays, respectively for $i = 1$ and $i = 2$, which are equal up to pointers from P-questions to O-answers.

$$\begin{array}{c} \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ q_0 \quad q_1 \quad a_1 \quad q_1 \quad a_1 \quad q_2 \end{array} \quad \begin{array}{c} \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ q_0 \quad q_1 \quad a_1 \quad q_1 \quad a_1 \quad q_2 \end{array}$$

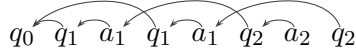
Remark 7.4. In the conference version of the paper [17] we suggested that justification pointers of the above kind be represented with numerical indices encoding the target of the pointer inside the current P-view. More precisely, one could enumerate (starting from 0) all question-enabling O-answers in the P-view. Then pointers from P-questions to O-answers could be encoded by decorating the P-question with the index of the O-answer. The plays

above could then be coded as $q_0q_1a_1q_1a_1q_2^0$ and $q_0q_1a_1q_1a_1q_2^1$ respectively. Unfortunately, there exists terms that generate plays where such indices would be unbounded, such as

$$\text{let } g_1 = f() \text{ in (while } h() \text{ do let } g_2 = f() \text{ in } g_2()); g_1() : \text{unit}$$

where $f : \text{unit} \rightarrow \text{unit} \rightarrow \text{unit}$ and $h : \text{unit} \rightarrow \text{int}$. Because the number of loop iterations is unrestricted, the number needed to represent the justification pointer corresponding to the rightmost occurrence of $g_1()$ cannot be bounded. Consequently, the representation scheme proposed in [17] would lead to an infinite alphabet. Next we show that this problem can be overcome, though.

The above-mentioned defect can be patched with the help of a different representation scheme based on annotating targets and sources of justification pointers, with \circ and \bullet respectively. We shall use the same two symbols for each pointer. This can lead to ambiguities if many pointers are represented at the same time in a single string. However, to avoid that, we are going to use multiple strings to represent a single play. More precisely, these will be strings corresponding to the underlying sequence of moves in which each pointer may or may not be represented, i.e. plays featuring n pointers from P-questions to O-answers will be represented by 2^n encoded strings. For example, to represent



we shall use the following four strings

$$\begin{array}{ll} q_0 q_1 a_1 q_1 a_1 q_2 a_2 q_2 & q_0 q_1 a_1^\circ q_1 a_1 q_2^\bullet a_2 q_2 \\ q_0 q_1 a_1 q_1 a_1^\circ q_2 a_2 q_2^\bullet & q_0 q_1 a_1^\circ q_1 a_1^\circ q_2^\bullet a_2 q_2^\bullet \end{array}$$

Note that the last string represents pointers ambiguously, though the second and third strings identify them uniquely. We could have achieved the same effect using strings in which only one pointer is represented [9] but, from the technical point of view, it is easier to include all possible pointer/no-pointer combinations.

7.3. Regular-language interpretation. In order to translate IA_\circ -terms into regular languages representing their game semantics, we restrict our translation to terms in a canonical shape, to be defined next. Any IA_\circ -term can be converted effectively to such a form and the conversion preserves denotation.

The canonical forms are defined by the following grammar. We use types as superscripts, whenever we want to highlight the type of an identifier (u, v, x, y, z range over identifier names). Note that the only identifiers in canonical form are those of base type, represented by x^β below.

$$\begin{aligned} \mathbb{C} ::= & () \mid i \mid x^\beta \mid x^\beta \oplus y^\beta \mid \text{if } x^\beta \text{ then } \mathbb{C} \text{ else } \mathbb{C} \mid x^{\text{var}} := y^{\text{int}} \mid !x^{\text{var}} \mid \lambda x^\theta. \mathbb{C} \mid \\ & \text{mkvar}(\lambda x^{\text{unit}}. \mathbb{C}, \lambda y^{\text{int}}. \mathbb{C}) \mid \text{new } x^{\text{var}} \text{ in } \mathbb{C} \mid \text{while } \mathbb{C} \text{ do } \mathbb{C} \mid \text{let } x^\beta = \mathbb{C} \text{ in } \mathbb{C} \mid \\ & \text{let } x = zy^\beta \text{ in } \mathbb{C} \mid \text{let } x = z \text{ mkvar}(\lambda u^{\text{unit}}. \mathbb{C}, \lambda v^{\text{int}}. \mathbb{C}) \text{ in } \mathbb{C} \mid \text{let } x = z(\lambda x^\theta. \mathbb{C}) \text{ in } \mathbb{C} \end{aligned}$$

Lemma 7.5. Let $\Gamma \vdash M : \theta$ be an IA_\circ -term. There is an IA_\circ -term $\Gamma \vdash N : \theta$ in canonical form, effectively constructible from M , such that $\llbracket \Gamma \vdash M \rrbracket = \llbracket \Gamma \vdash N \rrbracket$.

Proof. N can be obtained via a series of η -expansions, β -reductions and commuting conversions involving let and if. We present a detailed argument in Appendix B. \square

A useful feature of the canonical form is that the problems with pointers can be related to the syntactic shape: they concern references to let-bound identifiers x^θ such that θ is *not* a base type (i.e. $\theta = \text{var}$ or θ is a function type). Below we state our representability theorem for IA_\circ^{2+} -terms. The definition of \mathcal{A}_M is actually too generous, as we shall only need \circ, \bullet to decorate P-questions enabled by O-answers.

Proposition 7.6. Suppose $\Gamma \vdash M : \theta$ is an IA_\circ^{2+} -term. Let $\mathcal{A}_M = M_A + (M_A \times \{\circ, \bullet\})$, where $A = \llbracket \Gamma \vdash \theta \rrbracket$. Let $\mathcal{C}_{\Gamma \vdash M}$ be the set of non-empty spinal complete plays from $\llbracket \Gamma \vdash M : \theta \rrbracket$. Then $\mathcal{C}_{\Gamma \vdash M}$ can be represented as a regular language over a *finite* subset of \mathcal{A}_M .

Proof. $\mathcal{C}_{\Gamma \vdash M}$ can be decomposed as $\sum_{i \in I_A} (i \mathcal{C}_{\Gamma \vdash M}^i)$. Obviously $\mathcal{C}_{\Gamma \vdash M}$ is regular if, and only if, any $\mathcal{C}_{\Gamma \vdash M}^i$ is regular ($i \in I_A$). Hence, it suffices to show that $\mathcal{C}_{\Gamma \vdash M}^i$ is regular for any relevant i .

As already discussed, the regular-language representations, which we shall also refer to by $\mathcal{C}_{\Gamma \vdash M}^i$, will consist of plays in which individual pointers may or may not be represented. However, because all possibilities are covered, this will yield a faithful representation of the induced complete plays. We proceed by induction on the structure of canonical forms. Let i_Γ range over $I_{\llbracket \Gamma \rrbracket}$.

- $\mathcal{C}_{\Gamma \vdash ()}^{i_\Gamma} = \star$
- $\mathcal{C}_{\Gamma \vdash j}^{i_\Gamma} = j$
- $\mathcal{C}_{\Gamma, x; \beta \vdash x; \beta}^{(i_\Gamma, jx)} = j$
- $\mathcal{C}_{\Gamma, x; \text{int}, y; \text{int} \vdash x \oplus y}^{(i_\Gamma, jx, ky)} = j \oplus k$
- $\mathcal{C}_{\Gamma, x; \text{int} \vdash \text{if } x \text{ then } M_1 \text{ else } M_0}^{(i_\Gamma, jx)} = \mathcal{C}_{\Gamma, x \vdash M_{h(j)}}^{(i_\Gamma, jx)}$, where $h(j) = \begin{cases} 0 & j = 0 \\ 1 & j > 0 \end{cases}$
- $\mathcal{C}_{\Gamma, x; \text{var}, y; \text{int} \vdash x := y}^{(i_\Gamma, \star x, jy)} = \text{write}(j)_x \text{ok}_x \star$
- $\mathcal{C}_{\Gamma, x; \text{var} \vdash !x}^{(i_\Gamma, \star x)} = \text{read}_x (\sum_{j=0}^N jxj)$
- $\mathcal{C}_{\Gamma \vdash \text{mkvar}(\lambda x^{\text{unit}}, M_1, \lambda y^{\text{int}}, M_2)}^{i_\Gamma} = \star(\epsilon + \text{read } \mathcal{C}_{\Gamma, x \vdash M_1}^{(i_\Gamma, \star x)} + \sum_{j=0}^N \text{write}(j) \mathcal{C}_{\Gamma, y \vdash M_2}^{(i_\Gamma, jy)} [\text{ok}/\star])$
- $\mathcal{C}_{\Gamma \vdash \lambda x^\theta. M}^{i_\Gamma} = \star(\epsilon + \sum_{i \in I_{\llbracket \theta \rrbracket}} i \mathcal{C}_{\Gamma, x \vdash M}^{(i_\Gamma, ix)} [m'_x/m_x])$
- $\mathcal{C}_{\Gamma \vdash \text{new } x \text{ in } M}^{i_\Gamma} = (\mathcal{C}_{\Gamma, x \vdash M}^{(i_\Gamma, \star x)} \cap \mathcal{C}')[\epsilon/m_x]$ where, writing \parallel for the shuffle operator on strings,

$$\mathcal{C}' = (\mathcal{A}' \setminus (M_{\llbracket \text{var} \rrbracket})_x)^* \parallel ((\text{read}_x 0_x)^* (\sum_{j=0}^N \text{write}(j)_x \text{ok}_x (\text{read}_x jx)^*))^*$$

and \mathcal{A}' is the finite alphabet used to represent $\Gamma, x : \text{var} \vdash M$.

The substitution $[m'_x/m_x]$ highlights the fact that the moves associated with x have to be bijectively relabelled, because the copy of θ moved from the left- to the right-hand side of the context. $[\epsilon/m_x]$ stands for erasure of all moves associated with x . Obviously, these (homomorphic) operations preserve regularity. Note that the clause for $\lambda x^\theta. M$ is correct because we consider spinal plays only.

For $\Gamma \vdash M : \beta$ we sometimes refer to components determined by the following decomposition.

$$\mathcal{C}_{\Gamma \vdash M; \beta}^{i_\Gamma} = \sum_{j \in I_{\llbracket \beta \rrbracket}} (\mathcal{C}_{\Gamma \vdash M}^{i_\Gamma, j} j)$$

The components $\mathcal{C}_{\Gamma \vdash M}^{i_{\Gamma}, j}$ can be extracted from $\mathcal{C}_{\Gamma \vdash M; \beta}^{i_{\Gamma}}$ by applying operations preserving regularity (intersection, erasure), so the latter is regular iff each of the former is.

- $\mathcal{C}_{\Gamma \vdash \text{while } M \text{ do } N}^{i_{\Gamma}} = (\sum_{j=1}^N \mathcal{C}_{\Gamma \vdash M}^{i_{\Gamma}, j} \mathcal{C}_{\Gamma \vdash N}^{i_{\Gamma}, \star})^* \mathcal{C}_{\Gamma \vdash M}^{i_{\Gamma}, 0}$
- $\mathcal{C}_{\Gamma \vdash \text{let } x^{\beta} = M \text{ in } N}^{i_{\Gamma}} = \sum_{j \in I_{[\beta]}} \mathcal{C}_{\Gamma \vdash M}^{i_{\Gamma}, j} \mathcal{C}_{\Gamma, x \vdash N}^{(i_{\Gamma}, j, x)}$

The remaining cases are those of let-bindings of the form $\text{let } x = z(\dots) \text{ in } \dots$. First we explain some notation used throughout. Consider the following context $\Gamma, z : \theta' \rightarrow \theta, x : \theta$. We shall refer to moves contributed by $x : \theta$ with m_x . If we want to range solely over O- or P-moves from the component, we use o_x and p_x respectively. Moreover, we use $m_{z,x}, o_{z,x}, p_{z,x}$ to refer to copies of m_x, o_x, p_x in the $z : \theta' \rightarrow \theta$ component. The most common operation performed using this notation will be the relabelling of m_x to $m_{z,x}$. If θ is a function type, then there is a unique P-question q_x enabled by the initial move \star_x . Whenever we have a separate substitution rule for q_x , the rule for m_x or p_x will not apply to q_x . In most cases we will want to substitute $q_{z,x}$ or $q_{z,x}^{\bullet}$ for q_x . In the latter case, $q_{z,x}^{\bullet}$ stands for $q_{z,x}$ annotated to represent the source of a pointer to a source move represented by $\star_{z,x}^{\circ}$.

First we tackle cases where the bound value is of function type, i.e. those related to possibly ambiguous pointer reconstruction. Note that we include plays with and without pointer representations.

- $\Gamma, z : \beta \rightarrow (\theta_1 \rightarrow \theta_2), y : \beta \vdash \text{let } x = zy \text{ in } N$ given $\Gamma, z, y, x : \theta_1 \rightarrow \theta_2 \vdash N$

$$\begin{aligned} \mathcal{C}_{\Gamma, z, y \vdash \text{let } \dots \text{ in } \dots}^{(i_{\Gamma}, \star_z, j_y)} &= j_z \star_{z,x}^{\circ} \mathcal{C}_{\Gamma, z, y, x \vdash N}^{(i_{\Gamma}, \star_z, j_y, \star_x)} [q_{z,x}^{\bullet}/q_x, m_{z,x}/m_x] \\ &+ j_z \star_{z,x} \mathcal{C}_{\Gamma, z, y, x \vdash N}^{(i_{\Gamma}, \star_z, j_y, \star_x)} [q_{z,x}/q_x, m_{z,x}/m_x] \end{aligned}$$

- $\Gamma, z : (\beta_1 \rightarrow \beta_2) \rightarrow (\theta_1 \rightarrow \theta_2) \vdash \text{let } x = z(\lambda y^{\beta_1}. M) \text{ in } N$ given $\Gamma, z, y : \beta_1 \vdash M : \beta_2$ and $\Gamma, z, x : \theta_1 \rightarrow \theta_2 \vdash N$

$$\begin{aligned} \mathcal{C}_{\Gamma, z \vdash \text{let } \dots \text{ in } \dots}^{(i_{\Gamma}, \star_z)} &= q_z \mathcal{C}' \star_{z,x}^{\circ} \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma}, \star_z, \star_x)} [q_{z,x}^{\bullet} \mathcal{C}'/q_x, p_{z,x} \mathcal{C}'/p_x, o_{z,x}/o_x] \\ &+ q_z \mathcal{C}' \star_{z,x} \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma}, \star_z, \star_x)} [q_{z,x} \mathcal{C}'/q_x, p_{z,x} \mathcal{C}'/p_x, o_{z,x}/o_x] \end{aligned}$$

where $\mathcal{C}' = (\sum_{i \in I_{[\beta_1]}} i_z (\sum_{j \in I_{[\beta_2]}} \mathcal{C}_{\Gamma, z, y \vdash M}^{(i_{\Gamma}, \star_z, i_y), j} j_z))^*$

- $\Gamma, z : \text{var} \rightarrow (\theta_1 \rightarrow \theta_2) \vdash \text{let } x = z \text{ mkvar}(\lambda u^{\text{unit}}. M_1, \lambda v^{\text{int}}. M_2) \text{ in } N$ given $\Gamma, z, u \vdash M_1$, as well as $\Gamma, z, v \vdash M_2$ and $\Gamma, z, x : \theta_1 \rightarrow \theta_2 \vdash N$

$$\begin{aligned} \mathcal{C}_{\Gamma, z \vdash \text{let } \dots \text{ in } \dots}^{(i_{\Gamma}, \star_z)} &= q_z \mathcal{C}' \star_{z,x}^{\circ} \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma}, \star_z, \star_x)} [q_{z,x}^{\bullet} \mathcal{C}'/q_x, p_{z,x} \mathcal{C}'/p_x, o_{z,x}/o_x] \\ &+ q_z \mathcal{C}' \star_{z,x} \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma}, \star_z, \star_x)} [q_{z,x} \mathcal{C}'/q_x, p_{z,x} \mathcal{C}'/p_x, o_{z,x}/o_x], \end{aligned}$$

where $\mathcal{C}' = (\text{read}_z (\sum_{j=0}^N \mathcal{C}_{\Gamma, z, u \vdash M_1}^{(i_{\Gamma}, \star_z, \star_u), j} j_z) + \sum_{j=0}^N \text{write}(j)_z \mathcal{C}_{\Gamma, z, v \vdash M_2}^{(i_{\Gamma}, \star_z, j_v), \star} \text{ok}_z))^*$

To understand the second formula (the third case is analogous) observe that, after $\star_{z,x}$ has been played, plays for $\text{let } \dots \text{ in } \dots$ are plays from N interleaved with possible detours to $\lambda x^{\beta_1}. M$: such a detour can be triggered by i_z from β_1 each time the second move (q_z) is O-visible. Moreover, provided q_z is O-visible, such a detour can also take place between q_z and $\star_{z,x}$. The following auxiliary lemma will help us analyze when detours can occur.

Lemma 7.7. Let A be a prearena, $s \in P_A$ be non-empty and P_i — the set of P-moves enabled by the initial move of s . Let s' be a prefix of s containing at least two moves. Then the O-view of s' contains exactly one move from P_i .

Proof. Any non-initial move must be either in P_i or hereditarily enabled by a move from P_i . By visibility the O-view of s' any prefix of s must thus contain a move from P_i . Because moves from P_i are enabled by the initial move, they are always the second moves in O-views. Hence, no two moves from P_i can occur in the same O-view. \square

Let us apply the lemma to the denotation of M . Because β_2 is a base type it follows that at any time non-trivial O-views will contain a move from Γ enabled by the initial move or the answer from β_2 (which completes the play). Returning to the play for let \dots in \dots , this means that during a detour the second move q_z will be hidden from O-view until the detour is completed, i.e. a single detour has to be completed before the next one begins. Hence, \mathcal{C}' has the form $(\dots)^*$.

Also by the lemma above, once the play after $\star_{z,x}$ progresses, the second move q_z will be O-visible if, and only if, $q_{z,x}$ (which $\star_{z,x}$ enables) is. Thus detours will be possible exactly after P plays a P -move hereditarily justified by $\star_{z,x}$, which corresponds to P playing a P -move hereditarily justified by q_x in N (hence the substitutions $p_{z,x}\mathcal{C}'/p_x$ restricted to P -moves). A special case is then that of $q_{z,x}$ which, as a question enabled by an answer, should be represented both with and without a pointer.

The above three cases cover all scenarios (that can arise in $\mathbf{IA}_{\circlearrowleft}^{2+}$) in which z 's type is of the form $\theta \rightarrow (\theta_1 \rightarrow \theta_2)$. The cases where $z : \theta \rightarrow \text{var}$ are analogous except that one needs to use q_x to range over $\text{read}_x, \text{write}(0)_x, \dots, \text{write}(N)_x$ rather than the single move enabled by \star_x . It remains to consider cases of $z : \theta \rightarrow \beta$. The bound values are of base type, so no new pointer indices need to be introduced.

- $\Gamma, z : \beta' \rightarrow \beta, y : \beta' \vdash \text{let } x = zy \text{ in } N$ given $\Gamma, z, y, x : \beta \vdash N$

$$\mathcal{C}_{\Gamma, z, y \vdash \text{let in}}^{(i_{\Gamma, \star z, j y})} = j_z \left(\sum_{k \in I_{[\beta]}} k_z \mathcal{C}_{\Gamma, z, x, y \vdash N}^{(i_{\Gamma, \star z, j y, k x})} \right)$$

- $\Gamma, z : (\beta_1 \rightarrow \beta_2) \rightarrow \beta \vdash \text{let } x = z(\lambda y^{\beta_1}. M) \text{ in } N$ given $\Gamma, z, y : \beta_1 \vdash M : \beta_2$ and $\Gamma, z, x : \beta \vdash N$

$$\mathcal{C}_{\Gamma, z \vdash \text{let in}}^{(i_{\Gamma, \star z})} = q_z \mathcal{C}' \left(\sum_{k \in I_{[\beta]}} k_z \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma, \star z, k x})} \right)$$

where $\mathcal{C}' = (\sum_{i \in I_{[\beta_1]}} i_z (\sum_{j \in I_{[\beta_2]}} \mathcal{C}_{\Gamma, z, y \vdash M}^{(i_{\Gamma, \star z, i y}, j)} j_z))^*$

- $\Gamma, z : \text{var} \rightarrow \beta \vdash \text{let } x = z \text{mkvar}(\lambda u^{\text{unit}}. M_1, \lambda v^{\text{int}}. M_2) \text{ in } N$ given $\Gamma, z, u \vdash M_1, \Gamma, z, v \vdash M_2$ and $\Gamma, z, x \vdash N$

$$\mathcal{C}_{\Gamma, z \vdash \text{let in}}^{(i_{\Gamma, \star z})} = q_z \mathcal{C}' \left(\sum_{k \in I_{[\beta]}} k_z \mathcal{C}_{\Gamma, z, x \vdash N}^{(i_{\Gamma, \star z, k x})} \right)$$

where $\mathcal{C}' = (\text{read}_z (\sum_{j=0}^N \mathcal{C}_{\Gamma, z, u \vdash M_1}^{(i_{\Gamma, \star z, \star u}, j)} j_z) + \sum_{j=0}^N \text{write}(j)_z \mathcal{C}_{\Gamma, z, v \vdash M_2}^{(i_{\Gamma, \star z, j v}, \star)} \text{ok}_z)^*$. \square

Theorem 7.8. Program equivalence of $\mathbf{IA}_{\circlearrowleft}^{2+}$ -terms is decidable.

We remark that adding dynamic memory allocation in the form of `ref` to $\mathbf{IA}_{\circlearrowleft}^{2+}$, or its second-order sublanguage, results in undecidability [15]. Hence, at second order, block structure is “strictly weaker” than scope extrusion.

8. SUMMARY

In this paper we have introduced the notion of block-innocence that has been linked with call-by-value Idealized Algol in a sequence of results. Thanks to the faithfulness of block-innocence, we could investigate the interplay between type theory, functional computation and stateful computation with block structure and dynamic allocation respectively. We have also shown a new decidability result for a carefully designed fragment of IA_{cbv} . Its extension to product types poses no particular difficulty. In fact, it suffices to follow the way we have tackled the `var` type, which is itself a product type. The result thus extends those from [7] and is a step forward towards a full classification of decidable fragments of IA_{cbv} : the language $\text{IA}_{\text{cbv}}^{2+}$ we considered features all second-order types and some third-order types, while finitary IA_{cbv} is known to be undecidable at order 5 [14]. Interestingly, $\text{IA}_{\text{cbv}}^{2+}$ features restrictions that are compatible with the use of higher-order types in PASCAL [12], in which procedure parameters cannot be procedures with procedure parameters. An interesting topic for future work would be to characterise the uniformity inherent in block-innocence in more abstract, possibly category-theoretic, terms.

ACKNOWLEDGEMENT

We would like to thank the anonymous referees for their constructive comments and feedback.

REFERENCES

- [1] S. Abramsky, R. Jagadeesan, and P. Malacaria. Full abstraction for PCF. *Information and Computation*, 163:409–470, 2000.
- [2] S. Abramsky and G. McCusker. Call-by-value games. In *Proceedings of CSL*, volume 1414 of *Lecture Notes in Computer Science*, pages 1–17. Springer-Verlag, 1997.
- [3] S. Abramsky and G. McCusker. Linearity, sharing and state: a fully abstract game semantics for Idealized Algol with active expressions. In P. W. O’Hearn and R. D. Tennent, editors, *Algol-like languages*, pages 297–329. Birkhäuser, 1997.
- [4] S. Abramsky and G. McCusker. Full abstraction for Idealized Algol with passive expressions. *Theoretical Computer Science*, 227:3–42, 1999.
- [5] C. Cotton-Barratt, D. Hopkins, A. S. Murawski, and C.-H. L. Ong. Fragments of ML decidable by nested data class memory automata. In *Proceedings of FOSSACS’15*, volume 9034 of *Lecture Notes in Computer Science*, pages 249–263. Springer, 2015.
- [6] M. J. Gabbay and A. M. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13:341–363, 2002.
- [7] D. R. Ghica. Regular-language semantics for a call-by-value programming language. In *Proceedings of MFPS*, volume 45 of *Electronic Notes in Computer Science*. Elsevier, 2001.
- [8] K. Honda and N. Yoshida. Game-theoretic analysis of call-by-value computation. *Theoretical Computer Science*, 221(1–2):393–456, 1999.
- [9] D. Hopkins, A. S. Murawski, and C.-H. L. Ong. A fragment of ML decidable by visibly pushdown automata. In *Proceedings of ICALP*, volume 6756 of *Lecture Notes in Computer Science*, pages 149–161. Springer, 2011.
- [10] J. M. E. Hyland and C.-H. L. Ong. On Full Abstraction for PCF: I. Models, observables and the full abstraction problem, II. Dialogue games and innocent strategies, III. A fully abstract and universal game model. *Information and Computation*, 163(2):285–408, 2000.
- [11] R. Milner, M. Tofte, and R. Harper. *The Definition of Standard ML*. The MIT Press, Cambridge, Massachusetts, 1990.
- [12] J. C. Mitchell. *Concepts in programming languages*. Cambridge University Press, 2002.
- [13] E. Moggi. Notions of computation and monads. *Information and Computation*, 93:55–92, 1991.

- [14] A. S. Murawski. About the undecidability of program equivalence in finitary languages with state. *ACM Transactions on Computational Logic*, 6(4):701–726, 2005.
- [15] A. S. Murawski. Functions with local state: regularity and undecidability. *Theoretical Computer Science*, 338(1/3):315–349, 2005.
- [16] A. S. Murawski, C.-H. L. Ong, and I. Walukiewicz. Idealized Algol with ground recursion and DPDA equivalence. In *Proceedings of ICALP*, volume 3580 of *Lecture Notes in Computer Science*, pages 917–929. Springer, 2005.
- [17] A. S. Murawski and N. Tzevelekos. Block structure vs scope extrusion: between innocence and omniscience. In *Proceedings of FOSSACS*, volume 6014 of *Lecture Notes in Computer Science*, pages 33–47. Springer-Verlag, 2010.
- [18] F. Oles. Type algebras, functor categories and block structure. In M. Nivat and J. C. Reynolds, editors, *Algebraic Methods in Semantics*, pages 543–573. Cambridge University Press, 1985.
- [19] C.-H. L. Ong. Observational equivalence of 3rd-order Idealized Algol is decidable. In *Proceedings of IEEE Symposium on Logic in Computer Science*, pages 245–256. Computer Society Press, 2002.
- [20] A. M. Pitts and I. Stark. On the observable properties of higher order functions that dynamically create local names, or: What’s new? In *Proc. 18th Int. Symp. on Math. Foundations of Computer Science*, pages 122–141. Springer-Verlag, 1993. LNCS Vol. 711.
- [21] G. D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.
- [22] J. Power. Semantics for local computational effects. *Electr. Notes Theor. Comput. Sci.*, 158:355–371, 2006.
- [23] J. C. Reynolds. The essence of Algol. In J. W. de Bakker and J.C. van Vliet, editors, *Algorithmic Languages*, pages 345–372. North Holland, 1981.
- [24] I. D. B. Stark. *Names and Higher-Order Functions*. PhD thesis, University of Cambridge Computing Laboratory, 1995. Technical Report No. 363.
- [25] N. Tzevelekos. Full abstraction for nominal general references. *Logical Methods in Computer Science*, 5(3), 2009.

APPENDIX

APPENDIX A. S-PLAYS

In this section we use the term “move” and “S-move” interchangeably.

Lemma A.1 (Prev-PA). If $s = \dots m^\Sigma a_P^T \dots$ is an S-play then, for any α ,

- (a) if $\alpha \in \nu(T)$ then $\alpha \in \nu(\Sigma)$,
- (b) if $\alpha \in \nu(\Sigma \setminus T)$ then α is closed in $s_{<a_P^T}$.

Moreover, $T \leq_p \Sigma$ and therefore $\Sigma \setminus (\Sigma \setminus T) \leq_p T$ and $\Sigma \setminus T \leq_s \Sigma$.

Proof. Let $s = s_1 q_0^{\Sigma_0} \overleftarrow{s_2} a^T \dots$. As $q_0^{\Sigma_0}$ is the pending-Q in $s_1 q_0^{\Sigma_0} s_2$, we have that s is in fact of the form:

$$s_1 q_0^{\Sigma_0} \overleftarrow{q_1^{T_1}} \dots a_1^{\Sigma_1} \overleftarrow{q_2^{T_2}} \dots a_2^{\Sigma_2} \dots q_j^{T_j} \overleftarrow{a_j^{\Sigma_j}} a^T$$

For (a), by Just-P we have that $\alpha \in \nu(\Sigma_0)$ and therefore α is not closed in $s_1 m_0^{\Sigma_0} s_2$. Hence, by Prev-PQ(b), $\alpha \in \nu(T_1)$ and thus, by Just-O, $\alpha \in \nu(\Sigma_1)$. Repeating this argument j times we obtain $\alpha \in \nu(\Sigma_j)$, i.e. $\alpha \in \nu(\Sigma)$.

For (b), let $\alpha \in \nu(\Sigma) = \nu(\Sigma_j)$ be open in $s_{<a^T}$. We claim that then $\alpha \in \nu(\Sigma_0)$ and therefore $\alpha \in \nu(T)$ by Just-P. We have that $\alpha \in \nu(T_j)$, by Just-O. Moreover, since there are no open questions in $q_j^{T_j} \dots a_j^{\Sigma_j}$, we have that α is open in $s_{<q_j^{T_j}}$, so $\alpha \in \nu(s_{<q_j^{T_j}})$ and thus, by Prev-PQ(a), $\alpha \in \text{dom}(\Sigma_{j-1})$. Applying this argument j times we obtain $\alpha \in \nu(\Sigma_0)$.

Finally, we show by induction that $\Sigma_0 \leq_p \Sigma_i$, for all $0 \leq i \leq j$. For the inductive step, we have that $\Sigma_i \setminus (\Sigma_i \setminus T_{i+1}) \leq_p T_{i+1} \leq_p \Sigma_{i+1}$. By IH, $\Sigma_0 \leq_p \Sigma_i$. Moreover, if $\alpha \in \nu(\Sigma_i \setminus T_{i+1})$ then $\alpha \notin \nu(\Sigma_0)$, by Prev-PQ(b), so $\Sigma_0 \leq_p \Sigma_i \setminus (\Sigma_i \setminus T_{i+1})$, $\therefore \Sigma_0 \leq_p \Sigma_{i+1}$. Thus, $T \leq_p \Sigma_0 \leq_p \Sigma_j = \Sigma$. \square

Lemma A.2 (Block Form). If s is an S-play then $\lceil s \rceil$ is in *block-form*: for any α , we have $\lceil s \rceil = s_1 s_2 s_3$, where

- $\alpha \notin \nu(s_1) \cup \nu(s_3)$ and $\forall m^\Sigma \in s_2. \alpha \in \nu(\Sigma)$,
- if $s_2 \neq \epsilon$ then its first element is the move introducing α in s .

Proof. We do induction on $|s|$, with the cases of $|s| \leq 1$ being trivial. If $s = s' o^\Sigma$ then, by IH, $\lceil s \rceil = s'' p^T o^\Sigma$, some $s'' p^T$ in block-form, and $\nu(T) = \nu(\Sigma)$, which imply that $\lceil s \rceil$ is in block-form. If $s = s' p^\Sigma$ then $\lceil s \rceil = \lceil s' \rceil p^\Sigma$ with $\lceil s' \rceil = s_1 s_2 s_3$ in block-form by IH. If $\alpha \notin \text{dom}(\Sigma)$ then OK. Otherwise, if α does not appear in the last move in $s_1 s_2 s_3$ then, by Prev, p^Σ in fact introduces α in s and therefore $\lceil s \rceil$ has block-form. \square

Lemma A.3. Let $s = s_1 o^\Sigma p^T s_2$ be an S-play with $\alpha \in \nu(\Sigma) \setminus \nu(T)$. Then, for any $s'_2 \sqsubseteq_p s_2$, $\alpha \notin \nu(\lceil s_1 o^\Sigma p^T s'_2 \rceil)$.

Proof. We do induction on $|s'_2|$. For the base case, by the previous lemma we have that $\lceil s_1 o^\Sigma \rceil$ has block-form; in particular, it ends in a block of moves which contains the move introducing α in s , and all moves in the block contain α in their stores. Hence, since the justifier of p^T , say $o'^{\Sigma'}$, occurs in $\lceil s_1 o^\Sigma \rceil$ and $\alpha \notin \nu(\Sigma')$ (by Just and the fact that $\alpha \notin \nu(T)$), we have that $o'^{\Sigma'}$ occurs in s before the move introducing α in it and therefore $\alpha \notin \nu(\lceil s_1 o^\Sigma p^T \rceil)$. Now, if $s'_2 = s''_2 o'^{\Sigma'}$ then we need to show that $\alpha \notin \nu(\lceil s_1 o^\Sigma p^T s''_2 o'^{\Sigma'} \rceil)$ given by IH that $\alpha \notin \nu(\lceil s_1 o^\Sigma p^T s''_2 \rceil)$, which immediately follows from Just and Visibility. Finally, if $s'_2 = s''_2 p^{T'}$ then, by IH, $\alpha \notin \nu(s''_2)$ and therefore, by Prev, $\alpha \notin \nu(T')$. Let $p^{T'}$ be justified by some $o'^{\Sigma'}$ in s . If $o'^{\Sigma'}$ occurs in s''_2 then our argument follows directly from the IH. Otherwise, arguing as before, $o'^{\Sigma'}$ occurs in s before the introduction of α and therefore $\alpha \notin \nu(\lceil s_1 o^\Sigma p^T s''_2 p^{T'} \rceil)$. \square

Corollary A.4 (Close). If $s = s_1 o^\Sigma p^T s_2$ is an S-play with $\alpha \in \nu(\Sigma) \setminus \nu(T)$ then $\alpha \notin \nu(s_2)$. \square

Proof of Lemma 4.10. For (a), assuming WLOG p is a P-move in AB and taking $s = s' n^{T'} p^{\Sigma'}$, by definition of the interaction and Prev we have that if $\alpha \in \nu(\Sigma \setminus T)$ then $\alpha \in \nu(\Sigma' \setminus T')$, hence $\alpha \notin \nu(s' n^{T'})$ and therefore $\alpha \notin \nu(un^T)$.

For (b), we do induction on $|s||t|$, base case is trivial. Now, if $s||t = up^\Sigma$ with p a generalised P-move then $\lceil s||t \rceil = \lceil u \rceil p^\Sigma$ and, by IH, $\lceil u \rceil$ has block form, say $u_1 u_2 u_3$. If $\alpha \notin \nu(\Sigma)$ then OK. Otherwise, if α does not appear in the last move of $\lceil u \rceil$ then p^Σ is in fact the move introducing α in $s||t$, so OK. Otherwise, $u_3 = \epsilon$ and therefore $\lceil s||t \rceil$ in block form. If $s||t = uo^T$ with o an O-move in AC then $\lceil s||t \rceil = u'o^T$ for some u' in block-form, and the last move in u' has domain $\text{dom}(T)$. This implies that $s||t$ is in block-form.

For (c), assuming WLOG p is a P-move in AB and taking $s = s' n^{T'} p^{\Sigma'}$, by definition of the interaction we have $\alpha \in \nu(T' \setminus \Sigma')$ so, by Prev, α is closed in $s' n^{T'}$. Now suppose α is open in un^T , that is, there is an open question $q_1^{\Sigma_1}$ in un^T with $\alpha \in \nu(\Sigma_1)$. Then, if $q_0^{\Sigma_0}$ is the pending question of un^T then $\alpha \in \nu(\Sigma_0)$: $\lceil un^T \rceil$ has block-form, by (b), and $q_0^{\Sigma_0}$ appears in it, thus if $\alpha \notin \nu(\Sigma_0)$ then $q_0^{\Sigma_0}$ would precede the move introducing α in un^T and hence it would precede $q_1^{\Sigma_1}$ too. As the interaction ends in an O-move in AB , $q_0^{\Sigma_0}$ is the pending

question of $un^T \uparrow AB$. Let $q_0^{\Sigma_0}$ be the move in s corresponding to $q_0^{\Sigma_0}$. $q_0^{\Sigma_0}$ is the pending question in $s'n^{T'}$ and therefore it appears in $\ulcorner s'n^{T'} \urcorner$. Moreover, since α is closed in $s'n^{T'}$, $\alpha \notin \nu(\Sigma_0')$, which means that $q_0^{\Sigma_0}$ occurs in $\ulcorner s'n^{T'} \urcorner$ before the move introducing α in s . But then $\alpha \notin \nu(\Sigma_0)$, a contradiction.

For (d), we do induction on $|s||t|$, the base case being trivial. If m^Σ is an O-move in AC then the claim is obvious. So assume WLOG m^Σ is a P-move in AB and consider $\ulcorner s||t \uparrow AB \urcorner_{\geq n^T}$, and take two consecutive moves $m_1^{\Sigma_1} m_2^{\Sigma_2}$ in it, and assume $m_2^{\Sigma_2}$ be a P-move in AB . Then these are consecutive also in $s||t \uparrow AB$ and hence, by switching, also in $s||t$. Let $\alpha \in \nu(\Sigma_1 \setminus \Sigma_2)$. By definition of interaction, this dropping of α happens in s too, at the respective consecutive moves $m_1^{\Sigma_1'} m_2^{\Sigma_2'}$, assuming $s = \dots n^{T'} \dots m_1^{\Sigma_1'} m_2^{\Sigma_2'} \dots m^{\Sigma'}$. We can see that, as $\ulcorner s \urcorner$ is in block-form, $\alpha \notin \nu(\Sigma')$ and therefore, by Just, $\alpha \notin \nu(T')$. More than that, $n^{T'}$ occurs in s before the move introducing α in s , thus $\alpha \notin \nu(T)$. We therefore have $\Sigma_1 \setminus (\Sigma_1 \setminus \Sigma_2) \leq_p \Sigma_2$ and $\nu(\Sigma_1 \setminus \Sigma_2) \cap \nu(T) = \emptyset$. On the other hand, if $m_2^{\Sigma_2}$ is an O-move in AB then $m_1^{\Sigma_1}$ is its justifier and therefore, by IH, $\Sigma_1 \leq_p \Sigma_2$. Thus, for every move $m''^{\Sigma''}$ in $\ulcorner s||t \uparrow AB \urcorner_{\geq n^T}$, we have $T \leq_p \Sigma''$ and hence $T \leq_p \Sigma$. Finally, if m^Σ is an answer then, since $\ulcorner s||t \uparrow AB \urcorner$ satisfies well-bracketing, we have that $\ulcorner s||t \uparrow AB \urcorner_{\geq n^T} = n^{T_0} q_1^{\Sigma_1} a_1^{T_1} \dots q_j^{\Sigma_j} a_j^{T_j} m^{\Sigma_{j+1}}$ with $\text{dom}(\Sigma_i) = \text{dom}(T_i)$ for each $1 \leq i \leq j$ by IH. Assuming $s_{\geq n^{T'}} = n^{T_0} q_1^{\Sigma_1'} \dots a_1^{T_1'} \dots q_j^{\Sigma_j'} \dots a_j^{T_j'} m^{\Sigma_{j+1}'}$, if $\alpha \in \nu(\Sigma_{i+1} \setminus T_i)$ for some i then $\alpha \in \nu(\Sigma_{i+1}' \setminus T_i')$ and therefore $\alpha \notin \nu(T_0') = \nu(\Sigma_{j+1}')$. But the latter implies that $\alpha \in \nu(T_{i'}' \setminus \Sigma_{i'+1}')$ for some $i < i' \leq j$ and therefore $\alpha \notin \nu(\Sigma_{i'+1})$. Thus, $\nu(\Sigma \setminus T) = \emptyset$.

For (e), we replay the proof of lemma A.3, that is, we show that, for every $u_2' \sqsubseteq_p u_2$, $\alpha \notin \nu(\ulcorner u_1 n^T p^\Sigma u_2' \urcorner)$. We do induction on $|u_2'|$. For the base case, by (b) we have that $\ulcorner u_1 n^T \urcorner$ has block-form; in particular, it ends in a block of moves which contains the move introducing α in u , and all moves in the block contain α in their stores. Hence, since the justifier of p^Σ , say $n^{T'}$, occurs in $\ulcorner u_1 n^T \urcorner$ and $\alpha \notin \nu(T')$ (by (d) and the fact that $\alpha \notin \nu(\Sigma)$), we have that $n^{T'}$ occurs in u before the move introducing α in it and therefore $\alpha \notin \nu(\ulcorner u_1 n^T p^\Sigma \urcorner)$. Now, if $u_2' = u_2'' o^{T'}$ (an O-move in AC) then we need to show that $\alpha \notin \nu(\ulcorner u_1 m^T p^\Sigma u_2'' o^{T'} \urcorner)$ given by IH that $\alpha \notin \nu(\ulcorner u_1 m^T p^\Sigma u_2'' \urcorner)$, which immediately follows from Visibility. Finally, if $u_2' = u_2'' p'^{\Sigma'}$ (a generalised P-move) then, by IH, $\alpha \notin \nu(u_2'')$ and therefore, by (a), $\alpha \notin \nu(\Sigma')$. Let $p'^{\Sigma'}$ be justified by some $m^{T'}$ in u . If $m^{T'}$ occurs in u_2'' then our argument follows directly from the IH. Otherwise, arguing as before, $m^{T'}$ occurs in u before the introduction of α and therefore $\alpha \notin \nu(\ulcorner u_1 m^T p^\Sigma u_2'' p'^{\Sigma'} \urcorner)$.

For (f), suppose α appears in a B-move n^T of u . By (e), and because α reappears in m^Σ , we have that α also appears in the move following n^T in u . Applying this reasoning repeatedly, we obtain that α appears in $u \uparrow AC$ after n^T .

For (g), we do induction on $|s||t|$, the base case being trivial. Now assume $s = s'm^{\Sigma'}$. If m is an O-move then $\Sigma' \leq \Sigma$ follows from (d) and from the IH applied to the subsequence ending in the justifier of m^Σ . Moreover, if $\alpha \in \nu(\Sigma')$ then $\Sigma'(\alpha)$ is determined by the last appearance of α in s' , say $n^{T'}$. By IH, the corresponding move of $s' || t$ has store T , with $T(\alpha) = T'(\alpha)$. But then, by inspection of the definition of interaction, any move in $s || t$ occurring after $n^{T'}$ does not change the value of α , hence $\Sigma(\alpha) = T(\alpha) = \Sigma'(\alpha)$. If m is a P-move preceded by $n^{T'}$ then, using the IH, we have

$$\Sigma' = T'[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T') \leq (sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T') = \Sigma.$$

From the above, and using again the IH, we obtain also that $\Sigma[\Sigma'] = \Sigma$. Moreover, if $\alpha \in \nu(\text{st}(t))$ then, by IH, if the last B -move of $s \parallel t$ has store Σ_B then $\Sigma_B(\alpha) = \text{st}(t)(\alpha)$ and the value of α cannot be changed by subsequent A -moves. Hence, if $\alpha \in \nu(\Sigma)$ then $\Sigma(\alpha) = \text{st}(t)(\alpha)$. The case of $t = t'm^{\Sigma'}$ is treated dually.

For (h), the last two moves come from the same sequence, say s , so let $s = s'n^{T'}p^{\Sigma'}$. We have that

$$\begin{aligned} T \setminus \Sigma &= (sn^{T'} \cdot t) \setminus ((sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma') + (\Sigma' \setminus T')) \\ &= (sn^{T'} \cdot t) \setminus ((sn^{T'} \cdot t)[\Sigma'] \setminus (T' \setminus \Sigma')) = (T' \setminus \Sigma')[sn^{T'} \cdot t] \end{aligned}$$

where the last equality holds because $T' \setminus \Sigma' \leq T' \leq T = sn^{T'} \cdot t$, by (g). Hence, $T \setminus (T \setminus \Sigma) = T \setminus (T' \setminus \Sigma') \leq_p \Sigma$. We still need to show that $T' \setminus \Sigma' \leq_s T$. Let α_1, α_2 be consecutive names in $\text{dom}(T)$ such that $\alpha_1 \in \nu(T' \setminus \Sigma')$. Then, the point of introduction of α_2 in un^T does not precede that of α_1 , say $q_1^{\Sigma_1}$. Now, by (c), α_1 is closed in un^T so, using also (b), the latter has the form $u'q_1^{\Sigma_1} \cdots a_1^{T_1} q_2^{\Sigma_2} \cdots a_2^{T_2} \cdots q_j^{\Sigma_j} \cdots a_j^{T_j}$. Thus, by (d), the point of introduction of α_2 has to be one of the $q_i^{\Sigma_i}$'s. This implies that $\alpha_1, \alpha_2 \in \nu(s)$ and therefore α_1, α_2 are consecutive also in T' . But then $\alpha_1 \in \nu(T' \setminus \Sigma')$ implies $\alpha_2 \in \nu(T' \setminus \Sigma')$ too. \square

Proof of lemma 4.12. Let us write Σ_{ij} for $\Sigma_i \setminus \Sigma_j$. Then, the LHS is:

$$\begin{aligned} L &= \Sigma_1[\Phi(\Sigma_3, \Sigma_4, \Sigma_5)] \setminus (\Sigma_2 \setminus \Phi(\Sigma_3, \Sigma_4, \Sigma_5)) + \Phi(\Sigma_3, \Sigma_4, \Sigma_5) \setminus \Sigma_2 \\ &= \Sigma_1[\Phi(\Sigma_3, \Sigma_4, \Sigma_5)] \setminus (\Sigma_2 \setminus (\Sigma_3[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54})) + (\Sigma_3[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54}) \setminus \Sigma_2 \\ &= \Sigma_1[\Phi(\Sigma_3, \Sigma_4, \Sigma_5)] \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45})) + (\Sigma_3[\Sigma_5] \setminus \Sigma_{45}) \setminus \Sigma_2 + \Sigma_{54} \end{aligned}$$

where the last equality holds because of (a). The first constituent above is:

$$L_1 = \Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54}] \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45})) = \Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45}))$$

On the other hand, the RHS is:

$$\begin{aligned} R &= \Phi(\Sigma_1, \Sigma_2, \Sigma_3)[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54} = (\Sigma_1[\Sigma_3] \setminus \Sigma_{23} + \Sigma_{32})[\Sigma_5] \setminus \Sigma_{45} + \Sigma_{54} \\ &= ((\Sigma_1[\Sigma_3] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] + (\Sigma_{32} \setminus \Sigma_{45})[\Sigma_5] + \Sigma_{54} \\ &= ((\Sigma_1[\Sigma_3] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] + (\Sigma_3[\Sigma_5] \setminus \Sigma_{45}) \setminus \Sigma_2 + \Sigma_{54} \end{aligned}$$

The first constituent above is:

$$\begin{aligned} R_1 &= ((\Sigma_1[\Sigma_3 \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] = ((\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45})[\Sigma_5] \\ &= (\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}] \setminus \Sigma_{23}) \setminus \Sigma_{45} \end{aligned}$$

For the last equality, let α be in the domain of the resulting store. Then $\alpha \in \nu(\Sigma_5) \cap \nu(\Sigma_1)$, $\therefore \alpha \in \nu(\Sigma_4) \cap \nu(\Sigma_1)$, so $\alpha \in \nu(\Sigma_2)$ by (b). Moreover, $\alpha \notin \nu(\Sigma_{23})$, $\therefore \alpha \in \nu(\Sigma_3)$. Now, let us write Σ for $\Sigma_1[\Sigma_3[\Sigma_5] \setminus \Sigma_{45}]$. We need to show that

$$\Sigma \setminus (\Sigma_2 \setminus (\Sigma_3 \setminus \Sigma_{45})) = (\Sigma \setminus (\Sigma_2 \setminus \Sigma_3)) \setminus \Sigma_{45}$$

and, in fact, it suffices to show that these stores, say Σ_L and Σ^R , have the same domain. By elementary computation,

- $\alpha \in \nu(\Sigma_L)$ iff $(\alpha \in \nu(\Sigma) \wedge \alpha \notin \nu(\Sigma_2)) \vee (\alpha \in \nu(\Sigma) \wedge \alpha \in \nu(\Sigma_3) \wedge \alpha \notin \nu(\Sigma_{45}))$,
- $\alpha \in \nu(\Sigma^R)$ iff $(\alpha \in \nu(\Sigma) \wedge \alpha \notin \nu(\Sigma_2) \wedge \alpha \notin \nu(\Sigma_{45})) \vee (\alpha \in \nu(\Sigma) \wedge \alpha \in \nu(\Sigma_3) \wedge \alpha \notin \nu(\Sigma_{45}))$.

But note now that $\alpha \in \nu(\Sigma) \wedge \alpha \notin \nu(\Sigma_2)$ implies that $\alpha \notin \nu(\Sigma_4)$, by (b), so $\alpha \notin \nu(\Sigma_{45})$. \square

APPENDIX B. PROOF OF LEMMA 7.5

We prove two auxiliary results first, which are special cases of Lemma 7.5.

Lemma B.1. Any identifier x^θ satisfies Lemma 7.5. Moreover, the canonical form is of the form $\lambda y_1^\theta.\mathbb{C}$ when $\theta \equiv \theta_1 \rightarrow \theta_2$ and of the shape $\text{mkvar}(\lambda y^{\text{unit}}.\mathbb{C}, \lambda z^{\text{int}}.\mathbb{C})$ if $\theta \equiv \text{var}$.

Proof. Induction with respect to type structure. If θ is a base type, x^θ is already in canonical form. x^{var} can be converted to one using the rule

$$x^{\text{var}} \longrightarrow \text{mkvar}(\lambda u^{\text{int}}.x := u, \lambda v^{\text{unit}}.!x)$$

For $\theta \equiv \theta_1 \rightarrow \theta_2$ we use the rule

$$x^{\theta_1 \rightarrow \theta_2} \longrightarrow \lambda z^{\theta_1}.\text{let } v^{\theta_2} = xz^{\theta_1} \text{ in } v$$

and appeal to the inductive hypothesis for z^{θ_1} and v^{θ_2} . \square

Lemma B.2. Suppose C_1, C_2 are canonical forms. Then let $y^\theta = C_1$ in C_2 , if typable, satisfies Lemma 7.5.

Proof. Induction with respect to type structure. If θ is a base type, the term is already in canonical form. If θ is not a base type, C_1 can take one of the following three shapes: $\text{mkvar}(\lambda x^{\text{unit}}.\mathbb{C}, \lambda y^{\text{int}}.\mathbb{C})$, $\lambda x_1^{\theta_1}.\mathbb{C}$, if x^β then \mathbb{C} else \mathbb{C} or let \dots in \mathbb{C} .

We first focus on the first two of them to which the remaining two cases will be reduced later.

- Suppose $C_1 \equiv \text{mkvar}(\lambda x_1^{\text{unit}}.C_{11}, \lambda x_2^{\text{int}}.C_{12})$. Then $\theta \equiv \text{var}$. Since C_2 is in canonical form, y can only occur in it as part of a canonical subterm of the form $y^{\text{var}} := z^{\text{int}}$ or $!y$. Hence, after substitution for y , we will obtain non-canonical subterms of the shape $\text{mkvar}(\lambda x_1^{\text{unit}}.C_{11}, \lambda x_2^{\text{int}}.C_{12}) := z$ and $!(\text{mkvar}(\lambda x_1^{\text{unit}}.C_{11}, \lambda x_2^{\text{int}}.C_{12}))$. Using the rules

$$\begin{aligned} !\text{mkvar}(\lambda u^{\text{unit}}.D_1, \lambda v^{\text{int}}.D_2) &\longrightarrow D_1[()/u] \\ \text{mkvar}(\lambda u^{\text{unit}}.D_1, \lambda v^{\text{int}}.D_2) := z &\longrightarrow D_2[z/v] \end{aligned}$$

we can easily convert them (and thus the whole term) to canonical form.

- Suppose $C_1 \equiv \lambda x_1^{\theta_1}.C_3$ and $\theta \equiv \theta_1 \rightarrow \theta_2$. Let us substitute C_1 for the rightmost occurrence of y in C_2 . This will create a non-canonical subterm in C_2 of the form let $x^{\theta_2} = (\lambda x_1^{\theta_1}.C_3)C_4$ in $C_5 \equiv \text{let } x^{\theta_2} = (\text{let } x_1^{\theta_1} = C_4 \text{ in } C_3) \text{ in } C_5$. By inductive hypothesis for θ_1 , let $x_1^{\theta_1} = C_4$ in C_3 can be converted to canonical form, say, C_6 . Consequently, the non-canonical subterm let $x^{\theta_2} = (\lambda x_1^{\theta_1}.C_3)C_4$ in C_5 can be transformed into the form let $x^{\theta_2} = C_6$ in C_5 , which — by inductive hypothesis for θ_2 — can also be converted to canonical form. Thus, we have shown how to recover canonical forms after substitution for the rightmost occurrence of y . Because of the choice of the rightmost occurrence, the transformation does not involve terms containing other occurrences of y , so it will also decrease their overall number in C_2 by one. Consequently, by repeated substitution for rightmost occurrences one can eventually arrive at a canonical form for let $y^\theta = (\lambda x_1^{\theta_1}.C_3)$ in C_2 .

For the remaining two cases it suffices to take advantage of the following conversions before referring to the two cases above.

$$\begin{aligned} \text{let } y &= (\text{if } x \text{ then } D_1 \text{ else } D_0) \text{ in } E \longrightarrow \text{if } x \text{ then } (\text{let } y = D_1 \text{ in } E) \text{ else } (\text{let } y = D_0 \text{ in } E) \\ \text{let } y &= (\text{let } x = D \text{ in } E) \text{ in } F \longrightarrow \text{let } x = D \text{ in } (\text{let } y = E \text{ in } F) \end{aligned} \quad \square$$

Now we are ready to prove Lemma 7.5 by induction on term structure. The base cases of $()$, i are trivial. That of x^θ follows from Lemma B.1.

The following inductive cases follow directly from the inductive hypothesis: $\text{new } x \text{ in } M$, $\lambda x^\theta.M$, $\text{while } M \text{ do } M$. The cases of $M_1 \oplus M_2$ and $\text{if } M \text{ then } N_1 \text{ else } N_0$ are only slightly more difficult. After invoking the inductive hypothesis one needs to apply the rules given below. Note that in this case all the let -bindings are of base type.

$$\begin{aligned} D_1 \oplus D_2 &\longrightarrow \text{let } x = D_1 \text{ in } (\text{let } y = D_2 \text{ in } (x \oplus y)) \\ \text{if } D \text{ then } D_1 \text{ else } D_0 &\longrightarrow \text{let } x = D \text{ in } (\text{if } x \text{ then } D_1 \text{ else } D_0) \end{aligned}$$

For $!M$ and $M := N$ we take advantage of the fact that a canonical form of type var can only take three shapes: $\text{mkvar}(\lambda x^{\text{unit}}.\mathbb{C}, \lambda y^{\text{int}}.\mathbb{C})$, $\text{if } x^\beta \text{ then } \mathbb{C} \text{ else } \mathbb{C}$ or $\text{let } \dots \text{ in } \mathbb{C}$. An appeal to the inductive hypothesis for M and N and the conversions given below will then yield the canonical forms for $!M$ and $M := N$.

$$\begin{aligned} !\text{mkvar}(\lambda u^{\text{unit}}.D_1, \lambda v^{\text{int}}.D_2) &\longrightarrow D_1[()/u] \\ \text{mkvar}(\lambda u^{\text{unit}}.D_1, \lambda v^{\text{int}}.D_2) := E &\longrightarrow \text{let } x^{\text{int}} = E \text{ in } D_2[x/v] \\ !(\text{if } x \text{ then } D_1 \text{ else } D_0) &\longrightarrow \text{if } x \text{ then } !D_1 \text{ else } !D_0 \\ (\text{if } x \text{ then } D_1 \text{ else } D_0) := D &\longrightarrow \text{if } x \text{ then } (D_1 := D) \text{ else } (D_0 := D) \\ !(\text{let } x = D \text{ in } E) &\longrightarrow \text{let } x = D \text{ in } !E \\ (\text{let } x = D \text{ in } E) := F &\longrightarrow \text{let } x = D \text{ in } (E := F) \end{aligned}$$

To convert $\text{mkvar}(M, N)$ to canonical form we observe that a canonical form of function type can take the following shapes: $\lambda x^\theta.\mathbb{C}$, $\text{if } x^\beta \text{ then } \mathbb{C} \text{ else } \mathbb{C}$ or $\text{let } \dots \text{ in } \mathbb{C}$. Hence, by appealing to the inductive hypothesis and then repeatedly applying the rules below we will arrive at a canonical form.

$$\begin{aligned} \text{mkvar}(\text{if } x \text{ then } D_1 \text{ else } D_0, E) &\longrightarrow \text{if } x \text{ then } \text{mkvar}(D_1, E) \text{ else } \text{mkvar}(D_0, E) \\ \text{mkvar}(\text{let } x = M \text{ in } D, E) &\longrightarrow \text{let } x = M \text{ in } \text{mkvar}(D, E) \\ \text{mkvar}(\lambda u^{\text{unit}}.D, \text{if } x \text{ then } E_1 \text{ else } E_0) &\longrightarrow \text{if } x \text{ then } \text{mkvar}(\lambda u^{\text{unit}}.D, E_1) \text{ else } \text{mkvar}(\lambda u^{\text{unit}}.D, E_0) \\ \text{mkvar}(\lambda u^{\text{unit}}.D, \text{let } x = M \text{ in } E) &\longrightarrow \text{let } x = M \text{ in } \text{mkvar}(\lambda u^{\text{unit}}.D, E) \end{aligned}$$

Finally, we handle application MN . First we apply the inductive hypothesis to both terms. Then we use the rules below to reveal the λ -abstraction inside the canonical form of M .

$$\begin{aligned} (\text{if } x \text{ then } D_1 \text{ else } D_0)E &\longrightarrow \text{if } x \text{ then } (D_1E) \text{ else } (D_2E) \\ (\text{let } x = D \text{ in } E)F &\longrightarrow \text{let } x = D \text{ in } (EF) \end{aligned}$$

Now it suffices to be able to deal with terms of the form $(\lambda x^\theta.C_1)C_2 \equiv \text{let } x = C_2 \text{ in } C_1$, and this is exactly what Lemma B.2 does.

All our transformation preserve denotations: the proofs are simple exercises in the use of Moggi's monadic approach [13] to modelling call-by-value languages (the store-free game model is an instance of the monadic framework). \square