Logical Methods in Computer Science
Vol. 13(3:7)2017, pp. 1–39
https://lmcs.episciences.org/

Submitted Oct. 03, 2016
Published Jul. 28, 2017

ALGEBRAIC AND LOGICAL DESCRIPTIONS OF GENERALIZED TREES

BRUNO COURCELLE

LaBRI, CNRS, 351 Cours de la Libération, 33405 Talence, France
e-mail address: courcell@labri.fr

ABSTRACT. *Quasi-trees* generalize trees in that the unique "path" between two nodes may be infinite and have any countable order type. They are used to define the rank-width of a countable graph in such a way that it is equal to the least upper-bound of the rank-widths of its finite induced subgraphs. *Join-trees* are the corresponding directed trees. They are useful to define the modular decomposition of a countable graph. We also consider *ordered join-trees*, that generalize rooted trees equipped with a linear order on the set of sons of each node. We define algebras with finitely many operations that generate (via infinite terms) these generalized trees. We prove that the associated regular objects (those defined by regular terms) are exactly the ones that are the unique models of monadic second-order sentences. These results use and generalize a similar result by W. Thomas for countable linear orders.

INTRODUCTION

We define and study countable generalized trees, called *quasi-trees,* such that the unique "path" between two nodes may be infinite and have any order type, in particular that of rational numbers. Our motivation comes from the notion of *rank-width*, a complexity measure of finite graphs investigated first in [21] and [22]. Rank-width is based on graph decompositions formalized with finite undirected trees of maximal degree at most 3. In order to extend it to countable graphs in such a way that *the compactness property* holds, *i.e.*, that the rank-width of a countable graph is the least upper bound of those of its finite induced subgraphs, we base decompositions on *quasi-trees*[1] [11]. Quasi-trees arise as least upper bounds of increasing sequences of finite trees, $H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq \dots$, where H_{i+1} is obtained from H_i by the addition of a new node, either linked to an existing one by a new

2012 ACM CCS: [**Theory of computation**]: Logic; Decision problems ; [**Mathematics of computing**]: Trees; Model theory;
2010 Mathematics Subject Classification: 05C05, 68R10.
Key words and phrases: Rank-width, quasi-tree, join-tree, ordered tree, algebra, regular term, monadic second-order logic.
This work has been supported by the French National Research Agency (ANR) within the IdEx Bordeaux program "Investments for the future", CPU, ANR-10-IDEX-03-02.

[1] Compactness does not hold if one uses trees. For a comparison, the natural extension of tree-width to countable graphs has the compactness property [19] without needing quasi-trees.

edge or inserted on an existing edge. If one inserts infinitely many nodes on an edge of some H_i, then, the least upper bound is not a tree but a quasi-tree.

Join-trees can be seen as directed quasi-trees. A join-tree is a partial order $(N, \leq)$ such that every two elements have a least upper bound (called their *join*) and each set $\{y \mid y \geq x\}$ is linearly ordered. The modular decomposition of a countable graph is based on an *ordered join-tree* [13].

Our objective is to obtain finitary descriptions (usable in algorithms) of generalized trees that are of the following three types: join-trees, ordered join-trees and quasi-trees. For this purpose, we will define, for each type of generalized tree, an algebra based on finitely many operations such that the finite and infinite terms over these operations define all generalized trees of this type. The *regular generalized trees* are those defined by *regular terms*, *i.e.* that have finitely many different subterms, equivalently, that are the unique solutions of certain finite equation systems. We will prove that a generalized tree is regular if and only if it is *monadic second-order definable*, *i.e.*, is the unique finite or countable model (up to isomorphism) of a monadic second-order sentence.

As a special case, we have linear orders. A countable linear order whose elements are labelled by letters from a finite alphabet is called an *arrangement*. The linear order of a *regular arrangement* is the left-right order of the leaves of the tree representing a *regular term*, equivalently, the lexicographic ordering of the words of a regular language. Regular arrangements were first defined and studied in [8] and [18], and their monadic second-order definability was proved in [23]. We will use the latter result for proving its extension to our generalized trees.

The study of regular linear orders has been continued by Bloom and Ésik in [1, 2]. They have also studied the *algebraic linear orders*, defined similarly from *algebraic trees* (infinite terms that are solutions of certain first-order equation systems, cf. [9]) or equivalently, as lexicographic orderings of the words of deterministic context-free languages [3, 4].

In Sections 1 and 2, we review definitions and basic results. In Section 3, we first study binary join-trees and then, we extend the definitions and results concerning them to all join-trees. In Section 4, we study ordered join-trees, and, in Section 5, we study quasi-trees. An introductory article on these results is [12].

1. Orders, trees and terms

All sets, trees and logical structures are finite or countably infinite. We denote by $X \uplus Y$ the union of sets X and Y if they are disjoint. Isomorphism of ordered sets, trees and other logical structures is denoted by $\simeq$. The restriction of a relation R or a function f defined on a set V to a subset W of V is denoted by $R \upharpoonright W$ or $f \upharpoonright W$ respectively.

For partial orders $\leq, \preceq, \sqsubseteq, \ldots$ we denote respectively by $<, \prec, \sqsubset, \ldots$ the corresponding strict orders and $X < Y$ means that $x < y$ for every $x \in X$ and $y \in Y$.

Let $(V, \leq)$ be a partial order. The least upper bound of x and y is denoted by $x \sqcup y$ if it exists and is called their *join*. The notation $x \bot y$ means that x and y are incomparable. A *line*[2] is a subset Y of V that is linearly ordered and satisfies the following *convexity property*: if $x, z \in Y$, $y \in V$ and $x \leq y \leq z$, then $y \in Y$. Particular notations for convex sets (not necessarily linearly ordered) are $[x, y]$ denoting $\{z \mid x \leq z \leq y\}$, $]x, y]$ denoting

[2] In [11] we call *line* a linearly ordered subset, without imposing the convexity property.

$\{z \mid x < z \leq y\}$, $]-\infty, x]$ denoting $\{y \mid y \leq x\}$ (even if V is finite), $]x, +\infty[$ denoting $\{y \mid x < y\}$ etc. If $X \subseteq V$, then $\downarrow(X)$ is the union of the sets $]-\infty, x]$ for x in X.

The first infinite ordinal and the linear order $(\mathbb{N}, \leq)$ are denoted by ω.

Let A be a finite set that is linearly ordered by $\leq$, and A^* be the set of finite words over A; the empty word is ε. This set is linearly ordered by the lexicographic order $\leq_{lex}$ defined by $u \leq_{lex} v$ if and only if $v = uw$ or, $u = wax$ and $u = wby$ for some w, x, y in A^* and a, b in A such that $a < b$. Every finite or countable linear order is isomorphic to $(L, \leq_{lex})$ for some set $L \subseteq \{0,1\}^*$ that is *prefix-free*, which means that, if $u, uv \in L$ where $v \in \{0,1\}^*$, then $v = \varepsilon$ (Theorem 1.7 of [8]). The case where L is regular has been studied in [1, 2, 8, 18, 23].

1.1. **Trees.** A *tree* is a possibly empty, finite or countable, undirected graph that is connected and has no cycles. Hence, it has neither loops nor parallel edges (it has no two edges with same ends). The set of nodes of a tree T is N_T.

A *rooted tree* is a nonempty tree equipped with a distinguished node called its *root*. The *level* of a node x is the number of edges of the path between it and the root and $Sons(x)$ denotes the set of its sons. We define on N_T the partial order $\leq_T$ such that $x \leq_T y$ if and only if y is on the unique path between x and the root. The least upper bound of x and y, denoted by $x \sqcup_T y$, is their least common ancestor. We will specify a rooted tree T by $(N_T, \leq_T)$ and we will omit the index T when the considered tree is clear. For a node x of T, the *subtree issued from* x, denoted by T/x, is defined as $(N_{T/x}, \leq_T \restriction N_{T/x})$ where $N_{T/x} :=]-\infty, x]$.

A partial order $(N, \leq)$ is $(N_T, \leq_T)$ for some rooted tree T if and only if it has a largest element max and for each $x \in N$, the set $[x, max]$ is finite and linearly ordered. These conditions imply that any two nodes have a join.

An *ordered tree* is a rooted tree such that each set $Sons(x)$ is linearly ordered by an order $\sqsubseteq_x$.

1.2. **Finite and infinite terms.** Let F be a finite set of operations, each f in F being given with an arity $\rho(f)$. We call (F, ρ) a *signature*. The maximal arity of a symbol is denoted by $\rho(F)$. A *term over* F is finite or infinite. We denote by $T^\infty(F)$ the set of all terms over F and by $T(F)$ the set of finite ones. A typical example of an infinite term, easily describable linearly, is, with f binary and a and b nullary, the term $t_\infty := f(a, f(b, f(a, f(b, f(\dots))))))$ that is the unique solution in $T^\infty(F)$ of the equation $t = f(a, f(b, t))$.

Positions in terms are designated by Dewey words[3] over $\{1, \dots, \rho(F)\}$ considered as an alphabet. The set $Pos(t)$ of positions of a term t is ordered by $\leq_t$, the reversal of the prefix order on words. A term t can be seen as a labelled, ordered and rooted tree whose set of nodes is $Pos(t)$. We have $Pos(t_\infty) = 2^* \uplus 2^*1$, where 2^* is the set of occurrences of f, $(22)^*1$ is the set of occurrences of a and $(22)^*21$ is that of b.

There is a canonical F-algebra structure on $T^\infty(F)$, of which $T(F)$ is a subalgebra. If $\mathbb{M} = \langle M, (f_\mathbb{M})_{f \in F} \rangle$ is an F-algebra, a *value mapping* is a homomorphism $h : T^\infty(F) \to \mathbb{M}$. Its restriction to finite terms is uniquely defined. In some cases, we will use algebras with two sorts. The corresponding modifications of the definitions are straightforward, see [17] for details.

[3] For the term $t = f(a, g(b, c))$ taken as an example, ε denotes the occurrence of f and the unique occurrences of a, g, b and c are denoted respectively by the words 1,2,21 and 22.

The partial order on terms. Let F contain a special nullary symbol Ω intended to be the least term. We define on $T(F)$ a partial order $\leqslant$ by the following induction: $\Omega \leqslant t$ for any $t \in T(F)$, and $f(t_1, \ldots, t_k) \leqslant g(t'_1, \ldots, t'_{k'})$ if and only if $k = k'$, $f = g$ and $t_i \leqslant t'_i$ for $i = 1, .., k$.

For terms in $T^{\infty}(F)$, the definition (subsuming the previous one) is: $t \leqslant t'$ if and only if $Pos(t) \subseteq Pos(t')$ and every occurrence in t of a symbol in $F - \{\Omega\}$ is an occurrence in t' of the same symbol (an occurrence in t of Ω is an occurrence in t' of any symbol).

Every increasing sequence of terms has a least upper bound. More details on this order can be found in [9, 17].

If $\mathbb{M} = \langle M, (f_{\mathbb{M}})_{f \in F} \rangle$ is a partially ordered F-algebra, whose order is ω-complete (increasing sequences have least upper bounds) and whose operations are ω-continous (they preserve such least upper bounds), then, one can define the value in $\mathbb{M}$ of an infinite term as the least upper bound of the values of the finite smaller terms [9, 17]. However, this approach fails for our algebras of generalized trees, because no appropriate partial order can be defined, as proved in Section 6 of [8]. Instead of orders, this article uses category theory. This categorical setting could be used here but direct constructions of generalized trees from terms (in Definitions 3.15, 3.28 and 4.9) are simpler and better formalizable in logic.

Regular terms. A term $t \in T^{\infty}(F)$ as *regular* if there is a mapping h from $Pos(t)$ into a finite set Q and a mapping $\tau : Q \to F \times Seq(Q)$ (where $Seq(Q)$ denotes the set of finite sequences of elements of Q) such that, if u is an occurrence of a symbol f of arity k, then $\tau(h(u)) = (f, (h(u_1), \ldots, h(u_k)))$ where $(u_1, \ldots, u_k)$ is the sequence of sons of u.

Intuitively, τ is the transition function of a top-down deterministic automaton with set of states Q; $h(\varepsilon)$ is its initial (root) state and h defines its unique run. This is equivalent to requiring that t has finitely many different subterms, or is a component of a finite system of equations that has a unique solution in $T^{\infty}(F)$. (The set Q can be taken as the set of unknowns of such a system, see [9].)

The above term t_{∞} is regular with $Q := \{1, 2, 3, 4\}$, $\tau(1) = (f, (2, 3))$, $\tau(2) = (a, ())$, $\tau(3) = (f, (4, 1))$, $\tau(4) = (b, ())$.

We associate with a term t the relational structure $\lfloor t \rfloor := (Pos(t), \leq_t, (br_i)_{1 \leq i \leq \rho(F)}, (lab_f)_{f \in F})$ where $br_i(u)$ is true if and only if u is the i-th son of his father and $lab_f(u)$ is true if and only if f occurs at position u. A term t can be reconstructed in a unique way from any relational structure isomorphic to $\lfloor t \rfloor$.

A term t is regular if and only if $\lfloor t \rfloor$ is MS definable, *i.e.*, is, up to isomorphism, the unique model of a monadic second-order sentence. This result is due to Rabin [25], see Thomas [24].

1.3. **Arrangements and labelled sets.** We review a notion introduced in [8] and further studied in [18, 23]. Let X be a set. A linear order $(V, \leq)$ equipped with a labelling mapping $lab : V \to X$ is called an *arrangement over* X. It is *simple* if lab is injective. We denote by $\mathcal{A}(X)$ the set of arrangements over X. We will generalize arrangements to tree structures.

An arrangement over a finite set X considered as an alphabet can be considered as a generalized word. A linear order $(V, \leq)$ is identified with the simple arrangement $(V, \leq, Id_V)$ such that $Id_V(v) := v$ for each $v \in V$. In the sequel, Id denotes the identity function on any set.

An *isomorphism of arrangements* $i : (V, \leq, lab) \to (V', \leq', lab')$ is an order preserving bijection $i : V \to V'$ such that $lab' \circ i = lab$. Isomorphism is denoted by $\simeq$.

If $w = (V, \leq, lab) \in \mathcal{A}(X)$ and $r : X \to Y$, then, $\overline{r}(w) := (V, \leq, r \circ lab)$ is an arrangement over Y. If r maps V into Y, then $\overline{r}((V, \leq))$ is the arrangement $(V, \leq, r)$ over Y since we identify $(V, \leq)$ to the simple arrangement $(V, \leq, Id)$.

The concatenation of linear orders yields a concatenation of arrangements denoted by $\bullet$. We denote by Ω the empty arrangement and by a the one reduced to a single occurrence of $a \in X$. Clearly, $w \bullet \Omega = \Omega \bullet w = w$ for every $w \in \mathcal{A}(X)$. The infinite word $w = a^\omega$ is the arrangement over $\{a\}$ with underlying order ω; it is described by the equation $w = a \bullet w$. Similarly, the arrangement $w = a^\eta$ over $\{a\}$ with underlying linear order $(\mathbb{Q}, \leq)$ (that of rational numbers) is described by the equation $w = w \bullet (a \bullet w)$.

Let X be a set of first-order variables (they are nullary symbols) and $t \in T^\infty(\{\bullet, \Omega\} \cup X)$. Hence, $Pos(t) \subseteq \{1, 2\}^*$. The *value* of t is the arrangement $val(t) := (\mathrm{Occ}(t, X), \leq_{lex}, lab)$ where $\mathrm{Occ}(t, X)$ is the set of positions of the elements of X and $lab(u)$ is the symbol of X occurring at position u. We say that t *denotes* w if w is isomorphic to $val(t)$, and that w is the *frontier* of the syntactic tree of t [8].

For an example, $t_\bullet := \bullet(a, \bullet(b, \bullet(a, \bullet(b, \bullet(\ldots\ldots\ldots))))))$ denotes the infinite word $abab\ldots$. Its value is defined from $\mathrm{Occ}(t_\bullet, \{a, b\}) = 2^*1$, lexicographically ordered (we have $1 < 21 < 221 < \ldots$) by taking $lab(2^i1) := a$ if i is even and $lab(2^i1) := b$ if i is odd. The arrangements a^ω and a^η are denoted respectively by t_1 and t_2 that are the unique solutions in $T^\infty(\{\bullet, \Omega, a\})$ of the equations $t_1 = a \bullet t_1$ and $t_2 = t_2 \bullet (a \bullet t_2)$.

An arrangement is *regular* if it is denoted by a regular term. The term $t_\bullet$ is regular. The arrangements a^ω and a^η are regular[4].

An arrangement is regular if and only if it is a component of the *initial*[5] *solution of a regular system of equations* over F, or also, the value of a *regular expression* in the sense of [18]. We will use the result of [23] that an arrangement over a finite alphabet is regular if and only if is *monadic second-order definable*[6]. (We review monadic second-order logic in the next section). For this result, we represent an arrangement $w = (V, \leq, lab)$ over a finite set X by the relational structure $\lfloor w \rfloor := (V, \leq, (lab_a)_{a \in X})$ where $lab_a(u)$ is true if and only if $lab(u) = a$.

If r maps X to Y and $w \in \mathcal{A}(X)$ is regular, then $\overline{r}(w)$ is regular. This is clear from the definitions because the substitution of $r(x)$ for $x \in X$ in a regular term in $T^\infty(\{\bullet, \Omega\} \cup X)$ yields a regular term [9].

An X-*labelled set* is a pair $m = (V, lab)$ where $lab : V \to X$, or, equivalently, a relational structure $(V, (lab_a)_{a \in X})$ where each element of V belongs to a unique set lab_a. We denote by $set(w)$ the X-labelled set obtained by forgetting the linear order of an arrangement w over X. Up to isomorphism, an X-labelled set m is defined by the cardinalities in $\mathbb{N} \cup \{\omega\}$ of the sets lab_a, hence is a finite or countable *multiset of elements of* X, in other words, a mapping that indicates for each $a \in X$ the number, in $\mathbb{N} \cup \{\omega\}$, of its occurrences in m.

If X is finite, each X-labelled set is MS_{fin}-definable, *i.e.*, is the unique, finite or countably infinite model up to isomorphism of a sentence of *monadic second-order logic* extended with a set predicate $Fin(U)$ expressing that a set U is finite. It is also *regular*, *i.e.*, is $set(val(t))$ for

[4] The subalgebra of regular arrangements is characterized by equational axioms in [2].

[5] in the sense of category theory, see [8].

[6] The article [7] establishes that a set of arrangements is recognizable if and only if it is monadic second-order definable.

some regular term in $T^\infty(\{\bullet,\Omega\}\cup X)$. The notion of regularity is thus trivial for X-labelled sets when X is finite.

2. Monadic second-order logic and related notions.

Monadic second-order logic[7] extends first-order logic by the use of *set variables* X, Y, Z ... denoting subsets of the domain of the considered logical structure, and the atomic formulas $x \in X$ expressing membership of x in X. We call *first-order* a formula where set variables are not quantified. For example, a first-order formula can express that $X \subseteq Y$. A *sentence* is a formula without free variables.

Let $\mathcal{R}$ be a finite set of relation symbols, each symbol R being given with an arity $\rho(R)$. We call it a *relational signature*. For every set of variables $\mathcal{W}$, we denote by $MS(\mathcal{R},\mathcal{W})$ the set of MS formulas written with $\mathcal{R}$ and free variables in $\mathcal{W}$. An *$\mathcal{R}$-structure* is a tuple $S = (D_S, (R_S)_{R\in\mathcal{R}})$ where D_S is a finite or countably infinite set, called its *domain,* and each R_S is a relation on D_S of arity $\rho(R)$. A property P of $\mathcal{R}$-structures is *monadic second-order expressible* if it is equivalent to the validity, in every $\mathcal{R}$-structure S, of a monadic second-order sentence φ, which we denote by $S \models \varphi$.

For example, a graph G without parallel edges can be identified with the $\{edg\}$-structure (V_G, edg_G) where V_G is its vertex set and $edg_G(x, y)$ means that there is an edge from x to y, or between x and y if G is undirected. To take an example, 3-colorability is expressed by the MS sentence:

$$\exists X,Y\big[X\cap Y=\emptyset \wedge \neg\exists u,v(edg(u,v)\wedge[(u\in X\wedge v\in X)\vee \\ (u\in Y\wedge v\in Y)\wedge(u\notin X\cup Y\wedge v\notin X\cup Y)])\big].$$

Many properties of partial orders $(N,\leq)$ can also be expressed by MS formulas. Here are examples that will be useful in our proofs.

(a) The formula $Lin(X)$ defined as $\forall x,y.[(x\in X\wedge y\in X)\Longrightarrow(x\leq y\vee y\leq x)]$ expresses that a subset X of N, partially ordered by $\leq$, is linearly ordered.

(b) The formula $Lin(X)\wedge\exists a,b.[\min(X,a)\wedge\max(X,b)\wedge\theta(X,a,b)]$ expresses that X is linearly ordered and finite, where $\min(X,a)$ and $\max(X,b)$ are first-order formulas expressing respectively that X has a least element a and a largest one b, and $\theta(X,a,b)$ is an MS formula expressing, (1) that each element x of X except b has a successor c in X (i.e., c is the least element of $\{y\in X \mid y>x\}$), and (2), that $(a,b)\in Suc^*$, where Suc is the above defined successor relation (depending on X) and Suc^* is its reflexive and transitive closure.

Property (b) is expressed by the MS formula:

$$\forall U\,[U\subseteq X\wedge a\in U\wedge\forall x,y((x\in U\wedge(x,y)\in Suc)\Longrightarrow y\in U)\Longrightarrow b\in U]\,.$$

First-order formulas expressing $U\subseteq X$, $(x,y)\in Suc$ and Property (a) are easy to build. Without a linear order, the finiteness of a set X is not MS expressible. It is thus useful, in some cases, to enrich MS logic with a *finiteness predicate* $Fin(X)$ expressing that the set X is finite. We denote by MS_{fin} the corresponding extension of MS logic.

If S is a relational structure $(N,\leq,(br_i)_{1\leq i\leq\rho(F)},(lab_f)_{f\in F})$ isomorphic to the structure $\lfloor t\rfloor$ representing a term $t\in T^\infty(F)$, then a linear order $\sqsubseteq$ on N is definable by a first-order

[7] *MS* will abreviate *monadic second-order* in the sequel.

formula as follows:

$$x \sqsubseteq y :\Longleftrightarrow x \leq y \vee (x \perp y \text{ and } x \text{ is below the } i\text{-th son of } x \sqcup y \text{ and } y \text{ is below the } j\text{-th son of } x \sqcup y \text{ where } i < j).$$

The definability of linear orders by MS formulas is studied in [6].

Monadic second-order transductions are transformations of logical structures specified by MS or MS_{fin} formulas. We will use them in the proofs of Theorems 3.21, 3.30, 4.11, 5.8 and 5.11. For these proofs, we will only need very simple transductions, said to be *noncopying and parameterless* in [14]. We call them *MS-transductions.*

Let $\mathcal{R}$ and $\mathcal{R}'$ be two relational signatures. A *definition scheme* of type $\mathcal{R} \to \mathcal{R}'$ is a tuple of formulas of the form $\mathcal{D} = \langle \chi, \delta, (\theta_R)_{R \in \mathcal{R}'} \rangle$ such that $\chi \in MS(\mathcal{R})$, $\delta \in MS(\mathcal{R}, \{x\})$ and $\theta_R \in MS(\mathcal{R}, \{x_1, \dots, x_{\rho(R)}\})$ for each R in $\mathcal{R}'$. We define $\widehat{\mathcal{D}}(S) := S' = (D_{S'}, (R_{S'})_{R \in \mathcal{R}'})$ as follows:

– S' is defined if and only if $S \models \chi$,
– $D_{S'}$ is the set of elements d of D_S such that $S \models \delta(d)$,
– $R_{S'}$ is the set of tuples $(d_1, \dots, d_{\rho(R)})$ of elements of D_S such that $S \models \theta_R(d_1, \dots, d_{\rho(R)})$.

Our main tool is the following (well-known) result:

Theorem 2.1. *Let $\mathcal{D}$ be a definition scheme as above and $\varphi \in MS_{fin}(\mathcal{R}', \mathcal{W})$. There exists a formula $\varphi^{\mathcal{D}} \in MS_{fin}(\mathcal{R}, \mathcal{W})$ such that, for every $\mathcal{R}$-structure S, for every $\mathcal{W}$-assignment ν in D_S, we have $(S, \nu) \models \varphi^{\mathcal{D}}$ if and only if*

(1) *$S \models \chi$ (so that $\widehat{\mathcal{D}}(S) = S'$ is well-defined),*
(2) *ν is an $\mathcal{W}$-assignment in $D_{S'}$ (*i.e.*, $\nu(x) \in D_{S'}$ and $\nu(X) \subseteq D_{S'}$ for $x, X \in \mathcal{W}$) and*
(3) *$(S', \nu) \models \varphi$.*

Proof. The proof is given in [14] (*Backwards Translation Theorem*, Theorem 7.10) for finite structures, hence the finiteness predicate $Fin(X)$ is of no use. However, it works as well for infinite structures and formulas written with the predicate $Fin(X)$ that translates back to itself (under the assumption that $\nu(X) \subseteq D_{S'}$).

The formula $\varphi^{\mathcal{D}}$ is the conjunction of χ, a formula expressing Property (2) and a formula φ' obtained from φ by replacing each atomic formula $R(x_1, \dots, x_r)$ by $\theta_R(x_1, \dots, x_{\rho(R)})$, that is, by its definition given by $\mathcal{D}$. If φ is a sentence, then $\mathcal{W} = \emptyset$ and Property (2) is trivially true. □

It follows that, if the monadic theory of a class of structures $\mathcal{S}$ is decidable (which means that one can decide whether a given sentence is true in all structures of $\mathcal{S}$) and $\mathcal{S}' = \widehat{\mathcal{D}}(\mathcal{S})$ for some definition scheme $\mathcal{D}$, then the monadic theory of $\mathcal{S}'$ is decidable, because $S' \models \varphi$ for all S' in $\mathcal{S}'$ if and only if $S \models \chi \Longrightarrow \varphi^{\mathcal{D}}$ for all S in $\mathcal{S}$.

3. Join-trees

Join-trees have been used in [13] for defining the modular decomposition of countable graphs.

3.1. **Join-trees, join-forests and their structurings.** Join-trees are defined as particular partial-orders. Finite nonempty join-trees correspond to finite rooted trees.

Definition 3.1 (Join-tree)**.**

(a) A *join-tree*[8] is a pair $J = (N, \leq)$ such that:
   (1) N is a possibly empty, finite or countable set called the set of *nodes*,
   (2) $\leq$ is a partial order on N such that, for every node x, the set $[x, +\infty[$ (the set of nodes $y \geq x$) is linearly ordered,
   (3) every two nodes x and y have a join $x \sqcup y$.

   A minimal node is a *leaf*. If N has a largest element, we call it the *root* of J. The set of strict upper bounds of a nonempty set $X \subseteq N$ is a line L; if L has a smallest element, we denote it by $\widehat{X}$ and we say that $\widehat{X}$ is *the top* of X. Note that $\widehat{X} \notin X$.

(b) A *join-forest* is a pair $J = (N, \leq)$ that satisfies Conditions (1), (2) and the following weakening of (3):
   (3') if two nodes have an upper bound, they have a join.

   The relation that two nodes have a join is an equivalence. Let N_s for $s \in S$ be its equivalence classes and $J_s := (N_s, \leq\restriction N_s)$, more simply denoted by $(N_s, \leq)$ by leaving implicit the restriction to N_s. Then each J_s is a join-tree, and J is the union of these pairwise disjoint join-trees, called its *components.*

(c) A join-forest $J = (N, \leq)$ is *included in* a join-forest $J' = (N', \leq')$, denoted by $J \subseteq J'$, if $N \subseteq N'$, $\leq$ is $\leq'\restriction N$ and $\sqcup$ is $\sqcup' \restriction N$; if J and J' are join-trees, we say also that J is a *subjoin-tree of* J'.

Definition 3.2 (Direction and degree)**.** Let $J = (N, \leq)$ be a join-forest, and x be one of its nodes. Let $\sim$be the equivalence relation on $]-\infty, x[$ such that $z \sim y$ if and only if $z \sqcup y < x$. Each equivalence class C is called a*direction of* J *relative to* x, and we have $\widehat{C} = x$. The set of directions relative to x is denoted by $Dir(x)$ and the *degree* of x is the number of its directions. The leaves are the nodes of degree 0. A join-tree is *binary* if its nodes have degree at most 2. We call it a *BJ-tree* for short.

For concatenating vertically two join-trees, we need that every join-tree has a distinguished "branch", a line, that we call an axis. As we want to construct join-trees with operations including concatenation, all subtrees must be of the same type, hence must have axes. It follows that a join-tree will be partionned into lines, one of them being its axis, the others being the axes of its subtrees. We call such a partition a structuring.

Definition 3.3 (Structured join-trees and join-forests)**.**

(a) Let $J = (N, \leq)$ be a join-tree. A *structuring* of J is a set $\mathcal{U}$ of nonempty lines forming a partition of N that satisfies some conditions, stated with the following notation : if $x \in N$, then $U(x)$ denotes the line of $\mathcal{U}$ containing x, $U_{-}(x) := U(x) \cap]-\infty, x[$ and $U_{+}(x) := U(x) \cap [x, +\infty[$. (The set $[x, +\infty[$ has no top but it can have a greatest element). The conditions are :
   (1) exactly one line A of $\mathcal{U}$ is *upwards closed* (*i.e.*, $[x, +\infty[\subseteq A$ if $x \in A$), hence, has no strict upper bound and no top; we call it the *axis*; each other line U has a top $\widehat{U}$,
   (2) for each x in N, the sequence $y_0, y_1, y_2, \ldots$ such that $y_0 = x$, $y_{i+1} = \widehat{U(y_i)}$ is finite; its last element is $y_k \in A$ (y_{k+1} is undefined) and we call k the *depth* of x.

[8] The article [20] defines a *tree* as a partial order of any cardinality that satisfies Condition (2). A join-forest is a tree in that sense.

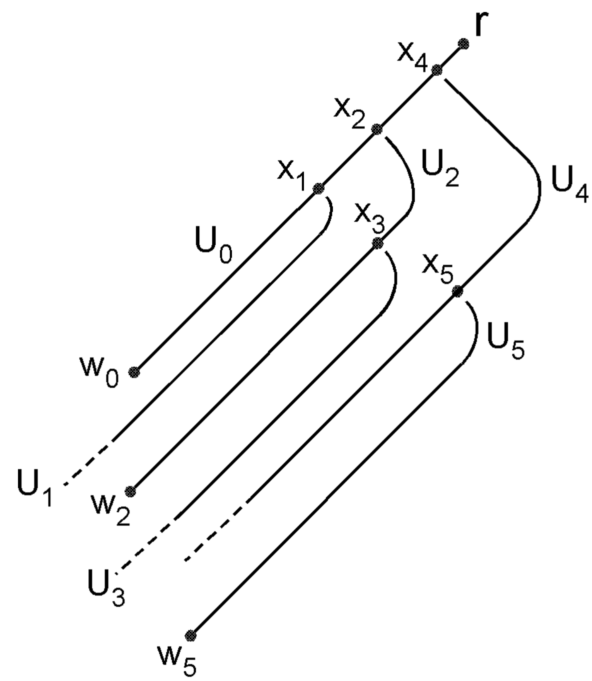

Figure 1: A structured binary join-tree.

The nodes on the axis are those at depth 0. The lines $[y_i, y_{i+1}[$ for $i \in [0, k-1]$ and $[y_k, +\infty[$ are convex subsets of pairwise distinct lines of $\mathcal{U}$. We have

$$[x, +\infty[= [y_0, y_1[\uplus [y_1, y_2[\uplus \cdots \uplus [y_k, +\infty[,$$

where $[y_i, y_{i+1}[= U_+(y_i)$ for each $i < k$, $[y_k, +\infty[= U_+(y_k) \subseteq A$ and the depth of y_i is $k - i$.

We call such a triple $(N, \leq, \mathcal{U})$ a *structured join-tree*, an *SJ-tree* for short. Every linear order is an SJ-tree whose elements are all of depth 0.

Remark. If $x < A$ for some x, then A has a smallest element, which is the node y_k of Condition 2) (because if $z \in A$ is smaller than y_k, then $x < z$, which contradicts the observation that $[y_{k-1}, y_k[\subseteq U(y_{k-1})$ and $U(y_{k-1}) \cap A = \emptyset$).

(b) Let $J = (N, \leq)$ be a join-forest whose components are J_s, $s \in S$. A *structuring* of J is a set $\mathcal{U}$ of nonempty lines forming a partition of N such that, if $\mathcal{U}_s$ is the set of lines of $\mathcal{U}$ included in N_s (every line of $\mathcal{U}$ is included in some N_s), then each triple $(N_s, \leq, \mathcal{U}_s)$ is a structuring of J_s.

Example 3.4. Figure 1 shows a structuring $\{U_0, \ldots, U_5\}$ of a binary join-tree. The axis is U_0. The directions relative to x_2 are $U_{0-}(x_2) \cup U_1$ and $U_2 \cup U_3$. The maximal depth of a node is 2.

Proposition 3.5. *Every join-tree and, more generally, every join-forest has a structuring.*

Proof. Let $J = (N, \leq)$ be a join-tree. Let us choose an enumeration of N and a maximal line B_0 ; it is upwards closed. For each $i > 0$, we choose a maximal line B_i containing the first node not in $B_{i-1}\cup\cdots\cup B_0$. We define $U_0 := B_0$ and, for $i > 0$, $U_i := B_i - (U_{i-1} \uplus \cdots \uplus U_0) = B_i - (B_{i-1}\cup\cdots\cup B_0)$. We define $\mathcal{U}$ as the set of lines U_i. It is a structuring of J. The axis is U_0. If J is a join-forest, it has a structuring that is the union of structurings of its components. □

Remark. Since each line B_i is maximal, if U_i has a smallest element, this element is a node of degree 0 in J.

In view of our use of monadic second-order logic, we give a description of SJ-trees by relational structures.

Definition 3.6 (SJ-trees as relational structures)**.** If $J = (N, \leq, \mathcal{U})$ is an SJ-tree, we define $S(J)$ as the relational structure $(N, \leq, N_0, N_1)$ such that N_0 is the set of nodes at even depth and $N_1 := N - N_0$ is the set of those at odd depth. (N_0 and N_1 are sets but we will also consider them as unary relations).

Proposition 3.7.

(1) *There is an MS formula $\varphi(N_0, N_1)$ expressing that a relational structure $(N, \leq, N_0, N_1)$ is $S(J)$ for some SJ-tree $J = (N, \leq, \mathcal{U})$.*

(2) *There exist MS formulas $\theta_{Ax}(X, N_0, N_1)$ and $\theta(u, U, N_0, N_1)$ expressing, respectively, in a structure $(N, \leq, N_0, N_1) = S((N, \leq, \mathcal{U}))$, that X is the axis and that $U \in \mathcal{U} \wedge u = \widehat{U}$.*

Proof. Let $J = (N, \leq)$ be a join-tree and $X \subseteq N$. We say that X is *laminar* if, for all $x, y, z \in X$, if $[x, z] \cup [y, z] \subseteq X$ (where $x < z$ and $y < z$), then $[x, z] \subseteq [y, z]$ or $[y, z] \subseteq [x, z]$ (the intervals $[x, z]$ and $[y, z]$ are relative to J). This condition implies that the lines of J that are included in X and are maximal with this condition form a partition of X whose parts will be called its *components*.

It is clear from the definitions that, if $J = (N, \leq, \mathcal{U})$ is an SJ-tree and $S(J) = (N, \leq, N_0, N_1)$, then the sets N_0 and N_1 are laminar, $\mathcal{U}$ is the set of their components and the axis A is a component of N_0.

(1) That a partial order $(N, \leq)$ is a join-tree is first-order expressible. The formula $\varphi(N_0, N_1)$ will include this condition.

Let $J = (N, \leq)$ be a join-tree, N the union of two disjoint laminar sets N_0 and N_1, and $\mathcal{U}$ the set of their components. Then, $J = (N, \leq, \mathcal{U})$ is an SJ-tree such that $S(J) = (N, \leq, N_0, N_1)$ if and only if:

(i) every component of N_1 has a top in N_0 and every component of N_0 except one has a top in N_1,

(ii) for each U in $\mathcal{U}$, the sequence $U_0, U_1, \ldots$ of lines of $\mathcal{U}$ such that $U_0 = U$, $\widehat{U_0} \in U_1, \ldots, \widehat{U_i} \in U_{i+1}$ terminates at some U_k that has no top, hence is included in N_0.

These conditions are necessary. As they rephrase Definition 3.3, they are also sufficient. The integer k in Condition (ii) is the common depth of all nodes in U.

That a set X is laminar is first-order expressible, and one can build an MS formula $\psi(U, X)$ expressing that U is a component of X assumed to be laminar. This formula can be used to express that N is the union of two disjoint laminar sets N_0 and N_1 that satisfy Conditions (i) and (ii). For expressing Condition (ii), we define, for each U in $\mathcal{U}$, a set of nodes W as follows: it is the least set such that $\widehat{U} \in W$, and, for each $w \in W$, the top of $U(w)$ belongs to W if it is defined (where $U(w)$ is the unique set in $\mathcal{U}$ that contains w). The set W is linearly ordered (it consists of $\widehat{U_0} < \cdots < \widehat{U_i} \ldots$) and Condition (ii) says that it must be finite. To write the formula, we use the observation made in Section 2 that the finiteness of a linearly ordered set is MS expressible.

(2) The construction of φ actually uses the MS formulas $\theta_{Ax}(X, N_0, N_1)$ and $\theta(u, U, N_0, N_1)$. □

3.2. **Description schemes of structured binary join-trees.** In order to introduce technicalities step by step, we first consider binary join-trees. They are actually sufficient for defining the rank-width of a countable graph. See Section 5.

Definition 3.8 (Structured binary join-trees)**.** Let $J = (N, \leq)$ be a binary join-tree, a *BJ-tree* in short. A *structuring* of J is a set $\mathcal{U}$ of lines satisfying the conditions of Definition 3.3 and, furthermore:

(i) if the axis A has a smallest element, then its degree is 0 or 1,

(ii) each $x \in N$ is the top of at most one set $U \in \mathcal{U}$, denoted by U^x, and $U^x := \emptyset$ if x is the top of no $U \in \mathcal{U}$.

We call $(N, \leq, \mathcal{U})$ a *structured binary join-tree*, an *SBJ-tree* in short.

Proposition 3.9.

(1) *Every BJ-tree J has a structuring.*

(2) *The class of stuctures $S(J)$ for SBJ-trees J is monadic second-order definable.*

Proof.

(1) We use the construction of Proposition 3.5 for $J = (N, \leq)$. The remark following it implies that, if the axis $A = U_0$ has a smallest element, this element has degree 0. It implies also that, if $\widehat{U_i} = x$, then x cannot have degree 0 in the BJ-tree J_{i-1} induced by $U_{i-1} \uplus \cdots \uplus U_0$ because each line B_i is chosen maximal; furthermore, it cannot have degree 2 or more in J_{i-1} because J is binary. Hence it has degree 1 in J_{i-1}. It follows that x is the top of no line U_j for $j < i$. Hence (ii) holds and the construction yields an SBJ-tree $(N, \leq, \mathcal{U})$.

(2) The formula φ of Proposition 3.7 can be modified so as to express that $(N, \leq, N_0, N_1)$ is $S(J)$ for some SBJ-tree J. □

Definition 3.10 (Description schemes for SBJ-trees)**.**

(a) A *description scheme for an SBJ-tree*, in short an *SBJ-scheme,* is a triple $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$ such that Q is a set called the set of *states*, $w_{Ax} \in \mathcal{A}(Q)$ (is an arrangement over Q) and $w_q \in \mathcal{A}(Q)$ for each q. It is *regular* if Q is finite and the arrangements w_{Ax} and w_q are regular.

(b) We recall that a linear order $(V, \leq)$ is identified with the arrangement $(V, \leq, Id)$. If $W \subseteq V$ and $r : V \to Q$, then $\overline{r}((W, \leq))$ is the arrangement $(W, \leq \restriction W, r \restriction W) \in \mathcal{A}(Q)$ that we will denote more simply by $(W, \leq, r)$ leaving implicit the restrictions of $\leq$ and r to W.

An SBJ-scheme Δ *describes* an SBJ-tree $J = (N, \leq, \mathcal{U})$ whose axis is A if there exists a mapping $r : N \to Q$ that we call a *run*, such that:

$$\overline{r}((A, \leq)) \simeq w_{Ax} \text{ and } \overline{r}((U^x, \leq)) \simeq w_{r(x)} \text{ for every } x \in N.$$

We will also say that Δ *describes* the BJ-tree $fgs(J) := (N, \leq)$, where *fgs* makes an SBJ-tree into a BJ-tree by *forgetting its structuring.* The mapping r need not be surjective, this means that some elements of Q and the corresponding arrangements may be useless, and thus can be removed from Δ.

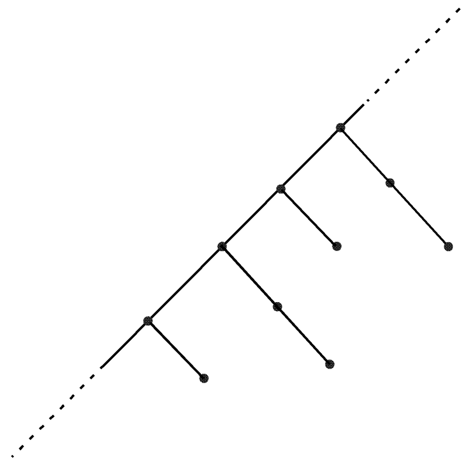

Figure 2: A binary join-tree.

For an example, let $\Delta = (Q, w_{Ax}, (w_q)_{q\in Q})$ be the SBJ-scheme such that $Q = \{a, b, c\}$, $w_{Ax} := (\mathbb{Z}, \leq, \ell)$ where $\ell(i) = a$ if i is even and $\ell(i) = b$ if i is odd, $w_a := \{c\}, w_b := cc$ (two nodes labelled by c) and $w_c = \Omega$. It describes the BJ-tree of Figure 2.

Proposition 3.11.
(1) *Every SBJ–tree is descibed by some SBJ-scheme.*
(2) *Every SBJ-scheme Δ describes a unique SBJ-tree, where unicity is up to isomorphism.*

Proof.
(1) Each SBJ-tree $J = (N, \leq, \mathcal{U})$ has a *standard* description scheme

$$\Delta(J) := (N, (A, \leq), ((U^x, \leq))_{x\in N}).$$

The run is the identity mapping $r : N \to N$ showing that $\Delta(J)$ describes J.

(2) We will denote by $\mathit{Unf}(\Delta)$ the SBJ-tree described by Δ and call it called the *unfolding of* Δ (see the remark following the proof about terminology). Let $\Delta = (Q, w_{Ax}, (w_q)_{q\in Q})$ be an SBJ-scheme, defined with arrangements $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$ and $w_q = (V_q, \preceq, lab_q)$ such that, without loss of generality, the sets V_{Ax} and V_q are pairwise disjoint and the same symbol $\preceq$ denotes their orders. We construct $(N, \leq, \mathcal{U}) = \mathit{Unf}(\Delta)$ as follows.
   (a) N is the set of finite nonempty sequences $(v_0, v_1, \ldots, v_k)$ such that $v_0 \in V_{Ax}, v_i \in V_{q_i}$ for $1 \leq i \leq k$, where $q_1 = lab_{Ax}(v_0)$, $q_2 = lab_{q_1}(v_1)$, $\ldots$, $q_k = lab_{q_{k-1}}(v_{k-1})$.
   (b) $(v_0, v_1, \ldots, v_k) \leq (v'_0, v'_1, \ldots, v'_j)$ if and only if $k \geq j$, $(v_0, v_1, \ldots, v_{j-1}) = (v'_0, v'_1, \ldots, v'_{j-1})$ and $v_j \preceq v'_j$.
   (c) The axis A is the set of one-element sequences (v) for $v \in V_{Ax}$; for $x = (v_0, v_1, \ldots, v_k)$, we define $U(x)$ as the set of sequences $(v_0, v_1, \ldots, v_{k-1}, v)$ such that $v \in V_{q_k}$, hence, we have $\widehat{U(x)} = (v_0, v_1, \ldots, v_{k-1})$.

   Note that $(v_0, \ldots, v_k) < (v_0, \ldots, v_j)$ if $j < k$ and that $(v_0, \ldots, v_{k-1}, v_k) \leq (v_0, \ldots, v_{k-1}, v)$ if and only if $v_k \preceq v$. We claim that Δ describes $(N, \leq, \mathcal{U})$. For proving that, we define a run $r : N \to Q$ as follows:
   – if $x \in A$, then $x = (v)$ for some $v \in V_{Ax}$ and $r(x) := lab_{Ax}(v)$;
   – if $x \in N$ has depth $k \geq 1$, then $x = (v_0, v_1, \ldots, v_k)$ for some $v_0, v_1, \ldots, v_k$ as in (a) and $r(x) := lab_{q_k}(v_k)$.

It follows that $\overline{r}((A,\leq)) \simeq w_{Ax}$ and that, for $x = (v_0, v_1, \ldots, v_k)$ (of depth k), we have $\overline{r}((U^x,\leq)) \simeq w_{q_k} = w_{r(x)}$, which proves the claim.

We now prove unicity. Assume that Δ describes $J = (N, \leq, \mathcal{U})$ with axis A and also $J' = (N', \leq', \mathcal{U}')$ with axis A', by means of runs $r : N \to Q$ and $r' : N' \to Q$. We construct an isomorphism $h : J \to J'$ as the common extension of bijections $h_k : N_k \to N'_k$, where N_k (resp. N'_k) is the set of nodes of J (resp. of J') of depth at most k, and such that they map $\leq$ to $\leq'$, and the lines of $\mathcal{U}$ to those of $\mathcal{U}'$ of same depth, and finally, $r' \circ h_k = r \upharpoonright N_k$.

– *Case* $k = 0$. We have:

$$\overline{r}((A,\leq)) = (A, \leq, r) \simeq w_{Ax} \simeq \overline{r'}((A',\leq)) = (A', \leq', r')$$

which gives the order preserving bijection $h_0 : N_0 = A \to N'_0 = A'$ such that $r' \circ h_0 = r \upharpoonright N_0$.

– *Case* $k > 0$. We assume inductively that h_{k-1} has been constructed.
Let $U \in \mathcal{U}$ be such that $x = \widehat{U}$ has depth $k-1$; hence, $U \cap N_{k-1} = \emptyset$. Then $(U, \leq, r) \simeq w_{r(x)}$. Let $x' = h_{k-1}(x)$; we have $r'(x') = r(x)$. Hence there is $U' \in \mathcal{U}'$ such that $x' = \widehat{U'}$, $U' \cap N'_{k-1} = \emptyset$ and$(U', \leq', r') \simeq w_{r'(x')} = w_{r(x)}$. Hence, there is an order preserving bijection $h_U : U \to U'$ such that $r' \circ h_U = r \upharpoonright U$.
We define h_k as the common extension of the injective mappings h_{k-1} and h_U such that $U \in \mathcal{U}$ and the depth of $\widehat{U}$ is $k-1$. These mappings have pairwise disjoint domains whose union is N_k.

The extension to N of all these mappings h_k is the desired isomorphism h. □

Remarks.

(1) We call *unfolding* the transformation of Δ into $Unf(\Delta)$ because it generalizes the unfolding of a directed graph G into a finite or countable rooted tree. The unfolding is done from a particular vertex s of G, and the nodes of the tree are the sequences of the form $(x_0, \ldots, x_k)$ such that $s = x_0$ and there is a directed edge in G from x_i to x_{i+1}, for each $i < k$. If Δ is such that the arrangements w_{Ax} and w_q are reduced to a single element, the corresponding directed graph has all its vertices of outdegree one and the tree resulting from the unfolding consists of one infinite path: the SBJ-tree $Unf(\Delta)$ is the order type ω^- of negative integers and the sets in $\mathcal{U}$ are singletons.

(2) An SBJ-scheme Δ describing an SBJ-tree J can be seen as a *quotient of* $\Delta(J)$. We define quotients in terms of surjective mappings.

Let $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$ and $s : Q \to Q'$ be surjective such that, if $s(q) = s(q')$, then $\overline{s}(w_q) \simeq \overline{s}(w_{q'})$. We define $\Delta' := (Q', \overline{s}(w_{Ax}), (w'_p)_{p \in Q'})$ where $w'_{s(q)} \simeq \overline{s}(w_q)$ for each q in Q. If Δ describes J via a run $r : N \to Q$, then Δ' describes J via $s \circ r : N \to Q'$. We say that Δ' is *the quotient of* Δ by the equivalence $\approx$ on Q such that $q \approx q'$ if and only if $s(q) = s(q')$, and we denote it by $\Delta/\approx$. If Δ is regular, then $\Delta/\approx$ is regular and its set of sates is smaller than that of Δ unless s is a bijection.

Let us now start from an SBJ-tree $J = (N, \leq, \mathcal{U})$ with axis A. For $x \in N$, let J_x be the SBJ-tree $(N_x, \leq_x, \mathcal{U}_x)$ such that $N_x := \downarrow(U^x)$, $\leq_x$ is the restriction of $\leq$ to N_x, and $\mathcal{U}_x := \{U^x\} \cup \{U^y \mid y \in N_x\} - \{\emptyset\}$. Its axis is U^x. For the example of Figure 1, we have $\mathcal{U}_{x_2} = \{U_2, U_3\}$ and U_2 is the axis.

From J, we define as follows a canonical SBJ-scheme $\Delta(J)/\approx$ based on the equivalence $\approx$ on N such that $x \approx x'$ if and only if $J_x \simeq J_{x'}$. Let s be the surjective mapping :

$N \to N' := N/\approx$. If $J_x \simeq J_y$ by an isomorphism $h : N_x \to N_y$, then $(U^x, \leq) \simeq (U^y, \leq)$ by $h \upharpoonright U^x : U^x \to U^y$ and furthermore, if $w \in N_x$, then $J_w \simeq J_{h(w)}$ by $h \upharpoonright N_w : N_w \to N_{h(w)}$. It follows that $\overline{s}((U^x, \leq)) \simeq \overline{s}((U^y, \leq))$, hence, the quotient SBJ-scheme $\Delta(J)/\approx := (N', \overline{s}((A, \leq)), (w'_p)_{p \in N'})$ such that $w'_p \simeq \overline{s}((U^x, \leq))$ if $s(x) = p$ is well-defined and describes J.

Let $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$ describe J via a surjective run $r : N \to Q$ and consider the equivalence relation on Q such that $q \equiv q'$ if and only if there exist x, y such that $r(x) = q$, $r(y) = q'$ and $J_x \simeq J_y$. It is well-defined because if $r(x) = r(y) = q$, then $\overline{r}((U^x, \leq)) \simeq \overline{r}((U^y, \leq)) \simeq w_q$, and furthermore $J_x \simeq J_y$: one constructs an isomorphism h that extends the one between $\overline{r}((U^x, \leq))$ and $\overline{r}((U^y, \leq))$ (this construction uses an induction on the depth of u in J_x for defining $h(u)$). Hence, $\Delta/\equiv$ is well-defined and describe J via the run r' such that $r'(x)$ is the equivalence class of $r(x)$ with respect to $\equiv$. It follows that $\Delta/\equiv$ is isomorphic to $\Delta(J)/\approx$. If J is regular, then $\Delta(J)/\approx$ is the unique regular SBJ-scheme describing J that has a minimum number of states. As usual, unicity is up to isomorphism. This construction is similar to that of the minimal deterministic automaton of a regular language, defined from its quotients (see *e.g.,* [26], chapter I.3.3).

Proposition 3.12. *A BJ-tree is monadic second-order definable if it is described by a regular SBJ-scheme.*

Proof. That $J = (N, \leq)$ is a BJ-tree is first-order expressible. Assume that $J = \mathit{fgs}(J')$ where $J' = (N, \leq, \mathcal{U}) \simeq \mathit{Unf}(\Delta)$ for some regular SBJ-scheme $\Delta = (Q, w_{Ax}, (w_q)_{q \in Q})$ such that $Q = \{1, \ldots, m\}$. Let r be the corresponding mapping: $N \to Q$ (cf. Definition 3.10(b)). For each $q \in Q$, let ψ_q be an MS sentence that characterizes w_q, up to isomorphism, by the main result of [23]. Similarly, ψ_{Ax} characterizes w_{Ax}. We claim that a relational structure $(X, \leq)$ is isomorphic to J if and only if there exist subsets $N_0, N_1, M_1, \ldots, M_m$ of X such that:

(i) $(X, \leq)$ is a BJ-tree and $(X, \leq, N_0, N_1) = S(J'')$ for some SBJ-tree $J'' = (X, \leq, \mathcal{U}')$,
(ii) $(M_1, \ldots, M_m)$ is a partition of X; we let r' maps each $x \in X$ to the unique $q \in Q$ such that $x \in M_q$,
(iii) for every $q \in Q$ and $x \in M_q$, the arrangement $\overline{r'}((U^x, \leq))$ over Q (where $U^x \in \mathcal{U}'$) is isomorphic to w_q,
(iv) the arrangement $\overline{r'}((A', \leq))$ over Q where A' is the axis of J'' is isomorphic to w_{Ax}.

Conditions (ii)-(iv) express that Δ describes J'', hence that J'' is isomorphic to J', and so that $(X, \leq) \simeq \mathit{fgs}(J') = J$.

By Proposition 3.9, Condition (i) is expressed by an MS formula $\varphi(N_0, N_1)$, and the property $U \in \mathcal{U} \wedge x = \widehat{U}$ is expressed in terms of N_0, N_1 by an MS formula $\theta(x, U, N_0, N_1)$. Conditions (iii) and (iv) are expressed by means of the MS sentences ψ_{Ax} and ψ_q suitably adapted to take $N_0, N_1, M_1, \ldots, M_m$ as arguments. Hence, J is (up to isomorphism) the unique model of an MS sentence of the form:

$$\exists N_0, N_1. \left[\varphi(N_0, N_1) \wedge \exists M_1, \ldots, M_m. \varphi'(N_0, N_1, M_1, \ldots, M_m)\right]$$

where φ' expresses conditions (ii)-(iv). □

Theorem 3.21 will establish a converse.

3.3. **The algebra of binary join-trees.** We define three operations on structured binary join-trees (SBJ-trees in short). The finite and infinite terms over these operations will define all SBJ-trees.

Definition 3.13 (Operations on SBJ-trees)**.**

– *Concatenation along axes.*
Let $J = (N, \leq, \mathcal{U})$ and $J' = (N', \leq', \mathcal{U}')$ be disjoint SBJ-trees, with respective axes A and A'. We define:

$$J \bullet J' := (N \uplus N', \leq'', \mathcal{U}'') \text{ where}$$
$$x \leq'' y :\Longleftrightarrow x \leq y \vee x \leq' y \vee (x \in N \wedge y \in A'),$$
$$\mathcal{U}'' := \{A \uplus A'\} \uplus (\mathcal{U} - \{A\}) \uplus (\mathcal{U}' - \{A'\}).$$

It follows that $J \bullet J'$ is an SBJ-tree with axis $A \uplus A'$; the depth of a node in $J \bullet J'$ is the same as in J or J'. This operation generalizes the concatenation of linear orders: if $(N, \leq)$ and $(N', \leq')$ are disjoint linear orders, then the SBJ-tree $(N, \leq, \{N\}) \bullet (N', \leq', \{N'\})$ corresponds to the concatenation of $(N, \leq)$ and $(N', \leq')$ usually denoted by $(N, \leq) + (N', \leq')$, cf. [20]. If $K = (M, \leq, \mathcal{V})$ is an SBJ-tree with axis B, and $B = A \uplus A'$ such that $A < A'$, then $K = J \bullet J'$ where:

$N :=\downarrow(A), N' := M - N,$
$\mathcal{U}$ is the set of lines of $\mathcal{V}$ included in $N - A$ together with A,
$\mathcal{U}'$ is the set of lines of $\mathcal{V}$ included in $N' - A'$ together with A' and
the orders of J and J' are the restrictions of $\leq$ to N and N'.

– *The empty SBJ-tree.*
The nullary symbol Ω denotes the empty SBJ-tree.

– *Extension.*
Let $J = (N, \leq, \mathcal{U})$ be an SBJ-tree, and $u \notin N$. Then:

$$ext_u(J) := (N \uplus \{u\}, \leq', \{\{u\}\} \uplus \mathcal{U}) \text{ where}$$
$$x \leq' y :\Longleftrightarrow x \leq y \vee y = u,$$

the axis is $\{u\}$. Clearly, $ext_u(J)$ is an SBJ-tree. The depth of $v \in N$ is its depth in J plus 1. The axis of J is turned into an "ordinary line" of the structuring of $ext_u(J)$ with top equal to u. When handling SBJ-trees up to isomorphism, we use the notation $ext(J)$ instead of $ext_u(J)$.

– *Forgetting structuring.*
If J is an SBJ-tree as above, $fgs(J) := (N, \leq)$ is the underlying BJ-tree (binary join-tree), where *fgs forgets the structuring.*

Anticipating the sequel, we observe that a linear order $a_1 < \cdots < a_n$, identified with the SBJ-tree $(\{a_1, \dots, a_n\}, \leq, \{\{a_1, \dots, a_n\}\})$ is defined by the term $t := ext_{a_1}(\Omega) \bullet ext_{a_2}(\Omega) \bullet \cdots \bullet ext_{a_n}(\Omega)$. The binary (it is even "unary") join-tree $(\{a_1, \dots, a_n\}, \leq)$ is defined by the term $fgs(t)$ and also, in a different way, by the term $fgs(ext_{a_n}(ext_{a_{n-1}}(\dots(ext_{a_1}(\Omega)))..)))$.

Definition 3.14 (The algebra $\mathbb{SBJT}$)**.** We let F be the signature $\{\bullet, ext, \Omega\}$. We obtain an algebra $\mathbb{SBJT}$ whose domain is the set of isomorphism classes of SBJ-trees. Concatenation is associative with neutral element Ω.

Definition 3.15 (The value of a term)**.**

(a) In order to define the value of a term t in $T^\infty(F)$, we compare its positions by the following equivalence relation:

$u \approx v$ if and only if every position w such that $u <_t w \leq_t u \sqcup_t v$ or $v <_t w \leq_t u \sqcup_t v$ is an occurrence of $\bullet$.

We will also use the lexicographic order $\leq_{lex}$ (positions are Dewey words). If w is an occurrence of a binary symbol, then $s_1(w)$ is its first (left) son and $s_2(w)$ its second (right) one.

(b) We define the *value* $val(t) := (N, \leq, \mathcal{U})$ of t in $T^\infty(F)$ as follows:
– $N := \mathrm{Occ}(t, ext)$, the set of occurences in t of ext,
– $u \leq v :\Longleftrightarrow u \leq_t w \leq_{lex} v$ for some $w \in N$ such that $w \approx v$,
– $\mathcal{U}$ is the set of equivalence classes of $\approx$.

Equivalently, we have:

$u \leq v :\Longleftrightarrow u \leq_t v$ or $u \leq_t s_1(u \sqcup_t v)$, $v \leq_t s_2(u \sqcup_t v)$ and $v \approx u \sqcup_t v$ (the position $u \sqcup_t v$ is an occurrence of $\bullet$),

and so (we recall that $\perp$ denotes incomparability):

$u \perp v :\Longleftrightarrow u \leq_t s_1(u \sqcup_t v)$, $v \leq_t s_2(u \sqcup_t v)$ and there is an occurrence of ext between v and $u \sqcup_t v$ or vice-versa by exchanging u and v.

(c) We now consider terms t written with the operations ext_a (such that a is the node created by applying this operation). For each a, the operation ext_a must have at most one occurrence in t. Assuming this condition satisfied, then $val(t) := (N, \leq, \mathcal{U})$, where
– N is the set of nodes a such that ext_a has an occurence in t that we will denote by u_a,
– $a \approx b :\Longleftrightarrow u_a \approx u_b$, with $\approx$ as in (a),
– $a \leq b :\Longleftrightarrow u_a \leq u_b$, with $\leq$ as in (b),
– $\mathcal{U}$ is the set of equivalence classes of $\approx$.

Clearly, the mapping val in (b) is a value mapping $T^\infty(F) \to \mathbb{SBJT}$.

We say that t *denotes* an SBJ-tree J if J is isomorphic to $val(t)$, and, in this case, we also say that $fgs(t)$ denotes the BJ-tree $fgs(J)$.

Note that we do not define the value of term as the least upper bound of the values of its finite subterms. We could use a notion of least upper bound based on category theory as in [8], at the cost of heavy definitions. Our simpler definition shows furthermore that the mapping associating the join-tree $(N, \leq)$ with $\lfloor t \rfloor$ for $t \in T^\infty(F)$ is an MS-transduction (cf. Section 2) defined by $\mathcal{D} = \langle \chi, \delta, \theta_\leq \rangle$ where χ expresses that the considered input structure S is isomorphic to $\lfloor t \rfloor$ for some $t \in T^\infty(F)$, $\delta(x)$ is $lab_{ext}(x)$ (expressing that $x \in N$) and $\theta_\leq(x, y)$ expresses that $x \leq y$, cf. Definition 3.15(b).

Examples 3.16.

(1) The term t_0 that is the unique solution in $T^\infty(F)$ of the equation $t_0 = t_0 \bullet t_0$ denotes the empty SBJ-tree Ω.

(2) Figure 3 shows a finite SBJ-tree J whose structuring consists of $U_0, \ldots, U_5$, and U_0 is the axis. The linear order on U_0 can be described by the word $fedca$ (with $f < e < d < \ldots$). Similarly, $U_1 = b, U_2 = hg, U_3 = i$, $U_4 = kj$ and $U_5 = m$.

Let us examine the term t of Figure 4 that denotes J. A function symbol ext_u specifies the node u of J, and we also denote by u its occurrence, a position of t (hence b denotes position 21). The occurrences of $\bullet$ and Ω are denoted by Dewey words. For

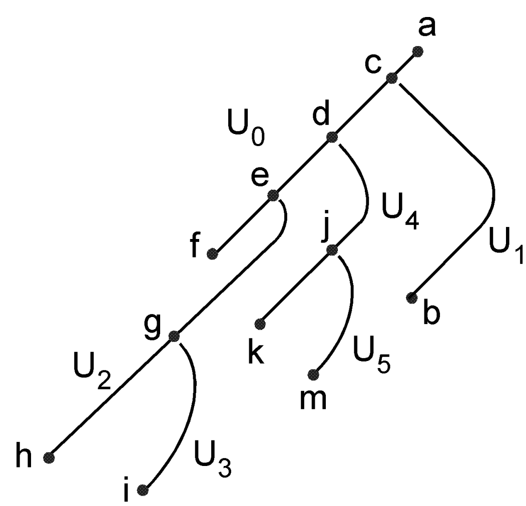

Figure 3: A finite SBJ-tree J.

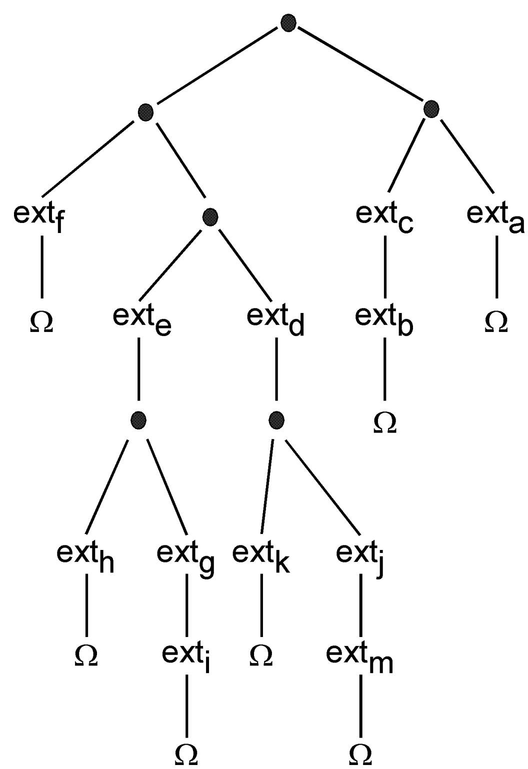

Figure 4: A term t denoting J.

example, the occurrences of $\bullet$ above the symbols *ext* are the words $\varepsilon, 1, 2, 12$. The set $\{\varepsilon, 1, 2, 12, f, e, d, c, a\}$ is an equivalence class of $\approx$. Another one is $\{1221, k, j\}$. Each line U_i is the set of positions of the *ext* symbols in some equivalence class of $\approx$. Let us now examine how each line is ordered.

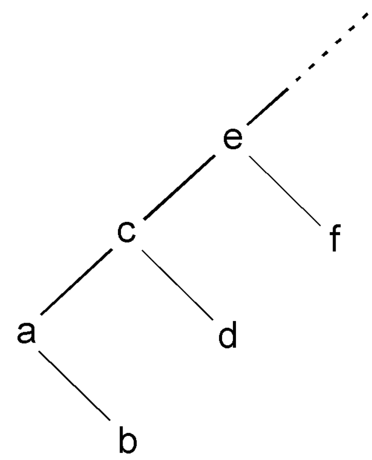

Figure 5: The SBJ-tree $val(t_1)$.

The case where $u < v$ holds because $u <_t v$ is illustrated, to take a few cases, by $i < g, g < e, m < j$ and $j < d$. The case where $u < v$ holds because $u \bot_t v$, $u \leq_t s_1(u \sqcup_t v)$, $v \leq_t s_2(u \sqcup_t v)$ and $v \approx u \sqcup_t v$ is illustrated by $f < e, e < d, d < c$ and $i < d$. We have $i < d$ because $i \sqcup_t d = 12$, $i <_t s_1(12)$, $d \leq_t s_2(12)$ and $d \approx 12$. We do not have $i < j$ because j is not $\approx$-equivalent to 12, whereas $i \sqcup_t j = 12$, $i <_t s_1(12)$ and $j \leq_t s_2(12)$. This case illustrates the characterization of $\bot$ Definition 3.15(c).

(3) Let t_1 be the solution in $T^\infty(F)$ of the equation $t_1 = ext(ext(\Omega)) \bullet t_1$. We write it by naming $a, a', b, b', c, c' \ldots$ the nodes created by the operations ext, hence, $t_1 = ext_a(ext_{a'}(\Omega)) \bullet (ext_b(ext_{b'}(\Omega)) \bullet (ext_c(ext_{c'}(\Omega)) \bullet \ldots)))$. This term and its value are shown in Figure 5. The bold edges link nodes in the axis. The nodes a' and c' are incomparable because the corresponding occurrences of ext, that are 111 and 2211, have least common ancestor ε and 221 is an occurrence of ext between 2211 and ε.

(4) The following BJ-tree is defined by Fraïssé in [16] (Section 10.5.3). We let $W := (Seq_+(\mathbb{Q}), \preceq)$ where $Seq_+(\mathbb{Q})$ is the set of nonempty sequences of rational numbers, partially ordered as follows:$(x_n, \ldots, x_0) \preceq (y_m, \ldots, y_0)$ if and only if $n \geq m$, $(x_{m-1}, \ldots, x_0) = (y_{m-1}, \ldots, y_0)$ and $x_m \leq y_m$. In particular, $\prec$ is the transitive closure of $\prec_0 \cup \prec_1$ where $(x_{p+1}, x_p, \ldots, x_0) \prec_0 (x_p, \ldots, x_0)$ and $(y, x_{p-1}, \ldots, x_0) \prec_1 (z, x_{p-1}, \ldots, x_0)$ if $y < z$. It is easy to check that W is a BJ-tree. In particular, two nodes $(x_n, \ldots, x_0)$ and $(y_m, \ldots, y_0)$ are incomparable if and only if $(y_m, \ldots, y_0) = (y_m, \ldots, y_{p+1}, x_p, \ldots, x_0)$ and $y_{p+1} \neq x_{p+1}$ for some $p < n, m$. In this case, their join is $(\min\{y_{p+1}, x_{p+1}\}, x_p, \ldots, x_0)$. The two directions relative to a node $x = (x_p, \ldots, x_0)$ are:

$\partial_0(x) := \{(y_m, \ldots, y_{p+1}, x_p, \ldots, x_0) \mid n > p, y_m, \ldots, y_{p+1} \in \mathbb{Q}\}$ and,

$\partial_1(x) := \{(y_m, \ldots, y_{p+1}, x'_p, \ldots, x_0) \mid n \geq p, x'_p, y_m, \ldots, y_{p+1} \in \mathbb{Q}, x'_p < x_p\}$.

A structuring $\mathcal{U}$ of W consists of the sets $\{(x_n, \ldots, x_0) \mid x_n \in \mathbb{Q}\}$ for each (possibly empty) sequence $(x_{n-1}, \ldots, x_0)$. The set of one element sequences (r) for $r \in \mathbb{Q}$ is the axis, and $U_-(x) \subseteq \partial_1(x)$ for all $x \in Seq_+(\mathbb{Q})$.

The proof in [16] that every finite or countable generalized tree in the sense of [20] (*i.e.*, partial order satisfying Condition 2) of Definition 3.1(a)) is isomorphic to $(X, \preceq\restriction X)$ for some subset X of $Seq_+(\mathbb{Q})$ uses implicitely the structuring $\mathcal{U}$. Our description of the two directions of a node shows that $W = W \bullet (ext(W) \bullet W)$, hence, that W is denoted by the regular term $t \in T^\infty(F)$ such that $t = t \bullet (ext(t) \bullet t)$.

Definition 3.17 (The description scheme associated with a term)**.**

(1) Let $t \in T^{\infty}(F)$ and $u \in Pos(t)$. We denote by $\mathrm{Max}(t, ext, u)$ the set of maximal occurrences of ext in t that are below u or equal to it. Positions are denoted by Dewey words, hence, these sets are linearly ordered by $\leq_{lex}$. We denote by $W(t,u)$ the simple arrangement $(\mathrm{Max}(t, ext, u), \leq_{lex})$. Let $J = (N, \leq, \mathcal{U})$ be the value of t (cf. Definition 3.15) and x be an occurrence of ext with son u. We have $(U^x, \leq)$ $= (\mathrm{Max}(t, ext, u), \leq_{lex})$. For the term t in Example 3.16(2), see Figure 4, we have $W(t,\varepsilon) = fedca, W(t,1) = fed, W(t,1211) = hg$. For t_1 in Example 3.16(3), we have $W(t_1,\varepsilon) = abc\ldots, W(t_1,1) = a, W(t_1,11) = a'$ and $W(t_1,111) = \Omega$.

(2) We define $\Delta(t)$ as the SBJ-scheme $(\mathrm{Occ}(t, ext), W(t,\varepsilon), (W(t, s(x)))_{x \in \mathrm{Occ}(t,ext)})$ where $s(x)$ is the unique son of an occurrence x of ext. We obtain

$$\Delta(t_1) = (2^*1 \uplus 2^*11, abc\ldots, (w_x)_{x \in \mathrm{Occ}(t_1,ext)})$$

with $w_1 = a'$, $w_{21} = b'$, $\ldots$, $w_{11} = \Omega$, $w_{211} = \Omega$, $\ldots$ for the term t_1 of Example 3.16(3).

Lemma 3.18. *If $t \in T^{\infty}(F)$, then $val(t)$ is described by $\Delta(t)$.*

Proof. Let $val(t) = (N, \leq, \mathcal{U})$. The conditions of Definition 3.10(b) hold with the identity on $\mathrm{Occ}(t, ext)$ as mapping r because $(U^x, \leq) = (\mathrm{Max}(t, ext,\ s(x)), \leq_{lex})$ as observed in Definition 3.17(a). □

Proposition 3.19. *Every SBJ-tree is the value of a term.*

Proof. Let $J = (N, \leq, \mathcal{U})$ be an SBJ-tree. For each k, we let J_k be the SBJ-tree $(N_k, \leq, \mathcal{U}_k)$ where N_k is the set of nodes of depth at most k and $\mathcal{U}_k$ is the set of lines $U \in \mathcal{U}$ of depth at most k. By induction on k, we define for each k a term t_k that defines J_k such that $t_k \lll t_{k'}$ if $k < k'$, and then, the least upper bound of the terms t_k is the desired term t whose value is J. We define terms using the symbols ext_a where a names the node created by the corresponding occurrence of the extension operation.

If $k = 0$, then $J_0 = (A, \leq, \{A\})$. There exists a term $t \in T^{\infty}(\{\bullet\}, Ext_A)$ whose value is J_0, where Ext_A is the set of terms $ext_a(\Omega)$ for $a \in A$ (we use Ext_A as a set of nullary symbols). We use here Theorem 2.3 of [8], that follows immediately from the representation of a linear order by the lexicographic order on a prefix-free language[9] recalled in Section 1.

Let $k \geq 1$, where t_{k-1} defines J_{k-1}. Then J_k is obtained from J_{k-1} by adding below some nodes x at depth $k-1$ the line U^x (if $U^x = \emptyset$, there is nothing to add below x). Let $t_x \in T^{\infty}(\{\bullet\}, Ext_{U^x})$ whose value is $(U^x, \leq)$. We obtain t_k by replacing in t_{k-1} each subterm $ext_x(\Omega)$ by $ext_x(t_x)$, for x at depth $k-1$ such that $U^x \neq \emptyset$. It is clear that $t_{k-1} \lll t_k$ and that the least upperbound of the terms t_k defines J. □

For an example, we apply this construction to the SBJ-tree J of Figure 3. For defining J_0, we can take:

$$t_0 = ((ext_f(\Omega) \bullet ext_e(\Omega)) \bullet ext_d(\Omega)) \bullet (ext_c(\Omega) \bullet ext_a(\Omega)).$$

To obtain t_1, we replace $ext_e(\Omega)$ by $ext_e(ext_h(\Omega) \bullet ext_g(\Omega))$, $ext_d(\Omega)$ by $ext_d(ext_k(\Omega) \bullet ext_j(\Omega))$ and $ext_c(\Omega)$ by $ext_c(ext_b(\Omega))$, which gives:

$$t_1 = ((ext_f(\Omega) \bullet ext_e(ext_h(\Omega) \bullet ext_g(\Omega))) \bullet ext_d(ext_k(\Omega) \bullet ext_j(\Omega)) \bullet (ext_c(ext_b(\Omega)) \bullet ext_a(\Omega)).$$

Then, we obtain t_2 that defines J by replacing $ext_g(\Omega)$ by $ext_g(ext_i(\Omega))$ and $ext_j(\Omega)$ by $ext_j(ext_m(\Omega))$.

[9] Also used in the related paper [11].

3.4. **Regular binary join-trees.** As said in the introduction, the regular objects are those defined by regular terms. We apply this meta-definition to binary join-trees and their structurings.

Definition 3.20 (Regular BJ- and SBJ-trees)**.** A BJ-tree (resp. an SBJ-tree) T is *regular* if it is denoted by $fgs(t)$ (resp. by t) where t is a regular term in $T^{\infty}(F)$.

Theorem 3.21. *The following properties of a BJ-tree J are equivalent:*
(1) *J is regular,*
(2) *J is described by a regular scheme,*
(3) *J is MS definable.*

Proof.
(1)$\Longrightarrow$(2) Let $J = fgs(J')$ with J' denoted by a regular term t in $T^{\infty}(F)$. Let $h : Pos(t) \to Q$ and τ be as in the definition of a regular term in Section 1. Without loss of generality, we can assume that $h(Pos(t)) = Q$. If this is not the case, we replace Q by $h(Pos(t))$ and τ by its restriction to this set.

Claim.
(a) For each $u \in Pos(t)$, the arrangement $\overline{h}(W(t,u)) = (\mathrm{Max}(t, ext, u), \leq_{lex}, h)$ over Q is regular.
(b) If u' is another position in t and $h(u') = h(u)$, then $t/u' = t/u$ and furthermore[10] $\overline{h}(W(t,u')) \simeq \overline{h}(W(t,u))$.

Leaving its routine proof, we define $\Delta := (Q, w_{Ax}, (w_q)_{q\in Q})$ as follows:
(i) $w_{Ax} := \overline{h}(W(t,\varepsilon))$,
(ii) if $q \in Q$, then $w_q := \overline{h}(W(t, s(u)))$ where $s(u)$ is the unique son of an occurrence u of ext such that $h(u) = q$; if v is another occurrence of ext such that $h(v) = q$, then $h(s(v)) = h(s(u))$ and so by the claim, $\overline{h}(W(t, s(v))) \simeq \overline{h}(W(t, s(u)))$. Hence, w_q is well-defined up to isomorphism.

Informally, Δ is obtained from $\Delta(t)$ by replacing the labelling mapping Id of the arrangements $W(t,u)$ by h, so that these arrangements are turned into arrangements $\overline{h}(W(t,u))$ over Q. Clearly, Δ is a regular scheme. As mapping r showing that it describes J' (cf. Definition 3.10), hence also J, we take the resriction of h to $\mathrm{Occ}(t, ext)$ that is the set of nodes of $J' = val(t)$.

(2)$\Longrightarrow$(3) is proved in Proposition 3.12.

(3)$\Longrightarrow$(1) By Definition 3.15, the mapping α that transforms the relational structure $\lfloor t \rfloor$ for t in $T^{\infty}(F)$ into the BJ-tree $J = (N, \leq) = fgs(val(t))$ is an MS-transduction because an MS formula can identify the nodes of J among the positions of t and another one can define $\leq$.

Let $J = (N, \leq)$ be an MS definable BJ-tree. It is, up to isomorphism, the unique model of an MS sentence β. It follows by a standard argument[11] that the set of terms t in $T^{\infty}(F)$ such that $\alpha(\lfloor t \rfloor) \models \beta$ is MS definable and thus, contains a regular term, a result by Rabin [24, 25]. This term denotes J, hence J is regular. □

[10] Unless $u = u'$, the sets $\mathrm{Max}(t, ext, u)$ and $\mathrm{Max}(t, ext, u')$ are not equal, so that the arrangements $\overline{h}(W(t,u))$ and $\overline{h}(W(t,u'))$ are isomorphic but not equal.

[11] If α is an MS-transduction and β is an MS sentence, then the set of structures S such that $\alpha(S) \models \beta$ is MS-definable (Theorem 2.1).

Corollary 3.22. *The isomorphism problem for regular BJ-trees is decidable.*

Proof. A regular BJ-tree can be given, either by a regular term, a regular scheme or an MS sentence. The proof of Theorem 3.21 is effective : algorithms can convert any of these specifications into another one. Hence, two regular BJ-trees can be given, one by an MS sentence β, the other by a regular term t. They are isomorphic if and only if $\alpha(\lfloor t \rfloor) \models \beta$ (cf. the proof of (3)$\Longrightarrow$(1) of Theorem 3.21) if and only if $\lfloor t \rfloor \models \beta'$ where β' obtained by applying Theorem 2.1 to the sentence β and the transduction α. This is decidable [24, 25]. $\square$

3.5. **Logical and algebraic descriptions of join-trees.** We now extend to join-trees the definitions and results of the previous sections. Structured join-trees are defined in Section 3.1 (Definition 3.3). We extend to them the definitions and results of Sections 3.2-3.4. A first novelty is that the argument of the extension operation *ext* will be an SJ-forest, equivalently a set of SJ-trees, instead of a single SBJ-tree. We will need an algebra with two sorts, the sort of SJ-trees and that of SJ-forests. A second difference consists in the use in monadic second-order formulas of a finiteness predicate (cf. Section 2).

Definition 3.23 (Description schemes for SJ-trees)**.**

(a) A *description scheme for an SJ-tree*, in short an *SJ-scheme,* is a 5-tuple $\Delta = (Q, D, w_{Ax}, (m_q)_{q\in Q}, (w_d)_{d\in D})$ such that Q, D are sets, $w_{Ax} \in \mathcal{A}(Q)$, $w_d \in \mathcal{A}(Q)$ for each $d \in D$ and $m_q = (M_q, lab_q)$ is a D-labelled set (cf. Section 2) for each $q \in Q$. Without loss of generality, we will assume that the domains V_{Ax} and V_d of the arrangements w_{Ax}, w_d and the sets M_q are pairwise disjoint, because these arrangements and labelled sets will be used up to isomorphism. Informally, M_q encodes the different lines U such that $\widehat{U} = x$ where x is labelled by q, and each of these lines is defined, up to isomorphism, by the arrangement w_d where d is its label in D, defined by lab_q.

We say that Δ is *regular* if $Q \cup D$ is finite and the arrangements w_{Ax} and w_d are regular. The finiteness of D implies that each D-labelled set m_q is regular.

(b) Let $J = (N, \leq, \mathcal{U})$ be an SJ-tree with axis A; for each $x \in N$, we denote by $\mathcal{U}^x$ the set of lines $U \in \mathcal{U}$ such that $\widehat{U} = x$. In the example of Figure 3, we have $\mathcal{U}^d = \{U_4\}$. An SBJ-scheme Δ as in a) *describes* J if there exist mappings $r : N \to Q$ and $\widetilde{r} : \mathcal{U} - \{A\} \to D$ such that:

(b.1) the arrangement $(A, \leq, r)$ over Q is isomorphic to w_{Ax},

(b.2) for each $x \in N$, the D-labelled set[12] $(\mathcal{U}^x, \widetilde{r})$ is isomorphic to $m_{r(x)}$,

(b.3) for each $U \in \mathcal{U} - \{A\}$, the arrangement $(U, \leq, r)$ over Q is isomorphic to $w_{\widetilde{r}(U)}$.

We will also say that Δ *describes* the join-tree $fgs(J) := (N, \leq)$, obtained from J by forgetting the structuring.

Proposition 3.24.

(1) *Every SJ-tree is described by some SJ-scheme.*
(2) *Every SJ-scheme Δ describes a unique SJ-tree $Unf(\Delta)$ where unicity is up to isomorphism.*

Proof. We extend the proof of Proposition 3.11.

[12] $\mathcal{U}^x$ is a set of subsets of N and $\widetilde{r}$ replaces each set in $\mathcal{U}^x$ by some $d \in D$. Hence, $\widetilde{r}(\mathcal{U}^x \dots)$ is a multiset of elements of D.

(1) Each SJ-tree $J = (N, \leq, \mathcal{U})$ has a *standard* description scheme $\Delta(J) := (N, \mathcal{U} - \{A\}, (A, \leq), ((\mathcal{U}^x, Id))_{x \in N}, ((U, \leq))_{U \in \mathcal{U} - \{A\}})$. The identity mappings $N \to N$ and $\mathcal{U} - \{A\} \to \mathcal{U} - \{A\}$ show that $\Delta(J)$ describes J.

(2) Let $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ be an SJ-scheme, defined with arrangements $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$ and $w_d = (V_d, \preceq, lab_d)$, and labelled sets $m_q = (M_q, lab_q)$ such that the sets V_{Ax}, V_d and M_q are pairwise disjoint and the same symbol $\preceq$ denotes the orders of the arrangements w_{Ax} and w_d. We construct $Unf(\Delta) := (N, \leq, \mathcal{U})$ as follows.

   (a) N is the set of finite nonempty sequences $(v_0, s_1, v_1, s_2, \ldots, s_k, v_k)$ such that:
   $v_0 \in V_{Ax}, v_i \in V_{d_i}$ and $s_i \in M_{q_{i-1}}$ for $1 \leq i \leq k$, where
   $q_0 = lab_{Ax}(v_0)$, $d_1 = lab_{q_0}(s_1)$, $q_1 = lab_{d_1}(v_1)$, $d_2 = lab_{q_1}(s_2)$, $\ldots$,
   $q_i = lab_{d_i}(v_i)$, $d_{i+1} = lab_{q_i}(s_{i+1})$ for $1 \leq i \leq k-1$.

   (b) $(v_0, s_1, v_1, \ldots, s_k, v_k) \leq (v'_0, s'_1, v'_1, \ldots, s'_j, v'_j)$ if and only if
   $k \geq j$, $(v_0, s_1, v_1, \ldots, s_j) = (v'_0, s'_1, v'_1, \ldots, s'_j)$ and $v_j \preceq v'_j$ $(v_j, v'_j \in V_{d_j})$.

   (c) the axis A is the set of one-element sequences (v) for $v \in V_{Ax}$ and, for $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$, $U(x)$ is the set of sequences in N of the form $(v_0, s_1, v_1, s_2, \ldots, s_k, v)$ for $v \in V_{d_k}$, so that $\widehat{U(x)} = (v_0, s_1, v_1, \ldots, s_{k-1}, v_{k-1})$.

   Note that $(v_0, s_1, v_1, \ldots, v_k) < (v_0, s_1, v_1, \ldots, v_j)$ if $j < k$ and that $(v_0, s_1, v_1, \ldots, s_k, v_k) \leq (v_0, s_1, v_1, \ldots, s_k, v)$ if and only if $v_k \preceq v$. In order to prove that Δ describes J, we define $r : N \to Q$ and $\widetilde{r} : \mathcal{U} - \{A\} \to D$ as follows:
   – if $x \in A$, then $x = (v)$ for some $v \in V_{Ax}$ and $r(x) := lab_{Ax}(v)$;
   – if $x \in N$ has depth $k \geq 1$, then $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$ for some $v_0, s_1, \ldots, s_k, v_k$ and $r(x) := lab_{d_k}(v_k)$;
   – if $U \in \mathcal{U} - \{A\}$, then $U = U(x)$ for some $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$, $k \geq 1$, and $\widetilde{r}(U) := d_k$.

   We check the three conditions of 3.23(b). We have $(A, \leq, r) \simeq w_{Ax}$, hence (b.1) holds. For checking (b.2), we consider $x = (v_0, s_1, v_1, \ldots, s_k, v_k) \in N, k \geq 1$. The sets U in $\mathcal{U}^x$ are those of the form $\{(v_0, s_1, v_1, \ldots, s_k, v_k, s, v) \mid v \in V_{d_{k+1}}\}$ for all $s \in M_{q_k}$ where $q_k = lab_{d_k}(v_k) = r(x)$, hence (b.2) holds. For checking (b.3), we let $U = U(x)$ for some $x = (v_0, s_1, v_1, \ldots, s_k, v_k), k \geq 1$; it is the set of sequences $(v_0, s_1, v_1, s_2, \ldots, s_k, v)$ for $v \in V_{d_k}$ ordered by $\preceq$ on the last components. Hence, $(U, \leq, lab_{d_k})$ is isomorphic to w_{d_k}, which proves the property since $\widetilde{r}(U) := d_k$.

   Unicity is proved as in Proposition 3.11. □

As for SBJ-trees, every SJ-tree is described by a canonical SJ-scheme, that is regular and has a minimum number of states if the SJ-tree is regular. The following proposition extends Proposition 3.12.

Proposition 3.25. *A join-tree is MS_{fin}-definable if it is described by a regular SJ-scheme.*

Proof. Let $(N, \leq)$ be a join-tree J (this property is first-order expressible). Assume that $J = fgs(J')$ where $J' = (N, \leq, \mathcal{U}) \simeq Unf(\Delta)$ for some regular SJ-scheme $\Delta = (Q, D, w_{Ax}, (m_q)_{q \in Q}, (w_d)_{d \in D})$ such that $Q = \{1, \ldots, m\}$ and $D = \{1, \ldots, p\}$. Let r,$\widetilde{r}$ be the corresponding mappings (cf. Definition 3.23(b)). For each $d \in D$, let ψ_d be an MS sentence that characterizes w_d up to isomorphism, by the main result of [23]. Similarly, ψ_{Ax} characterizes w_{Ax}.

A D-labelled set m_q is described up to isomorphism by a p-tuple $(m_q^1, \ldots, m_q^p)$ where m_q^j is the number (possibly ω) of elements having label j. By Proposition 3.7, there is a bipartition (N_0, N_1) of N that describes the structuring $\mathcal{U}$; from this bipartition, we can define the axis A, the lines forming $\mathcal{U}$ and the node $\widehat{U}$ for each $U \in \mathcal{U} - \{A\}$ by MS formulas. There is a partition $(Y_1, \dots, Y_m)$ of N that describes r by $Y_q := r^{-1}(q)$. There is a partition $(Z_1, \dots, Z_p)$ where Z_j is the union of the lines $U \in \mathcal{U} - \{A\}$ such that $\widetilde{r}(U) = j$.

Consider a relational structure $(X, \leq, N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$. By MS formulas, one can express the following properties:

(i) $(X, \leq, N_0, N_1)$ is $S(J'')$ for some SJ-tree $J'' = (X, \leq, \mathcal{U}')$; its axis is denoted by A',
(ii) $(Y_1, \dots, Y_m)$ is a partition of X; we let $r(x) := q$ if and only if $x \in Y_q$,
(iii) $(Z_1, \dots, Z_p)$ is a partition of X such that each Z_j is a union of sets $U \in \mathcal{U}' - \{A'\}$ such that $(U, \leq, r) \simeq w_j$,
(iv) $(A', \leq, r) \simeq w_{Ax}$,
(v) for each $q \in Q$ and $x \in Y_q$, the number of lines $U \in \mathcal{U}'^x$ that are contained in Z_j is m_q^j.

These formulas are constructed as follows: $\varphi(N_0, N_1)$ for (i) is from Proposition 3.7. The formula for (ii) is standard. All other formulas are constructed so as to express the desired properties when (i) and (ii) do hold. For (iii), we use a suitable adaptation of ψ_i and the fact from Proposition 3.7 that, if (i) holds, we can define from (N_0, N_1), by MS formulas, the axis A', the lines forming $\mathcal{U}'$ and the node $\widehat{U}$ for each $U \in \mathcal{U}'$. The mapping r is given by $(Y_1, \dots, Y_m)$. For (iv), we do as for (iii) with ψ_{Ax}.

For (v), we do as follows. We write an MS formula $\gamma(x, N_0, N_1, Z, W)$ expressing that W consists of one node of each set $U \in \mathcal{U}' - \{A'\}$ that is contained in Z and is such that $\widehat{U} = x$. For any x and Z, all sets W satisfying $\gamma(x, N_0, N_1, Z, W)$ have same cardinality. Then, Property (v) holds if and only if, for all $q = 1, \dots, m$, $x \in Y_q$ and $j = 1, \dots, p$, if $\gamma(x, N_0, N_1, Z_j, W)$ holds, then W has cardinality m_q^j. If some number m_q^j is ω, we need the finiteness predicate $Fin(W)$ to express this condition[13].

Let $\beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$ express conditions (ii)-(v) in $(X, \leq)$. If a join-tree $(X, \leq)$ satisfies $\varphi(N_0, N_1) \wedge \beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p)$, it has a structuring $\mathcal{U}'$ described by N_0, N_1: we let $J'' := (X, \leq, \mathcal{U}')$. The sets $Y_1, \dots, Y_m, Z_1, \dots, Z_p$ yield a scheme Δ that describes J'' (by Conditions (iii)-(v)), hence J'' is isomorphic to J' by the unicity property of Proposition (3.24), and so, we have $(X, \leq) \simeq fgs(J') = J$.

Hence, J is (up to isomorphism) the unique model of the MS_{fin} sentence:

$$\exists N_0, N_1 (\varphi(N_0, N_1) \wedge \exists Y_1, \dots, Y_m, Z_1, \dots, Z_p . \beta(N_0, N_1, Y_1, \dots, Y_m, Z_1, \dots, Z_p))). \quad \square$$

Theorem 3.30 will establish a converse.

Definition 3.26 (Operations on SJ-trees and SJ-forests)**.** We recall from Definition 3.1 that a join-forest is the union of disjoint join-trees. A structured join-forest (an SJ-forest, cf. Definition 3.3) is the union of disjoint SJ-trees. It has no axis (each of its components has an axis, but we do not single out any of them). We will use objects of three types: join-trees, SJ-trees and SJ-forests, but a 2-sorted algebra will suffice (similarly as above for $\mathbb{SBJT}$, we have not introduced a separate sort for BJ-trees). The two sorts are $\boldsymbol{t}$ for SJ-trees and $\boldsymbol{f}$ for SJ-forests.

– *Concatenation of SJ-trees along axes.* The *concatenation* $J \bullet J'$ of disjoint SJ-trees J and J' is defined exactly as in Definition 3.13 for SBJ-trees.
– *The empty SJ-tree* is denoted by the nullary symbol $\Omega_{\boldsymbol{t}}$.

[13] If the nodes of J have degree at most $a \in \mathbb{N}$, then $m_i^j \leq a$ for all i, j and the finiteness predicate is not needed, hence, J is MS definable.

– *Extension of an SJ-forest into an SJ-tree.* Let $J = (N, \leq, \mathcal{U})$ be an SJ-forest and $u \notin N$. Then $ext_u(J)$ is an SJ-tree defined as in Definition 3.13. When handling SJ-trees up to isomorphism, we will use the notation $ext(J)$ instead of $ext_u(J)$.
– *The empty SJ-forest* is denoted by the nullary symbol $\Omega_{\boldsymbol{f}}$.
– *Making an SJ-tree into an SJ-forest.*
This is done by the unary operation *mkf* that is actually the identity on the triples that define SJ-trees.
– *The union of two disjoint SJ-forests* is denoted by $\uplus$.

The types of these operations are thus:

$$\bullet : \boldsymbol{t} \times \boldsymbol{t} \to \boldsymbol{t}, \qquad \Omega_{\boldsymbol{t}} : \boldsymbol{t}, \qquad ext : \boldsymbol{f} \to \boldsymbol{t},$$
$$\uplus : \boldsymbol{f} \times \boldsymbol{f} \to \boldsymbol{f}, \qquad \Omega_{\boldsymbol{f}} : \boldsymbol{f}, \qquad mkf : \boldsymbol{t} \to \boldsymbol{f}.$$

In addition, we have, as in Definition 3.13 the *Forgetting the structuring:* If J is an SJ-tree, $fgs(J)$ is the underlying join-tree.

Definition 3.27 (The algebra $\mathbb{SJT}$)**.** We let F' be the 2-sorted signature $\{\bullet, \uplus, ext, mkf, \Omega_{\boldsymbol{t}}, \Omega_{\boldsymbol{f}}\}$ where the types of these six operations are as above. We obtain an F'-algebra $\mathbb{SJT}$ whose domains are the sets of isomorphism classes of SJ-trees and of SJ-forests. Concatenation is associative with neutral element $\Omega_{\boldsymbol{t}}$ and disjoint union is associative and commutative with neutral element $\Omega_{\boldsymbol{f}}$.

Definition 3.28 (The value of a term)**.** The definition is actually identical to that for SBJ-trees (Definition 3.15). We recall it for the reader's convenience. The equivalence relation $\approx$ is as in this definition. The *value* $val(t) = (N, \leq, \mathcal{U})$ of $t \in T^{\infty}(F')$ is defined as follows:
– $N := \mathrm{Occ}(t, ext)$, the set of occurences in t of ext,
– $u \leq v :\Longleftrightarrow u \leq_t w \leq_{lex} v$ for some $w \in N$ such that $w \approx v$,
– $\mathcal{U}$ is the set of equivalence classes of $\approx$.

If t has sort $\boldsymbol{t}$ (resp. $\boldsymbol{f}$) then $val(t)$ is an SJ-tree (resp. an SJ-forest). It is clear that we have a value mapping: $T^{\infty}(F') \to \mathbb{SJT}$.

For terms t written with the operations ext_a, then $val(t) := (N, \leq, \mathcal{U})$ where:
– N is the set of nodes a such that ext_a has an occurence in t, actually a unique one, that we will denote by u_a,
– $a \leq b :\Longleftrightarrow u_a \leq u_b$,
– $a \approx b :\Longleftrightarrow u_a \approx u_b$, and
– $\mathcal{U}$ is the set of equivalence classes of $\approx$.

Definition 3.29 (Regular join-trees)**.** A join-tree (resp. an SJ-tree) T is *regular* if it is denoted by $fgs(t)$ (resp. by t) where t is a regular term in $T^{\infty}(F')$ of sort $\boldsymbol{t}$.

Theorem 3.30. *The following properties of a join-tree J are equivalent:*
(1) *J is regular,*
(2) *J is described by a regular scheme,*
(3) *J is MS_{fin}-definable.*

Proof. (1)$\Longrightarrow$(2). Similar to that of Theorem 3.21.
(2)$\Longrightarrow$ (3) By Proposition 3.25.

(3) $\Longrightarrow$(1) As in the proof of Theorem 3.21, the mapping α that transforms the relational structure $\lfloor t \rfloor$ for t in $T^\infty(F')_{\boldsymbol{t}}$ (the set of terms in $T^\infty(F')$ of sort $\boldsymbol{t}$) into the join-tree $J = (N, \leq) = fgs(val(t))$ is an MS-transduction. Let $J = (N, \leq)$ be an $\mathrm{MS}_{\mathit{fin}}$-definable join-tree. It is, up to isomorphism, the unique model of an $\mathrm{MS}_{\mathit{fin}}$ sentence β. The set L of terms t in $T^\infty(F')_{\boldsymbol{t}}$ such that $\alpha(\lfloor t \rfloor) \models \beta$ is thus $\mathrm{MS}_{\mathit{fin}}$-definable. However, since the relational structures $\lfloor t \rfloor$ have MS definable linear orderings, L is also MS definable (see Section 2), hence, it contains a regular term. This term denotes J, hence J is regular. □

The same proof as for Corollary 3.22 yields:

Corollary 3.31. *The isomorphism problem for regular join-trees is decidable.*

The rooted trees of unbounded degree, without order on the sets of sons of their nodes are the join-trees defined by the terms in $T^\infty(F' - \{\bullet\})_{\boldsymbol{t}}$. Theorem 3.30 and Corollary 3.31 hold for them.

4. Ordered join-trees

Definition 4.1 (Ordered join-trees and join-hedges)**.** Let $(N, \leq)$ be a join-forest. A direction relative to a node x is a maximal subset C of $]-\infty, x[$ such that $y \sqcup z < x$ for all $y, z \in C$ (cf. Definition 3.2). The set of directions relative to x is denoted by $Dir(x)$. The notation $x \perp y$ means that x and y are incomparable with respect to $\leq$, so that $x < x \sqcup y$ and $y < x \sqcup y$ if $x \perp y$ and $x \sqcup y$ is defined.

(a) We say that a join-tree $J = (N, \leq)$ is *ordered* (is an *OJ-tree*) if each set $Dir(x)$ is equipped with a linear order $\sqsubseteq_x$. (In this way, we generalize the notion of an ordered tree, cf. Section 1.) From these orders, we define a single linear order $\sqsubseteq$ on N as follows:

$$x \sqsubseteq y \text{ if and only if } x \leq y \text{ or, } x \perp y \text{ and } \delta \sqsubset_{x \sqcup y} \delta'$$
$$\text{where } \delta, \delta' \in Dir(x \sqcup y), x \in \delta \text{ and } y \in \delta'.$$

(b) The linear order $\sqsubseteq$ satisfies the following properties, for all x, y, x', y':
   (i) $x \leq y$ implies $x \sqsubseteq y$,
   (ii) if $x \leq y$, $x' \leq y'$ and $y \perp y'$, then $x \sqsubset x'$ if and only if $y \sqsubset y'$.

Claim. If $J = (N, \leq)$ is a join-tree and $\sqsubseteq$ is a linear order on N satisfying conditions (i) and (ii), then J is ordered by the family of orders $(\sqsubseteq_x)_{x \in N}$ such that, for all δ, δ' in $Dir(x)$, we have $\delta \sqsubseteq_x \delta'$ if and only if $\delta = \delta'$ or $y \sqsubset y'$ for some $y \in \delta$ and $y' \in \delta'$ (if and only if $\delta = \delta'$ or $y \sqsubset y'$ for all $y \in \delta$ and $y' \in \delta'$).

Proof Sketch. Consider different directions $\delta, \delta' \in Dir(x)$ such that $y \sqsubset y'$ for some $y \in \delta$ and $y' \in \delta'$. We have also $y_1 \sqsubset y'_1$ for any $y_1 \in \delta$ and $y'_1 \in \delta'$ because $(y \sqcup y_1) < x, (y' \sqcup y'_1) < x$ and $(y \sqcup y_1) \perp (y' \sqcup y'_1)$, hence, Condition (ii) implies that $y \sqcup y_1 \sqsubset y' \sqcup y'_1$ and $y_1 \sqsubset y'_1$.

Hence, each relation $\sqsubseteq_x$ is a linear order on $Dir(x)$. It is clear that $\sqsubseteq$ is derived from the relations $\sqsubseteq_x$ by (a). □

It follows that an ordered join-tree can be equivalently defined as a triple $(N, \leq, \sqsubseteq)$ such that $(N, \leq)$ is a join-tree and $\sqsubseteq$ is a linear order that satisfies Conditions (i) and (ii). These conditions are first-order expressible.

(c) We define a *join-hedge* as a triple $H = (N, \leq, \sqsubseteq)$ such that $(N, \leq)$ is a join-forest and $\sqsubseteq$ is a linear order that satisfies Conditions (i) and (ii). Let J_s, for $s \in S$, be the join-trees composing $(N, \leq)$. Each of them is ordered by $\sqsubseteq$ according to the above claim, and the index set S is linearly ordered by $\sqsubseteq_S$ such that $s \sqsubset_S s'$ if and only if $s \neq s'$ and $x \sqsubset y$ for all nodes x of J_s and y of $J_{s'}$. Hence H is also a simple arrangement of pairwise disjoint join-trees.

Definition 4.2 (Structured join-hedges and structured ordered join-trees)**.**

(a) A *structured join-hedge*, an *SJ-hedge* in short, is a 4-tuple $J = (N, \leq, \sqsubseteq, \mathcal{U})$ such that $(N, \leq, \sqsubseteq)$ is a join-hedge and $\mathcal{U}$ is a structuring of the join-forest $(N, \leq)$.

A structured ordered join-tree could be defined in the same way, as an OJ-tree $(N, \leq, \sqsubseteq)$ equipped with a structuring $\mathcal{U}$. However, we will need a refinement in order to define the operations that construct ordered join-trees and join-hedges (cf. Definition 4.8 and Remark 4.12 below).

(b) Let J be an OJ-tree $(N, \leq, \sqsubseteq)$ and $\mathcal{U}$ be a structuring of $(N, \leq)$. For each node x, the set $Dir(x)$ of its directions consists of the following sets:
– the sets $\downarrow(U)$ for each line $U \in \mathcal{U}^x$ (we recall that $\downarrow(U) := \{y \mid y \leq z$ for some $z \in U\}$),
– the set $\downarrow(U_-(x))$ (cf. Definition 3.3) if $U_-(x)$ is not empty; in this case we call it the *central direction* of x.

If x is the smallest element of $U(x)$, it has no central direction but $\mathcal{U}^x$ may be nonempty. It is clear that $\downarrow(U) \cap \downarrow(U') = \emptyset$ if U and U' are distinct lines in $\mathcal{U}^x$. We get a linear order on $\mathcal{U}^x$ based on that on directions, that we also denote by $\sqsubseteq_x$: we have $U \sqsubset_x U'$ if and only if $y \sqsubset y'$ for all $y \in U$ and $y' \in U'$.

(c) A *structured ordered join-tree* (an *SOJ-tree*) is a tuple $(N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ such that $(N, \leq, \sqsubseteq)$ is an OJ-tree and $\mathcal{U} := \{A\} \uplus \mathcal{U}^- \uplus \mathcal{U}^+$ is a structuring of $(N, \leq)$ with axis A, such that, for each node x: if $U \in \mathcal{U}^x \cap \mathcal{U}^-$ and $U' \in \mathcal{U}^x \cap \mathcal{U}^+$, then $U \sqsubset_x U'$ and furthermore, if x has a central direction δ, then $U \sqsubset_x \delta \sqsubset_x U'$.

We define then $Dir^-(x)$ as the set of directions $\downarrow(U)$ for $U \in \mathcal{U}^x \cap \mathcal{U}^-$ and, similarly, $Dir^+(x)$ with $U \in \mathcal{U}^x \cap \mathcal{U}^+$.

Let $U \in \mathcal{U}$ and $x \notin U$ be such that $[x, +\infty[\cap U \neq \emptyset$. By Condition (2) of Definition 3.3(a), there is a node y_i in U for some $i > 0$ (we use the notation of that definition). We say that x is *to the left* (resp. *to the right*) of U if, for some direction δ relative to y_i, we have $x \in \delta \in Dir^-(y_i)$ (resp. $x \in \delta \in Dir^+(y_i)$).

As in Propositions 3.5 and 3.9, we have:

Proposition 4.3. *Every join-hedge and every ordered join-tree has a structuring.*

Proof. For a join-hedge $(N, \leq, \sqsubseteq)$, we take any structuring $\mathcal{U}$ of the join-forest $(N, \leq)$. Let $(N, \leq, \sqsubseteq)$ be an OJ-tree and $\mathcal{U}$ be any structuring of the join-tree $(N, \leq)$. Let A be its axis. In order to define $\mathcal{U}^-$ and $\mathcal{U}^+$, we need only partition each set $\mathcal{U}^x$ into two sets $\mathcal{U}^x \cap \mathcal{U}^-$ and $\mathcal{U}^x \cap \mathcal{U}^+$. If x has a central direction δ, we let $\mathcal{U}^x \cap \mathcal{U}^-$ consist of the lines U in $\mathcal{U}^x$ such that $\downarrow(U) \sqsubset_x \delta$, and $\mathcal{U}^x \cap \mathcal{U}^+$ consist of those such that $\delta \sqsubset_x \downarrow(U)$. Otherwise, we let $\mathcal{U}^+$ contain[14] $\mathcal{U}^x$ so that $\mathcal{U}^x \cap \mathcal{U}^- = \emptyset$. □

[14] We might alternatively partition $\mathcal{U}^x$ into any two sets $\mathcal{U}^x \cap \mathcal{U}^-$ and $\mathcal{U}^x \cap \mathcal{U}^+$ such that $\mathcal{U}^x \cap \mathcal{U}^- \sqsubset_x \mathcal{U}^x \cap \mathcal{U}^+$.

We now establish the MS definability of these structurings. If $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ is an SOJ-tree, we define $S(J)$ as the structure $(N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+)$ such that A is the axis, N_0^- (resp. N_0^+) is the union of the lines $U \in \mathcal{U}^-$ (resp. $U \in \mathcal{U}^+$) of even depth and N_1^- (resp. N_1^+) is the union of the lines $U \in \mathcal{U}^-$ (resp. $U \in \mathcal{U}^+$) of odd depth.

Proposition 4.4. (1) *There is an MS formula* $\varphi(A, N_0^-, N_0^+, N_1^-, N_1^+)$ *expressing that a structure* $S = (N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+)$ *is* $S(J)$ *for some SOJ-tree* $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$.

(2) *There exists an MS formula* $\theta^-(u, U, N_0^-, N_0^+, N_1^-, N_1^+)$ *expressing in a structure* $(N, \leq, \sqsubseteq, A, N_0^-, N_0^+, N_1^-, N_1^+) = S(N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ *that* $U \in \mathcal{U}^- \wedge u = \widehat{U}$*; similarly, there exists an MS formula* $\theta^+(u, U, N_0^-, N_0^+, N_1^-, N_1^+)$ *expressing that* $U \in \mathcal{U}^+ \wedge u = \widehat{U}$.

Proof. Easy modification of the proof of Proposition 3.7. □

Definition 4.5 (Description schemes for SOJ-trees)**.** (a) A *description scheme for an SOJ-tree*, in short an *SOJ-scheme,* is a 6-tuple $\Delta = (Q, D, w_{Ax}, (w_q^-)_{q\in Q}, (w_q^+)_{q\in Q}, (w_d)_{d\in D})$ such that Q, D are sets, called respectively the set of *states* and of *directions*, $w_{Ax} \in \mathcal{A}(Q)$, $(w_d)_{d\in D}$ is a family of arrangements over Q and $(w_q^-)_{q\in Q}$ and $(w_q^+)_{q\in Q}$ are families of arrangements over D. Without loss of generality, we will assume that the domains of these arrangements are pairwise disjoint, and the same symbol $\preceq$ denotes their orders. Informally, $(w_q^-)_{q\in Q}$ and $(w_q^+)_{q\in Q}$ encodes the sets of lines, ordered by $\sqsubseteq_x$ of the two sets $Dir^-(x)$ and $Dir^+(x)$ where x is labelled by q.

We say that Δ is *regular* if $Q \cup D$ is finite and the arrangements w_{Ax}, w_d, w_q^- and w_q^+ are regular.

(b) Let $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ be an SOJ-tree. An SOJ-scheme Δ as in (a) *describes* J if there exist mappings $r : N \to Q$ and $\widetilde{r} : \mathcal{U}^- \cup \mathcal{U}^+ \to D$ such that:
(b.1) $(A, \leq, r) \simeq w_{Ax}$,
(b.2) for each $x \in N$, the arrangement $(\mathcal{U}^x \cap \mathcal{U}^-, \sqsubseteq_x, \widetilde{r})$ over D is isomorphic to $w_{r(x)}^-$,
(b.3) for each $x \in N$, the arrangement $(\mathcal{U}^x \cap \mathcal{U}^+, \sqsubseteq_x, \widetilde{r})$ over D is isomorphic to $w_{r(x)}^+$,
(b.4) for each $U \in \mathcal{U}^- \cup \mathcal{U}^+$, the arrangement $(U, \leq, r)$ over Q is isomorphic to $w_{\widetilde{r}(U)}$.

We also say that Δ *describes* the OJ-tree $fgs(J) := (N, \leq, \sqsubseteq)$ where *fgs forgets the structuring.*

Proposition 4.6. (1) *Every SOJ-tree is described by some SOJ-scheme.*
(2) *Every SOJ-scheme describes an SOJ-tree that is unique up to isomorphism.*

Proof. (1) The proof is similar to those of Propositions 3.11 and 3.24.

(2) Let $\Delta = (Q, D, w_{Ax}, (w_q^-)_{q\in Q}, (w_q^+)_{q\in Q}, (w_d)_{d\in D})$ be an SOJ-scheme, defined with arrangements $w_{Ax} = (V_{Ax}, \preceq, lab_{Ax})$, $w_d = (V_d, \preceq, lab_d)$, $w_q^- = (W_q^-, \preceq, lab_q)$ and $w_q^+ = (W_q^+, \preceq, lab_q)$ such that the sets V_{Ax}, V_d, W_q^- and W_q^+ are pairwise disjoint. Furthermore, we extend $\prec$ by letting $s \prec s'$ for all $s \in W_q^-, s' \in W_q^+$ and $q \in Q$. We construct $J = Unf(\Delta) = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ as follows. Clauses a) to d) are essentially as in Proposition 3.24.

a) N is the set of finite nonempty sequences $(v_0, s_1, v_1, s_2, \ldots, s_k, v_k)$ such that:
$v_0 \in V_{Ax}, v_i \in V_{d_i}$ and $s_i \in W_{q_{i-1}}^- \cup W_{q_{i-1}}^+$ for $1 \leq i \leq k$, where
$q_0 = lab_{Ax}(v_0)$, $d_1 = lab_{q_0}(s_1)$, $q_1 = lab_{d_1}(v_1)$, $d_2 = lab_{q_1}(s_2)$, $\ldots$,
$q_i = lab_{d_i}(v_i)$, $d_{i+1} = lab_{q_i}(s_{i+1})$ for $1 \leq i \leq k-1$.

b) $(v_0, s_1, v_1, \ldots, s_k, v_k) \leq (v'_0, s'_1, v'_1, \ldots, s'_j, v'_j)$ if and only if:
$k \geq j$, $(v_0, s_1, v_1, \ldots, s_j) = (v'_0, s'_1, v'_1, \ldots, s'_j)$ and $v_j \preceq v'_j$ $(v_j, v'_j \in V_{d_j})$.

c) The axis A is the set of one-element sequences (v) for $v \in V_{Ax}$.

d) If $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$, the line $U(x)$ is the set of sequences $(v_0, s_1, v_1, s_2, \ldots, s_k, v)$ for $v \in V_{d_k}$; it belongs to $\mathcal{U}^-$ if $s_k \in W^-_{q_{k-1}}$ and to $\mathcal{U}^+$ if $s_k \in W^+_{q_{k-1}}$; in both cases, $\widehat{U(x)} = (v_0, s_1, v_1, \ldots, s_{k-1}, v_{k-1})$.

e) $x = (v_0, s_1, v_1, \ldots, s_k, v_k) \sqsubseteq y = (v'_0, s'_1, v'_1, \ldots, s'_j, v'_j)$ if and only if either $x \leq y$ or, for some $\ell < \{j, k\}$, we have
e.1) $(v_0, s_1, v_1, \ldots, v_\ell) = (v'_0, s'_1, v'_1, \ldots, v'_\ell)$ and $s_{\ell+1} \prec s'_{\ell+1}$,
e.2) or $(v_0, s_1, v_1, \ldots, s_\ell) = (v'_0, s'_1, v'_1, \ldots, s'_\ell)$, $s_{\ell+1} \in W^-_{q_\ell}$ and $v'_\ell \prec v_\ell$,
e.3) or $(v_0, s_1, v_1, \ldots, s_\ell) = (v'_0, s'_1, v'_1, \ldots, s'_\ell)$, $s'_{\ell+1} \in W^+_{q_\ell}$ and $v_\ell \prec v'_\ell$.
In Case e.1), x and y are in different directions of $z := (v_0, s_1, v_1, \ldots, v_\ell)$ that are not its central direction; in Case e.2), x is to the left of the central direction δ of z and $y \leq u$ where $u := (v_0, s_1, v_1, \ldots, v'_\ell)$ is here below z on δ; in Case e.3), y is to the right of the central direction δ' of u and $x \leq z$ where z is below u on δ'.

In order to prove that Δ describes J, we define $r : N \to Q$ and $\widetilde{r} : \mathcal{U}^- \cup \mathcal{U}^+ \to D$ as follows:
– if $x \in A$, then $x = (v)$ for some $v \in V_{Ax}$ and $r(x) := lab_{Ax}(v)$;
– if $x \in N$ has depth $k \geq 1$, then $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$ for some $v_0, s_1, \ldots, s_k, v_k$ and $r(x) := lab_{d_k}(v_k)$;
– if $U \in \mathcal{U}^- \cup \mathcal{U}^+$, then $U = U(x)$ for some $x = (v_0, s_1, v_1, \ldots, s_k, v_k)$, and $\widetilde{r}(U) := d_k$.

In the last case, as $d_k = lab_{q_{k-1}}(s_k)$, it depends only on s_k and v_{k-1} (via q_{k-1}). It follows that $\widetilde{r}(U)$ is the same if we consider U as $U(y)$ with $y = (v_0, s_1, v_1, \ldots, s_k, v)$ hence, is well-defined.

We check the four conditions of Definition 4.5(b). We have $(A, \leq, r) \simeq w_{Ax}$, hence (b.1) holds. For (b.2) and (b.3), we consider $x = (v_0, s_1, v_1, \ldots, s_k, v_k) \in N$. The sets U in $\mathcal{U}^x$ are those of the form $\{(v_0, s_1, v_1, \ldots, s_k, v_k, s, v) \mid v \in V_{d_{k+1}}\}$ for all $s \in W^-_{q_k} \cup W^+_{q_k}$ where $q_k = lab_{d_k}(v_k) = r(x)$, hence (b.2) and (b.3) hold.

For checking (b.4), we let $U = U(x)$ for some $x = (v_0, s_1, v_1, \ldots, s_k, v_k), k > 0$; then U is the set of sequences $(v_0, s_1, v_1, s_2, \ldots, s_k, v)$ such that $v \in V_{d_k}$ ordered by $\preceq$ on the last components. Hence, $(U, \leq, lab_{d_k})$ is isomorphic to w_{d_k}, which proves the property since $\widetilde{r}(U) := d_k$.

Unicity is proved as in Proposition 3.11. □

Proposition 4.7. *An SOJ-tree is MS definable if it is described by a regular SOJ-scheme.*

Proof. Similar to the proofs of Propositions 3.12 and 3.25. □

Note that, we need not the finiteness predicate as in Proposition 3.25 because we deal with arrangements that are linearly ordered structures, and not with labelled sets. Next we define an algebra $\mathbb{SOJT}$ with two sorts: $\boldsymbol{t}$ for SOJ-trees and $\boldsymbol{h}$ for SJ-hedges.

Definition 4.8 (Operations on SOJ-trees and SJ-hedges)**.**
– *Concatenation of SOJ-trees along axes.*

Let $J_1 = (N_1, \leq_1, \sqsubseteq_1, A_1, \mathcal{U}_1^-, \mathcal{U}_1^+)$ and $J_2 = (N_2, \leq_2, \sqsubseteq_2, A_2, \mathcal{U}_2^-, \mathcal{U}_2^+)$ be disjoint SOJ-trees. We define their concatenation as follows:

$$J_1 \bullet J_2 := (N_1 \uplus N_2, \leq, \sqsubseteq, A_1 \uplus A_2, \mathcal{U}_1^- \uplus \mathcal{U}_2^-, \mathcal{U}_1^+ \uplus \mathcal{U}_2^+) \text{ where}$$
$$x \leq y :\Longleftrightarrow x \leq_1 y \vee x \leq_2 y \vee (x \in N_1 \wedge y \in A_2),$$
$$x \sqsubseteq y :\Longleftrightarrow x \leq y \vee x \sqsubseteq_1 y \vee x \sqsubseteq_2 y,$$
$$\vee (x \bot y \wedge x \in N_1 \wedge y \in N_2 \wedge y \in U \in \mathcal{U}_2^+ \cap \mathcal{U}_2^{x \sqcup y})$$
$$\vee (x \bot y \wedge x \in N_2 \wedge y \in N_1 \wedge x \in U \in \mathcal{U}_2^- \cap \mathcal{U}_2^{x \sqcup y}), \text{ for some } U.$$

The relations $x \bot y$ and $x \sqcup y$ are relative to $\leq$. It is clear that $J_1 \bullet J_2$ is an SOJ-tree. Its axis is $A_1 \uplus A_2, \mathcal{U}^+ = \mathcal{U}_1^+ \uplus \mathcal{U}_2^+$ and $\mathcal{U}^- = \mathcal{U}_1^- \uplus \mathcal{U}_2^-$. *The empty SOJ-tree* is denoted by the nullary symbol $\Omega_{\boldsymbol{t}}$.

– *Extension of two SJ-hedges into a single SOJ-tree.*

Let $H_1 = (N_1, \leq_1, \sqsubseteq_1, \mathcal{U}_1)$ and $H_2 = (N_2, \leq_2, \sqsubseteq_2, \mathcal{U}_2)$ be disjoint SJ-hedges and $u \notin N_1 \uplus N_2$. Then:

$$ext_u(H_1, H_2) := (N_1 \uplus N_2 \uplus \{u\}, \leq, \sqsubseteq, \{u\}, \mathcal{U}_1, \mathcal{U}_2), \text{ where}$$
$$x \leq y :\Longleftrightarrow x \leq_1 y \vee x \leq_2 y \vee y = u,$$
$$x \sqsubseteq y :\Longleftrightarrow x \leq y \vee x \sqsubseteq_1 y \vee x \sqsubseteq_2 y \vee (x \in N_1 \wedge y \in N_2).$$

Clearly, $ext_u(H_1, H_2)$ is an SOJ-tree, where u has no central direction. When handling SOJ-trees and SJ-hedges up to isomorphism, we replace the notation $ext_u(H_1, H_2)$ by $ext(H_1, H_2)$.

– *The empty SJ-hedge* is denoted by the nullary symbol $\Omega_{\boldsymbol{h}}$.

– *Making an SOJ-tree into an SJ-hedge.*

This is done by the unary operation *mkh* such that, if $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ is an SOJ-tree, then

$$mkh(J) := (N, \leq, \sqsubseteq, \{A\} \uplus \mathcal{U}^- \uplus \mathcal{U}^+).$$

Similarly as *fgs*, this operation forgets some information, here, it merges three sets. Note that in $mkh(J)$, we distinguish neither $\mathcal{U}^-$ from $\mathcal{U}^+$ nor the axis A from the other lines.

– *The concatenation of two disjoint SJ-hedges.*

Let $H_1 = (N_1, \leq_1, \sqsubseteq_1, \mathcal{U}_1)$ and $H_2 = (N_2, \leq_2, \sqsubseteq_2, \mathcal{U}_2)$ be disjoint SJ-hedges. Their "horizontal" concatenation is:

$$H_1 \otimes H_2 := (N_1 \uplus N_2, \leq_1 \uplus \leq_2, \sqsubseteq, \mathcal{U}_1 \uplus \mathcal{U}_2) \text{ where}$$
$$x \sqsubseteq y :\Longleftrightarrow x \sqsubseteq_1 y \vee x \sqsubseteq_2 y \vee (x \in N_1 \wedge y \in N_2).$$

We let F'' be the 2-sorted signature $\{\bullet, \otimes, ext, mkh, \Omega_{\boldsymbol{t}}, \Omega_{\boldsymbol{h}}\}$ whose operation types are:

$$\bullet : \boldsymbol{t} \times \boldsymbol{t} \to \boldsymbol{t}, \qquad \Omega_{\boldsymbol{t}} : \boldsymbol{t}, \qquad ext : \boldsymbol{h} \times \boldsymbol{h} \to \boldsymbol{t},$$
$$\otimes : \boldsymbol{h} \times \boldsymbol{h} \to \boldsymbol{h}, \qquad \Omega_{\boldsymbol{h}} : \boldsymbol{h}, \qquad mkh : \boldsymbol{t} \to \boldsymbol{h}.$$

In addition, we have, as in Definitions 3.13 and 3.26:

– *Forgetting the structuring:* If $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ is an SOJ-tree, then $fgs(J) := (N, \leq, \sqsubseteq)$ is the underlying OJ-tree.

Definition 4.9 (The value of a term)**.** If u is an occurrence of a binary symbol in a term t, we denote by $s_1(u)$ its first son and by $s_2(u)$ the second one (cf. Definition 3.15). The value

$val(t) := (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ of a term $t \in T^\infty(F'')_{\boldsymbol{t}}$ is an SOJ-tree defined in a similar way[15] as for $t \in T^\infty(F')_{\boldsymbol{t}}$, cf. Definitions 3.15 and 3.28:

– $N := \mathrm{Occ}(t, ext)$,

– $x \leq y :\Longleftrightarrow x \leq_t w \leq_{lex} y$ for some $w \in N$ such that $w \approx y$,

– $A := \mathrm{Max}(t, ext, \varepsilon)$,

where $\approx$ is the equivalence relation on N defined as in Definition 3.15(a):

– $\mathcal{U}^-$ is the set of equivalence classes of $\approx$ of nodes in $\mathrm{Max}(t, ext, s_1(u))$ for some occurrence u of ext,

– $\mathcal{U}^+$ is the set of equivalence classes of $\approx$ of nodes in $\mathrm{Max}(t, ext, s_2(u))$ for some occurrence u of ext.

Hence, $U(x) \in \mathcal{U}^-$ if $x \leq_t s_1(\widehat{U(x)})$ and $U(x) \in \mathcal{U}^+$ if $x \leq_t s_2(\widehat{U(x)})$.

Next we define $x \sqsubseteq y :\Longleftrightarrow x \leq y$ or $x \perp y$ ($\perp$ is relative to $\leq$, not to $\leq_t$) and we have one of the following cases:

(i) $x \sqcup_t y$ is an occurrence of $\otimes$ or ext, $x \leq_t s_1(x \sqcup_t y)$ and $y \leq_t s_2(x \sqcup_t y)$,

(ii) $x \sqcup_t y$ is an occurrence of $\bullet$, $x \leq_t s_1(x \sqcup_t y)$ and $y \leq_t s_2(z)$ where z is the unique maximal occurrence of ext such that $y <_t z \leq_t s_2(x \sqcup_t y)$,

(iii) $x \sqcup_t y$ is an occurrence of $\bullet$, $y \leq_t s_1(x \sqcup_t y)$ and $x \leq_t s_1(z)$ where z is the unique maximal occurrence of ext such that $x <_t z \leq_t s_2(x \sqcup_t y)$.

If $t \in T^\infty(F'')_{\boldsymbol{h}}$ its value $val(t)$ is $(N, \leq, \sqsubseteq, \mathcal{U})$ with $(N, \leq, \sqsubseteq)$ defined as above and $\mathcal{U}$ as in Definition 3.28.

Claim. (1) The mapping val is a value mapping $T^\infty(F'') :\to \mathbb{SOJT}$.
(2) The transformation α of $\lfloor t \rfloor$ into $(N, \leq, \sqsubseteq)$ is an MS-transduction.

Proof. (1) is clear from the definitions and (2) holds because the conditions of Definition 4.9 are expressible in $\lfloor t \rfloor$ by MS formulas. □

Example 4.10. We now illustrate this definition. Figure 6 shows a term T where A, B, C and D are subterms of type $\boldsymbol{t}$ and E, F and G are subterms of type $\boldsymbol{h}$. They contain occurrences of ext that define nodes x, x', y, y', w, z and z' of $val(T)$.

The OJ-tree $val(T)$ is shown on Figure 7, where we designate by A, B, ..., G the trees and hedges defined by the terms $A, B, \ldots, G$. We have the following comparisons for $<$:

– $\{z, z', u\} < v$, because $\{z, z'\} <_T v$, $u <_{lex} v$ and $u \approx v$,

– $\{y, y', w\} < u$, because $\{y, y', w\} <_T u$,

– $x \leq \{u, v\}$ because $x \leq_T a <_{lex} \{u, v\}$ and $a \approx u \approx v$ where a is the root position of A,

– $v < x'$ if and only if x' is on X, the axis of B, because in this case, $v \approx x'$ and otherwise v and x' are incomparable with respect to $\leq$; in all cases we have $v <_{lex} x'$.

For $\sqsubset$ we have: $z \sqsubset y \sqsubset y' \sqsubset x \sqsubset w \sqsubset u \sqsubset z' \sqsubset v$ and $x' \sqsubset z$ if x' is to the left of X ; otherwise $v \sqsubset x'$. All inequalities for $<$ yield the corresponding inequalities for $\sqsubset$. We now compare z, y, y', x, w, z' that are pairwise incomparable for $<$.

– By Case (i) of Definition 4.9, we get $\{y, y'\} \sqsubset w, y \sqsubset y'$ and $z \sqsubset z'$.

– By Case (ii), we get $x \sqsubset w, \{x, w\} \prec z'$ and $\{y, y'\} \prec w$.

– By Case(iii) we get $\{z, y, y'\} \prec x$ and $z \prec \{y, y'\}$.

[15] Example 4.10 will illustrate this definition.

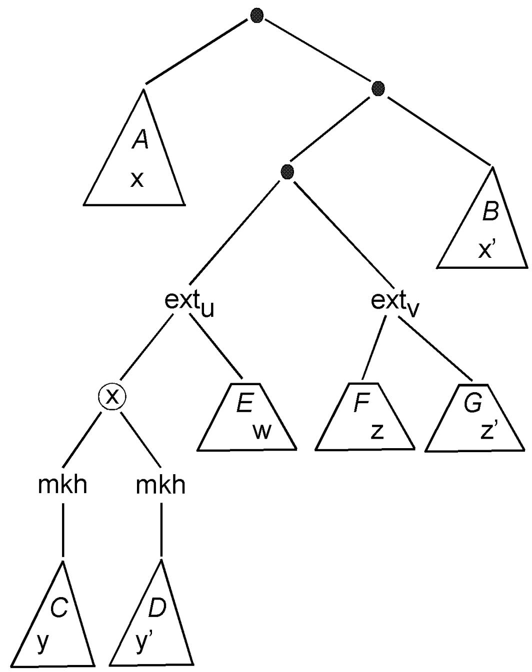

Figure 6: Term T of Example (4.10).

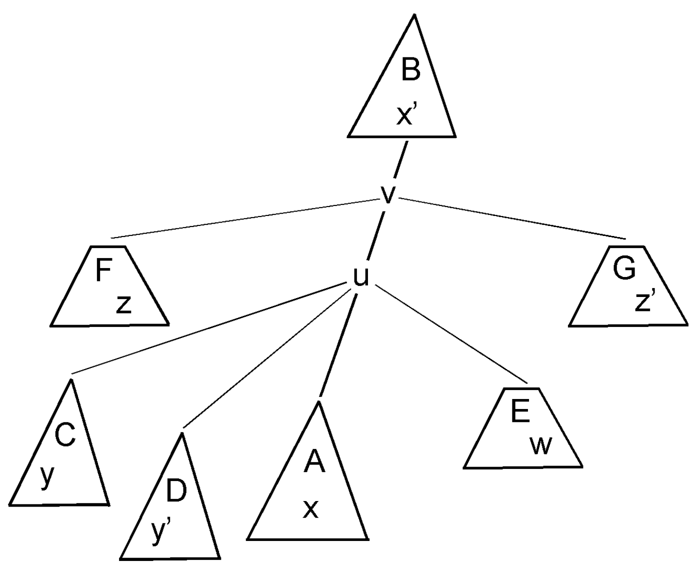

Figure 7: The OJ-tree $val(T)$ of Example (4.10).

Finally, if x' is to the left of X, then Case (iii) gives $x' \sqsubset z$, and if it to its right, then Case (ii) gives $z \sqsubset x'$.

Theorem 4.11. *The following properties of an OJ-tree J are equivalent :*

(1) *J is regular,*

(2) *J is described by a regular SOJ-scheme,*

(3) *J is MS definable.*

Proof. The proof is similar to that of Theorem 3.21. We only indicate some differences.

(1)$\Longrightarrow$(3): Follows from Proposition 4.7.

(2)$\Longrightarrow$(1): As observed in Definition 4.9 (cf. the claim), the mapping α that transforms the relational structure $\lfloor t \rfloor$ for t in $T^{\infty}(F'')_{\boldsymbol{t}}$ into the OJ-tree $(N, \leq, \sqsubseteq) = fgs(val(t))$ is an MS-transduction. Let $J = (N, \leq, \sqsubseteq)$ be an MS definable OJ-tree. It is, up to isomorphism, the unique model of an MS sentence β. The set of terms t in $T^{\infty}(F'')_{\boldsymbol{t}}$ such that $\alpha(\lfloor t \rfloor) \models \beta$ is thus MS definable, hence, it contains a regular term. This term denotes J, hence J is regular. $\square$

As in Corollaries 3.22 and 3.31, we deduce that the isomorphism problem for regular OJ-trees is decidable.

Remark 4.12 (An alternative notion of SOJ-tree)**.** We present a variant of Definition 4.2. If $J = (N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ is an SOJ-tree, Definition 4.2(c) shows that, for each $x \in N$, the partition $(\mathcal{U}^x \cap \mathcal{U}^-, \mathcal{U}^x \cap \mathcal{U}^+)$ of $\mathcal{U}^x$ is defined in a unique way from $\sqsubseteq$ and the structuring $\mathcal{U} := \{A\} \uplus \mathcal{U}^- \uplus \mathcal{U}^+$ of $(N, \leq)$, except if x has no central direction (cf. Proposition 4.3). This partition is useful only when x is the minimal element of A, denoted by $\min(A)$ when it exists. To see that, we consider J and another structuring of the same OJ-tree, $J' = (N, \leq, \sqsubseteq, A, \mathcal{U}'^-, \mathcal{U}'^+)$, such that $\mathcal{U}'^x = \mathcal{U}^x$ for each node $x \neq \min(A)$ and $\mathcal{U}^x \cap \mathcal{U}^- \neq \mathcal{U}'^x \cap \mathcal{U}^-$ if $x = \min(A)$. Let K be a nonempty SOJ-tree. Then $K \bullet J$ is not equal to $K \bullet J'$.

We could thus define an SOJ-tree as a tuple $S = (N, \leq, \sqsubseteq, \mathcal{U}', \mathcal{U}^-_{Ax}, \mathcal{U}^+_{Ax})$ such that $(N, \leq, \sqsubseteq)$ is an OJ-tree, $\mathcal{U} := \mathcal{U}' \uplus \mathcal{U}^-_{Ax} \uplus \mathcal{U}^+_{Ax}$ is a structuring of $(N, \leq)$ with axis A (belonging to $\mathcal{U}'$), $\mathcal{U}^-_{Ax} \uplus \mathcal{U}^+_{Ax} = \emptyset$ if A has no minimal element, and, otherwise, $\mathcal{U}^-_{Ax} \uplus \mathcal{U}^+_{Ax} = \mathcal{U}^{\min(A)}$ and $U \sqsubset U'$ for all $U \in \mathcal{U}^-_{Ax}$ and $U' \in \mathcal{U}^+_{Ax}$. Then, the structure S corresponding to an SOJ-tree $(N, \leq, \sqsubseteq, A, \mathcal{U}^-, \mathcal{U}^+)$ is $(N, \leq, \sqsubseteq, \mathcal{U}', \mathcal{U}^-_{Ax}, \mathcal{U}^+_{Ax})$ where:

– $\mathcal{U}^-_{Ax} := \mathcal{U}^- \cap \mathcal{U}^{\min(A)}$ and $\mathcal{U}^+_{Ax} := \mathcal{U}^+ \cap \mathcal{U}^{\min(A)}$ if A has a minimal element,
– $\mathcal{U}^-_{Ax} := \mathcal{U}^+_{Ax} := \emptyset$ otherwise, and, in both cases,
– $\mathcal{U}' := \{A\} \cup \mathcal{U}^- \cup \mathcal{U}^+ - (\mathcal{U}^-_{Ax} \cup \mathcal{U}^+_{Ax})$.

It is not difficult conversely to construct $(A, \mathcal{U}^-, \mathcal{U}^+)$ from $(N, \leq, \sqsubseteq, \mathcal{U}', \mathcal{U}^-_{Ax}, \mathcal{U}^+_{Ax})$ and to redefine the operations of Definition 4.8 in terms of the structures S as above. This alternative definition of SOJ-trees contains no redundant information. However, we found the initial definition easier to handle in our logical setting.

5. Quasi-trees

Quasi-trees can be viewed intuitively as "undirected join-trees". As in [11], we define them in terms of a ternary betweenness relation. Their use for defining rank-width is reviewed at the end of the section.

Definition 5.1 (Betweenness)**.**

(a) Let $L = (X, \leq)$ be a linear order. Its *betweenness relation* is the ternary relation on X such that $B_L(x, y, z)$ holds if and only if $x < y < z$ or $z < y < x$. It is empty if X has less than 3 elements.

(b) If T is a tree, its *betweenness relation* is the ternary relation on N_T, such that $B_T(x, y, z)$ holds if and only if x, y, z are pairwise distinct and y is on the unique path between x and z. If R is a rooted tree and $T = Und(R)$ is the tree obtained from R by forgetting its root and edge directions, then $B_T(x, y, z) :\Longleftrightarrow x, y, z$ are pairwise distinct and, either $x <_R y \leq_R x \sqcup_R z$ or $z <_R y \leq_R x \sqcup_R z$.

(c) If B is a ternary relation on a set X, and $x, y \in X$, then $[x, y]_B := \{x, y\} \cup \{z \in X \mid B(x, z, y)\}$. This set is finite if $B = B_T$ for some tree T.

Proposition 5.2 (cf. [11])**.**

(1) *The betweenness relation B of a linear order $(X, \leq)$ satisfies the following properties for all $x, y, z, u \in X$.*
A1: $B(x, y, z) \Rightarrow x \neq y \neq z \neq x$.
A2: $B(x, y, z) \Rightarrow B(z, y, x)$.
A3: $B(x, y, z) \Rightarrow \neg B(x, z, y)$.
A4: $B(x, y, z) \wedge B(y, z, u) \Rightarrow B(x, y, u) \wedge B(x, z, u)$.
A5: $B(x, y, z) \wedge B(x, u, y) \Rightarrow B(x, u, z) \wedge B(u, y, z)$.
A6: $B(x, y, z) \wedge B(x, u, z) \Rightarrow y = u \vee [B(x, u, y) \wedge B(u, y, z)] \vee [B(x, y, u) \wedge B(y, u, z)]$.
A7': $x \neq y \neq z \neq x \Rightarrow B(x, y, z) \vee B(x, z, y) \vee B(y, x, z)$.

(2) *The betweenness relation B of a tree T satisfies the properties A1-A6 for all x, y, z, u in N_T together with the following weakening of A7':*
A7: $x \neq y \neq z \neq x \Rightarrow \begin{array}{l} B(x, y, z) \vee B(x, z, y) \vee B(y, x, z) \vee \\ \exists w.(B(x, w, y) \wedge B(y, w, z) \wedge B(x, w, z)) \end{array}.$

Proposition 5.3. *Let B be a ternary relation on a set X that satisfies properties A1-A7' for all x, y, z, u in X. Let a and b be distinct elements of X. There is a unique linear order $L = (X, \leq)$ such that $a < b$ and $B_L = B$. It is quantifier-free definable in the logical structure (X, B, a, b).*

Proof. Let X, B, a, b be as in the statement. Let us enumerate X as $x_1 = a$, $x_2 = b$, $x_3, \ldots, x_n, \ldots$ Let $X_n := \{x_1, \ldots, x_n\}$ for $n \geq 3$. Observe that $B \upharpoonright X_n$ satisfies properties A1-A7'. We prove by induction on n the existence and unicity of a linear order $L_n = (X_n, \leq)$ such that $a < b$ and $B_{L_n} = B \upharpoonright X_n$.

– *Basis*: $n = 3$. The conclusion follows from A7'.
– *Induction case*: We assume the conclusion true for n.

Claim. If $B(x_i, x_{n+1}, x_j)$ holds for some $i < j \leq n$, then, there is a unique pair k, l such that $k < l \leq n$, $B(x_k, x_{n+1}, x_l)$ holds and $[x_k, x_l]_L \cap X_{n+1} = \{x_k, x_{n+1}, x_l\}$.

Proof. Assume that $x_m \in [x_i, x_j]_L \cap X_{n+1} - \{x_i, x_{n+1}, x_j\}$, which implies $m \leq n$. Then, by A6, we have $B(x_i, x_{n+1}, x_m)$ or $B(x_m, x_{n+1}, x_j)$ and we can repeat the argument with (x_i, x_m) or (x_m, x_j) instead of (i, j). Furthermore, the considered set, $[x_i, x_m]_L \cap X_{n+1}$ or $[x_m, x_j]_L \cap X_{n+1}$ has less elements than $[x_i, x_j]_L \cap X_{n+1}$.Hence, we must obtain k, l such that $[x_k, x_l]_L \cap X_{n+1} = \{x_k, x_{n+1}, x_l\}$ as desired. □

In this case, there is a unique way to extend L_n into L_{n+1}: we put x_{n+1} between x_k and x_l. There is another case.

Claim. If $B(x_i, x_{n+1}, x_j)$ holds for no $i < j \leq n$, then there is a unique k such that $k \leq n$, $B(x_l, x_k, x_{n+1})$ holds for some $l \leq n$, and $[x_k, x_{n+1}]_L \cap X_{n+1} = \{x_k, x_{n+1}\}$. The element x_k is *extremal* in L_n, that is, either maximal or minimal. □

The proof is similar. In this case, there is a unique way to extend L_n into L_{n+1}: we put x_{n+1} after x_k if it is maximal in L_n or before it if it is minimal. By taking the union of all orders L_n, we get the desired and unique linear order on X, that we will denote by $\leq_{a,b}$. We now define it by a first-order formula.

We first observe a particularly simple case. If there are no $u, v \in X$ such that $B(u, b, v)$ holds, we have $x \leq_{a,b} y \Longleftrightarrow x = y \vee y = b \vee B(x, y, b))$. A similar description can be given if there are no u, v such that $B(u, a, v)$ holds. From (X, B, a, b) as in the statement, we define the following binary relation:

$$Z(x,y) :\Longleftrightarrow x \neq y \wedge$$
$$[(B(x,a,b) \wedge \neg B(y,x,a)) \vee (x = a \wedge \neg B(y,a,b)) \vee (B(a,x,b) \wedge \neg B(y,x,b)) \vee$$
$$(x = b \wedge B(a,b,y)) \vee (B(a,b,x) \wedge B(b,x,y))].$$

It is easy to see that $x <_{a,b} y$ implies $Z(x, y)$. In particular, $Z(a, b)$ holds by the clause $x = a \wedge \neg B(y, a, b)$ with $y = b$. For the converse, assume that $Z(x, y)$ holds and $x <_{a,b} y$ does not. Then, we have $y <_{a,b} x$ because $\leq_{a,b}$ is a linear order. By looking at the different relative positions of x, y, a and b, we get a contradiction. Hence $x \leq_{a,b} y$ if and only if $x = y \vee Z(x, y)$, which is expressed by a quantifier-free formula $\xi(a, b, x, y)$. □

Remark 5.4. If there are no $u, v \in X$ such that $B(u, b, v)$ holds, then:

$$Z(x,y) \Longleftrightarrow x \neq y \wedge [(B(x,a,b) \wedge \neg B(y,x,a)) \vee (x = a \wedge \neg B(y,a,b)) \vee$$
$$(B(a,x,b) \wedge \neg B(y,x,b))]$$

which is equivalent to $y = b \vee B(x, y, b))$ as one can (painfully) check by using axioms A1-A7'.

Definition 5.5 (Quasi-trees)**.**

(a) A *quasi-tree* is a structure $S = (N, B)$ such that B is a ternary relation on N, called the set of *nodes*, that satisfies conditions A1-A7 (a definition from [11]). To avoid uninteresting special cases, we also require that N has at least 3 nodes.

Lemma 11 of [11] shows that in a quasi-tree, the four cases of the conclusion of A7 are exclusive and that, in the fourth one, there is a unique node w satisfying $B(x, w, y) \wedge B(y, w, z) \wedge B(x, w, z)$, which we denote by $M_S(x, y, z)$.

A *leaf* (of S) is a node z such that $B(x, z, y)$ holds for no x, y. A *line* is set of nodes L such that $[x, y]_B \subseteq L$ if $x, y \in L$ and $(L, B \upharpoonright L)$ satisfies A7'. We say that S is *discrete* if each set $[x, y]_B$ is finite. A quasi-tree $S = (N, B)$ is *a subquasi-tree of* a quasi-tree $S' = (N', B')$, which we denote by $S \subseteq S'$, if $N \subseteq N'$ and $B = B' \upharpoonright N$. This condition implies that $M_S = M_{S'} \upharpoonright N$.

(b) From a join-tree $J = (N, \leq)$, we define a ternary relation B_J on N by:

$$B_J(x,y,z) :\Longleftrightarrow x \neq y \neq z \neq x \text{ and } (x < y \leq x \sqcup z) \vee (z < y \leq x \sqcup z).$$

Proposition 5.6.

(1) *The structure $qt(J) := (N, B_J)$ associated with a join-tree $J = (N, \leq)$ having at least 3 nodes is a quasi-tree. Every line of J is a line of $qt(J)$. If J is a subjoin-tree of J', then $qt(J)$ is a subquasi-tree of $qt(J')$.*

(2) *Every quasi-tree S is $qt(J)$ for some join-tree J.*

(3) *A quasi-tree is discrete if and only if it is $qt(J)$ for some rooted tree J.*

Proof.

(1) Let $J = (N, \leq)$ be a join-tree with at least 3 nodes.

If it is finite, then it is $(N_T, \leq_T)$ for a finite rooted tree T, and $qt(J)$ is a finite quasi-tree by Proposition 5.2(b).

Otherwise consider distinct elements x, y, z, u of N. We want to prove that they satisfy A1-A7. There is a set $N' \subseteq N$ of cardinality at most 7 that contains x, y, z, u and is closed under $\sqcup$. Then $J' = (N', \leq\restriction N')$ is a finite join-tree, $J' \subseteq J$ and $qt(J') = (N', B_J \restriction N')$ is a quasi-tree by the initial observation, so that x, y, z, u satisfy A1-A7 for $B = B_{J'}$ hence, also for B_J. (The node w that may be necessary to satisfy A7 may have to be chosen in the set $\{x \sqcup y, x \sqcup z, x \sqcup u, \dots\}$). As x, y, z, u are arbitrary, A1-A7 hold for B_J and all $x, y, z, u \in N$. Hence, (N, B_J) is a quasi-tree.

That every line of J is a line of $qt(J)$ follows from the definitions. (The converse does not hold.) The assertion about subjoin-trees is also easy to prove.

(2) Let $S = (N, B)$ be a quasi-tree and r be any element of N. We define $x \leq_r y :\Longleftrightarrow y \in [x, r]_B$. Then $(N, \leq_r)$ is a join-tree J with root r and $S = qt(J)$ by Lemma 14 of [11].

(3) A quasi-tree $S = qt(J)$ is discrete if J is rooted tree. Conversely, if S is a discrete quasi-tree, then $S = qt(T)$ for some tree T by Proposition 17 of [11]. By choosing any node as a root, one makes T into a rooted tree, and its betweenness relation is that of T. □

We say that a quasi-tree S is *described* by an SJ-scheme if this scheme describes a join-tree J such that $qt(J) = S$. It is *regular* if it is $qt(J)$ for some regular join-tree J.

Proposition 5.7. *A quasi-tree is MS_{fin}-definable if it is described by a regular SJ-scheme.*

Proof. We first prove a technical result.

Claim. There exists a first-order formula $\mu(L, a, b, u, v)$ such that, for every join-tree $J = (N, \leq)$, if $S = qt(J) = (N, B)$, then there is a subset L of N and elements a, b of N such that, for every $u, v \in N$, $(N, B) \models \mu(L, a, b, u, v)$ if and only if $u \leq v$.

Proof of the claim. The formula $\mu(L, a, b, u, v)$ will be defined as $u = v \vee \mu_1(L, a, b, u, v) \vee \mu_2(L, a, b, u, v)$ so as to handle two exclusive cases relative to $J = (N, \leq)$.

– Case $J = (N, \leq)$ has a root r. Then, $u < v$ if and only if $v = r$ or $B(u, v, r)$. Hence, we define $\mu_1(L, a, b, u, v)$ as $L = \emptyset \wedge a = b \wedge (v = b \vee B(u, v, b))$.
– Case J has no root. It has a line L that is *upwards closed*, i.e., such that $y \in L$ if $x \leq y$ and $x \in L$. This line has no maximal element (otherwise its maximal element would be a root of J) and is infinite. Moreover, for every $u \in N$, we have $u < x$ for some $x \in L$ (to prove that, consider $u \sqcup y$ where y is any element of L). For all $u, v \in N$ we have:

$$u < v \Longleftrightarrow \exists x, y \in L[x < y \wedge B(u, v, x) \wedge B(v, x, y)].$$

If $u < v$ we have $u < v < x < y$ for some x, y in L. Hence, we have $B(u, v, x) \wedge B(v, x, y)$. Assume for the converse that $x < y \wedge B(u, v, x) \wedge B(v, x, y)$ for some x, y in L. We first prove that $u, v < x$. Since $B = B_J$, $B(v, x, y) \Longleftrightarrow v < x \leq v \sqcup y \vee y < x \leq y \sqcup v$. As $x < y$, we must have $v < x$. Axiom A4 gives $B(u, x, y)$, from which we get similarly $u < x$. From $B(u, v, x)$ we get $u < v \leq u \sqcup x$ or $x < v \leq x \sqcup v$. As $v < x$, we have $u < v$. Let $a, b \in L$ such that $a < b$. Proposition 5.3 is applicable to $(L, B \restriction L)$ that satisfies Conditions A1-A7'. Hence the quantifier-free formula $\xi(a, b, x, y)$ defines $x <_{a,b} y$ for $x, y \in L$. We define $\mu_2(L, a, b, u, v)$ as $\exists x, y \in L[a, b \in L \wedge \xi(a, b, x, y) \wedge B(u, v, x) \wedge B(v, x, y)]$.

We now complete the proof. If $J = (N, \leq)$ has a root r, we choose $L = \emptyset \wedge a = b = r$, $\mu_2(L, a, b, u, v)$ is false and $\mu_1(L, a, b, u, v)$ is equivalent to $u < v$. If J has no root, we let L

be an upwards closed line, and $a, b \in L$ such that $a < b$. Then $\mu_1(L, a, b, u, v)$ is false and $\mu_2(L, a, b, u, v)$ is equivalent to $u < v$. □

We let $\mu'(L, a, b, u, v)$ be $u = v \vee \mu(L, a, b, u, v)$. For proving the main assertion, we let $S = qt(J)$ be a quasi-tree defined from a regular join-tree J and ψ be the MS_{fin} sentence expressing that a structure $(N, \leq)$ is a join-tree isomorphic to J; this sentence exists by Theorem 3.30. Let φ be the MS_{fin} sentence expressing that a structure (D, B) is a quasi-tree that satisfies $\exists L, a, b(\psi' \wedge$"$B$ is the betweenness relation of the order relation $\leq$ defined by μ'") where ψ' is obtained from ψ by replacing each atomic formula $x \leq y$ by $\mu'(L, a, b, x, y)$.

Then $qt(J)$ satisfies φ by the claim. If conversely, (D, B) satisfies φ for some L, a and b, then $(D, \leq)$ is a join-tree J', where $x \leq y$ is defined in (D, B) by $\mu'(L, a, b, x, y)$ (because ψ' holds), $(D, B) = qt(J')$ (because B is the betweenness relation of $\leq$) and $J' \simeq J$ (because of ψ'). Hence, $(D, B) \simeq qt(J)$. Hence $qt(J)$ is characterized by φ up to isomorphism. □

The next theorem establishes a converse. As algebra for quasi-trees, we take the algebra $\mathbb{SJT}$ of join-trees together with the (external) *operation qt* (similar to *fgs*) that makes a join-tree into a quasi-tree.

Theorem 5.8. *The following properties of a quasi-tree S are equivalent:*
(1) *S is regular,*
(2) *S is described by a regular SJ-scheme,*
(3) *S is* MS_{fin} *definable.*

Furthermore, the isomorphism problem of regular quasi-trees is decidable.

Proof.
(1)⇒(2): The proof is similar to that of Theorem 3.21.
(2)⇒(3): By Proposition 5.7.
(3)⇒(1): The mapping α that transforms the relational structure $\lfloor t \rfloor$ for t in $T^{\infty}(F')_{\boldsymbol{t}}$ into the quasi-tree $S = qt(fgs(val(t)))$ is an MS-transduction by Definitions 4.9 (cf. the claim) and 5.5(b). The proof continues as in Theorem 3.21.

The decidability of the isomorphism problem is as in Corollary 3.22. □

We make these results more precise for subcubic quasi-trees: they are useful for defining the rank-width of countable graphs, as we will recall.

Definition 5.9 (Directions)**.** Let $S = (N, B)$ be a quasi-tree and x a node of S.

(a) We say that $y, z \in N - \{x\}$ are the *same direction relative to* x (or *of* x) if, either $y = z$ or $B(y, z, x)$ or $B(z, y, x)$ or $B(y, u, x) \wedge B(z, u, x)$ for some node u (a definition from [11]). Equivalently, $y \sqcup_x z <_x x$ ($<_x$ is as in Proposition 5.6(2)). Hence, if $B(y, x, z)$ holds, then y and z are in different directions relative to x. This relation is an equivalence, denoted by $y \sim_x z$, and its classes are the *directions of* x.
(b) The *degree* of x is the number of classes of $\sim_x$. A node has degree 1 if and only if it is a leaf. We say that S is *subcubic* if its nodes have degree at most 3. If $S = qt(T)$ for a tree T, then a direction of x is associated with each neighbour y of x and is the set of nodes of the connected component of $T - \{x\}$ that contains y.
(c) If $S = qt(J)$ for a join tree $J = (N, \leq)$, then, the directions of x in S are those of x in J together with the set $N -]-\infty, x]$ if it is not empty. It follows that S is subcubic if J is a BJ-tree.

Lemma 5.10. *Every subcubic quasi-tree is* $qt(fgs(J))$ *for some SBJ-tree* J.

Proof. Let S be a subcubic quasi-tree. Then $S = qt(J)$ for some join-tree J. We choose a maximal line L of J and $a, b \in L$ such that $a < b$. By Proposition 5.7, the partial order $\leq$ of J is defined by $\mu'(L, a, b, x, y)$. The method of Proposition 3.5 with $U_0 := L$, gives structuring K of J, making it into an SBJ-tree as defined in Definition 3.8. □

Theorem 5.11. *The following properties of a subcubic quasi-tree S are equivalent :*
(1) *S is regular,*
(2) *S is described by a regular SBJ-scheme,*
(3) *S is MS definable.*

Proof. By Lemma 5.10 and Proposition 3.19, every subcubic quasi-tree S is $qt(fgs(val(t)))$ for some term $t \in T^{\infty}(F)$.

Property (1) means that $S = qt(fgs(val(t)))$ for some regular term in $T^{\infty}(F')_{\mathbf{t}}$. Let (1') mean that $S = qt(fgs(val(t)))$ for some regular term in $T^{\infty}(F)$. Then (1')$\Longrightarrow$(2) by the similar implication in Theorem 3.21.

(2)$\Longrightarrow$(3) by the similar implication in Theorem 3.21 and the observation that, in a quasi-tree S, the SBJ-trees J such that $S = qt(fgs(J))$ can be specified by MS formulas in terms of a 5-tuple (A, N_0, N_1, a, b) satisfying the formula $\varphi'(A, N_0, N_1, a, b)$ of the proof of Proposition 5.7.

(3)$\Longrightarrow$(1') by the observation that the mapping α that transforms the relational structure $\lfloor t \rfloor$ for t in $T^{\infty}(F)$ into the subcubic quasi-tree $qt(fgs(val(t)))$ is an MS-transduction. The proof goes then as in Theorem 3.21.

The implication (1')$\Longrightarrow$(1) is trivial and (1) implies that S is MS_{fin} definable by Proposition 5.7. But a term $t \in T^{\infty}(F)$ that defines S is MS definable, and the relational structure representing a term has an MS definable linear order. It follows that S has an MS definable linear order, hence that S is MS definable by the facts recalled in Section 2. □

We now review the use of quasi-trees for rank-width [11], a width measure first defined and investigated in [21] and [22] for finite graphs.

Definition 5.12 (Rank-width for countable graphs)**.** We consider finite or countable, loop-free, undirected graphs without parallel edges. The *adjacency matrix* of such a graph G is $M_G : V_G \times V_G \to \{0, 1\}$ with $M_G[x, y] = 1$ if and only if x and y are adjacent. If U and W are disjoint sets of vertices, $M_G[U, W]$ is the matrix that is the restriction of M_G to $U \times W$. The *rank* (over $GF(2)$) of $M_G[U, W]$ is the maximum cardinality of an independent set of rows (equivalently, of columns) and is denoted by $rk(M_G[U, W])$; it belongs to $\mathbb{N} \cup \{\omega\}$. We take $rk(M_G[\emptyset, W]) = rk(M_G[U, \emptyset]) := 0$. If $X \uplus Y$ is infinite, then $rk(M_G[X, Y]) = \sup\{rk(M_G[U, W]) \mid U \subseteq X, W \subseteq Y$ and, U and W are finite$\}$.

A *discrete layout* of a graph G is an unrooted tree T of maximal degree 3 whose set of leaves is V_G. Its *rank* is the least upper-bound of the ranks $rk(M_G[X \cap V_G, X^c \cap V_G])$ such that X and $X^c := N_T - X$ are the two connected components of T minus one edge. The *discrete rank-width* of G, denoted by $rwd^{dis}(G)$, is the smallest rank of a discrete layout. If G is finite, this value is the rank-width defined in [21]. By using for countable graphs G quasi-trees with nodes of maximal degree 3 instead of trees, one obtains their *rank-with* $rwd(G)$ (cf. [11] for details). We have $rwd(G) \leq rwd^{dis}(G)$. The notation $G \subseteq_i H$ means that G is an induced subgraph of H.

Theorem 5.13 (cf. [11])**.** *For every graph G:*
(1) *if $H \subseteq_i G$, then $rwd(H) \leq rwd(G)$ and $rwd^{dis}(H) \leq rwd^{dis}(G)$,*
(2) Compactness*: $rwd(G) = \mathrm{Sup}\{rwd(H) \mid H \subseteq_i G$ and H is finite$\}$,*
(3) Compactness with gap *: $rwd^{dis}(G) \leq 2 \cdot \mathrm{Sup}\{rwd(H) \mid H \subseteq_i G$ and H is finite$\}$.*

The *gap function* in (3) is $n \mapsto 2n$, showing a weak form of compactness. The proof of (2) uses Koenig's Lemma, and consists in taking G as the union of an increasing sequence of finite induced subgraphs. The desired layout of G is obtained from an increasing sequence of finite layouts of finite induced subgraphs where nodes are successively added. The union of these layouts is in general a quasi-tree and not a tree.

6. Conclusion

We have defined regular join-trees of different kinds and regular quasi-trees from regular terms. These terms have finitary descriptions. Other infinite terms have finitary descriptions: the algebraic ones [9] and more generally, those of Caucal's hierarchy [5]. Such terms also yield effective (algorithmically usable) notions of join-trees and quasi-trees. It is unclear whether the corresponding isomorphism problems are decidable[16].

The article [7] establishes that a *set of arrangements* is recognizable if and only if it is MS definable. One might wish to extend this result to sets of join-trees and quasi-trees. An appropriate notion of recognizability must be defined.

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