ON SESSIONS AND INFINITE DATA

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ABSTRACT. We define a novel calculus that combines a call-by-name functional core with session-based communication primitives. We develop a typing discipline that guarantees both normalisation of expressions and progress of processes and that uncovers an unexpected interplay between evaluation and communication.

1. INTRODUCTION

Infinite computations have long lost their negative connotation. Two paradigmatic contexts in which they appear naturally are reactive systems [22, 1] and lazy functional programming. The former contemplates the use of infinite computations in order to capture *non-transformational* computations, that is computations that cannot be expressed in terms of transformations from inputs to outputs; rather, computations of reactive systems are naturally modelled in terms of ongoing interactions with the environment. Lazy functional programming is acknowledged as a paradigm that fosters software modularity [17] and enables programmers to specify computations over possibly infinite data structures in elegant and concise ways. Nowadays, the synergy between these two contexts has a wide range of potential applications, including stream-processing networks, real-time sensor monitoring, and internet-based media services.

Nonetheless, not all diverging programs – those engaged in an infinite sequence of possibly intertwined computations and communications – are necessarily useful. There exist degenerate forms of divergence where programs do not produce results, in terms of observable data or performed communications. We investigate this issue by proposing a calculus for expressing computations over possibly infinite data types and involving message passing. The

calculus – called SID, after Sessions with Infinite Data – combines a call-by-name functional core (inspired by Haskell) with multi-threading and session-based communication primitives.

In the remainder of this section we provide an informal introduction to SID and its key features by means of a few examples. The formal definition of the calculus, of the type system, and its properties are given in the rest of the paper. A simple instance of computation producing an infinite data structure is given by

$$\texttt{from } x \ = \langle x, \texttt{from } (x+1)\rangle$$

where the function $\texttt{from}$ applied to a number n produces the stream (infinite list)

$$\langle n, \langle n+1, \langle n+2, \cdots \rangle\rangle\rangle$$

of integers starting from n. We can think of this list as abstracting the frames of a video stream or the samples taken from a sensor.

The key issue we want to address is how infinite data can be exchanged between communicating threads. The most straightforward way of doing this in SID is to take advantage of lazy evaluation. For instance, the SID process

$$x \Leftarrow \left(\texttt{send } c^+ \ (\texttt{from } 0)\right) \ggg f \quad | \quad y \Leftarrow \texttt{recv } c^- \ggg g$$

represents two threads x and y running in parallel and connected by a session c, of which thread x owns one endpoint c^+ and thread y the corresponding peer c^-. Thread x sends a stream of natural numbers on c^+ and continues as $f \ c^+$, where f is left unspecified. Thread y receives the stream from c^- and continues as $(g \ \langle \texttt{from } 0, c^-\rangle)$. The *bind* operator $\_ \ggg \_$ models sequential composition and has the same semantics as in Haskell, i.e. it passes the result of performing the left action to the (parametrised) right action. The result of sending a message on the endpoint a^+ is the endpoint itself, while the result of receiving a message from the endpoint a^- is a pair consisting of the message and the endpoint. In this example, the *whole stream* is sent *at once* in a single interaction between x and y. This behaviour is made possible by the fact that SID evaluates expressions *lazily*: the message ($\texttt{from } 0$) is not evaluated until it is used by the receiver.

In principle, exchanging "infinite" messages such as ($\texttt{from } 0$) between different threads is no big deal. In the real world, though, this interaction poses non-trivial challenges: the message consists in fact of a mixture of data (the parts of the messages that have already been evaluated, like the constant 0) and code (which lazily computes the remaining parts when necessary, like $\texttt{from}$). This observation suggests an alternative, more viable modelling of this interaction whereby the sender unpacks the stream element-wise, sends each element of the stream as a separate message, and the receiver gradually reconstructs the stream as each element arrives at destination. This modelling is intuitively simpler to realise (especially in a distributed setting) because the messages exchanged at each communication are basic values rather than a mixture of data and code. In SID we can model this as a process

$$prod \Leftarrow \texttt{stream}_0 \ c^+ \ (\texttt{from } 0) \quad | \quad cons \Leftarrow \texttt{display}_0 \ c^-$$

where the functions $\texttt{stream}_0$ and $\texttt{display}_0$ are defined as:

$$
\begin{aligned}
\texttt{stream}_0 \ y \ \langle x, xs \rangle \ &= \ \texttt{send } y \ x \ggg \lambda y'.\texttt{stream}_0 \ y' \ xs \\
\texttt{display}_0 \ y \ &= \ \texttt{recv } y \ggg \lambda\langle z, y'\rangle.\texttt{display}_0 \ y' \ggg \lambda zs.g \ \langle z, zs \rangle
\end{aligned}
\tag{1.1}
$$

The syntax $\lambda\langle \_, \_ \rangle.e$ is just syntactic sugar for a function that performs pattern matching on the argument, which must be a pair, in order to access its components. In $\texttt{stream}_0$, pattern matching is used for accessing and sending each element of the stream separately.

In $\mathtt{display_0}$, the pair $\langle z, y' \rangle$ contains the received head z of the stream along with the continuation y' of the session endpoint from which the element has been received. The recursive call $\mathtt{display_0}\ y'$ retrieves the tail of the stream zs, which is then combined with the head z and passed as an argument to g.

The code of $\mathtt{display_0}$ looks reasonable at first, but conceals a subtle and catastrophic pitfall: the recursive call $\mathtt{display_0}\ y'$ is in charge of receiving the *whole* tail zs, which is an infinite stream itself, and therefore it involves an infinite number of synchronisations with the producing thread! This means that $\mathtt{display_0}$ will hopelessly diverge striving to receive the whole stream before releasing control to g. This is a known problem which has led to the development of primitives (such as $\mathtt{unsafeInterleaveIO}$ in Haskell or $\mathtt{delayIO}$ in [30]) that allow the execution of I/O actions to interleave with their continuation. In this paper, we call such primitive $\mathtt{future}$, since its semantics is also akin to that of *future variables* [33]. Intuitively, an expression $\mathtt{future}\ e \gg= \lambda x.f\ x$ allows to evaluate $f\ x$ even if e, which typically involves I/O, has not been completely performed. The variable x acts as a placeholder for the result of e; if f needs to inspect the structure of x, its evaluation is suspended until e produces enough data. Using $\mathtt{future}$ we can amend the definitions of $\mathtt{stream_0}$ and $\mathtt{display_0}$ thus

$$\begin{aligned}
\mathtt{stream}\ y\ \langle x, xs \rangle &= \mathtt{send}\ y\ x \gg= \lambda y'.\mathtt{future}\ (\mathtt{stream}\ y'\ xs) \\
\mathtt{display}\ y &= \mathtt{recv}\ y \gg= \lambda \langle z, y' \rangle.\mathtt{future}\ (\mathtt{display}\ y') \gg= \lambda zs.g\ \langle z, zs \rangle
\end{aligned} \qquad (1.2)$$

where $\mathtt{display}$ allows g to start processing the stream as its elements come through the connection with the producer thread. The type system that we develop in this paper allows us to reason on sessions involving the exchange of infinite data and when such exchanges can be done "productively". In particular, our type system flags $\mathtt{stream_0}$ and $\mathtt{display_0}$ in (1.1) as ill-typed, while it accepts $\mathtt{stream}$ and $\mathtt{display}$ in (1.2) as well-typed. To do so, the type system uses a modal operator $\bullet$ which guarantees that the number of communications is finite if the number of generated threads is finite. As hinted by the examples (1.1) and (1.2), this operator plays a major role in the type of $\mathtt{future}$.

We remark that SID does not force exchanged messages to be basic, nor does it prevent exchanging infinite streams in one shot. The purpose of SID is to enable the modelling of systems where communications and infinite data structures are intertwined and to study a typing discipline that guarantees the preservation of productivity in this setting.

Contributions and Outline. The SID calculus, defined in Section 2, combines in an original way standard constructs from the λ-calculus and process algebras with session types in the spirit of [16, 14]. The type system, given in Section 3, has the novelty of using the modal operator $\bullet$ to control the recursion of programs that perform communications. To the best of our knowledge, the interplay between $\bullet$ and the type of $\mathtt{future}$ is investigated here for the first time. The properties of our framework, presented in Section 4 and Section 5, include subject reduction (Theorem 4.6 and Theorem 5.16), normalisation of expressions (Theorem 4.13), progress and confluence of processes (Theorems 5.23, 5.24). Sections 6 and 7 discuss related and future work, respectively. Appendixes contain the proofs of three theorems.

Publication History. This paper is a thoroughly revised and extended version of [37] and its companion technical report [38]. There are three substantial improvements compared to previous versions of the paper. First, we give a much simplified definition of well-polarisation (Definition 5.3) resulting in simpler and cleaner proofs. Second, we have strengthened the

progress theorem (Theorem 5.23) and as a consequence part of its proof is new. Finally, the strong normalisation of the reduction without rules [R-OPEN] and [R-FUTURE] (Theorem 5.22) appears here for the first time.

2. THE SID CALCULUS

We use an infinite set of *channels* a, b, c and a disjoint, infinite set of *variables* x, y. We distinguish between two kinds of channels: *shared channels* are public service identifiers that can only be used to initiate sessions; *session channels* represent private sessions on which the actual communications take place. We distinguish the two *endpoints* of a session channel c by means of a *polarity* $p \in \{+, -\}$ and write them as c^+ and c^-. We write $\overline{p}$ for the dual polarity of p, where $\overline{+} = -$ and $\overline{-} = +$, and we say that c^p is the *peer endpoint* of $c^{\overline{p}}$. A *bindable name* X is either a channel or a variable and a *name* u is either a bindable name or an endpoint.

The syntax of *expressions* and *processes* is given in Table 1. In addition to the usual constructs of the λ-calculus, expressions include constants, ranged over by k, and pair splitting. Constants are the unitary value unit, the pair constructor pair, the primitives for session initiation and communication open, send, and recv [16, 14], the monadic operations return and bind [30], and a primitive future to defer computations [29, 28]. We do not need a primitive constant for the fixed point operator because it can be expressed and typed inside the language. For simplicity, we do not include primitives for branching and selection typically found in session calculi. They are straightforward to add and do not invalidate any of the results. Expressions are subject to the usual conventions of the λ-calculus. In particular, we assume that the bodies of abstractions extend as much as possible to the right, that applications associate to the left, and we use parentheses to disambiguate the notation when necessary. Following established notation, we write $\langle e, f \rangle$ in place of pair e f, and $\lambda\langle x_1, x_2 \rangle.e$ in place of $\lambda x.$split x as x_1, x_2 in e, and $e \ggg f$ in place of bind e f. As usual, we assume that the infix operator $\ggg$ is right-associative.

A process can be either the idle process 0 that performs no actions, a thread $x \Leftarrow e$ with name x and body e that evaluates the body and binds the result to variable x, a server a e that waits for session initiations on the shared channel a and spawns a new thread computing e at each connection, the parallel composition of processes, and the restriction of a bindable

Table 1: Syntax of expressions and processes.

$e ::=$		**Expression**	$P ::=$		**Process**
	k	(constant)		0	(idle process)
$\mid$	u	(name)	$\mid$	$x \Leftarrow e$	(thread)
$\mid$	$\lambda x.e$	(abstraction)	$\mid$	server a e	(server)
$\mid$	ee	(application)	$\mid$	$P \mid P$	(parallel)
$\mid$	split e as x, y in e	(pair splitting)	$\mid$	$(\nu X)P$	(restriction)

k ::= unit $\mid$ pair $\mid$ open $\mid$ send $\mid$ recv $\mid$ future $\mid$ return $\mid$ bind

$X ::= a \mid x$

Table 2: Reduction semantics of expressions and processes.

Reduction of expressions

[R-BETA]
$$(\lambda x.e)\; f \longrightarrow e\{f/x\}$$

[R-BIND]
$$\texttt{return}\; e \gg= f \longrightarrow fe$$

[R-SPLIT]
$$\texttt{split}\; \langle e_1, e_2 \rangle \; \texttt{as}\; x, y \; \texttt{in}\; f \longrightarrow f\{e_1, e_2/x, y\}$$

[R-CTXT]
$$\frac{e \longrightarrow f}{\mathcal{E}[e] \longrightarrow \mathcal{E}[f]}$$

Reduction of processes

[R-OPEN]
$$\texttt{server}\; a\; e \mid x \Leftarrow \mathcal{C}[\texttt{open}\; a] \longrightarrow \texttt{server}\; a\; e \mid (\nu cy)(x \Leftarrow \mathcal{C}[\texttt{return}\; c^+] \mid y \Leftarrow e\; c^-)$$

[R-COMM]
$$x \Leftarrow \mathcal{C}[\texttt{send}\; a^p\; e] \mid y \Leftarrow \mathcal{C}'[\texttt{recv}\; a^{\overline{p}}] \longrightarrow x \Leftarrow \mathcal{C}[\texttt{return}\; a^p] \mid y \Leftarrow \mathcal{C}'[\texttt{return}\; \langle e, a^{\overline{p}} \rangle]$$

[R-FUTURE]
$$x \Leftarrow \mathcal{C}[\texttt{future}\; e] \longrightarrow (\nu y)(x \Leftarrow \mathcal{C}[\texttt{return}\; y] \mid y \Leftarrow e)$$

[R-RETURN]
$$(\nu x)(x \Leftarrow \texttt{return}\; e \mid P) \longrightarrow P\{e/x\}$$

[R-THREAD]
$$\frac{e \longrightarrow f}{x \Leftarrow e \longrightarrow x \Leftarrow f}$$

[R-NEW]
$$\frac{P \longrightarrow Q}{(\nu X)P \longrightarrow (\nu X)Q}$$

[R-PAR]
$$\frac{P \longrightarrow Q}{P \mid R \longrightarrow Q \mid R}$$

[R-CONG]
$$\frac{P \equiv P' \longrightarrow Q' \equiv Q}{P \longrightarrow Q}$$

name. In processes, restrictions bind tighter than parallel composition and we may abbreviate $(\nu X_1)\cdots(\nu X_n)P$ with $(\nu X_1 \cdots X_n)P$.

We have that $\texttt{split}\; e\; \texttt{as}\; x, y\; \texttt{in}\; f$ binds both x and y in f and $(\nu a)P$ binds any occurrence of the endpoints a^+ and a^- or of the shared channel a within P. The definitions of *free* and *bound* names follow as expected. We identify expressions and processes up to renaming of bound names.

The operational semantics of expressions is defined in the upper half of Table 2. Expressions reduce according to a standard *call-by-name* semantics, for which we define the *evaluation contexts for expressions* below:

$$\mathcal{E} ::= [\,] \mid \mathcal{E}e \mid \texttt{split}\; \mathcal{E}\; \texttt{as}\; x, y\; \texttt{in}\; e \mid \texttt{open}\; \mathcal{E} \mid \texttt{send}\; \mathcal{E} \mid \texttt{recv}\; \mathcal{E} \mid \texttt{bind}\; \mathcal{E}$$

Note that evaluation contexts do not allow to reduce pair components or an expression e in $\lambda x.e$, $\texttt{bind}\; f\; e$, $\texttt{return}\; e$, $\texttt{future}\; e$ and $\texttt{send}\; a^p\; e$. We say that e is in *normal form* if there is no f such that $e \longrightarrow f$.

The operational semantics of processes is given by a structural congruence relation $\equiv$ (which we leave undetailed since it is essentially the same as that of the π-calculus [34]) and a reduction relation, defined in the bottom half of Table 2. The *evaluation contexts for processes* are defined as

$$\mathcal{C} ::= [\,] \mid \mathcal{C} \gg= e$$

and force the left-to-right execution of monadic actions, as usual.

Rules [R-OPEN] and [R-COMM] model session initiation and communication, respectively. According to [R-OPEN], a client thread opens a connection with a server a. In the reduct, a fresh session channel c is created, the `open` in the client is replaced by the endpoint c^+ wrapped in the constructor `return`. Moreover, a copy of the server is spawned into a new thread that has a fresh name y and a body which is the application of the expression e (provided by the server) to c^-. This follows a continuation-passing style since e is a function expecting the end-point of a channel. So client and server can communicate using the private channel c. According to [R-COMM], two threads communicate if one is ready to send some message e on a session endpoint a^p and the other is waiting for a message from the peer endpoint $a^{\bar{p}}$. As in [14], the result for the sender is the same session endpoint and the result for the receiver is a pair consisting of the received message and the session endpoint. The difference is that in our case the results have to be wrapped in the constructor `return` for monadic actions.

Rules [R-FUTURE] and [R-RETURN] deal with futures. The former spawns an I/O action e in a separate thread y, so that the spawner is able to reduce (using [R-BIND]) even if e has not been executed yet. The name y of the spawned thread is used as a placeholder for the value yielded by e. Rule [R-RETURN] deals with a future variable x that has been evaluated to `return` e. In this case, x can be replaced by e everywhere within its scope. Note that the rule replaces in a single step the variable x in an arbitrary parallel composition of threads running on possibly different hosts. In this respect, the practical realisation of this rule may appear critical, if at all possible. In fact, since the replaced value is immutable, the reduction rule can be implemented without synchronising all the threads that are affected by the replacement, for example by means of a broadcast or multicast communication.

Rule [R-THREAD] lifts reduction of expressions to reduction of threads. The remaining rules close reduction under restrictions, parallel compositions, and structural congruence, as expected. Hereafter, we write $\longrightarrow^*$ for the reflexive, transitive closure of $\longrightarrow$.

As an example, let

$$Q \; = \; (\nu prod\, cons\, a\, c)(P \mid \texttt{server}\ a\ \texttt{display})$$

where

$$P \; = \; prod \Leftarrow \texttt{stream}\ c^+\ (\texttt{from}\ 0) \mid cons \Leftarrow \texttt{display}\ c^-$$

is the process discussed in the introduction. It is easy to verify that

$$P_0 \; = \; (\nu\, prod\, a)(prod \Leftarrow \texttt{open}\ a \ggg \lambda y.\texttt{stream}\ y\ (\texttt{from}\ 0) \mid \texttt{server}\ a\ \texttt{display})$$

reduces to process Q.

3. TYPING SID

We now develop a typing discipline for SID. The challenge comes from the fact that the calculus allows a mixture of pure computations (handling data) and impure computations (doing I/O). In particular, SID programs can manipulate potentially infinite data while performing I/O operations that produce/consume pieces of such data as shown by the examples of Section 1. Some ingredients of the type system are easily identified from the syntax of the calculus. We have a core type language with unit, products, and arrows. As in [14], we distinguish between *unlimited* and *linear* arrows for there sometimes is the need to specify that certain functions must be applied exactly once. As in Haskell [30, 28], we use

the `IO` type constructor to denote monadic I/O actions. For shared and session channels we respectively introduce channel types and session types [16]. Finally, following [25], we introduce the *delay* type constructor $\bullet$, so that an expression of type $\bullet t$ denotes a value of type t that is available "at the next moment in time". This constructor is key to control recursion and attain normalisation of expressions. Moreover, the type constructors $\bullet$ and `IO` interact in non-trivial ways as shown later by the type of `future`.

Table 3: Syntax of Pre-types and Pre-session types.

$t ::=^{coind}$		**Pre-type**	$T ::=^{coind}$		**Pre-session type**
	B	(basic type)		end	(end)
$\mid$	T	(session type)	$\mid$	$?t.T$	(input)
$\mid$	$\langle T \rangle$	(shared channel type)	$\mid$	$!t.T$	(output)
$\mid$	$t \times t$	(product)	$\mid$	$\bullet T$	(delay)
$\mid$	$t \to t$	(arrow)			
$\mid$	$t \multimap t$	(linear arrow)			
$\mid$	IO t	(input/output)			
$\mid$	$\bullet t$	(delay)			

3.1. **Types.** The syntax of *pre-types* and *pre-session types* is given by the grammar in Table 3, whose productions are meant to be interpreted coinductively. A pre-(session) type is a possibly infinite tree, where each internal node is labelled by a type constructor and has as many children as the arity of the constructor. The leaves of the tree (if any) are labelled by either basic types or `end`. We use a coinductive syntax to describe the type of infinite data structures (such as streams) and arbitrarily long protocols, e.g. the one between *prod* and *cons* in Section 1.

We distinguish between unlimited pre-types (those denoting expressions that can be used any number of times) from linear pre-types (those denoting expressions that must be used exactly once). Let lin be the smallest predicate defined by

$$\mathsf{lin}(?t.T) \qquad \mathsf{lin}(!t.T) \qquad \mathsf{lin}(t \multimap s) \qquad \mathsf{lin}(\mathtt{IO}\ t) \qquad \frac{\mathsf{lin}(t)}{\mathsf{lin}(t \times s)} \qquad \frac{\mathsf{lin}(s)}{\mathsf{lin}(t \times s)} \qquad \frac{\mathsf{lin}(t)}{\mathsf{lin}(\bullet t)}$$

We say that t is *linear* if $\mathsf{lin}(t)$ holds and that t is *unlimited*, written $\mathsf{un}(t)$, otherwise. Note that all I/O actions are linear, since they may involve communications on session channels which are linear resources.

Definition 3.1 (Types). A pre-(session) type t is a *(session) type* if:

(1) For each sub-term $t_1 \to t_2$ of t such that $\mathsf{un}(t_2)$ we have $\mathsf{un}(t_1)$.
(2) For each sub-term $t_1 \multimap t_2$ of t we have $\mathsf{lin}(t_2)$.
(3) The tree representation of t is regular, namely it has finitely many distinct sub-trees.
(4) Every infinite path in the tree representation of t has infinitely many $\bullet$'s.

All conditions except possibly 4 are natural. Condition 1 essentially says that unlimited functions are *pure*, namely they do not contain and they cannot erase communications. Indeed, an unlimited function (one that does not contain linear names) that accepts a linear argument should return a linear result. Condition 2 states that a linear function (one that may contain linear names) always yields a linear result. This is necessary to keep track of

the presence of linear names in the function, even when the function is applied and its linear arrow type eliminated. For example, consider z of type $\mathtt{Nat} \multimap \mathtt{Nat}$ and both y and w of type $\mathtt{Nat}$, then without Condition 2 we could type $(\lambda x.y)(z\ w)$ with $\mathtt{Nat}$. This would be incorrect, because it discharges the expression $(z\ w)$ involving the linear name z. Condition 3 implies that we only consider types admitting a finite representation, for example using the well-known "μ notation" for expressing recursive types (for the relation between regular trees and recursive types we refer to [31, Chapter 20]). We define infinite types as trees satisfying a given recursive equation, for which the existence and uniqueness of a solution follow from known results [10]. For example, there are unique pre-types $\mathtt{S}'_{\mathtt{Nat}}$, $\mathtt{S}_{\mathtt{Nat}}$, and $\bullet^\infty$ that respectively satisfy the equations $\mathtt{S}'_{\mathtt{Nat}} = \mathtt{Nat} \times \mathtt{S}'_{\mathtt{Nat}}$, $\mathtt{S}_{\mathtt{Nat}} = \mathtt{Nat} \times \bullet\mathtt{S}_{\mathtt{Nat}}$, and $\bullet^\infty = \bullet\bullet^\infty$. *En passant*, note that linearity is decidable on types due to Condition 3. The fact that lin has been defined above as the *smallest* predicate that satisfies certain axioms and rules is crucial. In particular, $\bullet^\infty$ is not linear.

Condition 4 intuitively means that not all parts of an infinite data structure can be available at once: those whose type is prefixed by a $\bullet$ are "delayed" in the sense that recursive calls on them must be deeper. For example, $\mathtt{S}_{\mathtt{Nat}}$ is a type that denotes streams of natural numbers where each subsequent element of the stream is delayed by one $\bullet$ compared to its predecessor. Instead $\mathtt{S}'_{\mathtt{Nat}}$ is not a type: it would denote an infinite stream of natural numbers, whose elements are all available right away. Similarly, $\mathtt{Out}_{\mathtt{Nat}}$ and $\mathtt{In}_{\mathtt{Nat}}$ defined by $\mathtt{Out}_{\mathtt{Nat}} = !\mathtt{Nat}.\bullet\,\mathtt{Out}_{\mathtt{Nat}}$ and $\mathtt{In}_{\mathtt{Nat}} = ?\mathtt{Nat}.\bullet\,\mathtt{In}_{\mathtt{Nat}}$ are session types, while $\mathtt{Out}'_{\mathtt{Nat}}$ and $\mathtt{In}'_{\mathtt{Nat}}$ defined by $\mathtt{Out}'_{\mathtt{Nat}} = !\mathtt{Nat}.\mathtt{Out}'_{\mathtt{Nat}}$ and $\mathtt{In}'_{\mathtt{Nat}} = ?\mathtt{Nat}.\mathtt{In}'_{\mathtt{Nat}}$ are not. The type $\bullet^\infty$ is somehow degenerate in that it contains no actual data constructors. Unsurprisingly, we will see that non-normalising terms such as $\Omega = (\lambda x.x\ x)(\lambda x.x\ x)$ can only be typed with $\bullet^\infty$. Without Condition 4, Ω could be given any type.

We adopt the usual conventions of parentheses. Arrow types associate to the right. We assume the following precedence among type constructors: $\bullet$, $\mathtt{IO}$, $\times$, followed by $\to$ and $\multimap$ with the same (and lowest) precedence. We also need a notion of duality to relate the session types associated with peer endpoints. Our definition extends the one of [16] in the obvious way to delayed types. More precisely, the *dual* of a session type T is the session type $\overline{T}$ coinductively defined by the equations:

$$\overline{\mathtt{end}} = \mathtt{end} \qquad \overline{?t.T} = !t.\overline{T} \qquad \overline{!t.T} = ?t.\overline{T} \qquad \overline{\bullet T} = \bullet\overline{T}$$

Sometimes we will write $\bullet^n t$ in place of $\underbrace{\bullet \cdots \bullet}_{n\text{-times}} t$.

3.2. Typing Rules for Expressions.

First we assign types to constants:

$\mathtt{unit}$	$: \mathtt{Unit}$	$\mathtt{send}$	$: !t.T \to t \multimap \mathtt{IO}\ T$	$\mathtt{pair} : t \to s \multimap t \times s$	if $\mathsf{lin}(t)$
$\mathtt{return}$	$: t \to \mathtt{IO}\ t$	$\mathtt{recv}$	$: ?t.T \to \mathtt{IO}\ (t \times T)$	$\mathtt{pair} : t \to s \to t \times s$	if $\mathsf{un}(t)$
$\mathtt{open}$	$: \langle T \rangle \to \mathtt{IO}\ T$	$\mathtt{future}$	$: \bullet^n(\mathtt{IO}\ t) \to \mathtt{IO}\ \bullet^n t$	$\mathtt{bind} : \mathtt{IO}\ t \to (t \multimap \mathtt{IO}\ s) \multimap \mathtt{IO}\ s$	

Each constant $\mathtt{k} \neq \mathtt{unit}$ is polymorphic and we use $\mathsf{types}(\mathtt{k})$ to denote the set of types assigned to $\mathtt{k}$, e.g. $\mathsf{types}(\mathtt{return}) = \bigcup_t \{t \to \mathtt{IO}\ t\}$.

The types of $\mathtt{unit}$ and $\mathtt{return}$ are as expected. The type schema of $\mathtt{bind}$ is similar to the type it has in Haskell, except for the two linear arrows. The leftmost linear arrow allows linear functions as the second argument of $\mathtt{bind}$. The rightmost linear arrow is needed to satisfy Condition 1 of Definition 3.1, being $\mathtt{IO}\ t$ linear. The type of $\mathtt{pair}$ is also familiar, except that the second arrow is linear or unlimited depending on the first element of the pair. If the first element of the pair is a linear expression, then it can (and actually must) be used

Table 4: Typing rules for expressions.

$$\frac{[\bullet\mathrm{I}]}{\Gamma \vdash e : t}{\Gamma \vdash e : \bullet t} \qquad \frac{[\textsc{const}]}{\Gamma \vdash \mathtt{k} : t} \; \mathsf{un}(\Gamma), t \in \mathsf{types}(\mathtt{k}) \qquad \frac{[\textsc{axiom}]}{\Gamma, u : t \vdash u : t} \; \mathsf{un}(\Gamma)$$

$$\frac{[\to\mathrm{I}]}{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n(t \to s)} \; \mathsf{un}(\Gamma) \qquad \frac{[\to\mathrm{E}]}{\Gamma_1 \vdash e_1 : \bullet^n(t \to s) \qquad \Gamma_2 \vdash e_2 : \bullet^n t}{\Gamma_1 + \Gamma_2 \vdash e_1 e_2 : \bullet^n s} \qquad \frac{[\multimap\mathrm{I}]}{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n(t \multimap s)}$$

$$\frac{[\multimap\mathrm{E}]}{\Gamma_1 \vdash e_1 : \bullet^n(t \multimap s) \qquad \Gamma_2 \vdash e_2 : \bullet^n t}{\Gamma_1 + \Gamma_2 \vdash e_1 e_2 : \bullet^n s} \qquad \frac{[\times\mathrm{E}]}{\Gamma_1 \vdash e : \bullet^n(t_1 \times t_2) \qquad \Gamma_2, x : \bullet^n t_1, y : \bullet^n t_2 \vdash f : \bullet^n s}{\Gamma_1 + \Gamma_2 \vdash \mathtt{split}\ e\ \mathtt{as}\ x, y\ \mathtt{in}\ f : \bullet^n s}$$

for creating exactly one pair. The types of `send` and `recv` are almost the same as in [14], except that these primitives return I/O actions instead of performing them as side effects. The type of `open` is standard and obviously justified by its operational semantics. The most interesting type is that of `future`, which commutes delays and the `IO` type constructor. Intuitively, `future` applied to a delayed I/O action returns an immediate I/O that yields a delayed expression. This fits with the semantics of `future`, since its argument is evaluated in a separate thread and the one invoking `future` can proceed immediately with a placeholder for the delayed expression. If the body of the new thread reduces to `return` e, then e substitutes the placeholder.

The typing judgements for expressions have the shape $\Gamma \vdash e : t$, where *typing environments* (for used resources) Γ are mappings from variables to types, from shared channels to shared channel types, and from endpoints to session types:

$$\Gamma \quad ::= \quad \emptyset \quad | \quad \Gamma, x : t \quad | \quad \Gamma, a : \langle T \rangle \quad | \quad \Gamma, a^p : T$$

The domain of Γ, written $\mathsf{dom}(\Gamma)$, is defined as expected. A typing environment Γ is *linear*, notation $\mathsf{lin}(\Gamma)$, if there is $u : t \in \Gamma$ such that $\mathsf{lin}(t)$; otherwise Γ is *unlimited*, notation $\mathsf{un}(\Gamma)$. As in [14], we use a (partial) combination operator $+$ for environments that prevents names with linear types from being used more than once. Formally the environment $\Gamma + \Gamma'$ is defined inductively on Γ' by

$$\begin{aligned} \Gamma + \emptyset &= \Gamma \\ \Gamma + (\Gamma', u : t) &= (\Gamma + \Gamma') + u : t \end{aligned} \quad \text{where} \quad \Gamma + u : t = \begin{cases} \Gamma, u : t & \text{if } u \notin \mathsf{dom}(\Gamma), \\ \Gamma & \text{if } u : t \in \Gamma \text{ and } \mathsf{un}(t), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The typing axioms and rules for expressions are given in Table 4. The side condition $\mathsf{un}(\Gamma)$ in [const], [axiom], and [→I] is standard [14]. The typing rules differ from the ones in [14] on two crucial details. First of all, each rule allows for an arbitrary delay in front of the types of the entities involved. Intuitively, the number of $\bullet$'s represents the delay at which a value becomes available. So for example, rule [→I] says that a function which accepts an argument x of type t delayed by n and produces a result of type s delayed by the same n has type $\bullet^n(t \to s)$, that is a function delayed by n that maps elements of t into elements of s. The second difference with respect to the type system in [14] is the presence of rule [$\bullet$I], which allows to further delay a value of type t. Crucially, it is not possible to *anticipate* a

delayed value: if it is known that a value will only be available with delay n, then it will also be available with any delay $m \geq n$, but not earlier.

3.3. Examples of Type Derivations for Expressions.
Using rule [●I] and the recursive type $s = \bullet s \to t$, we can derive that the fixed point combinator

$$\mathtt{fix} = \lambda y.(\lambda x.y \ (x \ x))(\lambda x.y \ (x \ x))$$

has type $(\bullet t \to t) \to t$ by assigning the type $s \to t$ to the first occurrence of $\lambda x.y \ (x \ x)$ and the type $\bullet s \to t$ to the second one [25].

It is possible to derive the following types for the functions in Section 1:

$$\mathtt{from} : \mathtt{Nat} \to \mathtt{S_{Nat}} \quad \mathtt{stream} : \mathtt{Out_{Nat}} \to \mathtt{S_{Nat}} \to \mathtt{IO} \ \bullet^\infty \quad \mathtt{display} : \mathtt{In_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}$$

where, in the derivation for $\mathtt{display}$, we assume type $\mathtt{S_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}$ for g. We show the most interesting parts of this derivation. We use the following rules, which are easily obtained from those in Table 4 and the types of the constants.

[FIX]
$$\frac{\Gamma, x : \bullet t \vdash e : t}{\Gamma \vdash \mathtt{fix} \ \lambda x.e : t} \ \mathsf{un}(\Gamma)$$

[BIND]
$$\frac{\Gamma_1 \vdash e_1 : \bullet^n(\mathtt{IO} \ t) \qquad \Gamma_2 \vdash e_2 : \bullet^n(t \multimap \mathtt{IO} \ s)}{\Gamma_1 + \Gamma_2 \vdash e_1 \ggg= e_2 : \bullet^n \mathtt{IO} \ s}$$

[FUTURE]
$$\frac{\Gamma \vdash e : \bullet^{n+m}\mathtt{IO} \ t}{\Gamma \vdash \mathtt{future} \ e : \bullet^n \mathtt{IO} \ \bullet^m t}$$

[× → I]
$$\frac{\Gamma, x_1 : \bullet^n t_1, x_2 : \bullet^n t_2 \vdash e : \bullet^n s}{\Gamma \vdash \lambda\langle x_1, x_2\rangle.e : \bullet^n(t_1 \times t_2 \to s)} \ \mathsf{un}(\Gamma)$$

In order to type $\mathtt{display}$ we desugar its recursive definition as $\mathtt{display} = \mathtt{fix} \ (\lambda x.\lambda y.e)$, where

$$e \ = \ e_1 \ggg= e_2 \qquad \begin{array}{rcl} e_1 & = & \mathtt{recv} \ y \\ e_2 & = & \lambda\langle z, y'\rangle.e_3 \ggg= e_4 \end{array} \qquad \begin{array}{rcl} e_3 & = & \mathtt{future}\big(x \ y'\big) \\ e_4 & = & \lambda zs.g\langle z, zs\rangle \end{array}$$

We derive

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{\vdots}{\Gamma_2 \vdash e_1 : \mathtt{IO} \ (\mathtt{Nat} \times \bullet\mathtt{In_{Nat}})} \quad [\times \to \mathrm{I}] \cfrac{\cfrac{\nabla}{\Gamma, \Gamma_1, \Gamma_3, \Gamma_4 \vdash e_3 \ggg= e_4 : \mathtt{IO} \ \mathtt{S_{Nat}}}}{\Gamma, \Gamma_1 \vdash e_2 : (\mathtt{Nat} \times \bullet\mathtt{In_{Nat}}) \to \mathtt{IO} \ \mathtt{S_{Nat}}}
      }{\text{[BIND]} \quad \Gamma, \Gamma_1, y : \mathtt{In_{Nat}} \vdash e : \mathtt{IO} \ \mathtt{S_{Nat}}}
    }{\text{[→I]} \quad \Gamma, \Gamma_1 \vdash \lambda y.e : \mathtt{In_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}}
  }{\text{[FIX]} \quad \Gamma \vdash \mathtt{display} : \mathtt{In_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}}
}{}
$$

where $\Gamma = g : \mathtt{S_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}$, $\Gamma_1 = x : \bullet(\mathtt{In_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}})$, $\Gamma_2 = y : \mathtt{In_{Nat}}$, $\Gamma_3 = y' : \bullet\mathtt{In_{Nat}}$ and $\Gamma_4 = z : \mathtt{Nat}$. The derivation ∇ is as follows.

$$
\text{[BIND]} \cfrac{
  \text{[FUTURE]} \cfrac{
    \text{[→E]} \cfrac{\Gamma_1 \vdash x : \bullet(\mathtt{In_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}) \qquad \Gamma_3 \vdash y' : \bullet\mathtt{In_{Nat}}}{\Gamma_1, \Gamma_3 \vdash x \ y' : \bullet\mathtt{IO} \ \mathtt{S_{Nat}}}
  }{\Gamma_1, \Gamma_3 \vdash e_3 : \mathtt{IO} \ \bullet \mathtt{S_{Nat}}} \qquad \cfrac{\vdots}{\Gamma, \Gamma_4 \vdash e_4 : \bullet\mathtt{S_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}}
}{\Gamma, \Gamma_1, \Gamma_3, \Gamma_4 \vdash e_3 \ggg= e_4 : \mathtt{IO} \ \mathtt{S_{Nat}}}
$$

Note that the types of the premises of [→E] in the above derivation have a $\bullet$ constructor in front. Moreover, $\mathtt{future}$ has a type that pushes the $\bullet$ inside the $\mathtt{IO}$; this is crucial for typing e_4 with $(\bullet\mathtt{S_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}})$. We can assign the type $\bullet\mathtt{S_{Nat}} \to \mathtt{IO} \ \mathtt{S_{Nat}}$ to e_4 by guarding the argument z of type $\bullet\mathtt{S_{Nat}}$ under the constructor $\mathtt{pair}$. Without $\mathtt{future}$, the expression $x \ y' \ggg= e_4$ has type $\bullet\mathtt{IO} \ \mathtt{S_{Nat}}$ and for this reason $\mathtt{display_0}$ cannot be typed.

Controlling guardedness of recursion is subtle as it could require types with several bullets. For example, let $e = \mathtt{split}\ ys\ \mathtt{as}\ y, zs\ \mathtt{in}\ (s\ zs)$ and consider the function

$$\mathtt{skip} = \mathtt{fix}\ \lambda s.\lambda\langle x, ys\rangle.\langle x, e\rangle$$

that deletes the elements at even positions of a stream. Function $\mathtt{skip}$ has type $\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}$, where $\mathtt{S2_{Nat}} = \mathtt{Nat} \times \bullet\bullet\, \mathtt{S2_{Nat}}$. We derive

$$
\cfrac{
\text{[FIX]}\ \cfrac{
\text{[$\to$I]}\ \cfrac{
\text{[$\times$I]}\ \cfrac{\Gamma \vdash x : \mathtt{Nat} \qquad \cfrac{\nabla}{\Gamma \vdash e : \bullet\bullet\, \mathtt{S2_{Nat}}}}{\Gamma \vdash \langle x, e\rangle : \mathtt{S2_{Nat}}}
}{s : \bullet(\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}) \vdash \lambda\langle x, ys\rangle.\langle x, e\rangle : \mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}}
}{\vdash \mathtt{skip} : \mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}}
}{}
$$

where $\Gamma = s : \bullet(\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}), x : \mathtt{Nat}, ys : \bullet\mathtt{S_{Nat}}$, rule [$\times$I] is

$$
\text{[$\times$I]}\ \cfrac{\Gamma_1 \vdash e_1 : \bullet^n t \qquad \Gamma_2 \vdash e_2 : \bullet^n s}{\Gamma_1 + \Gamma_2 \vdash \langle e_1, e_2\rangle : \bullet^n(t \times s)}
$$

and the type derivation ∇ is

$$
\text{[$\times$E]}\ \cfrac{\Gamma \vdash ys : \bullet(\mathtt{Nat} \times \bullet\mathtt{S_{Nat}}) \qquad \text{[$\to$E]}\ \cfrac{\text{[$\bullet$I]}\ \cfrac{\Gamma' \vdash s : \bullet(\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}})}{\Gamma' \vdash s : \bullet\bullet\,(\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}})} \qquad \Gamma' \vdash zs : \bullet\bullet\, \mathtt{S_{Nat}}}{\Gamma' \vdash s\ zs : \bullet\bullet\, \mathtt{S2_{Nat}}}}{\Gamma \vdash e : \bullet\bullet\, \mathtt{S2_{Nat}}}
$$

where $\Gamma' = s : \bullet(\mathtt{S_{Nat}} \to \mathtt{S2_{Nat}}), y : \bullet\mathtt{Nat}, zs : \bullet\bullet\, \mathtt{S_{Nat}}$. Note that in the above derivation, the first premise of [$\to$E] has two $\bullet$'s in front of the arrow type. The same derivation can be done in the system of [25]. Instead [3] uses clock variables and [7] uses one constant to type this example as a particular case of lifting guarded recursive data to coinductive data.

3.4. **Typing Rules for Processes.** The typing judgements for processes have the shape $\Gamma \vdash P \triangleright \Delta$, where Γ is a typing environment as before, while Δ is a *resource environment*, keeping track of the resources defined in P. In particular, Δ maps the names of threads and servers in P to their types and it is defined by

$$\Delta \quad ::= \quad \emptyset \quad | \quad \Delta, x : t \quad | \quad \Delta, a : \langle T\rangle$$

Table 5 gives the typing rules for processes. A thread is well-typed if so is its body, which must be an I/O action. The type of a thread is that of the result of its body, where the delay moves from the I/O action to the result. The side condition makes sure that the thread is unable to use the very value that it is supposed to produce. The resulting environment for defined resources associates the name of the thread with the type of the action of its body. A server is well-typed if so is its body e, which must be a function from the dual of T to an I/O action. This agrees with the reduction rule of the server, where the application of e to an endpoint becomes the body of a new thread each time the server is invoked. It is natural to forbid occurrences of free variables and linear channels in server bodies. This is assured by the condition $\mathsf{shared}(\Gamma)$, which requires Γ to contain only shared channels. Clearly $\mathsf{shared}(\Gamma)$ implies $\mathsf{un}(\Gamma)$, and then we can type the body e with a non linear arrow. The type of the new thread (which will be t if e has type $\overline{T} \to \mathtt{IO}\ t$) must be unlimited, since a server can be invoked an arbitrary number of times. The environment $\Gamma + a : \langle T\rangle$ in the conclusion of

Table 5: Typing rules for processes.

$$
\begin{array}{l}
[\text{THREAD}] \\[4pt]
\dfrac{\Gamma \vdash e : \bullet^n(\texttt{IO } t)}{\Gamma \vdash x \Leftarrow e \ \triangleright \ x : \bullet^n t} \ x \notin \mathsf{dom}(\Gamma)
\end{array}
\qquad
\begin{array}{l}
[\text{SERVER}] \\[4pt]
\dfrac{\Gamma \vdash e : \overline{T} \to \texttt{IO } t}{\Gamma + a : \langle T\rangle \vdash \texttt{server } a \ e \ \triangleright \ a : \langle T\rangle} \ \begin{array}{l}\mathsf{shared}(\Gamma)\\ \mathsf{un}(t)\end{array}
\end{array}
$$

$$
\begin{array}{l}
[\text{PAR}] \\[4pt]
\dfrac{\Gamma_1 \vdash P_1 \ \triangleright \ \Delta_1 \qquad \Gamma_2 \vdash P_2 \ \triangleright \ \Delta_2}{\Gamma_1 + \Gamma_2 \vdash P_1 \mid P_2 \ \triangleright \ \Delta_1, \Delta_2}
\end{array}
\qquad
\begin{array}{l}
[\text{SESSION}] \\[4pt]
\dfrac{\Gamma, a^p : T, a^{\overline{p}} : \overline{T} \vdash P \ \triangleright \ \Delta}{\Gamma \vdash (\nu a)P \ \triangleright \ \Delta}
\end{array}
\qquad
\begin{array}{l}
[\text{NEW}] \\[4pt]
\dfrac{\Gamma, X : t \vdash P \ \triangleright \ \Delta, X : t}{\Gamma \vdash (\nu X)P \ \triangleright \ \Delta}
\end{array}
$$

the rule makes sure that the type of the server as seen by its clients is consistent with its definition.

The remaining rules are conventional. In a parallel composition we require that the sets of entities (threads and servers) defined by P_1 and P_2 are disjoint. This is enforced by the fact that the respective resource environments Δ_1 and Δ_2 are combined using the operator $\_, \_$ which (as usual) implicitly requires that $\mathsf{dom}(\Delta_1) \cap \mathsf{dom}(\Delta_2) = \emptyset$. The restriction of a session channel a introduces associations for both its endpoints a^+ and a^- in the typing environment with dual session types, as usual. Finally, the restriction of a bindable name X introduces associations in both the typing and the resource environment with the same type t. This makes sure that in P there is exactly one definition for X, which can be either a variable which names a thread or a shared channel which names a server, and that every usage of X is consistent with its definition.

3.5. **Example of Type Derivation for Processes.** Let $\texttt{inc} : \texttt{Nat} \to \texttt{Nat}$ be the increment function on natural numbers, and consider

$$
\begin{array}{rll}
\texttt{incStream } x & = & \texttt{recv } x \ggg \\
& & \lambda\langle y, x'\rangle.\texttt{future}\big(\texttt{incStream } x'\big) \ggg \qquad\qquad (3.1) \\
& & \lambda z.\texttt{return } \langle \texttt{inc } y, z\rangle
\end{array}
$$

which receives natural numbers in a channel x, increments them by one and returns them in a stream. Note that the function $\texttt{incStream}$ in (3.1) is the function $\texttt{display}$ in (1.2) once g is instantiated with $\lambda\langle x_1, x_2\rangle.\texttt{return } \langle \texttt{inc } x_1, x_2\rangle$. Then, the process

$$
x \Leftarrow \texttt{stream } c^+ \ (\texttt{from } 0) \quad \mid \quad y \Leftarrow (\texttt{incStream } c^-) \ggg \texttt{send } b^+ \quad \mid \quad z \Leftarrow \texttt{recv } b^- \quad (3.2)
$$

sends on channel b the whole sequence of integers starting from 1. We show part of a type derivation for the thread named y in (3.2).

$$
\dfrac{\dfrac{\begin{array}{c} c^- : \texttt{In}_{\texttt{Nat}} \vdash \texttt{incStream } c^- : \texttt{IO } \texttt{S}_{\texttt{Nat}} \\ b^+ : \texttt{!S}_{\texttt{Nat}}.\texttt{end} \vdash \texttt{send } b^+ : \texttt{IO } \texttt{S}_{\texttt{Nat}} \to \texttt{IO } \texttt{end} \end{array}}{c^- : \texttt{In}_{\texttt{Nat}}, b^+ : \texttt{!S}_{\texttt{Nat}}.\texttt{end} \vdash (\texttt{incStream } c^-) \ggg \texttt{send } b^+ : \texttt{IO } \texttt{end}} \ [\text{BIND}]}{c^- : \texttt{In}_{\texttt{Nat}}, b^+ : \texttt{!S}_{\texttt{Nat}}.\texttt{end} \vdash y \Leftarrow (\texttt{incStream } c^-) \ggg \texttt{send } b^+ \ \triangleright \ y : \texttt{end}} \ [\text{THREAD}]
$$

4. Properties of Typeable Expressions

This section is devoted to the proof of the two most relevant properties of typeable expressions, which are subject reduction (reduction of expressions preserves their types) and normalisation. As informally motivated in Section 3, the type constructor $\bullet$ controls recursion and guarantees normalisation of any expression that has a type different from $\bullet^\infty$.

4.1. **Subject Reduction for Expressions.** The proof of subject reduction for expressions (Theorem 4.6) is standard except for the fact that we are using the modal operator $\bullet$. For this, we need Lemma 4.1 below, which says that the type of an expression should be delayed as much as the types in the environment. This property reflects the fact that we can only move forward in time. For example, from $x : t \vdash \lambda y.x : s \to t$ we can deduce that $x : \bullet t \vdash \lambda y.x : \bullet(s \to t)$, but we cannot deduce $x : \bullet t \vdash \lambda y.x : s \to t$. Notably we can derive $x : \bullet t, y : \bullet t \to s \vdash yx : s$, i.e. the environment can contain types more delayed than the type of the expression.

Lemma 4.1 (Delay). *If $\Gamma \vdash e : t$, then $\Gamma_1, \bullet\Gamma_2 \vdash e : \bullet t$ for $\Gamma_1, \Gamma_2 = \Gamma$.*

Proof. By induction on the derivation. $\qquad\square$

The following property tells that, if an expression contains an endpoint or a variable with a linear type, then the type of that expression should be linear. For example, it is not possible to assign the unlimited type $\texttt{Nat} \to \texttt{IO end}$ to the function $\texttt{lc} = \lambda x.\texttt{send } a^p\ x$ which contains the free endpoint a^p of type $\texttt{!Nat.end}$. Otherwise, $\texttt{lc}$ could be erased in $(\lambda x.\texttt{unit})\ \texttt{lc}$ or duplicated in $(\lambda x.\langle x, x\rangle)\ \texttt{lc}$.

Lemma 4.2. *If $\Gamma \vdash e : t$ and $\mathsf{un}(t)$, then $\mathsf{un}(\Gamma)$.*

Proof. The proof is by induction on the derivation of $\Gamma \vdash e : t$. The case of $[\to\text{E}]$ uses Condition 1 and the case of $[\multimap\text{E}]$ uses Condition 2 of Definition 3.1. $\qquad\square$

The following three lemmas are standard in proofs of subject reduction.

Lemma 4.3 (Inversion for Expressions).
(1) *If $\Gamma \vdash \texttt{k} : t$, then $t = \bullet^n t'$ and $t' \in \mathsf{types}(\texttt{k})$ with $\mathsf{un}(\Gamma)$.*
(2) *If $\Gamma \vdash u : t$, then $t = \bullet^n t'$ and $\Gamma = \Gamma', u : t'$ with $\mathsf{un}(\Gamma')$.*
(3) *If $\Gamma \vdash \lambda x.e : t$ and $\mathsf{un}(\Gamma)$, then either $t = \bullet^n(t_1 \to t_2)$ or $t = \bullet^n(t_1 \multimap t_2)$ and $\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2$.*
(4) *If $\Gamma \vdash \lambda x.e : t$ and $\mathsf{lin}(\Gamma)$, then $t = \bullet^n(t_1 \multimap t_2)$ and $\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2$.*
(5) *If $\Gamma \vdash e_1 e_2 : t$, then $t = \bullet^n t_2$ and $\Gamma = \Gamma_1 + \Gamma_2$ with $\Gamma_2 \vdash e_2 : \bullet^n t_1$ and either $\Gamma_1 \vdash e_1 : \bullet^n(t_1 \to t_2)$ or $\Gamma_1 \vdash e_1 : \bullet^n(t_1 \multimap t_2)$.*
(6) *If $\Gamma \vdash \texttt{split } e \texttt{ as } x, y \texttt{ in } f : t$, then $\Gamma = \Gamma_1 + \Gamma_2$ and $t = \bullet^n t'$ with $\Gamma_1 \vdash e : \bullet^n(t_1 \times t_2)$ and $\Gamma_2, x : \bullet^n t_1, y : \bullet^n t_2 \vdash f : \bullet^n t'$.*

Proof. By case analysis and induction on the derivation. We only show Item 3 which is interesting because we need to shift the environment in time and apply Lemma 4.1. A derivation of $\Gamma \vdash \lambda x.e : t$ ends with an application of either $[\to\text{I}]$, $[\multimap\text{I}]$ or $[\bullet\text{I}]$. For the first two cases, the proof is immediate. If the last applied rule is $[\bullet\text{I}]$, then $t = \bullet t'$ and we have

$$\frac{\Gamma \vdash \lambda x.e : t'}{\Gamma \vdash \lambda x.e : \bullet t'}$$

By induction $t' = \bullet^n(t_1 \to t_2)$ or $t' = \bullet^n(t_1 \multimap t_2)$ and $\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2$. Hence,

$$t = \bullet t' = \bullet^{n+1}(t_1 \to t_2) \text{ or } t = \bullet t' = \bullet^{n+1}(t_1 \multimap t_2)$$

By Lemma 4.1 we have that $\Gamma, x : \bullet^{n+1} t_1 \vdash e : \bullet^{n+1} t_2$. $\qquad\square$

Lemma 4.4 (Substitution). *If $\Gamma_1, x : s \vdash e : t$ and $\Gamma_2 \vdash f : s$ and $\Gamma_1 + \Gamma_2$ is defined, then*

$$\Gamma_1 + \Gamma_2 \vdash e\{f/x\} : t.$$

Proof. By induction on the structure of expressions. We only consider the case $e = k$, to show the application of Lemma 4.2. It follows from Item 1 of Lemma 4.3 that $\mathsf{un}(\Gamma_1, x : s)$ and $t = \bullet^n t'$ with $t' = \mathsf{types}(k)$. From $\mathsf{un}(s)$, $\Gamma_2 \vdash f : s$ and Lemma 4.2, we derive $\mathsf{un}(\Gamma_2)$, and therefore $\Gamma_1 + \Gamma_2 \vdash k : t$ by [CONST] and $k\{f/x\} = k$. $\qquad\square$

Lemma 4.5 (Evaluation Contexts for Expressions). *If $\Gamma \vdash \mathcal{E}[e] : t$, then $\Gamma = \Gamma_1 + \Gamma_2$ and $\Gamma_1, x : s \vdash \mathcal{E}[x] : t$ and $\Gamma_2 \vdash e : s$ for some s.*

Proof. By induction on the structure of $\mathcal{E}$. $\qquad\square$

Theorem 4.6 (Subject Reduction for Expressions). *If $\Gamma \vdash e : t$ and $e \longrightarrow e'$, then $\Gamma \vdash e' : t$.*

Proof. By induction on the definition of $\longrightarrow$. Lemma 4.5 is useful for rule [R-CTXT]. We only consider the case $(\lambda x.e)\, f \longrightarrow e\{f/x\}$. Suppose $\Gamma \vdash (\lambda x.e)\, f : t$. By Item 5 of Lemma 4.3 $t = \bullet^n t_2$ and $\Gamma = \Gamma_1 + \Gamma_2$ and

$$\Gamma_2 \vdash f : \bullet^n t_1 \quad \text{and either} \quad \Gamma_1 \vdash \lambda x.e : \bullet^n(t_1 \to t_2) \quad \text{or} \quad \Gamma_1 \vdash \lambda x.e : \bullet^n(t_1 \multimap t_2)$$

In both cases, it follows from Item 3 of Lemma 4.3 that

$$\Gamma_1, x : \bullet^n t_1 \vdash e : \bullet^n t_2 \tag{4.1}$$

By applying Lemma 4.4 to (4.1) we get $\Gamma \vdash e\{f/x\} : \bullet^n t_2$. $\qquad\square$

4.2. **Normalisation of Expressions.** In this section we prove that any typeable expression whose type is different from $\bullet^\infty$ reduces to a normal form (Theorem 4.13). For this, we define a type interpretation indexed on the set of natural numbers for dealing with the temporal operator $\bullet$. The time is discrete and represented using the set of natural numbers. The semantics reflects the fact that one $\bullet$ corresponds to one unit of time by shifting the interpretation from i to $i + 1$. A similar interpretation of the modal operator with indexed sets is given in [25]. For simplicity we consider only `Unit` as basic type, the addition of other basic types is easy.

Before introducing the type interpretation, we give a few definitions. Let $\mathfrak{E}$ be the set of expressions. We define the following subsets of $\mathfrak{E}$:

$$\mathfrak{N} = \{e \mid e \longrightarrow^* f \ \& \ f \text{ is a normal form}\}$$
$$\mathfrak{N}_v = \{e \mid e \longrightarrow^* \mathcal{E}[x] \ \& \ x \text{ is a variable}\}$$
$$\mathfrak{N}_{IO} = \{e \mid e \longrightarrow^* \mathcal{C}[e_0] \ \& \ e_0 \in \{\mathtt{send}\ a^p\ e_1, \mathtt{recv}\ a^p, \mathtt{open}\ a, \mathtt{future}\ e_1\}\}$$

The sets $\mathfrak{N}_v$ and $\mathfrak{N}_{IO}$ are sets of expressions which reduce to normal forms of particular shapes. They are disjoint and both subsets of $\mathfrak{N}$. We will do induction on the rank of types. For `Unit`, session types, and shared channel types the rank is always 0. For the other types, the rank measures the depth of all what we can observe at time 0. We could also compute it by taking the maximal 0-length of all the paths in the tree representation of the type, where the 0-length of a path is the number of type constructors different from $\bullet$ from the root to a leaf or to a $\bullet$.

Definition 4.7 (Rank of a Type). The rank of a type t (notation $rank(t)$) is defined as follows.

$$rank(\text{Unit}) = rank(T) = rank(\langle T \rangle) = rank(\bullet t) = 0$$
$$rank(\text{IO } t) = rank(t) + 1$$
$$rank(t \times s) = max(rank(t), rank(s)) + 1$$
$$rank(t \to s) = max(rank(t), rank(s)) + 1$$
$$rank(t \multimap s) = max(rank(t), rank(s)) + 1$$

The rank is well defined (and finite) because the tree representation of a type cannot have an infinite branch with no $\bullet$'s at all (Condition 4 in Definition 3.1) and $rank(\bullet t)$ is set to 0.

We now define the type interpretation $[\![t]\!] \in \mathbb{N} \to \mathcal{P}(\mathfrak{E})$, which is an indexed set, where $\mathbb{N}$ is the set of natural numbers and $\mathcal{P}$ is the powerset constructor.

Definition 4.8 (Type Interpretation). We define $[\![t]\!]_i \subseteq \mathfrak{E}$ by induction on $(i, rank(t))$.

$$
\begin{aligned}
[\![\text{Unit}]\!]_i &= \mathfrak{N}_v \cup \{e \mid e \longrightarrow^* \text{unit}\} \\
[\![T]\!]_i &= \mathfrak{N}_v \cup \{e \mid e \longrightarrow^* a^p\} \\
[\![\langle T \rangle]\!]_i &= \mathfrak{N}_v \cup \{e \mid e \longrightarrow^* a\} \\
[\![t \times s]\!]_i &= \mathfrak{N}_v \cup \{e \mid e \longrightarrow^* \langle e_1, e_2 \rangle \text{ and } e_1 \in [\![t]\!]_i \text{ and } e_2 \in [\![s]\!]_i\} \\
[\![t \to s]\!]_i &= [\![t \multimap s]\!]_i \\
&= \mathfrak{N}_v \cup \{e \mid e \longrightarrow^* \lambda x.f \text{ and } ee' \in [\![s]\!]_j \ \forall e' \in [\![t]\!]_j, j \leq i\} \\
&\quad \cup \{e \mid e \longrightarrow^* \mathcal{E}[\mathsf{k}] \text{ and } ee' \in [\![s]\!]_j \ \forall e' \in [\![t]\!]_j, j \leq i\} \\
[\![\text{IO } t]\!]_i &= \mathfrak{N}_v \cup \mathfrak{N}_{IO} \cup \{e \mid e \longrightarrow^* \text{return } e' \text{ and } e' \in [\![t]\!]_i\} \\
[\![\bullet t]\!]_0 &= \mathfrak{E} \\
[\![\bullet t]\!]_{i+1} &= [\![t]\!]_i
\end{aligned}
$$

Note that $[\![\bullet^\infty]\!]_i = \mathfrak{E}$ for all $i \in \mathbb{N}$. In the interpretation of the arrow type, the requirement "for all $j \leq i$" (and not just "for all i") is crucial for dealing with the contra-variance of the arrow type in the proof of Item 3 of Lemma 4.10.

The next properties of the type interpretation are expected.

Lemma 4.9.
(1) $[\![\bullet^n t]\!]_i = \mathfrak{E}$ if $i < n$.
(2) $[\![\bullet^n t]\!]_i = [\![t]\!]_{i-n}$ if $i \geq n$.

Proof. Both items are proved by induction on n. $\qquad\square$

Lemma 4.10.
(1) *For all types t and $i \in \mathbb{N}$, we have $\mathfrak{N}_v \subseteq [\![t]\!]_i$.*
(2) *If $t \neq \bullet s$, then $[\![\bullet^{n+1} t]\!]_{n+1} \subseteq \mathfrak{N}$.*
(3) *For all $i \in \mathbb{N}$, $[\![t]\!]_{i+1} \subseteq [\![t]\!]_i$.*
(4) *If $t \neq \bullet^\infty$, then $\bigcap_{i \in \mathbb{N}} [\![t]\!]_i \subseteq \mathfrak{N}$.*

Proof. (Item 1). By induction on i and doing case analysis on the shape of the type. All cases are trivial except when the type is $\bullet t$.

(Item 2). Using Item 2 of Lemma 4.9.

(Item 3). By induction on $(i, rank(t))$. Suppose $e \in [\![t \to s]\!]_{i+1}$. Then $ee' \in [\![s]\!]_j$ for $j \leq i + 1$. This is equivalent to saying that $ee' \in [\![s]\!]_{j'+1}$ for $j' \leq i$. By induction hypothesis $[\![s]\!]_{j'+1} \subseteq [\![s]\!]_{j'}$. Hence, $e \in [\![t \to s]\!]_i$. The remaining cases are easy.

(Item 4). All the cases are trivial except for a type starting by $\bullet$. Since $t \neq \bullet^\infty$, we have that $t = \bullet^{n+1}s$ and $s \neq \bullet s'$. It follows from Item 2 that $[\![\bullet^{n+1}s]\!]_{n+1} \subseteq \mathfrak{N}$ and hence

$$\bigcap_{i \in \mathbb{N}} [\![t]\!]_i \subseteq [\![t]\!]_{n+1} = [\![\bullet^{n+1}s]\!]_{n+1} \subseteq \mathfrak{N}. \qquad \square$$

In order to deal with open expressions we resort to substitution functions, as usual. A substitution function is a mapping from (a finite set of) variables to $\mathfrak{E}$. We use δ to range over substitution functions. Substitution functions allows us to extend the semantics to typing judgements (notation $\Gamma \models_i e : t$).

Definition 4.11 (Typing Judgement Interpretation)**.** Let δ be a substitution function.
(1) $\delta \models_i \Gamma$ if $\delta(x) \in [\![t]\!]_i$ for all $x : t \in \Gamma$.
(2) $\Gamma \models_i e : t$ if $\delta(e) \in [\![t]\!]_i$ for all $\delta \models_i \Gamma$.

As expected we can show the soundness of our type system with respect to the indexed semantics.

Theorem 4.12 (Soundness)**.** *If $\Gamma \vdash e : t$, then $\Gamma \models_i e : t$ for all $i \in \mathbb{N}$.*

The proof of this theorem by induction on $\Gamma \vdash e : t$ can be found in Appendix A.

Theorem 4.13 (Normalisation of Typeable Expressions)**.** *If $\Gamma \vdash e : t$ and $t \neq \bullet^\infty$, then e reduces (in zero or more steps) to a normal form.*

Proof. It follows from Theorem 4.12 that

$$\Gamma \models_i e : t \qquad\qquad (4.2)$$

for all $i \in \mathbb{N}$. Let id be the identity substitution and suppose $x : s \in \Gamma$. Then

$$id(x) = x \quad \in \mathfrak{N}_v$$
$$\subseteq [\![s]\!]_i \quad \text{by Item 1 of Lemma 4.10.}$$

This means that $id \models_i \Gamma$ for all $i \in \mathbb{N}$. From (4.2) we have that $id(e) = e \in [\![t]\!]_i$ for all i. Hence,

$$e \in \bigcap_{i \in \mathbb{N}} [\![t]\!]_i$$

It follows from Item 4 of Lemma 4.10 that $e \in \mathfrak{N}$. $\qquad \square$

Notice that there are normalising expressions that cannot be typed, for example $\lambda x.\Omega\mathbf{I}$, where Ω is defined at the end of Section 3.1 and $\mathbf{I} = \lambda z.z$. In fact Ω has type $\bullet^\infty$ and by previous theorem it cannot have other types, and this implies that the application $\Omega\mathbf{I}$ has no type.

5. PROPERTIES OF REACHABLE PROCESSES

In general, processes lack subject reduction. For example, the process

$$(\nu xy)(x \Leftarrow \texttt{return } y \mid y \Leftarrow \texttt{return } x) \tag{5.1}$$

is well-typed by assigning both x and y any unlimited type, but its reduct

$$(\nu x)(x \Leftarrow \texttt{return } x)$$

is ill-typed because the thread name x occurs free in its body (*cf.* the side condition of rule
[THREAD]). Another paradigmatic example is

$$x \Leftarrow \texttt{send } a^+ y \mid y \Leftarrow \texttt{recv } a^- \tag{5.2}$$

which is well-typed in the environment $a^+ : \mathord{!}t.\texttt{end}, a^- : \mathord{?}t.\texttt{end}$ where $t = \bullet(t \times \texttt{end})$ and
which reduces to $x \Leftarrow \texttt{return } a^+ \mid y \Leftarrow \texttt{return } \langle y, a^- \rangle$. Again, the reduct is ill-typed because
the thread name y occurs free in its body. In general, these examples show that the reduction
rules [R-RETURN] and [R-COMM] can violate the side condition of the typing rule [THREAD], which
requires that a future variable is never defined in terms of itself.

Another source of problems is the fact that, as in many session calculi [4, 8], there
exist well-typed processes that are (or reduce to) configurations where mutual dependencies
between sessions and/or thread names prevent progress. For instance, both

$$(\nu xyab)(x \Leftarrow \texttt{send } a^+ \ 4 \ggg \lambda x.\texttt{recv } b^- \mid y \Leftarrow \texttt{send } b^+ \ 2 \ggg \lambda x.\texttt{recv } a^-) \tag{5.3}$$

$$(\nu xa)(x \Leftarrow \texttt{recv } a^- \ggg \lambda \langle y, z \rangle.\texttt{send } a^+ \ y) \tag{5.4}$$

are well-typed but also deadlocked.

The point is that none of the troublesome processes (including those shown above)
is relevant to us, because they cannot be obtained by reducing a so-called *initial* process
modelling the beginning of a computation. A closed, well-typed process P is *initial* if

$$P \equiv \quad (\nu xa_1 \cdots a_m)(x \Leftarrow e \mid \texttt{server } a_1 \ e_1 \mid \cdots \mid \texttt{server } a_m \ e_m)$$

namely if it refers to no undefined names and if it consists of one thread x – usually called
"main" in most programming languages – and an arbitrary number of servers that are
necessary for the computation. In particular, typeability guarantees that all bodies reduce to
normal forms and all open's refer to existing servers. Clearly, an initial process is typeable
in the empty environment.

We call *reachable* all processes that can be obtained by reducing an initial process. A
reachable process may have several threads running in parallel, resulting from either service
invocation or future's.

This section is organised as follows. Section 5.1 defines the set of well-polarised processes,
which includes the set of reachable processes. Subject reduction for reachable processes then
follows from subject reduction for well-polarised processes (Section 5.2). Well-polarisation of
reachable processes is also used in Section 5.3 to show progress and in Section 5.4 to show
confluence.

5.1. **Well-polarised Processes.** The most original and critical aspect of the following
proofs is to check that reachable processes do not have circular dependencies on session
channels and variables. The absence of circularities can be properly formalised by means of
a judgement that characterises the sharing of names among threads, inspired by the typing
of the parallel composition given in [20]. Intuitively, the notion of *well-polarisation* captures

the following properties of reachable processes and makes them suitable for proving subject reduction, progress and confluence:

(1) two threads can share at most one session channel;
(2) distinct endpoints of a session channel always occur in different threads;
(3) if the name of one thread occurs in the body of another thread, then these threads cannot share session channels nor can the first thread mention the second.

Note that (5.1) and (5.2) violate condition (3), (5.3) violates condition (1), and (5.4) violates condition (2). In order to define well-polarised processes, we need a few auxiliary notions. To begin with, we define functions to extract bounds, threads and servers from processes.

Definition 5.1 (Bounds, Threads, Servers). We define

$$\mathsf{bounds}(P) = \{X_1, \ldots, X_n\} \quad \mathsf{threads}(P) = Q \quad \mathsf{servers}(P) = R$$

if $P \equiv (\nu X_1 \cdots X_n)(Q \mid R)$, $Q \mid R$ does not contain restrictions, Q is *thread-only* (namely, it is a parallel composition of threads), and R is *server-only* (namely, it is a parallel composition of servers).

Next, we define a mapping that computes the set of polarised names occurring free in an expression or parallel composition of threads.

Definition 5.2 (Polarised Names). Let $\mathcal{N}$ be defined on expressions and thread-only processes by:

$$\begin{array}{rcl} \mathcal{N}(e) & = & \{a^p \mid a^p \in \mathsf{fn}(e)\} \cup \{x^+ \mid x \in \mathsf{fn}(e)\} \\ \mathcal{N}(x \Leftarrow e) & = & \{x^-\} \cup \mathcal{N}(e) \\ \mathcal{N}(P \mid Q) & = & \mathcal{N}(P) \cup \mathcal{N}(Q) \end{array}$$

Let $\mathcal{A}, \mathcal{B}$ be sets of polarised variables and endpoints. We say that $\mathcal{A}$ and $\mathcal{B}$ are *independent*, notation $\mathcal{A} \# \mathcal{B}$, if for every $X^p \in \mathcal{A}$ and $X^q \in \mathcal{B}$ we have $p = q$. Then $\mathcal{A} \# \mathcal{A}$ implies that $\mathcal{A}$ cannot contain the same name with opposite polarities.

Definition 5.3 (Well-polarised Processes). Let $\models P$ be the least predicate on thread-only processes such that

$$\begin{array}{ccc} [\text{WP-EMPTY}] & [\text{WP-THREAD}] & [\text{WP-PAR}] \\[4pt] \overline{\models 0} & \dfrac{x^+ \notin \mathcal{N}(e) \quad \mathcal{N}(e) \# \mathcal{N}(e)}{\models x \Leftarrow e} & \dfrac{\models P \quad \models Q}{\models P \mid Q} \; \mathcal{N}(P) \setminus \{X^p\} \# \mathcal{N}(Q) \setminus \{X^{\overline{p}}\} \end{array}$$

We say that P is *well-polarised* if $\models Q$ for some $Q \equiv \mathsf{threads}(P)$.

Note that the variable X in [WP-PAR] is existentially quantified. The empty process is trivially well-polarised and a thread $x \Leftarrow e$ is well-polarised if e does not contain references to both a^p and $a^{\overline{p}}$, nor to the thread name x. A parallel composition $P \mid Q$ is well-polarised if there is at most one variable or endpoint that occurs with opposite polarities in $\mathcal{N}(P)$ and $\mathcal{N}(Q)$. This means that either:

(1) P contains a^p and Q contains $a^{\overline{p}}$,
(2) P has a thread labelled x and Q has a thread whose body contains x (or vice versa),
(3) $\mathcal{N}(P) \# \mathcal{N}(Q)$, i.e. $\mathcal{N}(P)$ and $\mathcal{N}(Q)$ do not share names with opposite polarities.

Note that $\models P$ can hold even if P cannot be typed, for example $\models x \Leftarrow \Omega \mathbf{I}$, see the end of Section 4.2. Well-polarisation cannot be incorporated into the typing rules because

it is not closed under structural equivalence, i.e. $\models P$ and $P \equiv P'$ do not imply $\models P'$. A counterexample is shown below:

$$P = (x_1 \Leftarrow \texttt{return } 1 \mid y_2 \Leftarrow \texttt{return } x_1) \mid (x_2 \Leftarrow \texttt{return } y_1 \mid y_1 \Leftarrow \texttt{return } 2)$$
$$P' = (x_1 \Leftarrow \texttt{return } 1 \mid x_2 \Leftarrow \texttt{return } y_1) \mid (y_1 \Leftarrow \texttt{return } 2 \mid y_2 \Leftarrow \texttt{return } x_1)$$

Definition 5.4. We write $P \subseteq Q$ if $P \in \mathcal{S}(Q)$, where $\mathcal{S}$ is defined by

$$\mathcal{S}(0) = \{0\} \qquad \mathcal{S}(x \Leftarrow e) = \{x \Leftarrow e\}$$
$$\mathcal{S}(P_1 \mid P_2) = \mathcal{S}(P_1) \cup \mathcal{S}(P_2) \cup \{P_1' \mid P_2' \mid P_1' \in \mathcal{S}(P_1) \text{ and } P_2' \in \mathcal{S}(P_2)\}$$

We write $P \subset Q$ if $P \subseteq Q$ and $P \neq Q$.

Note that, if $P \subset Q$, then all threads of P respect the syntactic structure of Q. This means that if we represent the processes as trees, P is a sub-tree of Q. This is important because $\models$ is not necessarily preserved by structural equivalence.

Lemma 5.5. *If $\models P$ and $Q \subseteq P$, then $\models Q$.*

Proof. By induction on the derivation of $\models P$. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The proof that well-polarisation of typeable processes is preserved by reductions (Theorem 5.9) is a bit involved because $P_0 \longrightarrow Q_0$ and $\models P$ with $P \equiv \mathsf{threads}(P_0)$ do not imply that $\models Q$ for an arbitrary $Q \equiv \mathsf{threads}(Q_0)$. We will prove a variant of the above property: if $P_0 \longrightarrow Q_0$ and $\models P$ with $P \equiv \mathsf{threads}(P_0)$, then there exists Q' with $Q' \equiv \mathsf{threads}(Q_0)$ such that $\models Q'$. The problem lies on the reduction rules [R-COMM] and [R-RETURN].

Example 5.6. This example shows that using the rule [R-COMM] we can obtain Q from P such that $\models P$, but $\not\models Q$.

$$P = (x \Leftarrow \texttt{send } a^+ \ z \mid z \Leftarrow \texttt{return } 1) \mid y \Leftarrow \texttt{recv } a^-$$
$$Q = (x \Leftarrow \texttt{return } a^+ \mid z \Leftarrow \texttt{return } 1) \mid y \Leftarrow \texttt{return } \langle z, a^- \rangle$$

By re-arranging the threads of Q we get a process Q' such that $\models Q'$:

$$Q' = (x \Leftarrow \texttt{return } a^+ \mid y \Leftarrow \texttt{return } \langle z, a^- \rangle) \mid z \Leftarrow \texttt{return } 1$$

The rule [R-RETURN] has a similar problem as illustrated by the following example.

Example 5.7. Take $P_0 = (\nu x)P$ and

$$\begin{aligned}
P \ &= \ ((x \Leftarrow \texttt{return } \langle z_1, z_2 \rangle \mid z_1 \Leftarrow \texttt{return } z_2) \mid z_2 \Leftarrow \texttt{return } 1) \mid \\
&\quad\ (y \Leftarrow \texttt{send } a^+ x \mid u \Leftarrow \texttt{recv } a^-) \\
Q \ &= \ (z_1 \Leftarrow \texttt{return } z_2 \mid z_2 \Leftarrow \texttt{return } 1) \mid \\
&\quad\ (y \Leftarrow \texttt{send } a^+ \ \langle z_1, z_2 \rangle \mid u \Leftarrow \texttt{recv } a^-)
\end{aligned}$$

Then $\models P$ but $\not\models Q$. We have that $\models Q'$ and $Q \equiv Q'$ where

$$Q' = ((y \Leftarrow \texttt{send } a^+ \ \langle z_1, z_2 \rangle \mid u \Leftarrow \texttt{recv } a^-) \mid z_1 \Leftarrow \texttt{return } z_2) \mid z_2 \Leftarrow \texttt{return } 1$$

The details for finding a $Q' \equiv \mathsf{threads}(Q_0)$ for any Q_0 such that $P_0 \longrightarrow Q_0$ and $\models P$ with $P \equiv \mathsf{threads}(P_0)$ are given in Appendix B. Here we only give the formalisation of the properties (1), (2) and (3) listed at the beginning of this section.

Lemma 5.8.
(1) *Let $\models P \mid Q$ and either $X \neq Y$ or $p \neq q$. If $X^p, Y^q \in \mathcal{N}(P)$ and $X^{\bar{p}} \in \mathcal{N}(Q)$, then $Y^{\bar{q}} \notin \mathcal{N}(Q)$. Similarly, if $X^p \in \mathcal{N}(P)$ and $X^{\bar{p}}, Y^q \in \mathcal{N}(Q)$, then $Y^{\bar{q}} \notin \mathcal{N}(P)$.*
(2) *If $\models P$ and $x \Leftarrow e \subset P$ and $y \Leftarrow f \subset P$ and x occurs in f, then y cannot occur in e.*

(3) *If $\models P$ and $x \Leftarrow e \subset P$ and $y \Leftarrow f \subset P$ and a^p occurs in e and $a^{\overline{p}}$ occurs in f, then y cannot occur in e.*

Proof. Item 1 is easy to verify.

(Item 2). There exists a point in the derivation of $\models P$ where we split the two threads $x \Leftarrow e$ and $y \Leftarrow f$. This means that there is $P_1 \mid P_2 \subseteq P$ such that $x \Leftarrow e \subseteq P_1$ and $y \Leftarrow f \subseteq P_2$ (or vice versa). Lemma 5.5 implies $\models P_1 \mid P_2$, $\models P_1$ and $\models P_2$. Hence, $x^- \in \mathcal{N}(P_1)$ and $y^-, x^+ \in \mathcal{N}(P_2)$, because we assume that x occurs in f. By Item 1 $y^+ \notin \mathcal{N}(P_1)$, which means that y cannot occur in e.

(Item 3). Similar to the previous item. $\qquad\square$

The interest in well-polarisation comes from its preservation by reduction of typeable processes, as stated in the following theorem whose proof can be found in Appendix B.

Theorem 5.9. *If $P \longrightarrow P'$ and P is typeable and well-polarised, then P' is well-polarised too.*

As an immediate consequence we have that reachable processes are well-polarised, since an initial process is trivially well-polarised.

Corollary 5.10. *Each reachable process is well-polarised.*

5.2. **Subject Reduction for Reachable Processes.** The following three lemmas are for processes as Lemmas 4.3, 4.4 and 4.5 are for expressions.

Lemma 5.11 (Inversion for Processes)**.**
(1) *If $\Gamma \vdash x \Leftarrow e \triangleright \Delta$, then $\Delta = x : \bullet^n t$ with $\Gamma \vdash e : \bullet^n(\texttt{IO } t)$ and $x \notin \mathsf{dom}(\Gamma)$.*
(2) *If $\Gamma \vdash \texttt{server } a\ e \triangleright \Delta$, then $\Gamma = \Gamma', a : \langle T \rangle$ and $\Delta = a : \langle T \rangle$ with $\Gamma \vdash e : (\overline{T} \to \texttt{IO } t)$ and $\mathsf{shared}(\Gamma)$ and $\mathsf{un}(t)$.*
(3) *If $\Gamma \vdash P_1 \mid P_2 \triangleright \Delta$, then $\Gamma = \Gamma_1 + \Gamma_2$ and $\Delta = \Delta_1, \Delta_2$ with $\Gamma_1 \vdash P_1 \triangleright \Delta_1$ and $\Gamma_2 \vdash P_2 \triangleright \Delta_2$.*
(4) *If $\Gamma \vdash (\nu a)P \triangleright \Delta$, then either $\Gamma, a^p : T, a^{\overline{p}} : \overline{T} \vdash P \triangleright \Delta$ or $\Gamma, a : \langle T \rangle \vdash P \triangleright \Delta, a : \langle T \rangle$.*
(5) *If $\Gamma \vdash (\nu x)P \triangleright \Delta$, then $\Gamma, x : t \vdash P \triangleright \Delta, x : t$.*

Proof. By case analysis on the derivation. $\qquad\square$

Lemma 5.12 (Substitution)**.** *Let $\Gamma_1, x : t \vdash P \triangleright \Delta$ with $x \notin \mathsf{dom}(\Delta)$ and $\Gamma_2 \vdash e : t$ and $\Gamma_1 + \Gamma_2$ be defined and $\mathsf{dom}(\Gamma_2) \cap \mathsf{dom}(\Delta) = \emptyset$. Then $\Gamma_1 + \Gamma_2 \vdash P\{e/x\} \triangleright \Delta$.*

Proof. By induction on the structure of processes. We only discuss the case of rule [THREAD]. The interesting observation is that we need to use the hypothesis $\mathsf{dom}(\Gamma_2) \cap \mathsf{dom}(\Delta) = \emptyset$ to ensure that the name of the thread does not belong to its own body. We also use Lemma 4.4 to type the body of the thread itself. $\qquad\square$

Lemma 5.13 (Evaluation Contexts for Processes)**.** *If $\Gamma \vdash \mathcal{C}[e] : \bullet^n(\texttt{IO } s)$, then $\Gamma = \Gamma_1 + \Gamma_2$ and $\Gamma_1, x : \bullet^n(\texttt{IO } t) \vdash \mathcal{C}[x] : \bullet^n(\texttt{IO } s)$ and $\Gamma_2 \vdash e : \bullet^n(\texttt{IO } t)$.*

Proof. By induction on the structure of evaluation contexts for processes. $\qquad\square$

A useful consequence of the previous lemma is the following property of contexts filled by communication expressions.

Lemma 5.14. *If* $\Gamma, a^p : \bullet^n T \vdash \mathcal{C}[\texttt{send } a^p \ e] : \bullet^m(\texttt{IO } t)$ *or* $\Gamma, a^p : \bullet^n T \vdash \mathcal{C}[\texttt{recv } a^p] : \bullet^m(\texttt{IO } t)$, *then* $n \leq m$.

Proof. We only consider the case $\Gamma, a^p : \bullet^n T \vdash \mathcal{C}[\texttt{send } a^p \ e] : \bullet^m(\texttt{IO } t)$. By Lemma 5.13 $\Gamma = \Gamma_1 + \Gamma_2$ and $\Gamma_1, x : \bullet^m(\texttt{IO } s) \vdash \mathcal{C}[x] : \bullet^m(\texttt{IO } t)$ and $\Gamma_2, a^p : \bullet^n T \vdash \texttt{send } a^p \ e : \bullet^m(\texttt{IO } s)$. Since rule [→E] requires at least n bullets in front of the type of send we get $n \leq m$. $\square$

We say that an environment Γ is *balanced* if $a^p : T \in \Gamma$ and $a^{\overline{p}} : S \in \Gamma$ imply $T = \overline{S}$. We can now state subject reduction of well-polarised processes. The proof of this theorem is the content of Appendix C.

Theorem 5.15 (Subject Reduction for Well-polarised Processes). *Let* Γ *be balanced and* P *be well-polarised. If* $P \longrightarrow P'$ *and* $\Gamma \vdash P \ \triangleright \ \Delta$, *then there is balanced environment* Γ' *such that* $\Gamma' \vdash P' \ \triangleright \ \Delta$.

Theorem 5.16 (Subject Reduction for Reachable Processes). *All reachable processes are typeable.*

Proof. This follows from Corollary 5.10 and Theorem 5.15, observing that the empty session environment is balanced. $\square$

5.3. **Progress of Reachable Processes.** We now turn our attention to the progress property (Theorem 5.23). A computation stops when there are no threads left. Recall that the reduction rule [R-RETURN] (*cf.* Table 2) erases threads. Since servers are permanent we say that a process P is *final* if

$$P \equiv (\nu a_1 \cdots a_m)(\texttt{server } a_1 \ e_1 \mid \cdots \mid \texttt{server } a_m \ e_m)$$

In particular, the idle process is final, since m can be 0.

The following lemma gives fundamental features of linear types, which play an important role in the proof of progress.

Lemma 5.17 (Linearity).
(1) *If* $\Gamma, u : t \vdash e : s$ *and* $\textsf{lin}(t)$, *then* u *occurs exactly once in* e.
(2) *If* $\Gamma, u : t \vdash P \ \triangleright \ \Delta$ *and* $\textsf{lin}(t)$, *then there exists exactly one thread* $x \Leftarrow e$ *of* P *where* u *occurs only once in* e *and* u *occurs as name of another thread if* $u : t \in \Delta$ *and nowhere else.*

Proof. Both items are proved by induction on derivations. $\square$

The following properties of typeable processes are handy in the proof of progress.

Lemma 5.18. *Let P be typeable. Then:*

(1) *If $x \in \mathsf{bounds}(P)$ and $y \Leftarrow \mathcal{E}[x] \subseteq \mathsf{threads}(P)$, then $x \Leftarrow e \subset \mathsf{threads}(P)$.*
(2) *If $a \in \mathsf{bounds}(P)$ and $x \Leftarrow \mathcal{C}[\mathtt{open}\ a] \subseteq \mathsf{threads}(P)$, then $\mathtt{server}\ a\ e \subseteq \mathsf{servers}(P)$.*
(3) *If $a \in \mathsf{bounds}(P)$ and $x \Leftarrow \mathcal{C}[\mathtt{send}\ a^p\ e] \subseteq \mathsf{threads}(P)$, then $y \Leftarrow f \subset \mathsf{threads}(P)$, where $a^{\overline{p}}$ only occurs in expression f and the typing environment for $\mathsf{threads}(P)$ contains both $a^p : !t.T$ and $a^{\overline{p}} : ?t.\overline{T}$.*
(4) *If $a \in \mathsf{bounds}(P)$ and $x \Leftarrow \mathcal{C}[\mathtt{recv}\ a^p] \subseteq \mathsf{threads}(P)$, then $y \Leftarrow f \subset \mathsf{threads}(P)$, where $a^{\overline{p}}$ only occurs in expression f and the typing environment for $\mathsf{threads}(P)$ contains both $a^p : ?t.T$ and $a^{\overline{p}} : !t.\overline{T}$.*

Proof. (Item 1) and (Item 2). To type the restriction of x (or a), we need to use rule [NEW], which requires x (or a) to occur in the resource environment. Rule [THREAD] is the only rule that puts the name of a thread in the resource environment. Rule [SERVER] is the only rule that puts the name of a server in the resource context.

(Item 3). To type the restriction of a, we need to use rule [SESSION], which requires the environment to contain dual session types for a^p and $a^{\overline{p}}$. Since a^p is an argument of $\mathtt{send}$, its type is of the form $!t.T$ and hence, $a^{\overline{p}}$ should have type $?t.\overline{T}$. The fact that $a^{\overline{p}}$ occurs in only one thread follows from Item 2 of Lemma 5.17.

(Item 4). The proof is similar to Item 3. □

The proof of Theorem 5.23 requires to define a standard precedence between threads and show that this relation is acyclic. Informally, a thread precedes another one if the first thread must be evaluated before the second one. The simpler case is when the body of one thread is an evaluation context containing the name of another thread, i.e. $x \Leftarrow e$ precedes $y \Leftarrow \mathcal{E}[x]$. In the remaining cases the bodies of the threads are the normal forms $\mathcal{C}[\mathtt{send}\ a^p\ e]$ or $\mathcal{C}[\mathtt{recv}\ a^p]$ which have to wait for $a^{\overline{p}}$ to be inside an evaluation context. This is formalised in the following definition.

Definition 5.19 (Precedence).

(1) The endpoint a^p is *ready* in e if
$$e \in \{\mathcal{C}[\mathtt{send}\ a^p\ f], \mathcal{C}[\mathtt{recv}\ a^p]\}$$

(2) The endpoint a^p is *blocked* in e if one of the following conditions holds:
  (a) $e = \mathcal{C}[\mathtt{send}\ b^q\ f]$ and $a \neq b$ and a^p occurs in $\mathcal{C}$ or in f;
  (b) $e = \mathcal{C}[\mathtt{recv}\ b^q]$ and $a \neq b$ and a^p occurs in $\mathcal{C}$;
  (c) $e = \mathcal{E}[x]$ and a^p occurs in $\mathcal{E}$.
(3) The expression e *precedes* the expression f (notation $e \prec f$) if a^p is ready in f while $a^{\overline{p}}$ is blocked in e.
(4) The thread $x \Leftarrow e$ *precedes* the thread $y \Leftarrow f$ (notation $x \Leftarrow e \prec y \Leftarrow f$) if either $e \prec f$ or $f = \mathcal{E}[x]$.

Note that a channel is either ready or blocked in a typeable expression.

The following lemma follows easily from the definition of $\prec$. The proof of the third item uses Lemma 5.17 and typeability of P.

Lemma 5.20. *Let P be a reachable process and $x \Leftarrow e \mid y \Leftarrow f \subseteq P$ and $x \Leftarrow e \prec y \Leftarrow f$. Then there is X such that:*

(1) $X^p \in \mathcal{N}(x \Leftarrow e)$ and $X^{\overline{p}} \in \mathcal{N}(y \Leftarrow f)$.

(2) *If X is a variable, then $X = x$.*
(3) *If $X = a$, then a^p is blocked in e while $a^{\bar{p}}$ is ready in f. Moreover, these are the only occurrences of a^p and $a^{\bar{p}}$ in P.*

A process P is *acyclic* if the precedence between the threads in $\mathsf{threads}(P)$ has no cycles. As we will see in the proof of Theorem 5.23 acyclicity is a crucial property to assure progress. We can show that each reachable process is acyclic.

Lemma 5.21. *Each reachable process is acyclic.*

Proof. Suppose towards a contradiction that a reachable process P contains a cycle and consider $Q \subseteq \mathsf{threads}(P)$ such that Q includes all the threads involved in that cycle. By Lemma 5.5 $\models Q$.

Suppose first that $Q = x \Leftarrow e$. Then $x \Leftarrow e \prec x \Leftarrow e$. It follows from Item 1 of Lemma 5.20 that both X^p and $X^{\bar{p}}$ occur in $x \Leftarrow e$. This contradicts $\models Q$.

Suppose now that $Q = Q_1 \mid Q_2$. Since Q contain all the threads involved in the cycle, there are two threads $x_1 \Leftarrow e_1$ and $y_1 \Leftarrow f_1$ of Q_1 and two threads $x_2 \Leftarrow e_2$ and $y_2 \Leftarrow f_2$ of Q_2 such that

$$x_1 \Leftarrow e_1 \prec x_2 \Leftarrow e_2$$
$$y_1 \Leftarrow f_1 \succ y_2 \Leftarrow f_2$$

Item 1 of Lemma 5.20 gives $X^p \in \mathcal{N}(x_1 \Leftarrow e_1)$, $X^{\bar{p}} \in \mathcal{N}(x_2 \Leftarrow e_2)$ and $Y^q \in \mathcal{N}(y_1 \Leftarrow f_1)$, $Y^{\bar{q}} \in \mathcal{N}(y_2 \Leftarrow f_2)$. Item 1 of Lemma 5.8 requires $X = Y$ and $p = q$. Suppose $X = Y$ is a variable. Then it follows from Item 2 of Lemma 5.20 that $x_1 = X = Y = y_2$. This contradicts the typeability of the process P, since the typing rule [PAR] guarantees that all threads have different names. Suppose now that $X = a$. Then $a^p \in \mathcal{N}(e_1)$, $a^{\bar{p}} \in \mathcal{N}(e_2)$ and $a^p \in \mathcal{N}(f_1)$, $a^{\bar{p}} \in \mathcal{N}(f_2)$. It follows from Item 3 of Lemma 5.20 that a^p and $a^{\bar{p}}$ occur only once in P. This is possible only if $e_1 = f_1$ and $e_2 = f_2$. Item 3 of Lemma 5.20 implies that a^p is blocked in e_1 while ready in f_1, and $a^{\bar{p}}$ is ready in e_2 while blocked in f_2. This is absurdum since $e_1 = f_1$ and $e_2 = f_2$. □

For the proof of progress it is useful to consider the reduction $\longrightarrow^-$ without rules [R-OPEN] and [R-FUTURE] and to show that it is strongly normalising for typeable processes. The process $\mathsf{display}_0$ of Section 1 which has an infinite $\longrightarrow^-$-reduction sequence is rejected by our type system.

Theorem 5.22 (Strong Normalisation of $\longrightarrow^-$). *The reduction $\longrightarrow^-$ on typeable processes is strongly normalising.*

Proof. The proof requires some definitions for getting a weight of typeable processes which decreases by reduction. For $t \neq \bullet^\infty$ we define the function $\mathsf{delay}(t)$ which counts the number of initial bullets in a type t as follows.

$$\mathsf{delay}(t) = \begin{cases} 1 + \mathsf{delay}(t') & \text{if } t = \bullet t' \\ 0 & \text{otherwise} \end{cases}$$

We extend delay to resource environments by

$$\mathsf{delay}(x_1 : t_1, \ldots, x_n : t_n) = max\{\mathsf{delay}(t_i) \mid 1 \leq i \leq n\}$$

Let $T \neq \bullet^\infty$ and $m \in \mathbb{N}$. We define the function $\mathsf{ct}(m, T)$ that counts the number of ? and ! in a session type T only until time m as follows.

$$\mathsf{ct}(m, \mathsf{end}) = 0 \qquad \mathsf{ct}(m, ?t.T) = \mathsf{ct}(m, !t.T) = 1 + \mathsf{ct}(m, T)$$
$$\mathsf{ct}(0, \bullet T) = 0 \qquad \mathsf{ct}(m + 1, \bullet T) = \mathsf{ct}(m, T)$$

We extend the function ct to type environments by $\mathsf{ct}(m, \Gamma) = \mathsf{ct}(m, T_1) + \ldots + \mathsf{ct}(m, T_n)$, where $a_1^+ : T_1, \ldots, a_n^+ : T_n$ are the type declarations for the positive session channels occurring in Γ.

Let $\mathsf{ns}(e)$ be the number of reduction steps to reach the normal form of e. We define the *weight of the typeable process P for the environments Δ and Γ* by

$$\mathsf{wt}(P, \Delta, \Gamma) = (k, \mathsf{ct}(\mathsf{delay}(\Delta), \Gamma), \mathsf{ns}(e_1) + \ldots + \mathsf{ns}(e_k))$$

where $\mathsf{threads}(P) \equiv Q = x_1 \Leftarrow e_1 \mid \ldots \mid x_k \Leftarrow e_k$ and $\Gamma \vdash Q \ \triangleright \ \Delta$.

We will prove that if $P \longrightarrow^- P'$ and $\Gamma \vdash \mathsf{threads}(P) \ \triangleright \ \Delta$ and $\Gamma' \vdash \mathsf{threads}(P') \ \triangleright \ \Delta$, where the derivation $\Gamma' \vdash \mathsf{threads}(P') \ \triangleright \ \Delta$ is obtained from $\Gamma \vdash \mathsf{threads}(P) \ \triangleright \ \Delta$ as in the proof of the Subject Reduction Theorem, then $\mathsf{wt}(P, \Delta, \Gamma) > \mathsf{wt}(P', \Delta, \Gamma')$.

The only interesting case is $\mathsf{threads}(P) \equiv x \Leftarrow \mathcal{C}[\mathtt{send}\ a^+\ e] \mid y \Leftarrow \mathcal{C}'[\mathtt{recv}\ a^-] \mid Q$ and $\mathsf{threads}(P') \equiv x \Leftarrow \mathcal{C}[\mathtt{return}\ a^+] \mid y \Leftarrow \mathcal{C}'[\mathtt{return}\ \langle e, a^- \rangle] \mid Q$. The first components of $\mathsf{wt}(P, \Delta, \Gamma)$ and $\mathsf{wt}(P', \Delta, \Gamma')$ are equal since the number of threads does not change. We will prove that the second component decreases. By the Inversion Lemma Γ must contain suitable session types for a^+, a^-. We can then assume

$$\Gamma = a^+ : \bullet^n(!t.T), a^- : \bullet^n(?t.\overline{T}), \Gamma_0$$

We get $\Gamma' = a^+ : \bullet^n T, a^- : \bullet^n \overline{T}, \Gamma_0$. Let $m = \mathsf{delay}(\Delta)$. By the Inversion Lemma if the type of $\mathcal{C}[\mathtt{send}\ a^+\ e]$ is $\bullet^{m'}\mathtt{IO}\ s$, then Δ contains $x : \bullet^{m'}s$. By Lemma 5.14 $n \leq m'$, which implies $n \leq m$. From $\mathsf{ct}(m, \bullet^n(!t.T)) = \mathsf{ct}(m - n, !t.T) = 1 + \mathsf{ct}(m - n, T)$ and $\mathsf{ct}(m, \bullet^n T) = \mathsf{ct}(m - n, T)$ we conclude $\mathsf{ct}(m, \Gamma) > \mathsf{ct}(m, \Gamma')$ as desired. $\square$

As a consequence of the above theorem, every infinite reduction of a typeable process spawns infinitely many threads.

Theorem 5.23 (Progress of Reachable Processes). *A reachable process either reduces or it is final. Moreover a non-terminating reachable process reduces in a finite number of steps to a process to which one of the rules [R-OPEN] or [R-FUTURE] must be applied.*

Proof. If a process has no thread, then it is final. In discussing the other cases we omit to mention the application of rules [R-NEW], [R-PAR] and [R-CONG].

If a process has a thread whose body is a reducible expression, then the process is reducible by rule [R-THREAD]. If a process has a thread whose body is $\mathcal{C}[\mathtt{future}\ e]$, then the process is reducible by rule [R-FUTURE]. If a process has a thread whose body is $\mathtt{return}\ e$, then the process is reducible by rule [R-RETURN]. If a process has a thread whose body is $\mathcal{C}[\mathtt{open}\ a]$, then by Item 2 of Lemma 5.18 the process has a server named a. Therefore the process is reducible by rule [R-OPEN].

Otherwise all the bodies of the threads of the process are of the shapes $\mathcal{C}[\mathtt{send}\ a^p\ e]$, $\mathcal{C}[\mathtt{recv}\ a^p]$ and $\mathcal{E}[x]$. Since reachable processes are well-polarised, Lemma 5.21 assures that there is at least one minimal thread in the precedence order, let it be $x \Leftarrow e$. The expression e cannot be $\mathcal{E}[y]$, since Item 1 of Lemma 5.18 implies that the process should have one thread $y \Leftarrow f$; and by definition of precedence $y \Leftarrow f \ \prec \ x \Leftarrow e$, which contradicts the minimality of $x \Leftarrow e$. Let $e = \mathcal{C}[\mathtt{send}\ a^p\ e']$. Item 3 of Lemma 5.18 implies that the process should have one thread $y \Leftarrow f$ and $a^{\overline{p}}$ occurs in f. The expression f can be neither of the following:

- $\mathcal{C}'[\mathtt{send}\ b^q\ f']$ with $b \neq a$ and $a^{\overline{p}}$ occurring in $\mathcal{C}'$ or f'
- $\mathcal{C}'[\mathtt{recv}\ b^q]$ with $b \neq a$ and $a^{\overline{p}}$ occurring in $\mathcal{C}'$
- $\mathcal{E}[z]$ with $a^{\overline{p}}$ occurring in $\mathcal{E}$

since we would get $y \Leftarrow f \prec x \Leftarrow e$. Then f can only be either $\mathcal{C}'[\texttt{send } a^{\overline{p}} \; f']$ or $\mathcal{C}'[\texttt{recv } a^{\overline{p}}]$. Since reachable processes are typeable, Item 3 of Lemma 5.18 gives type $?t.\overline{T}$ for $a^{\overline{p}}$, so we have $f = \mathcal{C}'[\texttt{recv } a^{\overline{p}}]$. The process can then be reduced using rule [R-COMM]. The proof for the case $e = \mathcal{C}[\texttt{recv } a^p]$ is similar and it uses Item 4 of Lemma 5.18.

Theorem 5.22 assures that infinite applications of rules [R-OPEN] and [R-FUTURE] are needed to get infinite computations. $\hfill\square$

Let P_0 and Q be defined as at the end of Section 2. Note that P_0 is typeable, and indeed an initial process. Hence, by Theorems 5.16 and 5.23, process Q is typeable and has progress.

We now show two initial processes whose progress is somewhat degenerate. The first one realises an infinite sequence of *delegations* (the act of sending an endpoint as a message), thereby postponing the use of the endpoint forever:

$$\texttt{badserver} \;\; = \;\; (\nu xab)(x \Leftarrow \texttt{open } a \ggg \texttt{loop1} \,|$$
$$\texttt{server } a \; \lambda y.\texttt{open } b \ggg \texttt{loop2 } y \;|\; \texttt{server } b \; \texttt{recv})$$

where

$$\texttt{loop1} \;\; = \;\; \texttt{fix } \lambda f.\lambda x.\texttt{recv } x \ggg \lambda y.\texttt{split } y \texttt{ as } y_1, y_2 \texttt{ in send } y_2 \; y_1 \ggg$$
$$\lambda z.\texttt{future } (fz)$$

$$\texttt{loop2} \;\; = \;\; \texttt{fix } \lambda g.\lambda yx.\texttt{send } x \; y \ggg \lambda z.\texttt{recv } z \ggg$$
$$\lambda u.\texttt{split } u \texttt{ as } u_1, u_2 \texttt{ in future } (gu_1u_2)$$

We have that $\texttt{loop1} : \mathsf{RS}_t \to \texttt{IO } \bullet^\infty$ and $\texttt{loop2} : t \to \mathsf{SR}_t \multimap \texttt{IO } \bullet^\infty$, where $\mathsf{RS}_t = ?t.!t. \bullet \mathsf{RS}_t$ and $\mathsf{SR}_t = !t.?t. \bullet \mathsf{SR}_t$. Since no communication ever takes place on the session created with server b, $\texttt{badserver}$ violates lock freedom, which is progress in [12].

The second example is the initial process $(\nu x)(x \Leftarrow \Omega_{\texttt{future}})$, where $\Omega_{\texttt{future}} = \texttt{fix future}$. This process only creates new threads.

5.4. **Confluence of Reachable Processes.** In this section we prove that the reduction relation is confluent on reachable processes. The proof is trivial for expressions, since there is only one redex at each reduction step. However, for processes we may have several redexes to contract at a time and the proof requires to analyse these possibilities. Once again well-polarisation plays a crucial role in the proof. The fact that we can mix pure evaluations and communications and still preserve determinism is of practical interest.

Notice that typeability forbids processes where the same variable can be replaced by different expressions, like the process

$$(\nu x)(x \Leftarrow \texttt{return } 0 \;|\; x \Leftarrow \texttt{return } 1 \;|\; y \Leftarrow \texttt{stream } c^+ \; (\texttt{from } x))$$

which reduces to both $y \Leftarrow \texttt{stream } c^+ \; (\texttt{from } 0)$ and $y \Leftarrow \texttt{stream } c^+ \; (\texttt{from } 1)$.

Theorem 5.24 (Confluence of Reachable Processes)**.** *Let P be a reachable process. If $P \longrightarrow P_1$ and $P \longrightarrow P_2$, then either $P_1 \equiv P_2$ or there is P_3 such that $P_1 \longrightarrow P_3$ and $P_2 \longrightarrow P_3$.*

Proof. The proof proceeds by case analysis.

(1) Suppose rule [R-RETURN] is not applied. Since the redexes are non-overlapping, it is easy to see that $P \equiv Q_1 \;|\; Q_2$ and $P_1 \equiv Q_1' \;|\; Q_2$ and $P_2 \equiv Q_1 \;|\; Q_2'$ from $Q_1 \longrightarrow Q_1'$ and $Q_2 \longrightarrow Q_2'$. The common reduct is then $Q_1' \;|\; Q_2'$.

(2) Let $P \equiv (\nu xy)(x \Leftarrow \mathtt{return}\ e \mid y \Leftarrow \mathtt{return}\ f \mid R)$ and suppose we apply rule [R-RETURN] in both directions. Typing rule [PAR] implies $x \neq y$. Since P is reachable, and then well-polarised by Corollary 5.10, we cannot have both $y \in \mathsf{fv}(e)$ and $x \in \mathsf{fv}(f)$ by Item 2 of Lemma 5.8. Suppose $y \notin \mathsf{fv}(e)$. Then

$$P_1 \equiv (\nu x)(x \Leftarrow \mathtt{return}\ e \mid R\{f/y\})\ \text{and}$$
$$P_2 \equiv (\nu y)(y \Leftarrow \mathtt{return}\ (f\{e/x\}) \mid R\{e/x\})$$

and the common reduct of P_1 and P_2 is $R\{e/x\}\{f\{e/x\}/y\}$.

(3) Let $P \equiv (\nu z)(x \Leftarrow \mathcal{C}[\mathtt{send}\ a^p\ e] \mid y \Leftarrow \mathcal{C}'[\mathtt{recv}\ a^{\overline{p}}] \mid z \Leftarrow \mathtt{return}\ f \mid R)$ and suppose that in one direction we apply [R-RETURN] and in the other direction we apply [R-COMM]. Then

$$P_1 \equiv x \Leftarrow \mathcal{C}\{f/z\}[\mathtt{send}\ a^p\ e\{f/z\}] \mid y \Leftarrow \mathcal{C}'\{f/z\}[\mathtt{recv}\ a^{\overline{p}}] \mid R\{f/z\}$$
$$P_2 \equiv (\nu z)(x \Leftarrow \mathcal{C}[\mathtt{return}\ a^p] \mid y \Leftarrow \mathcal{C}'[\mathtt{return}\ \langle e, a^{\overline{p}}\rangle] \mid z \Leftarrow \mathtt{return}\ f \mid R)$$

It is easy to see that P_1 and P_2 have the common reduct:

$$x \Leftarrow \mathcal{C}\{f/z\}[\mathtt{return}\ a^p] \mid y \Leftarrow \mathcal{C}'\{f/z\}[\mathtt{return}\ \langle e\{f/z\}, a^{\overline{p}}\rangle] \mid R\{f/z\}$$

(4) The remaining cases are similar to the last one. $\square$

6. Related Work

To the best of our knowledge, SID is the first calculus that combines *session-based* communication primitives [16, 41] with a *call-by-need* operational semantics [42, 2, 23].

There are many calculi with functional and concurrent features, one of the more interesting ones being Boudol's blue calculus [5]. In the context of communication-centric calculi, infinite data are explicitly considered in [21, 11] and "implicitly" handled in [39, 40], where recursive/coinductive sessions are used to encode infinite communications.

Toninho et al. [39] integrate the Curry-Howard interpretation of linear sequent calculus as session-typed processes in a functional language. The main construct is a contextual monad encapsulating open concurrent computations, which can be communicated between processes in the style of higher-order processes. This allows for example to construct a stream transducer. In the same framework [40] handles infinite data by encoding them as coinductive sessions.

Lindley and Morris [21] combine recursive and co-recursive data types with communication primitives. They have fold and unfold over both recursive and corecursive session types instead of a general fixed point operator. The constructors in and out witness the isomorphism of recursion and corecursion. The operational semantics is call-by-value, but sending code is allowed because fold and unfold are values.

SSCC [11] offers an explicit primitive to deal with streams. Our language enables the modelling of more intricate interactions between infinite data structures and infinite communications. Besides, the type system of SSCC considers only finite sessions types and does not guarantee progress of processes.

Following [25], we use a modal operator $\bullet$ to restrict the application of the fixed point operator and exclude degenerate forms of divergence. This paper is an improvement over past typed lambda calculi with a temporal modal operator in two respects. Firstly, we do not need any subtyping relation as in [25] and secondly SID programs are not cluttered with constructs for the introduction and elimination of individuals of type $\bullet$ as in [18, 36, 19, 3, 6, 7]. A weak criterion to ensure productivity of infinite data is the *guardedness condition* [9]. We do

not need such condition because we can type more normalising expressions (such as `display` in (1.2)) using the modal operator •.

Futures originated in functional programming as annotations for implicitly parallelising programs [15]. Different operational semantics for an idealised functional language with futures are discusses in [13].

The papers more related to ours are [26] and [33]. The call-by-value calculus of [26] models Alice [32], a concurrent extension of standard ML [24], where synchronisation is based on futures as placeholders for values. A linear type system assures safety. The call-by-need λ-calculus in [33] provides a semantic foundation for the concurrent Haskell extended with futures. It shows the correctness of several program transformations using contextual semantics. Our calculus shares threads with these calculi. A main difference is the way in which the threads interact: through thread names and cells in [26] and through shared memory in form of Haskell's mutable variable and a global heap of shared expressions in [33]. Recursion is obtained by allowing the body of a thread to contain the thread name in [26] and by recursive heaps in [33].

In the session calculi literature, the word "progress" has two different meanings. Sometimes it is synonym of deadlock freedom [4], at other times it means lock freedom, i.e. that each offered communication in an open session eventually happens [12, 27, 8]. Reachable SID processes cannot be stuck, and if they do not terminate they generate new threads infinitely often. This means that the property of progress satisfied by our calculus is stronger than that of [4] and weaker than that of [12, 27, 8].

7. Conclusions

This paper studies the interaction between communications and infinite data structures by means of a calculus that combines sessions with lazy evaluation. A distinguished feature of SID is the possibility of modelling computations in which infinite communications interleave with the production and consumption of infinite data (*cf.* the examples in Section 1). Our examples considered infinite streams for simplicity. However, more general infinite data structures can be handled in SID. An evaluation of the expressiveness of SID in dealing with (distributed) algorithms based on such structures is scope for future investigations.

The typing discipline we have developed for SID guarantees normalisation of expressions with a type other than $•^\infty$ and progress of (reachable) processes, besides the standard properties of sessions (communication safety, protocol fidelity, determinism). The type system crucially relies on a modal operator • which has been used in a number of previous works [25, 18, 36, 6] to ensure productivity of well-typed expressions. In this paper, we have uncovered for the first time some intriguing interactions between this operator and the typing of impure expressions with the monadic `IO` type constructor. Conventionally, the type of `future` primitive is simply `IO` $t \to$ `IO` t and says nothing about the semantics of the primitive itself. In our type system, the type of `future` reveals its effect as an operator that turns a delayed computation into another that can be performed immediately, but which produces a delayed result.

As observed at the end of Section 6 and formalised in Theorem 5.23, our notion of progress sits somehow in between deadlock and lock freedom. It would be desirable to strengthen the type system so as to guarantee the (eventual) execution of all pending communications and exclude, for instance, the degenerate examples discussed in Section 5. This is relatively easy to achieve in conventional process calculi, where expressions only consist of names or

ground values [4, 27, 8], but it is far more challenging in the case of SID, where expressions embed the λ-calculus. We conjecture that one critical condition to be imposed is to forbid postponing linear computations, namely restricting the application of [•I] to non-linear types. Investigations in this direction are left for future work.

Another obvious development, which is key to the practical applicability of our theory, is the definition of a type inference algorithm for our type system. First steps in this direction have already been taken in [35] by solving type inference for the pure part of SID (without IO and concurrency) combining unification of types with integer linear programming.

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Appendix A. Proof of Theorem 4.12

Lemma A.1. (1) *Let $e \longrightarrow e'$. Then $e \in [\![t]\!]_i$ iff $e' \in [\![t]\!]_i$ for all $i \in \mathbb{N}$ and type t.*
(2) *If $\mathtt{k} : t$ and $t \in \mathsf{types}(\mathtt{k})$, then $\mathtt{k} \in \bigcap_{i \in \mathbb{N}} [\![t]\!]_i$.*

Proof. (Item 1). By induction on $(i, rank(t))$.
    (Item 2). We only consider the case $\mathtt{k} = \mathtt{bind}$ and prove that

$$\mathtt{bind} \in [\![\mathtt{IO}\ t \to (t \multimap \mathtt{IO}\ s) \multimap \mathtt{IO}\ s]\!]_i$$

Suppose $e_1 \in [\![\mathtt{IO}\ t]\!]_j$ and $e_2 \in [\![t \multimap \mathtt{IO}\ s]\!]_j$ for $j \le i$. We show that $\mathtt{bind}\ e_1\ e_2 \in [\![\mathtt{IO}\ s]\!]_i$. By definition of $[\![\mathtt{IO}\ t]\!]_j$ we have three cases:

(1) Case $e_1 \in \mathfrak{N}_v$. Hence $\mathtt{bind}\ e_1\ e_2 \longrightarrow^* \mathtt{bind}\ \mathcal{E}[x]\ e_2$. Taking $\mathcal{E}'[x] = \mathtt{bind}\ \mathcal{E}[x]\ e_2$ we have that $\mathtt{bind}\ e_1\ e_2 \in \mathfrak{N}_v$ and $\mathfrak{N}_v \subseteq [\![\mathtt{IO}\ s]\!]_j$ by Item 1 of Lemma 4.10.
(2) Case $e_1 \in \mathfrak{N}_{IO}$. Hence $e_1 \longrightarrow^* \mathcal{C}[e_0]$ and $e_0 \in \{\mathtt{send}\ a^p\ e_1', \mathtt{recv}\ a^p, \mathtt{open}\ a, \mathtt{future}\ e_1'\}$. Then

$$\mathtt{bind}\ e_1\ e_2 \longrightarrow^* \mathtt{bind}\ \mathcal{C}[e_0]\ e_2 \in \mathfrak{N}_{IO}$$

which implies $\mathtt{bind}\ \mathcal{C}[e_0]\ e_2 \in [\![\mathtt{IO}\ s]\!]_j$ by definition of $[\![\mathtt{IO}\ s]\!]_j$. By Item 1 we conclude that $\mathtt{bind}\ e_1\ e_2 \in [\![\mathtt{IO}\ s]\!]_j$.
(3) Case $e_1 \longrightarrow^* \mathtt{return}\ e_1'$ and $e_1' \in [\![t]\!]_j$. This gives $e_2\ e_1' \in [\![\mathtt{IO}\ s]\!]_j$. Since

$$\mathtt{bind}\ e_1\ e_2 \longrightarrow^* \mathtt{bind}\ (\mathtt{return}\ e_1')\ e_2 \longrightarrow e_2\ e_1'$$

we conclude that $\mathtt{bind}\ e_1\ e_2 \in [\![\mathtt{IO}\ s]\!]_j$ by Item 1. $\qquad\square$

Lemma A.2. (1) *If $\delta \models_i \Gamma_1 + \Gamma_2$, then $\delta \models_i \Gamma_1$ and $\delta \models_i \Gamma_2$.*
(2) *If $\delta \models_i \Gamma$, then $\delta \models_j \Gamma$ for all $j \le i$.*

Proof. Item 1 is an easy consequence of Definition 4.11.
    Item 2 follows from Item 3 of Lemma 4.10. $\qquad\square$

Proof of Theorem 4.12. We prove that $\Gamma \models_i e : t$ for all $i \in \mathbb{N}$ by induction on $\Gamma \vdash e : t$. We only show some interesting cases.

Rule [CONST].

It follows from Item 2 of Lemma A.1.

Rule [•I]. The derivation ends with the rule:

$$\frac{\Gamma \vdash e : t}{\Gamma \vdash e : \bullet t}\ ^{[\bullet \mathrm{I}]}$$

Suppose $i = 0$. Then

$$\delta(e) \in [\![t]\!]_0 = \mathfrak{E}$$

Suppose $i > 0$ and $\delta \models_i \Gamma$. It follows from Item 2 of Lemma A.2 that $\delta \models_{i-1} \Gamma$. By induction hypothesis $\Gamma \models_j e : t$ for all $j \in \mathbb{N}$. In particular $\Gamma \models_{i-1} e : t$. Hence $\delta(e) \in [\![t]\!]_{i-1}$ and

$$\delta(e) \in [\![t]\!]_{i-1} = [\![\bullet t]\!]_i$$

Rule [→E].

The derivation ends with the rule:

$$\frac{\Gamma_1 \vdash e_1 : \bullet^n(s \to t) \qquad \Gamma_2 \vdash e_2 : \bullet^n s}{\Gamma_1 + \Gamma_2 \vdash e_1 e_2 : \bullet^n t}$$

with $\Gamma = \Gamma_1 + \Gamma_2$ and $e = e_1 e_2$. By induction hypothesis for all $i \in \mathbb{N}$

$$\Gamma_1 \models_i e_1 : \bullet^n(s \to t) \tag{A.1}$$

$$\Gamma_2 \models_i e_2 : \bullet^n s \tag{A.2}$$

We have two cases:

(1) Case $i < n$. By Item 1 of Lemma 4.9 $[\![\bullet^n t]\!]_i = \mathfrak{E}$. We trivially get

$$\delta(e_1 e_2) \in [\![\bullet^n t]\!]_i$$

(2) Case $i \geq n$. Suppose that $\delta \models_i \Gamma$. It follows from Item 1 of Lemma A.2 that $\delta \models_i \Gamma_1$ and $\delta \models_i \Gamma_2$.

$$\begin{aligned} \delta(e_1) &\in & [\![\bullet^n(s \to t)]\!]_i \text{ by (A.1)} \\ &= & [\![(s \to t)]\!]_{i-n} \text{ by Item 2 of Lemma 4.9} \end{aligned} \tag{A.3}$$

$$\begin{aligned} \delta(e_2) &\in & [\![\bullet^n s]\!]_i \text{ by (A.2)} \\ &= & [\![s]\!]_{i-n} \text{ by Item 2 of Lemma 4.9} \end{aligned} \tag{A.4}$$

By Definition of $[\![(s \to t)]\!]_{i-n}$ and (A.3) there are two possibilities:

(a) Case $\delta(e_1) \in \mathfrak{N}_v$. Then

$$\delta(e_1 e_2) = \delta(e_1)\delta(e_2) \longrightarrow^* \mathcal{E}[x]\delta(e_2) \tag{A.5}$$

Hence

$$\begin{aligned} \delta(e_1 e_2) &\in \mathfrak{N}_v & \text{by (A.5)} \\ &\subseteq [\![\bullet^n t]\!]_i & \text{by Item 1 of Lemma 4.10.} \end{aligned}$$

(b) Case $\delta(e_1) \longrightarrow^* \lambda x.e'$ or $\delta(e_1) \longrightarrow^* \mathcal{E}[\mathsf{k}]$. We also have that

$$\delta(e_1)e'' \in [\![t]\!]_{i-n} \quad \forall e'' \in [\![s]\!]_{i-n}$$

In particular (A.4) implies

$$\delta(e_1 e_2) = \delta(e_1)\delta(e_2) \in [\![t]\!]_{i-n}$$

Since $[\![t]\!]_{i-n} = [\![\bullet^n t]\!]_i$ by Item 2 of Lemma 4.9 we are done.

Rule [→I].

The derivation ends with the rule:

$$\frac{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n(t \to s)}$$

By induction hypothesis for all $i \in \mathbb{N}$

$$\Gamma, x : \bullet^n t \models_i e : \bullet^n s \tag{A.6}$$

We have two cases:

(1) Case $i < n$. By Item 1 of Lemma 4.9 $[\![\bullet^n(t \to s)]\!]_i = \mathfrak{E}$. We trivially get

$$\delta(\lambda x.e) \in [\![\bullet^n(t \to s)]\!]_i$$

(2) Case $i \geq n$. Suppose that $\delta \models_i \Gamma$. By Item 2 of Lemma 4.9 it is enough to prove that

$$\delta(\lambda x.e) \in [\![t \to s]\!]_{i-n}$$

For this suppose $f \in [\![t]\!]_j$ for $j \leq i - n$. We consider the substitution function defined as $\delta_0 = \delta \cup \{(x, f)\}$. We have that

$$\delta_0 \models_{j+n} \Gamma, x : \bullet^n t \tag{A.7}$$

because
  (a) $\delta_0(x) = f \in [\![t]\!]_j = [\![\bullet^n t]\!]_{j+n}$ by Item 2 of Lemma 4.9.
  (b) $\delta_0 \models_{j+n} \Gamma$ by Item 2 of Lemma A.2 and the fact that $\delta_0 \models_i \Gamma$.
It follows from (A.6) and (A.7) that

$$\delta_0(e) \in [\![\bullet^n s]\!]_{j+n} \tag{A.8}$$

Therefore we obtain

$$\begin{aligned}
(\lambda x.e)f \longrightarrow \delta(e)\{f/x\} = \delta_0(e) \quad &\in [\![\bullet^n s]\!]_{j+n} \quad &\text{by (A.8)}\\
&= [\![s]\!]_j \quad &\text{by Item 2 of Lemma 4.9}
\end{aligned}$$

By Item 1 of Lemma A.1 we conclude

$$(\lambda x.e)f \quad \in [\![s]\!]_j. \hspace{4cm} \square$$

APPENDIX B. PROOF OF THEOREM 5.9

We use $x \Leftarrow e \subseteq_1 P$ as short for $x \Leftarrow e \subseteq P$ and there is only one thread named x in P. If $x \Leftarrow e \subseteq_1 P$ we denote by $P[Q / x \Leftarrow e]$ the replacement of the *unique* occurrence of the thread $x \Leftarrow e$ by the process Q in the process P. In particular, if $x \Leftarrow \mathcal{C}[\texttt{send } a^p e] \subseteq_1 P$ we will abbreviate

$$P[x \Leftarrow \mathcal{C}[\texttt{return } a^p] / x \Leftarrow \mathcal{C}[\texttt{send } a^p e]] \text{ as } P[\texttt{s } a^p e x].$$

Similarly, if $x \Leftarrow \mathcal{C}[\texttt{recv } a^p] \subseteq_1 P$ we will abbreviate

$$P[x \Leftarrow \mathcal{C}[\texttt{return } \langle e, a^p \rangle] / x \Leftarrow \mathcal{C}[\texttt{recv } a^p]] \text{ as } P[\texttt{r } a^p e x].$$

Notice that in both cases $\mathcal{C}$ and a^p are uniquely determined by the body of the thread named x, while the expression e occurs for $\texttt{send}$ but not for $\texttt{recv}$. Writing e as argument of both $\texttt{s}$ and $\texttt{r}$ allows us to easily express the exchanged message. These replacements are useful to find the right re-arrangements of threads which are derivable after applying the rule [R-COMM] to P. Informally, the derivation of $\models P$ must contain a sub-derivation of the shape

$$\frac{\overset{\text{[WP-PAR]}}{\models P_1 \qquad \models P_2}}{\models P_1 \mid P_2} \; \mathcal{N}(P_1) \setminus \{a^p\} \; \# \; \mathcal{N}(P_2) \setminus \{a^{\bar{p}}\}$$

with $x \Leftarrow \mathcal{C}[\texttt{send } a^p e] \subseteq_1 P_1$ and $y \Leftarrow \mathcal{C}'[\texttt{recv } a^{\bar{p}}] \subseteq_1 P_2$. We build the desired process by replacing $x \Leftarrow \mathcal{C}[\texttt{send } a^p e]$ with $x \Leftarrow \mathcal{C}[\texttt{return } a^p] \mid P_2[\texttt{r } a^{\bar{p}} e y]$ in P_1. Consider the processes defined in Example 5.6 and the reduction $P \longrightarrow Q$ using rule [R-COMM]. Let

$$P_1 = x \Leftarrow \texttt{send } a^+ z \mid z \Leftarrow \texttt{return } 1 \qquad P_2 = y \Leftarrow \texttt{recv } a^-$$

then $P = P_1 \mid P_2$. The process Q' such that $Q' \equiv Q$ and $\models Q'$ is obtained by replacing in P_1 the thread $x \Leftarrow \texttt{send } a^+ z$ by the process $x \Leftarrow \texttt{return } a^+ \mid y \Leftarrow \texttt{return}\langle z, a^- \rangle$, i.e.

$$Q' = (x \Leftarrow \texttt{return } a^+ \mid y \Leftarrow \texttt{return}\langle z, a^- \rangle) \mid z \Leftarrow \texttt{return } 1$$

Lemma B.1. *Let* $x \Leftarrow \mathcal{C}[\texttt{recv } a^p] \subseteq_1 P$ *and* $x^+ \notin \mathcal{N}(e)$. *If* $\models P$ *and* $\mathcal{N}(e) \# \mathcal{N}(e)$ *and* $\mathcal{N}(P) \# \mathcal{N}(e)$, *then* $\models P [\texttt{r } a^p \ e \ x]$.

Proof. By induction on the derivation of $\models P$. $\qquad\qquad\qquad\qquad\qquad\square$

Lemma B.2. *Let* $x \Leftarrow \mathcal{C}[\texttt{send } a^p \ e] \subseteq_1 P$ *and* $y \Leftarrow \mathcal{C}'[\texttt{recv } a^{\overline{p}}] \subseteq_1 Q$ *and* a^p *occurs only once in* P. *If* $\models P \mid Q$, *then there is* R *such that* $\models R$ *and*

$$R \equiv P [\texttt{s } a^p \ e \ x] \mid Q [\texttt{r } a^{\overline{p}} \ e \ y]$$

Proof. It follows from $\models P \mid Q$ that $\models P$ and $\models Q$ and

$$\mathcal{N}(P) \setminus \{a^p\} \# \mathcal{N}(Q) \setminus \{a^{\overline{p}}\} \tag{B.1}$$

and

$$\mathcal{N}(e) \# \mathcal{N}(e) \tag{B.2}$$

We do induction on $\models P$.

Suppose the last rule in the derivation is [WP-THREAD]. Then

$$\frac{}{[\text{WP-THREAD}] \quad \models x \Leftarrow \mathcal{C}[\texttt{send } a^p \ e]} \quad \begin{array}{l} x^+ \notin \mathcal{N}(\mathcal{C}[\texttt{send } a^p \ e]) \\ \mathcal{N}(\mathcal{C}[\texttt{send } a^p \ e]) \# \mathcal{N}(\mathcal{C}[\texttt{send } a^p \ e]) \end{array}$$

For the thread obtained applying the replacement $[\texttt{s } a^p \ e \ x]$ we derive:

$$\frac{}{[\text{WP-THREAD}] \quad \models x \Leftarrow \mathcal{C}[\texttt{return } a^p]} \quad \begin{array}{l} x^+ \notin \mathcal{N}(\mathcal{C}[\texttt{return } a^p]) \\ \mathcal{N}(\mathcal{C}[\texttt{return } a^p]) \# \mathcal{N}(\mathcal{C}[\texttt{return } a^p]) \end{array} \tag{B.3}$$

From (B.1) and $a^p \notin \mathcal{N}(e)$ we get $\mathcal{N}(Q) \# \mathcal{N}(e)$. It follows from this, (B.2) and Lemma B.1 that

$$\models Q [\texttt{r } a^{\overline{p}} \ e \ y] \tag{B.4}$$

The condition $\mathcal{N}(\mathcal{C}[\texttt{send } a^p \ e]) \# \mathcal{N}(\mathcal{C}[\texttt{send } a^p \ e])$ implies $\mathcal{N}(\mathcal{C}[\texttt{return } a^p]) \# \mathcal{N}(e)$. This together with (B.1) gives $\mathcal{N}(\mathcal{C}[\texttt{return } a^p]) \setminus \{a^p\} \# \mathcal{N}(Q [\texttt{r } a^{\overline{p}} \ e \ y]) \setminus \{a^{\overline{p}}\}$. Applying [WP-PAR] to (B.3) and (B.4) we derive:

$$\models x \Leftarrow \mathcal{C}[\texttt{return } a^p] \mid Q [\texttt{r } a^{\overline{p}} \ e \ y]$$

Suppose the last rule in the derivation is

$$\frac{[\text{WP-PAR}]}{\models P_1 \quad \models P_2} \quad \mathcal{N}(P_1) \setminus \{X^p\} \# \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\} \tag{B.5}$$

and $x \Leftarrow \mathcal{C}[\texttt{send } a^p \ e] \subseteq_1 P_1$. By induction hypothesis $\models R_1$ for some

$$R_1 \equiv P_1 [\texttt{s } a^p \ e \ x] \mid Q [\texttt{r } a^{\overline{p}} \ e \ y]$$

It follows from (B.1) and $a^p \notin \mathcal{N}(P_2)$ that $\mathcal{N}(Q) \# \mathcal{N}(P_2)$. Since $\mathcal{N}(R_1) = \mathcal{N}(P_1) \cup \mathcal{N}(Q)$ and using the side condition of (B.5), we get $\mathcal{N}(R_1) \setminus \{X^p\} \# \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\}$. We can apply [WP-PAR] and derive

$$\models R_1 \mid P_2$$

Clearly,

$$R_1 \mid P_2 \equiv (P_1 \mid P_2) [\texttt{s } a^p \ e \ x] \mid Q [\texttt{r } a^{\overline{p}} \ e \ y] \qquad\qquad\square$$

Lemma B.3. *Let* $x \Leftarrow \mathcal{C}[\texttt{send } a^p \ e] \subseteq_1 P$ *and* $y \Leftarrow \mathcal{C}'[\texttt{recv } a^{\overline{p}}] \subseteq_1 P$ *and* a^p *occurs only once in* P. *If* $\models P$, *then there exists* Q *such that* $\models Q$ *and* $Q \equiv P [\texttt{s } a^p \ e \ x] [\texttt{r } a^{\overline{p}} \ e \ y]$.

Proof. By induction on $\models P$. We only show the most interesting case:

$$\frac{[\textsc{wp-par}]}{\models P_1 \qquad \models P_2}{\models P_1 \mid P_2} \; \mathcal{N}(P_1) \setminus \{a^p\} \; \# \; \mathcal{N}(P_2) \setminus \{a^{\overline{p}}\}$$

By Lemma B.2, there is Q such that $\models Q$ and

$$Q \equiv P_1 \, [\mathsf{s} \; a^p \; e \; x] \mid P_2 \, [\mathsf{r} \; a^{\overline{p}} \; e \; y] = (P_1 \mid P_2) \, [\mathsf{s} \; a^p \; e \; x] \, [\mathsf{r} \; a^{\overline{p}} \; e \; y] \qquad \square$$

We now give some lemmas in order to find a right re-arrangements of the threads which are derivable after applying the rule [R-RETURN] to $(\nu x)P$. Informally, if $x^+ \in \mathcal{N}(P)$, the derivation of $\models P$ must contain a sub-derivation of the shape

$$\frac{[\textsc{wp-par}]}{\models P_1 \qquad \models P_2}{\models P_1 \mid P_2} \; \mathcal{N}(P_1) \setminus \{x^-\} \; \# \; \mathcal{N}(P_2) \setminus \{x^+\}$$

with $x \Leftarrow \mathtt{return} \; e \subseteq_1 P_1$. If $x^+ \notin \mathcal{N}(P_1)$, then the desired process is obtained by replacing $x \Leftarrow \mathtt{return} \; e$ with $P_2\{e/x\}$ in P_1. Otherwise we need to parenthesise differently $P_1 \mid P_2$ in order to satisfy this condition. Consider the process P of Example 5.7 which we write as $P = P_1 \mid P_2$ where

$$P_1 = (x \Leftarrow \mathtt{return} \; \langle z_1, z_2 \rangle \mid z_1 \Leftarrow \mathtt{return} \; z_2) \mid z_2 \Leftarrow \mathtt{return} \; 1$$
$$P_2 = y \Leftarrow \mathtt{send} \; a^+ x \mid u \Leftarrow \mathtt{recv} \; a^-$$

Let $(\nu x)P \longrightarrow Q$ using rule [R-RETURN]. The process Q' such that $Q' \equiv Q$ and $\models Q'$ is obtained by replacing in P_1 the thread $x \Leftarrow \mathtt{return} \; \langle z_1, z_2 \rangle$ by the process $Q_1 = P_2\{\langle z_1, z_2 \rangle / x\}$, i.e.

$$Q' = P_1 \, [\, Q_1 \, / \, x \Leftarrow \mathtt{return} \; \langle z_1, z_2 \rangle \,]$$

Lemma B.4. *Let* $x \Leftarrow \mathtt{return} \; e \subseteq_1 P$ *and* $\models P$. *If* $\models Q$ *and* $\mathcal{N}(P) \setminus \{x^-\} \; \# \; \mathcal{N}(Q) \setminus \mathcal{N}(e)$, *then* $\models P \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,]$.

Proof. By induction on $\models P$. Suppose $P = x \Leftarrow \mathtt{return} \; e$ and the derivation of $\models P$ is:

$$\frac{[\textsc{wp-thread}]}{\models x \Leftarrow \mathtt{return} \; e} \; \frac{x^+ \notin \mathcal{N}(e)}{\mathcal{N}(e) \; \# \; \mathcal{N}(e)}$$

In this case $\models P \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,]$, since $P \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,] = Q$.

Suppose $P = P_1 \mid P_2$ and the derivation of $\models P$ ends with the rule:

$$\frac{[\textsc{wp-par}]}{\models P_1 \qquad \models P_2}{\models P_1 \mid P_2} \; \mathcal{N}(P_1) \setminus \{X^p\} \; \# \; \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\} \qquad (\text{B.6})$$

Let $x \Leftarrow \mathtt{return} \; e \subseteq_1 P_1$ and $X \neq x$. By induction hypothesis $\models P_1 \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,]$ since $\mathcal{N}(P_1) \subset \mathcal{N}(P)$ and $\mathcal{N}(P) \setminus \{x^-\} \; \# \; \mathcal{N}(Q) \setminus \mathcal{N}(e)$ imply $\mathcal{N}(P_1) \setminus \{x^-\} \; \# \; \mathcal{N}(Q) \setminus \mathcal{N}(e)$. Now we apply [WP-PAR] using this new premise:

$$\frac{[\textsc{wp-par}]}{\models P_1 \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,] \qquad \models P_2}{\models P_1 \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,] \mid P_2} \; \mathcal{N}(P_1 \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,]) \setminus \{X^p\} \; \# \; \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\}$$
$$(\text{B.7})$$

We need to prove that the side condition of (B.7) holds. Since

$$\mathcal{N}(P_1 \, [\, Q \, / \, x \Leftarrow \mathtt{return} \; e \,]) \setminus \{X^p\} \subseteq \mathcal{N}(P_1) \setminus \{X^p\} \cup \mathcal{N}(Q) \setminus (\mathcal{N}(e) \cup \{X^p\})$$

it is enough to show that $\mathcal{N}(Q) \setminus (\mathcal{N}(e) \cup \{X^p\}) \mathrel{\#} \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\}$. This is a consequence of $\mathcal{N}(Q) \setminus \mathcal{N}(e) \mathrel{\#} \mathcal{N}(P_2)$, being $\mathcal{N}(P) \setminus \{x^-\} \mathrel{\#} \mathcal{N}(Q) \setminus \mathcal{N}(e)$ and $\mathcal{N}(P_2) \subseteq \mathcal{N}(P) \setminus \{x^-\}$. The case $X = x$ is similar and simpler than the previous one. □

Lemma B.5. *If* $\models P \mid Q$ *and* $x \Leftarrow e \subset_1 P \mid Q$ *and* $x^+ \in \mathcal{N}(P)$ *and* $x^+ \in \mathcal{N}(Q)$, *then there are* P', Q' *such that* $P' \mid Q' \equiv P \mid Q$ *and* $\models P' \mid Q'$ *and* $x \Leftarrow e \subseteq_1 P'$ *and* $x^+ \notin \mathcal{N}(P')$.

Proof. We assume $x \Leftarrow e \subseteq_1 P$, the proof for $x \Leftarrow e \subseteq_1 Q$ being symmetric. The derivation of $\models P \mid Q$ must end by:

$$[\text{WP-PAR}] \quad \frac{\models P \qquad \models Q}{\models P \mid Q} \ \mathcal{N}(P) \setminus \{x^-\} \mathrel{\#} \mathcal{N}(Q) \setminus \{x^+\} \tag{B.8}$$

The proof is by induction on the derivation of $\models P$. From $x \Leftarrow e \subseteq_1 P$ and $x^+ \in \mathcal{N}(P)$ and $x^+ \notin \mathcal{N}(e)$ we get $P \equiv P_1 \mid P_2$. Let $x \Leftarrow e \subseteq_1 P_1$, then the derivation of $\models P$ must end by:

$$[\text{WP-PAR}] \quad \frac{\models P_1 \qquad \models P_2}{\models P_1 \mid P_2} \ \mathcal{N}(P_1) \setminus \{x^-\} \mathrel{\#} \mathcal{N}(P_2) \setminus \{x^+\} \tag{B.9}$$

If $x^+ \notin \mathcal{N}(P_1)$ we can choose $P' = P_1$ and $Q' = P_2 \mid Q$. In fact we can derive:

$$[\text{WP-PAR}] \ \frac{\models P_1 \qquad [\text{WP-PAR}] \ \dfrac{\models P_2 \qquad \models Q}{\models P_2 \mid Q} \ \mathcal{N}(P_2) \mathrel{\#} \mathcal{N}(Q)}{\models P_1 \mid (P_2 \mid Q)} \ \mathcal{N}(P_1) \setminus \{x^-\} \mathrel{\#} \mathcal{N}(P_2 \mid Q) \setminus \{x^+\}$$

The first side condition follows from $x^- \notin \mathcal{N}(P_2)$, $\mathcal{N}(P_2) \subseteq \mathcal{N}(P)$, and the side condition of (B.8). The second side condition follows from $\mathcal{N}(P_1) \subseteq \mathcal{N}(P)$ and the side conditions of (B.8), (B.9).

If $x^+ \in \mathcal{N}(P_1)$ by induction there are P_1', P_2' such that $P_1' \mid P_2' \equiv P_1 \mid P_2$ and $\models P_1' \mid P_2'$ and $x \Leftarrow e \subseteq_1 P_1'$ and $x^+ \notin \mathcal{N}(P_1')$. We can choose $P' = P_1'$ and $Q' = P_2' \mid Q$. In fact we can derive:

$$[\text{WP-PAR}] \ \frac{\models P_1' \qquad [\text{WP-PAR}] \ \dfrac{\models P_2' \qquad \models Q}{\models P_2' \mid Q} \ \mathcal{N}(P_2') \mathrel{\#} \mathcal{N}(Q)}{\models P_1' \mid (P_2' \mid Q)} \ \mathcal{N}(P_1') \setminus \{x^-\} \mathrel{\#} \mathcal{N}(P_2' \mid Q) \setminus \{x^+\}$$

The first side condition follows from $x^- \notin \mathcal{N}(P_2')$, $\mathcal{N}(P_2') \subseteq \mathcal{N}(P)$, and the side condition of (B.8). Observe that $\models P_1' \mid P_2'$ implies that

$$\mathcal{N}(P_1') \setminus \{x^-\} \mathrel{\#} \mathcal{N}(P_2') \setminus \{x^+\} \tag{B.10}$$

Then the second side condition follows from $\mathcal{N}(P_1') \subseteq \mathcal{N}(P)$, (B.10), and the side condition of (B.8). □

Lemma B.6. *Let* $\models P$.
(1) *If* $x^- \notin \mathcal{N}(P)$ *and* $\mathcal{N}(e) \# \mathcal{N}(e)$ *and* $\mathcal{N}(P) \# \mathcal{N}(e)$, *then* $\models P\{e/x\}$.
(2) *If* $x \Leftarrow \texttt{return}\ e \subseteq_1 P$, *then* $\models Q\{e/x\}$ *for some* Q *such that* $Q \mid x \Leftarrow \texttt{return}\ e \equiv P$.

Proof. Both items are proved by induction on the derivation of $\models P$.
    (Item 1). We show only the case of [WP-THREAD]. Suppose that

$$[\text{WP-THREAD}] \frac{}{\models y \Leftarrow f} \quad \begin{array}{l} y^+ \notin \mathcal{N}(f) \\ \mathcal{N}(f) \# \mathcal{N}(f) \end{array}$$

We can do the following inference:

$$[\text{WP-THREAD}] \frac{}{\models y \Leftarrow f\{e/x\}} \quad \begin{array}{l} y^+ \notin \mathcal{N}(f\{e/x\}) \\ \mathcal{N}(f\{e/x\}) \# \mathcal{N}(f\{e/x\}) \end{array}$$

The side condition $y^+ \notin \mathcal{N}(f\{e/x\})$ holds because $y^+ \notin \mathcal{N}(f)$ and $\mathcal{N}(P) = \mathcal{N}(f) \cup \{y^-\}$ and $\mathcal{N}(f) \cup \{y^-\} \# \mathcal{N}(e)$. The side condition $\mathcal{N}(f\{e/x\}) \# \mathcal{N}(f\{e/x\})$ holds because $\mathcal{N}(P) \# \mathcal{N}(e)$ and $\mathcal{N}(e) \# \mathcal{N}(e)$.
    (Item 2). If $x^+ \notin \mathcal{N}(P)$ we can choose $Q = P\left[\,0\,/\,x \Leftarrow \texttt{return}\ e\,\right]$ by rule [WP-EMPTY] and Lemma B.4. Otherwise suppose $P = P_1 \mid P_2$ and the last rule of the derivation is:

$$[\text{WP-PAR}] \frac{\models P_1 \quad \models P_2}{\models P_1 \mid P_2} \quad \mathcal{N}(P_1) \setminus \{X^p\} \# \mathcal{N}(P_2) \setminus \{X^{\overline{p}}\} \tag{B.11}$$

We can assume $x \Leftarrow \texttt{return}\ e \subseteq_1 P_1$ since the case $x \Leftarrow \texttt{return}\ e \subseteq_1 P_2$ is symmetric. We distinguish three cases:
(1) Case $x^+ \notin \mathcal{N}(P_1)$. The key observation is that $X^p = x^-$ and $X^{\overline{p}} = x^+$. From the side condition of (B.11) and $\mathcal{N}(P_1) \setminus \{x^-\} \supseteq \mathcal{N}(e)$ and $x^+ \notin \mathcal{N}(e)$ we have $\mathcal{N}(P_2) \# \mathcal{N}(e)$. Since $\models x \Leftarrow \texttt{return}\ e$ implies $\mathcal{N}(e) \# \mathcal{N}(e)$, Item 1 gives $\models P_2\{e/x\}$. From this and $\mathcal{N}(P_1) \setminus \{x^-\} \# \mathcal{N}(P_2\{e/x\}) \setminus \mathcal{N}(e)$, it follows by Lemma B.4 that

$$\models P_1\left[\,P_2\{e/x\}\,/\,x \Leftarrow \texttt{return}\ e\,\right]$$

We can choose $Q = P_1\left[\,P_2\,/\,x \Leftarrow \texttt{return}\ e\,\right]$, since it is not difficult to check that

$$x \Leftarrow \texttt{return}\ e \mid P_1\left[\,P_2\,/\,x \Leftarrow \texttt{return}\ e\,\right] \equiv P_1 \mid P_2$$

(2) Case $x^+ \in \mathcal{N}(P_1)$ and $x^+ \in \mathcal{N}(P_2)$. By Lemma B.5 there are P_1' and P_2' such that $P_1 \mid P_2 \equiv P_1' \mid P_2'$ and $x \Leftarrow \texttt{return}\ e \subseteq_1 P_1'$ and $x^+ \notin \mathcal{N}(P_1')$. We can now proceed as in the previous case. Note that this case and the previous one are sort of "base cases" for which the induction hypothesis is not needed.
(3) Case $x^+ \notin \mathcal{N}(P_2)$. By induction hypothesis $\models Q_1\{e/x\}$ for some Q_1 such that $Q_1 \mid x \Leftarrow \texttt{return}\ e \equiv P_1$. We can apply rule [WP-PAR] to $\models Q_1\{e/x\}$ and $\models P_2$ since $\mathcal{N}(Q_1\{e/x\}) = \mathcal{N}(P_1) \setminus \{x^+, x^-\}$. So we conclude $\models Q_1\{e/x\} \mid P_2$.   □

    Since the definition of $\models$ is not invariant under $\equiv$, we cannot prove that the reduction preserves well-polarisation by induction on $\longrightarrow$. Instead, we use the following lemma, which immediately follows from the definition of $\longrightarrow$:

Lemma B.7 (Inversion of $\longrightarrow$). *If* $P \longrightarrow P'$, *then* $P \equiv (\nu X_1 \ldots X_n)P_0$ *and* $P' \equiv (\nu X_1 \ldots X_n)P_0'$ *and one of the following cases hold:*
(1) $P_0 = \texttt{server}\ a\ e \mid x \Leftarrow \mathcal{C}[\texttt{open}\ a] \mid Q$ *and*
    $P_0' = \texttt{server}\ a\ e \mid (\nu cy)(x \Leftarrow \mathcal{C}[\texttt{return}\ c^+] \mid y \Leftarrow e\ c^-) \mid Q$.

(2) $P_0 = x \Leftarrow \mathcal{C}[\text{send } a^p \ e] \mid y \Leftarrow \mathcal{C}'[\text{recv } a^{\overline{p}}] \mid Q$ and
 $P_0' = x \Leftarrow \mathcal{C}[\text{return } a^p] \mid y \Leftarrow \mathcal{C}'[\text{return } \langle e, a^{\overline{p}} \rangle] \mid Q$.
(3) $P_0 = x \Leftarrow \mathcal{C}[\text{future } e] \mid Q$ and $P_0' = (\nu y)(x \Leftarrow \mathcal{C}[\text{return } y] \mid y \Leftarrow e) \mid Q$.
(4) $P_0 = (\nu x)(x \Leftarrow \text{return } e \mid Q)$ and $P_0' = Q\{e/x\}$.
(5) $P_0 = x \Leftarrow e \mid Q$ and $P_0' = x \Leftarrow e' \mid Q$ with $e \longrightarrow e'$.

Proof of Theorem 5.9. Well-polarisation of P implies that

$$P \equiv (\nu X_1 \ldots X_n)(Q \mid R)$$

where $\{X_1, \ldots, X_n\} = \text{bounds}(P)$, $Q \equiv \text{threads}(P)$ and $\models Q$ and $R = \text{servers}(P)$. Using Lemma B.7, we analyse cases according to the shapes of P, Q and R. We only show the interesting cases.

(1) Case $x \Leftarrow \mathcal{C}[\text{open } a] \subseteq Q$ and $\text{server } a \ e \subseteq R$. Hence,

$$P' \equiv (\nu X_1 \ldots X_n cy)(Q[x \Leftarrow \mathcal{C}[\text{return } c^+] / x \Leftarrow \mathcal{C}[\text{open } a]] \mid y \Leftarrow e \ c^- \mid R)$$

It is easy to show that

$$\models Q[x \Leftarrow \mathcal{C}[\text{return } c^+] / x \Leftarrow \mathcal{C}[\text{open } a]]$$

Since P is typeable, $\mathcal{N}(e) = \emptyset$ and

$$\models y \Leftarrow ec^-$$

Using [WP-PAR], we obtain that

$$\models Q[x \Leftarrow \mathcal{C}[\text{return } c^+] / x \Leftarrow \mathcal{C}[\text{open } a]] \mid y \Leftarrow ec^-$$

Hence, P' is well-polarised.

(2) Case $Q \equiv x \Leftarrow \mathcal{C}[\text{send } a^p \ e] \mid y \Leftarrow \mathcal{C}'[\text{recv } a^{\overline{p}}] \mid Q_0$. Then,

$$P' \equiv (\nu X_1 \ldots X_n)(Q' \mid R)$$

where $Q' = x \Leftarrow \mathcal{C}[\text{return } a^p] \mid y \Leftarrow \mathcal{C}'[\text{return } \langle e, a^{\overline{p}} \rangle] \mid Q_0$. Typeability of P implies that a^p occurs only once and that the above threads are the unique ones named x and y in P. By Lemma B.3 there exists Q'' such that $Q'' \equiv Q'$ and $\models Q''$. Then P' is well-polarised.

(3) Case $X_n = x$ and $Q \equiv x \Leftarrow \text{return } e \mid Q_0$. Then

$$P' \equiv (\nu X_1 \ldots X_{n-1})(Q_0\{e/x\} \mid R)$$

Typeability of P implies that the above thread is the only one named x in P. It follows from Item 2 of Lemma B.6 that $\models Q_0'\{e/x\}$ for some $Q_0' \equiv Q_0$ and hence P' is well-polarised. $\square$

Appendix C. Proof of Theorem 5.15

Proof of Theorem 5.15. The proof is by induction on the definition of $\longrightarrow$. We only show the most interesting cases.

Case $(\nu x)(x \Leftarrow \text{return } e \mid P) \longrightarrow P\{e/x\}$.

Let P_1 and P_2 be such that $P \equiv P_1 \mid P_2$ and P_1 contains all and only the threads in whose bodies the variable x occur. It follows from $\Gamma \vdash (\nu x)(x \Leftarrow \text{return } e \mid P) \triangleright \Delta$ and the

Inversion Lemmas for Processes and Expressions (Items 1, 3 and 5 of Lemma 5.11 and Items 1 and 5 of Lemma 4.3) that

$$\Gamma_0 \vdash e : t \qquad \Gamma_1, x : t \vdash P_1 \,\triangleright\, \Delta_1 \qquad \Gamma_2 \vdash P_2 \,\triangleright\, \Delta_2$$

where $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$ and $\Delta = \Delta_1 + \Delta_2$. Since $(\nu x)(x \Leftarrow \mathtt{return}\ e \mid P)$ is well-polarised, if $y \Leftarrow f$ is in P_1 (i.e. x occurs in f), then y cannot occur in e by Item 2 of Lemma 5.8. Hence, $\mathsf{dom}(\Gamma_1) \cap \mathsf{dom}(\Delta_1) = \emptyset$. Then we can apply Lemma 5.12 to P_1 and obtain

$$\Gamma_0 + \Gamma_1 \vdash P_1\{e/x\} \,\triangleright\, \Delta_1$$

By rule [PAR] we derive

$$\Gamma_0 + \Gamma_1 + \Gamma_2 \vdash P_1\{e/x\} \mid P_2 \,\triangleright\, \Delta$$

Case $x \Leftarrow \mathcal{C}_1[\mathtt{send}\ a^p\ e] \mid y \Leftarrow \mathcal{C}_2[\mathtt{recv}\ a^{\overline{p}}] \longrightarrow x \Leftarrow \mathcal{C}_1[\mathtt{return}\ a^p] \mid y \Leftarrow \mathcal{C}_2[\mathtt{return}\ \langle e, a^{\overline{p}} \rangle].$

It follows from $\Gamma \vdash x \Leftarrow \mathcal{C}_1[\mathtt{send}\ a^p\ e] \mid y \Leftarrow \mathcal{C}_2[\mathtt{recv}\ a^{\overline{p}}] \,\triangleright\, \Delta$ and the Inversion Lemmas for Processes (Items 1 and 3 of Lemma 5.11) that $\Gamma = \Gamma_1 + \Gamma_2$ and $\Delta = x : \bullet^{n_1} t_1, y : \bullet^{n_2} t_2$ and

$$\Gamma_1 \vdash \mathcal{C}_1[\mathtt{send}\ a^p\ e] : \bullet^{n_1}(\mathtt{IO}\ t_1) \tag{C.1}$$

$$\Gamma_2 \vdash \mathcal{C}_2[\mathtt{recv}\ a^{\overline{p}}] : \bullet^{n_2}(\mathtt{IO}\ t_2) \tag{C.2}$$

Using the fact that Γ is balanced, it is not difficult to show that

$$a^p : \bullet^m(!t.T) \in \Gamma_1$$
$$a^{\overline{p}} : \bullet^m(?t.\overline{T}) \in \Gamma_2$$

for some m such that $m \le n_1$ and $m \le n_2$ by Lemma 5.14. By applying Lemma 5.13 to (C.1), we have $\Gamma_1 = \Gamma_3 + \Gamma_4, a^p : \bullet^m(!t.T)$ with

$$\Gamma_3, z : \bullet^{n_1}\mathtt{IO}\ T \vdash \mathcal{C}_1[z] : \bullet^{n_1}(\mathtt{IO}\ t_1)$$
$$\Gamma_4, a^p : \bullet^m(!t.T) \vdash \mathtt{send}\ a^p\ e : \bullet^{n_1}(\mathtt{IO}\ T) \tag{C.3}$$

Items 1, 2 and 5 of Lemma 4.3 give

$$\Gamma_4 \vdash e : \bullet^{n_1} t \tag{C.4}$$

Using rules [CONST], [AXIOM], [$\bullet$I], [$\to$E] being $m \le n_1$ we derive

$$a^p : \bullet^m T \vdash \mathtt{return}\ a^p : \bullet^{n_1}(\mathtt{IO}\ T) \tag{C.5}$$

By applying Lemma 4.4 to (C.3) and (C.5) we get

$$\Gamma_3, a^p : \bullet^m T \vdash \mathcal{C}_1[\mathtt{return}\ a^p] : \bullet^{n_1}(\mathtt{IO}\ t_1)$$

hence by [THREAD] we derive

$$\Gamma_3, a^p : \bullet^m T \vdash x \Leftarrow \mathcal{C}_1[\mathtt{return}\ a^p] \,\triangleright\, x : \bullet^{n_1} t_1 \tag{C.6}$$

By applying Lemma 5.13 to (C.2)

$$\Gamma_5, z : \bullet^{n_2}\mathtt{IO}\ (t \times \overline{T}) \vdash \mathcal{C}_2[z] : \bullet^{n_2}(\mathtt{IO}\ t_2)$$
$$a^{\overline{p}} : \bullet^m?t.\overline{T} \vdash \mathtt{recv}\ a^{\overline{p}} : \bullet^{n_2}\mathtt{IO}\ (t \times \overline{T}) \tag{C.7}$$

for $\Gamma_2 = \Gamma_5, a^{\overline{p}} : \bullet^m?t.\overline{T}$. From (C.4) and $m \le n_2$ using rules [CONST], [AXIOM], [$\bullet$I], [$\to$E] we derive

$$\Gamma_4, a^{\overline{p}} : \bullet^m\overline{T} \vdash \mathtt{return}\ \langle e, a^{\overline{p}} \rangle : \bullet^{n_2}\mathtt{IO}\ (t \times \overline{T}) \tag{C.8}$$

Applying Lemma 4.4 to (C.7) and (C.8) it follows that

$$\Gamma_4 + \Gamma_5, a^{\overline{p}} : \bullet^m\overline{T} \vdash \mathcal{C}_2[\mathtt{return}\ \langle e, a^{\overline{p}} \rangle] : \bullet^{n_2}(\mathtt{IO}\ t_2) \tag{C.9}$$

From well-polarisation and Item 3 of Lemma 5.8, y cannot occur in e. Then we can apply rule [THREAD] to (C.9) deriving

$$\Gamma_4 + \Gamma_5, a^{\overline{p}} : \bullet^m \overline{T} \vdash y \Leftarrow \mathcal{C}_2[\texttt{return } \langle e, a^{\overline{p}} \rangle] \; \triangleright \; y : \bullet^{n_2} t_2 \qquad (C.10)$$

By applying rule [PAR] to (C.6) and (C.10) we conclude

$$\Gamma_3 + \Gamma_4 + \Gamma_5, a^p : \bullet^m T, a^{\overline{p}} : \bullet^m \overline{T} \vdash x \Leftarrow \mathcal{C}_1[\texttt{return } a^p] \mid y \Leftarrow \mathcal{C}_2[\texttt{return } \langle e, a^{\overline{p}} \rangle] \; \triangleright \; \Delta$$

where $\Gamma_3 + \Gamma_4 + \Gamma_5, a^p : \bullet^m T, a^{\overline{p}} : \bullet^m \overline{T}$ is balanced. $\qquad \square$